KO-RINGS AND J-GROUPS OF LENS SPACES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

By

Mehmet Kırdar

February, 1998

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy.

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ABSTRACT

KO-RINGS AND J-GROUPS OF LENS SPACES

Mehmet Kırdar
Ph. D. in Mathematics
Advisor: Prof. Dr. İbrahim Dibağ
February, 1998

In this thesis, we make the explicit computation of the real K-theory of lens spaces and making use of these results and Adams conjecture, we describe their J-groups in terms of generators and relations. These computations give nice by-products on some geometrical problems related to lens spaces. We show that J-groups of lens spaces approximate localized J-groups of complex projective spaces. We also make connections of the J-computations with the classical cross-section problem and the James numbers conjecture. Many difficult geometric problems remain open. The results are related to some arithmetic on representations of cyclic groups over fields and the Atiyah-Segal isomormhism. Eventually, we are interested in representations over rings, in connection with Algebraic K-theory. This turns out to be a very non-trivial arithmetic problem related to number theory.

Keywords : Topological K-Theory, Lens spaces, Representations of cyclic groups, J-groups, $Im(J)$-theory, Algebraic K-theory.

ÖZET

LENS UZAYLARININ KO-HALKALARI VE J-GRUPLARI

Mehmet Kırdar
Matematik Bölümü Doktora
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Şubat, 1998

Bu tezde, lens uzaylarının gerçel K-teorisinin açık bir hesabını ve ayrıca bu sonuçları ve Adams konjektürünü kullanarak bu uzayların J-gruplarının bir betimlemesini, bu gurupları tanımlayan elemanlar ve üzerlerindeki ilişkiler yoluyla, yapıyoruz. Lens uzayların J-gruplarının, complex projektif uzayların lokal J-gruplarına yaklaştıklarını gösteriyoruz. Ayrıca, J-hesaplarının, klasik cross-section problemine ve James sayıları konjektürüne olan bağlantılarını yapıyoruz. Birçok zor geometrik problemi açık bırakıyoruz. Bütün sonuçlar siklik grupların alanlar üzerindeki temsilleri üzerinde olan aritmetik ve bir Atiyah-Segal isomorfizmi uygulaması olarak görülebilir. Doğal olarak, halkalar üzerindeki temsilleri cebirsel K-teorisi gözüyle merak ediyoruz. Bunun sayı teorisiyle ilgili zor bir aritmetik problemi olduğunu farkediyoruz.

Anahtar Kelimeler : Topolojik K-teorisi, lens uzayları, siklik grupların temsilleri, J-grubları, $Im(J)$-teorisi, cebirsel K-teorisi.

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I thank all my friends with whom I shared everything; good times, bad times and hopes.

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Chapter 1

Introduction

Topological K-theory TKT is roughly the classical theory which aims at calculating the homotopy groups $[X, BGL(n, \Lambda)]$ of maps from X to the general linear group of invertible $n \times n$ matrices over Λ where Λ is the field of real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ or queternions $\mathbb{H}$, n is preferably infinity and X is a CW-complex. Breakthrough came by Bott for infinite fields in the late 50's: calculation of the coefficient groups, i.e.,

$$K\Lambda_i(pt.) = \pi_i(BGL(\infty, \Lambda)).$$

Quillen did an analogous computation for finite field $\mathbb{F}_q$ with a modification (+-construction!), at the beginning of 70's. In general, he gave definition-construction of Algebraic K-theory (AKT) of rings as the homotopy groups of some appropriate spectra, compatible with the classical algebraic K_i-functors, $i \leq 2$. AKT is n hot subject today and has been developed extensively in connection with other theories. We should say here that these homotopy computations are unfortunately quite nontrivial and topological. For general CW-complexes, there are Atiyah-Hirzebruch spectral sequences (AHSS), gluing the local data to the problem considered. For topological K-theory and its applications, see e.g., [32], [12], [43]. For algebraic K-theory, you may see the site http://www.math.uiuc.edu/K-theory.

When we put a classifying space $X = BG$ of a (connected Lie or preferably finite) group on the left, a striking topological isomorphism occur due to Atiyah and Segal [15]:

$$R\Lambda(G)\hat{} \cong [BG, \mathbb{Z} \times BGL(\Lambda)]$$

adding some topology to representation theory where $R\Lambda(G)\hat{}$ is the Λ-representation ring of G completed at its augmentation ideal. An analogous result was proved by

Rector, [50]. for finite fields $\mathbb{F}_q$ using the Quillen's spectrum for these fields.

We want to make a very short philosophical digression for any interested reader. The macrocosm of the whole never-ending theory can be shown (with no comment!) by the schematic

$$SH \to AKT \to TKT$$

where arrows are supposed to denote the decrease in complexity (at least in terms of arithmetic) and where SH: stable homotopy, AKT: algebraic K-theory. TKT: topological K-theory. Ordinary cohomology enters in the spectral sequences gluing local data and one can put rational ordinary cohomology on the right of TKT. This schematic can be realized as a part of the realm of generalized (co)homology theories which in turn is a part of homological algebra. For an account of generalized (co)homology theories and stable homotopy, see [1]. Notice that TKT and AKT are related to linearity in this macrocosm.

The notion of fibre homotopy equivalence, stemming from a homotopy problem, cross-section problem of fibrations, caused $Im(J)$-theories to arise. They are defined by the fibre sequences of some stable cohomology operation maps $\psi^k - 1$ on TKT-spectra. $Im(J)$, fantastically, provides a feed-back to the left, i.e., to AKT and SH. For the former definitions and examples, see [32]. [2], [3], [4]; for advanced theories, see [36], [19]. The Atiyah-Segal isomorphism seems to have an analogue for the J-groups between $J\Lambda(BG)$ and $J\Lambda(G)$ which is defined as in [16].

This work is an effort to understand further TKT which is the most understood part of the above schematic and its connections with representation theory and also the feed-back mentioned above via working on a particular problem:

The main purpose of this thesis is to compute the real K-theory and J-groups of lens spaces. Lens spaces are finite skeletons of classifying spaces of cyclic groups. We thus also solve an elementary algebra problem due to the Atuyah-Segal isomorphism: Describe the structure of the I-adic completion of real representation rings of cyclic groups. These computations have been almost known to many people worked in these problems. For example, see [30], [16]. [39], [29]. I will make them explicit stressing on the relations between the generators, but will not intend to do direct summand decomposition of these groups since this is quite combinatorial and I could not see a way to give this a precise topological meaning. We will have some known by-products of these computations to some geometrical problems and also will make some connections with SH and AKT.

In Chapter 2, we will give a very brief survey of the materials which we require partially in the forthcoming chapters.

In Chapter 3, we will describe $\tilde{KO}(L^k(p^n))$ and make some applications related to the $\tilde{KO}$-order of the powers of the canonical complex line bundle. We have also a striking explanation of the results in terms of the representation groups of $\mathbb{Z}_{p^n}$ and say something more on representations. An amusing observation from these computations is to see the Bott 8- periodicity in the I-adic filtration of the real representation ring of these groups.

In Chapter 4, we describe J-groups of $L^k(p^n)$. We also make some comments on the group structure of $\tilde{J}(L^k(p^n))$ and show that for large n, these groups are isomorphic to $\tilde{J}$-group of projective space $\mathbb{C}P^k$ localized at p and thus saturate. This is in fact a consequence of the approximation of projective spaces by lens spaces at p. In particular, we will be interested in the $\tilde{J}$-order of the canonical complex line bundle over lens spaces and the cross-section problem related, as for the projective spaces it is done.

In Chapter 5, we explain shortly two related problems. This will be a little obscure but maybe a motivation to force the computations.

In conclusion, we will try to determine the directions of further study of the results both in TKT and towards the left.

Chapter 2

Background Material

In what follows Λ will represent the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Spaces and maps will be from the category *cw* of finite CW-complexes and homotopy classes of continuous maps when not indicated otherwise. I will assume that the reader is familiar with the basics of (generalized) (co)homology theories like me. I will make a brief and superficial survey of the related materials. In fact, we will need a little of this survey for our results. Because of that, it seems unnecessary to label anything, and the materials will be used or quoted implicitly in the following chapters.

2.1 K-cohomology

Let $K\Lambda(X)$ be the (Grothendieck) ring obtained by ring completion of the semi-ring $Vect_\Lambda(X)$ of isomorphism classes of Λ-vector bundles over X with addition induced by Whitney sum and multiplication induced by tensor product of vector bundles. $K\Lambda$ is a contravariant functor from *cw* to the category of commutative rings with unit. We have a natural splitting $K\Lambda(X) \cong \mathbb{Z} \oplus \tilde{K}\Lambda(X)$ induced by the augmentation map sending a bundle to its (virtual) dimension. We call $\tilde{K}\Lambda(X)$ the reduced ring. Instead of $K\Lambda$, we write K when $\Lambda = \mathbb{C}$, KO when $\Lambda = \mathbb{R}$.

Realification, complexification and complex conjugation of bundles yield the corresponding homomorphisms on the $K\Lambda$-rings:

$$c : KO(X) \to K(X),$$
$$r : KO(X) \to K(X),$$
$$t : K(X) \to K(X).$$

Here, c and t are ring homomorphisms, whereas in general r is not. From the corresponding facts from representations, it follows immediately that $rc = 2$ and $cr = 1 + t$.

We define the classifying space BU_Λ of the infinite classical group $U_\Lambda = \cup_{k\geq 1} U_\Lambda(k)$ by

$$BU_\Lambda = \cup_{k\geq 1} G_k(\Lambda^{2k})$$

with inductive topology, where $G_k(\Lambda^{2k})$ is the Grassmann manifold of k-dimensional subspaces in Λ^{2k}. We denote BU_Λ by BO in the real case, by BU in the complex case. Similarly, we can define the oriented version BSU_Λ classifying SU_Λ.

The functor $K\Lambda$ is representable by the homotopy classes of maps into the H-space $\mathbb{Z} \times BU_\Lambda$, see [32] :

$$K\Lambda = [-, \mathbb{Z} \times BU_\Lambda].$$

This equivalence of functors gives us a way to extend the definition of the functor $K\Lambda$ from finite to infinite complexes.
We define the n-th cohomology groups of $X \in cw$ and the pair $(X, Y) \in cw^2$ by

$$K\Lambda^{-n}(X) = \tilde{K}\Lambda(S^n X^+),$$

$$K\Lambda^{-n}(X,Y) = \tilde{K}\Lambda(S^n(X/Y))$$

respectively for $n \geq 0$ where S denotes the suspension and superscript $+$ stands for the adjunction of a base point. Notice that $K\Lambda^0(X) = K\Lambda(X)$. These definitions are traditional for all kinds of generalized cohomology theories, but a little confusing. It makes sense to take X non-compact CW-complex (e.g. $X = Y^-$, deleting a point from $Y \in cw$ so that as if $+$ fills it), disregarding the fact that K-cohomology is homotopy invariant. We also define reduced cohomology groups of a CW-complex X by

$$\tilde{K}\Lambda^{-n}(X) = \tilde{K}\Lambda(S^n X)$$

We can extend these definitions to negative integers due to the Bott periodicity:

$$\mathbb{Z} \times BU_\Lambda \simeq \Omega^i BU_\Lambda$$

where $i = 2$ when $\Lambda = \mathbb{C}$, $i = 8$ when $\Lambda = \mathbb{R}$. In particular, we have $\Omega BU_\Lambda \simeq U_\Lambda$.

The above homotopy equivalences define the Ω-spectrum $K\Lambda$. For the definition of spectrum and examples of some spectra, see [1]. As a consequence, this spectrum defines the corresponding generalized (co)homology theory: $K\Lambda$-(co)homology. It satisfies the first six axioms of Eilenberg and Steenrod except the dimension axiom, see [1]. Thus, $K\Lambda(X)$ makes up the *zero*-th part of the cohomology ring $K\Lambda^*(X)$ and is of particular interest.

The main purpose of this thesis is the computation of the real K-theory of the standard lens spaces *mod* m, $m \in \mathbb{Z}^+$, see Chapter 3. Classical examples of computations are those made for $X = \mathbb{R}P^k$, $\mathbb{C}P^k$, $\mathbb{H}P^k$, real, complex and quaternionic projective spaces respectively. As another exercise, one can take finite skeletons of any other classifying space of finite groups. As we will see in Section 1.2, generators are coming from the representations and theoretically these rings can be computable easily. On the other hand, for the relations on the generators one should collect enough information which can be obtained by means of some topology.

K-theory of compact Lie groups and their homogeneous spaces has been also a subject of interest. If G is a compact connected Lie group with $\pi_1(G)$ torsion-free, the ring $K^*(G)$ is computed by the Theorem of Hodgkin [31]: Let $\beta : R(G) \to K^1(G)$ be the map which takes any representation $\rho : G \to U(n)$ to $\beta(\rho) \in K^1(G) = [G, U]$ by means of the inclusion $U(n) \subset U$. Then

$$K^*(G) \cong \Lambda(Im\beta),$$

the exterior algebra on the module of primitive elements generated by the elements of the form $\beta(\rho)$. When G is semisimple then, images of well-known irreducible representations of the Lie group G form a basis for this algebra. Similar results for real K-theory was obtained by Seymour, [53].

Fortunately, the topology (=homotopy structure) of the loop space BU_Λ has been well-understood, thanks to the Bott periodicity. The coefficient groups $\tilde{K\Lambda}(S^n)$ of these cohomology theories, $\Lambda = \mathbb{C}, \mathbb{R}$, are given by the following classical table:

n *mod* 8	0	1	2	3	4	5	6	7
$K^{-n}(pt.)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$KO^{-n}(pt.)$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0

These groups are parts of the coefficient rings $\pi_*(K\Lambda) = K\Lambda^*(pt.)$. For the complex case we have $\pi_*(K) = \mathbb{Z}[u, u^{-1}]$ where $u \in K^{-2}(pt.)$ is the Bott generator. For the real case, $\pi_*(KO) = \mathbb{Z}[\alpha, \beta]/ < 2\alpha, \alpha^3, \alpha\beta, \beta^2 - 4 >$ where $\alpha \in KO^{-1}(pt.)$ and $\beta \in KO^{-4}(pt.)$, see [53]. In [13], a deep connection is established between these rings and Clifford modules and thus the vector-field problem on spheres.

Associated to these cohomology theories, there are Atiyah-Hirzebruch spectral sequences, $AHSS$, (E_r, d_r) such that for every $X \in cw$,

$$E_2^{p,q} \cong H^p(X; K\Lambda^q(pt.))$$

and

$$E_\infty^{p,q} \cong \frac{ker[K\Lambda^{p+q}(X) \to K\Lambda^{p+q}(X^{p-1})]}{ker[K\Lambda^{p+q}(X) \to K\Lambda^{p+q}(X^p)]} = G^{p,q}K\Lambda(X)$$

where X^p denotes the p-skeleton of X. These spectral sequences are natural.

We shall say that an element $x \in \tilde{K}\Lambda(X)$ has filtration $\geq a$ if x is the image of an element in $\tilde{K}\Lambda(X/X^{a-1})$ and a if it is exactly in $E_\infty^{a,-a}$. One can show that if x, y have filtrations $\geq a, b$ then xy has filtration $\geq a + b$ and $x + y$ has filtration $\geq min(a, b)$.

For $\Lambda = \mathbb{C}$, if $H^{odd}(X; \mathbb{Z}) = 0$, then the spectral sequence collapses and hence $H^{even}(X; \mathbb{Z})$ is isomorphic to the graded ring $GK(X)$ associated to $K^0(X)$. This, for example, give the computation of $K^*(\mathbb{C}P^k)$ for the k-dimensional complex projective space. This is a consequence of a more general fact: We have a map, called Chern character,

$$ch(X) : K^*(X) \to H^*(X; \mathbb{Q})$$

defined by the Chern classes of vector bundles. In modern language, this means that there is a quotient map $K \to H\mathbb{Q}$ between the corresponding spectra which induces $ch(X)$ for any $X \in cw$. If $K^*(X)$ has no torsion then $ch(X)$ is a monomorphism, [14]. This map was used by Adams several times in his K-theory works and in the definition of e-invariant for spheres.

One can construct equivariant versions of the topological K-theories. Considering G-equivariant vector bundles over a finite G-CW-complex X for a compact Lie group G, we obtain functors $K\Lambda_G$. They have similar properties and constructions which respect G-actions. For a good account of equivariant topological K-theory, see [51]. We note only two special cases. If X is a free G-space then

$$K\Lambda_G(X) \cong K\Lambda(X/G).$$

If G acts trivially on X then

$$K_G(X) \cong K(X) \otimes R(G)$$

where $R(G)$ is the complex representation ring of G. Decomposition of $KO_G(X)$ under trivial action is a little different, see [51].

One can consider generalized (co)homology theories with coefficients. Let G be an

Abelian group. We define the Moore spectrum of type G to be a spectrum MG with $\pi_{-n}(MG) = 0$, $H_n(MG) = 0$, $n > 0$ and $\pi_0(MG) = G$. Now for any spectrum E, we define corresponding spectrum with coefficients G to be $EG = E \wedge MG$. This makes easier to understand and to work with the cohomology theories. The most frequent of this is the localization of spectra at a prime, i.e., introducing the coefficients $\mathbb{Z}_{(p)}$ where p is a prime. Thus, for a spectrum E, we construct $E_{(p)} = E \wedge M\mathbb{Z}_{(p)}$ with $\pi_*(E_{(p)}) = \pi_*(E) \otimes \mathbb{Z}_{(p)}$, inverting primes other than p, i.e., killing non-p torsion. For example, we localize K at a prime p and we get the splitting $K_{(p)} = \vee_{i=1}^{p-1} S^i G$ where G is a smaller spectrum which can be more suitable for computational purposes. A striking application of introducing coefficients is to introduce a ring of coefficients containing $\frac{1}{2}$ to K in which case K can not be distinguished from two copies of KO, e.g., $K\mathbb{Z}[1/2] \simeq KO\mathbb{Z}[1/2] \vee KO\mathbb{Z}[1/2]$. One can introduce also finite coeffecients $\mathbb{Z}/m$. See remarks in Section 2.6 for homotopy with finite coefficients.

Another 'strange' thing we can do is the p-completion of a spectrum E, taking the inverse homotopy limit of E with finite coeffecients $mod\ p^n$ exactly as in the definition of p-adic integers:

$$E\hat{} = ho\varprojlim E \wedge M\mathbb{Z}/p^n.$$

In fact, these constructions are examples of a more general process: localization of a spectrum with respect to another spectrum, developed largely by Bousfield. These constructions are quite technical and we are interested only in their algebraic consequences. See [1], [49] for details.

2.2 K-theory of Classifying Spaces and Representations of Groups

Let G be a finite group (or more generally a compact Lie group). There exists a universal principal G-bundle $EG \to BG$ which classifies principal G-bundles over paracompact topological spaces. EG is contractible and BG, called the classifying space of G, is simply the quotient EG/G. BG leads to the alternative definition of the (co)homology of G : $H(G, R) = H(BG; R)$ for R a commutative ring of coefficients. We also note the homotopy equivalence $\Omega BG \simeq G$. For more details, see [32], [17].

