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Calibrated American option pricing by stochastic linear programming

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We propose an approach for computing the arbitrage-free interval for the price of an American option in discrete incomplete market models via linear programming. The main idea is built replicating strategies that use both the basic asset and some European derivatives available on the market for trading. This method goes under the name of calibrated option pricing and it has given significant results for European options. Here, we extend the analysis to American options showing that the arbitrage-free interval can be characterized in terms of martingale measures and that it gets significantly reduced with respect to the non-calibrated case.

Keywords: American option; incomplete market; arbitrage-free interval; calibrated option pricing; dual theory; martingale measures

1. Introduction

In this work, we apply a linear programming method to price American options in a discrete and incomplete market model. The linear programming theory has been used in contexts of completeness by Naik [1], Ortu [2] and by Baccara et al. [3]. As it is well known, we are no longer able to exhibit a unique price for derivatives. In the literature an arbitrage-free interval is usually given by characterizing its endpoints as infimum and supremum of the expected values of the pay-off with respect to a family of martingale measures. Hence, an appropriate criterion to select a specific measure (minimal variance, minimal martingale measure etc.) is used (see Föllmer and Schweizer [4] and Frittelli [5]).

A possible alternative approach is to study the values of all possible admissible investment strategies, trying to select those that replicate an arbitrage-free pricing of the derivative (see Cont [6]). This leads to setting up two optimization problems, known as the buyer's and seller's problem. The optimal values of these problems represent the maximum price and the minimum price that allow, respectively the buyer and the seller of the contract to exploit an arbitrage opportunity. Once they pass these levels they are in no arbitrage conditions (see King [7]). The values of the optimal solutions of these two problems will give the endpoints of the arbitrage-free price interval for the considered derivative.

More precisely in incomplete arbitrage-free markets, the price of an option is not unique but should lie somewhere between the least cost of a super replication strategy (seller's price) and the greatest amount that the buyer would pay for it without facing the risk of a negative

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terminal wealth (buyer's price). When frictionless trading is possible, these bounds can be expressed as the supremum and infimum of the discounted expected future cash-flows of the option on the set of all pricing measures. Having focused on strategies, linear programming becomes the natural tool to look for those values in finite-state discrete-time markets, also providing a setting of easy implementation.

In practice, the resulting problems are often quite big and sometimes, even though the computational time is not high, they do not provide a significant outcome, meaning that the computed arbitrage-free interval is quite large.

To answer this problem, King et al. in [8] employ a modified approach, inspired by what actually happens in real financial markets. In conditions of incompleteness, since the basic assets are not sufficient to devise a replicating strategy for each derivative, agents incorporate derivatives in their hedging portfolios. Hence, King et al. introduce the so-called calibrated option pricing. In their work, for European options, they write new linear programming problems for the buyer and the seller, where they include in the hedging strategy the possibility of selling and buying other European options with respective bid and ask prices and maturities. The authors prove again that the optimal solutions may be characterized in terms of the average values with respect to martingale measures that may belong, now, to a much smaller set. This reduces sensibly the arbitrage-free interval and the advantage of the method is illustrated numerically.

In this paper, the hedging problem is modelled as a stochastic problem using the mathematical technique of conjugate duality or alternatively Lagrange duality (ref. Rockafellar [9]). The first to use this technique to price an American contingent claim in incomplete markets were Pennanen and King [10] and Flåm [11]. Pennanen and King, in particular, studied the linear programming problems for the buyer's and the seller's prices, but in the buyer's case their proof is valid only for the American options, a more general proof was given by Camci and Pinar [12] later on. The main contribution of the present paper is to show that the calibrated option bounds for American options can be computed by solving two linear programming problems, improving the numerical results. Moreover, we show that the end points of the no arbitrage interval may be characterized again by infimum and supremum of the expected pay-off with respect to an appropriate set of martingale measure. The main results are contained in Theorems 3.1 and 3.2. Especially, Theorem 3.1, while similar in spirit to the aforementioned results in the utilization of duality, requires a careful proof. We substantiate our results with numerical illustrations using real data.

The outline of the paper is as follows. Section 2 describes the market model. In Section 3, we study the calibrated option bounds for American contingent claims pointing out the relations between hedging and martingale measures in incomplete markets. In Section 4, we present some numerical results applied to the pricing of S&P500 options. Section 5 is devoted to the concluding remarks about the advantages and the weaknesses of the model.

2. Discrete market models

We consider the same discrete finite-dimensional market model as in King [7]. There are $J+1$ securities tradable at discrete times $k = 0, \dots, K$. We denote by $S_k = (S_k^0, \dots, S_k^J)$ the price process, whose first component represents the price of the riskless asset, thus it is always strictly positive. Thanks to this assumption, we can define the discount processes

$\beta_k = \frac{S_0^0}{S_k^0}$. Moreover, S_k is adapted with respect to a given filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_k\}, \mathbb{P})$, such that $|\Omega| < \infty$.

The market is arbitrage-free, frictionless and investors are small, comparatively to the market dimension.

To have a finite sample space, Ω simplifies the analysis and allows a natural description of the market model in terms of a scenario tree. Here, we assume that the tree is non-recombinant, which might be important in incomplete markets, where trading strategies are in general path dependent. The atoms of $\mathscr{F}_k$, denoted by $\mathcal{N}_k$, are the nodes of the scenario tree at time k. At time 0, the set $\mathcal{N}_0$ consists of the root node $n = 0$; since the tree is non-recombinant, each outcome is uniquely identified by its path, the nodes $n \in \mathcal{N}_K$ correspond one-to-one to the probability atoms $\omega \in \Omega$. The collection of all nodes will be denoted by $\mathcal{N} = \cup_{k=0}^{K} \mathcal{N}_k$. Since the scenario tree is non-recombinant, we have that for $k = 1, \ldots, K$, each node $n \in \mathcal{N}_k$ comes from a unique element $a(n) \in \mathcal{N}_{k-1}$, thus we can define the set $\mathcal{C}(n) = \{m \in \mathcal{N} | a(m) = n\} \in \mathcal{N}_{\mathrm{k}+1}$ of the first descendants of a node $n \in \mathcal{N}_k$ with $k = 1, \ldots, K$. We denote by $\mathcal{A}(n)$ the collection of ascendant nodes or path history of node including n itself and $\mathcal{D}(n)$ the set of descendant nodes of n, again including n itself. From now on, we focus our attention no longer on times but on nodes, hence we will denote by $S_n, n \in \mathcal{N}_k$ the value of the assets at each single node of the tree at time k.

