ISOMORPHIC CLASSIFICATION PROBLEM AND LINEAR TOPOLOGICAL INVARIANTS

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Dora Arslan
May 1995

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tarafından bağışlanmıştır

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ISOMORPHIC CLASSIFICATION PROBLEM AND LINEAR TOPOLOGICAL INVARIANTS

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M.S. in Mathematics
Supervisor: Prof. Dr. Mefharet Kocatepe
May 1995

We consider all possible isomorphisms of cartesian products of Dragilev spaces, and thanks to relations between the Dragilev functions of each factor try to show that if there exists such an isomorphism, then any factor on the left of the isomorphism is nearly isomorphic to the corresponding factor on the right. We also try to get a necessary condition for the isomorphisms of the tensor products of infinite type Dragilev spaces by the dual of an infinite type Montel power series space.

Keywords : Dragilev space, Montel power series space, rapidly increasing function, nearly isomorphic, weakly equivalent, different lacunarity.

ÖZET

İZOMORFİK SINIFLANDIRMA PROBLEMİ VE LİNEER TOPOLOJİK İNVARYANTLAR

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Matematik Yüksek Lisans
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Mayıs 1995

Dragilev uzaylarının kartezyen çarpımlarının mümkün olan bütün isomorfizmlerini ele aldık ve böyle bir isomorfizmin varlığı halinde çarpanların Dragilev fonksiyonları arsındaki ilişkiler yardımıyla izomorfizmin sol tarafındaki çarpanların her birinin sağ tarafta karşılık gelen bir çarpana hemen hemen izomorf olduğunu gösterdik. Ayrıca, sonsuz tipteki bir Dragilev uzayı ile gene sonsuz tipteki bir Montel kuvvet serisi uzayının tensör çarpımlarının izomorfizmi için gerek koşul bulmaya çalıştık.

Anahtar Kelimeler :Dragilev uzayı, Montel kuvvet serisi uzayı, hızlı artan fonksiyon, hemen hemen izomorf, zayıf eşdeğerlilik, değişik artış hızı.

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TABLE OF CONTENTS

1 Introduction **1**

2 Preliminaries and Linear Topological Invariants **3**

2.1 Definitions and Some Preliminary Results 3

2.2 Linear Topological Invariants 9

3 Some Cartesian Products of Power Series Spaces and Dragilev Spaces **14**

3.1 $E_0(a) \times L_f(b,\infty) \simeq E_0(\tilde{a}) \times L_f(\tilde{b},\infty)$ 14

3.2 $E_0(a) \times L_f(b,\infty) \simeq E_0(\tilde{a}) \times L_{\tilde{f}}(\tilde{b},\infty)$ 18

3.3 $L_f(a,0) \times E_\infty(b) \simeq L_{\tilde{f}}(\tilde{a},0) \times E_\infty(\tilde{b})$ 21

3.4 $L_{f_1}(a,0) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},\infty)$ 23

4 Cartesian Products of Various Types of Dragilev Spaces **27**

4.1 $L_{f_1}(a,\infty) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$ 30

4.1.1 $f_4 \approx f_1 \prec f_2 \approx f_3$ 33

4.1.2 $f_1 \approx f_2 \approx f_3 \approx f_4$ 37

4.1.3 Other Cases 38

4.2 $L_{f_1}(a,0) \times L_{f_2}(b,0) \simeq L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},0)$ 39

4.2.1 $f_3 \approx f_2 \prec f_1 \approx f_4$ 39

4.3 $L_{f_1}(a,1) \times L_{f_2}(b,1) \simeq L_{f_3}(\tilde{a},1) \times L_{f_4}(\tilde{b},1)$ 41

4.3.1 $f_3 \approx f_1 \prec f_2 \approx f_4$ 41

4.4 $L_{f_1}(a,-1) \times L_{f_2}(b,-1) \simeq L_{f_3}(\tilde{a},-1) \times L_{f_4}(\tilde{b},-1)$ 43

4.4.1 $f_1 \approx f_3 \prec f_2 \approx f_4$ 43

4.5 $L_{f_1}(a,\infty) \times L_{g_1}(b,0) \simeq L_{f_2}(\tilde{a},\infty) \times L_{g_2}(\tilde{b},0)$ 45

4.5.1 $f_1 \approx f_2 \prec g_1 \approx g_2$ 47

4.6 $L_{f_1}(a,1) \times L_{g_1}(b,0) \simeq L_{f_2}(\tilde{a},1) \times L_{g_2}(\tilde{b},0)$ 49

4.6.1 $f_1 \approx f_2 \prec g_1 \approx g_2$ 49

4.7 $L_{f_1}(a,-1) \times L_{g_1}(b,1) \simeq L_{f_2}(\tilde{a},-1) \times L_{g_2}(\ b,1)$ 50

4.7.1 $f_1 \approx f_2 \prec g_1 \approx g_2$ 50

4.8 $L_{f_1}(a,-1) \times L_{g_1}(b,\infty) \simeq L_{f_2}(\tilde{a},-1) \times L_{g_2}(\tilde{b},\infty)$ 52

4.8.1 $f_1 \approx f_2 \prec g_1 \approx g_2$ 52

5 Case of tensor product **54**

Chapter 1

Introduction

Linear topologic invariants (such as approximative and diametral dimensions) as a tool of isomorphic classification of non-normed linear topological spaces appeared in investigations of Pelczynski [14], Kolmogorof [9], Bessaga, Pelczynski, Rolewicz [15], Mitiagin [11], et al: they were also initiated by Gelfand [5].

Linear topological invariants are quite strong. Many of the results (some of which can also be proved by Riesz theory or some other methods) were proved by linear topological invariants.

V.P. Zahariuta has proved a number of theorems in [20]. But he achieved this by using Riesz theory. In [19], M. Yurdakul and V.P. Zahariuta have considered the cartesian products of Montel power series spaces and proved the already known property that if $E_0(a) \times E_\infty(b)$, $E_0(\tilde{a}) \times E_\infty(\tilde{b})$ are isomorphic, then $E_0(a)$ is near isomorphic to $E_0(\tilde{a})$ and $E_\infty(b)$ is near isomorphic to $E_\infty(\tilde{b})$ by using linear topological invariants. This gave us the idea that we may do the same thing for $L_f(a, 0)$ and $L_g(b, \infty)$ spaces.

Most of the results in this work are not new, because they proved in [20]. Our aim here is to give new proofs of these results by means of geometric invariants instead of Riesz theory. Also, in this work, there are some more results which are not covered in [20]. All these will enable us to test and understand the scope of using invariants to some extent.

On the other hand Ahonen introduced some properties of Dragilev spaces (for their different types) in [1]. By using these properties we tried to consider all possibile isomorphisms of cartesian products of different or same types of

Chapter 2

Preliminaries and Linear Topological Invariants

2.1 Definitions and Some Preliminary Results

Definition 1 Let X and Y be locally convex spaces (lcs's). A linear operator $T : X \to Y$ is called a *near-isomorphism* (as defined in [20]), if the following conditions are satisfied:

a)$T(X)$ is closed in Y and T is an open map from X onto $T(X)$,

b)$\alpha(T) = dim\ \mathrm{Ker}T < \infty$,

c)$\beta(T) = codim\ T(X) = dimY/T(X) < \infty$ (see [16]).

The lcs's X and Y are said to be *nearly isomorphic* if there exists a near isomorphism T from X onto Y.

If the locally convex spaces X and Y are isomorphic , we write $X \simeq Y$ and if they are nearly isomorphic we write $X \approx Y$. $\mathbb{N} = \{1, 2, ...\}$ denotes the set of all positive integers .

Let $A = (a_{i,p})_{i \in I, p \in \mathbb{N}}$ be a matrix of real numbers such that $0 \leq a_{i,p} \leq a_{i,p+1}$ for each p and for each index i in the countable set I (most of the time we shall let $I = \mathbb{N}$). The Köthe space $K(A)$, defined by the matrix A , is the locally convex spaces of all sequences $x = (x_i)$ of scalars such that

It is well known that (see [4]), if f is an increasing logarithmically convex function, then f is either rapidly increasing or slowly increasing and $L_f(\alpha, r)$ is isomorphic to a power series space if and only if f is slowly increasing. So we shall assume that f is rapidly increasing. In this case, there are four classes of Dragilev spaces corresponding to $r = \infty$, 1, 0, -1. If ϕ is a rapidly increasing function , then for each $A > 0$,

$$\lim_{t\to\infty} \frac{\phi^{-1}(At)}{\phi^{-1}(t)} = 1.$$

Lemma 2.1 *(* **Dragilev** *[4]) Let f be an increasing, logarithmically convex function. Then, for every $a > 1$,*

$$\lim_{t\to\infty} \frac{f(at)}{f(t)} = \tau_f(a)$$

exists. Moreover, either
a)$\tau(a) = \infty$, or
b)$\tau(a) < \infty$ for $1 < a < \infty$ and $\tau(a) \leq \tau(b)$ for $a < b$,
and $\lim \tau(a) = \infty$.

Lemma 2.2 *Let h be a (not necessarily logarithmically convex) slowly increasing, odd function. Then*

$$\lim_{a\to 1} \tau_h(a) = 1.$$

We shall use the relations "$\prec$" and "$\approx$" between f_1, f_2, where f_1 and f_2 are functions satisfying the conditions of Definition 2 and $-\infty < r \leq \infty$. We shall say $f_1 \prec f_2$, if $f_1^{-1}\, f_2$ is rapidly increasing, and $f_1 \approx f_2$, if both $f_1^{-1}\, f_2$ and $f_2^{-1}\, f_1$ are slowly increasing.

Theorem 2.1 *Let f, g be rapidly increasing, logarithmically convex Dragilev functions, and $f^{-1}\, g$ be logarithmically convex and slowly increasing function. Let $r = \infty$ or 0. Then*

$$L_f(a, r) \simeq L_g(b, r) \text{ if and only if } f^{-1}\, g(b) \asymp a.$$

Let $r = 1$ or -1. Then

$$L_f(a, r) \simeq L_g(b, r) \text{ if and only if } \lim_{i\to\infty} \frac{f^{-1}\, g(b_i)}{a_i} = 1.$$

Therefore we have shown that if we have the isomorphism $L_f(a,r) \simeq L_g(b,r)$, then $\forall p\, \exists q\, \exists C > 0$ such that

$$a_{i,p} \le C\, b_{i,q} \text{ and } b_{i,p} \le C\, a_{i,q}.$$

Then with $D = \ln C$,

$$\begin{aligned}
a_{i,p} \le C\, b_{i,q} &\Rightarrow f((1-\tfrac{1}{p})a_i) \le D + g((1-\tfrac{1}{q})b_i) \le g((1-\tfrac{1}{q+1})b_i) \\
&\Rightarrow 1-\tfrac{1}{p} \le \tfrac{f^{-1}\, g((1-\frac{1}{q+1})b_i)}{a_i} \le \tfrac{f^{-1}\, g(b_i)}{a_i} \\
b_{i,p} \le C\, a_{i,q} &\Rightarrow g((1-\tfrac{1}{p})b_i) \le D + f((1-\tfrac{1}{q})a_i) \le f((1-\tfrac{1}{q+1})a_i) \\
&\Rightarrow \tfrac{f^{-1}\, g((1-\frac{1}{p})b_i)}{f^{-1}\, g(b_i)} \tfrac{f^{-1}\, g(b_i)}{a_i} \le 1-\tfrac{1}{q+1} \le 1
\end{aligned}$$

Thus $1-\dfrac{1}{p} \le \dfrac{f^{-1}\, g(b_i)d}{a_i} \le \dfrac{1}{\tau_{f^{-1}\, g}(1-\frac{1}{p})}$. Since $\lim_{t\to 1} \tau_{f^{-1}\, g}(t) = 1$, by choosing p large enough, we get

$$\lim_{i\to\infty} \frac{f^{-1}\, g(b_i)}{a_i} = 1.$$

Conversly, let $\lim_{i\to\infty} \frac{f^{-1}\, g(b_i)}{a_i} = 1$. We will show $\forall p\, \exists q\, \exists C$ such that

$$e^{f((1-\frac{1}{p})a_i)} \le C\, e^{g((1-\frac{1}{q})b_i)} \quad \text{and} \quad e^{g((1-\frac{1}{p})b_i)} \le C\, e^{f((1-\frac{1}{q})a_i)}.$$

Assume the contrary. Then there exists p, for any q, there exist subsequences (a_{i_k}), (b_{i_k}) such that

$$f((1-\frac{1}{p})a_{i_k}) > D + g((1-\frac{1}{q})b_{i_k}).$$

Then

$$\begin{aligned}
& f((1-\tfrac{1}{p})a_{i_k}) > g((1-\tfrac{1}{q})b_{i_k}) \\
\Longrightarrow\ & 1-\tfrac{1}{p}\ \tfrac{f^{-1}\, g((1-\frac{1}{q})b_{i_k})}{f^{-1}\, g(b_{i_k})} \tfrac{f^{-1}\, g(b_{i_k})}{a_{i_k}} \\
\Longrightarrow\ & 1-\tfrac{1}{p} > \lim_{i\to\infty} \frac{f^{-1}\, g((1-\frac{1}{q})b_{i_k})}{f^{-1}\, g(b_{i_k})} = \tau_{f^{-1}\, g}(1-\frac{1}{q}).
\end{aligned}$$

But this is not possible. Because, $\lim_{q\to\infty} \tau_{f^{-1}\, g}(1-\frac{1}{q}) = 1$.•

Let $a = (a_i)$ be a positive sequence and $1 \le \tau < t < \infty$. Then $M_a(t,\tau) := |\{i : \tau < a_i \le t\}|$ and $m_a(t) := |\{i : a_i \le t\}|$ where $|\cdot|$ denotes the cardinality of the set. If $a_i \nearrow \infty$, then $M_a(t,\tau) = m_a(t) - m_a(\tau)$. Let $b = (b_i)$ be another positive sequence. If there is a constant $\Delta > 0$ such that

$$M_a(t,\tau) \le M_b(\Delta t, \frac{\tau}{\Delta}) \text{ and } M_b(t,\tau) \le M_a(\Delta t, \frac{\tau}{\Delta}),$$

then the counting functions M_a, M_b are said to be *equivalent* and we write $M_a \approx M_b$ (see e.g. [12],[13]).

The following tecnical lemma will be used throughout this thesis. It was proved in [19]. Since it plays a crucial role in our work, we present its proof here.

Lemma 2.3 *Let $a = (a_i)$ and $\tilde{a} = (\tilde{a}_i)$ be increasing sequences such that $a_i \nearrow \infty$, $\tilde{a}_i \nearrow \infty$. Assume $\exists C \geq 1\ \exists \tau_0 > 0\ :\ \tau_0 \leq \tau \leq t \Rightarrow M_a(t,\tau) \leq M_{\tilde{a}}(Ct, \frac{\tau}{C})$ and $M_{\tilde{a}}(t,\tau) \leq M_a(Ct, \frac{\tau}{C})$. Then there is $k \in \mathbb{Z}$ such that $a_{i+k} \asymp \tilde{a}_i$.*

Proof Since (a_i) and $(\tilde{a}_i)$ are increasing, the inequalities are equivalent to:

$$\exists C \geq 1\ \exists \tau_0\ :\ \tau_0 \leq \tau \leq t \Rightarrow \begin{array}{ll} m_a(t) - m_a(\tau) \leq m_{\tilde{a}}(Ct) - m_{\tilde{a}}(\frac{\tau}{C}) & (i) \\ m_{\tilde{a}}(t) - m_{\tilde{a}}(\tau) \leq m_a(Ct) - m_a(\frac{\tau}{C}) & (ii) \end{array}$$

Putting $\tau = \tau_0$ in (i), we get

$$\forall t \geq \tau_0\ :\ m_a(t) - m_{\tilde{a}}(Ct) \leq m_a(\tau_0) - m_{\tilde{a}}(\frac{\tau_0}{C}).$$

So, $m_a(\cdot) - m_{\tilde{a}}(C\cdot)$ is an integer valued function which is bounded above. So it attains its maximum k at some $t = t_1 \geq \tau_0$. Thus

$$m_a(t_1) - m_{\tilde{a}}(Ct_1) = k \quad ...(A)$$

and

$$\forall t \geq \tau_0 \quad m_a(t) \leq m_{\tilde{a}}(Ct) + k \quad ...(I).$$

Putting $\tau = Ct_1$ in (ii), we get

$$\forall t \geq Ct_1 \quad m_{\tilde{a}}(t) + k \leq m_a(Ct) \quad ...(II)$$

Since the inequalities (I) and (II) are symmetrical in k and $-k$, we may assume $k \geq 0$. (I) can be written as

$$|\{i : a_{i+k} \leq t\}| = m_a(t) - k \leq m_{\tilde{a}}(Ct) = |\{j : \tilde{a}_j \leq Ct\}|.$$

Now let n be large enough so that $a_{n+k} \geq \tau_0$ and let $t = a_{n+k}$. Then the set on the left hand side has n elements. Thus the set on the right hand

side has at least n elements. Since $(\tilde{a}_j)$ is increasing, we have $\tilde{a}_n \leq Ct$, i.e. $\frac{1}{C} \leq \frac{a_{n+k}}{\tilde{a}_n}$.

(II) means $|\{i : \tilde{a}_i \leq t\}| + k \leq |\{j : a_j \leq Ct\}|$. Let n be large enough so that $\tilde{a}_n \geq Ct_1$ and let $t = \tilde{a}_n$. Then left hand side of (II) is $n+k$. So the set on the right hand side has at least $n+k$ elements. Since (a_j) is increasing, we get $a_{n+k} \geq Ct$, i.e. $\frac{a_{n+k}}{\tilde{a}_n} \leq C$.

Let X be a lcs with an absolute basis $e = (e_i)$ and $a = (a_i)$ be a positive sequence of reals. For $1 \leq p \leq \infty$, we denote by $B_p^e(a)$ the *weighted l^p-ball* in it with respect to the basis e and by $B^e(a)$ the ball $B_1^e(a)$. If (e_i) is the sequence which is zero at each coordinate except the ith where it is 1, we omit e and simply write $B_p(a)$ or $B(a)$. So

$$B_p^e(a) = B_p^e(a_i) = \{x = (\xi_i) \in X : \sum |\xi_i|^p a_i^p \leq 1\}.$$

2.2 Linear Topological Invariants

We will use the following simplest characteristic function for a pair of absolutely convex subsets in X :

$$\beta(V, U) = \sup\{\dim L : L \cap U \subset V\}$$

where L stands for a finite dimensional subspace of X (i.e. $L \in \mathcal{F}(X)$ where $\mathcal{F}(X)$ is the set of all finite dimensional subspaces of X). If $V_1 \subset V_2$ and $U_2 \subset U_1$ then $\beta(V_1, U_1) \leq \beta(V_2, U_2)$.

Lemma 2.4 *Let the positive sequences $a = (a_i), b = (b_i)$ be such that $\dfrac{b_i}{a_i} \to \infty$ and $V = B_p(b), U = B_p(a)$. Then $\beta(V, U) = |\{i : \dfrac{b_i}{a_i} \leq 1\}|$.*

Lemma 2.5 *Let $U = B(a)$, $V = B(b)$ and $c = (c_i)$, $d = (d_i)$ where*

$$c_i = \max\{a_i, b_i\},\ d_i = \min\{a_i, b_i\}.$$

Then $B(c) \subset U \cap V \subset B(c/2)$ and $B(d) = \overline{\Gamma}(U \cup V)$.

In Chapter 2 we will use basic invariants together with interpolation methods, i.e. for a given pair of neighborhoods U_p, U_q in the Köthe space $K(A)$, $A = (a_{ip})$ we consider an intermediate interpolative neighborhood

$$U_p^{1-\alpha}U_q^{\alpha} = \{x = (x_i) \in K(A) : \sum_i |x_i| a_{ip}^{1-\alpha} a_{iq}^{\alpha} \leq 1\}, \quad 0 \leq \alpha \leq 1.$$

To construct such neighborhoods, the following very simple interpolation lemma is applicable (see e.g. [2]);

Lemma 2.6 *If* $T \in L(l^1(a^0),\ l^1(b^0)) \cap L(l^1(a^1), l^1(b^1))$ *then* $T \in L(l^1(a^{\alpha}),\ l^1(b^{\alpha}))$ *where* $0 \leq \alpha \leq 1$, $a_j^{\alpha} = (a_j^0)^{1-\alpha}(a_j^1)^{\alpha}$ *and* $b_j^{\alpha} = (b_j^0)^{1-\alpha}(b_j^1)^{\alpha}$. *Further* $\|T\|_{\alpha} \leq \max\{\|T\|_0,\ \|T\|_1\}$.

Lemma 2.7 *Consider the balls* $B(a^p)$, $a^p = (a_{ip})$, $p = 1,2,3,4$ *in a lcs* X. *Then*

$$(i) \qquad |\{i : \frac{\max\{a_{i3},\ a_{i2}\}}{\min\{a_{i2}, a_{i1}\}} \leq 1\}| = |\{i : \frac{a_{i3}}{a_{i2}} \leq 1, \frac{a_{i2}}{a_{i1}} \leq 1\}|$$

$$\leq \beta\left(B^e(a^2) \cap B^e(a^3),\ \overline{\Gamma}(B^e(a^1) \cup B^e(a^2))\right).$$

$$(ii) \qquad \beta\left(B^e(a^2) \cap B^e(a^3),\ \overline{\Gamma}(B^e(a^1) \cup B^e(a^4))\right)$$

$$\leq |\{i : \frac{\frac{1}{2}\max\{a_{i3}, a_{i2}\}}{\min\{a_{i4}, a_{i1}\}} \leq 1\}| \leq |\{i : \frac{a_{i3}}{a_{i4}} \leq 2, \frac{a_{i2}}{a_{i1}} \leq 2\}|.$$

The proofs of the above lemmas can be found in [19]. Also, detailed informations can be found in [23], [24], [25].