By definition $K\Lambda(BG) = [BG, \mathbb{Z} \times BU_\Lambda]$. In general, it is not true that

$$K\Lambda(B) = \varprojlim K\Lambda(B^n)$$

for a CW-complex B with n-th skeleton B^n. But, in our situation it turns out that lim^1 term vanishes in the Milnor exact sequence and $K\Lambda(BG)$ can be defined that way, see [15] for notation and details.

Suppose V is a finite dimensional Λ-representation of G. Then we have the Λ-vector bundle $EG \times_G V \to EG \times_G * = BG$ over BG with fibre V. This gives us a map $\alpha_G : R\Lambda(G) \to K\Lambda(G)$ where $R\Lambda(G)$ is the the Grothendieck ring of representations of G over Λ. Let I_G be the kernel of the augmentation map which sends a representation to its (virtual) dimension and define the I-adic completion of the representation ring to be

$$R\Lambda(G)\hat{} = \varprojlim R\Lambda(G)/I_G^n.$$

When G is a p-group, the I-adic completion is the same as p-adic completion, i.e., $R\Lambda(G)\hat{} \cong \mathbb{Z} \oplus (\hat{\mathbb{Z}}_p \otimes \tilde{R\Lambda}(G))$, where $\hat{\mathbb{Z}}_p$ denotes the ring of p-adic integers. In general the obvious map, the constant sequence map, $q : R\Lambda(G) \to R\Lambda(G)\hat{}$ has non-trivial kernel, e.g., $R(\mathbb{Z}/6)$.

We have the following topological isomorphism, [8], [15]:

$$\alpha_G\hat{} : R\Lambda(G)\hat{} \xrightarrow{\cong} K\Lambda(BG).$$

This simply says that $K\Lambda(BG)$ is the set of formal sums of representations and those in $ker(q)$ give vector bundles which are contractible to infinity, i.e., have infinite filtration. In fact, as sets of formal sums, basically, $R\Lambda(G) = K\Lambda(BG)$, i.e., we are complicating matters with topology!

For other parts $K\Lambda^i(BG)$, $i \neq 0$, of the cohomology ring, there are similar isomorphisms, see [15] again. For example $K^1(BG) = 0$, completing the ring $K^*(BG)$, because of Bott 2-periodicity.

Under the isomorphism above, $AHSS$ in K-theory gives the Atiyah spectral sequence $H^*(G, \mathbb{Z}) \Longrightarrow R(G)\hat{}$ going from the integral cohomology ring of G to the completed representation ring, [8], in a different context.

It is interesting to analyze the map α_G using different coefficients. Let $K(X; \hat{\mathbb{Z}}_p)$ be the K-theory with coefficients $\hat{\mathbb{Z}}_p$. Then there is a unique $\hat{\mathbb{Z}}_p$-linear map $\alpha_G\hat{} : \hat{\mathbb{Z}}_p \otimes \tilde{R}(G) \to \tilde{K}(BG; \hat{\mathbb{Z}}_p)$ which lifts α_G. If G is a finite p-group then, this map is an isomorphism, [6]. Thus $\tilde{K}(BG) \to \tilde{K}(BG; \hat{\mathbb{Z}}_p)$ is an isomorphism and the filtration topology coincides with the p-adic topology, due to the remark above on the I-adic completion of $R(G)$ for G a p-group. We have the same argument for the real case because of the inclusion $RO(G) \subset R(G)$.

An analogue of the theorem of Atiyah is the Segal conjecture which takes in place of $K\Lambda$-spectrum and $R\Lambda(G)$, the sphere spectrum S and the Burnside ring $A(G)$. The transfer maps, [17, Page 60], induce an isomorphism

$$A(G)^{\wedge} \to \pi_S^0(BG).$$

This turned out to be a much more difficult problem since the cohomotopy groups of BG and $A(G)$ are difficult to compute. For other spectra, e.g., the complex cobordism spectrum MU, there seems analogous formulations.

We want to recall a consequence of Thom isomorphism in equivariant K-theory, which demonstrates the connection of $K\Lambda$ to $R\Lambda$: Let E be a finite dimensional complex G-module such that G acts freely on $S(E)$. We have the Atiyah exact sequence, [12],

$$0 \to K^1(S(E)/G) \to R(G) \xrightarrow{\Phi} R(G) \xrightarrow{\theta} K^0(S(E)/G) \to 0$$

where Φ is multiplication by $\sum_{i=0}^{dim E}(-1)^i\Lambda^i E$ and θ is the usual construction which assigns to a representation $\rho \in R(G)$ of degree d the vector bundle $S(E) \times_G \mathbb{C}^d$. Note that when E is infinite, we should get the Atiyah isomorphism $R(G)^{\wedge} = K(BG)$.
For the real case, let E be a real G-module of dimension $8k$, $k \geq 1$ such that G acts freely on $S(E)$. Then, we have the half short exact sequence below:

$$RO(G) \xrightarrow{\Phi} RO(G) \xrightarrow{\theta} KO(S(E)/G) \to 0$$

where the maps are as in the complex case.

2.3 Adams Operations and Characteristic Classes

It seems to the author that this section deserves more interest compared to others. Attached to a spectrum E, thus to a generalized cohomology theory, there are stable and unstable cohomology operations which are natural transformations from a subring of $E^*(X)$ to $E^*(X)$. The study of these operations are beyond of the scope of this thesis and we will consider only one kind of unstable operations, namely Adams operations which are natural ring homomorphisms $\psi_\Lambda^k : K\Lambda(X) \to K\Lambda(X)$, $k \in \mathbb{N}$.

They are constructed by the action of symmetric groups (or cyclic groups) on the tensor powers of vector bundles, generalizing Frobenius automorphism which is encountered in various contexts: Due to [9], there is an isomorphism

$$\Delta = \oplus_{k\geq 1}\Delta_k : R_* = \oplus_{k\geq 1} Hom(R(S_k), \mathbb{Z}) \to Sym[t_1, t_2, ...].$$

Let $\psi^k \in R_*$ be such that $\Delta_k(\psi^k) = \sum_{i=1}^{\infty} t_i^k$. In terms of exterior power generators of R_*, we have $\psi^k = Q_k(\lambda^1, \lambda^2, ..., \lambda^k)$ where Q_k is the Newton polynomial which expresses the kth power sum in terms of elementary symmetric functions. We define Adams operations by applying ψ^k to the exterior powers of vector bundles and extending to virtual bundles linearly.

Adams operations come out more naturally in the theory of special λ-rings. A special λ-ring is a commutative ring with unity and countable set of maps $\lambda^n : R \to R$ such that for all $x, y \in R$,
(i) $\lambda^0(x) = 1$. (ii) $\lambda^1(x) = x$. (iii) $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$.
Examples of special λ-rings are $K\Lambda(X)$ and $R\Lambda(G)$. Let Op be the ring of natural operations on the category of special λ-rings as defined in [16]. It turns out that $Op \cong \mathbb{Z}[\lambda^1, \lambda^2, ...]$.
Let t be an indeterminate. and for $x \in R$ define $\lambda_t(x) = \sum_{n\geq 0} \lambda^n(x)t^n$. Alternatively, Adams operations are defined by

$$\psi_{-t}(x) = t\frac{d}{dt}(\lambda_t(x))/\lambda_t(x) \text{ where } \psi_t(x) = \sum_{n\geq 0} \psi^n(x)t^n$$

The computation of $K\Lambda(X)$ or $R\Lambda(G)$, in one sense, is to understand the special λ-ring structure in terms of generators.
By the way, we note the Grothendick γ-operations which are defined by $\gamma_t(x) = \sum_{n\geq 0} \gamma^n t^n = \lambda_{t/1-t}(x)$. They enter in the problem of immersion and embedding of manifolds, [8].
Some important properties of ψ^k are :

(i) $\psi^k\psi^l = \psi^{kl} = \psi^l\psi^k$
(ii) If ξ is a line bundle, $\psi^k(\xi) = \xi^k$
(iii) If p is prime, $\psi^p(x) = x^p \ (mod\ p)$
(iv) $\psi^k_{\mathbb{C}} c = c\psi^k_{\mathbb{R}}$
(v) If $u \in \tilde{K}(S^{2n})$, $\psi^k_{\mathbb{C}}(u) = k^n u$.

Adams gives a periodicity theorem for the operations ψ^k, [3]. If $x \in K\Lambda(X)$ and $m \in \mathbb{Z}$ then the value of $\psi^k_\Lambda(x)$ in $K\Lambda(X)/mK\Lambda(X)$ is periodic in k with period m^e where e depends on X and Λ but not on x and m. A more precise and obvious periodicity occurs in $R\Lambda(G)$, [16]. Let N be the order of G. Then $\psi^{k+N}_\Lambda = \psi^k_\Lambda$. This periodicity is carried into $K\Lambda(BG)$ by the Atiyah-Segal isomorphism.

We will now sketch the construction of Bott characteristic classes ρ^k_Λ. As Stiefel-Whitney characteristic classes are related to Steenrod operations *mod* 2, or Conner-Floyd characteristic classes to Novikov operations, these classes are related to Adams operations in a similar way, with the difference that the first two operations are stable. Let ξ be a vector bundle over X with structural group $U(n)$ if it is complex or $Spin(8n)$ if it is real, then we have the Thom isomorphism

$$\phi_\Lambda : K\Lambda(X) \to \tilde{K}\Lambda(T(\xi))$$

where $T(\xi) = D(\xi)/S(\xi)$ is the Thom complex of ξ. ψ^k_Λ does not commute with ϕ and we define the corrective elements $\rho^k_\Lambda(E) = \phi^{-1}\psi^k_\Lambda\phi(1) \in K\Lambda(X)$. The classes ρ^k_Λ are induced by the following virtual characters of the corresponding structural groups :

$$\prod_{1\leq r\leq n} \frac{z_r^k - 1}{z_r - 1} \quad \text{and} \quad \prod_{1\leq r\leq 4n} \frac{z_r^{k/2} - z_r^{-k/2}}{z_r^{1/2} - z_r^{-1/2}}$$

for $U(n)$ and $Spin(8n)$ respectively, see [2] for more details. From these characters, one can compute ρ^k_Λ for line bundles easily.

ρ^k_Λ are homomorphisms from addition to multiplication, thus we can extend the definition of these classes to virtual bundles at the price of introducing denominators. Define $Q_k = \{n/k^m | n, m \in \mathbb{Z}\}$. Then, if w is a virtual bundle, $\rho^k_\Lambda(w)$ is defined as an element of $K\Lambda(X) \otimes Q_k$. Furthermore, in the real case, we can extend the definition from $Spin(8n)$-bundles to $SO(2n)$-bundles, [2].

We can represent the Adams operation ψ^k_Λ, as a map from BU_Λ to itself. Using Yoneda's Lemma, since $\tilde{K}\Lambda$ is a representable functor, ψ^k_Λ can be taken as an element of $[BU_\Lambda, BU_\Lambda]$ by induction over the skeletons of BU_Λ, [17], although the real case is a little tempting.

As we will see, there are various applications in homotopy theory related in some way to the kernel or cokernel of $\psi^k - k^n$ for some k and n where ψ^k is the unstable Adams operation in $K\Lambda$-theory. Rationally, $Ker(\psi^k - k^n)$ is a cohomology theory. But, this is no longer true over $\mathbb{Z}$ or p-locally since $Ker(\psi^k - k^n)$ is not exact. Taking $k^{-n}\psi^k$ on the $2n$-th skeleton of the spectrum $K\Lambda_{(p)}$ induces a stable operation ψ^k. Thus $\psi^k - 1$ is a stable self map on $K\Lambda_{(p)}$ or on the connective version of $K\Lambda_{(p)}$. The cofibre sequence associated to this map for some proper choice of k and spectrum defines the required cohomology theories, [36], [38]. We will introduce two theories related to J-homomorphism in the next section in which the stable map $\psi^k - 1$ shows up.

The most elegant use of $\psi^k - 1$ is in the computation of Algebraic K-theory of finite fields by Quillen, [17] : For a ring R, consider $GL(R)$, general linear group with coefficients R. Quillen defined $BGL(R)^+$ which is homologically same as $BGL(R)$ but with fundamental group $GL(R)/E(R)$ where $E(R)$ is the commutator subgroup of $GL(R)$ and the groups

$$K_i(R) = \pi_i(K_0(R) \times BGL(R)^+).$$

Let $\psi^q - 1 : BU \to BU$, q a prime power and take the fibre of this map $F\psi^k$ (homotopy fixed point set of ψ^k). Quillen proved that $BGL(\mathbb{F}_q)^+ \simeq F\psi^q$ and thus computed $K_{2j}(\mathbb{F}_q) = 0$ and $K_{2j-1} = \mathbb{Z}/(q^{2j} - 1)$. Furthermore, $\mathbb{Z} \times BGL(\mathbb{F}_q)^+$ is an infinite loop space and thus defines a spectrum $K\mathbb{F}_q$, which is closely related to an $Im(J)$ spectrum, [46, Theorem 2.9].

2.4 The Groups $J(X)$

Let $X \in cw$ and define $T(X)$ to be the subgroup of $\tilde{KO}(X)$, generated by the elements of the form $[\xi] - [\eta]$ where ξ and η are orthogonal bundles whose associated sphere bundles are fibre homotopy equivalent. See [32] for the notion of fibre homotopy equivalence. Then the set of all stable fibre homotopy classes of vector bundles denoted by $J(X)$ is the quotient $KO(X)/T(X)$. There is a natural quotient surjection $J : KO(X) \to J(X)$ as a group homomorphism, [32]. We have the usual decomposition $J(X) = \mathbb{Z} \oplus \tilde{J}(X)$ where $\tilde{J}$ stands for the reduced J-groups. We note that $\tilde{J}(X)$ is a finite group.
Similarly, one can define $J_{\mathbb{C}}(X)$ as a quotient of $K(X)$. The real version carries information related to stable homotopy.

Adams gave an upper bound $J''(X)$ by means of the Adams conjecture which states that if $X \in cw$ and $x \in KO(X)$ and $k \in \mathbb{Z}$, then $\exists n \in \mathbb{Z}$ such that

$$k^n(\psi^k_{\mathbb{R}} - 1)x = 0$$

in $J(X)$. See [23] for an elementary proof of the Adams conjecture. He also gives a lower bound $J'(X)$ for $J(X)$ and showed that for $X \in cw$, $J''(X) = J(X) = J'(X)$, [3]. We give a modern reformulation of this, [19] :
Take $KSO_{(p)}$ the (p)-local KSO-theory and $k \in \mathbb{Z}^+$ with $(p, k) = 1$. Adams showed that there is a commutative diagram

$$\begin{tikzcd}
\tilde{KSO}(X)_{(p)} \arrow[r, "\psi^k - 1"] \arrow[d, "\rho^k"] & \tilde{KSO}(X)_{(p)} \arrow[d, "\rho^k"] \\
1 + \tilde{KSO}(X)_{(p)} \arrow[r, "\psi^k/1"] & 1 + \tilde{KSO}(X)_{(p)}
\end{tikzcd}$$

which exhibits the exponential property of ρ^k and ρ^k induces an isomorphism $\rho^k : Im(\psi^k - 1) \cong Im(\psi^k/1)$ where $\psi^k/1$ is the map which sends x to $\psi^k(x)/x$. Then

$$J''(X)_{(p)} = KO(X)_{(p)}/(\psi^k - 1)(\tilde{KSO}(X)_{(p)}) \qquad \text{and}$$
$$J'(X)_{(p)} = KO(X)_{(p)}/(\rho^k)^{-1}((\psi^k/1)(1 + \tilde{KSO}(X)_{(p)}))$$

yield the above equalities.

Classical examples of $\tilde{J}$-groups of some well-known spaces are those for spheres which give the image of the stable J-homomorphism as explained in the next section ; those for real projective spaces and quoternionic projective spaces. For complex projective spaces, the group structure is not known. We will see in Chapter 4 that $\tilde{J}$-groups of lens spaces $mod\ p^n$, n large, approximate these groups when localized at p and thus we will be able to give a description of the p-local $\tilde{J}$-groups of the complex projective spaces in terms of generators and relations. One should work on the relations to obtain the direct summand decomposition of these groups. This is combinatorially very involved. See [39], [29].

The order of the canonical line bundle in the $\tilde{J}$-group of a projective space is important because of its relation to the cross-section problem of a Λ-Stiefel fibering and some stable homotopy problems. The following geometric fact is of fundamental importance for this: Let $X \in cw$ and ξ be a vector bundle over X. $J(\xi) = 0$ means that the Thom spaces $T(\xi)$ and $T(0) = X^+$ have the same S-type, i.e. the same stable homotopy type, [32].

The isomorphism between $RO(G)^{\wedge}$ and $KO(BG)$ seems to have an analogue for the J-groups. In [16], for a finite group G, J-equivalence of representations is defined in the following way: Let U and V be two orthogonal (or unitary) representations, then $U \sim V$ if there exist maps $\theta : S(U) \to S(V)$ and $\phi : S(V) \to S(U)$ of degree prime to $|G|$, the order of G. Define $J(G) = RO(G)/\sim$.

Let $N = |G|$ and Γ_N be the Galois group of $\mathbb{Q}(w)$ over $\mathbb{Q}$ where w is a primitive Nth root of unity. Γ_N operates on $R(G)$: Let χ be a (virtual) complex character and $\alpha \in \Gamma_N$ such that $\alpha(w) = w^k$. Then $\alpha\chi = \psi^k(\chi)$. This shows, in particular, the periodicity $\psi^N = \epsilon$ where ϵ is the augmentation map of $R(G)$. This action can be restricted to $RO(G)$ by the complexification map $c : RO(G) \to R(G)$ which is a monomorphism. It is easy to see that the action of the subgroup $\{\pm 1\}$ of Γ_N is trivial on $RO(G)$ so that the action factors through the quotient group $\Gamma'_N = \Gamma_N/\{\pm 1\}$. Let $RO(G)_{\Gamma_N}$ be the quotient ring. The canonical map $RO(G) \to J(G)$ factors through $RO(G)_{\Gamma_N}$. Let $\mu : RO(G)_{\Gamma_N} \to J(G)$ be the induced map. It is shown in [16] that for a p-group G ($p \neq 2$), μ is an isomorphism. We will see that this is true also for $G = \mathbb{Z}_{2^n}$. This isomorphism gives a connection of J-theory to representation theory.

We want to point out a deep theory developed largely by K. Knapp. $J(X)_{(p)}$ can be realized as a part of a (co)homology theory, namely p-local complex $Im(J)$-theory, [36]. The spectrum Ad of this theory is defined in the following way : Let p be a fixed prime, choose $k \in \mathbb{Z}^+$ generating $(\mathbb{Z}/p^2)^*$for p odd, $k = 3$ for $p = 2$ and let $\psi^k : K \to K$ be the stable Adams operation in $K = K_{(p)}$, p-local complex K-theory. Then Ad is defined by the cofibre sequence

$$... \to Ad \xrightarrow{D} K \xrightarrow{\psi^k - 1} K \xrightarrow{\Delta} SAd \to ...$$

Similarly, one can define and develop the real version of this cohomology theory restricting the stable Adams operation $\psi^k - 1$ to $KO_{(p)}$.