The probability measure $\mathbb{P}$ gives weight $p_n^K > 0$ to each node $n \in \mathcal{N}_K$ so that $\sum_{n \in \mathcal{N}_{\mathrm{K}}} p_n^K = 1$. The probability of each node $n \in \mathcal{N} \setminus \mathcal{N}_{\mathrm{K}}$ is determined by conditioning so $p_n^k = \sum_{m \in \mathcal{C}(n)} p_m^{k+1}$ for $k = 0, \ldots, K-1$. Hence, each non-leaf node has a probability mass equal to the combined mass of its child nodes. The expected value of S_k given $\mathbb{P}$ is given by the finite sum

$$\mathbb{E}^{\mathbb{P}}[S_k] = \sum_{n \in \mathcal{N}_K} p_n^k S_n.$$

The conditional expectation of S_{k+1} on $\mathcal{N}_k$ is hence obtained by

$$\mathbb{E}^{\mathbb{P}}[S_{k+1} | \mathcal{N}_k] = \sum_{m \in \mathcal{C}(n)} \frac{p_m^{k+1}}{p_n^k} S_m.$$

Definition 2.1 A probability measure $\mathbb{Q} = \{q_n^K\}_{n \in \mathcal{N}_K}$, such that

$$\beta_k S_k = \mathbb{E}^{\mathbb{Q}}[\beta_{k+1} S_{k+1} | \mathcal{N}_k] \ (k \leq K-1)$$

is called a martingale probability measure.

A measure $\mathbb{Q}$ is said to be equivalent to $\mathbb{P}$ if $q_n^k > 0$. To simplify the notations, from now on we write p_n for the probability, i.e. we will omit the index k.

A trading strategy is an $\mathbb{R}^{J+1}$-valued $\{\mathscr{F}_k\}_{k=0}^{K}$-adapted process $\theta = (\theta_k^0, \ldots, \theta_k^J)_{k=0}^{K}$, where the value of θ_k^j is the fraction of security j held in the portfolio during the period $(k, k+1]$. The value of the portfolio $\theta_k = (\theta_k^0, \ldots, \theta_k^J)$ at time k is the scalar product

$$S_k \cdot \theta_k := \sum_{j=0}^{J} S_k^j \theta_k^j.$$

An arbitrage opportunity is the possibility to find a trading strategy which starts from zero initial wealth and whose final value is positive with positive probability. In mathematical terms, this means that there exists a trading strategy θ such that

$$\begin{aligned} S_0 \cdot \theta_0 &= 0, \\ S_k \cdot (\theta_k - \theta_{k-1}) &= 0, \; k = 1, \ldots, K, \\ S_K \cdot \theta_K &\geq 0, \; \mathbb{P} - a.s., \\ \mathbb{E}^{\mathbb{P}}[S_K \cdot \theta_K] &> 0. \end{aligned}$$

By the fundamental theorem of asset pricing (see [7]) the absence of arbitrage is equivalent to the existence of a martingale measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ for the discounted prices $\beta_k S_k$. We will denote the set of martingale measures for the discounted price processes by $\mathcal{M}$. From [9] in an arbitrage-free market, the set of equivalent martingale measures is exactly the relative interior, ri-$\mathcal{M}$, of $\mathcal{M}$.

3. The calibrated option bounds for American options

An American contingent claim associated with a real-valued stochastic process $X = \{X_k\}_{k=0}^K$ is a security whose owner can, at any stage $k = 0, \ldots, K$, choose to take X_k euros, after which the security expires. Pennanen and King in [10] and Camci and Pinar in [12] characterize the end points of the arbitrage-free interval for an American contingent claim with payoff X by appropriate linear programming problems constructed considering the point of view of the buyer and of the seller.

We want to evaluate an American option in a market where other European options are available for trading (using the same methodology introduced by King et al. in [8] for European pricing problems). It is natural for an investor in a real market to try to include these in the hedging strategies. If everything remains unchanged, this can only improve the investor's situation. In particular, these tools should make the buyer's price higher and the seller's price lower, thus the arbitrage interval becomes smaller. With this in mind, let us see how the linear programming problems produced in [10] have to be modified.

Let $G^h, h = 1, \ldots, H$ be European contingent claims with bid-ask prices $C_b^h \leq C_a^h$ and pay-offs G_n^h. The first step is to consider the buyer's point of view, he is interested in finding the maximum amount one could pay for it without the risk of having negative terminal wealth. If he includes those derivatives in the admissible hedging strategy, the price V can be characterized by the following optimization problem:

$$\begin{aligned} &\max_{V,\theta,e,\xi_+,\xi_-} \; V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 + C_a \cdot \xi_+ - C_b \cdot \xi_- = X_0 e_0 - V, \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = G_n \cdot (\xi_+ - \xi_-) + X_n e_n \qquad n \in \mathcal{N}_k, k \geq 1 \\ & S_n \cdot \theta_n \geq 0 \qquad n \in \mathcal{N}_K, \\ & \sum_{m \in \mathcal{A}(n)} e_m \leq 1 \qquad n \in \mathcal{N}_K \\ & e_n \in \{0, 1\} \qquad n \in \mathcal{N}_k, k \geq 0, \\ & \xi_+, \xi_- \geq 0, \\ & \theta, e, \xi_+, \xi_- \text{ are } \mathscr{F}_k\text{-adapted,} \end{aligned} \tag{1}$$

where ξ_+^h and ξ_-^h are the bought and sold amounts of the European option indexed h with pay-off G^h at time $k >= 1$ and e_k denotes the amount of the American contingent claim exercised at time k. The constraints on e mean that the claim is exercised at most one time. Every number below the optimal value of this problem is not an arbitrage-free price. Indeed, if the claim could be bought for a price lower than V, then the agent could pocket the difference and, following the strategy of the optimal solution of (1), still obtain a non-negative terminal wealth. On the other hand, to buy the claim at a price above the optimal value does not lead to arbitrage opportunity, indeed this is the maximum price that allows the buyer to have a positive final wealth. The optimum value is called buyer's price of X.

We can describe the exercise strategy for an American contingent claim through stopping times instead of using the variables e. The relation $e_n = 1$ for some $n \in \mathcal{N}_k \Leftrightarrow \tau_n = k$, defines a one-to-one correspondence between stopping times and processes $e \in E$, where

$$E = \left\{ e | e \text{ is } \mathscr{F}_k\text{-adapted, } \sum_{m \in \mathcal{A}(n)} e_m \le 1 \text{ for all } n \in \mathcal{N}_K \text{ and } e \in \{0, 1\} \mathbb{P}\text{-a.s.} \right\}.$$

We will denote with $\mathcal{T}$ the set of all stopping times between 0 and K.