Remark If the balls $U_p = B^e(a^p)$, $p = 1,2,3$, $V_q = B^f(b^q)$, $q = 1,2,3,4$ in a lcs X are such that $V_1 \subset U_1, V_4 \subset U_2 \subset V_2, U_3 \subset V_3$ then by lemma 1.4 we have

$$|\{i : \frac{a_{i3}}{a_{i2}} \leq 1, \frac{a_{i2}}{a_{i1}} \leq 1\}| \leq |\{j : \frac{b_{j3}}{b_{j4}} \leq 2, \frac{b_{j2}}{b_{j1}} \leq 2\}| .$$

Let (e_i) be the canonical basis in $X = K(C)$, $C = (c_{ip})$ and $Y = K(D)$, $D = (d_{ip})$. If $T : Y \to X$ is an isomorphism, by the continuity of T^{-1}, given p_1, there are $\tilde{p}$ and $c > 0$ such that

$$U_{\tilde{p}} \subset cT(V_{p_1}) \Rightarrow \frac{1}{cp_1}U_{\tilde{p}} \subset \frac{1}{p_1}T(V_{p_1}).$$

Choose now $p \geq \tilde{p}$ such that $\frac{1}{p} \leq \frac{1}{cp_1}$, i.e. $p \geq cp_1$.

$$p \geq \tilde{p} \Rightarrow U_p \subset U_{\tilde{p}} \Rightarrow \frac{1}{p}U_p \subset \frac{1}{cp_1}U_{\tilde{p}} \ \ (\subset \frac{1}{p_1}T(V_{p_1})) \Rightarrow \frac{1}{p}U_p \subset \frac{1}{p_1}T(V_{p_1}).$$

Then given q $(q \geq p)$ we have

$$\frac{1}{q}U_q \subset \frac{1}{p}U_p,$$

and by continuity of T, given q $(q \geq p)$, there are $\tilde{q}$ and $C > 0$ such that

$$V_{\tilde{q}} \subset CT^{-1}(U_q) \Rightarrow \frac{1}{Cq}V_{\tilde{q}} \subset \frac{1}{q}T^{-1}(U_q).$$

Choose $q_1 \geq \tilde{q}$ such that $\frac{1}{q_1} \leq \frac{1}{Cq}$, i.e. $q_1 \geq Cq$.

$$q_1 \geq \tilde{q} \Rightarrow V_{q_1} \subset V_{\tilde{q}} \Rightarrow \frac{1}{q_1}T(V_{q_1} \subset \frac{1}{Cq}V_{\tilde{q}}(\subset \frac{1}{q}T^{-1}(Uq) \Rightarrow \frac{1}{q_1}T(V_{q_1}) \subset \frac{1}{q}U_q.$$

Up to now, we get $\forall p_1\, \exists p\, \forall q\, \exists q_1$:

$$\frac{1}{q}T(V_{q1}) \subset \frac{1}{q}U_q \subset \frac{1}{p}U_p \subset \frac{1}{p_1}T(V_{p_1}).$$

By repeating this we get $\forall p_1\, \exists p\, \forall q\, \exists q_1\, \forall r_1\, \exists r\, \forall s\, \exists s_1 \ldots$ such that

$$\ldots\frac{1}{s_1}T(V_{s_1}) \subset \frac{1}{s}U_s \subset \frac{1}{r}U_r \subset \frac{1}{r_1}T(V_{r_1}) \subset \frac{1}{q_1}T(V_{q_1}) \subset \frac{1}{q}U_q \subset \frac{1}{p}U_p \subset \frac{1}{p_1}T(V_{p_1}).$$

If we denote the neigborhoods of X by U_i's and the neigborhoods of Y by V_j's , then given $p_1 \in \mathbb{N}$ we can find $p, q_1, q, \tilde{q}_1, r_1, r, s, s_1$ with the chain

$$s_1 > s > 2r > 2r_1 > 4\tilde{q}_1 > 4q > 4q_1 > 8p > 8p_1$$

and positive constants $c_1,\ c_2,\ c_3,\ c_4,\ c_5$ such that

$$c_1U_p \subset T(V_{p_1}),\ c_3T(V_{\tilde{q}_1}) \subset U_q \subset \frac{1}{c_2}T(V_{q_1}),\ c_4U_r \subset T(V_{r_1}),\ c_5T(V_{s_1}) \subset U_s.$$

Now consider the following sets

$$\begin{aligned}
W_1 &= \tau U_s, & \tilde{W}_1 &= \tau T(V_{s1}) \\
W_2 &= U_q, & \tilde{W}_2 &= T(V_{q_1}) \\
W_3 &= U_p^{1/2}U_r^{1/2} \cap tU_r, & \tilde{W}_3 &= T(_{p_1})^{1/2}T(V_{r_1})^{1/2} \cap tT(V_{r_1}) \\
& & \tilde{W}_4 &= T(V_{\tilde{q}_1})
\end{aligned} \tag{2.2}$$

where $t, \tau \in \mathbb{R}^+$. Define now

$$\begin{aligned}
\beta_1 &:= \beta\left(W_2 \cap W_3, \overline{\Gamma}(W_1 \cup W_2)\right), \\
\beta_2 &:= \beta\left(c(\tilde{W}_2 \cap \tilde{W}_3), \overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4)\right), \\
I_1 &:= \{i : \frac{\max\{c_{ip}^{1/2} c_{ir}^{1/2}, \frac{1}{t} c_{ir}\}}{c_{iq}} \leq 1, \frac{c_{iq}}{\frac{1}{\tau} c_{is}} \leq 1\} \\
&= \{i : c_{ip} c_{ir} \leq c_{iq}^2, \frac{c_{ir}}{c_{iq}} \leq t, \frac{c_{is}}{c_{iq}} \geq \tau\}, \\
I_2 &:= \{i : \frac{\frac{1}{2c} \max\{d_{ip_1}^{1/2} d_{ir_1}^{1/2}, \frac{1}{t} d_{ir_1}\} d_{i\tilde{q}_1}}{\leq} 1, \frac{\frac{1}{2c} d_{iq_1}}{\frac{1}{\tau} d_{is_1}} \leq 1\} \\
&= \{i : d_{ip_1} d_{ir_1} \leq 4c^2 d_{i\tilde{q}_1}^2, \frac{d_{ir_1}}{d_{i\tilde{q}_1}} \leq 2ct, \frac{d_{is_1}}{d_{iq_1}} \geq \frac{\tau}{2c}\}.
\end{aligned} \tag{2.3}$$

(here $c > 0$ is some constant). Since $W_2 \cap W_3 \subset \tilde{c}_1(\tilde{W}_2 \cap \tilde{W}_3)$ for some constant $\tilde{c}_1 > 0$ and $\tilde{c}_2 \overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4) \subset \overline{\Gamma}(W_1 \cup W_2)$ for some constant $\tilde{c}_2 > 0$ we have $\beta_1 \leq \beta_2$. We also have $|I_1| \leq \beta_1$ by lemma 1.4(i) and $\beta_2 \leq |I_2|$ by lemma 1.4(ii). Thus $|I_1| \leq |I_2|$.

On the other hand given $p \in \mathbb{N}$ we can find p_1, q_1, q, $\tilde{q}_1$, r, r_1, s_1, s with the chain

$$s > s_1 > 2r_1 > 2r > 4\tilde{q}_1 > 4q > 4q_1 > 8p_1 > 8p$$

and costants c_1, c_2, c_3, c_4, $c_5 > 0$

$$c_1 V_{p_1} \subset T^{-1}(U_p),\ c_3 V_{\tilde{q}_1} \subset T^{-1}(U_q) \subset \frac{1}{c_2} V_{q_1},\ c_4 V_{r_1} \subset T^{-1}(U_r),\ c_5 T^{-1}(U_s) \subset V_{s_1}.$$

Consider now the sets

$$\begin{aligned}
W_1 &= \overline{\Gamma}((T^{-1}(U_p))^{1/2}(T^{-1}(U_r))^{1/2} \cup tT^{-1}(U_r)), \\
W_2 &= T^{-1}(U_q), \\
W_3 &= \tau T^{-1}(U_s), \\
\tilde{W}_1 &= \overline{\Gamma}(V_{p_1}^{1/2} V_{r_1}^{1/2} \cup tV_{r_1}), \\
\tilde{W}_2 &= V_{q_1}, \\
\tilde{W}_3 &= \tau V_{s_1} \\
\tilde{W}_4 &= V_{\tilde{q}_1}
\end{aligned} \tag{2.4}$$

where $t, \tau \in \mathbb{R}^+$. Define now

$$
\begin{aligned}
\beta &:= \beta(W_2 \cap W_3\,,\ \overline{\Gamma}(W_1 \cup W_2)),\\
\tilde{\beta} &:= \beta(c(\tilde{W}_2 \cap \tilde{W}_3), \overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4)),\\
I &:= \{i : \frac{\frac{1}{\tau}c_{is}}{c_{iq}} \leq 1,\ \frac{c_{iq}}{\min\{c_{ip}^{1/2} c_{ir}^{1/2}\}},\ \frac{1}{t}c_{ir} \leq 1\}\\
&= \{i : c_{iq}^2 \leq c_{ip}c_{ir}, \frac{c_{is}}{c_{iq}} \leq \tau, \frac{c_{ir}}{c_{iq}} \geq t\},\\
\tilde{I} &:= \{i : \frac{\frac{1}{2c}\frac{1}{\tau}d_{is_1}}{d_{i\tilde{q}_1}} \leq 1, \frac{\frac{1}{2c}d_{iq_1}}{\min\{d_{ip_1}^{1/2} d_{ir_1}^{1/2}, \frac{1}{t}d_{ir_1}\}} \leq 1\}\\
&= \{i : d_{iq_1}^2 \leq 4c^2 d_{ip_1} d_{r_1}, \frac{d_{is_1}}{d_{i\tilde{q}_1}} \leq 2c\tau, \frac{d_{ir_1}}{d_{iq_1}} \geq \frac{t}{2c}\}.
\end{aligned}
\tag{2.5}
$$

(here $c > 0$ is some constant). Since $W_2 \cap W_3 \subset \tilde{c}_1(\tilde{W}_2 \cap \tilde{W}_3)$ for some constant $\tilde{c}_1 > 0$ and $\tilde{c}_2\overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4) \subset \overline{\Gamma}(W_1 \cup W_2)$ for some constant $\tilde{c}_2 > 0$ we have $\beta \leq \tilde{\beta}$. We also have $|I| \leq \beta$ by lemma 1.4(i) and $\tilde{\beta} \leq |\tilde{I}|$ by lemma 1.4(ii). Then $|I| \leq |\tilde{I}|$.

In the upcoming chapters we will have further estimates on $|I_1|$, $|I_2|$, $|I|$ and $|\tilde{I}|$

Chapter 3

Some Cartesian Products of Power Series Spaces and Dragilev Spaces

In this chapter, we consider the cartesian product of a Dragilev space and a power series space or cartesian product of two Dragilev spaces. We show that the method used in [10] can be used in these cases.

In this chapter, we assume that all Dragilev functions are rapidly increasing, and for any two considered Dragilev function, for example f and $\tilde{f}$, $f^{-1}\tilde{f}$ and $\tilde{f}^{-1}f$ are slowly increasing.

3.1 $E_0(a) \times L_f(b,\infty) \simeq E_0(\tilde{a}) \times L_f(\tilde{b},\infty)$

Let $K(C) = E_0(a) \times L_f(b,\infty)$, $K(D) = E_0(\tilde{a}) \times L_f(\tilde{b},\infty)$ and suppose $T : K(D) \longrightarrow K(C)$ is an isomorphism where $C = (c_{ip}), D = (d_{ip})$. Then

$$c_{ip} = \begin{cases} e^{-\frac{1}{p}a_k} & \text{if } i = 2k-1 \\ e^{f(pb_k)} & \text{if } i = 2k \end{cases} \quad , \quad d_{ip} = \begin{cases} e^{-\frac{1}{p}\tilde{a}_k} & \text{if } i = 2k-1 \\ e^{f(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

Given $p_1 \in \mathbb{N}$, by Remark, we can find a positive integer $p > p_1$ and a constant $c_1 > 0$ such that $c_1 U_p \subset T(V_{p_1})$. Let $q_1 > 2p$, find $\tilde{q}_1 > q > q_1 (> 2p)$ and constants $c_2 > 0, c_3 > 0$ such that $c_3 T(V_{\tilde{q}_1}) \subset U_q \subset \frac{1}{c_2} T(V_{q_1})$. Let $r_1 > 2\tilde{q}_1$, find $r > r_1$ and a constant $c_4 > 0$ such that $c_4 U_r \subset T(V_{r_1})$. For $s > 2r$, find $s_1 > s$ and a constant $c_5 > 0$ such that $c_5 T(V_{s_1}) \subset U_s$.

Now, consider (2.2). If $i = 2k$ in the set I_1 in (2.3), then $c_{ip} = e^{f(pb_k)}$. So

the first inequality in the set becomes

$$e^{f(pb_k)}e^{f(rb_k)} \leq e^{2f(qb_k)} \iff f(pb_k) + f(rb_k) \leq 2f(qb_k)$$
$$\iff \frac{f(pb_k)}{f(qb_k)} + \frac{f(rb_k)}{f(qb_k)} \leq 2.$$

Let $qb_k = x$, then

$$\lim \frac{f(rb_k)}{f(qb_k)} = \lim \frac{f(\frac{r}{q}x)}{f(x)} = \infty$$

since f is rapidly increasing. This shows us that the case $i = 2k$ is not possible. But for $i = 2k-1$ the same inequality gives

$$(-\frac{1}{p} - \frac{1}{r})a_k \leq -\frac{2}{q}a_k$$

which is possible for every value of k. So we take $i = 2k-1$ and $c_{ip} = e^{-\frac{1}{p}a_k}$. Then from the other two inequalities

$$\frac{c_{ir}}{c_{iq}} \leq t,\ \frac{c_{is}}{c_{iq}} \geq \tau \iff e^{(\frac{1}{q} - \frac{1}{r})a_k} \leq t,\ e^{(\frac{1}{q} - \frac{1}{s})a_k} \geq \tau$$
$$\iff (\frac{1}{q} - \frac{1}{r})a_k \leq \ln t,\ (\frac{1}{q} - \frac{1}{s})a_k \geq \ln \tau$$

i.e.

$$|I_1| = M_a\left(\frac{\ln t}{\frac{1}{q} - \frac{1}{r}}, \frac{\ln \tau}{\frac{1}{q} - \frac{1}{s}}\right).$$

But $|I_1| \leq \beta_1$ by lemma 1.4(i).

Similarly, the first inequality in I_2 cannot hold for $i = 2k$. Same inequality gives

$$\frac{1}{p_1} + \frac{1}{r_1} \leq \ln 4c^2 \frac{2}{\tilde{q}_1}$$

for $i = 2k-1$. So we take $i = 2k-1$ and $d_{ip} = e^{-\frac{1}{p}\tilde{a}_k}$. Then from the last two inequalities :

$$e^{-\frac{1}{r_1}\tilde{a}_k} \leq e^{-\frac{1}{\tilde{q}_1}\tilde{a}_k} 2ct \quad , \quad e^{-\frac{1}{s_1}\tilde{a}_k} \geq \frac{\tau}{2c} e^{-\frac{1}{q_1}\tilde{a}_k}$$
$$\iff (\frac{1}{\tilde{q}_1} - \frac{1}{r_1})\tilde{a}_k \leq \ln(2ct) \quad , \quad (\frac{1}{q_1} - \frac{1}{s_1}\tilde{a}_k) \geq \ln(\frac{\tau}{2c})$$
$$\iff \tilde{a}_k \leq \frac{\ln(2ct)}{\frac{1}{\tilde{q}_1} - \frac{1}{s_1}} \quad , \quad \tilde{a}_k \geq \frac{\ln(\frac{\tau}{2c})}{\frac{1}{q_1} - \frac{1}{s_1}}$$

i.e., $|I_2| = M_{\tilde{a}}\left(\frac{\ln(2ct)}{\frac{1}{\tilde{q}_1}-\frac{1}{r_1}}, \frac{\ln(\frac{\tau}{2c})}{\frac{1}{q_1}-\frac{1}{s_1}}\right)$. But, $|I_2| \geq \beta_2$ by the lemma 1.4(ii). So, $|I_1| \leq \beta_1 \leq \beta_2 \leq |I_2|$. Hence we have

$$M_a\left(\frac{\ln t}{\frac{1}{q}-\frac{1}{r}}, \frac{\ln \tau}{\frac{1}{q}-\frac{1}{s}}\right) \leq M_{\tilde{a}}\left(\frac{\ln(2ct)}{\frac{1}{\tilde{q}_1}-\frac{1}{r_1}}, \frac{\ln(\frac{\tau}{2c})}{\frac{1}{q_1}-\frac{1}{s_1}}\right).$$

Therefore we have shown that $\exists\, c > 0$ such that $M_a(t\,,\,\tau) \leq M_{\tilde{a}}(ct\,,\,\frac{\tau}{c})$. If $i = 2k-1$ then $c_{ip} = e^{-\frac{1}{p}a_k}$ and the first inequality in the set I gives $\frac{1}{p}+\frac{1}{r} \leq \frac{2}{q}$ which is impossible, because $q > 2p$, $r > 2q$. So we take $i = 2k$ and $c_{ip} = e^{f(pb_k)}$. Then from the last two inequalities we get

$$\begin{aligned} & e^{f(sb_k)-f(qb_k)} \leq \tau \quad , \quad e^{f(rb_k)-f(qb_k)} \geq t \\ \Longleftrightarrow\ & f(sb_k)-f(qb_k) \leq \ln\tau \quad , \quad f(rb_k)-f(qb_k) \geq \ln t \\ \Longleftarrow\ & \qquad f(sb_k) \leq \ln\tau \quad , \quad f((r-1)b_k) \geq \ln t \end{aligned}$$

(since f is rapidly increasing $f(rb_k)-f(qb_k) \geq f((r-1)b_k)$). Hence we have

$$M_b\left(\frac{f^{-1}(\ln\tau)}{s}, \frac{f^{-1}(\ln t)}{r-1}\right) \leq |I| \leq \beta.$$

Let us now consider

$$\begin{aligned} \tilde{I} &= \{i : \frac{\frac{1}{2c}d_{is_1}}{d_{i\tilde{q}_1}} \leq 1,\ \frac{\frac{1}{2c}d_{iq_1}}{\min\{d_{ip_1}^{1/2}d_{ir_1}^{1/2},\ \frac{1}{t}d_{ir_1}\}} \leq 1\} \\ &= \{i : d_{iq_1}^2 \leq 4c^2 d_{ip_1}d_{ir_1},\ \frac{d_{is_1}}{d_{i\tilde{q}_1}} \leq 2c\tau,\ \frac{d_{ir_1}}{d_{q_1}} \leq \frac{t}{2c}\}. \end{aligned}$$

If $i = 2k-1$, then the first inequality in the set $\tilde{I}$ is possible only for finitely many values of k. So we take $i = 2k$ and $d_{ip} = e^{f(p\tilde{b}_k)}$. Then from the last two inequalities we have

$$\begin{aligned} & e^{f(s_1\tilde{b}_k)-f(\tilde{q}_1\tilde{b}_k)} \leq 2c\tau,\ e^{f(r_1\tilde{b}_k)-f(q_1\tilde{b}_k)} \leq \frac{t}{2c} \\ \Longleftrightarrow\ & f(s_1\tilde{b}_k)-f(\tilde{q}_1\tilde{b}_k) \leq \ln(2c\tau),\ f(r_1\tilde{b}_k)-f(q_1\tilde{b}_k) \geq \ln(\frac{t}{2c}) \\ \Longrightarrow\ & \qquad f((s_1-1)\tilde{b}_k) \leq \ln(2c\tau),\ f(r_1\tilde{b}_k) \geq \ln(\frac{t}{2c}) \end{aligned}$$

(since f is rapidly increasing, $f((s_1-1)\tilde{b}_k) \leq f(s_1\tilde{b}_k)-f(\tilde{q}_1\tilde{b}_k)$). So,

$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{b}}\left(\frac{f^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f^{-1}(\ln\frac{t}{2c})}{r_1}\right).$$

Hence we have

$$M_b\left(\frac{f^{-1}(\ln\tau)}{s}, \frac{f^{-1}(\ln t)}{r-1}\right) \leq M_{\tilde{b}}\left(\frac{f^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f^{-1}(\ln\frac{t}{2c})}{r_1}\right).$$

Let $\frac{f^{-1}(\ln\tau)}{s} = T$. Then $\tau = e^{f(sT)}$ and

$$\begin{aligned}\frac{f^{-1}(\ln(2c\tau))}{s_1-1} &= \frac{f^{-1}(\ln 2c + f(sT)}{s_1-1}\\ &\leq \frac{f^{-1}f((s+1)T)}{s_1-1}\\ &= \frac{(s+1)T}{s_1-1}\\ &\leq K_1T \ (\text{ for } K_1 \geq \frac{s+1}{s_1-1}).\end{aligned}$$

Let now $\frac{f^{-1}(\ln t)}{r-1} = L$. Then $t = e^{f((r-1)L)}$ and

$$\begin{aligned}\frac{f^{-1}(\ln\frac{t}{2c})}{r_1} &= \frac{f^{-1}(f((r-1)L) - \ln 2c)}{r_1}\\ &\geq \frac{f^{-1}(f(r-2)L)}{r_1}\\ &= \frac{(r-2)L}{r_1}\\ &\geq \frac{L}{K_2} \ (\text{ for } K_2 > \frac{r_1}{r-2}).\end{aligned}$$

So, for $K > \max\{K_1, K_2\}$ we have

$$M_b(T, L) \leq M_{\tilde{b}}(KT, \frac{L}{K})$$

If we change the roles of U_i, V_j's in Remark, we similarly obtain the following symmetric inequalities : There is $c > 0$ such that $M_{\tilde{a}}(t, \tau) \leq M_a(ct, \frac{\tau}{c})$ for large values of τ and there is a constant, again we call, $K > 0$ such that $M_{\tilde{b}}(T, L) \leq M_b(KT, \frac{L}{K})$ for large values of L. So by lemma 2.3 there are integers k_1 and k_2 such that $a_{i+k_1} \asymp \tilde{a}_i$ and $b_{i+k_2} \asymp \tilde{b}_i$.

Therefore by theorem 2.1 we have

$$\begin{gathered}\text{If } E_0(a) \times L_f(b, \infty) \simeq E_0(\tilde{a}) \times L_f(\tilde{b}, \infty)\\ \text{then } E_0(a) \approx E_0(\tilde{a}) \ , \ L_f(b, \infty) \approx L_f(\tilde{b}, \infty).\end{gathered}$$

Note We observe that to prove our claim, it is sufficient to show the equivalence of the counting functions M. We shall do so in the rest of this work.