Sometimes, we call theories defined that way, $Im(J)$-theories, following Knapp and the whole subject $Im(J)$-theory.

Coefficient ring $Ad_*(S^0)$ can be computed easily from this sequence. As a (co)homology theory, properties of Ad are surveyed in [36]. Also some examples of $Im(J)$-groups, in particular, using the close connection between representation theory and K-theory, a complete description of $Ad_*(BG)$ for a finite group G are given. Using the universal coefficient formula for Ad-theory, $Ad^*(BG)$ almost can be determined, e.g. for $n \neq 0$ $Ad^{2n+1}(BG) \cong Ad_{2n-1}(BG)$. We have an important problem at $n = 0$, see [19, Section 6] and Section 4.2 of this thesis.
The relation between the groups $J(X)$ of stable fibre-homotopy equivalence classes of sphere bundles of vector bundles and $Im(J)$-theory is the following, [19] : For an odd prime p and $X \in cw$, there is a natural isomorphism

$$\tilde{J}(X)_{(p)} \cong im(\Delta) \subset Ad^1(X).$$

where Δ is as given in the above cofibre-sequence and takes the bundle $F : X \to K$ to $\Delta \circ f : X \to SAd$.

We want to point out also the equivariant version of J-groups. Considering G-vector bundles over a finite G-CW complex X we can define $J_G(X)$ as a quotient of $KO_G(X)$. We say that two G-bundles $p : E_i \to X$ are G-stable fibre homotopy equivalent, if for some G-module V there exists a G-fibre homotopy equivalence

$f : S(E_1 \oplus V) \to S(E_2 \oplus V)$. Let $T_G(X)$ be the subgroup of $KO_G(X)$ consisting of the elements $E_1 - E_2$ where $E_1 \simeq E_2$. We define $J_G(X) = KO_G(X)/T_G(X)$.
In [26, Section 11], some intermediate J-groups, $J_G^{loc}(X)$, are introduced. They are computable using the action of the Adams operations on $KO_G(X)$. Let $r \in \mathbb{Z}^+$ be a generator of p-adic units (mod ± 1 if $p = 2$) and X a finite G-CW-complex. Then the following sequence is exact:

$$KO_G(X)_{(p)} \xrightarrow{\psi^r_{\mathbb{R}} - 1} KO_G(X)_{(p)} \xrightarrow{J} J_G^{loc}(X)_{(p)}.$$

When $G = \{1\}$, this gives an immediate computation of $J(X)_{(p)}$, since J is onto. An application of this exact sequence is the direct computation of the p-primary part of the Atiyah-Todd number, [48].

2.5 $Im(J)$-Theory and v_1-part of stable homotopy

The ultimate goal of Algebraic topology is to compute the groups $[X, Y]$, homotopy classes of maps from X to Y, for say $X, Y \in cw$. This is too much difficult in general. We define stable homotopy groups by

$$\{X, Y\} = lim_{n\to\infty}[S^n X, S^n Y].$$

Due to its close relations to algebra, stable homotopy became popular in the late times of algebraic topology. This led to modern foundations of the theory in which the category is roughly the category of spectra with morphisms homotopy classes of maps between spectra. See [27] for details and the analogy of the structures in algebra.

When $X = S^r$ and $Y = S^0$, we obtain stable homotopy groups of spheres π_r^S which constitute a ring π_*^S. The main technical tool, which fascinated the homotopy theorists ever since, was introduced by Adams: There is a spectral sequence (ASS) converging to p-component of π_*^S with

$$E_2^{s,t} = Ext_A^{s,t}(\mathbb{F}_p, \mathbb{F}_p).$$

where $A = H\mathbb{F}_p^*(H\mathbb{F}_p)$ is the mod p Steenrod Algebra. This is a device converting algebraic information coming from the Steenrod algebra into geometric information. The first line $s = 1$ of this spectral sequence is related to existence of division algebras. The second line is not well-understood! There is another spectral sequence which is more suitable for computations. This is the Adams-Novikov spectral sequence

($ANSS$) which takes instead of the Moore spectrum $H\mathbb{F}_p$, *mod* p Brown-Peterson spectrum BP which comes out after we localize MU at p. There has been a huge amount of work in these directions aiming at the computation of the differentials in the spectral sequences and detecting homotopy elements. Meanwhile, new techniques and spectra come out. See [49] for this permenant 'painfully esoteric' subject.

$Im(J)$ (linear part) is the best understood part of stable homotopy. This part is called the v_1-periodic part of stable homotopy, being related to the coefficient ring $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, ...]$ and $ANSS$. In fact, in $ANSS$ for π_*^S, the first line E_∞ is generated by the image of J-homomorphism for spheres, i.e., the first line of $ANSS$ is related to K-theory. It is believed that higher lines are related to higher (non-linear) cohomology theories corresponding to $v_2, v_3, ...$ so that stable homotopy splits into cohomology theories. Elliptic cohomology is an effort to grasp the v_2-part.

The J-homomorphism is introduced by G.W. Whitehead in 1942 for spheres in its unstable form by $J : \pi_i(O(n)) \to \pi_{n+i}(S^n)$. In stable form, it is induced by the inclusion of the classical transformation group O into the loop space $\Omega^\infty S^\infty$ of maps from S^∞ to itself:

$$J : \pi_i(O) \to \pi_i^S.$$

In the most general form, it is the map

$$J : \hat{KO}^{-1}(X) = [X, O] \to \hat{\pi}_S^0(X)$$

defined in the following way: Let $w \in KO^{-1}(X)$ represented by a map $f : X \to O(n)$ with adjoint $\hat{f} : X \times \mathbb{R}^n \to \mathbb{R}^n$, $\hat{f}(x, v) = f(x)(v)$. Then $\hat{f}$ induces a map

$$T(\hat{f}) : X^{\mathbb{R}^n} = X \times D^n / X \times S^{n-1} \to D^n / S^{n-1} = S^n.$$

We identify $X^{\mathbb{R}^n}$ with $S^n(X^+)$. The stable map is independent of all choices and defines the element $J(w) = [T(\hat{f})]$ in the *zero*-th group of the cohomotopy ring of X, $\pi_S^0(X) = \{X^+, S^0\}$ and, by subtracting the degree, in the reduced group. One can localize everything at p when desired.

Adams, [4], introduced the e-invariant homomorphism $e : \pi_i^S \to \mathbb{Q}/\mathbb{Z}$, which is basically a tool using ordinary cohomology with coefficients $\mathbb{Q}$ (the cohomology theory below the topological K-theory!) to show that a homotopy class is non-zero, and showed that $e \circ J$ is injective so that the image of J is a direct-summand of the stable homotopy ring π_*^S.

$Im(J)$-theory is followed and developed by Mahowald, Miller, Knapp and others. Basically, the whole point of view can be summarized by the deep Mahowald-Miller theorem: 'v_1-localization of stable homotopy is equivalent to $Im(J)$ with finite coefficients', see [19, Section 3].

It is of interest to investigate J-homomorphism for other CW-complexes. The main technical tool for this is the Adams conjecture. The conjecture in its modern form says that $J \circ (\psi^k - 1)$ is null-homotopic at p for $(k, p) = 1$ where $J : BU_{\Lambda(p)} \to B\Omega^\infty S^\infty_{(p)}$, the complex or the real J-map localized at p and $\psi^k - 1 : BU_{\Lambda(p)} \to BU_{\Lambda(p)}$ is the Adams map. This is equivalent to the fact that $\psi^k - 1$ can be lifted to a map

$$BU_{\Lambda(p)} \to (\Omega^\infty S^\infty / U_\Lambda)_{(p)}.$$

We note that this lift is not unique and there are still some open and difficult problems related. Of course, the algebraic consequence on vector bundles, i.e., roughly J''-definition of Adams, is independent from liftings. Using this conjecture, J-homomorphism can be extended to the various $Im(J)$-theories, see e.g., [19, Section 2] and [20]. The idea is the same: to grasp a summand of a (co)homotopy group to solve the related problem, if possible.

We want to point out a challenging problem where J-homomorphism plays a crucial role. Let $X \in cw$ and ξ be a real vector bundle over X. We have a natural inclusion map $i : S^n = (pt.)^n \to X^\xi$ of Thom complexes where $n = dim\xi$. Consider the homomorphism

$$i^* : \{X^\xi, S^n\} \to \{S^n, S^n\} = \mathbb{Z}$$

of stable cohomotopy groups. Then, the codegree of ξ, which we denote by $d(X, \xi)$, is defined to be the non-negative generator of image of i^*, i.e., $d(X, \xi)$ is the least positive integer r such that the map $S^n \to S^n$ of degree r can be stably extended to X^ξ. Since a stable fibre homotopy equivalence of bundles induces a stable homotopy equivalence of their Thom complexes, we may regard $d(X, -)$ as a function from $J(X)$ to $\mathbb{Z}$. The following result is the most we can get in general, see [55]: p is a divisor of $d(X, \xi)$ iff p is a divisor of the order of $J(\xi)$. This result is a consequence of the inverse Dold Theorem, see [24]. Codegrees of multiples of Hopf bundles over projective spaces are believed to be determined by J-homomorphism.

2.6 Topological K-theory and Algebraic K-theory

We give, for the sake of completeness, a fable mixture of connections between AKT and TKT which are not well-understood yet. AKT is quite deep compared to TKT and we appologize for any nonsense argument stemming from the author's misunderstanding. What makes AKT uncomphensible and confusing with TKT, roughly, lies in the fact that given a topological group G, one can consider it with discrete topology G^δ. Then the classifying space BG^δ is used to define the corresponding

AKT by means of Quillen's terrifying definitions whereas the topological classifying space BG is used for TKT and these definitions are quite distinct. Just to give a feeling we note that $K_1(\mathbb{R}) = \mathbb{R}^*$ but $K_1(\mathbb{R})_{top} = \mathbb{Z}_2$ (the components of $\mathbb{R}^*$!). Algebraic K_2-groups of fields are given by Matsumoto, and for higher groups, there are few known computations. See [56],[54] for $\mathbb{R}$ and $\mathbb{C}$. If we consider discrete objects, e.g., $\mathbb{Z}$; we get rid of the confusion, since there is one sensible topology to put on $GL\mathbb{Z}$: the discrete topology.

Let us give a survey on the effect of TKT in AKT.

Let S_n be symmetric group on n letters and define $S_\infty = \lim_{n\to\infty} S_n$. The elegant Barrat-Priddy-Quillen theorem tells us that

$$\Omega^\infty S^\infty \simeq \mathbb{Z} \times BS_\infty^+.$$

In fact, the infinite loop space $\Omega^\infty S^\infty$ can be obtained by the May/Segal machinery to category of finite sets, hence the slogan 'stable homotopy groups of spheres = K-theory of the category of finite sets', see [46].

We have the following natural maps combining all in K-theory

$$BS_\infty^+ \to BGL(\mathbb{Z})^+ \to BGL(\mathbb{R})^+ \to BO \simeq BGL(\mathbb{R})_{top}^+,$$

induced by the obvious group homomorphisms and $+$-construction where subscript $GL(\mathbb{R})_{top}$ denotes the corresponding topological group attached to the discrete group. Passing to the homotopy groups we have

$$\pi_j^S \to K_j\mathbb{Z} \to K_j\mathbb{R} \to K_j\mathbb{R}_{top}.$$

We know only the groups $K_j\mathbb{R}_{top} = \pi_j(BO)$ completely which we described as in TKT. The whole coposite is the KO-theory degree map. Let

$$BO \to BO \otimes \mathbb{Q} \simeq \prod K(\mathbb{Q}, 4n)$$

be rational localization map and $F\mathbb{R}$ be its fibre. Then $F\mathbb{R}$ is a retract of $BGL\mathbb{R}^+$ giving a computable part of $K_*\mathbb{R}$ and the obvious map $\Omega^\infty S^\infty \to F\mathbb{R}$ is $-e$ where e is the Adams e-invariant, [56]. Little is known for the other groups and maps and one generally prefers to look at them with finite coefficients. We obtain a part of $K_j\mathbb{Z}$, if we consider the composite $J(\pi_j O) \subset \pi_j^S \to K_j\mathbb{Z}$. For $j = 4s-1$, this is an injection, in particular direct summand in some cases (Quillen). The remaining parts ($\mathbb{Z}_2$-parts) of the classical J-homomorphism map to zero (Waldhausen). On the other hand we have Adams families μ_{8k+1} and μ_{8k+2}, coming from a connective version of $Im(J)$-theory, see [38], which map to $\mathbb{Z}_2$ accordingly. These parts also map to $K_*\mathbb{R}$ and $K_*\mathbb{C}$ in a way as explained in [56]. For complete information, see [46]. That is all the feed-back from topological K-theory to $K_*\mathbb{Z}$ and $K_*\mathbb{R}$.

In general instead of $\mathbb{Z}$, we can consider the ring of algebraic integers $\mathcal{O}_F$ in a number field F and look for $K_*\mathcal{O}_F$. Complete computation of these groups is an important problem and has applications in number theory. For example, $K_0\mathcal{O}_F$ is the ideal class group of F. See [37], for the applications in Kummer-Vandiver and Iwasawa conjectures about the cyclotomic fields. Unfortunately, topology of $BGL\mathcal{O}_F^+$ is complicated and for this reason, the problem has gone out of topology very much, being spoken in various sheaf cohomologies with nasty coefficients. See [46], for the Quillen-Lichtenbaum conjectures, which relate these groups with étale cohomology and for a conjectural calculation of $K_*\mathbb{Z}$. Recent results of Veovodsky with works of Bloch, Lichtenbaum, Suslin and othsers have effectively led to the proof of these conjectures letting to determine the p-adic homotopy of the algebraic K-theory spectra of number rings. These are quite technical and beyond TKT. We also point out works, which are closer to TKT, of Boksedt, Madsen and Rognes on the spectra of p-adics $BGL(\hat{\mathbb{Z}}_p)^+$.

Algebraic K-theory of integers can be a tool for integral representations of finite groups, preferably cyclic groups. This is the important fact from our simple point of view but obviously too far to be tangible. See Section 3.8 for a discussion.

We now turn to our main interest, i.e., Algebraic K-theory for the fields $\mathbb{F}_q$, $\mathbb{C}$ and $\mathbb{R}$. It turns out that things are still quite enigmatic.

$\mathbb{F}_q$-case is well-understood. We recall the spectrum $K\mathbb{F}_q$, introduced by Quillen who also calculated $K_*\mathbb{F}_q$ as in Section 2.3, where $q = l^v$ and l a prime. This spectrum is in fact very close to an $Im(J)$-theory spectrum. Let $j(q) = Ker(\psi^q : bu \to S^2bu)$ where bu is the connective version of K, the complex K-theory spectrum. Then a deep result is that $K\mathbb{F}_q\hat{} \cong j(q)\hat{}$, where $\hat{}$ stands for l'-completion, l' chosen as in the definition of Ad of Knapp, [46]. We will demonstrate this in Section 3.8, by the computations of $R\mathbb{F}_q(\mathbb{Z}_{p^n})\hat{}$. We also note that $K\mathbb{F}_q$, $v = 1$, enters in the sequence of maps above via the obvious homomorphism $\mathbb{Z} \to \mathbb{F}_l$ and is compatible with $Im(J)$-image.

We desire that AKT for fields $\mathbb{C}$ and $\mathbb{R}$, roughly, reduces to TKT we deal with. Due to Suslin, this is the case only if we consider them with finite coefficients. In general, $K_*(k)$ is almost unknown for an infinite field k and things are very deep. See the remarkable [54]. The following observation may give an idea: Let G be a topological group. Then the obvious map $BG^\delta \to BG$ induces homology equivalence with finite coeffecients, [45]. Therefore, it is reasonable to pass on finite coeffecients. Of course, doing this, we loose a lot of information, namely divisible groups.

Let X be a CW-complex or an infinite loop space, we define, whenever it makes sense, j-th homotopy group of X with coefficients $\mathbb{Z}/m$ by

$$\pi_j(X;\mathbb{Z}/m) = [Y_m^j, X]$$

where $Y_m^j = S^{j-1} \cup_m e^j$ is the *mod m* Moore space of dimension j. In spectrum level, we write

$$E/m = E \wedge M_m$$

for the spectrum E with coeffecients m, where M_m is the Moore spectrum *mod m*.

We use the following exact sequences, [18],

$$0 \to \pi_k(X) \otimes \mathbb{Z}/m \to \pi_k(X;\mathbb{Z}/m) \to Tor(\pi_{k-1}(X), \mathbb{Z}/m) \to 0$$

which are analogue of the universal coefficient theorems of singular homology. To get a good grasp on these groups, we should analyse the homotopy of Moore spaces. This is, of course, as difficult as π_S^*. The fundemental feature of this definition is related to the order of the self-map of Y_m, $1_{Y_m} : Y_m \to Y_m$. See Proposition 1.5 in the usual reference [18].

Since $Y_{p^n}^j$, $j \geq 0$ are sub-complexes of the infinite lens space *mod* p^n, i.e., $B\mathbb{Z}_{p^n} = L^\infty(p^n)$, it is interesting to compare $KO(L^\infty(p^n))$ with $K_*(\mathbb{R};\mathbb{Z}_{p^n}) = (M_{p^n} \wedge K\mathbb{R})_*(pt.)$ where M_{p^n} is the Moore spectrum *mod* p^n. Simalarly, one can consider $\mathbb{C}$ and finite fields $\mathbb{F}_q$. We can, hence, point out some superficial connections between our computations on lens spaces and AKT of fields. In fact, there is a more intrinsic relation between $BGL(\mathbb{R})^+$ and the classifying spaces $B\mathbb{Z}_m$ of cyclic groups, [56].

Finally, we note that we have the natural map, connecting the integral representation theory of a finite group G to $K\mathbb{Z}$-cohomology of its classifying space,

$$\alpha : R\mathbb{Z}(G) \to K\mathbb{Z}^0(BG)$$

for which we do not know yet if there is the analogue of the Atiyah-Segal isomorphism.

Chapter 3

$K\Lambda$-rings of lens spaces

An important class of manifolds with constant Riemannian curvature are the manifolds, called spherical space forms, of the form S^{n-1}/G where G is a finite group which acts freely and orthogonally on the $(n-1)$th sphere S^{n-1}. Every spherical space form $M = S^{n-1}/G$ is connected, compact, orientable manifold without boundary, with a CW-structure and such that $\pi_1(M) = G$ if $n > 2$ and $\pi_i(M) \cong \pi_i(S^{n-1})$ if $i \neq 1$. See [43] and [28] for the classification of spherical space forms. G is a spherical space form group iff it satisfies all pq conditions, i.e. all subgroups of order pq, $\forall p, q$ prime, are cyclic. In particular, this implies the sylow subgroups of G are either cyclic or quternion.