It is possible to relax the problem (1) to have a convex optimization problem. Indeed, if we replace the constraint $e_n \in \{0, 1\}$ with $e_n \ge 0$ we obtain

$$\begin{aligned} &\max_{V,\theta,e,\xi_+,\xi_-} V \\ s.t.\quad & S_0 \cdot \theta_0 + C_a \cdot \xi_+ - C_b \cdot \xi_- = X_0 e_0 - V, \\ & S_n \cdot (\theta_n - \theta_{a(n)}) = G_n \cdot (\xi_+ - \xi_-) + X_n e_n \qquad n \in \mathcal{N}_k, k \ge 1 \\ & S_n \cdot \theta_n \ge 0 \qquad n \in \mathcal{N}_K, \\ & \sum_{m \in \mathcal{A}(n)} e_m \le 1 \qquad n \in \mathcal{N}_K \\ & e, \xi_+, \xi_- \ge 0, \\ & \theta, e, \xi_+, \xi_- \text{ are } \mathscr{F}_k\text{-adapted.} \end{aligned} \tag{2}$$

This is equivalent to ask that $e \in \tilde{E}$ with

$$\tilde{E} = \left\{ e | e \text{ is } \mathscr{F}_k\text{-adapted, } \sum_{k=0}^{K} e_k \le 1 \text{ and } e \ge 0 \ \mathbb{P}\text{-a.s.} \right\}.$$

Note that E is the set of extreme points of $\tilde{E}$. The following result shows that this relaxation does not affect the buyer's price. Indeed, we may view multiple fractional exercise as the exercise of more than one claim at different times by the buyer and the following theorem shows that this decision does not lead to pay a higher initial value.

THEOREM 3.1

(a) *The set of solutions of* (2) *contains at least a solution of* (1). *In particular, the optimum value of* (2) *equals that of* (1).
(b) *In an arbitrage-free market, the buyer's price of X can be expressed as:*

$$\max_{\tau \in \mathcal{T}} \min_{\mathbb{Q} \in \mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}[\beta_\tau X_\tau] = \min_{\mathbb{Q} \in \mathcal{M}_C} \max_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}[\beta_\tau X_\tau],$$

where

$$\mathcal{M}_C = \left\{ \mathbb{Q} \in \mathcal{M} | C_b^h \leq \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=1}^{K} \beta_k G_k^h \right] \leq C_a^h,\ h \in \{1, \ldots, H\} \right\}.$$

In particular, the buyer's price is finite if and only if $\mathcal{M}_C \neq \emptyset$.

Proof (a) We begin noting that if $e_n = 0$ for each $n \in \mathcal{N}$ at the optimal solution of (2) (relaxed problem), then this is automatically an optimal solution also for the non-relaxed problem (1) and there is nothing to prove. Hence, we assume the $e_n > 0$ for at last one $n \in \mathcal{N}$.

Keeping e fixed in (2) and maximizing only with respect to V and θ, we reduce (2) to a linear programming problem for the buyer of a European contingent claim with pay-offs $\{X_k e_k\}_{k=0}^{K}$. Writing the Lagrangian of this problem (considering $x_n \geq 0$), we have:

$$\begin{aligned} L(V, \theta, x, w) = {} & V - w_0(S_0\theta_0 + C_a \cdot \xi_+ - C_b \cdot \xi_- - X_0 e_0 + V) \\ & - \sum_{k=1}^{K} \sum_{n \in \mathcal{N}_k} w_n \left[S_n(\theta_n - \theta_{a(n)}) - X_n e_n - G_n(\xi_+ - \xi_-) \right] + \sum_{n \in \mathcal{N}_K} x_n S_n \theta_n \\ = {} & (1 - w_0)V + \sum_{k=0}^{K} \sum_{n \in \mathcal{N}_k} w_n X_n e_n \\ & + \sum_{n \in \mathcal{N}_K} (x_n - w_n) S_n \theta_n - \sum_{k=0}^{K-1} \sum_{n \in \mathcal{N}_k} \left[S_n w_n - \sum_{m \in \mathcal{C}(n)} w_m S_m \right] \theta_n \\ & + \left[\sum_{k=1}^{K} \sum_{n \in \mathcal{N}_k} w_n G_n - C_a \right] \xi_+ - \left[\sum_{k=1}^{K} \sum_{n \in \mathcal{N}_k} w_n G_n - C_b \right] \xi_-, \end{aligned}$$

deducing that $w_n = x_n$ for all $n \in \mathcal{N}_K$, we obtain the following dual problem

$$\begin{aligned} & \min_{w} \sum_{k=0}^{K} \sum_{n \in \mathcal{N}_k} w_n e_n X_n \\ \text{s.t.} \quad & w_0 = 1, \\ & S_n w_n = \sum_{m A(n)} S_m w_m \quad n \in \mathcal{N}_k, k = 0, \ldots, K-1 \\ & \sum_{n \in \mathcal{N}} G_n^h w_n \leq C_a^h \quad h = 1, \ldots, H, \\ & \sum_{n \in \mathcal{N}} G_n^h w_n \geq C_b^h \quad h = 1, \ldots, H, \\ & w_n \geq 0 \qquad n \in \mathcal{N}_K. \end{aligned}$$

The first and the fourth constraint imply that $w_n = \beta_n \tilde{q}_n$, where β_n is the discount factor and $\mathbb{Q} = \{\tilde{q}_n\}_{n \in \mathcal{N}_k, k=0,\ldots,K}$ is a probability measure in $\mathcal{M}_C$.