3.2 $E_0(a) \times L_f(b,\infty) \simeq E_0(\tilde{a}) \times L_{\tilde{f}}(\tilde{b},\infty)$

Let $K(C) = E_0(a) \times L_f(b,\infty), K(D) = E_0(\tilde{a}) \times L_{\tilde{f}}(\tilde{b},\infty)$ and $T : K(D) \longrightarrow K(C)$ be an isomorphism where $C = (c_{ip}), D = (d_{ip})$. Then

$$c_{ip} = \begin{cases} e^{-\frac{1}{p}a_k} & \text{if } i = 2k-1 \\ e^{f(pb_k)} & \text{if } i = 2k \end{cases} \quad , \quad d_{ip} = \begin{cases} e^{-\frac{1}{p}\tilde{a}_k} & \text{if } i = 2k-1 \\ e^{\tilde{f}(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

If $i = 2k$ then the first inequality in the set I_1 becomes

$$e^{f(pb_k)}e^{f(rb_k)} \leq e^{2f(qb_k)} \quad \text{or equivalently} \quad f(pb_k) + f(rb_k) \leq 2f(qb_k)$$

which is not possible. Because, otherwise

$$\frac{f(pb_k)}{f(qb_k)} + \frac{f(rb_k)}{f(qb_k)} \leq 2$$

and if we take limit on both sides

$$\lim \frac{f(pb_k)}{f(qb_k)} + \underbrace{\lim \frac{f(rb_k)}{f(qb_k)}}_{\infty} \leq 2.$$

So we take $i = 2k-1$ and $c_{ip} = e^{-\frac{1}{p}a_k}$. Then from the other two inequalities we get

$$\frac{\ln \tau}{\frac{1}{q} - \frac{1}{s}} \leq a_k \leq \frac{\ln t}{\frac{1}{q} - \frac{1}{r}} \qquad \text{i.e.} \quad |I_1| = M_a = \left(\frac{\ln t}{\frac{1}{q} - \frac{1}{s}}, \frac{\ln \tau}{\frac{1}{q} - \frac{1}{s}} \right).$$

Similarly, the first inequality in the set I_2 is impossible for $i = 2k$. So we take $i = 2k-1$ and $d_{ip} = e^{-\frac{1}{p}\tilde{a}_k}$. Then from the other two inequalities we get

$$\frac{\ln \frac{\tau}{2c}}{\frac{1}{q_1} - \frac{1}{s_1}} \leq \tilde{a}_k \leq \frac{\ln 2ct}{\frac{1}{\tilde{q}_1} - \frac{1}{r_1}} \qquad \text{i.e.} \quad |I_2| = M_{\tilde{a}} \left(\frac{\ln 2ct}{\frac{1}{\tilde{q}_1} - \frac{1}{r_1}}, \frac{\ln \frac{\tau}{2c}}{\frac{1}{q_1} - \frac{1}{s_1}} \right).$$

Since $|I_1| \leq \beta_1 \leq \beta_2 \leq |I_2|$ we consequently have

$$M_a \left(\frac{\ln t}{\frac{1}{q} - \frac{1}{r}}, \frac{\ln \tau}{\frac{1}{q} - \frac{1}{s}} \right) \leq M_{\tilde{a}} \left(\frac{\ln 2ct}{\frac{1}{\tilde{q}_1} - \frac{1}{r_1}}, \frac{\ln \frac{\tau}{2c}}{\frac{1}{q_1} - \frac{1}{s_1}} \right).$$

Therefore we have shown that $\exists c > 0$ such that

$$M_a(t,\tau) \leq M_{\tilde{a}}(ct, \frac{\tau}{c})$$

Now, since

$$|I| = |\{i : c_{iq}^2 \leq c_{ip}c_{ir},\ \frac{c_{is}}{c_{iq}} \leq \tau,\ \frac{c_{ir}}{c_{iq}} \geq t\}| \leq \beta$$

by lemma 1.4(ii), if $i = 2k-1$ then the first inequality in the set I gives

$$\frac{1}{p} + \frac{1}{r} \leq \frac{2}{q}$$

which is not possible. So we take $i = 2k$ and $c_{ip} = e^{f(pb_k)}$. Then from the other two inequalities we have

$$e^{f(sb_k) - f(qb_k)} \leq \tau \quad , \quad e^{f(rb_k) - f(qb_k)} \geq t.$$

$$\Longleftarrow \quad f(sb_k) \leq \ln \tau \quad , \quad f((r-1)b_k) \geq \ln t$$

$$\Longleftrightarrow \qquad \frac{f^{-1}(\ln t)}{r-1} \leq b_k \leq \frac{f^{-1}(\ln \tau)}{s}$$

and hence $$M_b\left(\frac{f^{-1}(\ln \tau)}{s}, \frac{f^{-1}(\ln t)}{r-1}\right) \leq |I|.$$

Note that $f((r-1)b_k) \leq f(rb_k) - f(qb_k)$ since f is rapidly increasing.

On the other hand if $i = 2k$ then the firt inequality in the set $\tilde{I}$ gives

$$\frac{1}{p_1} + \frac{1}{r_1} - \frac{2}{q_1} \leq \frac{\ln(4c^2)}{\tilde{a_k}}$$

which is possible only for finitely many values of k. So we take $i = 2k$ and $d_{ip} = e^{\tilde{f}(p\tilde{b_k})}$. Then from the other two inequalities we get

$$e^{\tilde{f}(s_1\tilde{b_k}) - \tilde{f}(\tilde{q_1}\tilde{b_k})} \leq 2c\tau, \quad e^{\tilde{f}(r_1\tilde{b_k}) - \tilde{f}(q_1\tilde{b_k})} \geq \tfrac{t}{2c}.$$

$$\Longleftrightarrow \tilde{f}(s_1\tilde{b_k}) - \tilde{f}(\tilde{q_1}\tilde{b_k}) \leq \ln 2c\tau, \quad \tilde{f}(r_1\tilde{b_k}) - \tilde{f}(q_1\tilde{b_k}) \geq \ln \tfrac{t}{2c}$$

$$\Longrightarrow \qquad \tilde{f}((s_1-1)\tilde{b_k}) \leq \ln 2c\tau, \qquad \tilde{f}(r_1\tilde{b_k}) \geq \ln \tfrac{t}{2c}$$

Thus $$|\tilde{I}| \leq M_{\tilde{b}}\left(\frac{\tilde{f}^{-1}(\ln 2c\tau)}{s_1 - 1}, \frac{\tilde{f}^{-1}(\ln \frac{t}{2c})}{r_1}\right).$$

Note that $\tilde{f}((s_1-1)\tilde{b_k}) \leq \tilde{f}(s_1\tilde{b_k}) - \tilde{f}(\tilde{q_1}\tilde{b_k})$ since $\tilde{f}$ is rapidly increasing.

Let now $T = \dfrac{f^{-1}(\ln\tau)}{s}$, i.e. $\tau = e^{f(sT)}$. Then

$$\frac{\tilde{f}^{-1}(\ln 2c\tau)}{s_1-1} = \frac{\tilde{f}^{-1}(\ln 2c + f(sT))}{s_1-1}$$
$$\leq \frac{\tilde{f}^{-1}((s+1)T)}{s_1-1} = \frac{\tilde{f}^{-1}f((s+1)T)}{\tilde{f}^{-1}f(T)}\frac{\tilde{f}^{-1}f(T)}{s_1-1} \leq K_1\tilde{f}^{-1}f(T)...(*)$$

for $K_1 > \lim_{T\to\infty}\dfrac{\tilde{f}^{-1}f((s+1)T)}{\tilde{f}^{-1}f(T)(s_1-1)}$. Similarly for $L = \dfrac{f^{-1}(\ln t)}{r}$, i.e. $t = e^{f(rL)}$

$$\frac{\tilde{f}^{-1}(\ln\frac{t}{2c})}{r_1} = \frac{\tilde{f}^{-1}(\ln\frac{e^{f(rL)}}{2c})}{r_1} = \frac{\tilde{f}^{-1}(f(rL)-\ln 2c)}{r_1}$$
$$\geq \frac{\tilde{f}^{-1}f((r-1)L)}{r_1} = \frac{\tilde{f}^{-1}f((r-1)L)}{\tilde{f}^{-1}f(L)}\frac{\tilde{f}^{-1}f(L)}{r_1} \geq \frac{1}{K_2}\tilde{f}^{-1}f(L)...(**)$$

for $K_2 > \lim_{L\to\infty}\dfrac{r_1\tilde{f}^{-1}f(L)}{\tilde{f}^{-1}f((r-1)L)}$.

If we take $K \geq \max\{K_1, K_2\}$ then both of $(*)$ and $(**)$ will hold. So we have

$$M_b(T,L) \leq M_{\tilde{b}}(K\tilde{f}^{-1}f(T), \frac{1}{K}\tilde{f}^{-1}f(L)).$$

Now choose η so large that $\tau_{\tilde{f}^{-1}f}(K) > \eta$. Then

$$M_{\tilde{b}}(K\tilde{f}^{-1}f(T), \frac{1}{K}\tilde{f}^{-1}f(L)) \leq M_{\tilde{b}}(\tilde{f}^{-1}f(\eta T), \tilde{f}^{-1}f(\frac{L}{\eta})).$$

Since $M_{\tilde{b}}(\tilde{f}^{-1}f(\eta T)\,,\ \tilde{f}^{-1}\,f(\frac{L}{\eta})) = M_{f^{-1}\tilde{f}(\tilde{b})}(\eta T\,,\ \frac{L}{\eta})$ we have

$$M_b(T\,,\ L) \leq M_{f^{-1}\tilde{f}(\tilde{b})}(\eta T\,,\ \frac{L}{\eta}).$$

If we change the roles of U_i, V_j's above we get the following symmetric inequalities;

$$M_{\tilde{a}}\left(\frac{\ln t}{\frac{1}{q}-\frac{1}{r}}\,,\ \frac{\ln\tau}{\frac{1}{q}-\frac{1}{s}}\right) \leq M_a\left(\frac{\ln 2ct}{\frac{1}{\tilde{q_1}}-\frac{1}{r_1}}\,,\ \frac{\ln\frac{\tau}{2c}}{\frac{1}{q_1}-\frac{1}{s_1}}\right)\quad,$$
$$M_{\tilde{b}}\left(\frac{\tilde{f}^{-1}(\ln\tau)}{s}\,,\ \frac{\tilde{f}^{-1}(\ln t)}{r}\right) \leq M_b\left(\frac{f^{-1}(\ln 2c\tau)}{s_1}\,,\ \frac{f^{-1}(\ln\frac{t}{2c})}{r_1}\right).$$

By these together with their symmetric inequalities we say

$$M_a \approx M_{\tilde{a}} \qquad \text{and} \qquad M_b \approx M_{\tilde{b}}.$$

As a consequence of these we get $a_i \asymp \tilde{a}_{i+k_1}$ and $b_j \asymp f^{-1}\tilde{f}(\tilde{b}_{j+k_2})$. Then by theorem (2.1) we have

$$\begin{aligned} &\text{if } E_0(a) \times L_f(b,\infty) \simeq E_0(\tilde{a}) \times L_{\tilde{f}}(\tilde{b},\infty) \\ &\text{then } E_0(a) \approx E_0(\tilde{a}) \text{ and } L_f(b,\infty) \approx L_{\tilde{f}}(\tilde{b},\infty). \end{aligned}$$

3.3 $L_f(a,0) \times E_\infty(b) \simeq L_{\tilde{f}}(\tilde{a},0) \times E_\infty(\tilde{b})$

Let $K(C) = L_f(a,0) \times E_\infty(b), K(D) = L_{\tilde{f}}(\tilde{a},0) \times E_\infty(\tilde{b})$ and let T be an isomorphism from $K(D)$ to $K(C)$ where $D = (d_{ip}), C = (c_{ip})$. Then

$$c_{ip} = \begin{cases} e^{-f(\frac{1}{p}a_k)} & \text{if } i = 2k-1 \\ e^{pb_k} & \text{if } i = 2k \end{cases} \quad , \quad d_{ip} = \begin{cases} e^{-\tilde{f}(\frac{1}{p}\tilde{b}_k)} & \text{if } i = 2k-1 \\ e^{p\tilde{b}_k} & \text{if } i = 2k \end{cases}$$

If $i = 2k$ then the first inequality in the set I_1 gives $p + r \leq 2q$ which is impossible. So we take $i = 2k-1$ and $c_{ip} = e^{-f(\frac{1}{p}b_k)}$. Then from the other two inequalities we get

$$\begin{aligned} & f(\frac{1}{q}a_k) - f(\frac{1}{r}a_k) \leq \ln t \quad , \quad f(\frac{1}{q}a_k) - f(\frac{1}{s}a_k) \geq \ln \tau \\ \Longleftarrow \quad & f(\frac{1}{q}a_k) \leq \ln t \quad , \quad f(\frac{1}{q+1}a_k) \geq \ln \tau \\ \Longleftrightarrow \quad & (q+1)f^{-1}(\ln \tau) \leq a_k \leq qf^{-1}(\ln t) \end{aligned}$$

and hence $$M_a(qf^{-1}(\ln t), (q+1)f^{-1}(\ln \tau)) \leq |I_1|.$$

On the other hand if $i = 2k$ then the first inequality in the set I_2 gives $p_1 + r_1 - 2\tilde{q}_1 \leq \dfrac{\ln 4c^2}{\tilde{b}_k}$ which is possible only for finitely many k. So we take $i = 2k-1$ and $d_{ip} = e^{-\tilde{f}(\frac{1}{p}\tilde{a}_k)}$ then from the other two inequalities we get

$$\begin{aligned} & \tilde{f}(\frac{1}{\tilde{q}_1}\tilde{a}_k) - \tilde{f}(\frac{1}{r_1}\tilde{a}_k) \leq \ln 2ct \quad , \quad \tilde{f}(\frac{1}{q_1}\tilde{a}_k) - \tilde{f}(\frac{1}{s_1}\tilde{a}_k) \geq \ln \frac{\tau}{2c} \\ \Longrightarrow \quad & \tilde{f}(\frac{1}{\tilde{q}_1+1}\tilde{a}_k) \leq \ln 2ct \quad , \quad \tilde{f}(\frac{1}{q_1}\tilde{a}_k) \geq \ln \frac{\tau}{2c} \\ \Longleftrightarrow \quad & q_1\tilde{f}^{-1}(\ln \frac{\tau}{2c}) \leq \tilde{a}_k \leq (\tilde{q}_1+1)\tilde{f}^{-1}(\ln 2ct) \end{aligned}$$

and so $$|I_2| \le M_{\tilde{a}}((\tilde{q}_1+1)\tilde{f}(\ln 2ct)\,,\ q_1\tilde{f}(\ln\frac{\tau}{2c}).$$

Since $|I_1| \le \beta_1 \le \beta_2 \le |I_2|$ have

$$M_a(qf^{-1}(\ln t)\,,\ (q+1)f^{-1}(\ln\tau)) \le M_{\tilde{a}}((\tilde{q}_1+1)\tilde{f}^{-1}(\ln 2ct)\,,\ q_1\tilde{f}^{-1}(\ln\frac{\tau}{2c})).$$

Let $T = qf^{-1}(\ln t)$ and $L = (q+1)f^{-1}(\ln\tau)$. Then for

$$K > \max\{(\tilde{q}_1+1)\lim_{T\to\infty}\frac{\tilde{f}^{-1}f(\frac{T}{q-1})}{\tilde{f}^{-1}f(T)},\ \frac{1}{q_1}\lim_{L\to\infty}\frac{\tilde{f}^{-1}f(L)}{\tilde{f}^{-1}f(\frac{L}{q-1})}\}$$

we have $\quad M_a(T\,,\,L) \le M_{\tilde{a}}(K\tilde{f}^{-1}f(T)\,,\,\frac{1}{K}\tilde{f}^{-1}f(L))$.

Choose now η so large that $\tau_{\tilde{f}^{-1}f}(\eta) > K$. Then

$$M_a(T\,,\,L) \le M_{\tilde{a}}(\tilde{f}^{-1}f(\eta T)\,,\,\tilde{f}^{-1}f(\frac{L}{\eta})).$$

Since $M_{\tilde{a}}(\tilde{f}^{-1}f(\eta T)\,,\,\tilde{f}^{-1}f(\frac{L}{\eta}) = M_{f^{-1}\tilde{f}(\tilde{a})}(\eta T\,,\,\frac{L}{\eta})$ we have

$$M_a(T\,,\,L) \le M_{f^{-1}\tilde{f}(\tilde{a})}(\eta T\,,\,\frac{L}{\eta}).$$

If $i = 2k-1$ then the first inequality in the set I gives

$$2f(\frac{1}{q}a_k) \ge f(\frac{1}{p}a_k) + f(\frac{1}{r}a_k)$$

which is not possible since $f(\frac{1}{p}a_k) \ge f(\frac{1}{q}a_k)$. So we take $i = 2k$ and $c_{ip} = e^{pb_k}$. Then from the other two inequalities we get

$$(s-q)b_k \le \ln\tau \quad,\quad (r-q)b_k \ge \ln t$$
$$\iff \quad \frac{\ln t}{r-q} \le b_k \le \frac{\ln\tau}{s-q} \quad,\ \text{i.e.}\quad |I_1| = M_b\left(\frac{\ln\tau}{s-q}\,,\,\frac{\ln t}{r-q}\right).$$

On the other hand if $i = 2k-1$ then the first inequality in the set $\tilde{I}$ gives

$$\tilde{f}(\frac{1}{p}\tilde{a}_k) + \tilde{f}(\frac{1}{r_1}\tilde{a}_k) - 2\tilde{f}(\frac{1}{q_1}\tilde{a}_k) \le \ln 4c^2$$

which is possible only for some $k < k_0$. So we take $i = 2k$ and $d_{ip} = e^{p\tilde{b}_k}$. Then from the other two inequalities we get

$$(s_1-\tilde{q}_1)\tilde{b}_k \le \ln 2c\tau \quad,\quad (r_1-q_1)\tilde{b}_k \ge \ln\frac{t}{2c}$$
$$\iff \quad \frac{\ln\frac{t}{2c}}{r_1-q_1} \le \tilde{b}_k \le \frac{\ln 2c\tau}{s_1-\tilde{q}_1} \quad,\ \text{i.e}\quad |\tilde{I}| = M_{\tilde{b}}\left(\frac{\ln 2c\tau}{s_1-\tilde{q}_1}\,,\,\frac{\ln\frac{t}{2c}}{r_1-q_1}\right).$$

We consequenly have

$$M_b\left(\frac{\ln\tau}{s-q}, \frac{\ln t}{r-q}\right) \leq M_{\tilde{b}}\left(\frac{\ln 2c\tau}{s_1-\tilde{q}_1}, \frac{\ln\frac{t}{2c}}{r_1-q_1}\right).$$

Also, if we change the roles of U_i, V_j's in Remark we get the following symmetric inequalities ;

$$M_{f^{-1}\tilde{f}(\tilde{a})}(T,L) \leq M_a(\eta T, \tfrac{L}{\eta}),$$
$$M_{\tilde{b}}\left(\tfrac{\tau}{s-q}, \tfrac{\ln t}{r-q}\right) \leq M_b\left(\tfrac{\ln 2c\tau}{s_1-\tilde{q}_1}, \tfrac{\ln\frac{t}{2c}}{r_1-q_1}\right)$$

These with their symmetric inequalities give

$$M_a \approx M_{f^{-1}\tilde{f}(\tilde{a})} \qquad \text{and} \qquad M_b \approx M_{\tilde{b}}.$$

As a result we get $a_i \asymp f^{-1}\tilde{f}(\tilde{a}_{i+k_1})$ and $b_i \asymp \tilde{b}_{j+k_2}$ for some k_1, $k_2 \in \mathbb{Z}$ by 2.3. Then by theorem (2.1) we have

$$\text{if } L_f(a,0) \times E_\infty(b) \simeq L_{\tilde{f}}(\tilde{a},0) \times E_\infty(\tilde{b})$$
$$\text{then } L_f(a,0) \approx L_{\tilde{f}}(\tilde{a},0) \text{ and } E_\infty(b) \approx E_\infty(\tilde{b}).$$

The following case is included in this chapter since its proof is similar to the earlier proofs.