We shall consider one kind of these manifolds, standard Lens spaces $L^k(m)$ *mod* m. $m, k \in \mathbb{Z}^+$, which are defined as follows:
Let S^{2k+1} be the unit $(2k+1)$-sphere in $\mathbb{C}^{k+1}$, $S^{2k+1} = \{(z_0, z_1, ..., z_k) | \sum |z_i|^2 = 1\}$. Let γ be the rotation of order m and weight $(1,1,...,1)$ of S^{2k+1} given by

$$\gamma.(z_0, z_1, ..., z_k) = (e^{2\pi i/m} z_0, e^{2\pi i/m} z_1, ..., e^{2\pi i/m} z_k)$$

Then γ generates a transformation group $\mathbb{Z}_m =< \gamma >\subset S^1$ and S^{2k+1} is a free $\mathbb{Z}_m$-space. We define the standard Lens space *mod* m to be the quotient

$$L^k(m) = S^{2k+1}/\mathbb{Z}_m.$$

The K and KO rings of the standard lens space $L^k(m)$ are investigated by several authors. Especially KO-rings brought a lot of calculations, e.g. see [30] for the computations for the lens spaces modulo powers of 2 where the additive structure of these rings is given. For $m = 2$. $L^k(2) = \mathbb{R}P^k$ is the real projective space and $K\Lambda(\mathbb{R}P^k)$ are determined in [5]. This computation gave Adams the chance to solve the famous

vector field problem on spheres. When $m = p$ is an odd prime, the structure of the reduced K and KO-rings of $L^k(p)$ is given by T. Kambe, [33]. The additive groups $\tilde{K\Lambda}(L^k(p^r))$ where p is an odd prime are determined by N. Muhammed, [42]. For a good reference of the problem and the computations for $m = 4$ with applications see [40]. And for the whole subject, as a guide, we recommend [43].
The works cited above approach the problem by making the direct summand decompositions of the rings and determining the generators of components. This approach is quite messy. On the other hand this does not explain the basic ring structure of $\tilde{KO}(L^k(m))$. We will see that there are additive and multiplicative relations on the generators which contain all the information. They are so obvious that we will not need any complicated spectral sequence argument or any inductive analysis on skeletons of Lens spaces to understand the rings $\tilde{KO}(L^k(m))$.

The computation of $\tilde{K\Lambda}(L^k(m))$ is important since it brings a set of relations which may help to discover new results in homotopy theory and generalized (co)homology theories. To give an example, we recall the Milnor computation of the dual of the Steenrod algebra, [44], in which he used lens spaces *mod p* as test spaces.

As a classical application, we study the immersions and embeddings of lens spaces in Euclidean space using γ-operations.

As another application, we obtain lower bounds for the stable orders of some stunded lens spaces following H. Yang.

Finally, we will make a topological discussion on representation rings of cyclic groups over fields and rings.

3.1 $K\Lambda$-rings of Projective spaces

In this section, we give the description of $K\Lambda$-rings of projective spaces ΛP^k. Especially, the results for $\mathbb{C}P^k$ are important for us. Our main reference is [43]
Let ξ_k and η_k denote the classical line bundles over the projective space $\mathbb{R}P^k$ and $\mathbb{C}P^k$ respectively. Let $\lambda_k = [\xi_k] - 1 \in \tilde{KO}(\mathbb{R}P^k)$ and $\mu_k = [\eta_k] - 1 \in \tilde{K}(\mathbb{C}P^k)$ be their reductions and $w_k = r(\mu_k) \in \tilde{KO}(\mathbb{C}P^k)$ be the realification of μ_k. We will omit the subscripts when it is understood.

Proposition 3.1.1.
(*i*) $K(\mathbb{C}P^k) \cong \mathbb{Z}[\mu]/ < \mu^{k+1} >$, $K^1(\mathbb{C}P^k) = 0$.
(*ii*) The operations $\psi^p_{\mathbb{C}}$ are given by $\psi^p_{\mathbb{C}}(\mu) = (1+\mu)^p - 1$.
Proof. [43], [32], [7].

For real K-theory of $\mathbb{C}P^k$ see especially [7] where a detailed analysis on skeletons is given and also it is shown that the J-order of the Hopf bundle is exactly the Atiyah-Todd number.

Proposition 3.1.2.
(i) $KO(\mathbb{C}P^k) = \mathbb{Z}[w]/I$ where I is the ideal generated by the following elements depending on k :

$$\begin{array}{ll} 2w^{[k/2]+1}, w^{[k/2]+2} & k = 4s+1 \quad (s \geq 0) \\ w^{[k/2]+1} & \text{otherwise} \end{array}$$

(ii) The operations $\psi^p_{\mathbb{R}}$ are given by $\psi^p_{\mathbb{R}}(w) = T_p(w)$ where T_p is the unique polynomial with integral coefficients such that $T_p(z + z^{-1} - 2) = z^p + z^{-p} - 2$.
(iii) $c : KO(\mathbb{C}P^k) \to K(\mathbb{C}P^k)$ is monomorphism if $k \not\equiv 1$ $(mod\ 4)$.
Proof. [43], [7].

We set $\nu_k = c(\xi_k) - 1$. There is a natural map $q : \mathbb{R}^{2k+1} \to \mathbb{C}P^k$ and one can show that $q^!(\mu) = \nu$, [5]. We have the following, [43], [32], [5]:

Proposition 3.1.3.
(i) $K(\mathbb{R}P^k) = \mathbb{Z}[\nu]/ < \nu^2 + 2\nu, \nu^{[k/2]+1} >$. In particular $K(\mathbb{R}P^{2k+1}) \cong K(\mathbb{R}P^{2k})$ is a cyclic group of order 2^k.
(ii) The operations $\psi^p_{\mathbb{C}}$ are given by

$$\psi^p_{\mathbb{C}}(\nu) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \nu & \text{if } p \text{ is odd} \end{cases}$$

Proposition 3.1.4.
(i) $KO(\mathbb{R}P^k) = \mathbb{Z}[\lambda]/ < \lambda^2 + 2\lambda, \lambda^{f(k)+1} >$ where $f(k)$ is the number of integers q with $q \equiv 0, 1, 2$ or 4 $(mod\ 8)$ and $0 < q \leq k$. In particular, the group $KO(\mathbb{R}P^k)$ is cyclic of order $2^{f(k)}$.
(ii) The operations $\psi^p_{\mathbb{R}}$ are given by

$$\psi^p_{\mathbb{C}}(\lambda) = \begin{cases} 0 & \text{if } p \text{ is even} \\ \lambda & \text{if } p \text{ is odd} \end{cases}$$

For the $K\Lambda$-rings for quaternionic projective space $\mathbb{H}P^k$, see [43].

Let X be a CW-complex, we define $X^n_m = X^n/X^{m-1}$ which are called stunded spaces (related to X). One also would like to know the $K\Lambda$-rings for the stunded projective spaces ΛP^n_m. These rings are found roughly by equating the additive generators of filtration less then m to zero. But in general we need some modifications around filtration m. See [5],[7] for the real and the complex stunded projective spaces.

3.2 Topology of Lens spaces

We define a CW-structure on S^{2k+1} by letting

$$\sigma^{2k'+1} = \{(z_0, \ldots, z_{k'}, 0, \ldots, 0) \in S^{2k+1} | 0 < \ arg \ \ z_{k'} < \frac{2\pi}{p^n}\} \text{ and}$$

$$\sigma^{2k'} = \{(z_0, \ldots, z_{k'}, 0, \ldots, 0) \in S^{2k+1} | Im \ \ z_{k'} = 0\} \qquad (0 \leq k' \leq k)$$

and by taking all the translates of these cells under the action of $\mathbb{Z}_m$ so that we have a total of m cells at each dimension. The group $\mathbb{Z}_m$ permutes these cells and hence this CW-structure on S^{2k+1} induces a CW-structure on $L^k(m)$ which has a single cell at each dimension between 0 and $2k+1$.

Thus $L^k(m)$ has a cell structure

$$L^k(m) = e^0 \cup e^1 \cup \ldots \cup e^{2k} \cup e^{2k+1}$$

and its cohomology groups are given by, [40],

$$H^i(L^k(m); \mathbb{Z}) = \begin{cases} \mathbb{Z}_m & \text{for } i = 2, 4 \ldots, 2k \\ \mathbb{Z} & \text{for } i = 0, 2k+1 \\ 0 & \text{otherwise} \end{cases}$$

$$H^i(L^k(m); \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2 & \text{for } m \text{ even or } m \text{ odd and } i = 0, 2k+1 \\ 0 & \text{otherwise} \end{cases}$$

By definition, these give the cohomology groups with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients of the groups $\mathbb{Z}_m$, by taking limit as k tends to infinity. See [17], for the group cohomology computation and the ring structure.

We have the following important fact about the CW-structure of $L^k(m)$: Let $L^k_0(m)$ denote the $2k$-th skeleton of $L^k(m)$. Then $L^k_0(m)/L^{k-1}_0(m) = S^{2k-1} \cup_m e^{2k}$, i.e., each even cell is connected by a map of degree m. This is important in the analysis of the exact sequences for pairs of skeletons of $L^k(m)$.

As we noticed from cohomology considerations, even case is a little involved. Take $m = 2l$. Let η be the canonical complex line bundle (Hopf bundle) over $L^k(2l)$. Then, the first chern class of η, $y = c_1(\eta)$ is a generator of $H^2(L^k(2l); \mathbb{Z}) \cong \mathbb{Z}_{2l}$. Let $x \in H^1(L^k(2l); \mathbb{Z}_2) \cong \mathbb{Z}_2$ be the generator such that $\beta x = y$ where $\beta : H^1(L^k(2l); \mathbb{Z}_2) \to H^2(L^k(2l); \mathbb{Z})$ is the Bockstein homomorphism associated with the sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}_2 \to 0$. Let ξ be the non-trivial real line bundle with the first Stiefel-Whitney class x. We have the following :

Proposition 3.2.1. $c(\xi) = \eta^l$.
Proof. [40].

A representation-theoretic approach shows that the situation is most simple. The inclusion homomorphism $\mathbb{Z}_m \hookrightarrow S^1$ induces a map on classifying spaces, thus on finite skeletons $\pi : L^k(m) \to \mathbb{C}P^k$. Let $\eta : S^1 \to U(1)$ be the identity map. It defines by the usual construction a complex line bundle over $\mathbb{C}P^k$ and $L^k(m)$, both denoted by η. It is clear that $\pi^!(\eta) = \eta$. If $m = 2l$, the representation $\xi : \mathbb{Z}_{2l} \to O(1)$, $\xi(x) = (-1)^x$, defines ξ. We have $c(\xi) = \eta^l$.

3.3 K-Rings of $L^k(m)$

Let η_k be the canonical complex line bundle over $L^k(m)$ as defined above and $\mu_k = \eta_k - 1 \in \tilde{K}(L^k(m))$ be the reduction of η_k. We will omit subscripts when the dimension is understood.

Theorem 3.3.1.
(i) $K(L^k(m)) = \mathbb{Z}[\mu]/ < \mu^{k+1}, (1+\mu)^m - 1 >$
(ii) The operations $\psi^p_{\mathbb{C}}$ are given by

$$\psi^p_{\mathbb{C}}(\mu) = (1+\mu)^p - 1.$$

Proof : [42].

We give a quick proof using the Atiyah exact sequence in Section 2.2. We have the exact sequence below

$$0 \to K^1(L^k(m)) \to R(\mathbb{Z}_m) \xrightarrow{\Phi} R(\mathbb{Z}_m) \xrightarrow{\theta} K^0(L^k(m)) \to 0.$$

where $R(\mathbb{Z}_m) = \mathbb{Z}[\eta]/ < \eta^m - 1 >$. Φ is multiplication by

$$[\sum_{i=0}^{k+1}(-1)^i \wedge^i \mathbb{C}^{k+1}] = (1-\eta)^{k+1} = (-\mu)^{k+1}.$$

Hence (i) follows.
(ii) follows from Theorem 3.1.1 (ii).

Corollary 3.3.2.
(i) The canonical inclusion $i : L^k_0(m) \to L^k(m)$ induces isomorphism $i^! : \tilde{K}(L^k(m)) \to \tilde{K}(L^k_0(m))$.
(ii) $\#\tilde{K}(L^k(m)) = m^k$.
(iii) The order of μ^i, $1 \le i \le k$, is $\prod_{j=1}^s p^{m_j + [\frac{k-i}{p_j - 1}]}$ where $m = \prod_{j=1}^s p_j^{m_j}$ is the prime

decomposition of m.

Proof :

(i) follows from the exact sequence of the pair $(L^k(m), L_0^k(m))$,

(ii) follows from the AHSS and

(iii) follows from Proposition 3.3.3 (1) below and the computations for $m = p^n$ where p is a prime number using the relations $\mu^{k+1} = 0$ and $(1+\mu)^{p^n} = 1$. [43].

Let M be a spherical form S^{2k+1}/G where G acts freely on S^{2k+1} and G_p be the p-sylow subgroup of G. Let $M_p = S^{2k+1}/G_p$ and call M_p the spherical p-form associated to M. Using the AHSS and cohomological properties of group extensions, it is shown in [43, Section 5] that $K\Lambda$-rings of M are related to those of its associated p-forms M_p. In particular, we have :

Proposition 3.3.3. Let $m = \prod_{j=1}^{s} p_j^{m_j}$, $p_1 < p_2 < \ldots$ Then,

1. $\tilde{K}(L^k(m)) \cong \bigoplus_{j=1}^{s} \tilde{K}(L^k(p_j^{m_j}))$

2.

(i) if $k \not\equiv 0 \ (mod\ 4)$,

$$\tilde{KO}(L^k(m)) \cong \oplus_{j=1}^{s} \tilde{KO}(L^k(p_j^{m_j}))$$

(ii) if $k \equiv 0 \ (mod\ 4)$,

$$\tilde{KO}(L^k(m)) \cong \begin{cases} \mathbb{Z}_2 \oplus \bigoplus_{j=1}^{s} \tilde{KO}(L_0^k(p_j^{m_j})) & \text{if } p_1 > 2 \\ \tilde{KO}(L^k(2^{m_1})) \oplus \bigoplus_{j=2}^{s} \tilde{KO}(L_0^k(p_j^{m_j})) & \text{if } p_1 = 2 \end{cases}$$

where subscript 0 denote the $2k$-th skeleton.

Proposition 3.3.3 shows that it is enough to work on $K\Lambda$-rings of $L^k(p^n)$ and $L_0^k(p^n)$ where p is a prime number and $n \in \mathbb{Z}^+$.

Remark 3.3.4. Due to Theorem 3.3.1, $\tilde{K}(L^k(p^n))$ is generated, as a group, by μ, $\mu^2, \ldots, \mu^s$ where $s = min(k, p^n - 1)$. Since it is an Abelian group of order p^{kn}, we can write as a direct sum $\oplus_{i=1}^{s} G_i$ where $G_i \cong \mathbb{Z}_{m_i}$ whose generators are integral linear combinations of μ^i, $1 \leq i \leq s$. In [42], the precise values of integers m_i and the generators of G_i are given. In particular, the first $N = min(k, p-1)$ cyclic groups G_i are respectively generated by μ, $\mu^2, \ldots, \mu^N$. For this decomposition see [42].

3.4 KO-rings of Lens Spaces $L^k(2^n)$

Because of Proposition 3.3.3, we can restrict ourselves only to study of $\tilde{KO}(L^k(p^n))$, where p is a prime number. We can also assume that $n \geq 2$. $\tilde{KO}(L^k(p))$ is well-known. [5], [33].

In this section, we will deal with the case $p = 2$.
Let ξ be the non-trivial line bundle over $L^k(2^n)$, $n \geq 2$, which is described as in Section 3.2 and $\lambda = \xi - 1 \in \tilde{KO}(L^k(2^n))$. Let η be the canonical complex line bundle (the Hopf bundle) over $L^k(2^n)$ and $\mu = \eta - 1 \in \tilde{K}(L^k(2^n))$. Define $w = r(\mu) = r(\eta) - 2 \in \tilde{KO}(L^k(2^n))$ to be the realification of μ. We will see that λ and w generates $\tilde{KO}(L^k(2^n))$ by means of the relations between the corresponding representations. This is what we expect from the Atiyah-Segal isomorphism. See also [30] for the additive structure.
We have the following observation which shows that the ring $\tilde{KO}(L^k(2^n))$ is almost a subring of $\tilde{K}(L^k(2^n))$.

Proposition 3.4.1. $c : \tilde{KO}(L^{4t+3}(2^n)) \to \tilde{K}(L^{4t+3}(2^n))$, $t \geq 0$, is a monomorphism.
Proof. Since $c(\xi) = \eta^{2^{n-1}}$ by Proposition 3.3.1, we have

1. $c(\lambda) = 2^{n-1}\mu + higher\ order\ terms.$

From the fact that $cr = 1+t$, $c(w) = \eta+\eta^{-1}-2$ and $\eta c(w) = \eta^2-2\eta+1$. Substituting $\eta = 1 + \mu$, we have $c(w) = \mu^2/(1+\mu) = \mu^2 - \mu^3 + \ldots$, i.e.,

2. $c(w^i) = \mu^{2i} + h.o.t.$

Let $F = Imc$. Then F is the subgroup of $\tilde{K}(L^k(2^n))$, generated by the elements $c(\lambda^\epsilon w^i)$, where $i \geq 1$ and $\epsilon = 0, 1$. Then, it is clear from 1 and 2 and the relation $2^n\mu^i = 2^{n-1}\mu^{i+1} + h.o.t.$, which is obtained from Theorem 3.3.1 (i), that $|F| = 2^{(2t+1)n+2t+2}$. So $|\tilde{KO}(L^k(2^n))| \geq 2^{(2t+1)n+2t+2}$.
Consider the AHSS for $\tilde{KO}(L^k(2^n))$. The second level,

$$E_2^{p,-p} = \tilde{H}^p(L^{4t+3}(2^n); KO^{-p}(*)),\ 0 < p \leq 2k+1$$

is $\mathbb{Z}_2$ for $p \equiv 1, 2\ (8)$ and $\mathbb{Z}_{2^n}$ for $p \equiv 0\ (4)$. Hence, $|\tilde{KO}(L^k(2^n))| \leq 2^{(2t+1)n+2t+2}$.
We deduce that $|\tilde{KO}(L^k(2^n))| = 2^{(2t+1)n+2t+2}$ and hence, $c : \tilde{KO}(L^{4t+3}(2^n)) \to \tilde{K}(L^{k4t+3}(2^n))$ is a monomorphism.
As a corollary, we have the following

Proposition 3.4.2. $\lambda w = (\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)$ in $\tilde{KO}(L^k(2^n))$. In particular these relation has the following form:

$$\lambda w = 2^n w^2 + h.o.t.$$

Proof.