Hence, minimizing on the admissible w is equivalent to minimizing on the probability measures $\mathbb{Q} \in \mathcal{M}_C$, the optimum of the problem therefore can be written as:

$$\min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K} \beta_k X_k e_k\right].$$

We are interested in finding the optimal value of (2), but the values of the optimal solutions of the primal and dual problems coincide therefore it can be written as:

$$\max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K} \beta_k X_k e_k\right]. \tag{3}$$

Since $\min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K} \beta_k X_k e_k\right]$ is continuous in e_n, we have that the component in $\tilde{E}$ of the optimal solution of (2) is given by

$$\{e_n^*\}_{n\in\mathcal{N}} = \arg\max_{\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K} \beta_k X_k e_k\right]. \tag{4}$$

We want to show that $\{e_n^*\}$ is in E. For this reason, we go back to the problem (2) and construct its dual problem considering $z, x \geq 0$ and writing the associated Lagrangian

$$\begin{aligned}
&L(V,\theta,e,\xi_+,\xi_-,x,y,z) \\
&\quad = V - y_0(S_0\theta_0 + C_a\cdot\xi_+ - C_b\cdot\xi_- + V - X_0e_0) \\
&\qquad - \sum_{k=1}^{K}\sum_{n\in\mathcal{N}_k} y_n\left[S_n(\theta_n - \theta_{a(n)}) - G_n\cdot(\xi_+ - \xi_-) - X_ne_n\right] \\
&\qquad + \sum_{n\in\mathcal{N}_K} x_nS_n\theta_n + \sum_{n\in\mathcal{N}_K} z_n\left(1 - \sum_{m\in\mathcal{A}(n)} e_m\right) \\
&\quad = V(1-y_0) + \sum_{n\in\mathcal{N}_K}(x_n - y_n)S_n\theta_n - \sum_{k=0}^{K-1}\sum_{n\in\mathcal{N}_k}\theta_n\left(S_ny_n - \sum_{m\in\mathcal{C}(n)} S_my_m\right) \\
&\qquad + \sum_{n\in\mathcal{N}}\left(y_nX_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_K} z_m\right)e_n + \left(\sum_{n\in\mathcal{N}} G_ny_n - C_ay_0\right)\cdot\xi_+ \\
&\qquad - \left(\sum_{n\in\mathcal{N}} G_ny_n - C_by_0\right)\cdot\xi_- + \sum_{n\in\mathcal{N}_K} z_n.
\end{aligned}$$

From the second term we deduce $y_n = x_n$ and, since we know that $x_n \geq 0$ for all $n \in \mathcal{N}_K$, we get the dual problem

$$\begin{aligned}
&\min_{z,y} \sum_{n\in\mathcal{N}_K} z_n \\
\text{s.t.}\quad & y_0 = 1, \\
& S_n y_n = \sum_{m\in\mathcal{C}(n)} S_m y_m \qquad n\in\mathcal{N}_k, k=0,\dots,K-1, \\
& y_n X_n \le \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_K} z_m \qquad n\in\mathcal{N}, \\
& \sum_{n\in\mathcal{N}} G_n^h y_n \le C_a^h \qquad h=1,\dots,H, \\
& \sum_{n\in\mathcal{N}} G_n^h y_n \ge C_b^h \qquad h=1,\dots,H, \\
& z_n, y_n \ge 0 \qquad n\in\mathcal{N}_K.
\end{aligned} \tag{5}$$

Since the constraints are valid also for the riskless asset $S_n^0 > 0$ for all $n \in \mathcal{N}$ and $y_n \geq 0$ for $n \in \mathcal{N}_K$, from the second constraint we find recursively that $y_n \geq 0$ for all $n \in \mathcal{N}$. For any $n \in \mathcal{N}_{K-1}$, we have $y_n = \frac{\sum_{m\in\mathcal{C}(n)} S_m^0 y_m}{S_n^0} \geq 0$ since $m \in \mathcal{C}(n) \subseteq \mathcal{N}_K$, on the other hand starting from $n = 0$ and and applying the first constraint for all k, there must be at least an $n \in \mathcal{N}_k$ such that $y_n > 0$.

The constraints imply that if y_n is part of an admissible solution, $y_n = \beta_n q_n$, where β_n is the discount factor and $\mathbb{Q} = \{q_n\}_{n\in\mathcal{N}_k, k=0,\dots,K}$ is a martingale measure in $\mathcal{M}_C$. Thus, problem (5) becomes

$$\begin{aligned}
&\min_{z,q} \sum_{n\in\mathcal{N}_K} z_n \\
\text{s.t.}\quad & q_0 = 1, \\
& S_n \beta_n q_n = \sum_{m\in\mathcal{C}(n)} S_m \beta_m q_m \qquad n\in\mathcal{N}_k, k=0,\dots,K-1, \\
& \beta_n q_n X_n \le \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_K} z_m \qquad n\in\mathcal{N}, \\
& \mathbb{Q}\in\mathcal{M}_C, \\
& z_n \ge 0 \qquad n\in\mathcal{N}_K.
\end{aligned}$$

Let us consider the optimal solution that contains the e^* found before. From the initial part of the proof, we can suppose that $e_n^* > 0$ for some $n \in \mathcal{N}$, we want to prove that $e_n^* = 1$. Let us denote by k' the first time with some strictly positive e_n^* and let us denote by $n' \in \mathcal{N}_{k'}$ one of the nodes where this happens. By the complementary slackness theorem, it holds

$$\left(\beta_{n'} q_{n'} X_{n'} - \sum_{m\in\mathcal{D}(n')\cap\mathcal{N}_K} z_m\right) e_{n'}^* = 0,$$

thus, since $e_{n'}^* > 0$, we have

$$q_{n'} \beta_{n'} X_{n'} = \sum_{m\in\mathcal{D}(n')\cap\mathcal{N}_K} z_m.$$

Therefore, the objective function verifies

$$\begin{aligned}
\min_{z,q} \sum_{m\in\mathcal{N}_K} z_m &= \min_{z,q} \left(\sum_{m\in\mathcal{D}(n')\cap\mathcal{N}_K} z_m + \sum_{m\in\mathcal{N}_K\setminus\mathcal{D}(n')} z_m \right) \\
&= \min_{z,q} \left(q_{n'}\beta_{n'} X_{n'} + \sum_{m\in\mathcal{N}_K\setminus\mathcal{D}(n')} z_m \right).
\end{aligned} \tag{6}$$

Moreover, note that $\mathcal{D}(n) \cap \mathcal{N}_K \subset \mathcal{D}(n') \cap \mathcal{N}_K$ for all $n \in \mathcal{D}(n') \setminus \{n'\}$, so we can break the constraint $\beta_n q_n X_n \leq \sum_{m\in\mathcal{D}(n)\cap N_K} z_m \; n \in \mathcal{N}$ in three pats

$$\beta_{n'} q_{n'} X_{n'} = \sum_{m\in\mathcal{D}(n')\cap\mathcal{N}_K} z_m$$

$$\beta_n q_n X_n \leq \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_K} z_m \qquad n \in \mathcal{D}(n') \setminus \{n'\}$$

$$\beta_n q_n X_n \leq \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_K} z_m \qquad n \notin \mathcal{D}(n') \setminus \{n'\},$$

and we can eliminate the second constraint, since it is redundant it does not appear in the objective function. Using complementary slackness again we conclude $e_n^* = 0$ for $n \in \mathcal{D}(n') \setminus \{n'\}$.