3.4 $L_{f_1}(a,0) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},\infty)$

Let $K(C) = L_{f_1}(a,0) \times L_{f_2}(b,\infty)$, $K(D) = L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},\infty)$ and let T be an isomorphism from $K(D)$ to $K(C)$ where $D = (d_{ip}), C = (c_{ip})$. Then

$$c_{ip} = \begin{cases} e^{-f_1(\frac{1}{p}a_k)} & \text{if } i = 2k-1 \\ e^{f_2(pb_k)} & \text{if } i = 2k \end{cases}, \quad d_{ip} = \begin{cases} e^{-f3(\frac{1}{p}\tilde{b}_k)} & \text{if } i = 2k-1 \\ e^{f_4(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

If $i = 2k$ in the set

$$I_1 = \{i : c_{ip}c_{ir} \leq c_{iq}^2, \frac{c_{ir}}{c_{iq}} \leq t, \frac{c_{is}}{c_{iq}} \geq \tau\}.$$

then the first inequality gives

$$f_2(pb_k) + f_2(rb_k) \leq 2f_2(qb_k).$$

But $\dfrac{f_2(rb_k)}{f_2(qb_k)} \nearrow \infty$ since f_2 is rapidly increasing. So we take $i = 2k-1$ and $c_{ip} = e^{-f_1(\frac{1}{p}a_k)}$. Then from the other two inequalities we get

$$\begin{aligned}
& f_1(\frac{1}{q}a_k) - f_1(\frac{1}{r}a_k) \le t \quad , \quad f_1(\frac{1}{q}a_k) - f_1(\frac{1}{s}a_k) \ge \tau \\
\Longleftarrow \quad & f_1(\frac{1}{q}a_k) \le \ln t \quad , \quad f_1(\frac{1}{q+1}a_k) \ge \ln \tau \\
\Longleftrightarrow \quad & (q+1)f_1^{-1}(\ln \tau) \le \ a_k \ \le q f_1^{-1}(\ln t)
\end{aligned}$$

and hence $$M_a(qf_1^{-1}(\ln t), (q+1)f_1^{-1}(\ln \tau)) \le |I_1|.$$

On the other hand, if $i = 2k$ in the set

$$I_2 = \{i : d_{ip_1} d_{ir_1} \le 4c^2 d_{i\tilde{q}_1}, \ \frac{d_{ir_1}}{d_{i\tilde{q}_1}} \le 2ct, \ \frac{d_{is_1}}{d_{iq_1}} \ge \frac{\tau}{2c}\}$$

then the first inequality gives

$$\frac{f_4(p_1\tilde{b}_k)}{f_4(\tilde{q}_1\tilde{b}_k)} + \frac{f_4(r_1\tilde{b}_k)}{f_4(\tilde{q}_1\tilde{b}_k)} \le \frac{\ln 4c^2}{f_4(\tilde{q}_1\tilde{b}_k)} + 2 \ \ (\le \ln 4c^2 + 2 =: E)$$

which is not possible for every k since $\lim \dfrac{f_4(r_1\tilde{b}_k)}{f_4(\tilde{q}_1\tilde{b}_k)} = \infty$. So we take $i = 2k-1$ and $d_{ip} = e^{-f_3(\frac{1}{p}\tilde{b}_k)}$ and get from the other two inequalities that

$$\begin{aligned}
& f_3(\frac{1}{q_1}\tilde{a}_k) - f_3(\frac{1}{r_1}\tilde{a}_k) \le \ln 2ct \quad , \quad f_3(\frac{1}{q_1}\tilde{a}_k) - f_3(\frac{1}{s_1}\tilde{a}_k) \ge \ln \frac{\tau}{2c} \\
\Longrightarrow \quad & f_3(\frac{1}{\tilde{q}_1+1}\tilde{a}_k) \le \ln 2ct \quad , \quad f_3(\frac{1}{q_1}\tilde{a}_k) \ge \ln \frac{\tau}{2c} \\
\Longleftrightarrow \quad & q_1 f_3^{-1}(\ln \frac{\tau}{2c}) \le \ \tilde{a}_k \ \le (\tilde{q}_1+1)f_3^{-1}(\ln 2ct)
\end{aligned}$$

and hence $$|I_2| \le M_{\tilde{a}}((\tilde{q}_1+1)f_3^{-1}(\ln 2ct)\,,\ q_1 f_3^{-1}(\ln \frac{\tau}{2c})).$$

Since $|I_1| \le \beta_1 \le \beta_2 \le |I_2|$ by Remark we have

$$M_a(qf_1^{-1}(\ln t)\,,\ (q+1)f_1^{-1}(\ln \tau)) \le M_{\tilde{a}}((\tilde{q}_1+1)f_3^{-1}(\ln 2ct)\,,\ q_1 f_3^{-1}(\ln \frac{\tau}{2c})).$$

Consider now the set

$$I = \{i : c_{iq}^2 \le c_{ip}c_{ir}, \ \frac{c_{is}}{c_{iq}} \le \tau, \ \frac{c_{ir}}{c_{iq}} \ge t\}.$$

If $i = 2k-1$ then the first inequality in the set I gives

$$2f_1(\frac{1}{q}a_k) \leq f_1(\frac{1}{p}a_k) + f_1(\frac{1}{r}a_k) \quad (\leq 2f_1(\frac{1}{p}a_k))$$

which is not possible since $p < q$ and f is rapidly incresaing. So we take $i = 2k$ and $c_{ip} = e^{f_2(pb_k)}$. Then from the other two inequalities we get

$$\begin{aligned} & f_2(sb_k) - f_2(qb_k) \leq \ln\tau \quad , \quad f_2(rb_k) - f_2(qb_k) \geq \ln t \\ \Longleftarrow \quad & f_2(sb_k) \leq \ln\tau \quad , \quad f_2((r-1)b_k) \geq \ln t \\ \Longleftrightarrow \quad & \frac{f_2^{-1}(\ln t)}{r-1} \leq b_k \leq \frac{f_2^{-1}(\ln\tau)}{s} \end{aligned}$$

and hence
$$M_b\left(\frac{f_2^{-1}(\ln\tau)}{s}, \frac{f_2^{-1}(\ln t)}{r-1}\right) \leq |I|.$$

On the other hand for the set

$$\tilde{I} = \{i : d_{iq_1}^2 \leq 4c^2 d_{ip_1} d_{ir_1}, \frac{d_{is_1}}{d_{i\tilde{q_1}}} \leq 2c\tau, \frac{d_{ir_1}}{d_{iq_1}} \geq \frac{t}{2c}\}$$

if $i = 2k$, then the first inequality in the set gives

$$\frac{f_3(\frac{1}{p_1}\tilde{a_k})}{f_3(\frac{1}{q_1}\tilde{a_k})} + \frac{f_3(\frac{1}{r_1}\tilde{a_k})}{f_3(\frac{1}{q_1}\tilde{a_k})} \leq \ln 4c^2 + 2.$$

But $\lim \frac{f_3(\frac{1}{p_1}\tilde{a_k})}{f_3(\frac{1}{q_1}\tilde{a_k})} = \infty$. So we take $i = 2k$ and $d_{ip} = e^{f_4(p\tilde{b_k})}$. Then from the other two inequalities we get

$$\begin{aligned} & f_4(s_1\tilde{b_k}) - f_4(\tilde{q_1}\tilde{b_k}) \leq \ln 2c\tau \quad , \quad f_4(r_1\tilde{b_k}) - f_4(q_1\tilde{b_k}) \geq \ln\frac{t}{2c} \\ \Longrightarrow \quad & f_4((s_1-1)\tilde{b_k} \leq \ln 2c\tau \quad , \quad f_4(r_1\tilde{b_k}) \geq \ln\frac{t}{2c} \\ \Longleftrightarrow \quad & \frac{f_4^{-1}(\ln\frac{t}{2c})}{r_1} \leq \tilde{b_k} \leq \frac{f_4^{-1}(\ln 2c\tau)}{s_1-1} \end{aligned}$$

and hence
$$|\tilde{I}| \leq M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f_4^{-1}(\ln\frac{t}{2c})}{r_1}\right).$$

Since $|I| \leq \beta \leq \tilde{\beta} \leq |\tilde{I}|$ we consequently have

$$M_b\left(\frac{f_2^{-1}(\ln\tau)}{s}, \frac{f_2^{-1}(\ln t)}{r-1}\right) \leq M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f_4^{-1}(\ln\frac{t}{2c})}{r_1}\right).$$

Also, if we change the roles of U_i, V_j's in Remark we get the following symmetric inequalities ;

$$\begin{aligned} M_{\tilde{a}}(qf_3^{-1}(\ln t)\,,\,(q+1)f_3^{-1}(\ln \tau)) &\leq M_a((\tilde{q}_1+1)f_1^{-1}(\ln 2ct)\,,\,q_1 f_1^{-1}(\ln \frac{\tau}{2c})), \\ M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln \tau)}{s}\,,\,\frac{f_4^{-1}(\ln t)}{r-1}\right) &\leq M_b\left(\frac{f_2^{-1}(\ln 2c\tau)}{s_1-1}\,,\,\frac{f_2^{-1}(\ln \frac{t}{2c})}{r_1}\right). \end{aligned}$$

These with their symmetric inequalities we get

$$M_a \approx M_{f_1^{-1}f_3(\tilde{a})} \qquad \text{and} \qquad M_b \approx M_{f_2^{-1}f_4(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}f_3(\tilde{a}_{i+k_1})$ and $b_i \asymp f_2^{-1}f_4(\tilde{b}_{j+k_2})$ for some $k_1, k_2 \in \mathbb{Z}$ by 2.3. Then by theorem (2.1) we have

$$\begin{gathered} \text{if } L_{f_1}(a,0) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},\infty) \\ \text{then } L_{f_1}(a,0) \approx L_{f_3}(\tilde{a},0) \text{ and } L_{f_2}(b,\infty) \approx L_{f_2}(\tilde{b},\infty). \end{gathered}$$

Chapter 4

Cartesian Products of Various Types of Dragilev Spaces

In this chapter we shall consider all possible cases of the isomorphisms of cartesian products $L_f(\alpha, r) \times L_g(\beta, s)$. We begin with a definition, which was given in [1].

Definition A Köthe space $K(A)$ $A = (a_{ip})$ is said to have the property

$$\begin{array}{lll}
S_1^+(f), & \text{if} \quad \exists p\, \forall M\, M > 1\, \forall q\, \exists r & (!) \text{ holds,} \\
S_2^-(f), & \text{if} \quad \forall M\, M > 1\, \exists p\, \forall q\, \exists r & (!)^* \text{ holds,} \\
S_4^+(f), & \text{if} \quad \exists p\, \forall q\, \exists r\, \exists M\, M > 1 & (!) \text{ holds,} \\
S_4^-(f), & \text{if} \quad \exists p\, \forall q\, \exists r\, \forall M\, M > 1 & (!)^* \text{ holds,} \\
Q_1^+(f), & \text{if} \quad \forall p\, \exists q\, \forall r\, \exists M\, M > 1 & (!!) \text{ holds,} \\
Q_2^-(f), & \text{if} \quad \forall p\, \exists q\, \exists M\, M > 1\, \forall r & (!!)^* \text{ holds,} \\
Q_4^+(f), & \text{if} \quad \forall M\, M > 1\, \forall p\, \exists q\, \forall r & (!!) \text{ holds,} \\
Q_4^-(f), & \text{if} \quad \forall M\, M > 1\, \forall p\, \exists q\, \forall r & (!!)^* \text{ holds,}
\end{array}$$

where

$$\begin{array}{rl}
(!): & M\, f^{-1} \log(\frac{a_{iq}}{a_{ip}}) \leq f^{-1} \log(\frac{a_{ir}}{a_{iq}}) \quad \text{for every } i \in \mathbb{N}, \\
(!)^*: & f^{-1} \log(\frac{a_{iq}}{a_{ip}}) \leq M\, f^{-1} \log(\frac{a_{ir}}{a_{iq}}) \quad \text{for every } i \in \mathbb{N}, \\
(!!): & f^{-1} \log(\frac{a_{ir}}{a_{iq}}) \leq M\, f^{-1} \log(\frac{a_{iq}}{a_{ip}}) \quad \text{for every } i \in \mathbb{N}, \\
(!!)^*: & M\, f^{-1} \log(\frac{a_{ir}}{a_{iq}}) \leq f^{-1} \log(\frac{a_{iq}}{a_{ip}}) \quad \text{for every } i \in \mathbb{N}.
\end{array}$$

Note Subspaces of $L_f(\alpha, \infty)$, $L_f(\alpha, 1)$, $L_f(\alpha, 0)$, $L_f(\alpha, -1)$ respectively have $S_1^+(f)$, $S_4^+(f)$, $S_4^-(f)$, $S_2^-(f)$, and quotients of $L_f(\alpha, \infty)$, $L_f(\alpha, 1)$,

$L_f(\alpha,0)$, $L_f(\alpha,-1)$ respectively have $Q_1^+(f)$, $Q_4^+(f)$, $Q_4^-(f)$, $Q_2^-(f)$. Since we consider $L_f(\alpha,r)$ spaces in cartesian products, they are both subspaces and quotients (see [1]).

Lemma 4.1 *Following implications are always valid ;*

$S_1^+(f) \Rightarrow S_4^+(f)$, $S_2^-(f) \Rightarrow S_4^-(f)$ *and* $Q_4^-(f) \Rightarrow Q_2^-(f)$, $Q_4^+(f) \Rightarrow Q_1^+(f)$.

One of the linear topological invariants that we will consider in this chapter is the property $\mathcal{D}_\varphi$. This property was considered by Vogt [18] and Tidten [17] and called DN_φ by them (see also [6], [22],). Let φ be continuous, increasing function. A Fréchet space $(X, \|\cdot\|)$ is said to have property $\mathcal{D}_\varphi$ if

$$\exists p\, \forall q\, \exists r\, \exists C > 0 : \ \|x\|_q \leq \varphi(s)\|x\|_p + \frac{C}{s}\|x\|_r \ \ \forall s > 0,\ \forall x \in X.$$

Proposition 4.1 *Let $X = K(a_{i,p})$ be a Schwarz Köthe space. Then the following are equivalent.*

$$\begin{array}{ll} (i) & X \text{ has property } \mathcal{D}_\varphi \\ (ii) & \exists p\, \forall q\, \exists r\, \exists C > 0 : \ \dfrac{a_{i,q}}{a_{i,p}} \leq \varphi(\dfrac{a_{i,r}}{a_{i,q}}). \end{array}$$

Given a function φ as in the definition of property $\mathcal{D}_\varphi$ and $u > 0$, we define

$$\Phi(u) = \inf_{s>0}(\varphi(s) + \frac{u}{s})$$

Proposition 4.2 *Let $p < q < r, C > 0, K > 0$ and $2a_{iq} \leq a_{i,q+1}$ for some i. Then*

$$\begin{array}{ll} (i) & K\dfrac{a_{i,q}}{a_{i,p}} \leq \varphi(C\dfrac{a_{i,r}}{a_{i,q}}) \Rightarrow K\dfrac{a_{i,q}}{a_{i,p}} \leq \Phi(CK\dfrac{a_{i,r}}{a_{i,p}}) \\ (ii) & \dfrac{a_{i,q+1}}{a_{i,p}} \leq \Phi(C\dfrac{a_{i,r}}{a_{i,p}}) \Rightarrow \dfrac{a_{i,q}}{a_{i,p}} \leq \varphi(2C\dfrac{a_{i,r}}{a_{i,q}}). \end{array}$$

Proposition 4.3 *Let $p < r$ and*

$$V = \overline{\Gamma}(\bigcup_{k>0}(\frac{1}{\varphi(k)}U_p \cap kU_r)) \ \ , \ \ V' = B^e(a_{i,p}\Phi(\frac{a_{i,r}}{a_{i,p}}))$$

where $\overline{\Gamma}$ denotes the absolutely convex hull, and $\Phi(u) = \inf_{s>0}(\varphi(s) + (\frac{u}{s}))$ as defined before. Then $V' \subset V \subset 3V'$.

The proofs of above propositions can be found in [8].

Note For the spaces $L_f(\alpha,\infty), L_f(\alpha,1)$ if we take

$$\varphi_f = \exp f\frac{1}{M}f^{-1}\log,$$

and for the spaces $L_f(\alpha,0), L_f(\alpha,-1)$

$$\varphi_f = \exp fMf^{-1}\log,$$

we see that with the same quantifiers as in the definitions of $S_1^+(f),\ldots,S_4^-(f)$, the inequality $\frac{a_{i,q}}{a_{i,p}} \leq \varphi_f(\frac{a_{i,r}}{a_{i,q}})$ holds.

Now, we are going to start to analyse the cases of isomorphisms mentioned above. We shall use a similar method as in Chapter 1, but in this chapter we shall consider following neighborhoods

$$\begin{aligned}
W_1 &= \tau U_s,\\
W_2 &= U_q,\\
W_3 &= \overline{\Gamma}(\bigcup_{k>0}(\frac{1}{\varphi_f(k)}U_p \cap kU_r)) \cap tU_r\\
\tilde{W}_1 &= \tau T(V_{s_1}),\\
\tilde{W}_2 &= T(V_{q_1}),\\
\tilde{W}_3 &= \overline{\Gamma}(\bigcup_{k>0}(\frac{1}{\varphi_f(k)}T(V_{p_1}) \cap T(V_{r_1}))) \cap tT(V_{r_1})\\
\tilde{W}_4 &= T(V_{\tilde{q}_1})
\end{aligned} \tag{4.1}$$

instead of (1.2), and

$$\begin{aligned}
W_1 &= \overline{\Gamma}(\overline{\Gamma}(\bigcup_{k>0}\frac{1}{\varphi_f(k)}T^{-1}(U_p) \cap kT^{-1}(U_r)) \cup tT^{-1}(U_r)),\\
W_2 &= T^{-1}(U_q),\\
W_3 &= \tau T^{-1}(U_s),\\
\tilde{W}_1 &= \overline{\Gamma}(\overline{\Gamma}(\bigcup_{k>0}\frac{1}{\varphi_f(k)}V_{p_1} \cap kV_{r_1}) \cup tV_{r_1}),\\
\tilde{W}_2 &= V_{q_1},\\
\tilde{W}_3 &= \tau V_{s_1},\\
\tilde{W}_4 &= V_{\tilde{q}_1}
\end{aligned} \tag{4.2}$$

instead of (1.4). Then we will again have $\beta_1 \leq \beta_2$ and $\beta \leq \tilde{\beta}$.

We shall consider all possible relations of the considered Dragilev functions. But we will eliminate many of them by using the following facts (see [20]) ;

Definition We say that a pair of lcs's (X, Y) satisfies the condition $\mathcal{R}$ $((\mathcal{X}, \mathcal{Y}) \in \mathcal{R})$ if every linear continuous operator T from X to Y is compact.

Proposition 4.4 *Let $X = X_1 \times X_2$, $Y = Y_1 \times Y_2$ be lcs's, and $(X_1, Y_2) \in \mathcal{R}$, $(Y_1, X_2) \in \mathcal{R}$. Then $X \approx Y$ if and only if $X_1 \approx Y_1$ and $X_2 \approx Y_2$.*

Theorem 4.1 *Let f_1, f_2 and $\phi = f_1^{-1} \circ f_2$ be rapidly increasing. Then*

$$(L_{f_1}(a, r)\,,\ L_{f_2}(b, s)) \in \mathcal{R}$$

if $0 < r \leq \infty$, $0 < s \leq \infty$ and

$$(L_{f_2}(b, s)\,,\ L_{f_1}(a, r)) \in \mathcal{R}$$

if $-\infty < r \leq 0$, $-\infty < s \leq \infty$ independently of the choice of the sequences $a = (a_k)$, $b = (b_k)$.

4.1 $L_{f_1}(a, \infty) \times L_{f_2}(b, \infty) \simeq L_{f_3}(\tilde{a}, \infty) \times L_{f_4}(\tilde{b}, \infty)$

We show all possible relations between f_1, f_2, f_3, f_4 in the following table ;

	A1,A3	**(*)**	**B2,B3**	**B3**
	$f_1 \prec f_2 \prec f_3 \prec f_4$	$f_1 \prec f_2 \approx f_3 \prec f_4$	$f_3 \prec f_1 \prec f_2 \prec f_4$	$f_3 \approx f_1 \prec f_2 \prec f_4$
	B2		**D2,D4**	**(*)**
	$f_3 \prec f_1 \prec f_2 \approx f_4$	$\mathbf{f_3 \approx f_1 \prec f_2 \approx f_4}$	$f_3 \prec f_4 \prec f_1 \prec f_2$	$f_3 \prec f_4 \approx f_1 \prec f_2$
	B1,B4	**B4**	**B1**	
$f_1 \prec f_2$; $f_3 \prec f_4$	$f_1 \prec f_3 \prec f_4 \prec f_2$	$f_1 \approx f_3 \prec f_4 \prec f_2$	$f_1 \prec f_3 \prec f_4 \approx f_2$	$f_1 \approx f_3 \prec f_4 \approx f_2$
	B1,B3	**B3**	**(*)**	**B2,B4**
	$f_1 \prec f_3 \prec f_2 \prec f_4$	$f_1 \approx f_3 \prec f_2 \prec f_4$	$f_1 \prec f_3 \approx f_2 \prec f_4$	$f_3 \prec f_1 \prec f_4 \prec f_2$
	B4	**(*)**	**B2**	**B1**
	$f_3 \approx f_1 \prec f_4 \prec f_2$	$f_3 \prec f_1 \approx f_4 \prec f_2$	$f_3 \prec f_1 \prec f_4 \approx f_2$	$f_1 \prec f_3 \prec f_2 \approx f_4$

$f_1 \prec f_2$; $f_4 \prec f_3$	**A1,A3** $f_1 \prec f_2 \prec f_4 \prec f_3$	**A1** $f_1 \prec f_2 \approx f_4 \prec f_3$	**C1,C4** $f_4 \prec f_1 \prec f_2 \prec f_3$	**(*)** $f_4 \approx f_1 \prec f_2 \prec f_3$
	(*) $f_4 \prec f_1 \prec f_2 \approx f_3$	 $\mathbf{f_4 \approx f_1 \prec f_2 \approx f_3}$	**D2,D4** $f_4 \prec f_3 \prec f_1 \prec f_2$	**D4** $f_4 \prec f_3 \approx f_1 \prec f_2$
	B1,B4 $f_1 \prec f_4 \prec f_3 \prec f_2$	**(*)** $f_1 \approx f_4 \prec f_3 \prec f_2$	**(*)** $f_1 \prec f_4 \prec f_3 \approx f_2$	 $f_1 \approx f_4 \prec f_3 \approx f_2$
	D1,D4 $f_4 \prec f_1 \prec f_3 \prec f_2$	**(*)** $f_4 \approx f_1 \prec f_3 \prec f_2$	**D4** $f_4 \prec f_1 \approx f_3 \prec f_2$	**(*)** $f_4 \prec f_1 \prec f_3 \approx f_2$
	A1,A4 $f_1 \prec f_4 \prec f_2 \prec f_3$	**A1** $f_1 \prec f_4 \approx f_2 \prec f_3$	**(*)** $f_1 \prec f_4 \prec f_2 \approx f_3$	**(*)** $f_1 \approx f_4 \prec f_2 \prec f_3$
$f_1 \prec f_2$; $f_3 \approx f_4$	**A1,A3** $f_1 \prec f_2 \prec f_3 \approx f_4$	**(*)** $f_1 \prec f_2 \approx f_3 \approx f_4$	**D2,D4** $f_3 \approx f_4 \prec f_1 \prec f_2$	**(*)** $f_3 \approx f_4 \approx f_1 \prec f_2$
	B1,B4 $f_1 \prec f_3 \approx f_4 \prec f_2$			
$f_2 \prec f_1$; $f_3 \prec f_4$	**A1,A3** $f_2 \prec f_1 \prec f_3 \prec f_4$	**A3** $f_2 \prec f_1 \approx f_3 \prec f_4$	**B2,B4** $f_3 \prec f_2 \prec f_1 \prec f_4$	**(*)** $f_3 \approx f_2 \prec f_1 \prec f_4$
	 $f_3 \prec f_2 \prec f_1 \approx f_4$	 $\mathbf{f_3 \approx f_2 \prec f_1 \approx f_4}$	**D2,D4** $f_3 \prec f_4 \prec f_2 \prec f_1$	**D2** $f_3 \prec f_4 \approx f_2 \prec f_1$
	C2,C4 $f_2 \prec f_3 \prec f_4 \prec f_1$	**(*)** $f_2 \approx f_3 \prec f_4 \prec f_1$	**(*)** $f_2 \prec f_3 \prec f_4 \approx f_1$	 $f_2 \approx f_3 \prec f_4 \approx f_1$
	D2,D3 $f_3 \prec f_2 \prec f_4 \prec f_1$	**(*)** $f_3 \approx f_2 \prec f_4 \prec f_1$	**D2** $f_3 \prec f_2 \approx f_4 \prec f_1$	**(*)** $f_3 \prec f_2 \prec f_4 \approx f_1$
	A2,A3 $f_2 \prec f_3 \prec f_1 \prec f_4$	**A3** $f_2 \prec f_3 \approx f_1 \prec f_4$	**(*)** $f_2 \prec f_3 \prec f_1 \approx f_4$	**(*)** $f_2 \approx f_3 \prec f_1 \prec f_4$
$f_2 \prec f_1$; $f_4 \prec f_3$	**A1,A3** $f_2 \prec f_1 \prec f_4 \prec f_3$	**(*)** $f_2 \prec f_1 \approx f_4 \prec f_3$	**C1,C4** $f_4 \prec f_2 \prec f_1 \prec f_3$	**C1** $f_4 \approx f_2 \prec f_1 \prec f_3$
	C4 $f_4 \prec f_2 \prec f_1 \approx f_3$	 $\mathbf{f_4 \approx f_2 \prec f_1 \approx f_3}$	**D2,D4** $f_3 \prec f_4 \prec f_2 \prec f_1$	**D2** $f_3 \prec f_4 \approx f_2 \prec f_1$
	C2,C3 $f_2 \prec f_4 \prec f_3 \prec f_1$	**C2** $f_2 \approx f_4 \prec f_3 \prec f_1$	**C3** $f_2 \prec f_4 \prec f_1 \approx f_3$	 $f_2 \approx f_4 \prec f_3 \approx f_1$
	C2,C4 $f_4 \prec f_2 \prec f_3 \prec f_1$	**C2** $f_4 \approx f_2 \prec f_3 \prec f_1$	**(*)** $f_4 \prec f_2 \approx f_3 \prec f_1$	**C4** $f_4 \prec f_2 \prec f_3 \approx f_1$
	C1,C3 $f_2 \prec f_4 \prec f_1 \prec f_3$	**(*)** $f_2 \prec f_4 \approx f_1 \prec f_3$	**C3** $f_2 \prec f_4 \prec f_1 \approx f_3$	**C1** $f_2 \approx f_4 \prec f_1 \prec f_3$

$f_2 \prec f_1$; $f_3 \approx f_4$	**A1,A3** $f_2 \prec f_1 \prec f_3 \approx f_4$	(*) $f_2 \prec f_1 \approx f_3 \approx f_4$	**D2,D4** $f_3 \approx f_4 \prec f_2 \prec f1$	(*) $f_3 \approx f_4 \approx f_2 \prec f_1$
	C2,C3 $f_2 \prec f_3 \approx f_4 \prec f_1$			
$f_1 \approx f_2$; $f_3 \prec f_4$	**A1,A3** $f_1 \approx f_2 \prec f_3 \prec f_4$	(*) $f_1 \approx f_2 \approx f_3 \prec f_4$	**B2,B3** $f_3 \prec f_1 \approx f_2 \prec f_4$	**D2,D4** $f_3 \prec f_4 \prec f_1 \approx f_2$
	(*) $f_3 \prec f_4 \approx f_1 \approx f_2$			
$f_1 \approx f_2$; $f_4 \prec f_3$	**A1,A3** $f_1 \approx f_2 \prec f_4 \prec f_3$	(*) $f_1 \approx f_2 \approx f_4 \prec f_3$	(*) $f_4 \prec f_1 \approx f_2 \prec f_3$	**D2,D4** $f_4 \prec f_3 \prec f_1 \approx f_2$
	(*) $f_4 \prec f_3 \approx f_1 \approx f_2$			
$f_1 \approx f_2$; $f_3 \approx f_4$	**A1,A3** $f_1 \approx f_2 \prec f_3 \approx f_4$	$\mathbf{f_1 \approx f_2 \approx f_3 \approx f_4}$	**D2,D4** $f_3 \approx f_4 \prec f_1 \approx f_2$	

This table is valid for the next three isomorphism.