1. $c(\lambda w) = c(\lambda)c(w) = (\eta^{2^{n-1}} - 1)(\eta + \eta^{-1} - 2)$
   $= \eta^{2^{n-1}+1} + \eta^{2^{n-1}-1} - 2\eta^{2^{n-1}} - \eta - \eta^{-1} + 2.$
2. $c((\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)) = (\psi_{\mathbb{C}}^{2^{n-1}+1} - \psi_{\mathbb{C}}^{2^{n-1}} - 1)(c(w))$
   $= (\psi_{\mathbb{C}}^{2^{n-1}+1} - \psi_{\mathbb{C}}^{2^{n-1}} - 1)(\eta + \eta^{-1} - 2)$
   $= \eta^{2^{n-1}+1} + \eta^{-2^{n-1}-1} - 2 - \eta^{2^{n-1}} - \eta^{-2^{n-1}} + 2 - \eta - \eta^{-1} + 2$
   $= \eta^{2^{n-1}+1} + \eta^{2^{n-1}-1} - 2\eta^{2^{n-1}} - \eta - \eta^{-1} + 2.$

So, $c(\lambda w) = c((\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w))$. Let $k = 4t+3$. Then, c is monomorphism and we deduce that $\lambda w = (\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)$.
Let k be arbitrary and choose $k' = 4t+3$ such that $k' > k$. Consider the inclusion $j : L^k(2^n) \to L^{k'}(2^n)$. It is clear that by definitions of ξ and η, $j^!(\eta) = \eta$ and $j^!(\xi) = \xi$. $j^!$ commutes with $\div$ and $\otimes$ and thus $j^!(x) = x$, for all $x = f(\xi, \eta) \in \tilde{KO}(L^{k'}(2^n))$ and any polynomial f. Thus, the relation holds in $\tilde{KO}(L^k(2^n))$ for all k.

We recall from Theorem 3.1.2 that $\psi_{\mathbb{R}}^k(w) = T_k(w)$ where T_k is the unique polynomial with integral coefficients of degree k defined by $T_k(z + z^{-1} - 2) = z^k + z^{-k} - 2$.

Lemma 3.4.3. $T_k(w) = \sum_{j=1}^k \alpha_{k,j} w^j$ where $\alpha_{k,j} = \frac{\binom{k}{j}\binom{k+j-1}{j}}{\binom{2j-1}{j}}$.
Proof. [22].

The following lemma will be the key to the problem of dedecting the relations on powers of w.

Lemma 3.4.4. $r(\mu^k) = \sum_{r=0}^{[\frac{k}{2}]} \beta_{k,r} w^{k-r}$, where $\beta_{k,r} = \frac{\binom{2r}{r}\binom{k}{2r}}{\binom{k-1}{r}}$.

Proof : We prove by induction. It is true for $k = 1$. Assume that it is true for $< k$. Due to [7, Lemma A2], r commutes with ψ_Λ^k. Thus we have $r\psi_{\mathbb{C}}^k(\mu) = \psi_{\mathbb{R}}^k(w)$, i.e., $r((1+\mu)^k - 1) = \psi_{\mathbb{R}}^k(w)$. By induction,

$$r(\mu^k) = \sum_{j=1}^{k} \alpha_{k,j} w^j - \sum_{i=1}^{k-1} \binom{k}{i} \sum_{r=0}^{[\frac{i}{2}]} \beta_{i,r} w^{i-r}.$$

The coefficient of w^j in $r(\mu^k)$ is

$$\alpha_{k,j} - \sum_{r=0}^{k-j-1} \binom{k}{j+r} \beta_{j+r,r}.$$

If we equate it to $\beta_{k,k-j}$ and simplify, we find $\binom{k+j-1}{j} = \sum_{r=0}^{k-j} \binom{k-j}{r}\binom{2j-1}{j-r}$. By substituting $m = k+j-1$, $p = k-j$ we have $\binom{m}{j} = \sum_{r=0}^{p} \binom{p}{r}\binom{m-p}{j-r}$. This verifies the formula.

Corollary 3.4.5.

$$\sum_{p=0}^{k-2j} \sum_{r=p-[\frac{p}{2}]}^{p+j} \binom{2^n}{p+1} \beta_{2j+p,j+p-r} w^{j+r} = 0, \ 1 \leq j \leq [\tfrac{k}{2}].$$

in $\tilde{KO}(L^k(2^n))$. In particular, these relations have the following form:

$$2^{n+1} w^j = 2^{n-1} w^{j+1} + h.o.t.$$

Proof. These are simply the trivial relations $r(\mu^r\eta^{2^n} - \mu^r) = 0$. From $\eta^{2^n} = 1$, after the binomial expansion and multiplication by μ^{2j-1}, we have

$$\sum_{p=0}^{k-2j} \binom{2^n}{p+1} \mu^{2j+p} = 0.$$

Using Lemma 3.4.4, we realify both sides, then we have

$$\sum_{p=0}^{k-2j} \sum_{q=0}^{j+[\frac{k}{2}]} \beta_{2j+p,q} w^{2j+p-q} = 0.$$

Put $r = j + p - q$. It is clear from the expression that $j \le [\frac{k}{2}]$.
The coefficient of w^{j+r} of the relation is $\sum_{p=0}^{2r} \binom{2^n}{p+1} \beta_{2j+p,j+p-r}$. In particular, for $r = 0$, 2^{n+1} and for $r = 1$, $-2^{n-1} + 2^n(...)$. For $r \ge 2$, since $v_2(\binom{2^n}{2r}) \ge n + 3 - 2r$, it has at least $2^{n-2(r-1)}$ as a factor. Thus, we have $2^{n+1}w^j = 2^{n-1}w^{j+1} + h.o.t.$ which are annihilated faster than $2^{n-1}w^{j+1}$ by multiplication by 2.

We can collect all above to, [34],

Theorem 3.4.6. $KO(L^k(2^n)) = \mathbb{Z}[\lambda, w]/I$, $n \ge 2$, where I is the ideal generated by the following elements:

(i) $\psi_{\mathbb{R}}^{2^n+i}(w) - \psi_{\mathbb{R}}^{i}(w)$, $i \ge 0$, and terminating elements.
$w^{[\frac{k}{2}]+1}$, if $k \not\equiv 1 \ (mod\ 4)$.
$2w^{[\frac{k}{2}]+1}$ and $w^{[\frac{k}{2}]+2}$, if $k \equiv 1 (mod\ 4)$.

(ii) $2\lambda - \psi_{\mathbb{R}}^{2^{n-1}}(w)$, $\lambda^2 + 2\lambda$.

(iii) $\lambda w - (\psi_{\mathbb{R}}^{2^{n-1}+1} - \psi_{\mathbb{R}}^{2^{n-1}} - 1)(w)$.

Proof.
(i) The set of relations $r(\mu^i\eta^{2^n} - \mu^i) = 0$ $i \ge 0$, given in Corollary 3.4.5, are equivalent, by induction, to the set of relations $\psi_{\mathbb{R}}^{2^n+i}(w) - \psi_{\mathbb{R}}^{i}(w) = 0$:
It is clear for $i = 0$. Assume that it is true for $< i$. Put $\mu = \eta - 1$ in $r(\mu^i\eta^{2^n} - \mu^i) = 0$ and obtain $r(\eta^{2^n+i} - \eta^i) = r(\eta^{2^n+i} - 1) - r(\eta^i - 1) = 0$ using the assumption. We remind that these are the well-known periodicity relations which we mentioned in Section 2.2 and 2.3. The terminating relations on the powers of w come from Theorem 3.1.2 by pull-back via $\pi : L^k(2^n) \to \mathbb{C}P^k$.
(ii) follows from the relation $c(\xi) = \eta^{2^{n-1}}$. Since ξ is a real line bundle, $\xi^2 = 1$ and thus $(1 + \lambda)^2 = 1$.
(iii) is Proposition 3.4.2.

We will prove that these elements generate the ideal which defines the ring. Let (E_r, d_r) denote the AHSS for $\tilde{KO}(L^k(2^n))$. From the proof of Proposition 3.4.1, it follows that for $k = 4t + 3$, the differentials d_r, $r \ge 2$, on the main diagonal

$E_r^{p,-p}$ are zero. For other values of k this is also true in a standard way. Choose $k' = 4t+3$ such that $k' > k$ and let $j : L^k(2^n) \to L^{k'}(2^n)$ be the inclusion. $j^* : H^p(L^{k'}(2^n); KO^{-p}(*)) \to H^p(L^k(2^n); KO^{-p}(*))$ is epimorphism for $p \geq 0$. Let (E'_r, d'_r) be the AHSS for $\tilde{KO}(L^{k'}(2^n))$. By naturality of Atiyah-Hirzebruch spectral sequences, we have $d_r j^* = j^* d'_r$. Since, $d'_2 = 0$ and j^* is epimorphism on the diagonal of E'_2, we have $d_2 = 0$ and j^* is epimorphism on the diagonal E'_3. By induction, we prove our claim and also deduce that $j^! : \tilde{KO}(L^{k'}(2^n)) \to \tilde{KO}(L^k(2^n))$ is epimorphism. From this fact and Proposition 3.4.2 and Corollary 3.4.5, it is clear that the relations (i) and (iii) are the smallest relations.

Remark 3.4.7. If we put $n = 1$, then $2\lambda = \iota^!(w) = w$ by (ii). Thus, Theorem 3.4.6 reduces to Theorem 3.1.4, i.e. to the computation of $\tilde{KO}(\mathbb{R}P^{2k+1})$. This means that $n = 1$ case is also included.

Remark 3.4.8. It is interesting to look at the filtrations of the elements of $\tilde{KO}(L^k(2^n))$ in the spectral sequence. Consider the AHSS for $\tilde{KO}(\mathbb{C}P^k)$. Due to Lemma 2.4 in [7], $E_\infty^{8i+2j,-8i-2j}$ are generated by w^{2i+1}, $2w^{2i+1}$, w^{2i+2} for $j = 1, 2, 4$ respectively. Due to Theorem 3.4.6, this is the same for the spectral sequence of $\tilde{KO}(L^k(2^n))$. From relation (iii), we deduce that $E_\infty^{8i+1,-8i-1}$ is generated by λw^{2i} (or $2^n w^{2i}$).

Remark 3.4.9. It follows from Theorem 3.3.1 that $K(L^k(2^n)) \cong R(\mathbb{Z}_{2^n})/I^{k+1}$ where I is the augmentation ideal of $R(\mathbb{Z}_{2^n})$. We wonder if there is a similar quotient for $KO(L^k(2^n))$. We discuss only $k = 4t+3$, $t \geq 0$. From the Atiyah exact sequence $RO(\mathbb{Z}_{2^n}) \xrightarrow{\Phi} RO(\mathbb{Z}_{2^n}) \to KO(L^k(2^n)) \to 0$ and Theorem 3.4.6, we observe that Φ is multiplication by w^{2t+2} and hence, $KO(L^k(2^n)) = RO(\mathbb{Z}_{2^n})/ < w^{2t+2} >$. It is interesting to note that $I^{2t+2} =< w^{2t+2} >$ where I is the augmentation ideal of $RO(\mathbb{Z}_{2^n})$ for $n \geq 3$ but not for $n = 2$.

Corollary 3.4.10.
(i) For $k > 0$, the reduced KO-ring of the $2k$ skeleton $L_0^k(2^n)$ is given by

$$\tilde{KO}(L_0^k(2^n)) = \begin{cases} \tilde{KO}(L^k(2^n))/\mathbb{Z}_2 < 2^n w^{[\frac{k}{2}]} > & \text{if } k \equiv 0 \ (4) \\ \tilde{KO}(L^k(2^n)) & \text{otherwise.} \end{cases}$$

(ii) $\#\tilde{KO}(L^k(2^n)) = 2^{(n+1)[\frac{k}{2}]+1+\epsilon}$ where $\epsilon = 1$ if $k \equiv 1$ (4) and 0 otherwise.
(iii) The $\tilde{KO}$-order of w^i, $1 \leq i \leq [\frac{k}{2}]$, is $2^{n+2[\frac{k}{2}]+1-2i}$.
Proof.
(i) It follows from the exact sequence of the pair $(L^k(2^n), L_0^k(2^n))$ and the Atiyah-Hirzebruch spectral sequences of $L^k(2^n)$, $L_0^k(2^n)$ that the $\tilde{KO}$-rings are the same except for $k \equiv 0$ (4) and in that case a $\mathbb{Z}_2$ is cut in the top dimension generated by the class $2^n w^{[\frac{k}{2}]}$.

(ii) 1 for λ, $(n+1)[\frac{k}{2}]$ for w^i, $1 \leq i \leq [\frac{k}{2}]$ and ϵ for $w^{[\frac{k}{2}]+1}$.
(iii) follows from Corollary 3.4.5. See also [43].

3.5 KO-Rings of Lens Spaces $L^k(p^n)$, p odd

Let now p be an odd prime. There is no non-trivial real line bundle over $L^k(p^n)$ since $H^1(L^k(p^n);\mathbb{Z}_2) = 0$. We define μ and w as usual. The AHSS is fine. i.e.. the differentials map to zero. except for $k \equiv 0$ (4), in which case we have a $\mathbb{Z}_2$-summand, destroying the fact that $\tilde{KO}(L^k(p^n))$ is a p-group.

Lemma 3.5.1. The AHSS for $\tilde{KO}(L^k(p^n))$ collapses on the diagonal. If $k = 4t$ then $\tilde{KO}(L^k(p^n)) = \mathbb{Z}_2 \oplus \tilde{KO}(L_0^k(p^n))$ where the $\mathbb{Z}_2$-component is generated by an element ϵ satisfying $w\epsilon = \epsilon^2 = 2\epsilon = 0$. If $k \neq 4t$ then $\tilde{KO}(L^k(p^n)) \cong \tilde{KO}(L_0^k(p^n))$.
Proof. From the cohomological considerations in Section 3.2, $E_2^{p,-p} = \mathbb{Z}_{p^n}$ for $p \equiv 0$ (4) and otherwise is zero except when $k \not\equiv 0$ (4) and $p = 2k+1$ in which case $E_2^{2k+1,-2k+1} = H^{2k+1}(L^k(p^n));\mathbb{Z}_2) = \mathbb{Z}_2$ since $L^k(p^n)$ is orientable. It is clear that for $k \neq 4t$ we have no problem.
Let $k = 4t$ and consider the exact sequence of the pair $(L^k(p^n), L_0^k(p^n))$ in KO-cohomology. The connecting homomorphism $\delta : \tilde{KO}(L_0^k(p^n)) \rightarrow KO^1(L^k(p^n), L_0^k(p^n)) \cong KO^1(S^{2k+1}) = \tilde{KO}(S^{8t}) = \mathbb{Z}$ is zero since $\tilde{KO}(L_0^k(p^n))$ is a p-group from the AHSS for $\tilde{KO}(L_0^k(p^n))$. By suspension $\delta : KO^{-1}(L_0^k(p^n)) \rightarrow \tilde{KO}(L^k(p^n), L_0^k(p^n)) \cong \tilde{KO}(S^{2k+1}) \cong \mathbb{Z}_2$ is also zero. Thus, we have

$$0 \rightarrow \mathbb{Z}_2 \xrightarrow{j^!} \tilde{KO}(L^k(p^n)) \rightarrow \tilde{KO}(L_0^k(p^n)) \rightarrow 0$$

This shows that $\mathbb{Z}_2$ survives to infinity. Let $\epsilon = j^!(1)$. Then $\epsilon^2 = 2\epsilon = 0$. Since $w\epsilon$ has filtration $> 2k+1$. we have $w\epsilon = 0$ and thus the short exact sequence above splits.

Theorem 3.5.2 Let p be an odd prime and $n \geq 1$. Then, $KO(L_0^k(p^n)) = \mathbb{Z}[w]/I$ where I is the ideal generated by the elements $\psi_{\mathbb{R}}^{p^n+i}(w) - \psi_{\mathbb{R}}^i(w)$. $i \geq 0$ and the terminating element $w^{[\frac{k}{2}]+1}$.
Proof. From Lemma 3.5.1, it is clear that the AHSS for $\tilde{KO}(L_0^k(p^n))$ collapses. To see this in a different way, consider the complexification $c : \tilde{KO}(L_0^k(p^n)) \rightarrow \tilde{K}(L_0^k(p^n))$. As in Proposition 3.4.1, we have $c(w) = \mu^2 - \mu^3 + ...$ Then by Theorem 3.3.1, $|Im(c)| = p^{[\frac{k}{2}]n}$. On the other hand, by counting from the spectral sequence, $|\tilde{KO}(L_0^k(p^n))| \leq p^{[\frac{k}{2}]n}$. This shows what we want.
Therefore, to show that the periodicity relations are the required relations, it is enough to show that they can be reduced to equations in the form $p^n w^i = h.o.t.$

$i > 0$. The equation $\psi_{\mathbb{R}}^{p^n+i}(w) = \psi_{\mathbb{R}}^{i}(w)$ gives $(p^{2n} + 2p^n i)w = h.o.t.$ We should find $a, b \in \mathbb{Z}$ such that $a(p^{2n} + 2p^n) - (p^{2n} + 2p^n b) = p^n$, i.e., $(a-1)p^n + (a-b)2 = 1$. Since $(p^n, 2) = 1$, this is possible. Hence, we can get a relation of the form $p^n w = h.o.t.$ For higher powers of w, we make the same arguments. This shows that the periodicity relations are the smallest.

Corollary 3.5.3. The $\tilde{KO}$-order of w^i is $p^{n+[\frac{k-2i}{p-1}]}$.
Proof. Since c is monomorphism, this order is equal to the $\tilde{K}$-order of $c(w) = \mu^2 + h.o.t.$ which is equal to $\tilde{K}$-order of μ^2 which is equal to $p^{n+[\frac{k-2i}{p-1}]}$ by Corollary 3.3.2.

3.6 Embeddings and Immersions of Lens spaces

Let M be a compact N-dimensional manifold and let $\tau(M)$ denote its tangent bundle. Put $\tau_0(M) = \tau(M) - N \in \tilde{KO}(M)$. Consider the Grothendieck operations $\gamma^i : \tilde{KO}(X) \to \tilde{KO}(X)$ given in Section 2.3. A result of Atiyah is the following, [10]:

Theorem 3.6.1. M is immersed (embedded) in $\mathbb{R}^{n+k}$, then $\gamma^i(-\tau_0(M)) = 0$ for $i > k$ $(i \geq k)$.

We consider $L^k(m)$. It is well-known that $1 \oplus \tau(L^k(m)) = (k+1)r(\eta)$ [40]. Thus, $\tau_0(L^k(m)) = (k+1)w$.

Lemma 3.6.2. $\gamma_t(w) = 1 + wt - wt^2$.
Proof. [33].

Proposition 3.6.3. $L^k(m)$ can not be immersed (embedded) in $\mathbb{R}^{2k+2L(k,m)}$ (in $\mathbb{R}^{2k+2L(k,m)+1}$) where

$$L(k,m) = \max\{i | \binom{k+i}{i} w^i \neq 0\}$$

Proof.

$$\begin{aligned}
\gamma_t(-\tau_0(L^k(m))) &= \gamma_t(-(k+1)w) \\
&= \gamma_t(w)^{-k-1} && \text{since } \gamma_t \text{ is ring homomorphism [40].} \\
&= [1 + w(t - t^2)]^{-k-1} && \text{by Lemma 3.6.2} \\
&= \textstyle\sum_{i=0}^{\infty}(-1)^i \binom{k+i}{i} w^i (t - t^2)^i
\end{aligned}$$

Therefore, we obtain
$\gamma^i(-\tau_0(L^k(m))) \neq 0$, for $i = 2L(k,m)$ and
$\gamma^i(-\tau_0(L^k(m))) = 0$, for $i > 2L(k,m)$.