Let us consider k'' the first time with some strictly positive e_n^* in $\mathcal{N} \setminus \mathcal{D}(n')$ (i.e. without considering n') and let $n'' \in \mathcal{N}_{k''}$ be one of the nodes, such that $e_{n''}^* > 0$. Applying the slackness theorem, we can write (6) in the following form

$$\min_{z,q} \left(q_{n'}\beta_{n'}X_{n'} + q_{n''}\beta_{n''}X_{n''} + \sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m \right),$$

and as before we may say that $e_n^* = 0$ for all $n \in \mathcal{D}(n'') \setminus \{n''\}$.

Iterating our procedure, we find a finite set of nodes $\{n', n'', \dots, n^{(N)}\}$, with $N \leq |\mathcal{N}|$, such that $e_n^* = 0$ for $n \in \cup_{i=1}^N (\mathcal{D}(n^{(i)}) \setminus \{n^{(i)}\})$. Since these are the nodes corresponding to the first times with a strictly positive exercise, we can say that for $n \in \mathcal{N} \setminus (\cup_{i=1}^N \mathcal{D}(n^{(i)}))$ there aren't nodes such that $e_n^* > 0$, that is $e_n^* = 0$ except the nodes in the set $\{n', n'', \dots, n^{(N)}\}$. So we may conclude that (6) is equal to

$$\min_{z,q} \left(\sum_{i=1}^N q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}} + \sum_{m\in\mathcal{N}_K\setminus(\cup_{i=1}^N \mathcal{D}(n^{(i)}))} z_m \right).$$

Let us define the function $v : \mathcal{N} \to \{0, 1\}$ by

$$v_n = \begin{cases} 1 & n \in \{n', n'', \dots, n^{(N)}\} \\ 0 & \text{otherwise} \end{cases},$$

substituting in (6) we obtain

$$\min_{z,q} \left(\sum_{i=1}^N q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}} + \sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m \right).$$

Let us note that $\{v_n\} \in E$ and, since $e_n^* = 0$ except for the nodes in the set $\{n', n'', \dots, n^{(N)}\}$, we have that for all $n \in \mathcal{N}$

$$e_n^* \neq 0 \iff v(n) \neq 0.$$

Considering expression (3), we find

$$\max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}} \left[\sum_{k=0}^K \beta_k X_k e_k \right] = \max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{k=0}^K \sum_{n\in\mathcal{N}_k} q_n\beta_n X_n e_n$$

$$= \max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{k=0}^K \sum_{n\in\mathcal{N}_k} q_n\beta_n X_n v_n e_n = \max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^N q_n\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}}e_{n^{(i)}},$$

but (6) and (3) must be equal (the solutions are the optimal value of the same linear programming problem), so

$$\begin{aligned}
&\max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}}e_{n^{(i)}} \\
&= \min_{z,q}\left(\sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}} + \sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m\right) \\
&= \min_{q}\sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}} = v_{n^{(i)}} + \min_{z} \sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m.
\end{aligned}$$

Since we know $0 \le e_n \le 1$, it holds

$$\max_{e\in\tilde{E}} \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}}e_{n^{(i)}} \le \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}},$$

thus

$$\begin{aligned}
&\min_{q}\sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}} + \min_{z} \sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m \\
&\le \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}}.
\end{aligned}$$

We must have $\sum_{m\in\mathcal{N}_K\setminus(\mathcal{D}(n')\cup\mathcal{D}(n''))} z_m = 0$ (the constraints of the dual problem say that this quantity is non-negative), and it is also true that

$$\min_{q}\sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}}e_{n^{(i)}} = \min_{\mathbb{Q}\in\mathcal{M}_C} \sum_{i=1}^{N} q_{n^{(i)}}\beta_{n^{(i)}}X_{n^{(i)}}v_{n^{(i)}},$$

with $v_n \in \{0,1\}$, $0 \le e_n^* \le 1$, $e_n^* \ne 0 \iff v(n) \ne 0$ and $X_n, q_n, \beta_n \ge 0$, which finally implies $e_n = v_n$ for all $n \in \mathcal{N}$.

We found there is an optimal solution for (2) that is admissible for (1), but the set defined by the constraints of (2) is larger than the set defined by the constraints of (1), so this must be also an optimal solution for (1). In conclusion we can write the optimal value of (1) as

$$\max_{e\in E} \min_{\mathbb{Q}\in\mathcal{M}_C} \mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K} \beta_k X_k e_k\right],$$

and the correspondence between stopping times and the processes $e \in E$ gives the first part of the equality in (b).

(b) By Part (a), we can replace E by $\tilde{E}$ without changing the value of (3). Both $\tilde{E}$ and $\mathcal{M}_C$ are bounded convex sets, so we can change the order of max and min without affecting the value in (3). For each fixed $\mathbb{Q} \in \mathcal{M}$, the objective function is linear in e, thus its max is achieved on the boundary of $\tilde{E}$ that is for $e \in E$, yielding the second part of the equality. □

Let us consider the problem of the seller of the option, he wants to know what is the smallest amount of cash required by a strategy not allowing a loss. Moreover, he must hedge against any future exercise that may result from the contract. For this reason he is interested in an optimization problem given by

$$
\begin{aligned}
&\min_{V,\theta,\xi_+,\xi_-} \quad V \\
s.t.\quad & S_0\cdot\theta_0 + C_a\cdot\xi_+ - C_b\cdot\xi_- = V, \\
& S_n\cdot(\theta_n-\theta_{a(n)}) = G_n\cdot(\xi_+-\xi_-) \quad n\in\mathcal{N}_k,\, k\geq 1 \\
& S_n\cdot\theta_n \geq 0 \quad n\in\mathcal{N}_K, \\
& S_n\cdot\theta_n \geq X_n \quad n\in\mathcal{N}, \\
& \theta,\xi_+,\xi_- \text{ are } \mathscr{F}_k\text{-adapted.}
\end{aligned}
\tag{7}
$$

Thanks to the constraint $S_n\theta_n \geq X_n$, the solution of this problem hedges against any future exercise of the American contingent claim. The constraint $S_n\theta_n \geq 0\ n\in\mathcal{N}_K$ implies that any number below the optimum value of (7) is an arbitrage price for the seller. On the other hand, selling X for a lower price does not lead to arbitrage opportunities, since the optimal value of (7) is the smallest price that allows to have an investing strategy θ^* (that is a part of the optimal solution) that respects the constraint $S_n\theta_n \geq 0\ n\in\mathcal{N}_K$, i.e. selling X at a lower price means to force the seller to construct an investing strategy that can lead to a negative terminal wealth with a positive probability. The optimal value of (7) will be called the *seller's price*.