We eliminate many of the posibilities above. The remaining uneliminated cases are written in bold face. Those are the cases which will be discussed. We use some notations to show how we eliminated them. We wrote, for instance **A1,A3**, over some cases. Because by theorem 3.1, the assumption of above isomorphism fails whenever one of

$$\begin{array}{lll} \mathbf{A} & : & f_1 \prec f_4 \,;\, f_2 \prec f_3 \\ \mathbf{B} & : & f_1 \prec f_4 \,;\, f_3 \prec f_2 \\ \mathbf{C} & : & f_4 \prec f_1 \,;\, f_2 \prec f_3 \\ \mathbf{D} & : & f_4 \prec f_1 \,;\, f_3 \prec f_2 \end{array} \quad \text{holds and one of} \quad \begin{array}{lll} \mathbf{1} & : & f_1 \prec f_3 \\ \mathbf{2} & : & f_3 \prec f_1 \\ \mathbf{3} & : & f_2 \prec f_4 \\ \mathbf{4} & : & f_4 \prec f_2 \end{array} \quad \text{holds.}$$

Also we wrote (*), for instance, over $f_1 \prec f_2 \approx f_3 \prec f_4$. Because, for this case if we have the isomorphism $T : L_{f_1}(a,\infty) \times L_{f_2}(b,\infty) \to L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$, then $S = T|_{L_{f_1}(a,\infty)} :\to L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$ is also an isomorphism onto a closed subspace. On the other hand, for the projections $P_3 : L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty) \to L_{f_3}(\tilde{a},\infty)$ and $P_4 : L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$, $P_3 \circ S$ and $P_4 \circ S$ are compact since $f_1 \prec f_2 \approx f_3$. So $P_3 \circ S + P_4 \circ S$ should be compact. Then $(P_3 + P_4) \circ S$ is compact. But this means that S is compact since $P_3 + P_4$ is identity. This contradics the assumption T is an isomorphism.

Proposition 4.5 *Let $f_1 \prec f_2$. Then*

(i) $L_{f_1}(a,\infty)$ does not have $S_1^+(f_2)$,

(ii) $L_{f_2}(\tilde{a},\infty)$ does not have $Q_1^+(f_1)$.

Proof

(i) Assume that $L_{f_1}(a,\infty)$ has $S_1^+(f_2)$. Then $\exists p\, \forall M\ M > 1\ \forall q\, \exists r$ such that

$$M\, f_2^{-1} \log(e^{f_1(qa_k)} - f_1(pa_k)) \leq f_2^{-1} \log(e^{f_1(ra_k)} - f_1(qa_k))$$

$$\Longrightarrow \quad M\, f_2^{-1}\, f_1((q-1)a_k) \leq f_2^{-1}\, f_1(ra_k)$$

$$\Longrightarrow \quad M \leq \frac{f_2^{-1}\, f_1(ra_k)}{f_2^{-1}\, f_1((q-1)a_k)}.$$

But since $f_1 \prec f_2$, i.e. $f_1^{-1}\, f_2$ is rapidly increasing the right hand side tends to 1. This contradicts the assumption that $M > 1$. Thus $L_{f_1}(a,\infty)$ does not have $S_1^+(f_2)$.

(ii) Similarly as in proof of (i), if we assume the contrary of (ii), we obtain $\forall p\, \exists q\, \forall r\, \exists M\ M > 1$ such that

$$\frac{f_1^{-1}\, f_2((r-1)\tilde{a}_k)}{f_1^{-1}\, f_2(q\tilde{a}_k)} \leq M.$$

But this is not possible, because the left hand side tends to infinity.

For $K(C) = L_{f_1}(a,\infty) \times L_{f_2}(b,\infty)$, $K(D) = L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$,

$$c_{ip} = \begin{cases} e^{f_1(pa_k)} & \text{if } i = 2k-1 \\ e^{f_2(pb_k)} & \text{if } i = 2k \end{cases}, \quad d_{ip} = \begin{cases} e^{f_3(p\tilde{a}_k)} & \text{if } i = 2k-1 \\ e^{f_4(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

4.1.1 $f_4 \approx f_1 \prec f_2 \approx f_3$

In this case $f_4^{-1}\, f_1$, $f_1^{-1}\, f_4$, $f_3^{-1}\, f_2$, $f_2^{-1} f_3$ are all logarithmically convex and slowly increasing, and $f_1^{-1}\, f_2$, $f_1^{-1}\, f_3$, $f_4^{-1}\, f_2$, $f_4^{-1}\, f_3$ are rapidly increasing.

Then

$$L_{f_1}(a,\infty) \text{ part does not have } S_1^+(f_2),\ S_1^+(f_3),$$

$$L_{f_2}(b,\infty) \text{ part does not have } Q_1^+(f_1),\ Q_1^+(f_4),$$
$$L_{f_3}(\tilde{a},\infty) \text{ part does not have } Q_1^+(f_1),\ Q_1^+(f_4),$$
$$L_{f_4}(\tilde{b},\infty) \text{ part does not have } S_1^+(f_2),\ S_1^+(f_3).$$

Assume $K(D) = L_{f_3}(\tilde{a},\infty)\times L_{f_4}(\tilde{b},\infty)$ and $K(C) = L_{f_1}(a,\infty)\times L_{f_2}(b,\infty)$ areisomorphic where $C = (c_{ip}), D = (d_{ip})$,

$$c_{ip} = \begin{cases} e^{f_1(pa_k)} & \text{if } i = 2k-1 \\ e^{f_2(pb_k)} & \text{if } i = 2k \end{cases} \quad , \quad d_{ip} = \begin{cases} e^{f_3(p\tilde{a}_k)} & \text{if } i = 2k-1 \\ e^{f_4(p\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

Consider $\beta_1 = \beta(W_2 \cap W_3, \overline{\Gamma}(W_1 \cap W_2))$. Then

$$\begin{aligned}
\beta_1 \ &\geq \beta\left(B^e(c_{is}) \cap \overline{\Gamma}(B^e(c_{i,p}\Phi_{f_1}(\frac{c_{i,r}}{c_{i,p}})) \cap tB^e(c_{i,r}))\ ,\ \overline{\Gamma}(\tau B^e(c_{i,s}) \cup B^e(c_{i,q}))\right) \\
&\geq |\{i : \frac{\max\{c_{i,p}\Phi_{f_1}(\frac{c_{i,r}}{c_{i,p}}), \frac{1}{t}c_{i,r}\}}{c_{i,q}} \leq 1\ ,\ \frac{c_{i,q}}{\frac{1}{\tau}c_{i,s}} \leq 1\}| \\
&= |\{i : \Phi_{f_1}(\frac{c_{i,r}}{c_{i,p}}) \leq \frac{c_{i,q}}{c_{i,p}}\ ,\ \frac{c_{i,r}}{c_{i,q}} \leq t\ ,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| \\
&\geq |\{i : \underbrace{\varphi_{f_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_1^+(f_1)}\ ,\ \frac{c_{i,r}}{c_{i,q}} \leq t\ ,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| \ := |I_1|.
\end{aligned}$$

where $\varphi_{f_1} = \exp\ f_1\ \frac{1}{M}\ f_1^{-1}\ \log$.

If $i = 2k$ in I_1, then $c_{i,p} = e^{f_2(pb_k)}$, but $L_{f_2}(b,\infty)$ does not have $Q_1^+(f_1)$. So we take $i = 2k-1$ and $c_{i,p} = e^{f_1(pa_k)}$. Then the last two inequalities give

$$\begin{aligned}
& f_1(ra_k) - f_1(qa_k) \leq \ln t \quad , \quad f_1(sa_k) - f_1(qa_k) \geq \ln\tau \\
\Longleftarrow \quad & f_1(ra_k) \leq \ln t \quad , \quad f_1((s-1)a_k) \geq \ln\tau \\
\Longleftrightarrow \quad & \frac{f_1^{-1}(\ln\tau)}{s-1} \leq a_k \leq \frac{f_1^{-1}(\ln t)}{r}
\end{aligned}$$

and hence
$$M_a\left(\frac{f_1^{-1}(\ln t)}{r}, \frac{f_1^{-1}(\ln\tau)}{s-1}\right) \leq |I_1| \leq \beta_1.$$

On the other hand for $\beta_2 = \beta\left(c(\tilde{W}_2 \cap \tilde{W}_3)\ ,\ \overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4)\right)$,

$$\begin{aligned}
\beta_2 \leq \beta(c(B^e(d_{i,q_1}) \cap \overline{\Gamma}(3d_{i,p_1}\Phi_{f_1}(\frac{d_{i,r_1}}{d_{i,p_1}})) \cap tB^e(d_{i,r_1}))\ , \\
\overline{\Gamma}(\tau B^e(d_{i,s_1}) \cup B^e(d_{i,\tilde{q}_1})))
\end{aligned}$$

$$
\begin{aligned}
&\le |\{i : \frac{\frac{1}{2c}\max\{\frac{1}{3}d_{i,p_1}\Phi_{f_1}(\frac{d_{i,r_1}}{d_{i,p_1}}),\ \frac{1}{t}d_{i,r_1}\}}{d_{i,\tilde{q}_1}} \le 1,\ \frac{\frac{1}{2c}d_{i,q_1}}{\frac{1}{\tau}d_{i,s_1}} \le 1\}| \\
&= |\{i : \varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,p_1}}) \le 6c\frac{d_{i,\tilde{q}_1}}{d_{i,p_1}},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \le 2ct, \frac{d_{i,s_1}}{d_{i,q_1}} \ge \frac{\tau}{2c}\}| \\
&\le |\{i : \underbrace{\varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \le \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_1^+(f_1)},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \le 2ct, \frac{d_{i,s_1}}{d_{i,q_1}} \ge \frac{\tau}{2c}\}| := |I_2|
\end{aligned}
$$

where $\varphi_{f_1} = \exp : f_1 \frac{1}{M} f_1^{-1} \log$. Since $L_{f_3}(\tilde{a}, \infty)$ does not have $Q_1^+(f_1)$ we take $i = 2k$ in I_2. Then from the last two inequalities we see that

$$
\beta_2 \le |I_2| \le M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2ct)}{r_1 - 1}, \frac{f_4^{-1}(\ln \frac{\tau}{2c})}{s_1}\right).
$$

Therefore we have

$$
M_a\left(\frac{f_1^{-1}(\ln t)}{r}, \frac{f_1^{-1}(\ln \tau)}{s-1}\right) \le M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2ct)}{r_1 - 1}, \frac{f_4^{-1}(\ln \frac{\tau}{2c})}{s_1}\right)
$$

since $\beta_1 \le \beta_2$.

Consider now the sets 4.2 and replace f by f_2. Then for $\beta = \beta\left(W_2 \cap W_3, \overline{\Gamma}(W_1 \cup W_2)\right)$,

$$
\begin{aligned}
\beta \ &\ge \beta\left(B^e(c_{i,q}) \cap \tau B^e(c_{i,s}), \overline{\Gamma}(3B^e(c_{i,p}\Phi_{f_2}(\frac{c_{i,r}}{c_{i,p}})) \cup tB^e(c_{i,r}) \cup B^e(c_{i,q}))\right) \\
&\ge |\{i : \frac{\frac{1}{\tau}c_{i,s}}{c_{i,q}} \le 1, \frac{c_{i,q}}{\min\{\frac{1}{3}c_{i,p}\Phi_{f_2}(\frac{c_{i,r}}{c_{i,p}}), \frac{1}{t}c_{i,r}\}} \le 1\}| \\
&= |\{i : \frac{c_{i,s}}{c_{i,q}} \le \tau, \frac{c_{i,q}}{c_{i,p}\Phi(\frac{c_{i,r}}{c_{i,p}})} \le \frac{1}{3}, \frac{c_{i,q}}{c_{i,r}} \le \frac{1}{t}\}| \\
&= |\{i : \frac{c_{i,q}}{c_{i,p}} \le \frac{1}{3}\Phi_{f_2}'(\frac{c_{i,r}}{c_{i,p}}), \frac{c_{i,s}}{c_{i,q}} \le \tau, \frac{c_{i,r}}{c_{i,q}} \ge t\}| \\
&\ge |\{i : \frac{c_{i,q+1}}{c_{i,p}} \le \Phi_{f_2}(\frac{c_{i,r}}{c_{i,p}}), \frac{c_{i,s}}{c_{i,q}} \le \tau, \frac{c_{i,r}}{c_{i,q}} \ge t\}| \\
&\ge |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \le \varphi_{f_2}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_1^+(f_2)}, \frac{c_{i,s}}{c_{i,q}} \le \tau, \frac{c_{i,r}}{c_{i,q}} \ge t\}| := |I|.
\end{aligned}
$$

where $\varphi_{f_2} = \exp f_2 \frac{1}{M} f_2^{-1} \log$ Since $L_{f_1}(a, \infty)$ does not have $S_1^+(f_2)$ we take $i = 2k$. Then from the last two inequalities we get

$$
M_b\left(\frac{f_2^{-1}(\ln \tau)}{s}, \frac{f_2^{-1}(\ln t)}{r-1}\right) \le |I| \le \beta.
$$

On the other hand, since $\tilde{\beta} = \beta(c(\tilde{W}_2 \cap \tilde{W}_3), \overline{\Gamma}(\tilde{W}_1 \cup \tilde{W}_4)) \leq \beta'$ where

$$\beta' = \beta(c(B^e(d_{i,q_1}) \cap \tau B^e(d_{i,s_1}))\,,\ \overline{\Gamma}((\overline{\Gamma}(B^e(d_{i,p_1}\Phi_{f_2}(\frac{d_{i,r_1}}{d_{i,p_1}}))\cup \\ tB^e(d_{i,r_1}))) \cup B^e(d_{i,\tilde{q}_1}))),$$

and

$$\begin{aligned}
\beta' &\leq |\{i : \tfrac{\frac{1}{\tau}d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\,,\ \tfrac{d_{i,q_1}}{\min\{d_{i,p_1}\Phi_{f_2}(\frac{d_{i,r_1}}{d_{i,p_1}})\,,\,\frac{1}{t}d_{i,r_1}\}} \leq 2c\}| \\
&= |\{i : \frac{d_{is_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau\,,\ \frac{d_{i,q_1}}{d_{i,p_1}} \leq 2c\Phi_{f_2}(\frac{d_{i,r_1}}{d_{i,p_1}})\,,\ \frac{d_{i,q_1}}{d_{i,r_1}} \leq \frac{2c}{t}\}| \\
&= |\{i : \frac{d_{i,q_1}}{d_{i,p_1}} \leq 2c\Phi_{f_2}(\frac{d_{i,r_1}}{d_{i,p_1}})\,,\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau\,,\ \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| \\
&\leq |\{i : \frac{d_{i,q-1}}{d_{i,p_1}} \leq \Phi_{f_2}(\frac{d_{i,r_1}}{d_{i,p_1}})\,,\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau\,,\ \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| \\
&\leq |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \leq \varphi_{f_2}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_1^+(f_2)}\,,\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau\,,\ \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| := |\tilde{I}|
\end{aligned}$$

where $\varphi_{f_2} = \exp f_2 \frac{1}{M} f_2^{-1} \log$. Since $L_{f_4}(\tilde{b}, \infty)$ does not have $S_1^+(f_2)$ we take $i = 2k - 1$. Then by the last two inequalities we see that

$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2c\tau)}{s_1 - 1}, \frac{f_3^{-1}(\ln \frac{t}{2c})}{r_1}\right).$$

Therefore we have

$$M_b\left(\frac{f_2^{-1}(\ln \tau)}{s}, \frac{f_2^{-1}(\ln t)}{r-1}\right) \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2c\tau)}{s_1 - 1}, \frac{f_3^{-1}(\ln \frac{t}{2c})}{r_1}\right).$$

Let now $T = \dfrac{f_2^{-1}(\ln \tau)}{s}$. Then

$$\begin{aligned}
\frac{f_3^{-1}(\ln 2c\tau)}{s_1 - 1} = \frac{f_3^{-1}(\ln 2c + f_2(Ts))}{s_1 - 1} &\leq \frac{f_3^{-1}f_2(T(s+1))}{s_1 - 1} \\
&= \frac{f_3^{-1}f_2(T)}{s_1 - 1}\frac{f_3^{-1}f_2(T(s+1))}{f_3^{-1}f_2(T)} \leq Af_3^{-1}f_2(T)
\end{aligned}$$

for $A > \lim\limits_{T\to\infty} \dfrac{f_3^{-1}f_2(T(s+1))}{(s_1 - 1)f_3^{-1}f_2(T)}$. Note that since $f_3^{-1}f_2$ is slowly increasing we may find such a constant A.

Similarly for $L = \dfrac{f_2^{-1}(\ln t)}{r-1}$ we get

$$\frac{f_3^{-1}(\ln \frac{t}{2c})}{r_1} \geq \frac{1}{B}f_3^{-1}f_2(L)$$

for $B > \lim_{L\to\infty} \dfrac{r_1 f_3^{-1} f_2(L)}{f_3^{-1} f_2((r-2)L)}$. So for

$$K > \max\{ A, B \}$$

we have

$$M_b(T\,,\,L) \le M_{\tilde{a}}\left(K f_3^{-1} f_2(T)\,,\, \frac{f_3^{-1} f_2(L)}{K} \right).$$

Choose now M so large that $\tau_{f_3^{-1} f_2}(M) > K$. Then

$$M_b(T\,,\,L) \le M_{f_2^{-1} f_3(\tilde{a})}(MT\,,\,\frac{L}{M})$$

and similarly

$$M_a(T\,,\,L) \le M_{f_1^{-1} f_4(\tilde{b})}(MT\,,\,\frac{L}{M}).$$

Also, by the symmetry we have

$$M_{f_2^{-1} f_3(\tilde{a})}(T\,,\,L) \le M_b(MT\,,\,\frac{L}{M}),$$
$$M_{f_1^{-1} f_4(\tilde{b})}(T\,,\,L) \le M_a(MT\,,\,\frac{L}{M}).$$

Similarly as before, from these last four inequalities we obtain

$$b_i \asymp f_2^{-1} f_3(\tilde{a}_{i+k_1}) \qquad , \qquad a_j \asymp f_1^{-1} f_4(\tilde{b}_{j+k_2}).$$

Then by theorem (2.1) we have

if $L_{f_1}(a,\infty) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$, and $f_4 \approx f_1 \prec f_2 \approx f_3$
then $L_{f_1}(a,\infty) \approx L_{f_4}(\tilde{b},\infty)$, $L_{f_2}(b,\infty) \approx L_{f_3}(\tilde{a},\infty)$.

4.1.2 $f_1 \approx f_2 \approx f_3 \approx f_4$

Remark Let f, g be two rapidly increasing Dragilev functions such that $h = f^{-1}\, g$ and $h^{-1} = g^{-1}\, f$ are both logaritmically convex and slowly increasing. Then there are constants α, $\beta > 0$ such that

$$g(x) = f(x^\alpha e^\beta) \quad \text{and} \quad f(x) = g(x^{1/\alpha} e^{-\beta/\alpha}).$$

This case is quite different from the others. We may take in this case all the functions the same. We may say

$L_{f_1}(a,\infty) \times L_{f_2}(b,\infty) \simeq L_h(c,\infty)$ and $L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty) \simeq L_{\tilde{h}}(\tilde{c},\infty)$,

where $f_2(x) = f_1(x^\alpha e^\beta)$, $f_4(x) = f_3(x^{\tilde{\alpha}} e^{\tilde{\beta}})$ $h = f_1$, $\tilde{h} = f_3$, and

$$c = (c_{ip}) = \begin{cases} e^{f_1(pa_k)}, & i = 2k-1 \\ e^{f_1(p^\alpha b_k^\alpha e^\beta)}, & i = 2k \end{cases} = \begin{cases} e^{h(pa_k)}, & i = 2k-1 \\ e^{h(p^\alpha b_k^\alpha e^\beta)}, & i = 2k \end{cases}$$

and

$$\tilde{c} = (\tilde{c}_{ip}) = \begin{cases} e^{f_3(p\tilde{a}_k)}, & i = 2k-1 \\ e^{f_3(p^{\tilde{\alpha}} \tilde{b}_k^{\tilde{\alpha}} e^{\tilde{\beta}})}, & i = 2k \end{cases} = \begin{cases} e^{\tilde{h}(p\tilde{a}_k)}, & i = 2k-1 \\ e^{\tilde{h}(p^{\tilde{\alpha}} \tilde{b}_k^{\tilde{\alpha}} e^{\tilde{\beta}})}, & i = 2k \end{cases}.$$

We could take h and $\tilde{h}$ to be f_2 and f_4 respectively too. Moreover we could take h and $\tilde{h}$ to be f_1 and f_4 or f_2 and f_3 respectively. Hence we may get each of the following;

$$L_{f_1}(a,\infty) \approx L_{f_3}(\tilde{a},\infty) \text{ and } L_{f_2}(b,\infty) \approx L_{f_4}(\tilde{b},\infty),$$
$$L_{f_1}(a,\infty) \approx L_{f_4}(\tilde{b},\infty) \text{ and } L_{f_2}(b,\infty) \approx L_{f_3}(\tilde{a},\infty).$$

by the same method as before.

