On fibred biset functors with fibres of order prime and four

Nadia Romero

Mathematics Department, Bilkent University, Ankara, Turkey

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ABSTRACT

This note has two purposes: First, to present a counterexample to a conjecture parametrizing the simple modules over Green biset functors, appearing in an author’s previous article. This parametrization fails for the monomial Burnside ring over a cyclic group of order four. Second, to classify the simple modules for the monomial Burnside ring over a group of prime order, for which the above-mentioned parametrization holds.

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Introduction

This note presents a counterexample to a conjecture appearing in [5], parametrizing the simple modules over a Green biset functor. The conjecture generalized the classification of simple biset functors, as well as the classification of simple modules over Green functors appearing in Bouc [2]. It relied on the assumption that for a simple module over a Green biset functor its minimal groups should be isomorphic, which we will see is not generally true.

For a better understanding of this note, the reader is invited to take a look at [5], where he can acquaint himself with the context of modules over Green biset functors.

Given a Green biset functor \( A \), defined in a class of groups \( \mathcal{Z} \) closed under subquotients and direct products, and over a commutative ring with identity \( R \), one can define the category \( \mathcal{P}_A \). The objects of \( \mathcal{P}_A \) are the groups in \( \mathcal{Z} \), and given two groups \( G \) and \( H \) in \( \mathcal{Z} \), the set \( \text{Hom}_{\mathcal{P}_A}(G, H) \) is \( A(H \times G) \). Composition in \( \mathcal{P}_A \) is given through the product \( \times \) of the definition of a Green biset functor, that is, given \( \alpha \) in \( A(G \times H) \) and \( \beta \) in \( A(H \times K) \), the product \( \alpha \circ \beta \) is defined as

\[
A(\text{Def}_{G \times K}^{G \times \Delta(H) \times K} \circ \text{Res}_{G \times \Delta(H) \times K}^{G \times H \times K})(\alpha \times \beta).
\]
The identity element in $A(G \times G)$ is $A(\text{Ind}_{\Delta(G)}^{G \times G} \circ \text{Ind}_{\Delta(G)}^{A(G)}(\varepsilon_A))$, where $\varepsilon_A \in A(1)$ is the identity element of the definition of a Green biset functor. Even if this product may seem a bit strange, in many cases the category $P_A$ is already known and has been studied. For example, if $A$ is the Burnside ring functor, $P_A$ is the biset category defined in $Z$. It is proved in [5] that for any Green biset functor $A$, the category of $A$-modules is equivalent to the category of $R$-linear functors from $P_A$ to $R\text{-Mod}$, and it is through this equivalence that they are studied.

In Section 2 of [5], we defined $I_A(G)$ for a group $G$ in $Z$ as the submodule of $A(G \times G)$ generated by elements which can be factored through $\circ$ by groups in $Z$ of order smaller than $|G|$. We denote by $\hat{A}(G)$ the quotient $A(G \times G)/I_A(G)$. Conjecture 2.16 in [5] stated that the isomorphism classes of simple $A$-modules were in one-to-one correspondence with the equivalence classes of couples $(H, V)$ where $H$ is a group in $Z$ such that $\hat{A}(H) \neq 0$ and $V$ is a simple $\hat{A}(H)$-module. Two couples $(H, V)$ and $(G, W)$ are related if $H$ and $G$ are isomorphic and $V$ and $W$ are isomorphic as $\hat{A}(H)$-modules (the $\hat{A}(H)$-action on $W$ is defined in Section 4 of [5]). The correspondence assigned to the class of a simple $A$-module $S$, the class of the couple $(H, V)$ where $H$ is a minimal group for $S$ and $V = S(H)$. We will see in Section 2 that for the monomial Burnside ring over a cyclic group of order four and with coefficients in a field, we can find a simple module which has two non-isomorphic minimal groups.

For a finite abelian group $C$ and a finite group $G$, the monomial Burnside ring of $G$ with coefficients in $C$ is a particular case of the ring of monomial representations introduced by Dress [4]. Fibred biset functors were defined by Boltje and Coşkun as functors from the category in which the morphisms from a group $G$ to a group $H$ is the monomial Burnside ring of $H \times G$, they called these morphisms fibred bisets. This category is precisely $P_A$ when $A$ is the monomial Burnside ring functor, and so fibred biset functors coincide with $A$-modules for this functor. Boltje and Coşkun also considered the case in which $C$ may be an infinite abelian group, but we shall not consider this case. Unfortunately, there is no published material on the subject, I thank Laurence Barker and Olcay Coşkun for sharing this with me.

Another important element in this note will be the Yoneda–Dress construction of the Burnside ring functor $B$ at $C$, denoted by $B_C$. It assigns to a finite group $G$ the Burnside ring $B(G \times C)$, and it is a Green biset functor. Since the monomial Burnside ring of $G$ with coefficients in $C$ is a subgroup of $B_C(G)$, we will denote it by $B^1_C(G)$. We will see that there are various similarities between $B_C$ and $B^1_C$.

1. Definitions

All groups in this note will be finite.
$R$ will denote a commutative ring with identity.

Given a group $G$, we will denote its center by $Z(G)$. The Burnside ring of $G$ will be denoted by $B(G)$, and $RB(G)$ if it has coefficients in $R$.

Definition 1. Let $C$ be an abelian group and $G$ be any group. A finite $C$-free $(G \times C)$-set is called a $C$-fibred $G$-set.

A $C$-orbit of a $C$-fibred $G$-set is called a fibre.