As before, convex duality yields the following expression for the seller's price.

THEOREM 3.2 *In an arbitrage-free market, the calibrated seller's price of the option can be expressed as:*

$$\max_{\tau\in\mathcal{T}}\max_{\mathbb{Q}\in\mathcal{M}_C}\mathbb{E}^{\mathbb{Q}}[\beta_\tau X_\tau] = \max_{\mathbb{Q}\in\mathcal{M}_C}\max_{\tau\in\mathcal{T}}\mathbb{E}^{\mathbb{Q}}[\beta_\tau X_\tau].$$

Proof We begin by writing the Lagrangian of problem (7)

$$
\begin{aligned}
&L(V,\theta,\xi_+,\xi_-,x,y,z) \\
&\quad = V + y_0(S_0\theta_0 + C_a\cdot\xi_+ - C_b\cdot\xi_- - V) \\
&\qquad + \sum_{k=1}^{K}\sum_{n\in\mathcal{N}_k} y_n\left[S_n(\theta_n-\theta_{a(n)}) - G_n\cdot(\xi_+-\xi_-)\right] \\
&\qquad - \sum_{n\in\mathcal{N}_K} x_n S_n\theta_n + \sum_{k=1}^{K}\sum_{n\in\mathcal{N}_k} z_n(X_n - S_n\theta_n) \\
&\quad = V(1-y_0) + \sum_{n\in\mathcal{N}_K}(y_n - x_n - z_n)S_n\theta_n \\
&\qquad + \sum_{k=0}^{K}\sum_{n\in\mathcal{N}_k}^{K-1}\theta_n\left(S_n y_n - S_n z_n - \sum_{m\in\mathcal{A}(n)} S_m y_m\right) \\
&\qquad + \left(C_a y_0 - \sum_{n\in\mathcal{N}} G_n y_n\right)\xi_+ - \left(C_b y_0 - \sum_{n\in\mathcal{N}} G_n y_n\right)\xi_- + \sum_{k=0}^{K}\sum_{n\in\mathcal{N}_k} X_n z_n,
\end{aligned}
$$

where we are supposing that $z, x \geq 0$, so we get the dual

$$
\begin{aligned}
& \max_{y,z} \sum_{k=0}^{K} \textstyle\sum_{n\in\mathcal{N}_k} X_n z_n \\
\text{s.t.} \quad & y_0 = 1, \\
& S_n(y_n - z_n) = \sum_{m\in\mathcal{C}(n)} S_m y_m \quad n \in \mathcal{N}_k, k = 0, \dots, K-1, \\
& y_n - z_n \geq 0 \qquad n \in \mathcal{N}_K, \\
& \sum_{n\in\mathcal{N}} G_n y_n \leq C_a, \\
& \sum_{n\in\mathcal{N}} G_n y_n \geq C_b, \\
& z_n \geq 0 \qquad n \in \mathcal{N}.
\end{aligned}
\tag{8}
$$

Let us suppose that the set $\mathcal{M}_C \neq \emptyset$, thus there exists a strictly positive vector q, such that $S_n q_n = \sum_{m\in\mathcal{C}(n)} S_m q_m$. Then, we set $y_n = q_n$ and $z_n = 0$ for all $n \in \mathcal{N}$. Multiplying the resulting pairs (y, z) by $\frac{1}{y_0}$ one obtains a feasible solution of (8) that satisfies the strictly inequality $y_n - z_n > 0$ for all $n \in \mathcal{N}_K$, moreover the positivity of S^0 and the second constraint imply that $y_n - z_n > 0$ for all $n \in \mathcal{N}$. Since the optimal solution of a linear programming problem is achieved at the extreme points of the set defined by the constraints, we get that the seller's price equals the optimal value of

$$
\begin{aligned}
& \max_{y,z} \sum_{k=0}^{K} \sum_{n\in\mathcal{N}_k} X_n z_n \\
\text{s.t.} \quad & y_0 = 1, \\
& S_n = \sum_{m\in\mathcal{C}(n)} \frac{y_m}{y_n - z_n} S_m \quad n \in \mathcal{N}_k, k = 0, \dots, K-1, \\
& \sum_{n\in\mathcal{N}} G_n y_n \leq C_a, \\
& \sum_{n\in\mathcal{N}} G_n y_n \geq C_b, \\
& y_n > z_n \geq 0 \qquad n \in \mathcal{N}.
\end{aligned}
$$

The second constraint means that there exists a $\mathbb{Q} \in \mathrm{ri}\mathcal{M}_C$, such that

$$\frac{y_m}{y_n - z_n} = \frac{q_m \beta_m}{q_n \beta_n} \iff \frac{y_m}{q_m \beta_m} = \frac{y_n}{q_n \beta_n} - \frac{z_n}{q_n \beta_n}.$$

Applying the change of variables

$$f_n = \frac{y_n}{q_n \beta_n} \text{ and } e_n = \frac{z_n}{q_n \beta_n},$$

we can express the seller's price as

$$
\begin{aligned}
& \max_{y,z} \sum_{k=0}^{K} \sum_{n\in\mathcal{N}_k} X_n q_n \beta_n e_n \\
\text{s.t.} \quad & f_0 = 1, \\
& f_n = f_{a(n)} - e_{a(n)} \quad n \in \mathcal{N}_k, k = 1, \dots, K, \\
& f_n > e_n \geq 0 \qquad n \in \mathcal{N}, \\
& \mathbb{Q} \in \mathrm{ri}\mathcal{M}_C.
\end{aligned}
$$

Table 1. European call and put options on the S&P500.

Call				Put			
STR	MAT	C_b	C_a	STR	MAT	C_b	C_a
890	17	31.5	33.5	750	17	0.4	0.6
900	17	24.4	26.4	790	17	1	1.3
905	17	21.2	23.2	800	17	1.3	1.65
910	17	18.5	20.1	825	17	2.5	2.85
915	17	15.8	17.4	830	17	2.6	3.1
925	17	11.2	12.6	840	17	3.4	3.8
935	17	7.6	8.6	850	17	3.9	4.7
950	17	3.8	4.6	860	17	5.5	5.8
955	17	3	3.7	875	17	7.2	7.8
975	17	0.95	1.45	885	17	9.4	10.4
980	17	0.65	1.15	750	37	5.5	5.9
900	37	42.3	44.3	775	37	6.9	7.7
925	37	28.2	29.6	800	37	9.3	10
950	37	17.5	19	850	37	16.7	18.3
875	100	77.1	79.1	875	37	23	24.3
900	100	61.6	63.6	900	37	31	33
950	100	35.8	37.8	925	37	41.8	43.8
975	100	26	28	975	37	73	75
995	100	19.9	21.5	995	37	88.9	90.9
1025	100	12.6	14.2	650	100	5.7	6.7
1100	100	3.4	3.8	700	100	9.2	10.2
				750	100	14.7	15.8
				775	100	17.6	19.2
				800	100	21.7	23.7
				850	100	33.3	35.3
				875	100	40.9	42.9
				900	100	50.3	52.3