Corollary 3.6.4. $L^k(p^n)$ can not be immersed (embedded) in $\mathbb{R}^{2k+2L(k,p^n)}$

$(\mathbb{R}^{2k+2L(k,p^n)+1})$ where

$$L(k,p^n) = \max\{i|\binom{k+i}{i} \equiv 0 \ (mod\ p^{n+[\frac{k-2i}{p-1}]})\} \qquad \text{for } p \text{ odd}$$
$$L(k,2^n) = \max\{i|\binom{k+i}{i} \equiv 0 \ (mod\ 2^{n+2[\frac{k}{2}]+1-2i})\}$$

Proof. Follows from Corollary 3.4.10 and Corollary 3.5.3.

The cross-section problem of multiples of canonical line bundles over projective spaces is an important problem called generalized vector field problem. The immersion problem of a lens space $L^k(p^n)$ (like projective spaces) can be reduced to a cross-section problem for a bundle $mr(\eta)$ (the m-fold Whitney sum of $r(\eta)$) for some m depending on k, n. See [40, Theorem 7.7].

3.7 Stable Orders of Stunded Lens Spaces

The stable order of a space X, which is denoted by $|X|$, is defined to be the order of the stable identity map in the group $\{X,X\}$ of stable self-maps on X. The *mod p* stable order of X denoted by $|X|_p$ for a prime p is the order of this map in the group $\{X,X\}\otimes\mathbb{Z}_{(p)}$ where $\mathbb{Z}_{(p)}$ is the ring of integers localized at p.

If $|X|_p$ is finite, then $|X|_p x = 0$ for $x \in E^*(X)$ (or $E_*(X)$) where $E(-)$ is a reduced (co)homology theory localized at p. E^*-order of x gives a lower bound on $|X|_p$ for all $x \in E^*(X)$. We have an interesting application of this fact to stable order of stunded lens spaces. See [58], [59].

Let L^{2k+1} denote $L^k(p^n)$ and L^{2k} denote the $2k$-th skeleton $L_0^k(p^n)$. Define the stunded lens space $L_\alpha^\beta = L^\beta/L^{\alpha-1}$ taking the cells of dimension between α and β.

Lemma 3.7.1.
(i) $\tilde{K}$-order of $\mu^k \in \tilde{K}(L_{2k-1}^{2k+2m})$ is $p^{n+[\frac{m}{p-1}]}$, for any prime p.
(ii) Let $p=2$. $\tilde{KO}$-order of $w^k \in \tilde{KO}(L_{4k-1}^{4k+8l+4})$ is 2^{n+4l+3}.
Proof.
(i) follows from Corollary 3.3.2.
(ii) follows from Corollary 3.4.10. The spectral sequence for $\tilde{KO}(L_{4k-1}^{4k+8l+4})$ is fine and put $n := n$, $k := 2k+4l+2$, $i := k$ in $2^{n+2[\frac{k}{2}]+1-2i}$.

As a corollary we have

Theorem 3.7.2. $|L_{2k-1}^{2k+2m}|_p \geq p^{n+[\frac{m}{p-1}]}$ if p is odd; If $p=2$, $|L_{2k-1}^{2k+2m}|_2 \geq 2^{n+m+\epsilon}$ where $\epsilon = 1$ if k is even and $m \equiv 2\ (4)$ and $\epsilon = 0$ otherwise.

Remark 3.7.3. In [58] and [59], it is shown that the lower bounds given in Theorem

3.7.2 are actually the upper values, thus exact values. Of course, this is the difficult part of the problem. This is achieved by using the vanishing line of a mod p stable Adams spectral sequence.

3.8 Representations of Cyclic Groups

We have the following, by taking $k \to \infty$,

Theorem 3.8.1.

$$R\mathbb{C}(\mathbb{Z}_{p^n})\hat{} = \mathbb{Z}[\mu]/ < (1+\mu)^{p^n} - 1 >$$

$$R\mathbb{R}(\mathbb{Z}_{p^n})\hat{} = \begin{cases} \mathbb{Z}[\lambda, w]/I_e & \text{if } p = 2 \\ \mathbb{Z}[w]/I_o & \text{if } p \text{ is odd} \end{cases}$$

where I_e and I_o are ideals generated by the elements, except the terminating relations, given in Theorem 3.4.6 and Theorem 3.5.2 respectively.

We note the inclusion $c : RO(\mathbb{Z}_{p^n}) \subset R(\mathbb{Z}_{p^n})$, $\xi \to \eta^{2^{n-1}}$ and $r(\eta) \to \eta + \eta^{-1}$. We observe that the real case computation is related to the Galois invariance of $Gal(\mathbb{C}/\mathbb{R})$ in a fancy way. It produces the complex conjugation $t = \psi^{-1}$ and we compute the fix set of t.

One can consider Λ-representation rings when, e.g., $\Lambda = \mathbb{F}_q$, a finite field or $\Lambda = \mathbb{Q}$, the field of rational numbers. Instead of fields one can consider representations over any division ring in which case the problem becomes notoriously very difficult, but carries number theoretic information.

Let $q = l^v$, l a prime, $v \in \mathbb{Z}^+$. We recall from Section 2.3, the spectrum $K\mathbb{F}_q$ defined by the loop space $F\psi^q = Ker(\psi^q - 1 : BU \to BU)$. Rector proves an analogue of Atiyah's theorem, [36, Section 8],

$$R\mathbb{F}_q(G)\hat{} = [BG, \mathbb{Z} \times F\psi^k]$$

for a finite group G.

For an arbitrary p-group G, instead of using the spectral sequence, it follows from the defining sequence

$$K\mathbb{F}_q^0 \to \mathbb{Z} \times BU \overset{\psi^q - 1}{\to} BU$$

that, basically, $R\mathbb{F}_q(G)\hat{}$ consists of the characters χ such that $\psi^q(\chi) = \chi$. We observe also that if $p = l$ ($\mathbb{F}_p[G]$ is not semisimple!) then, we have nothing except trivial representations, due to periodicity. This is a consequence of the more general fact that modular $q = l^v$-characters are determined by l-regular components of group elements. Let $G = \mathbb{Z}_{p^n}$. It follows from the spectral sequence

$$H^i(BG; K_i(\mathbb{F}_q)) \Rightarrow R\mathbb{F}_q(BG)^\wedge$$

that $R\mathbb{F}_q(BG)^\wedge$ is generated by an element of dimension f which is the smallest number i such that $q^i \equiv 1(p)$.

The solution over $\mathbb{F}_q$ solves the problem over $\mathbb{Q}$, [46]. It is proved, by Segal, that $R\mathbb{Q}(G) = R\mathbb{F}_l(G)$ where l generates $(\mathbb{Z}/p^2)^*$, for p odd and G a p-group. Due to [47], for $p = 2$,

$$R\mathbb{Q}(BG) = \{\chi|\psi^3_{\mathbb{R}}(\chi) = \chi\} \subset RO(G)$$

where G is a 2-group. For cyclic groups, we have the easy deductions. This seems to be a consequence of the Galois invariance of the Galois group $Gal(\mathbb{C}(\zeta_p)/\mathbb{Q})$ where $\zeta_p = e^{\frac{2\pi i}{p}}$.

The computation of $\mathbb{Z}[\mathbb{Z}_{p^n}]$-modules, which are finiteley generated and projective over $\mathbb{Z}$, is a total mystery. See [21, Chapter 9] on integral representations and especially, §74 for the example $\mathbb{Z}_p$ and §81A. It is interesting that for $n \geq 3$, there are infinite number of indecomposable modules.

We sketch the approach of K-theory to the problem. There exists, probably, the usual AHSS

$$H^p(BG, \pi^{-p}(K\mathbb{Z})) \Rightarrow R\mathbb{Z}(G)^\wedge$$

for a finite group G. Here, we encounter the coefficient groups $\pi^{-p}(K\mathbb{Z}) = K_p(\mathbb{Z})$, i.e., $K_*(\mathbb{Z})$, the algebraic K-ring of $\mathbb{Z}$. We encounter coefficients which are related to Bernoulli numbers. This tells us that the problem is connected with number theory. In fact, we expect a finite part in the representation ring related to the class group of the corresponding group ring. The above hypothetical spectral sequence seems to give some information about the filtration of representations and some possible relations on them. But, of course, one should still work on the integral represantations to decide the generators and relations respecting the spectral sequence. This seems to be a very hard task. For the coeffecients, we note that topology of the loop space $BGL\mathbb{Z}^+$ is not well-understood compared to the loop spaces BO, BU or $BGL\mathbb{F}_l^+$. This is a hot and difficult subject and beyond of the scope of this thesis. Instead of $\mathbb{Z}$, one can take the ring of algebraic integers in a number field. On the other hand, algebraic K-groups of these rings which have also important consequences in number theory, are almost computed recently (1997).

In general, the problem is, [46],

Problem 3.8.2. To study the map

$$\theta : R\Lambda(G)^{\hat{}} \to K\Lambda^0(BG)$$

where Λ is a commutative ring. It is helpfull to invert a prime for getting closer to fields.

Conjecture 3.8.3. For G a p-group and $\Lambda = \mathbb{Z}[\frac{1}{p}]$, θ is an isomorphism.

Chapter 4

J-Groups of Lens Spaces

In this chapter, making use of the results on $\tilde{KO}(L^k(m))$ established in Chapter 3, we will study the groups $J(L^k(m))$. For reduced J-groups we will use the notation $\tilde{J}$ and we have the usual direct sum $J(X) \cong \mathbb{Z} \oplus \tilde{J}(X)$ via the map $J : KO(X) \to J(X)$ and the splitting $KO(X) = \mathbb{Z} \oplus \tilde{KO}(X)$.
By Proposition 3.3.3 and linearity of fibre homotopy equivalence, we can restrict ourselves to the study of $J(L^k(p^n))$, for p a prime.
We recall the J''-definition of Adams given in Section 2.4. For X a finite CW-complex, $J(X)$ can be defined by $KO(X)/KerJ$ where

$$KerJ = \sum_k (\bigcap_e k^e(\psi^k_{\mathbb{R}} - 1)\tilde{KO}(X)).$$

We also recall the J'-definition : Let X be a CW-complex. Then, $x = 0$ in $J(X)$ if and only if
(i) $w_1(x) = 0$ and
(ii) $\exists y \in \tilde{KO}(X)$ such that $\rho^k_{\mathbb{R}}(x) = \frac{\psi^k_{\mathbb{R}}(1+y)}{1+y} \in \tilde{KO}(X) \otimes Q_k$ for all $k \neq 0$.
In particular, x should be of even virtual dimension.

4.1 J-Groups of $L^k(p^n)$

We start with two remarks: Firstly, if $p = 2$ then there is a class of virtual dimension 1, namely $\lambda \in \tilde{KO}(L^k(2^n))$. This class, due to J'-definition, is not in $KerJ$. i.e. $J(\lambda) \neq 0$. In fact, a line bundle is trivial if and only if it is fibre homotopy trivial. Thus, if $KO(X)$ is generated by line bundles then $J(X) \cong KO(X)$, e.g., for

$X = \mathbb{R}P^k$, see [32].

Secondly, if p is odd and $k \equiv 0$ (4) then $\tilde{KO}(L^k(p^n))$ contains a $\mathbb{Z}_2$ summand generated by e which is pull-back from $\tilde{KO}(S^{2k+1}) \cong \mathbb{Z}_2$. We have $J(e) \neq 0$ since $\tilde{J}(S^{2k+1}) = \mathbb{Z}_2$. [2].

It follows from the definition above and the fact that $\tilde{KO}(L^k(p^n))$ is a p-group except for p odd and $k \equiv 0$ (4) and the remarks above that $KerJ$ is additively generated by the following elements

$$(\psi^k_{\mathbb{R}} - 1)(w^i) \quad \text{where} \quad (k,p) = 1$$

since if $(k,p) = 1$ then $\exists r$ such that $k^r \equiv 1$ (p^α) where p^α divides the order of the group $\tilde{KO}(L^k(p^n))$.

We can decrease the number of generators considerably, [25]. Let G_{p^n} be the group of units in $\mathbb{Z}_{p^n}$. We have

$$G_{p^n} = \begin{cases} \mathbb{Z}_{p^{n-1}(p-1)} & \text{if } p \text{ is odd} \\ \mathbb{Z}_2 \times \mathbb{Z}_{2^{n-2}} & \text{if } p = 2 \end{cases}$$

Let q be a generator of G_{p^n} if p is odd, a generator of second summand if $p = 2$. Then

Proposition 4.1.1. $KerJ$ is generated by the elements $(\psi^q_{\mathbb{R}} - 1)(w^i)$ $i \geq 1$.

Proof. It is enough to show that $J(u) = 0$ iff $u = (\psi^q_{\mathbb{R}} - 1)x$ for some $u, x \in \mathbb{Z}[w]$. Necessity part is clear. Let $u = (\psi^k_{\mathbb{R}} - 1)x$, $(k,p) = 1$ and $k = p_1 p_2 ... p_r$ be the prime decomposition of k. $\psi^k_{\mathbb{R}}$ are multiplicative and we can write

$$(\psi^k_{\mathbb{R}} - 1)x = (\psi^{p_1}_{\mathbb{R}} - 1)\psi^{p_2...p_r}(x) + (\psi^{p_2}_{\mathbb{R}} - 1)\psi^{p_3...p_r}(x) + ... + (\psi^{p_{r-1}}_{\mathbb{R}} - 1)\psi^{p_r}_{\mathbb{R}}(x).$$

Thus, it is enough to show for $k = p'$ a prime. Then $p' \equiv \pm q^m$ $(mod\ p^n)$ for some $m \in \mathbb{Z}^+$. Let η be the Hopf bundle over $L^k(p^m)$. Then, $\eta^{p'} = \eta^{\pm q^m}$ and by realification $\psi^{p'}_{\mathbb{R}}(w) = \psi^{q^m}_{\mathbb{R}}(w)$. Thus, $(\psi^{p'}_{\mathbb{R}} - 1)x = (\psi^{q^m}_{\mathbb{R}} - 1)x$ which is $(\psi^q_{\mathbb{R}} - 1)y$ for some $y \in \tilde{KO}(L^k(p^m))$ by the similar trick above.

Definition 4.1.2. Define $W'(k,n)$ and $W(k,n)$ to be the subgroups of $\tilde{KO}(L^k(p^n))$ and $\tilde{J}(L^k(p^n))$ respectively, generated by powers of w for a fixed prime p. $W'(k,n)$ is equal to $\tilde{KO}(L^k(p^n))$ for p odd and $k \not\equiv 0$ (4) and is a subgroup of index 2 otherwise and similarly for $W(k,n)$ and $\tilde{J}(L^k(p^n))$

As a corollary of Proposition 4.1.1, we have the following group isomorphism, [35], :

Theorem 4.1.3. $J(L^k(p^n)) = KO(L^k(p^n))/(\psi^q_{\mathbb{R}} - 1)W'(k,n)$.

Theorem 4.1.3 can be deduced easily using the exact sequence, [26].

$$KO_G(X)_{(p)} \xrightarrow{\psi^r_{\mathbb{R}} - 1} KO_G(X)_{(p)} \xrightarrow{J} J^{loc}_G(X)_{(p)}$$

where r generates the units of p-adic integers (modulo ± 1 when $p = 2$). Then r generates G_{p^n}. Let $G = \{1\}$ and $X = L^k(p^n)$ in the exact sequence above. We note that since we localize at p, the $\mathbb{Z}_2$-sum generated by ϵ when $k \equiv 0$ (4), p odd, disappears. For $p = 2$, we have $(\psi^r_{\mathbb{R}} - 1)\lambda = 0$, since r is odd. Thus we reproved the theorem.

Let $p = 2$. We can extend the result $J(\mathbb{R}P^n) = KO(\mathbb{R}P^n)$ which was proved by Adams in the vector field problem on spheres :

Corollary 4.1.4. $J(L^k(4)) = KO(L^k(4))$.
Proof. $q = 5 \equiv 1$ (mod 4). The result follows by periodicity given in Theorem 3.4.6 (i) and from Theorem 4.1.3.

It is clear from the KO & J relations that J-groups are smaller for $k \geq 3$, $p = 2$.

Corollary 4.1.5. $W(k, n)$ is generated (additively) by $\psi^{p^s}_{\mathbb{R}}(w)$, $0 \leq s \leq R(k, n) = \min(n - 1, r(k))$ where $p^{r(k)} \leq k + 1 < p^{r(k)+1}$.
Proof. It follows from Theorem 4.1.3 that $W(k, n)$ is generated by $\psi^{p^s}_{\mathbb{R}}(w)$, $s \geq 0$. We recall from Remark 3.3.4 that $\tilde{K}(L^k(p^n))$ is generated by μ^i, $i \leq \min(k, p^n - 1)$. Corollary follows from the fact that $r(\eta^i - 1) = \psi^i_{\mathbb{R}}(w)$.

Remark 4.1.6. Direct summand decomposition of $\tilde{J}(L^k(p^n))$ is a difficult problem and is related to Corollary 4.1.5. For p odd, it is shown in [39] that $W(k, n) \cong \bigoplus_{i=1}^{R(k,n)} \mathbb{Z}_{p^{a_s}}$ for some positive integers a_s, $0 \leq s \leq R(k, n)$ where $\mathbb{Z}_{p^{a_s}}$ is generated by the element $\psi^{p^s}_{\mathbb{R}}(w)$. For $\tilde{J}(L^k(2^n))$, similar results are obtained in [29]. The results are extremely messy arithmetic. We wonder if there is an geometric way to do that, i.e., a kind of inner-product operation on vector bundles which makes an orthogonalization.

Recall the definition of $J(G)$ for a finite group G from Section 2.4. We have the following result connecting the two theories :

Theorem 4.1.7. $J(\mathbb{Z}_{p^n}) = RO(\mathbb{Z}_{p^n})_{\Gamma_{p^n}}$.
Proof. Γ_{p^n} is isomorphic to $\mathbb{Z}^*_{p^n}$, via the map $\sigma \rightarrow i(\sigma)$ which is defined by $\sigma(\mu_{p^n}) = \mu_{p^n}^{i(\sigma)}$ where μ_{p^n} is a primitive p^nth root of unity. Then by definition of the action of Γ_{p^n} on $RO(G)$, $\psi^k_{\mathbb{R}}(w)$ and w define the same class in $RO(\mathbb{Z}_{p^n})_{\Gamma_{p^n}}$, for $(k, p) = 1$, where w denotes the reduced Hopf representation. Under the isomorphism $RO(\mathbb{Z}_{p^n}) \cong \tilde{KO}(L^\infty(p^n))$, the representation w corresponds to the reduced bundle w. From Theorem 4.1.3, we observe that $J(L^\infty(p^n)) \cong RO(\mathbb{Z}_{p^n})_{\Gamma_{p^n}}$. It also follows from definition of J-group for groups, [16] that $J(\mathbb{Z}_{p^n}) \cong J(L^\infty(p^n))$. The result follows by composing the two isomorphism.