The monomial Burnside ring for $G$ with coefficients in $C$, denoted by $B^1_C(G)$, is the abelian subgroup of $B(G \times C)$ generated by the $C$-fibred $G$-sets. We write $RB^1_C(G)$ if we are taking coefficients in $R$.

If $X$ is a $C$-fibred $G$-set, denote by $[X]$ its set of fibres. Then $G$ acts on $[X]$ and $X$ is $(G \times C)$-transitive if and only if $[X]$ is $G$-transitive. In this case, $[X]$ is isomorphic as $G$-set to $G/D$ for some $D \leq G$ and we can define a group homomorphism $\delta : D \to C$ such that if $D$ is the stabilizer of the orbit $C \alpha$, then $\alpha x = \delta(\alpha)x$ for all $\alpha \in D$. The subgroup $D$ and the morphism $\delta$ determine $X$, since $\text{Stab}_{G \times C}(x)$ is equal to $\{(\alpha, \delta(\alpha)^{-1}) \mid \alpha \in D\}$.
**Notation 2.** Given $D \leq G$ and $\delta : D \to C$ a group homomorphism, we will write $D_\delta$ for $\{(a, \delta(a)^{-1}) \mid a \in D\}$ and $CG/D_\delta$ for the C-fibre $G$-set $(G \times C)/D_\delta$. We will write $CG/D$ if $\delta$ is the trivial morphism. The morphism $\delta$ is called a $C$-subcharacter of $G$.

The $C$-subcharacters of $G$ admit an action of $G$ by conjugation $^g(D, \delta) = (^{gD}D, ^g\delta)$ and with this action we have:

**Remark 3.** (See 2.2 in Barker [1]) As an abelian group

$$B_1^C(G) = \bigoplus_{(D, \delta)} \mathbb{Z}[C\delta G/D]$$

where $(D, \delta)$ runs over a set of representatives of the $G$-classes of $C$-subcharacters of $G$.

The following notations are explained in more detail in Bouc [3]. Given $U$ an $(H, G)$-biset and $V$ a $(K, H)$-biset, the composition of $V$ and $U$ is denoted by $V \circ H U$. With this composition we know that if $H$ and $G$ are groups and $L \leq H \times G$, then the corresponding element in $RB(H \times G)$ satisfies the Bouc decomposition (2.3.26 in [3]):

$$\text{Ind}_D^H \times \text{Ind}_D^C \times \text{Ind}_D^{D/C} \text{Iso}(f) \times B/A \text{ Def}_{B/A}^B \times B \text{ Res}_B^C$$

with $C \leq D \leq H$, $A \leq B \leq G$ and $f : B/A \to D/C$ an isomorphism.

**Notation 4.** As it is done in [5], we will write $B_C$ for the Yoneda–Dress construction of the Burnside ring functor $B$ at $C$.

The functor $B_C$ is defined as follows. In objects, it sends a group $G$ to $B(G \times C)$. In arrows, for a $(G, H)$-biset $X$, the map $B_C(X) : B_C(H) \to B_C(G)$ is the linear extension of the correspondence $T \mapsto X \times_H T$, where $T$ is an $(H \times G)$-set and $X \times_H T$ has the natural action of $(G \times C)$-set coming from the action of $C$ on $T$.

We will denote by $T_{C-f}$ the subset of elements of $T$ in which $C$ acts freely. Clearly, it is an $H$-set.

**Lemma 5.** Assigning to each group $G$ the $\mathbb{Z}$-module $B_1^C(G)$ defines a Green biset functor.

**Proof.** We first prove it is a biset functor.

Let $G$ and $H$ be groups and $X$ be a finite $(G, H)$-biset. Let $T$ be a $C$-fibred $H$-set. We define $B_1^C(X)(T) = (B_C(X)(T))_{C-f}$.

To prove that composition is associative, let $Z$ be a $(K, G)$-biset. We must show

$$\left((Z \times_G X) \times_H T\right)_{C-f} \cong \left((Z \times_G (X \times_H T))_{C-f}\right)_{C-f}.$$

We claim that the right-hand side of this isomorphism is equal to $(Z \times_G (X \times_H T))_{C-f}$. To prove it, we prove that in general, if $W$ is a $(G \times C)$-set, then $(Z \times_G W)_{C-f}$ is equal to $(Z \times_G W)_{C-f}$. Let $[z, w]$ be an element in $(Z \times_G W)_{C-f}$. The element $[z, w]$ is an orbit for which any representative has the form $(zg^{-1}, gw)$ with $g \in G$. To prove that $gw$ is in $W_{C-f}$, suppose $cgw = gw$. Then, $[z, w] = [z, cw]$ and this is equal to $[z, w]$, so $c = 1$. The other inclusion is obvious.