The constraints on f and e mean that $e \geq 0$ and $\sum_{m\in\mathcal{A}(n)} e_n < 1$ for $n \in \mathcal{N}_K$. Thus, we can write the seller's problem as

$$\begin{aligned}
\max_{y,z}\ & \sum_{k=0}^{K} \sum_{n\in\mathcal{N}_k} X_n q_n \beta_n e_n \\
\text{s.t.}\quad & f_0 = 1, \\
& \sum_{m\in\mathcal{A}(n)} e_n < 1 \qquad n \in \mathcal{N}_k, k = 1, \ldots, K, \\
& f_n > e_n \geq 0 \qquad n \in \mathcal{N}, \\
& \mathbb{Q} \in \mathrm{ri}\mathcal{M}_C.
\end{aligned}$$

Since f is involved only in the third constraint and e_n must be necessarily less than 1, we can impose $f_n = 1$ without loss of generality.

Table 2. Non-calibrated and calibrated bounds with a three-period model.

		Non cal.			Cal		
STR	MAT	Bp	Sp	INT	Bp	Sp	INT
Call							
900	37	17.32	329.42	321.10	40.58	81.55	40,97
925	37	5.87	321.53	315.66	26.38	68.16	41.78
950	37	0.15	313.64	313.49	13.82	52.99	39.06
875	100	43.22	337.32	294.10	75.48	99.11	23.63
900	100	27.84	329.42	301.58	59.88	82.83	22.95
950	100	8.41	313.64	305.23	32.72	55.03	22.31
975	100	3.48	305.75	302.27	23.70	42.56	18.86
995	100	0.96	299.44	298.48	17.72	35.40	17.68
1025	100	7.74	292.15	284.41	8.01	28.13	20.12
1100	100	0.00	276.39	276.39	0.00	21.11	21.11
Put							
750	37	0.00	227.98	227.98	3.80	22.98	19.18
775	37	0.00	242.11	242.11	6.33	28.02	21.36
800	37	0.00	256.23	256.23	7.90	34.88	26.98
850	37	0.00	285.63	285.63	13.49	51.73	38.24
875	37	0.00	302.74	302.74	21.77	61.85	40.08
900	37	0.00	319.84	319.84	32.72	71.97	39.25
925	37	21.29	336.95	315.66	43.62	83.58	39.96
975	37	65.42	371.17	305.75	72.24	107.23	34.99
995	37	85.42	384.86	299.44	87.09	121.31	34.22
650	100	0.00	171.48	171.48	2.60	10.42	7.82
700	100	0.00	199.73	199.73	6.65	15.69	9.04
750	100	0.00	227.98	227.98	11.79	22.70	10.91
775	100	0.00	242.11	242.11	16.95	28.48	11.53
800	100	0.00	256.23	256.23	20.06	35.74	15.58
850	100	2.04	285.63	283.59	32.74	53.09	20.35
875	100	8.64	302.74	294.10	42.52	61.85	19.33
900	100	18.26	319.84	301.58	52.02	74.32	22.30

Noting that $\mathcal{M}_C$ and $\tilde{E}$ are convex sets, we can express the seller's price as

$$\sup_{e\in\tilde{E}}\sup_{\mathbb{Q}\in\mathcal{M}_C}\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K}\beta_k X_k e_k\right]=\sup_{\mathbb{Q}\in\mathcal{M}_C}\sup_{e\in\tilde{E}}\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K}\beta_k X_k e_k\right].$$

Since $\mathbb{E}^{\mathbb{Q}}\left[\sum_{k=0}^{K}\beta_k X_k e_k\right]$ is linear in e, the max over e is attained at an extreme point of $\tilde{E}$. Let us note that these extreme points are exactly E and, since E and $\mathcal{M}_C$ are closed sets, we obtain the desired expression. □

The real interval bounded by the buyer's and the seller's price is called the arbitrage-free interval, because every point that belongs to it is an arbitrage-free price for the claim. It is

Table 3. Calibrated bounds with a 4-period model and branching structure (20, 10, 10, 10).

Call					Put				
STR	MAT	Bp	Sp	INT	STR	MAT	Bp	Sp	INT
900	37	40.58	84.99	44.41	750	37	3.80	22.96	19.16
925	37	26.38	70.57	44.19	775	37	6.33	26.30	19.97
950	37	13.82	56.17	42.35	800	37	7.95	38.86	30.91
875	100	75.48	105.50	30.02	850	37	13.25	55.44	42.19
900	100	59.88	86.90	27.02	875	37	21.71	65.42	43.71
950	100	31.97	58.19	26.22	900	37	32.72	75.41	42.69
975	100	23.71	45.60	21.89	925	37	43.62	86.15	42.53
995	100	17.50	38.13	20.63	975	37	72.27	110.28	38.01
1025	100	7.97	30.47	22.50	995	37	86.49	122.57	36.08
1100	100	0.00	22.58	22.58	650	100	2.60	10.30	7.70
					700	100	6.65	16.45	9.80
					750	100	11.74	26.38	14.64
					775	100	16.95	32.42	15.47
					800	100	20.07	39.66	19.59
					850	100	32.74	56.95	24.21
					875	100	42.52	65.42	22.90
					900	100	52.02	80.34	28.32

not a priori clear whether the buyer's and seller's prices themselves are arbitrage-free or not. When problems (1) and (7) have the same optimal solution, the price of X is unique and the claim is said to be replicable. Finally, let us note that the interval determined by the optimal solutions of problems (1) and (7) is smaller than that defined by the non-calibrated American programming problems in [10] and [12], since the set $\mathcal{M}_C$ of calibrated martingale measures is smaller than $\mathcal{M}$. The keypoint is that the previous results hold and when we calibrate we fall into $\mathcal{M}$, hence the set $\mathcal{M}_C$ can be thought of as a set of martingale measures calibrated by the observed market prices.

