GEOMETRIC CHARACTERIZATION OF EXTENSION PROPERTY FOR MODEL COMPACT SETS

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Muhammed Altun
September 2000

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

GEOMETRIC CHARACTERIZATION OF EXTENSION PROPERTY FOR MODEL COMPACT SETS

Muhammed Altun
M. S. in Mathematics
Advisor: Assist. Prof. Alexander Goncharov
September 2000

In this work we examined the existence of a linear continuous extension operator for the space of Whitney functions given on subsets of the whole space. We studied the linear topological invariants, especially an invariant which topologically characterizes the existence of an extension operator. Finally, we gave necessary and sufficient conditions for the existence of an extension operator on some special type compact sets.

Keywords and Phrases: Fréchet spaces, Extension operator, Whitney functions, Linear Topological Invariants.

ÖZET

BAZI MODEL KOMPAKT KÜMELER İÇİN GENİŞLETME ÖZELLİĞİNİN GEOMETRİK KARAKTERİZASYONU

Muhammed Altun
Matematik Yuksek Lisans
Danışman: Doç. Dr. Alexander Goncharov
Eylül 2000

Bu çalışmada vektörel bir uzayın alt kümelerinde tanımlanmış olan Whitney fonksiyon uzaylarında lineer sürekli bir genişletme operatörünün var olma durumlarını inceledik. Ayrıca lineer topolojik invariantlar üzerinde, özellikle bir genişletme operatörünün var olma durumunu karakterize eden bir invariant üzerinde çalıştık. Son olarak bazı özel kompakt kümelerde bir genişletme operatörünün var olma durumu için yeter ve gerek şartları verdik.

Anahtar Kelimeler ve İfadeler: Fréchet uzayları, Genişletme operatörü, Whitney fonksiyonları, Lineer topolojik invariantlar.

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Contents

Chapter 1

Introduction

The development of differential calculus in the 20^{th} century has its origin in the work of Whitney on differentiable functions. The profound theorems proved during the last fifty years were motivated on the one hand by problems of Laurent Schwartz concerning division of distributions and differentiable functions, and on the other hand by the theory of singularities of differentiable mappings, developed at first by Thom and Whitney. Some of the most fundamental results are due to Schwartz's students Glaeser,Grothendick and Malgrange.

We will begin with an elementary theorem on differentable even functions, which introduces some important techniques and which provides a good illustration of the fundemental problems and the relationships among them.

Let U be an open set of $\mathbb{R}^n$. We denote by $\mathcal{E}^m(U)$ (respectively $\mathcal{E}(U)$) the algebra of m times continuously differentiable (respectively infinitely differentiable) functions in U, with the topology of uniform convergence of functions and all their partial derivatives on compact sets. This is the topology defined by the seminorms

$$|f|_m^K = sup\left\{|\frac{\partial^{|k|}f}{\partial x^k}(x)| : x \in K, |k| \leq m\right\},$$

where K is a compact subset of U (and m runs through $\mathbb{N}$ in the C^∞ case). Here $x = (x_1, ..., x_n)$, k denotes a multiindex $k = (k_1, ..., k_n) \in \mathbb{N}^n, |k| = k_1 + ... + k_n$ and

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} ... \partial x_n^{k_n}}.$$

We will sometimes use m for either a nonnegative integer or $+\infty$ and write

$\mathcal{E}^{+\infty}(U) = \mathcal{E}(U)$

Let $\mathcal{E}^m(\mathbb{R})_{even}$ be the closed subspace of $\mathcal{E}^m(\mathbb{R})$ consisting of even functions ($m \in \mathbb{N}$ or $m = +\infty$)

Theorem 1.1 *If $f(x)$ is a C^{2m} even function of one variable ($m \in \mathbb{N}$ or $m = +\infty$), then there exists a C^m function $g(y)$ such that $f(x) = g(x^2)$. In fact there exists a continuous linear operator $L : \mathcal{E}^{2m}(\mathbb{R})_{even} \longrightarrow \mathcal{E}^m(\mathbb{R})$ such that $f(x) = L(f)(x^2)$ for all $f \in \mathcal{E}^{2m}(\mathbb{R})_{even}$*

The first assertion is due to Whitney [25]. The second follows from the theorem of Seeley [20]. It will be clear that an analogous result holds for functions of several variables that are even in some of them.

The proof of the theorem can be given by using the following elemantary but important lemma.