It remains then to prove

$$\left((Z \times_G X) \times_H T\right)_{C-f} \cong \left((Z \times_G (X \times_H T))_{C-f}\right),$$

as $(K \times C)$-sets, which holds because $B_C$ is a biset functor.
Next we prove it is a Green biset functor.
Following Dress [4], we define the product

$$B^1_C(G) \times B^1_C(H) \to B^1_C(G \times H)$$

on the $C$-fibre $G$-set $T$ and the $C$-fibre $H$-set $Y$ as the set of $C$-orbits of $T \times Y$ with respect to the action $c(t, y) = (ct, c^{-1}y)$. The orbit of $(t, y)$ is denoted by $t \otimes y$. We extend this product by linearity and denote it by $T \otimes Y$. The action of $C$ in $t \otimes y$ is given by $ct \otimes y$ and so it is easy to see that $C$ acts freely on $T \otimes Y$. The identity element in $B^1_C(1)$ is the class of $C$. It is not hard to see that this product is associative and respects the identity element. To prove it is functorial, take $X$ a $(K, H)$-biset and $Z$ an $(L, G)$-biset. We must show that

$$((Z \times G T)_{C-f} \otimes (X \times_H Y)_{C-f}) \cong ((Z \times X) \times_G (T \otimes Y))_{C-f}$$

as $(K \times L \times C)$-sets. We can prove this in two steps: First, it is easy to observe that for any $C$-sets $N$ and $M$, the product $M_{C-f} \otimes N_{C-f}$ is isomorphic as $C$-set to $(M \otimes N)_{C-f}$. Then it remains to prove

$$(Z \times_G T) \otimes (X \times_H Y) \cong (Z \times X) \times_{G \times H} (T \otimes Y)$$

as $(K \times L \times C)$-sets. If $[z, t] \otimes [x, y]$ is an element on the left-hand side, then sending it to $[(z, x), t \otimes y]$ defines the desired isomorphism of $(K \times L \times C)$-sets. □

2. Fibred biset functors

The category $\mathcal{PP}_{RB^1_C}$, mentioned in the introduction and defined in Section 4 of [5], has for objects the class of all finite groups; the set of morphisms from $G$ to $H$ is the abelian group $RB^1_C(G \times G)$ and composition is given in the following way: If $T \in RB^1_C(G \times H)$ and $Y \in RB^1_C(H \times K)$, then $T \circ Y$ is given by restricting $T \otimes Y$ to $G \times \Delta(H) \times K$ and then deflating the result to $G \times K$. The identity element in $RB^1_C(G \times G)$ is the class of $G \times G)/\Delta(G)$. As it is done in Section 4.2 of [5], composition $\circ$ can be obtained by first taking the orbits of $T \times Y$ under the $(H \times C)$-action given by

$$(h, c)(t, y) = ((h, c)t, (h, c^{-1})y),$$

and then choosing the orbits in which $C$ acts freely.

Definition 6. From Proposition 2.11 in [5], the category of $RB^1_C$-modules is equivalent to the category of $R$-linear functors from $\mathcal{PP}_{RB^1_C}$ to $R$-Mod. These functors are called fibred biset functors.

Notation 7. Let $E$ be a subgroup of $H \times K \times C$. We will write $p_1(E)$, $p_2(E)$ and $p_3(E)$ for the projections of $E$ in $H$, $K$ and $C$ respectively; $p_{1,2}(E)$ will denote the projection over $H \times K$, and in the same way we define the other possible combinations of indices. We write $k_{ij}(E)$ for $\{h \in p_{ij}(E) | (h, 1, 1) \in E\}$. Similarly, we define $k_{2}(E)$, $k_{3}(E)$ and $k_{i,j}(E)$ for all possible combinations of $i$ and $j$.

The following formula was already known to Boltje and Coşkun. Here we prove it as an explicit expression of composition $\circ$ in the category $\mathcal{PP}_{RB^1_C}$. The proof follows the lines of Lemma 4.5 in [5].

The definition of the product $\ast$ can be found in Notation 2.19 of [3].

Lemma 8. Let $X = [C,V](G \times H)/\{x\} \in RB^1_C(G \times H)$ and $Y = [C,h](H \times K)/\{y\} \in RB^1_C(H \times K)$ be two transitive elements. Then the composition $X \circ Y \in RB^1_C(G \times K)$ in the category $\mathcal{PP}_{RB^1_C}$ is isomorphic to

$$\bigsqcup_{h \in S} C_{\nu,h}(G \times K)/(V \ast (h,1)U).$$
The notation is as follows: Let \([p_2(V) \setminus H/p_1(U)]\) be a set of representatives of the double cosets of \(p_2(V)\) and \(p_1(U)\) in \(H\), then \(S\) is the subset of elements \(h\) in \([p_2(V) \setminus H/p_1(U)]\) such that \(\nu(1, h')\mu(h'^h, 1) = 1\) for all \(h' \in k_2(V) \cap h'k_1(U)\); by \(\nu \mu^h\) we mean the morphism from \(V \ast (h, 1)U\) to \(C\) defined by \(\nu \mu^h(\gamma, k) = \nu(\gamma)\mu(h'^h, k)\) when \(h_1\) is an element in \(H\) such that \((g, h_1)\) in \(V\) and \((h_1, k)\) in \((h, 1)U\).

**Proof.** Notice that \(\nu \mu^h\) is a function if and only if \(\nu(1, h')\mu(h'^h, 1) = 1\) for all \(h' \in k_2(V) \cap h'k_1(U)\).

Let \(W\) be the \((G \times K \times C)\)-set obtained by taking the orbits of \(X \times Y\) under the action of \(H \times C\)

\[
(h, c)(x, y) = ((h, c)x, (h, c^{-1})y),
\]

for all \(c \in C, h \in H, x \in X, y \in Y\).