We will see the same case in the rest of this thesis. But the idea here is valid there too. So, we will not consider this case any more in the rest.

4.1.3 Other Cases

Under the assumption $L_{f_1}(a,\infty) \times L_{f_2}(b,\infty) \simeq L_{f_3}(\tilde{a},\infty) \times L_{f_4}(\tilde{b},\infty)$, for the cases

$$f_3 \approx f_1 \prec f_2 \approx f_4, \quad f_3 \approx f_2 \prec f_1 \approx f_4 \text{ and } f_4 \approx f_2 \prec f_1 \approx f_3$$

we respectively obtain

$$a_i \asymp f_1^{-1}\, f_3(\tilde{a}_{i+k_1}) \text{ and } b_i \asymp f_2^{-1}\, f_4(\tilde{b}_{i+k_2}),$$
$$b_i \asymp f_2^{-1}\, f_3(\tilde{a}_{i+l_1}) \text{ and } a_i \asymp f_1^{-1}\, f_4(\tilde{b}_{i+l_2}),$$
$$b_i \asymp f_2^{-1}\, f_4(\tilde{b}_{i+t_1}) \text{ and } a_i \asymp f_1^{-1}\, f_3(\tilde{a}_{i+t_2})$$

by using the same method as before, where k_1, k_2, l_1, l_2, t_1, t_2 are some integers.

So, for the rest, it will be enough to consider only one case for each considered isomorphism.

4.2 $L_{f_1}(a,0) \times L_{f_2}(b,0) \simeq L_{f_3}(\tilde{a},0) \times L_{f_4}(\tilde{b},0)$

Proposition 4.6 *Let $g_1 \prec g_2$. Then*
(i) $L_{g_1}(a,0)$ does not have $Q_4^-(g_2)$,
(ii) $L_{g_2}(\tilde{a},0)$ does not have $S_4^-(g_1)$.

Proof
(i) Assume that $L_{g_1}(a,0)$ has $Q_4^-(g_2)$. Then$\forall M$, $M > 1$, $\forall p\, \exists q\, \forall r$

$$M\, g_2^{-1}\, \log(e^{g_1(\frac{1}{q}a_k)-g_1(\frac{1}{r}a_k)}) \leq g_2^{-1}\, \log(e^{g_1(\frac{1}{p}a_k)-g_1(\frac{1}{r}a_k)})$$

$$\Longrightarrow \quad M\, g_2^{-1}\, g_1(\tfrac{1}{q+1}a_k) \leq g_2^{-1}\, g_1(\tfrac{1}{p}a_k)$$

$$\Longrightarrow \quad M \leq \frac{g_2^{-1}\, g_1(\frac{1}{p}a_k)}{g_2^{-1}\, g_1(\frac{1}{q+1}a_k)}.$$

But since $g_1 \prec g_2$, right hand side tends to 1. This contradicts the assumption $M > 1$. Thus, $L_{g_1}(a,0)$ does not have $Q_4^-(g_2)$.
(ii) The proof of (ii) is similar to (i).

The table in section 3.1 is valid here too.

The remaining cases after our eliminations are given below. In order to deal with these cases we will use

$$c_{ip} = \begin{cases} e^{-f_1(\frac{1}{p}a_k)} & \text{if } i = 2k-1 \\ e^{-f_2(\frac{1}{p}b_k)} & \text{if } i = 2k \end{cases} \quad , \quad d_{ip} = \begin{cases} e^{-f_3(\frac{1}{p}\tilde{a}_k)} & \text{if } i = 2k-1 \\ e^{-f_4(\frac{1}{p}\tilde{b}_k)} & \text{if } i = 2k \end{cases}$$

4.2.1 $f_3 \approx f_2 \prec f_1 \approx f_4$

For this case

$$\begin{aligned} L_{f_1}(a,0) &\text{ part dos not have } S_4^-(f_2),\ S_4^-(f_3), \\ L_{f_2}(b,0) &\text{ part dos not have } Q_4^-(f_1),\ Q_4^-(f_4), \\ L_{f_3}(\tilde{a},0) &\text{ part dos not have } Q_4^-(f_1),\ Q_4^-(f_4), \\ L_{f_4}(\tilde{b},0) &\text{ part dos not have } S_4^-(f_2),\ S_4^-(f_3). \end{aligned}$$

For $W_3 = \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{f_1}(k)}U_p \cap kU_r)\right) \cap tU_r$,

$\tilde{W}_3 \quad = \quad \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{f_1}(k)}T(V_{p_1}) \cap kT(V_{r_1}))\right) \cap tT(V_{r_1})$, where $\varphi_{f_1} =$ $\exp f_1 M f_1^{-1}$ log,in (4.1) we have

$$\beta_1 \geq |\{i : \underbrace{\varphi_{f_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_4^-(f_1)}, \frac{c_{i,r}}{c_{i,q}} \leq t, \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| := |I_1|,$$

$$\beta_2 \leq |\{i : \underbrace{\varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_4^-(f_1)}, \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct, \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| := |I_2|.$$

Since $L_{f_2}(b,0)$, $L_{f_3}(\tilde{a},0)$ do not have $Q_4^-(f_1)$ we take $i = 2k-1$ in I_1 and $i = 2k$ in I_2. Then we get

$$\beta_1 \geq |I_1| \geq M_a\left(q\, f_1^{-1}(\ln t)\,,\ (q+1)\, f_1^{-1}(\ln \tau)\right),$$

$$\beta_2 \leq |I_2| \leq M_{\tilde{b}}\left((\tilde{q}+1)\, f_4^{-1}(\ln 2ct)\,,\ q_1\, f_4^{-1}(\ln \frac{\tau}{2c})\right),$$

and hence

$$M_a\left(q\, f_1^{-1}(\ln t)\,,\ (q+1)\, f_1^{-1}(\ln \tau)\right) \leq M_{\tilde{b}}\left((\tilde{q}_1)\, f_4^{-1}(\ln 2ct)\,,\ q_1\, f_4^{-1}(\ln \frac{\tau}{2c})\right).$$

On the other hand, for

$$W_1 = \overline{\Gamma}\left(\bigcup_{k>0}\frac{1}{\varphi_{f_2}(k)}T^{-1}(U_p) \cap kT^{-1}(U_r)\right) \cup tT^{-1}(U_r,$$

$$\tilde{W}_1 = \overline{\Gamma}\left(\overline{\Gamma}\left(\bigcup_{k>0}\frac{1}{\varphi_{f_2}}V_{p_1} \cap kV_{r_1}\right) \cup tV_{r_1}\right)$$

where $\varphi_{f_2} = \exp f_2 M f_2^{-1}$ log in (4.2) we obtain

$$\beta \geq |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \leq \varphi_{f_2}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_4^-(f_2)}, \frac{c_{i,s}}{c_{i,q}} \leq \tau, \frac{c_{i,r}}{c_{i,q}} \geq t\}| := |I|,$$

$$\tilde{\beta} \leq |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \leq \varphi_{f_2}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_4^-(f_2)}, \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau, \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| := |\tilde{I}|.$$

Since $L_{f_1}(a,0), L_{f_4}(\tilde{b},0)$ do not have $S_4^-(f_2)$ we take $i = 2k$ in I and $i = 2k-1$ in $\tilde{I}$ and get

$$\beta \geq |I| \geq M_b\left(q\, f_2^{-1}(\ln \tau)\,,\ (q+1)\, f_2^{-1}(\ln t)\right),$$

$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{a}}\left((\tilde{q}_1+1)\, f_3^{-1}(\ln 2c\tau)\,,\ q_1\, f_3^{-1}(\ln \frac{t}{2c})\right),$$

and since $\beta \leq \tilde{\beta}$ we have

$$M_b\left(q\, f_2^{-1}(\ln \tau)\,,\ (q+1)\, f_2^{-1}(\ln t)\right) \leq M_{\tilde{a}}\left((\tilde{q}_1+1)\, f_3^{-1}(\ln 2c\tau)\,,\ q_1\, f_3^{-1}(\ln \frac{t}{2c})\right).$$

Because of symmetry, we therefore have

$$M_a \approx M_{f_1^{-1}\, f_4(\tilde{b})} \quad , \quad M_b \approx M_{f_2^{-1}\, f_3(\tilde{a})}.$$

As a result we get $a_i \asymp f_1^{-1}\, f_4(\tilde{b}_{i+k_1})$ and $b_j \asymp f_2^{-1}\, f_3(\tilde{a}_{j+k_2})$ for some $k_1,\ k_2 \in \mathbb{Z}$ by (2.3).

4.3 $L_{f_1}(a,1) \times L_{f_2}(b,1) \simeq L_{f_3}(\tilde{a},1) \times L_{f_4}(\tilde{b},1)$

Proposition 4.7 *Let $g_1 \prec g_2$. Then*

(i) $L_{g_1}(b,1)$ does not have $S_4^+(g_2)$,

(ii) $L_{g_2}(\tilde{b},1)$ does not have $Q_4^+(g_1)$.

Proof

(i) Assume that $L_{g_1}(b,1)$ has $S_4^+(g_2)$. Then $\exists p\, \forall q\, \exists r\, \exists M,\ M > 1$

$$M\, g_2^{-1}\, \log(e^{g_1((1-\frac{1}{q})b_k)-g_1((1-\frac{1}{p})b_k)}) \leq g_2^{-1}\, \log(e^{g_1((1-\frac{1}{r})b_k)-g_1((1-\frac{1}{q})b_k)}$$

$$\Longrightarrow \quad M\, g_2^{-1}\, g_1((1-\tfrac{1}{r})b_k) \leq g_2^{-1}\, g_1((1-\tfrac{1}{q-1})b_k)$$

$$\Longrightarrow \quad M \leq \frac{g_2^{-1}\, g_1((1-\frac{1}{r})b_k)}{g_2^{-1}\, g_1((1-\frac{1}{q-1})b_k)}.$$

Since $g_1 \prec g_2$, the right hand side tends to 1. But this contradicts the assumption that $M > 1$. Thus (i) must be true.

After eliminating many of the possible relations between the Dragilev functions f_1, f_2, f_3 and f_4, we only have 5 remaining cases :

4.3.1 $f_3 \approx f_1 \prec f_2 \approx f_4$

This case

$$L_{f_1}(a,1) \text{ part dos not have } S_4^+(f_2),\ S_4^+(f_4),$$

$$L_{f_2}(b,1) \text{ part dos not have } Q_4^+(f_1),\ Q_4^+(f_3),$$
$$L_{f_3}(\tilde{a},1) \text{ part dos not have } S_4^+(f_2),\ S_4^+(f_4),$$
$$L_{f_4}(\tilde{b},1) \text{ part dos not have } Q_4^+(f_1),\ Q_4^+(f_3),$$

and for f_1 (by using $\varphi_{f_1} = \exp f_1 \frac{1}{M} f_1^{-1} \log$) in (4.1) ,

$$\beta_1 \geq |\{i : \underbrace{\varphi_{f_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_4^+(f_1)},\ \frac{c_{i,r}}{c_{i,q}} \leq t,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| := |I_1|,$$

$$\beta_2 \leq |\{i : \underbrace{\varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_4^+(f_1)},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct,\ \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| := |I_2|.$$

Since $L_{f_2}(b,1)$, $L_{f_4}(\tilde{b},1)$ do not have $Q_4^+(f_1)$ we take $i = 2k-1$ in I_1 and in I_2. Then we get

$$\beta_1 \geq |I_1| \geq M_a\left(\frac{f_1^{-1}(\ln t)}{1-\frac{1}{r}},\ \frac{f_1^{-1}(\ln \tau)}{1-\frac{1}{s}}\right),$$
$$\beta_2 \leq |I_2| \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2ct)}{1-\frac{1}{r_1-1}},\ \frac{f_3^{-1}(\ln \frac{\tau}{2c})}{1-\frac{1}{s_1}}\right),$$

and hence $\quad M_a\left(\frac{f_1^{-1}(\ln t)}{1-\frac{1}{r}},\ \frac{f_1^{-1}(\ln \tau)}{1-\frac{1}{s}}\right) \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2ct)}{1-\frac{1}{r_1-1}},\ \frac{f_3^{-1}(\ln \frac{\tau}{2c})}{1-\frac{1}{s_1}}\right).$

Since $\beta_1 \leq \beta_2$.

On the other hand, by using $\varphi_{f_2} = \exp f_2 \frac{1}{M} f_2^{-1} \log$ in (4.2) ,

$$\beta \geq |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \leq \varphi_{f_2}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_4^+(f_2)},\ \frac{c_{i,s}}{c_{i,q}} \leq \tau,\ \frac{c_{i,r}}{c_{i,q}} \geq t\}| := |I|,$$

$$\tilde{\beta} \leq |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \leq \varphi_{f_2}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_4^+(f_2)},\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau,\ \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| := |\tilde{I}|.$$

Since $L_{f_1}(a,1)$, $L_{f_3}(\tilde{a},1)$ do not have $S_4^+(f_2)$ we take $i = 2k$ in I and in $\tilde{I}$. Then we get

$$\beta \geq |I| \geq M_b\left(\frac{f_2^{-1}(\ln \tau)}{1-\frac{1}{s}},\ \frac{f_2^{-1}(\ln t)}{1-\frac{1}{r-1}}\right),$$
$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}},\ \frac{f_4(\ln \frac{t}{2c})}{1-\frac{1}{r_1}}\right),$$

and hence $\quad M_b\left(\frac{f_2^{-1}(\ln\tau)}{1-\frac{1}{s}}, \frac{f_2^{-1}(\ln t)}{1-\frac{1}{r-1}}\right) \leq M_{\tilde{b}}\left(\frac{f_4^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}}, \frac{f_4(\ln\frac{t}{2c})}{1-\frac{1}{r_1}}\right).$

By symmetry, we therefore obtain

$$M_a \approx M_{f_1^{-1}\, f_3(\tilde{a})} \qquad , \qquad M_b \approx M_{f_2^{-1}\, f_4(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}\, f_3(\tilde{a}_{i+k_1})$ and $b_j \asymp f_2^{-1}\, f_4(\tilde{b}_{j+k_2})$ for some $k_1,\ k_2 \in \mathbb{Z}$ by (2.3).

4.4 $L_{f_1}(a,-1) \times L_{f_2}(b,-1) \simeq L_{f_3}(\tilde{a},-1) \times L_{f_4}(\tilde{b},-1)$

Proposition 4.8 *Let $f_1 \prec f_2$. Then (i) $L_{f_1}(a,-1)$ does not have $Q_2^-(f_2)$, (ii) $L_{f_2}(\tilde{a},-1)$ does not have $S_2^-(f_1)$.*

Proof If we assume the contrary of (i), we get

$$M \leq \frac{f_2^{-1}\, f_1((1+\frac{1}{p})a_k)}{f_2^{-1}\, f_1((1+\frac{1}{q+1})a_k)}.$$

Since $f_1 \prec f_2$ right hand side tends to 1. But this contradicts the assumption that $M > 1$. Thus, (i) must be true. The proof of (ii) is similar.

The remaining cases after elimination are same as before, that is:

4.4.1 $f_1 \approx f_3 \prec f_2 \approx f_4$

For this case

$$\begin{aligned} L_{f_1}(a,-1) &\text{ part dos not have } Q_2^-(f_2),\ Q_2^-(f_4),\\ L_{f_2}(b,-1) &\text{ part dos not have } S_2^-(f_1),\ S_2^-(f_3),\\ L_{f_3}(\tilde{a},-1) &\text{ part dos not have } Q_2^-(f_2),\ Q_2^-(f_4),\\ L_{f_4}(\tilde{b},-1) &\text{ part dos not have } S_2^-(f_1),\ S_2^-(f_3). \end{aligned}$$

By using f_2 (for $\varphi_{f_2} = \exp\ f_2\, M\, f_2^{-1}\ \log$) in (4.1)

$$\beta_1 \geq |\{i : \underbrace{\varphi_{f_2}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_2^-(f_2)}, \frac{c_{i,r}}{c_{i,q}} \leq t, \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| := |I_1|,$$

$$\beta_2 \leq |\{i : \underbrace{\varphi_{f_2}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_2^-(f_2)}, \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct, \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| := |I_2|.$$

Since $L_{f_1}(a,-1)$, $L_{f_3}(\tilde{a},-1)$ do not have $Q_2^-(f_2)$ we take $i = 2k$ in I_1, I_2. Then we see that

$$\beta_1 \geq |I_1| \geq M_b\left(\frac{f_2^{-1}(\ln t)}{1+\frac{1}{q}}, \frac{f_2^{-1}(\ln \tau)}{1+\frac{1}{q+1}}\right),$$

$$\beta_2 \leq |I_2| \leq M_b\left(\frac{f_4^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}}, \frac{f_4^{-1}(\ln \frac{\tau}{2c})}{1+\frac{1}{q_1}}\right),$$

and hence $$M_b\left(\frac{f_2^{-1}(\ln t)}{1+\frac{1}{q}}, \frac{f_2^{-1}(\ln \tau)}{1+\frac{1}{q+1}}\right) \leq M_b\left(\frac{f_4^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}}, \frac{f_4^{-1}(\ln \frac{\tau}{2c})}{1+\frac{1}{q_1}}\right)$$

since $\beta_1 \leq \beta_2$.

On the other hand for $\varphi_{f_1} = \exp f_1 M f_1^{-1} \log$ in (4.2) we get

$$\beta \geq |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \leq \varphi_{f_1}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_2^-(f_1)}, \frac{c_{i,s}}{c_{i,q}} \leq \tau, \frac{c_{i,r}}{c_{i,q}} \geq t\}| := |I|,$$

$$\tilde{\beta} \leq |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \leq \varphi_{f_1}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_2^-(f_1)}, \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau, \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| := |\tilde{I}|.$$

Since $L_{f_2}(b,-1)$, $L_{f_2}(\tilde{b},-1)$ do not have $S_2^-(f_1)$ we take $i = 2k-1$ in I, $\tilde{I}$. Then we get

$$\beta \geq |I| \geq M_a\left(\frac{f_1^{-1}(\ln \tau)}{1+\frac{1}{q}}, \frac{f_1^{-1}(\ln t)}{1+\frac{1}{q+1}}\right),$$

$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2c\tau)}{1+\frac{1}{\tilde{q}_1+1}}, \frac{f_3^{-1}(\ln \frac{t}{2c})}{1+\frac{1}{q_1}}\right),$$

and hence $$M_a\left(\frac{f_1^{-1}(\ln \tau)}{1+\frac{1}{q}}, \frac{f_1^{-1}(\ln t)}{1+\frac{1}{q+1}}\right) \leq M_{\tilde{a}}\left(\frac{f_3^{-1}(\ln 2c\tau)}{1+\frac{1}{\tilde{q}_1+1}}, \frac{f_3^{-1}(\ln \frac{t}{2c})}{1+\frac{1}{q_1}}\right)$$

since $\beta \leq \tilde{\beta}$.

Again by symmetry we therefore obtain

$$M_a \approx M_{f_1^{-1} f_3(\tilde{a})} \qquad , \qquad M_b \approx M_{f_2^{-1} f_4(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1} f_3(\tilde{a}_{i+k_1})$ and $b_j \asymp f_2^{-1} f_4(\tilde{b}_{j+k_2})$ for some $k_1\, k_2 \in \mathbb{Z}$ by (2.3).

4.5 $L_{f_1}(a,\infty) \times L_{g_1}(b,0) \simeq L_{f_2}(\tilde{a},\infty) \times L_{g_2}(\tilde{b},0)$

The table of considered possibilities in this case has some differences.