4.2 Order of $\tilde{J}(L^k(p^n))$

To compute the order of $\tilde{J}(L^k(p^n))$, we establish exact sequences analogous to the ones given in [7].The main difficulty is to prove the injectivity of the map $c^! : \tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) \to \tilde{J}(L^{2v}(p^n))$, whereas the corresponding result, e.g. [7, Lemma 4.9], is trivial for complex projective spaces. In [25], this difficulty is resolved using the transfer map $\tau : \tilde{KO}(L^k(p^n)) \to \tilde{KO}(L^k(p^{n+1}))$ with a number of preliminary results concerning binomial expansions.

Note : To get rid of $\mathbb{Z}_2$-component which occurs when p is odd and $k \equiv 0\ (4)$, we localize everything at p. We will omit the subscript (p) in the notation throughout the section, i.e., we will write $F(X)$ instead of $F(X)_{(p)}$.

We have the following result :

Proposition 4.2.1. Let p be a prime and $n, v \in \mathbb{Z}^+$ and $c : L^{2v}(p^n) \to L^{2v}(p^n)/L^{2v-2}(p^n)$ be the map collapsing $L^{2v-2}(p^n)$ to point. Then the induced homomorphism

$$c^! : \tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) \to \tilde{J}(L^{2v}(p^n))$$

is injective.
Proof : [25].

Corollary 4.2.2. We have an exact sequence,

$$0 \to \tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) \xrightarrow{c^!} \tilde{J}(L^{2v}(p^n)) \xrightarrow{i^!} \tilde{J}(L^{2v-2}(p^n)) \to 0$$

Proof : The exactness of the four terms on the right follows from [2, Theorem 3.12], [3, Theorem 1.1] and the Adams conjecture. The injectivity of $c^!$ follows from Proposition 4.2.1.

Proposition 4.2.3. $c^! : \tilde{J}(L^{4v+1}(p^n)/L^{4v}(p^n)) \to \tilde{J}(L^{4v+1}(p^n))$ is injective.
Proof : $\tilde{KO}(L^{4v+1}(p^n)/L^{4v}(p^n)) = \mathbb{Z}_2 \otimes \mathbb{Z}_{(p)}$ (zero when p is odd!) and is generated by $2^n w^{2v}$ when $p = 2$. The proof is identical with that of Proposition 4.2.1.

Corollary 4.2.4. The following sequence is exact,

$$0 \to \tilde{J}(L^{4v+1}(p^n)/L^{4v}(p^n)) \xrightarrow{c^!} \tilde{J}(L^{4v+1}(p^n)) \xrightarrow{i^!} \tilde{J}(L^{4v}(p^n)) \to 0$$

Proof : Identical with that of Corollary 4.2.2.

We define some numbers which we encounter in computations of $\tilde{J}$-groups of quotients of lens spaces.

Definition 4.2.5. We define as in [2, Section 2] numbers $m(t)$ by

For p odd, $v_p(m(t)) = \begin{cases} 0 & \text{if } t \not\equiv 0 \ (mod \ (p-1)) \\ 1 + v_p(t) & \text{if } t \equiv 0 \ (mod \ (p-1)) \end{cases}$

For $p = 2$, $v_2(m(t)) = \begin{cases} 1 & \text{if } t \not\equiv 0 \ (mod \ 2) \\ 2 + v_2(t) & \text{if } t \equiv 0 \ (mod \ 2) \end{cases}$

Definition 4.2.6. Let p be a prime and $v, n \in \mathbb{Z}^+$. Define

$$\epsilon(p, v, n) = \begin{cases} p^{min\ (n, v_p(m(2v)))} & \text{if } p \text{ is odd} \\ p^{min\ (n+1, v_p(m(2v)))} & \text{if } p = 2 \end{cases}$$

Note that $\epsilon(p, v, n) = 1$ if $v \not\equiv 0 \ (mod \ (p-1))$.

For $v \equiv 0 \ (mod \ (p-1))$, $\epsilon(p, v, n) = p^{\epsilon_p + (min\ (n, 1+v_p(2v)))}$ where

$$\epsilon_p = \begin{cases} 0 & \text{if } p \text{ is odd} \\ 1 & \text{if } p = 2 \end{cases}$$

Lemma 4.2.7. Let p be a prime ; $v, n \in \mathbb{Z}^+$, $n \geq 2$ if $p = 2$. Then

$$\tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) = \mathbb{Z}_{\epsilon(p,v,n)}.$$

Proof :

$$\tilde{KO}(L^{2v}(p^n)/L^{2v-2}(p^n)) = \begin{cases} \mathbb{Z}_{p^n} & \text{if } p \text{ is odd} \\ \mathbb{Z}_{2^{n+1}} & \text{if } p = 2 \end{cases}$$

and is generated by w^v. By [3, Theorem 1.1] and the Adams conjecture, $\tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) = \tilde{KO}(L^{2v}(p^n)/L^{2v-2}(p^n))/W$ where $W = \cap_f W_f$ where W_f is the subgroup generated by

$$\sum_{k \in \mathbb{Z}^+} k^{f(k)}(\psi^k_{\mathbb{R}} - 1)w^v = \sum_{k \in \mathbb{Z}^+} k^{f(k)}(k^{2v} - 1)w^v.$$

Let K_p be the principal ideal in $\mathbb{Z}$ generated by p^n if p is odd and by 2^{n+1} if $p = 2$. Let $\phi_p : \mathbb{Z} \to \mathbb{Z}/K_p = \tilde{KO}(L^{2v}(p^n)/L^{2v-2}(p^n))$ be the surjection. Define $W'_f = \phi_p^{-1}(W_f)$ and $W' = \cap_f W'_f = \phi_p^{-1}(W)$. Let $h(f, 2v)$ be the greatest common divisor of the integers $k^{f(k)}(k^{2v} - 1)$. Then W'_f is the principal ideal generated by $h(f, 2v)$ and by [1, Theorem 2.7], W_f is the principal ideal generated by $m(2v)$.
Then, we compute that $\tilde{J}(L^{2v}(p^n)/L^{2v-2}(p^n)) = \mathbb{Z}/(W' + K_p)$ and $W' + K_p$ is the principal ideal generated by $\epsilon(p, v, n)$.

Proposition 4.2.8. Let p be a prime and $v, n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then

$$| \tilde{J}(L^{2v}(p^n)) | = \begin{cases} \prod_{v'=1}^{v} \epsilon(p, v', n) & \text{if } p \text{ is odd} \\ 2\prod_{v'=1}^{v} \epsilon(2, v', n) & \text{if } p = 2 \end{cases}$$

Proof : It is an immediate consequence of Corollary 4.2.2 and 4.2.7.

Proposition 4.2.9. Let p be a prime ; $v, n \in \mathbb{Z}^+$. Then

$$| \tilde{J}(L^{4v+1}(p^n)) | = \begin{cases} | \tilde{J}(L^{4v}(p^n)) & \text{if } p \text{ is odd} \\ 2 \mid \tilde{J}(L^{4v}(p^n)) \mid & \text{if } p = 2 \end{cases}$$

Proof: It follows from the fact that $\tilde{KO}(L^{4v+1}(p^n)/L^{4v}(p^n)) = \mathbb{Z}_2 \otimes \mathbb{Z}_{(p)}$ and Corollary 4.2.4.

Proposition 4.2.10. $\tilde{J}(B_{8v+7}(\mathbb{Z}_{p^n})) = \tilde{J}(B_{8v+5}(\mathbb{Z}_{p^n}))$.
Proof : It follows from [2. Theorem 3.12] that there is an exact sequence,

$$\tilde{J}(B_{8v+7}(\mathbb{Z}_{p^n})/B_{8v+5}(\mathbb{Z}_{p^n})) \xrightarrow{c^!} \tilde{J}(B_{8v+7}(\mathbb{Z}_{p^n})) \xrightarrow{i^!} \tilde{J}(B_{8v+5}(\mathbb{Z}_{p^n})) \to 0.$$

Since $\tilde{KO}(B_{8v+7}(\mathbb{Z}_{p^n})/B_{8v+5}(\mathbb{Z}_{p^n})) = 0$ due to dimensions, it follows that $\tilde{J}(B_{8v+7}(\mathbb{Z}_{p^n})/B_{8v+5}(\mathbb{Z}_{p^n})) = 0$.

4.3 J-Approximation of Complex projective Spaces by Lens Spaces

In this section we study J-groups of complex projective spaces at a prime p. This will be very elementary, based on the counting given in the previous section.

Local J-groups of projective spaces are investigated in [19], using the Ad-spectrum, see Section 2.4. We give here some of their results to compare with our simpler description. Since $K^1(\mathbb{C}P^k) = 0$, it follows that $im(\Delta) = Ad^1(\mathbb{C}P^k)$ and thus, if p is odd then

$$\tilde{J}(\mathbb{C}P^k)_{(p)} \cong Ad^1(\mathbb{C}P^k).$$

In [19, Section 6], some remarks are made about the group structure of $Ad^1(\mathbb{C}P^k)$ for p odd. The group order of $Ad^1(\mathbb{C}P^k)$ follows directly from the defining sequence of Ad-theory by computing the determinant of $\psi^k - 1$, which is

$$v_p(|Ad^1(\mathbb{C}P^k)|) = \sum_{i=1}^{t}(1 + v_p(i)) = v_p((tp)!)$$

where $t = [\frac{k}{p-1}]$, [19]. We compute this order in Theorem 4.3.3 below using some exact sequences induced by cofibrations.
The number of cyclic summands in the Abelian group $Ad^1(\mathbb{C}P^k)$ for odd p is given

in [19] by $[\frac{log(k+1)}{log p}]$ which is equal to $r(k)$ for which $p^{r(k)} \leq k+1 < p^{r(k)+1}$. This is exactly the same value given in Remark 4.1.6 for $n \geq r(k)+1$. We will make this more precise. We will show that for all p, there exists an integer N_k such that for $n \geq N_k$, the group $\tilde{J}(\mathbb{C}P^k)_{(p)}$ and $W(k,n)$ are isomorphic, i.e., $W(k,n)$ is stable for $n \geq N_k$ and saturates to $\tilde{J}(\mathbb{C}P^k)_{(p)}$. In fact, this is clear for sufficiently large n from the KO and J relations, i.e., from Theorem 4.1.3, since J-relations dominate KO-relations when n gets larger. On the other hand we determine the exact saturation value of n in Theorem 4.3.3.

Let $\pi : L^k(p^n) \to \mathbb{C}P^k$ be the usual projection.

Observation 4.3.1. $\pi^!$ maps $\tilde{J}(\mathbb{C}P^k)_{(p)}$ onto $W(k,n)$.

Definition 4.3.2. For $k \in \mathbb{Z}^+$, define a number N_k by

$$v_p(N_k) = sup_{1 \leq r \leq [\frac{k}{p-1}]} 1 + v_p(r).$$

Theorem 4.3.3. $\pi^!$ maps $\tilde{J}(\mathbb{C}P^k)_{(p)}$ isomorphically onto $W(k,n)$ if and only if $n \geq v_p(N_k)$.

Proof : Due to 4.3.1, it is enough to compare the orders.

(i) Let $k = 2v$ be even.

By Proposition 4.2.8 and Observation 4.3.1, $\mid W(k,n) \mid = \prod_{v'=1}^{v} \epsilon(p,v',n)$. It follows from the proof of [7, Lemma 6.1] that there is a short exact sequence,

$$0 \to \tilde{J}(\mathbb{C}P^{2v}/\mathbb{C}P^{2v-2}) \to \tilde{J}(\mathbb{C}P^{2v}) \to \tilde{J}(\mathbb{C}P^{2v-2}) \to 0,$$

and from [7, Lemma 5.3] that $\tilde{J}(\mathbb{C}P^{2v}/\mathbb{C}P^{2v-2}) = \mathbb{Z}_{m(2v)}$ and hence that $\mid \tilde{J}(\mathbb{C}P^{2v})_{(p)} \mid = \prod_{v'=1}^{v} m_p(2v')$ where $m_p(2v')$ is the p-component of $m(2v')$.

Thus $\mid W(k,n) \mid = \mid \tilde{J}(\mathbb{C}P^k)_{(p)} \mid$ iff $\epsilon(p,v',n) = m(2v')$ for all $1 \leq v' \leq v$ iff $\epsilon(p,v',n) = m_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$ iff $n \geq 1 + v_p(2v')$ for all $1 \leq v' \leq v$ and $2v' \equiv 0 \pmod{p-1}$, and putting $2v' = r(p-1)$, iff $n \geq 1 + v_p(r)$ for all $1 \leq r \leq [\frac{2v}{p-1}]$, i.e., iff $n \geq v_p(N_k)$.

(ii) Let $k = 4v+1$.

$\mid \tilde{J}(\mathbb{C}P^k)_{(p)} \mid = \begin{cases} \mid \tilde{J}(\mathbb{C}P^{4v})_{(p)} \mid & \text{if } p \text{ is odd} \\ 2 \mid \tilde{J}(\mathbb{C}P^{4v})_{(2)} \mid & \text{if } p = 2 \end{cases}$ by [7, Lemma 6.2] and

$\mid W(k,n) \mid = \begin{cases} \mid W(4v,n) \mid & \text{if } p \text{ is odd} \\ 2 \mid W(4v,n) \mid & \text{if } p = 2 \end{cases}$ by Proposition 4.2.9 and the result follows from (i) above and the fact that $N_k = N_{4v}$.

(iii) Let $k = 4v+3$.

$\mid \tilde{J}(\mathbb{C}P^k)_{(p)} \mid = \mid \tilde{J}(\mathbb{C}P^{4v+2})_{(p)} \mid$ by [7, Lemma 6.2] and $\mid W(k,n) \mid = \mid W(4v+2,n) \mid$ by Proposition 4.2.10 and the result follows from (i) above and the fact that $N_k = N_{4v+2}$.

Corollary 4.3.4. Let $\pi : L^k(\mathbb{Z}_m) \to \mathbb{C}P^k$ be the standard projection. Then $\pi^!$ maps $\tilde{J}(\mathbb{C}P^k)_{(p)}$ isomorphically onto the subgroup of $\tilde{J}(L^k(m))$ generated by w iff N_k/m.

Stable co-degrees of vector-bundles enable us as in [24, Section 4] or [23, Definition 1.1.4] to define a degree-function q on $J(X)$; i.e., a function $q : \tilde{J}(X) \to \mathbb{Z}^+$ such that $u = 0$ in $\tilde{J}(X)$ iff $q(u) = 1$. The degree-function imposes on $\tilde{J}(X)$ an additional structure other than the usual algebraic structure. We now conjecture a stronger version of Theorem 4.3.3.

Conjecture 4.3.5. Let p be a prime. $n \in \mathbb{Z}^+$ and $n \geq 2$ if $p = 2$. Then the map $\pi^! : \tilde{J}(\mathbb{C}P^k) \to \tilde{J}(L^k(p^n))$ is a q-isomorphism.

Since $\mathbb{C}P^k$ and $L_0^k(p^n)$ are of the same dimension, it is more meaningful to compare their $\tilde{J}$-groups.

Lemma 4.3.6.

$$\tilde{J}(L^k(p^n)) = \begin{cases} \mathbb{Z}_2 \dotplus \tilde{J}(L_0^k(p^n)) & \text{if } k \equiv 0\ (4) \text{ and } p \text{ is odd} \\ \tilde{J}(L_0^k(2^n)) & \text{if } k \equiv 0\ (4),\ p = 2 \text{ and } v_p(N_k)/n \\ \tilde{J}(L_0^k(p^n)) & \text{if } k \not\equiv 0\ (4) \end{cases}$$

Proof : The case $k \not\equiv 0$ (4) follows from Corollary 3.4.10 (i) and Theorem 3.5.2. Let $k = 4t$. For p odd we have an extra $\mathbb{Z}_2$-summand since $J(u) \neq 0$ where u is as in Lemma 3.5.1. For $p = 2$, from Corollary 3.4.10 (i), we have an extra $\mathbb{Z}_2$-summand generated by $2^n w^{2t}$. If we choose n such that $v_p(N_k)/n$, then from Lemma 4.2.7 we observe that $J(2^n w^{2t}) = 0$ and thus we have no extra $\mathbb{Z}_2$-summand in $\tilde{J}(L^k(2^n))$.

Theorem 4.3.7. Let $v_p(N_k)/n$. Then

$$\tilde{J}(\mathbb{C}P^k)_{(p)} = \tilde{J}(L_0^k(p^n)) \cap W(k,n).$$

Proof : It follows from Theorem 4.3.3 and Lemma 4.3.6.

Remark 4.3.8. To obtain the group structure of $J(\mathbb{C}P^k)_{(p)}$ which is $W(k,n)$ in the stable region, we must obtain the direct summand decomposition of that group in terms of cyclic groups, determining a generator of each cyclic component. Decomposition of $\tilde{J}(L^k(p^n))$ is given in [39] for p odd and in [29] for $p = 2$ and some special cases of n. These works give eventually decompositions for $W(k,n)$ and $J(\mathbb{C}P^k)_{(p)}$. On the other hand, these articles contain huge amount of computations so that the results are not clear (See Remark 4.1.6). One can follow the following approach given in [19, Section 6]: We know that $\psi_{\mathbb{R}}^{p^i}(w)$, $i = 0, ..., [\log_p k]$ form a generating set of $\tilde{J}(\mathbb{C}P^k)_{(p)}$. The $\tilde{J}_{(p)}$-order of w is the p-primary part of the well-known Atiyah-Todd number M_{k+1},

$$v_p(M_{k+1}) = \max\{r + v_p(r) | 0 \leq r \leq [\frac{k}{p-1}]\}.$$

The simplest derivation of this is given in [41] where the $\tilde{J}_{(p)}$-order of $\psi_{\mathbb{R}}^{p^s}(w)$ is also determined as

$$|\psi_{\mathbb{R}}^{p^s}(w)| = \max\{r + v_p(r) | 0 \le r \le [\frac{k}{p^s(p-1)}]\} := c(k,s).$$

By the way, as a corollary, we can deduce

Corollary 4.3.9. $\psi_{\mathbb{R}}^{s}(w)$ is trivial in $\tilde{J}(\mathbb{C}P^k)_{(p)}$ for $p^s > \frac{k}{p-1}$.
Proof. $\frac{k}{p^s(p-1)} < 1$, so $c(k,s) = 0$.

The generating set $\{\psi_{\mathbb{R}}^{p^s}(w) | s \ge 0\}$ is in general not a basis for $\tilde{J}(\mathbb{C}P^k)_{(p)}$. Let C_k be the direct sum of cyclic groups of order $p^{c(k,s)}$ and generators h_s. Let $\phi_k : C_k \to \tilde{J}(\mathbb{C}P^k)_{(p)}$ be the map that send h_s to $\psi_{\mathbb{R}}^{p^s}(w)$. This map is always onto but may have non-trivial kernel. Comparing the orders of both groups gives the values of k where ϕ_k is isomorphism, e.g., $k = (p-1)t$ where $t = \sum_{i=0}^{b} \alpha_i p^i$ with $0 \le \alpha_i \le p-1$ and $\alpha_i \ne 0$, $\forall i$. In general the poreblem of determining the structure of $\tilde{J}(\mathbb{C}P^k)_{(p)}$ becomes combinatorially more and more involved as k grows and no general formula is known. But this is theoretically possible and basically lies in Theorem 4.1.3.