Now let \([(g, h, c)V_{\nu}, (h', k, c')U_{\mu}]\) be an element in \(W\). Then its orbit under the action of \(G \times K \times C\) is equal to the orbit of \([(1, 1)\nu, (h^{-1}h', 1, 1)U_{\mu}]\). From this it is not hard to see that the orbits of \(W\) are indexed by \([p_2(V) \setminus H/p_1(U)]\). To find the orbits in which \(C\) acts freely, suppose \(c \in C\) fixes \([(1, 1)\nu, (h, 1, 1)U_{\mu}]\). This means there exists \((h', c') \in H \times C\) such that

\[
(1, 1, c)V_{\nu} = (h', 1, c')V_{\nu} \quad \text{and} \quad (h, 1, 1)U_{\mu} = (h'h, 1, c^{-1}')U_{\mu}.
\]

Hence \(\nu(h', 1) = c^{-1}'c\) and \(\mu(h^{-1}h'h, 1) = c'.\) So that, \(c\) is equal to \(\mu(h^{-1}h'h, 1)\nu(h', 1),\) which gives us the condition on the set \(S.\)

The fact that the stabilizer on \(G \times K \times C\) of \([(1, 1, 1)\nu, (h, 1, 1)U_{\mu}]\) is the subgroup \((V \ast (h, 1)U)_{\nu \mu^h}\) follows as in the previous paragraph. □

The following lemma and corollary state for \(RB^*_C\) analogous results proved for \(RB_C\) in [5].

**Lemma 9.** Let \(X = C_\delta(G \times H)/D\) be a transitive element in \(RB^*_C(G \times H).\) Denote by \(e\) the natural transformation from \(RB\) to \(RB^*_C\) defined in a \(G\)-set \(X\) by \(e_G(X) = X \times C.\) Consider \(E = p_1(D), E' = E/k_1(D_\delta), F = p_2(D), F' = F/k_2(D_\delta).\) Then \(X\) can be decomposed in \(\mathcal{P}_{RB^*_C}\) as

\[
e_G \times E'((\text{Ind}_E^G \times E \text{ Res}_E^F) \circ \beta_1) \quad \text{and as} \quad \beta_2 \circ e_{F \times H}(\text{Def}_E^G \times F \text{ Res}_E^F)
\]

for some \(\beta_1 \in RB^*_C(E' \times H), \beta_2 \in RB^*_C(G \times F').\)

**Proof.** We will only prove the existence of the first decomposition, since the proof of the second one follows by analogy.

Observe that \(e_G \times E((\text{Ind}_E^G \times E \text{ Res}_E^F)\circ \beta_1)\) is the \(G\)-fibred \((G \times E')\)-set \(C(G \times E')/U\) where \(U = \{(g, gk_1(V_\delta)) \mid g \in E\}\).

Consider the isomorphism \(\sigma\) from \(p_1(D)/k_1(D)\) to \(p_2(D)/k_2(D)\), existing by Goursat’s Lemma 2.3.25 in [3]. Define \(\beta_1\) as \(C_\omega(E' \times H)/W\) where

\[
W = \{(gk_1(D_\delta), h) \mid \text{if } \sigma(gk_1(D)) = hk_2(D)\}
\]

and \(\omega : W \to C\) by \(\omega(gk_1(D_\delta), h) = \delta(g, h).\) That \(W\) is a group follows from \(k_1(D_\delta) \leq k_1(D)\). The extension of \(\delta\) to \(W\) is well defined, since it is not hard to see that \(k_1(D_\delta)\) is equal to \(k_1(\text{Ker}(\delta))\).

Also, since \(p_2(U) = p_1(W) = E'\) and \(k_2(U) = 1\), by the previous lemma, \(e_G \times E((\text{Ind}_E^G \times E \text{ Res}_E^F) \circ \beta_1)\) is isomorphic to \(C_\delta(G \times H)/(U \ast W)\). Finally, \(U \ast W = \{(g, h) \mid \sigma(gk_1(D)) = hk_2(D)\}\), and by Goursat’s Lemma, this is equal to \(D\). □

This decomposition leads us to the same conclusions we obtained from Lemma 4.8 of [5] for \(RB_C\). That is, if \(G\) and \(H\) have the same order \(n\) and \(C_\delta(G \times H)/D\) does not factor through \(\circ\) by a group.
of order smaller than \( n \), then we must have \( p_1(D) = G \), \( p_2(D) = H \), \( k_1(D_δ) = 1 \) and \( k_2(D_δ) = 1 \). In particular, Corollary 4.9 of the same reference is also valid, so we have:

**Corollary 10.** Let \( C \) be a group of prime order and \( S \) be a simple \( RB^1_C \)-module. If \( H \) and \( K \) are two minimal groups for \( S \), then they are isomorphic.

We will be back to the classification of simple \( RB^1_C \)-modules for \( C \) of prime order in the last section of the article. Now, we will find the counterexample mentioned in the introduction.

**2.1. The counterexample**

In Section 2 of [5], given a Green biset functor \( A \) defined in a class of groups \( Z \), we defined \( I_A(G) \) as the submodule of \( A(G \times G) \) generated by elements of the form \( a \circ b \), where \( a \) is in \( A(G \times K) \), \( b \) is in \( A(K \times G) \) and \( K \) is a group in \( Z \) of order smaller than \( |G| \). We denote by \( \hat{A}(G) \) the quotient \( A(G \times G)/I_A(G) \). From Section 4 of [5], we also know that if \( V \) is a simple \( \hat{A}(G) \)-module, we can construct a simple \( A \)-module that has \( G \) as a minimal group. This \( A \)-module is defined as the quotient \( L_{G/V} / J_{G/V} \), where \( L_{G/V} \) is defined as \( A(D \times G) \otimes_{A(G \times G)} V \) for \( D \in Z \) and \( L_{G/V}(a)(x \otimes v) = (a \circ x) \otimes v \) for \( a \in A(D' \times D) \). The subfunctor \( J_{G/V} \) is defined as

\[
J_{G/V}(G) = \left\{ \sum_{i=1}^{n} x_i \otimes n_i \mid \sum_{i=1}^{n} (y \circ x_i) \cdot n_i = 0 \forall y \in A(G \times D) \right\}.
\]

To construct the counterexample we will take coefficients in a field \( k \). We will find a group \( C \) and a simple \( kB^1_C \)-module \( S \) which has two non-isomorphic minimal groups.