$f_1 \prec f_2 : g_1 \prec g_2$	**B1,B3** $f_1 \prec f_2 \prec g_1 \prec g_2$	$\mathbf{S_4^-(g_1)}$ $f_1 \prec f_2 \approx g_1 \prec g_2$	**A1,A3** $g_1 \prec f_1 \prec f_2 \prec g_2$	**A1** $g_1 \approx f_1 \prec f_2 \prec g_2$
	A1,A3 $g_1 \prec f_1 \prec f_2 \approx g_2$	**A1** $g_1 \approx f_1 \prec f_2 \approx g_2$	**C1,C3** $g_1 \prec g_2 \prec f_1 \prec f_2$	$\mathbf{S_4^-(g_1)}$ $g_1 \prec g_2 \approx f_1 \prec f_2$
	A1,A3 $f_1 \prec g_1 \prec g_2 \prec f_2$	**A1,A3** $f_1 \approx g_1 \prec g_2 \prec f_2$	**A1,A3** $f_1 \prec g_1 \prec g_2 \approx f_2$	**A1,A3** $g_1 \prec f_1 \prec g_2 \prec f_2$
	A1,A3 $g_1 \approx f_1 \prec g_2 \prec f_2$	$\mathbf{S_4^-(g_1)}$ $g_1 \prec f_1 \approx g_2 \prec f_1$	**A1,A3** $f_1 \prec g_1 \prec f_2 \prec g_2$	$\mathbf{Q_1^+(f_1)}$ $f_1 \prec g_1 \approx f_2 \prec g_2$
$f_1 \prec f_2 \,;\, g_2 \prec g_1$	**B1,B4** $f_1 \prec f_2 \prec g_2 \prec g_1$	**B1,B4** $f_1 \prec f_2 \approx g_2 \prec g_1$	**D1,D4** $g_2 \prec f_1 \prec f_2 \prec g_1$	$\mathbf{S_4^-(g_2)}$ $g_2 \approx f_1 \prec f_2 \prec g_1$
	$\mathbf{S_4^-(g_2)}$ $g_2 \prec f_1 \prec f_2 \approx g_1$	$\mathbf{S_4^-(g_2)}$ $g_2 \approx f_1 \prec f_2 \approx g_1$	**C1,C4** $g_2 \prec g_1 \prec f_1 \prec f_2$	**C1,C4** $g_2 \prec g_1 \approx f_1 \prec f_2$
	A1,A4 $f_1 \prec g_2 \prec g_1 \prec f_2$	$\mathbf{S_4^-(g_2)}$ $f_1 \approx g_2 \prec g_1 \prec f_2$	$\mathbf{S_4^-(g_2)}$ $f_1 \prec g_2 \prec g_1 \approx f_2$	**C1,C4** $g_2 \prec f_1 \prec g_1 \prec f_2$
	$\mathbf{S_4^-(g_2)}$ $g_2 \approx f_1 \prec g_1 \prec f_2$	**C1,C4** $g_2 \prec f_1 \approx g_1 \prec f_2$	**B1,B4** $f_1 \prec g_2 \prec f_2 \prec g_1$	**B1,B4** $f_1 \prec g_2 \approx f_2 \prec g_1$
$f_1 \prec f_2 \,;\, g_1 \approx g_2$	**B1** $f_1 \prec f_2 \prec g_1 \approx g_2$	$\mathbf{Q_1^+(f_1)}$ $f_1 \prec f_2 \approx g_1 \approx g_2$	**C1** $g_1 \approx g_2 \prec f_1 \prec f_2$	$\mathbf{Q_1^+(f_1)}$ $g_1 \approx g_2 \approx f_1 \prec f_2$
	A1 $f_1 \prec g_1 \approx g_2 \prec f_2$			

$f_2 \prec f_1$; $g_1 \prec g_2$	**B2,B3** $f_2 \prec f_1 \approx g_1 \prec g_2$	**B2,B3** $f_2 \prec f_1 \approx g_1 \prec g_2$	**A2,A3** $g_1 \prec f_2 \prec f_1 \prec g_2$	$\mathbf{S_4^-(g_1)}$ $g_1 \approx f_2 \prec f_1 \prec g_2$
	$\mathbf{S_4^-(g_1)}$ $g_1 \prec f_2 \prec f_1 \approx g_2$	$\mathbf{S_4^-(g_1)}$ $g_1 \approx f_2 \prec f_1 \approx g_2$	**C2,C3** $g_1 \prec g_2 \prec f_2 \prec f_1$	**C2,C3** $g_1 \prec g_2 \approx f_2 \prec f_1$
	D2,D3 $f_2 \prec g_1 \prec g_2 \prec f_1$	$\mathbf{S_4^-(g_1)}$ $f_2 \approx g_1 \prec g_2 \prec f_1$	$\mathbf{S_4^-(g_1)}$ $f_2 \prec g_1 \prec g_2 \approx f_1$	**C2,C3** $g_1 \prec f_2 \prec g_2 \prec f_1$
	$\mathbf{S_4^-(g_1)}$ $g_1 \approx f_2 \prec g_2 \prec f_1$	**C2,C3** $g_1 \prec f_2 \approx g_2 \prec f_1$	**B2,B3** $f_2 \prec g_1 \prec f_1 \prec g_2$	**B2,B3** $f_2 \prec g_1 \approx f_2 \prec g_2$
$f_2 \prec f_1$; $g_2 \prec g_1$	**B2,B3** $f_2 \prec f_1 \prec g_2 \prec g_1$	$\mathbf{Q_1^+(f_2)}$ $f_2 \prec f_1 \approx g_2 \prec g_1$	**D2,D4** $g_2 \prec f_2 \prec f_1 \prec g_1$	**D2,D4** $g_2 \approx f_2 \prec f_1 \prec g_1$
	D2,D4 $g_2 \prec f_2 \prec f_1 \approx g_1$	**D2,D4** $g_2 \approx f_2 \prec f_1 \approx g_1$	**C2,C4** $g_2 \prec g_1 \prec f_2 \prec f_1$	$\mathbf{S_4^-(g_2)}$ $g_2 \prec g_1 \approx f_2 \prec f_1$
	D2,D4 $f_2 \prec g_2 \prec g_1 \prec f_1$	**D2,D4** $f_2 \approx g_2 \prec g_1 \prec f_1$	**D2,D4** $f_2 \prec g_2 \prec g_1 \approx f_1$	**D2,D4** $g_2 \prec f_2 \prec g_1 \prec f_1$
	D2,D4 $g_2 \approx f_2 \prec g_1 \prec f_1$	$\mathbf{S_4^-(g_2)}$ $g_2 \prec f_2 \approx g_1 \prec f_1$	**D2,D4** $f_2 \prec g_2 \prec f_1 \prec g_1$	$\mathbf{Q_1^+(f_2)}$ $f_2 \prec g_2 \approx f_1 \prec g_1$
$f_2 \prec f_1$; $g_1 \approx g_2$	**B2** $f_2 \prec f_1 \prec g_1 \approx g_2$	$\mathbf{Q_1^+(f_2)}$ $f_2 \prec f_1 \approx g_1 \approx g_2$	**C2** $g_1 \approx g_2 \prec f_2 \prec f_1$	$\mathbf{Q_1^+(f_2)}$ $g_1 \approx g_2 \approx f_2 \prec f_1$
	B2 $f_2 \prec g_1 \approx g_2 \prec f_1$			
$f_1 \approx f_2$; $g_1 \prec g_2$	**B3** $f_1 \approx f_2 \prec g_1 \prec g_2$	$\mathbf{S_4^-(g_1)}$ $f_1 \approx f_2 \approx g_1 \prec g_2$	**C3** $g_1 \prec g_2 \prec f_1 \approx f_2$	$\mathbf{S_4^-(g_1)}$ $g_1 \prec g_2 \approx f_1 \approx f_2$
	A3 $g_1 \prec f_1 \approx f_2 \prec g_2$			
$f_1 \approx f_2$; $g_2 \prec g_1$	**B4** $f_1 \approx f_2 \prec g_2 \prec g_1$	$\mathbf{S_4^-(g_2)}$ $f_1 \approx f_2 \approx g_2 \prec g_1$	**C4** $g_2 \prec g_1 \prec f_1 \approx f_2$	$\mathbf{S_4^-(g_2)}$ $g_2 \prec g_1 \approx f_1 \approx f_2$
	D4 $g_2 \prec f_1 \approx f_2 \prec g_1$			
$f_1 \approx f_2$; $g_1 \approx g_2$	$\mathbf{f_1 \approx f_2 \prec g_1 \approx g_2}$	$\mathbf{f_1 \approx f_2 \approx g_1 \approx g_2}$	$\mathbf{g_1 \approx g_2 \prec f_1 \approx f_2}$	

In order to eliminate many of the cases in above we wrote, for instance, $\mathbf{S_4^-(g_1)}$ over $f_1 \prec f_2 \approx g_1 \prec g_2$. By this, we mean that for the case $f_1 \prec f_2 \approx g_1 \prec g_2$ the property $\mathbf{S_4^-(g_1)}$ is not common on both sides of the isomorphism.

Because, firstly $L_f(\alpha,\infty)$ spaces always have the property $S_4^-(g)$. Indeed; otherwise for $p < q << r$, $M > 1$ and for some subsequence of indices,

$$\begin{aligned} g^{-1}\ \log(e^{f(qa_k)-f(pa_k)}) &> M\, g^{-1}\ \log(e^{f(ra_k)-f(qa_k)}) \\ \Longrightarrow g^{-1}\ f(qa_k) &> M\, g^{-1}\ f((r-1)a_k) \\ \Longrightarrow \tfrac{1}{M} &> \tfrac{g^{-1}\ f((r-1)a_k)}{g^{-1}\ f(qa_k)}. \end{aligned}$$

But the right hand side is at least 1. Since $M > 1$, this is a contradiction. Secondly, the space $L_{g_2}(\tilde{b},0)$ does not have the property $S_4^-(g_1)$ since $g_1 \prec g_2$. So on the left side of the isomorphism the property $S_4^-(g_1)$ is common, but on the right side it is not so. But if we have an isomorphism, then it must preserve such structural properties. Hence we eliminate this case.

We have 3 possible cases after all eliminations:

4.5.1 $f_1 \approx f_2 \prec g_1 \approx g_2$

Take $W_3 = \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{g_1}}U_p \cap kU_r)\right) \cap tU_r$,

$\tilde{W}_3 = \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{g_1}}T(V_{p_1}) \cap kT(V_{r_1})) \cap tT(V_{r_1})\right)$ where $\varphi_{g_1} = \exp g_1\, M\, g_1^{-1}\ \log$ in (4.1). Then we see that

$$\beta_1 \geq |\{i : \underbrace{\varphi_{g_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_4^-(g_1)},\ \frac{c_{i,r}}{c_{i,q}} \leq t,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| := |I_1|,$$

$$\beta_2 \leq |\{i : \underbrace{\varphi_{g_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_4^-(g_1)},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct,\ \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| := |I_2|.$$

$L_f(\alpha,\infty)$ spaces do not have the property $Q_4^-(g)$. So we take $i = 2k$ in I_1, I_2. Then we get that

$$\beta_1 \geq |I_1| \geq M_b\left(q\, g_1^{-1}(\ln t)\,,\ (q+1)g_1^{-1}(\ln \tau)\right),$$

$$\beta_2 \leq |I_2| \leq M_{\tilde{b}}\left((\tilde{q}_1+1)g_2^{-1}(\ln 2ct)\,,\ q_1\, g_2^{-1}(\ln \frac{\tau}{2c})\right),$$

and hence

$$M_b\left(q\, g_1^{-1}(\ln t)\,,\ (q+1)g_1^{-1}(\ln \tau)\right) \leq M_{\tilde{b}}\left((\tilde{q}_1+1)g_2^{-1}(\ln 2ct)\,,\ q_1\, g_2^{-1}(\ln \frac{\tau}{2c})\right) \quad (4.3)$$

since $\beta_1 \le \beta_2$.

On the other hand, for

$$W_1 = \overline{\Gamma}\left(\overline{\Gamma}(\bigcup_{k>0} \frac{1}{\varphi_{f_1}(k)} T^{-1}(U_p) \cap kT^{-1}(U_r)) \cup tT^{-1}(U_r)\right),$$

$$\tilde{W}_1 = \overline{\Gamma}\left(\overline{\Gamma}(\bigcup_{k>0} \frac{1}{\varphi_{f_1}(k)} V_{p_1} \cap kV_{r_1}) \cup tV_{r_1}\right)$$

in (4.2) where $\varphi_{f_1} = \exp\ f_1\ \frac{1}{M}\ f_1^{-1}\ \log$, we obtain

$$\beta \ge |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \le \varphi_{f_1}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_1^+(f_1)},\ \frac{c_{i,s}}{c_{i,q}} \le \tau,\ \frac{c_{i,r}}{c_{i,q}} \ge t\}| := |I|,$$

$$\tilde{\beta} \le |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \le \varphi_{f_1}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_1^+(f_1)},\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \le 2c\tau,\ \frac{d_{i,r_1}}{d_{i,q_1}} \ge \frac{t}{2c}\}| := |\tilde{I}|.$$

Since $L_{g_1}(b,0)$, $L_{g_2}(\tilde{b},0)$ do not have $S_1^+(f_1)$, we take $i = 2k$ in I, $\tilde{I}$ and obtain

$$\beta \ge |I| \ge M_a\left(\frac{f_1^{-1}(\ln\tau)}{s}, \frac{f_1^{-1}(\ln t)}{r-1}\right),$$

$$\tilde{\beta} \le |\tilde{I}| \le M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f_2^{-1}(\ln \frac{t}{2c})}{r_1}\right),$$

and hence

$$M_a\left(\frac{f_1^{-1}(\ln\tau)}{s}, \frac{f_1^{-1}(\ln t)}{r-1}\right) \le M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2c\tau)}{s_1-1}, \frac{f_2^{-1}(\ln \frac{t}{2c})}{r_1}\right). \tag{4.4}$$

(4.3), (4.4) with their symmetrical inequalities we therefore get

$$M_a \approx M_{f_1^{-1}\ f_2(\tilde{a})}, \qquad , \qquad M_b \approx M_{g_1^{-1}\ g_2(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}\ f_2(\tilde{a}_{i+k_1})$ and $b_j \asymp g_1^{-1}\ g_2(\tilde{b}_{j+k_2})$ for some $k_1\ k_2 \in \mathbb{Z}$ by (2.3).

4.6 $L_{f_1}(a,1) \times L_{g_1}(b,0) \simeq L_{f_2}(\tilde{a},1) \times L_{g_2}(\tilde{b},0)$

4.6.1 $f_1 \approx f_2 \prec g_1 \approx g_2$

Consider $W_3 = \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{g_1}(k)}U_p \cap kU_r)\right) \cap tU_r$,

$\tilde{W}_3 = \overline{\Gamma}\left(\bigcup_{k>0}(\frac{1}{\varphi_{g_1}(k)}T(V_{p_1}) \cap kT(V_{r_1}))\right) \cap tT(V_{r_1})$, where $\varphi_{g_1} = \exp g_1\, M\, g_1^{-1} \log$. Substituting them into (4.1) we get

$$\beta_1 \geq |\{i : \underbrace{\varphi_{g_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_4^-(g_1)},\ \frac{c_{i,r}}{c_{i,q}} \leq t,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| := |I_1|,$$

$$\beta_2 \leq |\{i : \underbrace{\varphi_{g_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_4^-(g_1)},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct,\ \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| := |I_2|.$$

Since $L_f(\alpha,1)$ spaces do not have the property $Q_4^-(g)$ we take $i = 2k$ in I_1, I_2 and obtain

$$\beta_1 \geq |I_1| \geq M_b\left(q\, g_1^{-1}(\ln t)\,,\ (q+1)\, g_1^{-1}(\ln \tau)\right),$$
$$\beta_2 \leq |I_2| \leq M_{\tilde{b}}\left((\tilde{q}_1+1)\, g_2^{-1}(\ln 2ct)\,,\ q_1\, g_2^1(\ln \frac{\tau}{2c})\right),$$

and hence

$$M_b\left(q\, g_1^{-1}(\ln t)\,,\ (q+1)\, g_1^{-1}(\ln \tau)\right) \leq M_{\tilde{b}}\left((\tilde{q}_1+1)\, g_2^{-1}(\ln 2ct)\,,\ q_1\, g_2^1(\ln \frac{\tau}{2c})\right)$$

since $\beta_1 \leq \beta_2$.

On the other hand, for $W_1 = \overline{\Gamma}\left(\overline{\Gamma}(\frac{1}{\varphi_{f_1}(k)}T^{-1}(U_p) \cap kT^{-1}(U_r)) \cup tT^{-1}(U_r)\right)$, $\tilde{W}_1 = \overline{\Gamma}\left(\overline{\Gamma}(\bigcup_{k>0}\frac{1}{\varphi_{f_1}(k)}V_{p_1} \cap kV_{r_1}) \cup tV_{r_1}\right)$ with $\varphi_{f_1} = \exp f_1 \frac{1}{M} f_1^{-1} \log$ in (4.2) we get

$$\beta \geq |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \leq \varphi_{f_1}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_4^+(f_1)},\ \frac{c_{i,s}}{c_{i,q}} \leq \tau,\ \frac{c_{i,r}}{c_{i,q}} \geq t\}| := |I|,$$

$$\tilde{\beta} \leq |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}} \leq \varphi_{f_1}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_4^+(f_1)},\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \leq 2c\tau,\ \frac{d_{i,r_1}}{d_{i,q_1}} \geq \frac{t}{2c}\}| := |\tilde{I}|.$$

Since $L_g(\alpha,0)$ spaces do not have the property $S_4^+(f)$ we take $i=2k-1$ in I, $\tilde{I}$ and see that

$$\beta \geq |I| \geq M_a\left(\frac{f_1^{-1}(\ln\tau)}{1-\frac{1}{s}}\,,\ \frac{f_1^{-1}(\ln t)}{1-\frac{1}{r-1}}\right),$$
$$\tilde{\beta} \leq |\tilde{I}| \leq M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}}\,,\ \frac{f_2^{-1}(\ln\frac{t}{2c})}{1-\frac{1}{r_1}}\right),$$

and hence $$M_a\left(\frac{f_1^{-1}(\ln\tau)}{1-\frac{1}{s}}\,,\ \frac{f_1^{-1}(\ln t)}{1-\frac{1}{r-1}}\right) \leq M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}}\,,\ \frac{f_2^{-1}(\ln\frac{t}{2c})}{1-\frac{1}{r_1}}\right)$$
since $\beta \leq \tilde{\beta}$.

Therefore, similarly as before we obtain

$$M_a \approx M_{f_1^{-1}\,f_2(\tilde{a})} \qquad , \qquad M_b \approx M_{g_1^{-1}\,g_2(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}\ f_2(\tilde{a}_{i+k_1})$ and $b_j \asymp g_1^{-1}\ g_2(\tilde{b}_{j+k_2})$ for some $k_1,\ k_2 \in \mathbb{Z}$ by (2.3).

4.7 $L_{f_1}(a,-1)\times L_{g_1}(b,1) \simeq L_{f_2}(\tilde{a},-1)\times L_{g_2}(\ b,1)$

4.7.1 $f_1 \approx f_2 \prec g_1 \approx g_2$

By taking $\varphi=\varphi_{f_1}=\exp\ f_1\ M\ f_1^{-1}\ \log$ in (4.1) we get

$$\beta_1 \geq |\{i : \underbrace{\varphi_{f_1}(\frac{c_{i,r+1}}{c_{i,q-1}}) \leq \frac{c_{i,q-1}}{c_{i,p}}}_{Q_2^-(f_1)}\,,\ \frac{c_{i,r}}{c_{i,q}} \leq t\,,\ \frac{c_{i,s}}{c_{i,q}} \geq \tau\}| \ : |I_1|,$$
$$\beta_2 \leq |\{i : \underbrace{\varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}}) \leq \frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_2^-(f_1)}\,,\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}} \leq 2ct\,,\ \frac{d_{i,s_1}}{d_{i,q_1}} \geq \frac{\tau}{2c}\}| \ : |I_2|.$$

Since $L_g(\alpha,1)$ spaces do not have the property $Q_2^-(f)$ we take $i=2k-1$ in I_1, I_2 and see that

$$\beta_1 \geq |I_1| \geq M_a\left(\frac{f_1^{-1}(\ln t)}{1+\frac{1}{}}\,,\ \frac{f_1^{-1}(\ln\tau)}{1+\frac{1}{q+1}}\right),$$
$$\beta_2 \leq |I_2| \leq M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}}\,,\ \frac{f_2^{-1}(\ln\frac{\tau}{2c})}{1+\frac{1}{q_1}}\right),$$

and hence $$M_a\left(\frac{f_1^{-1}(\ln t)}{1+\frac{1}{\ }},\ \frac{f_1^{-1}(\ln \tau)}{1+\frac{1}{q+1}}\right) \le M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}},\ \frac{f_2^{-1}(\ln \frac{\tau}{2c})}{1+\frac{1}{q_1}}\right)$$
since $\beta_1 \le \beta_2$.

We can do the same thing for $\varphi_{f_2} = \exp\ f_2\ M\ f_2^{-1}\ \log$.

On the other hand, by taking $\varphi_{g_1} = \exp\ g_1\ \frac{1}{M}\ g_1^{-1}\ \log$ in (4.2), we get

$$\beta \ge |\{i : \underbrace{\frac{c_{i,q+1}}{c_{i,p}} \le \varphi_{g_1}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_4^+(g_1)},\ \frac{c_{i,s}}{c_{i,q}} \le \tau,\ \frac{c_{i,r}}{c_{i,q}} \ge t\}| \ : |I|,$$

$$\tilde{\beta} \le |\{i : \underbrace{\frac{d_{i,q_1-2}}{d_{i,p}} \le \varphi_{g_1}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_4^+(g_1)},\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}} \le 2c\tau,\ \frac{d_{i,r_1}}{d_{i,q_1}} \ge \frac{t}{2c}\}| \ : |\tilde{I}|.$$

Since $L_f(\alpha,-1)$ spaces do not have the property $S_4^+(g)$, we take $i = 2k$ in I, $\tilde{I}$, and see that

$$\beta \ge |I| \ge M_b\left(\frac{g_1^{-1}(\ln \tau)}{1-\frac{1}{s}},\ \frac{g_1^{-1}(\ln t)}{1-\frac{1}{r-1}}\right),$$
$$\tilde{\beta} \le |\tilde{I}| \le M_{\tilde{b}}\left(\frac{g_2^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}},\ \frac{g_2^{-1}(\ln \frac{t}{2c})}{1-\frac{1}{r_1}}\right),$$

and hence $$M_b\left(\frac{g_1^{-1}(\ln \tau)}{1-\frac{1}{s}},\ \frac{g_1^{-1}(\ln t)}{1-\frac{1}{r-1}}\right) \le M_{\tilde{b}}\left(\frac{g_2^{-1}(\ln 2c\tau)}{1-\frac{1}{s_1-1}},\ \frac{g_2^{-1}(\ln \frac{t}{2c})}{1-\frac{1}{r_1}}\right)$$
since $\beta \le \tilde{\beta}$.

We can do the same thing for $\varphi_{g_2} = \exp\ g_2\ \frac{1}{M}\ g_2^{-1}\ \log$.

Therefore, by symmetry we have

$$M_a \approx M_{f_1^{-1}\ f_2(\tilde{a})} \quad , \quad M_b \approx M_{g_1^{-1}\ g_2(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}\ f_2(\tilde{a}_{k_1})$ and $b_j \asymp g_1^{-1}\ g_2(\tilde{b}_{j+k_2})$ for some $k_1,\ k_2 \in \mathbb{Z}$ by (2.3).