4.4 $\tilde{J}$-order of w

Let $b_k(n)$ denote the order of w in $\tilde{J}(L^k(p^n))$. $b_k(n)$ is obtained by solving KO and J-relations together. This turns out to be a difficult problem in the polynomial algebra $\mathbb{Z}[w]$, because of the reasons explained repeatedly in the previous sections. But we have from Theorem 4.1.3 and Theorem 4.3.3,

Theorem 4.4.1. $b_k(n)$ is an increasing sequence of n and saturates to $p^{v_p(M_{k+1})}$ at $n = N_k$.

Remark 4.4.2. We can make a change of basis

$$\{w^i\} \Longrightarrow \{\psi_{\mathbb{R}}^{i}(w)\}.$$

Under this transformation the terminating relations on powers of w can be expressed in terms of $\psi_{\mathbb{R}}^{p^s}(w)$ in the $\tilde{J}$-group. Then one can read off $b_k(n)$ from these transformed relations. This also helps to understand the group $\tilde{J}(\mathbb{C}P^k)_{(p)}$.

Before that, we should answer the following hard question: What is the geometric meaning of $\tilde{J}$-order of w in unstable region?

Remark 4.4.3. We recall that in stable region it is related to cross-sectioning of a complex Stiefel fibration, for $p = 2, n = 1$, to cross-sectioning of a real Stiefel fibration : Let

$$p : U_\Lambda(N)/U_\Lambda(N-k) \to S^{cN-1}$$

be the standard fibration where $c = 1$ if $\Lambda = \mathbb{R}$, $c = 2$ if $\Lambda = \mathbb{C}$. Note that S^{cN-1} which is realized in Λ^n admits orthogonal (resp. unitary) $(k-1)$-vector fields if and only if p admits a cross-section. This difficult problem was solved by methods of K-theory proving the power of algebraic topology, see [5], [7]. It is too short: p admits a cross-section if and only if $N \simeq N\xi_{k-1}(\Lambda)$, where $\xi_{k-1}(\Lambda)$ is the canonical line bundle over ΛP^{k-1}. i.e., if and only if $\tilde{J}$-order of $\xi_{k-1}(\Lambda) - 1$ divides N. Next thing one should do is to work on J-groups of projective spaces to compute the required orders.

Let $p = 2$. We will introduce a new fibration whose cross-section problem is related to J-order of the Hopf bundle over $L_k(2^n)$, thus giving a partial answer to the question we stated above.

Let $E_{m,k}$, $0 \leq k \leq m$ be the space of $\mathbb{Z}_{2^n}$-equivariant continuous maps $f : S^{2k+1} \to S^{2m+1}$, $n \geq 2$. Then we have the inclusions $E'_{m,k} \subset E_{m,k} \subset E''_{m,k}$ which correspond to maps invariant under $S^1 \supset \mathbb{Z}_{2^n} \supset \mathbb{Z}_2$ respectively.

Remark 4.4.4. Unfortunately, here, we can not restrict ourselves to linear maps, since then $E'_{m,k} = E_{m,k} = W_{m+1,k+1}$, the Stiefel manifold of $(k+1)$-tuples in $\mathbb{C}^{n+1}$. And the cross-section problem reduces to the one of a complex Stiefel fibration. This is trivial for us.

Evaluation at $x_0 = (1, 0, \ldots, 0) \in S^{2k+1}$ gives a fibration $p : E_{m,k} \to S^{2m+1}$ with fiber $E_{m-1,k-1}$. A cross-section of p gives $(2k+2)$ continuously-independent (We just imitate the notion of linear-independence which occurs when we consider linear maps!) vector fields on S^{2m+1} which is invariant under the action of $\mathbb{Z}_{2^n}$ on $\mathbb{R} \oplus TS^{2m+1} = \mathbb{R}^{2m+2} \times S^{2m+1}$. We have the following

Theorem 4.4.5. $p : E_{m,k} \to S^{2m+1}$ admits a cross-section if and only if $2m + 2 \simeq (m+1)w$.

Proof. We repeat here the proof of a result of Woodward replacing $\mathbb{Z}_2$ by $\mathbb{Z}_{2^n}$. In exactly the same way of [57, Lemma 1.2], we show that there is a homeomorphism

$$\phi : Map(S^{2m+1}, E_{m,k}) \to Map_{L^k(2^n)}(S(2m+2), S((m+1)w))$$

under which r-sections, i.e., maps $s : S^{2m+1} \to E_{m,k}$ with $ps : S^{2m+1} \to S^{2m+1}$ of degree r, correspond to fibre-preserving maps which are of degree r in each fibre. On the other hand, a fibre-preserving map between two sphere bundles of the same dimension, over a CW-complex, which is of degree 1 on each fibre is a fibre homotopy equivalence and conversely.

Corollary 4.4.6. $p : E_{m,k} \to S^{2m+1}$ admits a cross-section if and only if $m + 1 \equiv 0$ $(mod\ b_k(n))$.

Remark 4.4.7. We wonder if there is a subgroup or a quotient group $T(N)$ of the orthogonal group $O(N)$ which is related to this problem, i.e., with a map p : $T(2m + 2)/T(2m - 2k) \to S^{2m+1}$ such that the cross-section of p exists if and only $b_k(n)$ divides $m + 1$. This could be related to the equivariant Clifford algebras.

Chapter 5

Miscellaneous Problems

In this chapter. we will consider two problems which have close relationship to K-theory and $Im(J)$-theory computations of lens spaces. We have two important ideas of these applications. Firstly, lens spaces J-approximate the complex projective spaces by the results of the previous chapter and this seems to be extended to $Im(J)$-theory. More technically, we have a strong approximation in homotopy: The inclusion $\mathbb{Z}_{p^\infty} < S^1$ is dense and induces an equivalence

$$B\mathbb{Z}_{p^\infty}\hat{}_p \simeq \mathbb{C}P^\infty\hat{}_p$$

of p-adic completions of classifying spaces, [46]. Secondly, the Moore spaces $S^i \cup_{p^n} e^{i+1}$ are included as sub-complexes in the lens space $L^\infty(p^n)$.

5.1 Codegree of vector bundles over projective spaces

Let P^k denote the k-dimensional projective space $\mathbb{C}P^k$ and L^{2k+1} denote the $2k+1$-dimensional lens space *mod* p^r. $L^k(p^r)$ and L^{2k} denote its $2k$-th skeleton as usual, when p^r is understood. We define stunded spaces $P_k^n = P^n/P^{k-1}$ and $L_k^n = L^n/L^{k-1}$, $n \geq k \geq 0$.

Let $q : P_{n-k}^{n-1} \to P_{n-1}^{n-1} = S^{2n-2}$ be the projection collapsing cells of dimension $< 2n-2$. We define the unstable James number $\bar{U}(n,k)$ to be the index of $q_*(\tilde{\pi}_{2n-2}(P_{n-k}^{n-1}))$ in $q_*(\tilde{\pi}_{2n-2}(P_{n-1}^{n-1})) \cong \mathbb{Z}$ and the stable James number $U(n,k)$ to be the index of

$q_*(\tilde{\pi}^S_{2n-2}(P^{n-1}_{n-k}))$ in $q_*(\tilde{\pi}^S_{2n-2}(P^{n-1}_{n-1})) \cong \mathbb{Z}$. See [38, Proposition 2.2], for three equivalent definitions of these numbers. Similarly we can define the corresponding numbers for lens spaces and these saturate to the p-part of James numbers of projective spaces. The complete computation of these numbers is a challenging problem. We will deal with only $U(n,k)$ for projective spaces. Let $U_p(n,k)$ denote the p-exponent of $U(n,k)$. We can find a lower bound for $U_p(n,k)$ using K-theory and Adams operations, [38]. We have the following commutative diagram

$$\begin{tikzcd}
\tilde{\pi}^S_{2n-2}(P^{n-1}_{n-k})_{(p)} \arrow[r, "q_*"] \arrow[d, "h"] & \tilde{\pi}^S_{2n-2}(S^{2n-2})_{(p)} = \mathbb{Z}_{(p)} \arrow[d, "h=1"] \\
\tilde{kO}_{2n-2}(P^{n-1}_{n-k})_{(p)} \arrow[r, "q_*"] & \tilde{kO}_{2n-2}(S^{2n-2})_{(p)} = \mathbb{Z}_{(p)}
\end{tikzcd}$$

where h is the Hurewicz map to connective real K-homology kO_*. Define $U^j_p(n,k)$ to be the least integer $e \geq 0$ such that $p^e = q_*(x) \in \mathbb{Z}_{(p)}$ for some $x \in \tilde{kO}_{2n-2}(P^{n-1}_{n-k})_{(p)}$. One can replace kO-theory by connective $Im(J)$-theory j, defined as in [38, (4.1.7)], to define $U^j_p(n,k)$ and this is why the superscript j stands for. This $Im(J)$-theory j is an extension of the usual real $Im(J)$-theory J defined using KO and contains Adams families μ_r, $\eta\mu_r$, [4], for spheres and $p=2$. The computation of $U^j(n,k)$ is given in [38]. The conjecture is the following:

$$U(n,k) = U^j(n,k),$$

i.e., these numbers are determined only by K-theory. In fact, this is the most natural expectation from the diagram above.

First of all, from [38, Lemma 4.19], we observe that the direct summands of $\tilde{j}_{2n-2}(P^{n-1}_{n-k})$ related to Adams families map to zero on the right hand-side and they have nothing to do with the James numbers, but they probably give extra parts of the related homotopy. Thus we can use non-connective theories KO and J safely. We need a reformulation to make a connection to J-homomorphism.

Now recall the notion of the codegree of a (virtual) vector bundle given in Section 2.5. Let E be a real vector bundle over a connected CW-complex, we denote the codegree of E by $cd(E,X)$. This codegree depends only on the stable isomorphism class of E and so we can extend the notion to virtual bundles.

We also recall that stunded projective spaces (also lens spaces) are Thom spaces:

$$P^{n-1}_{n-k} = (P^{k-1})^{(n-k)H},$$

where $H = r(\eta)$ is the realification of the usual canonical line bundle. From this and using the canonical S-duality, it turns out that, [38],

Proposition 5.1.1. $U(n,k) = cd(-nH, P^{k-1})$.

Thus, we concentrate on the image of the restriction map

$$i^* : \tilde{\pi}_S^{-2n}((P^{k-1})^{-nH})_{(p)} \to \tilde{\pi}_S^{-2n}((pt)^{-nH})_{(p)} = \mathbb{Z}_{(p)}.$$

From the remarks about the codegree in Section 2.5 and Theorem 4.4.1, it is clear that

Corollary 5.1.2. Let p be a prime. Then $U_p(n,k) = 0$ if and only if $v_p(M_k) = 0$.

Consider the multiplicative cohomology theory J^* fitting into the long exact sequence of spectra

$$\ldots \to KO^{i-1}_{(p)} \xrightarrow{\Delta} J^i \xrightarrow{D} KO^i_{(p)} \xrightarrow{\psi^l_{\mathbb{R}}-1} KO^i_{(p)} \to \ldots$$

where l is 3 for $p = 2$ and a generator of $(\mathbb{Z}/p^2)^*$ for p odd.

A solution of Adams conjecture produces, for a connected CW-complex X with a basepoint a J-map:

$$j : J^0(X) \to \pi^0_S(X)_{(p)}$$

extending the classical J-map on $KO^{-1}(X)_{(p)}$ defined as in Section 2.5. This map is monomorphism either if p is odd or X is simply connected. Furthermore, it is linear if X is a suspension. We may restrict j to reduced groups. See [38].

From this fibre sequence, we have

$$\tilde{J}^0(S^0) \cong \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_{(2)} & \text{if } p = 2 \\ \mathbb{Z}_{(p)} & \text{if } p \text{ is odd} \end{cases}$$

Definition 5.1.3. $W^i(X) = Ker(\psi^l_{\mathbb{R}} - 1 : \tilde{KO}^i(X)_{(p)} \to \tilde{KO}^i(X)_{(p)})$.

Then, we have $W^0(S^0) = \mathbb{Z}_{(p)}$. We denote the restriction of the map j to $W^0(X)$ by the same letter j.

Solution of the James numbers conjecture depends on the existence of the following hypothetical commutative diagram:

$$\begin{array}{ccc} W^0(S^{2n}(P^{k-1})^{-nH}) & \xrightarrow{i^*} & W^0(S^{2n}(pt.)^{-nH}) = \mathbb{Z}_{(p)} \\ \downarrow j & & \downarrow j = 1 \\ \tilde{\pi}^0_S(S^{2n}(P^{k-1})^{-nH})_{(p)} & \xrightarrow{i^*} & \tilde{\pi}^0_S(S^{2n}(pt.)^{-nH})_{(p)} = \mathbb{Z}_{(p)} \end{array}$$

This is the point of the work of Knapp and Crabb on James numbers and it is quite 'obscure'. With this assumption, they observe that the image of j provides the necessary homotopy class which gives the lower bound $U^j(n,k)$ and thus proving the conjecture. For the computation of these numbers, we calculate a class in the kernel of

$$\psi^l_{\mathbb{R}} - 1 : \tilde{KO}(S^{2n}(P^{k-1})^{-nH}) \to \tilde{KO}(S^{2n}(P^{k-1})^{-nH}).$$

Suppose now that we have a Thom isomorphism

$$.u : KO^0(P^{k-1}) \to KO^{-2n}((P^{k-1})^{-nH})$$

which is the case when n is even. Let $g(w).u \in W^0(S^{2n}(P^{k-1})^{-nH})$. Then, using the fact that $\psi^l_{\mathbb{R}}(u) = \rho^l_{\mathbb{R}}(-nH).u$ we have

$$\rho^l_{\mathbb{R}}(-nH) = \frac{\psi^l_{\mathbb{R}}(g(w))}{g(w)}, \qquad (\theta_{-nH})$$

where the left-hand side is a fixed quantity $\in 1 + \tilde{KO}(P^{k-1})_{(p)}$. Due to [38], this is solvable by

$$g(w) = p^{UJ(n,k)}(\frac{\sqrt{w}/2}{sinh^{-1}(\sqrt{w}/2)})^n.$$

For odd n, there are some modifications to compute the kernel.

Due to [38, Remark 3.11], one can replace P^{k-1} by $L^{2k-2}(p^\infty)$ and the equation above needs an interpretation in view of our KO and J computations of lens spaces. We believe that the determination of the group structure of $W(k,n)$ for large n gives a solution to the equation (θ_{-nH}) above. In particular, the number of cyclic summands seems to be related to the James numbers, [20].

5.2 A filtration of $\tilde{KO}(L^k(p^n))$, $n \geq 2$

In this section, we only point out a connection of the results with AKT of fields. After Quillen defined the higher Algebraic K-rings and computed them for finite fields, it is conjectured that if F is an algebraicly closed field, $char\ F \neq p$, then algebraic K-theory spectrum of F should be the same as ordinary topological K-theory spectrum when considered with finite coefficients $\mathbb{Z}/p$; in particular, when completed at p. These hard conjectures are proved by Suslin. In particular, for the complex field, in terms of spectra, he proved that $K\mathbb{C}\hat{}_p \simeq bu\hat{}_p$ where $\hat{}_p$ denotes completion at a prime p. He also showed the same equivalence for the real field, i.e., $K\mathbb{R}\hat{}_p \simeq bo\hat{}_p$ which will be our main interest. Note that bo is the connective version of KO, i.e., equivalent to KO in homotopy ≥ 0 and $\pi_{-n}(bo) = 0$ for $n \in \mathbb{Z}^+$. For details, see [46], [54].

We want to compare the algebraic K-ring $K_*(\mathbb{R};\mathbb{Z}_{p^n})$ of the field $\mathbb{R}$ with $KO(B\mathbb{Z}_{p^n})$. From Suslin's results we have $K\mathbb{R}^{\hat{}} = b\hat{o}$ at any prime p. On the other hand, from the results of [6], for G a p-group, p-localization does not affect the KO-theory of BG, as explained in Section 2.2. Thus we have the isomorphism

$$K\mathbb{R}^{\hat{}}_p(B\mathbb{Z}_{p^n}) \cong KO(B\mathbb{Z}_{p^n})$$

of the *zero*-th part of the cohomologies, since $B\mathbb{Z}_{p^n}$ has cells only in positive dimensions. Let $Y^i_{p^n} = S^{i-1} \cup_{p^n} e^i$ denote the Moore space *mod* p^n of dimension i. From Section 4.3, we have the inclusions $i_k : Y^{2k} \subset B\mathbb{Z}_{p^n}$. Recall that

$$K_i(\mathbb{R};\mathbb{Z}_{p^n}) = \{Y^i, \mathbb{Z} \times K\mathbb{R}^0\}$$

and in particular, we can complete at p. Composing the elements of $KO(B\mathbb{Z}_{p^n})$ with i_k's on the left, we obtain elements in $K_{2k}(\mathbb{R};\mathbb{Z}_{p^n})$. It is not difficult to show that they are non-zero and generate the corresponding group, using the AHSS's and naturality. We have the following hypothetical computation

$k > 0$ *mod* 4	0	1	2	3
$K_{2k}(\mathbb{R};\mathbb{Z}_{2^n})$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_{2^n}$	0	$\mathbb{Z}_{2^n}$
$K_{2k}(\mathbb{R};\mathbb{Z}_{p^n})$, p *odd*	0	$\mathbb{Z}_{p^n}$	0	$\mathbb{Z}_{p^n}$

by filtration of $K\hat{O}(B\mathbb{Z}_{p^n})$ respecting the Moore spaces.

We complete this obscure discussion with a question

Question. How can the ring $\hat{K}_*(\mathbb{R};\mathbb{Z}_{p^n})$ be related to the ring $K\hat{O}(B\mathbb{Z}_{p^n})$ explicitly in terms of their structures; additive and multiplicative?

Chapter 6

Conclusion

What we have done so far is simply to demonstrate the Atiyah-Segal isomorphism which connects TKT to representation theory, by a simple example. Our computation was followed by the computation of the J-groups. These results give some classical applications. We also tried to establish some superficial connections of the results with SH, by means of the Adams conjecture, on a particular problem and with AKT.

The logical continuation for a topological K-theorist seems to consider classifying spaces of other groups, e.g.. symmetric groups.

$Im(J)$-theory is another continuation. We should perform the necessary algebraic computations, mentioned in the remarks of Chapter 4 and Section 5.1, in J-groups of lens spaces.

The most attractive continuation is to understand what's going on with integral matrices, i.e., representations over $\mathbb{Z}$. This computation is connected with number theory. AKT of $\mathbb{Z}$ is very difficult to compute compared to coefficient groups in TKT, since topology of $GL(\mathbb{Z})$ is complicated and is somewhat mysterious, exhibiting complexity of discrete objects and rich algebraic structure of rings. This also gives some appreciation of SH.

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