**Lemma 11.** Let \( C \) be a cyclic group and \( G \) and \( H \) be groups. Suppose that \( D \leq G \times H \) is such that \( p_1(D) = G \) and \( p_2(D) = H \). Let \( \delta : D \to C \) be a morphism of groups. We will write \( D^0 = \{(h, g) \mid (g, h) \in D\} \) and define \( \delta^0 : D^0 \to C \) as \( \delta^0(h, g) = \delta(g, h)^{-1} \). If \( X = C_\delta(G \times H)/D \) and \( X^0 = C_{\delta^0}(H \times G)/D^0 \), then \( X \circ X^0 \) is an idempotent in \( B^1_C(G \times G) \).

**Proof.** Since \( \delta(1, h)\delta^0(h, 1) = 1 \) for all \( h \in k_2(D) \), by Lemma 8 the composition \( X \circ X^0 \) is equal to \( W = C_\delta(G \times G)/D' \). Here, \( D' = D \ast D^0 \) and if \( (g_1, g_2) \in D' \) with \( h \in H \) being such that \( (g_1, h) \in D \) and \( (h, g_2) \in D^0 \), then \( \delta'(g_1, g_2) = \delta(g_1, h)\delta^0(h, g_2) \). From this it is not hard to see that \( D' = \{(g_1, g_2) \mid g_1g_2^{-1} \in k_1(D)\} \) and \( \delta'(g_1, g_2) = \delta(g_1g_2^{-1}, 1) \).

Observe that \( k_1(D') = k_2(D') = k_1(D) \) and clearly, \( \delta'(1, g)\delta'(g, 1) = 1 \) for all \( g \in k_1(D) \). In the same way, if \( g_1, g_2 \in G \) are such that there exists \( g \in G \) with \( (g_1, g) \in D' \) and \( (g, g_2) \in D' \) then \( \delta'(g_1, g)\delta'(g, g_2) = \delta(g_1g_2^{-1}, 1) \). Finally, \( p_1(D') = G \) since \( gg^{-1} \in k_1(D) \) for all \( g \in G \), and it is easy to see that \( D' \ast D' = D' \). So, Lemma 8 gives us \( W \circ W = W \). □

If now we find two non-isomorphic groups \( G \) and \( H \) having the same order, and a transitive element \( X = C_\delta(G \times H)/D \) in \( kB^1_C(G \times G) \) with \( p_1(D) = G \), \( p_2(D) = H \) and such that the class of \( W = X \circ X^0 \) is different from zero in \( kB^1_C(G) \), then we can construct a simple \( kB^1_C \)-module \( S \) which has \( G \) and \( H \) as minimal groups. By the previous lemma, \( W \) will be an idempotent in \( kB^1_C(G) \), so we can find \( V \) a simple \( kB^1_C(G) \)-module such that there exists \( v \in V \) with \( (X \circ X^0)v \neq 0 \). From the definition of \( S = S_{G/V} \), this implies \( S_{G/V}(H) \neq 0 \).

**Example 12.** Let \( C = \langle c \rangle \) be a group of order 4, \( G \) the quaternion group

\[
\langle x, y \mid x^4 = 1, \ xyx^{-1} = x^{-1}, \ x^2 = y^2 \rangle
\]
and \( H \) the dihedral group of order 8

\[
\langle a, b \mid a^4 = b^2 = 1, \ bab^{-1} = a^{-1} \rangle.
\]

Consider the subgroup of \( G \times H \) generated by \((x, a)\) and \((y, b)\), call it \( D \). The subgroup of \( D \) generated by \((x^{-1}, a)\) is a normal subgroup of order 4, and the quotient \( D/D_1 \) is isomorphic to \( C \) in such a way that we can define a morphism \( \delta : D \to C \) sending \((x, a)\) to \( c^2 \) and \((y, b)\) to \( c^{-1} \). It is easy to observe that \( p_1(D) = G \), \( p_2(D) = H \), \( k_1(D) = (x^2) \) and \( k_2(D) = (a^2) \). By the previous lemma, we have that \( X = \mathcal{C}_\delta(G \times H)/D \), then \( W = X \circ X^0 \) is an idempotent in \( k\mathcal{B}_1^1(G \times G) \). We will see now that the class of \( W \) in \( k\mathcal{B}_1^1(G) \) is different from 0.

Let \( D' = D \ast D^0 \) and \( \delta' : D' \to C \) be the morphism obtained from \( \delta \) as in the previous lemma. Suppose that \( W \) is in \( I_{k\mathcal{B}_1^1}(G) \). Since \( W \) is a transitive \((G \times G \times C)\)-set, this implies that there exists \( K \) a group of order smaller than 8, \( U \leq G \times K \) and \( V \leq K \times G \) such that \( D' = U \ast V \) (the conjugate of a group of the form \( U \ast V \) has again this form, so we can suppose \( D' = U \ast V \)), and group homomorphisms \( \mu : U \to C \) and \( v : V \to C \) such that \( \delta' = \mu v \) in the sense of Lemma 8.