4. Numerical tests with S&P500 options

In this section, we summarize the most meaningful numerical results obtained considering the 48 European option used by King et al. in their work. In particular, we consider the bid and ask closing prices of 21 European call and 27 European put options on the S&P500 index on September 10, 2002, in turn one of the European options will be considered American and all the others used for calibrating. In Table 1, we summarize our data-set, the columns labeled STR and MAT give the strike prices and maturities and those labeled with C_a and C_b give the bid and ask prices. To price each 'American' we solve the seller's and the buyer's problems 38 times, indeed we exclude the options with maturity 17 days because they are indistinguishable from the European case.

4.1. *A three-period model*

The risky asset is denoted by S^1 and it is the S&P500 index, while we assume the riskless asset to be constant. The period structure in the model is chosen according to the maturities of the options. That is, we assume that trading occurs at 0,17,37, and 100 days, which

Table 4. Calibrated bounds with a 4-period model and branching structure (30, 10, 10, 10).

Call					Put				
STR	MAT	Bp	Sp	INT	STR	MAT	Bp	Sp	INT
900	37	40.58	88.36	38.95	750	37	3.80	29.20	25.40
925	37	26.58	73.96	41.78	775	37	6.33	35.09	28.76
950	37	13.82	59.59	45.71	800	37	7.90	42.10	34.20
875	100	75.48	109.85	34.37	850	37	13.26	58.74	45.48
900	100	59.88	90.88	31.00	875	37	21.68	68.76	47.08
950	100	31.91	61.60	29.69	900	37	32.72	78.78	46.06
975	100	23.83	49.05	25.68	925	37	43.62	89.40	45.78
995	100	17.74	41.26	23.52	975	37	72.51	113.73	41.22
1025	100	7.99	33.73	25.74	995	37	87.02	126.04	39.02
1100	100	0.00	26.08	26.08	650	100	2.60	11.33	8.73
					700	100	6.65	19.58	12.93
					750	100	11.76	29.57	17.81
					775	100	16.95	35.64	18.69
					800	100	20.07	42.89	22.82
					850	100	32.74	60.45	27.71
					875	100	42.52	68.76	26.24
					900	100	52.02	84.64	32.62

are the expiration dates of the options in Table 1. The scenario tree is built by using the Gauss-Hermite procedure. We choose the branching structure (50, 10, 10). All the computations have been performed in GAMS 23.7, using CPLEX for the numerical solution and on a personal computer equipped with a Pentium Dual-Core 2.17 GHz and 3 GB RAM.

We begin considering in Table 2 the *Non-calibrated and calibrated* option bounds for American options. In the following table the columns Bp, Sp, INT give, respectively the buyer's price, the seller's price and the length of the arbitrage free interval. We note that the non-calibrated arbitrage interval can be quite large.

Comparing the calibrated columns of Table 2 with the table for European options in the work by King et al. we note that the buyer's problems have the same optimal value. We believe this happens because we are not considering in the money options, thus the buyer prefers to carry the options to maturity. If we observe the values of the variables e_n, we notice that the number of nodes of early exercise is irrelevant compared to the total number of nodes. It seems that the approximation works better with the negative moneyness, that is to say the more out of the money the option is and the smaller the interval is. This is in line with the fact that the more out of the money the option is and more the behavior of the option is similar to the European case.

In both calibrated and non-calibrated case the average time for the routines is about 15 seconds for the buyer's problem and about 10 seconds for the seller's problem.

4.2. *A four-period model*

We also studied the calibrated problem in a four-step model (the steps are in the days 8th, 17th, 37th and 100th). We considered the case of a scenario tree with a branching structure (20, 10, 10, 10) in Table 3 and (30, 10, 10, 10) in Table 4.

Table 5. Calibrated bounds with a 4-period model and branching structure (10, 10, 10, 30).

Call					Put				
STR	MAT	Bp	Sp	INT	STR	MAT	Bp	Sp	INT
900	37	40.58	83.37	40.38	750	37	3.80	22.96	19.16
925	37	26.38	68.24	41.86	775	37	6.33	28.26	21.93
950	37	13.82	53.98	40.16	800	37	7.90	35.37	27.47
875	100	75.48	100.87	25.39	850	37	13.26	52.38	39.12
900	100	59.88	83.86	23.98	875	37	21.68	62.58	40.90
950	100	31.91	56.02	24.11	900	37	32.72	72.79	40.07
975	100	23.83	43.61	19.78	925	37	43.62	84.00	40.38
995	100	17.74	36.33	18.59	975	37	72.51	108.30	35.79
1025	100	7.99	28.73	20.10	995	37	87.02	121.37	34.35
1100	100	0.00	22.09	22.09	650	100	2.60	11.26	8.66
					700	100	6.65	15.51	8.86
					750	100	11.76	22.70	10.94
					775	100	16.95	28.75	11.80
					800	100	20.07	36.23	16.16
					850	100	32.74	53.79	21.23
					875	100	42.52	62.58	20.06
					900	100	52.02	75.71	23.69

The average running time for the (20,10,10,10) tree is about 9 minutes for the buyer's problem and 7 minutes for the seller's problem and about 25 minutes for the buyer's problem and 20 minutes for the seller's problem for the (30,10,10,10) tree.

The average time for the (30,10,10,10) routines is instead about 25 minutes for the buyer's problem and about 20 minutes for the seller's problem.

We see that the option bounds do not improve generally. In particular, the buyer's problem has a solution similar to that of the three-stage model and the seller's price is higher, probably due to the constraint $S_n \cdot \theta_n \geq X_n,\ n \in \mathcal{N}$. The increase in the number of nodes in the seller's problem implies the strategies have to verify a higher number of constraints, thus their initial value will be higher, and this accounts also for the higher computational times. To confirm our conjecture we can see that in the case (10, 10, 10, 30) in Table 5.

5. Conclusions

The main idea of the paper is to extend the calibrated option pricing technique, originally used for European options by King et al. [8], to American contingent claims, whose arbitrage interval for the price is determined by appropriate buyer's and seller's stochastic linear programming problems such as in Pennanen and King [10] and Camci and Pinar [2]. This framework still allows a characterization of the end points of the arbitrage interval in terms of expectation with respect to an appropriate subset of martingale measures.

Numerically the set up linear programming problems are easy to be implemented and computational times still remain rather short for three-period branching structure. The reduction of the arbitrage interval is remarkable and the endpoints in many cases may provide a satisfactory approximation of the actual price.

Unfortunately an increase of the number of time periods does not necessarily correspond to a further reduction in size of the arbitrage interval. This is probably due to the fact that increasing the time steps we are at the same time increasing the range of the underlying. Also, we pay this refinement of periods in terms of computational times.

It would be desirable to test the procedure against a data-set of prices of actually traded American options, but these data are hardly free to retrieve.

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