4.8 $L_{f_1}(a,-1)\times L_{g_1}(b,\infty)\simeq L_{f_2}(\tilde{a},-1)\times L_{g_2}(\tilde{b},\infty)$

4.8.1 $f_1\approx f_2\prec g_1\approx g_2$

Consider (4.1), and take $\varphi=\varphi_{f_1}=\exp f_1\, M\, f_1^{-1}\log$. Then we get

$$\beta_1\geq|\{i:\underbrace{\varphi_{f_1}(\frac{c_{i,r+1}}{c_{i,q-1}})\leq\frac{c_{i,q-1}}{c_{i,p}}}_{Q_2^-(f_1)},\ \frac{c_{i,r}}{c_{i,q}}\leq t,\ \frac{c_{i,s}}{c_{i,q}}\geq\tau\}|\ :|I_1|,$$

$$\beta_2\leq|\{i:\underbrace{\varphi_{f_1}(\frac{d_{i,r_1}}{d_{i,\tilde{q}_1+1}})\leq\frac{d_{i,\tilde{q}_1+1}}{d_{i,p_1}}}_{Q_2^-(f_1)},\ \frac{d_{i,r_1}}{d_{i,\tilde{q}_1}}\leq 2ct,\ \frac{d_{i,s_1}}{d_{i,q_1}}\geq\frac{\tau}{2c}\}|\ :=|I_2|.$$

Since $L_g(\alpha,\infty)$ spaces do not have the property $Q_2^-(f)$, we take $i=2k-1$ in I_1, I_2 and see that

$$\beta_1\geq|I_1|\geq M_a\left(\frac{f_1^{-1}(\ln t)}{1+\frac{1}{q}},\ \frac{f_1^{-1}(\ln\tau)}{1+\frac{1}{q+1}}\right),$$

$$\beta_2\leq|I_2|\leq M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}},\ \frac{f_2^{-1}(\ln\frac{\tau}{2c})}{1+\frac{1}{q}}\right),$$

and hence $$M_a\left(\frac{f_1^{-1}(\ln t)}{1+\frac{1}{q}},\ \frac{f_1^{-1}(\ln\tau)}{1+\frac{1}{q+1}}\right)\leq M_{\tilde{a}}\left(\frac{f_2^{-1}(\ln 2ct)}{1+\frac{1}{\tilde{q}_1+1}},\ \frac{f_2^{-1}(\ln\frac{\tau}{2c})}{1+\frac{1}{q}}\right)$$

since $\beta_1\leq\beta_2$.

On the other hand, $\varphi_{g_1}=\exp g_1\,\frac{1}{M}\,g_1^{-1}\log$ in (4.2), we get

$$\beta\geq|\{i:\underbrace{\frac{c_{i,q+1}}{c_{i,p}}\leq\varphi_{g_1}(\frac{c_{i,r}}{c_{i,q+1}})}_{S_1^+(g_1)},\ \frac{c_{i,s}}{c_{i,q}}\leq\tau,\ \frac{c_{i,r}}{c_{i,q}}\geq t\}|\ :=|I|,$$

$$\tilde{\beta}\leq|\{i:\underbrace{\frac{d_{i,q_1-2}}{d_{i,p_1}}\leq\varphi_{g_1}(\frac{d_{i,r_1+1}}{d_{i,q_1-2}})}_{S_1^+(g_1)},\ \frac{d_{i,s_1}}{d_{i,\tilde{q}_1}}\leq 2c\tau,\ \frac{d_{i,r_1}}{d_{i,q_1}}\geq\frac{t}{2c}\}|\ :=|\tilde{I}|. \qquad (4.5)$$

Since $L_f(\alpha,-1)$ spaces do not have the property $S_1^+(g)$, we take $i=2k-1$ in I, $\tilde{I}$, and see that

$$\beta\geq|I|\geq M_b\left(\frac{g_1^{-1}(\ln\tau)}{s},\ \frac{g_1^{-1}(\ln\tau)}{r-1}\right),$$

$$\tilde{\beta}\leq|\tilde{I}|\leq M_{\tilde{b}}\left(\frac{g_2^{-1}(\ln 2c\tau)}{s_1-1},\ \frac{g_2^{-1}(\ln\frac{t}{2c})}{r_1}\right),$$

and hence $$M_b\left(\frac{g_1^{-1}(\ln\tau)}{s}\,,\ \frac{g_1^{-1}(\ln\tau)}{r-1}\right) \le M_{\tilde{b}}\left(\frac{g_2^{-1}(\ln 2c\tau)}{s_1-1}\,,\ \frac{g_2^{-1}(\ln\frac{t}{2c})}{r_1}\right)$$
since $\beta \le \tilde{\beta}$.

Therefore, by symmetry we have

$$M_a \approx M_{f_1^{-1}\, f_2(\tilde{a})} \qquad , \qquad M_b \approx M_{g_1^{-1}\, g_2(\tilde{b})}.$$

As a result we get $a_i \asymp f_1^{-1}\, f_2(\tilde{a}_{i+k_1})$ and $b_j \asymp g_1^{-1}\, g_2(\tilde{b}_{j+k_2})$ for some $k_1,\ k_2 \in \mathbb{Z}$ by (2.3).

Chapter 5

Case of tensor product

In this chapter, we tried to get the sufficient condition for the isomorphism $L_f(a,\infty)\hat{\otimes}E\prime_\infty(b) \simeq L_f(\tilde{a},\infty)\hat{\otimes}E\prime_\infty(\tilde{b})$ by a method similar to the method in [7]. In [7] they showed that $E_\infty(a)\hat{\otimes}E\prime_\infty(b) \simeq E_\infty(\tilde{a})\hat{\otimes}E\prime_\infty(\tilde{b})$ if and only if a and $\tilde{a}$, b and $\tilde{b}$ are identical in lacunarity respectively. Here, we tried to extend this result, but could obtain only sufficiency.

We shall use the following characteristic of lacunarity:

$$n_a(t,\tau) = \begin{cases} 1 & \text{if } M_a(t,\tau) > 0 \\ 0 & \text{otherwise} \end{cases}$$

for a given sequence $a = (a_i)$, $a_i \geq 1$.

For an arbitrary sequences $a = (a_i)$ and $b = (b_i)$, we say a and b have the same lacunarities or *are identical in lacunarity*, if

$$n_a(t,\tau) \leq n_b(Ct, \tfrac{\tau}{C}), \quad n_b(t,\tau) \leq n_a(Ct, \tfrac{\tau}{C}), \quad 1 \leq \tau \leq t < \infty$$

or, respectively,

$$m_a(t) \leq m_b(Ct), \quad m_b(t) \leq m_a(Ct), \quad t \leq 1$$

for a constant $C > 0$.

We shall use the following characteristics

$$\begin{aligned} \beta(W_1, W_2, W_3, W_4) &:= \beta(\overline{\Gamma}(W_1 \cup W_2, W_3 \cap W_4) \\ \beta(W_1, W_2, W_3) &:= \beta(W_1, W_2, W_3, W_2) \end{aligned}$$

for a given quadruple of absolutely convex sets W_i, $i = 1,2,3,4$. It is an immediate consequence of the definition that if $W_1 \subset V_1$, $W_2 \subset V_2$, $V_3 \subset W_3$, $V_4 \subset W_4$, then

$$\beta(V_1, V_2, V_3, V_4) \leq \beta(W_1, W_2, W_3, W_4).$$

Consider now $X = L_f(a,\infty)\widehat{\otimes}E\ell_\infty(b)$, $y = L_f(\tilde{a},\infty)\widehat{\otimes}E\ell_\infty(\tilde{b})$ where f is rapidly increasing. Let $a^{(\nu)} = (a_{ij}^{(\nu)})$, $b^{(\nu)} = (b_{kl}^{(\nu)})$ be the weighted sequences corresponding to sets A_{p_ν,q_ν} and B_{r_ν,s_ν} where

$$A_{p,q} = \{x \in X : \sum |\xi_{ij}| e^{pa_i - qb_j} \leq 1\}$$
$$B_{p,q} = \{x \in X : \sum |\eta_{kl}| e^{p\tilde{a}_k - q\tilde{b}_l} \leq 1\}$$

$$a_{ij}^{(\nu)} = e^{f(p_\nu a_i) - q_\nu b_j}, \quad (i,j) \in \mathbb{N}^2, \ \nu = 1,2,3,4,5,6,7,$$
$$b_{kl}^{(\nu)} = e^{f(r_\nu \tilde{a}_k) - s_\nu \tilde{b}_l}, \quad (k,l) \in \mathbb{N}^2, \ \nu = 1,2,3,4,5,6,7,8.$$

Proposition 5.1 *Let $a^{(p)} = (a_{ip})_{i\in I}$, $p = 1,2,3,4$. Then*

(i) $\beta(B^e(a^{(1)}), B^e(a^{(2)}), B^e(a^{(3)}), B^e(a^{(2)})) \geq \left|\{i : \frac{a_{i3}}{a_{i2}} \leq 1, \frac{a_{i2}}{a_{i1}} \leq 1\}\right|,$

(ii) $\beta(B^e(a^{(1)}), B^e(a^{(2)}), B^e(a^{(3)}), B^e(a^{(4)})) \leq \left|\{i : \frac{a_{i3}}{a_{i2}} \leq 2, \frac{a_{i4}}{a_{i1}} \leq 2\}\right|.$

So, for

$$\begin{aligned}
W_1 &= \overline{\Gamma}((A_{p_2,q_2})^{1/2}(A_{p_7,q_7})^{1/2} \cup e^\tau A_{p_7,q_7}),\\
W_2 &= W_4 = A_{p_4,q_4},\\
W_3 &= (A_{p_1,q_1})^{1/2}(A_{p_6,q_6})^{1/2} \cap e^t A_{p_6,q_6},\\
V_1 &= \overline{\Gamma}(\frac{1}{\sqrt{c_2c_7}}(B_{r_3,s_3})^{1/2}(B_{r_8,s_8})^{1/2} \cup (\frac{1}{c_7}e^\tau B_{r_8,s_8})),\\
V_2 &= \frac{1}{c_4}B_{r_5,s_5},\\
V_3 &= \sqrt{c_1c_6}(B_{r_1,s_1})^{1/2}(B_{r_6,s_6})^{1/2} \cap (c_6e^t B_{r_6,s_6}),\\
V_4 &= c_4 B_{r_4,s_4}
\end{aligned}$$

we define

$$\beta_P(t,\tau) := \beta(W_1, W_2, W_3, W_2), \quad P = (p_1, q_1; \ \dots ; p_7, q_7),$$
$$\beta_R(t,\tau) := \beta(V_1, V_2, V_3, V_4), \quad R = (r_1, s_1 : \ \dots ; r_8, s_8)$$

and we see that

$$\begin{aligned}
\beta_P(t,\tau) \;\geq\; & |\{(i,j): \frac{f(p_1a_i)+f(p_6a_i)}{2} - f(p_4a_i) - (\frac{q_1+q_6}{2} - q_4)b_j \leq 0, \\
& \frac{f(p_2a_i)+f(p_7a_i)}{2} - f(p_4a_i) - (\frac{q_2+q_7}{2} - q_4)b_j \geq 0, \\
& f(p_6a_i) - f(p_4a_i) - (q_4 - q_6)b_j \leq t, \\
& f(p_7a_i) - f(p_4a_i) + (q_4 - q_7)b_j \geq \tau\}|, \\
\beta_R(t,\tau) \;\leq\; & |\{(k,l): \frac{f(r_1\tilde{a}_k)+f(r_6\tilde{a}_k)}{2} - f(r_5\tilde{a}_k) - (\frac{s_1+s_6}{2} - s_5)\tilde{b}_l \leq \ln c_4\sqrt{c_1c_6}, \\
& \frac{f(r_3\tilde{a}_k)+f(r_8\tilde{a}_k)}{2} - f(r_4\tilde{a}_k) - (\frac{s_3-s_8}{2} - s_4)\tilde{b}_l \geq -\ln c_4\sqrt{c_2c_7}, \\
& f(r_6\tilde{a}_k) - f(r_5\tilde{a}_k) + (s_5 - s_6) \leq t + \ln c_4c_6, \\
& f(r_8\tilde{a}_k) - f(f_4\tilde{a}_k) + (s_4 - s_8) \geq \tau - \ln c_2c_7\}|.
\end{aligned}$$

Hence, for $p_6 < K < p_7$,

$$\frac{f(p_1a_i)+f(p_6a_i)}{2} - f(p_4a_i) \leq f(Ka_i) \leq \frac{f(p_2a_i)+f(p_7a_i)}{2} - f(p_4a_i),$$

$$f(p_6a_i) - f(p_4a_i) \leq f(Ka_i) \leq f(p_7a_i),$$

$$\frac{q_2-q_7}{2} - q_4 \leq q_2,\ \frac{q_1+q_6}{2} - q_4 \geq \frac{q_1}{4},\ q_4 - q_7 \geq \frac{q_4}{2} \geq \frac{p_8}{2},$$

$$f(\frac{r_6}{4}\tilde{a}_k) \leq \frac{f(r_1\tilde{a}_k)+f(r_6\tilde{a}_k)}{2} - f(r_5\tilde{a}_k),\ \frac{f(r_3\tilde{a}_k)+f(r_8\tilde{a}_k)}{2} - f(r_4\tilde{a}_k) \leq F(r_8\tilde{a}_k),$$

$$\frac{s_1+s_6}{2} - s_5 \leq s_1,\ \frac{s_3+s_8}{2} - s_4 \geq \frac{s_3}{2},\ f(r_6\tilde{a}_k) - f(r_5\tilde{a}_k) \geq f(\frac{r_6}{4}\tilde{a}_k),$$

$$f(r_8\tilde{a}_k) - f(r_4\tilde{a}_k) \leq f(r_8\tilde{a}_k),\ \frac{r_6}{2} \leq \frac{s_5}{2} \leq s_5 - s_6 \leq s_5,\ \frac{s_4}{2} \leq s_4 - s_8 \leq s_4,$$

we see that

$$\begin{aligned}
\beta_P(t,\tau) \;\geq\; & |\{(i,j): \frac{4}{q_1} \leq \frac{b_j}{f(Ka_i)+b_j} \leq \frac{1}{2q_2}, \\
& \tau \leq f(Ka_i) + b_j \leq \frac{t}{q_4}\}|, \\
\beta_R(t,\tau) \;\leq\; & |\{(k,l): \frac{1}{4s_1} \leq \frac{\tilde{b}_l}{f(\frac{r_6}{4}\tilde{a}_k)+\tilde{b}_l},\ \frac{\tilde{b}_l}{f(r_8\tilde{a}_k)+\tilde{b}_l} \leq \frac{4}{s_3}, \\
& f(r_8\tilde{a}_k) + \tilde{b}_l \geq \frac{\tau}{2s_4\ln c_2c_7},\ f(\frac{r_6}{4}\tilde{a}_k) + \tilde{b}_l \leq (\ln c_4c_6 + 1)t\}|.
\end{aligned}$$

Therefore, since $\beta_P(t,\tau) \leq \beta_R(t,\tau)$ we get the following ;

Proposition 5.2 *Let* $X = L_f(a,\infty)\hat{\otimes}E\ell_\infty(b)$, $Y = L_f(\tilde{a},\infty)\hat{\otimes}E\ell_\infty(\tilde{b})$ *and* $T : Y \to X$ *be an isomorphism.Let* $r_1 < p_1 < \ldots < r_8 < p_8 < q_8 < s_8 < \ldots < q_1 < s_1$ *be a fixed set of indices. Let* $p_6 < K < p_7$ *and* $\Delta = \max\{2s_4 \ln c_2c_7,\ q_4 \ln(c_4c_6+1)\}$. *Then,* $\exists\Delta\ \forall\tilde{\varepsilon}\ \exists\varepsilon\ \forall\delta\ \exists\tilde{\delta}$ *such that*

$$\begin{aligned}
|I| &:= |\{(i,j) : \delta \le \frac{b_j}{f(Ka_i)+b_j} \le \varepsilon,\ \tau \le f(Ka_i)+b_j \le t\}| \\
&\le |\{(k,l) : \tilde{\delta} \le \frac{\tilde{b}_l}{f(\frac{r_6}{4}\tilde{a}_k)+\tilde{b}_l},\ \frac{\tilde{b}_l}{f(r_8\tilde{a}_k)+\tilde{b}_l} \le \tilde{\varepsilon}, \\
&\qquad f(r_8\tilde{a}_k)+\tilde{b}_l \ge \frac{\tau}{\Delta},\ f(\frac{r_6}{4}\tilde{a}_k)+\tilde{b}_l \le \Delta t\}| =: |\tilde{I}|.
\end{aligned}$$

Now what we actually want to prove is the following :

Claim: Let $X = L_f(a,\infty)\hat{\otimes}E\ell_\infty(b)$, $Y = L_f(\tilde{a},\infty)\hat{\otimes}E\ell_\infty(\tilde{b})$ and $T : Y \to X$ be an isomorphism. Then a and $\tilde{a}$ are identical in lacunarity to b and $\tilde{b}$ respectively.

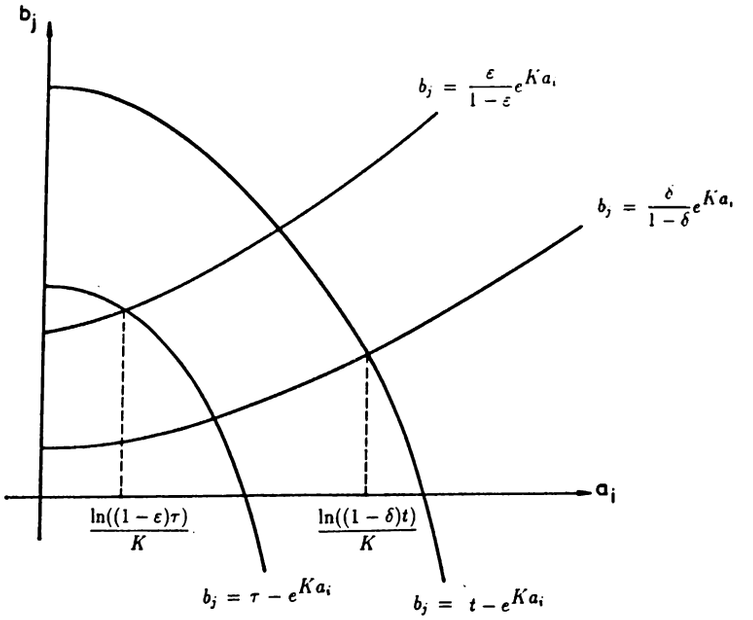
However, in our attempt to prove this we encountered some tecnical problems and so, we consider the special case $f(x) = e^x$.

Let us assume $X = L_f(a,\infty)\hat{\otimes}E\ell_\infty(b) \simeq L_f(\tilde{a},\infty)\hat{\otimes}E\ell_\infty(b) = Y$ where $f(x) = e^x$.

By the assumption and by the proposition (5.2), for $p_6 < K < p_7$, $\exists\Delta\ \forall\tilde{\varepsilon}\ \exists\varepsilon\ \forall\delta\ \exists\tilde{\delta}$ such that

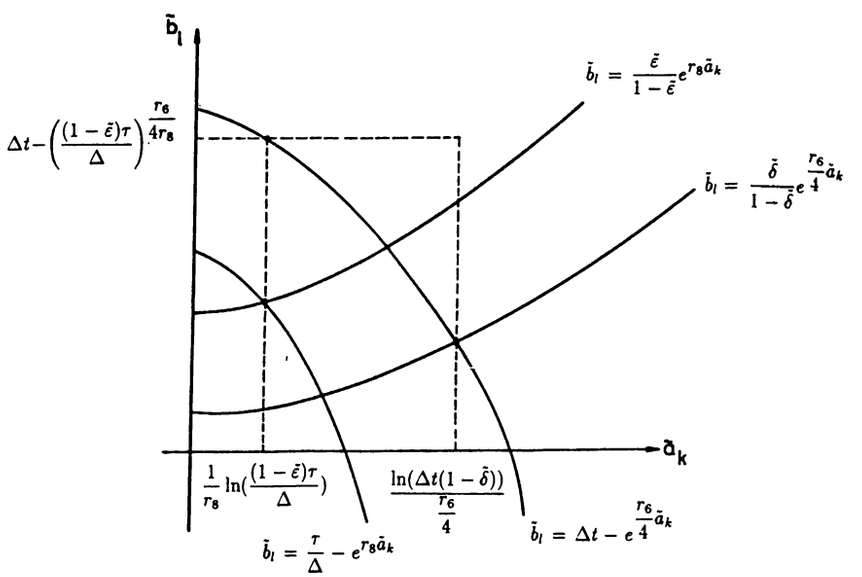
$$\begin{aligned}
|I| &= |\{(i,j) : \delta \le \frac{b_j}{e^{Ka_i}+b_j} \le \varepsilon,\ \tau \le e^{Ka_i}+b_j \le t\}| \\
&\le |\{(k,l) : \tilde{\delta} \le \frac{\tilde{b}_l}{e^{\frac{r_6}{4}\tilde{a}_k}+\tilde{b}_l},\ \frac{\tilde{b}_l}{e^{r_8\tilde{a}_k}+\tilde{b}_l} \le \tilde{\varepsilon}, \\
&\qquad e^{r_8\tilde{a}_k}+\tilde{b}_l \ge \frac{\tau}{\Delta},\ e^{\frac{r_6}{4}\tilde{a}_k}+\tilde{b}_l \le \Delta t\}| = |\tilde{I}|.
\end{aligned}$$

This inequality shows us that $|I|$ is approximately equal to the area between the curves $b_j = \frac{\delta}{1-\delta}e^{Ka_i}$, $b_j = \frac{\varepsilon}{1-\varepsilon}e^{Ka_i}$, $b_j = \tau - e^{Ka_i}$, $b_j = t - e^{Ka_i}$ in $a_i - b_j$ plane.



So $|I| \geq m_a\left(\frac{\ln((1-\delta)t)}{K}\right) - m_a\left(\frac{\ln((1-\varepsilon)\tau)}{K}\right)$.

Similarly, $|\tilde{I}|$ is approximately equal to the area between the curves $\tilde{b}_l = \frac{\tilde{\delta}}{1-\tilde{\delta}}e^{\frac{r_6}{4}\tilde{a}_k}$, $\tilde{b}_l = \frac{\tilde{\varepsilon}}{1-\tilde{\varepsilon}}e^{r_8\tilde{a}_k}$, $\tilde{b}_l = \frac{\tau}{\Delta} - e^{r_8\tilde{a}_k}$, $\tilde{b}_l = \Delta t - e^{\frac{r_6}{4}\tilde{a}_k}$ in $\tilde{a}_k - \tilde{b}_l$ plane.



So, $|\tilde{I}| \leq (\Delta t - \left(\frac{(1-\tilde{\varepsilon})\tau}{\Delta}\right)^{\frac{r_6}{4r_8}})\left[m_{\tilde{a}}\left(\frac{\ln(\Delta t(1-\tilde{\delta}))}{\frac{r_6}{4}}\right) - m_{\tilde{a}}\left(\frac{1}{r_8}\ln(\frac{(1-\tilde{\varepsilon})\tau}{\Delta})\right)\right]$.

Since we can obtain the symmetrical inequality by the same way, we

therefore have that a and $\tilde{a}$ are identical in lacunarity .$_\bullet$

In order to try to get this result for an arbitrary f we met with a tecnical problem for the estimation of coordinates of bigger rectangle. For example, to estimate the height of the rectangle we need to solve the equation $\frac{\tilde{\varepsilon}}{1-\tilde{\varepsilon}} f(r_8 \tilde{a}_k) = \Delta t - f(\frac{r_6}{4} \tilde{a}_k)$ for $\tilde{a}_k$ which is not possible. But we belive that by some approximation methods some points can be found very close to desired one approximately and then the same idea above can be followed.

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