Now, using point \( 2 \) of Lemma 2.3.22 in [3] and the fact that \( p_1(D') = p_1(D) \) and \( k_1(D') = k_1(D) \), we have that \( p_1(U) = G \) and \( k_1(U) \) can only have order one or two. Since \( p_1(U)/k_1(U) \) is isomorphic to \( p_2(U)/k_2(U) \) and the latter must have order smaller than 8, we obtain that \( k_1(U) \) has order two. This in turn implies that \( p_2(U)/k_2(U) \) has order 4, and since \( |p_2(U)| < 8 \), we have \( k_2(U) = 1 \). Hence, \( U \) is isomorphic to \( G \). Also, since \( k_1(U) = k_1(D') \), we have \( \mu(x^2, 1) = \delta(x^2, 1) \). Now, \( \delta(x^2, 1) \neq 1 \), but all morphisms from \( G \) to \( C \) send \( x^2 \) to 1, a contradiction.

### 2.2. Simple fibred biset functors with fibre of prime order

From now on \( C \) will be a group of prime order \( p \).

From Corollary 10, we have that Conjecture 2.16 of [5] holds for the functor \( R\mathcal{B}^1_0 \), the proof is a particular case of Proposition 4.2 in [5]. We will state this result after describing the structure of the algebra \( R\mathcal{B}^1_0(G) \) for a group \( G \).

We will see that if \( \mathcal{C}_\delta(G \times G)/D \) is a transitive \( C \)-fibred \((G \times G)\)-set the class of which is different from 0 in \( R\mathcal{B}^1_0(G) \), then \( D \) can only be of the form \( \{\sigma(g), g\} \mid g \in G \) for \( \sigma \) an automorphism of \( G \), or of the form \( \{(\omega(g)\zeta(c), g) \mid (g, c) \in G \times \mathcal{C}\} \) for \( \omega \) an automorphism of \( G \) and \( \zeta : C \to Z(G) \cap \Phi(G) \) an injective morphism of groups where \( \Phi(G) \) is the Frattini subgroup of \( G \). In the first case \( \delta \) will be any morphism from \( G \) to \( C \). In the second case \( \delta \) will assign \( c^{-1} \) to the couple \((\omega(g)\zeta(c), g), \) this is well defined since \( \zeta \) is injective. Of course, the second case can only occur if \( p \) divides \( |Z(G)| \).

If \( p \) does not divide \( |Z(G)| \), we will prove that \( R\mathcal{B}^1_0(G) \) is isomorphic to the group algebra \( \hat{R} \mathcal{G} \) where \( \hat{G} = \text{Hom}(G, C) \ast \text{Out}(G) \). If \( p \) divides \( |Z(G)| \), we will consider \( Y_G \) the set of injective morphisms \( \zeta : C \to Z(G) \cap \Phi(G) \) and then define \( \mathcal{Y}_G = \text{Out}(G) \times Y_G \). The \( R \)-module \( R\mathcal{Y}_G \) forms an \( R \)-algebra with the product

\[
(\omega, \zeta) \circ (\alpha, \chi) = \begin{cases} 
(\omega\alpha, \omega\chi) & \text{if } \zeta = \omega\chi, \\
0 & \text{otherwise}
\end{cases}
\]

for elements \( (\omega, \zeta) \) and \( (\alpha, \chi) \) in \( \mathcal{Y}_G \). The algebra \( R\mathcal{Y}_G \) can also be made into an \((\hat{R} \mathcal{G}, \hat{R} \mathcal{G})\)-bimodule. We could give the definitions of the actions now, and prove directly that \( R\mathcal{Y}_G \) is indeed an \((\hat{R} \mathcal{G}, \hat{R} \mathcal{G})\)-bimodule. Nonetheless, the nature of these actions is given by the structure of \( R\mathcal{B}^1_0(G) \), so they are best understood in the proof of the following lemma. The \( R \)-module \( R\mathcal{Y}_G \otimes R\hat{G} \) forms then an \( R \)-algebra.

Now suppose that \( G \) and \( H \) are two groups such that there exists an isomorphism \( \varphi : G \to H \). If \((t, \sigma)\) is a generator of \( \hat{G} \), then identifying \( \varphi\sigma\varphi^{-1} \) with its class in \( \text{Out}(H) \) we have that \((t\varphi^{-1}, \varphi\sigma\varphi^{-1}) \) is in \( RH \). On the other hand, if \((\omega, \zeta)\) is a generator in \( R\mathcal{Y}_G \), then \((\varphi\omega\varphi^{-1}, \varphi|Z(G)\zeta)\) is also in \( R\mathcal{Y}_H \).
Notation 13. Let $\mathcal{H}(G)$ be the group algebra $R\hat{G}$ if $p$ does not divide $|Z(G)|$ and $R\mathcal{Y}_C \oplus R\hat{G}$ in the other case.

We will write $\text{Seed}$ for the set of equivalence classes of couples $(G, V)$ where $G$ is a group and $V$ is a simple $\mathcal{H}(G)$-module. Two couples $(G, V)$ and $(H, W)$ are related if $G$ and $H$ are isomorphic, through an isomorphism $\varphi : G \to H$, and $V$ is isomorphic to $\varphi W$ as $\mathcal{H}(G)$-modules. Here $\varphi W$ denotes the $\mathcal{H}(G)$-module with action given through the elements defined in the previous paragraph.

With these observations, Proposition 4.2 in [5] can be written as follows.

Proposition 14. Let $S$ be the set of isomorphism classes of simple $RB_C^1$-modules. Then the elements of $S$ are in one-to-one correspondence with the elements of $\text{Seed}$ in the following way: Given $S$ a simple $RB_C^1$-module we associate to its isomorphism class the equivalence class of $(G, V)$ where $G$ is a minimal group of $S$ and $V = S(G)$. Given the class of a couple $(G, V)$, we associate the isomorphism class of the functor $S_{G,V}$ defined in the previous section.

It only remains to see that the algebra $RB_C^1(G)$ is isomorphic to $\mathcal{H}(G)$.

Lemma 15.

i) If $p$ does not divide $|Z(G)|$, then $RB_C^1(G)$ is isomorphic to the group algebra $R\hat{G}$.

ii) If $p$ divides $|Z(G)|$, then $RB_C^1(G)$ is isomorphic to $R\mathcal{Y}_C \oplus R\hat{G}$ as $R$-algebras.

Proof. Let $C_S(G \times G)/D$ be a transitive $C$-fibre $(G \times G)$-set the class of which is different from 0 in $RB_C^1(G)$. From Lemma 9 we have that $D_S$ must satisfy $p_1(D_S) = p_2(D_S) = G$ and $k_1(D_S) = k_2(D_S) = 1$.

Also, since $\delta$ is a function, we have that $k_3(D_S) = 1$. Goursat’s Lemma then implies that $D_S$ is isomorphic to $p_{2,3}(D_S)$, also isomorphic to $p_{1,3}(D_S)$. Since $C$ has prime order, we have two choices for $p_{2,3}(D_S)$, either it is of the form $G \times C$ or of the form $\{(g, t(g)) \mid g \in G, t : G \to C\}$, for some group homomorphism $t$.

By Goursat’s Lemma, if $p_{2,3}(D_S)$ is equal to $G \times C$, then

$$D_S = \{(\alpha(g, c), g, c) \mid (g, c) \in G \times C, \alpha : G \times C \to G\}$$

with $\alpha$ an epimorphism of groups. Since $k_2(D_S) = k_3(D_S) = 1$, we have that $\alpha(g, c) = \omega(g)\xi(c)$ with $\omega$ an automorphism of $G$ and $\xi$ and injective morphism from $C$ to $Z(G)$. In particular, if $p$ does not divide the order of $Z(G)$, then this case cannot occur.

Suppose that $p_{2,3}(D_S) = \{(g, t(g)) \mid g \in G, t : G \to C\}$, for a group homomorphism $t$. Goursat’s Lemma implies that there exists $\sigma$ an automorphism of $G$ such that $D_S = \{(\sigma(g), g, t(g)) \mid g \in G\}$. Hence $D = \Delta_\sigma(G)$ and $\delta(g_1, g_2) = t(g_2^{-1})$. We will then replace $\delta$ by $t$ and write $\Delta_{t,\sigma}$ for $C_S(G \times G)/D$ in this case. The isomorphism classes of these elements in $RB_C^1(G)$ form an $R$-basis for it, since Lemma 2.3.22 in [3] and Goursat’s Lemma imply that $\Delta_{t,\sigma}(G)$ cannot be written as $M \times N$ for any $M \leq G \times K$ and $N \leq K \times G$ with $K$ of order smaller than $|G|$. Let us see that we have a bijective correspondence between the basic elements $\{X_{t,\sigma}\}$ of $RB_C^1(G)$ and $\text{Hom}(G, C) \times \text{Out}(G)$. Any representative of the isomorphism class of $X_{t,\sigma}$ is of the form $X_{tc_1^{-1},c_2}\sigma_c$ where $c_1$ denotes the conjugation by some $g_1 \in G$ and $c_2^{-1}$ denotes the conjugation by some $g_2^{-1} \in G$. Since $C$ is abelian, $t^{-1}$ is equal to $t$, and the class of $\sigma$ in $\text{Out}(G)$ is the same as the class of $c_1\sigma c_2^{-1}$. On the other hand, if we take $\sigma_c g$ any representative of the class of an automorphism $\sigma$ in $\text{Out}(G)$, then $X_{t,\sigma} \cong X_{t,\sigma_c g}$.

It remains to see that this bijection is a morphism of rings. Using Lemma 8 it is easy to see that

$$X_{t_1,\sigma_1} \circ X_{t_2,\sigma_2} = X_{(t_1 \circ \sigma_2)t_2,\sigma_1\sigma_2}$$

and the product in $\hat{G}$ is precisely $(t_1, \sigma_1)(t_2, \sigma_2) = ((t_1 \circ \sigma_2)t_2, \sigma_1\sigma_2)$. 


This proves point i). From now on, we suppose that \( p \) divides \(|Z(G)|\).

As we said before, if \( p \) divides \(|Z(G)|\), then we can consider the case of \( C \)-fibred \((G \times G)\)-sets \( C_\delta(G \times G)/D \) such that \( p_{2,3}(D_\delta) = G \times C \). In this case, \( D_\delta \) equals

\[
\{ (\omega(g)\zeta(c), g, c) \mid (g, c) \in G \times C \}
\]

where \( \omega \) is an automorphism of \( G \) and \( \zeta \) is an injective morphism from \( C \) to \( Z(G) \). We will prove that the class of \( C_\delta(G \times G)/D \) in \( \hat{RB}^1_G(G) \) is different from 0 if and only if \( \text{Im} \zeta \subseteq Z(G) \cap \Phi(G) \), and we will write \( Y_{\omega,\zeta} \) for \( C_\delta(G \times G)/D \) in this case. The claim will be proved in two steps, first let us prove that the class of \( Y_{\omega,\zeta} \) in \( \hat{RB}^1_G(G) \) is different from 0 if and only if \( \mu|Z(G) \circ \zeta = 1 \) for every group homomorphism \( \mu : G \to C \). From Lemma 2.3.22 of [3] it is easy to see that \( D = \{(\omega(g)\zeta(c), g, c') \mid (g, c) \in G \times C \} \) is equal to \( M \times N \) for some \( M \subseteq G \times K \) and \( N \subseteq K \times G \) with \( K \) a group of order smaller than \(|G|\) if and only if \( K \) has order \(|G|/p \) and \( M \) and \( N \) are isomorphic to \( G \). Suppose now that there exist \( \mu : G \to C \) and \( \nu : G \to C \) such that \( \delta(g_1, g_2) = \mu(g_1)\nu(g_2) \), then in particular for every \( c \in C \),

\[
d(\zeta(c), 1) = c^{-1} = \mu(\zeta(c)).
\]

Conversely, if there exists \( \mu : G \to C \) such that \( \mu|Z(G) \circ \zeta \neq 1 \), then we can find \( \mu' : G \to C \) such that \( \mu'\zeta(c) = c^{-1} \) for all \( c \neq 1 \), and define \( \nu : G \to C \) as \( \nu(g) = \mu'(g)^{-1} \). So we have \( \mu'(\omega(g)\zeta(c))\nu(g) = c^{-1} \), which is equal to \( \delta(\omega(g)\zeta(c), g) \).

Now we prove that for \( \zeta : C \to Z(G) \), we have \( \text{Im} \zeta \subseteq \Phi(G) \) if and only if \( \mu|Z(G) \circ \zeta = 1 \) for every group homomorphism \( \mu : G \to C \) (thanks to the referee for this observation). Suppose \( \text{Im} \zeta \subseteq \Phi(G) \) and let \( \mu : G \to C \) be a morphism of groups. If there exists \( c \in C \) such that \( \mu\zeta(c) \neq 1 \), then \( \ker \mu \) is a normal subgroup of \( G \) of index \( p \) and so it is maximal. But clearly \( \zeta(c) \notin \ker \mu \), which is a contradiction. Now suppose that for all \( \mu : G \to C \) we have \( \mu \circ \zeta|Z(G) \neq 1 \). Let \( M \) be a maximal subgroup of \( G \) and \( c \) be a non-trivial element of \( \text{Im} \zeta = C' \). If \( c \notin M \), then \( C' \cap M = 1 \), and since \( C' \leq Z(G) \), we have that \( C'M \) is a subgroup of \( G \). Since \( M \) is maximal, \( G = C'M \). But this means that there exists \( \mu : G \to C \) such that \( \mu(c) \neq 1 \), a contradiction.

In a similar way as it is done in point i), we have a bijective correspondence between the isomorphism classes of elements \( Y_{\omega,\zeta} \) in \( \hat{RB}^1_G(G) \) and \( \hat{R}Y_G \). This establishes an isomorphism of \( R \)-modules between \( \hat{RB}^1_G(G) \) and \( \hat{R}Y_G \oplus \hat{R} \hat{G} \). Now we describe the algebra structure. The following calculations are made using Lemma 8, Lemma 9 and Lemma 2.3.22 in [3].

The composition of elements \( Y_{\omega,\zeta} \) is given by

\[
Y_{\omega,\zeta} \circ Y_{\alpha,\chi} = \begin{cases} Y_{\omega\alpha,\omega\chi} & \text{if } \zeta = \omega \chi, \\ 0 & \text{otherwise}. \end{cases}
\]

The product \( X_{t,\sigma} \circ Y_{\omega,\zeta} \) is different from 0 if and only if \( t\zeta(c)c \neq 1 \) for all \( c \neq 1 \). Then, if we let \( \text{Id}_C \) be the identity morphism of \( C \), we have that \((t\zeta)\text{Id}_C \) defines an automorphism on \( C \), which we will call \( r \). Given \( g \in G \) there exists only one \( c_g \in C \) such that \( t\omega(g) = r(c_g) \) and sending \( g \) to \( \omega(g)\zeta(c_g) \) defines an automorphism on \( G \), which we will call \( s \). We have

\[
X_{t,\sigma} \circ Y_{\omega,\zeta} = \begin{cases} Y_{\sigma s, s \zeta r^{-1}} & \text{if } r = (t\zeta)\text{Id}_C \text{ is an automorphism}, \\ 0 & \text{otherwise}. \end{cases}
\]

Using this formula on the indices defines a left action of \( \hat{R} \hat{G} \) on \( R Y_G \). On the other hand, \( Y_{\omega,\zeta} \circ X_{t,\sigma} \) is different from 0 if and only if \( \omega\sigma(g) \neq t(g) \) for all \( g \in G \), \( g \neq 1 \). Then sending \( g \in G \) to \( \omega\sigma(g)t(g) \) defines an automorphism in \( G \) and we have

\[
Y_{\omega,\zeta} \circ X_{t,\sigma} = \begin{cases} Y_{(\omega\sigma)t, t\zeta} & \text{if } (\omega\sigma)t \text{ is an automorphism}, \\ 0 & \text{otherwise}. \end{cases}
\]

With this we have the right action of \( \hat{R} \hat{G} \) on \( R Y_G \). It can be proved directly that with these actions \( R Y_G \oplus \hat{R} \hat{G} \) is an \( R \)-algebra, and it is clearly isomorphic to \( \hat{RB}^1_G(G) \). □
References