Lemma 1.2 *(Hadamard's lemma) If $f(x) = f(x_1, ..., x_n, x_{n+1}, ..., x_p)$ is a C^m function such that*

$$f(0, 0, ..., 0, x_{n+1}, ..., x_p) = 0$$

then there exists C^{m-1} functions $g_i(x_1, ..., x_p), 1 \leq i \leq n$, such that

$$f(x) = \sum_{i=1}^{n} x_i g_i(x)$$

Proof: By the fundamental theorem of calculus and the chain rule, we have

$$f(x) = \int_0^1 \frac{\partial f(tx_1, ..., tx_n, x_{n+1}, ..., x_p)}{\partial t} dt = \sum_{i=1}^{n} x_i g_i(x)$$

where

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, ..., tx_n, x_{n+1}, ..., x_p) dt$$

It is clear that the g_i defined in the proof of Lemma 1.2 depend in a continuous linear way on f.

Hadamard's lemma is a very simple type of division theorem for differentiable functions. In the C^∞ case, the assertion of the lemma is equivalent to the statement that the ideal in $\mathcal{E}(\mathbb{R}^p)$ generated by $x_1, ..., x_n$ is closed. Malgrange [14] proved that if U is an open subset of $\mathbb{R}^n$, then any ideal I in $\mathcal{E}(U)$ which is generated by finitely many analytic functions is closed. Malgrange's theorem has a more concrete formulation: a C^∞ function f on U belongs to I if and only if it "belongs formally to I". "Belongs formally to I" means that the formal Taylor series of f at each point of U is the formal Taylor series of some element of I. In fact according to Whitney's spectral theorem [26], the closure of any ideal I in $\mathcal{E}(U)$ equals the ideal of C^∞ functions which belong formally to I.

Proof of Theorem 1.1: Let $f(x)$ be a C^{2m} even function. There is a unique continuous function $g(y)$ defined in $[0,\infty)$ such that g is C^{2m} in $[0,\infty)$ and $f(x) = g(x^2)$. If $x \neq 0$, we have

$$\frac{dg^{(k)}(x^2)}{dx} = 2xg^{(k+1)}(x^2) \quad 0 \leq k < 2m$$

On the other hand we can use Hadamard's lemma to define $C^{2(m-k)}$ even functions h_k inductively as follows:

$$\begin{aligned} h_0 &= f \\ h'_k &= 2xh_{k+1}, \quad 0 \leq k < m \end{aligned}$$

It follows that $h_k(x) = g^{(k)}(x^2)$ outside the origin, so that each derivative $g^{(k)}, 0 \leq k \leq m$ can be continued up to the origin. We will prove that g is the restriction to $[0,\infty)$ of a C^m function defined on $\mathbb{R}$.

The problem of extending g to a differentiable function is a very special instance of Whitney's extension problem: When is a function f, defined in a closed subset X of $\mathbb{R}^n$, the restriction of a C^m function in $\mathbb{R}^n$? ([27],[28]). In fact we want to extend g in a continuous linear way. The existence of such an extension in the C^∞ case was first proved by Mityagin [17] and Seeley [20].

Let $\mathcal{E}^m([0,\infty))$ denote the space of continuous functions g in $[0,\infty)$ such that g is C^m in $(0,\infty)$ and all derivatives of $g|(0,\infty)$ extend continuously to $[0,\infty)$. Then $\mathcal{E}^m([0,\infty))$ has the structure of a Frechet space defined by the seminorms

$$|g|_m^K = \sup\{|g^k(y)| : y \in K, |k| \leq m\},$$

where K is a compact subset of $[0,\infty)$ (and m runs through $\mathbb{N}$ in the C^∞ case), and where g^k denotes the continuation of $(d^k/dy^k)(g|(0,\infty))$ to $[0,\infty)$.

The following theorem completes the proof of theorem 1.1.

Theorem 1.3 *There is a continuous linear extension operator*

$$E : \mathcal{E}^m([0,\infty)) \longrightarrow \mathcal{E}^m(\mathbb{R})$$

such that $E(g)|[0,\infty) = g$ *for all* $g \in \mathcal{E}^m([0,\infty))$.

Proof: Our problem is to define $E(g)(y)$ when $y < 0$. If $m = 0$ we can define $E(g)(y)$ by reflection in the origin : $E(g)(y) = g(-y), y < 0$. If $m = 1$ we can use a weighted sum of reflections. Consider

$$E(g)(y) = a_1 g(b_1 y) + a_2 g(b_2 y), \quad y < 0$$

Where $b_1, b_2 < 0$. Then $E(g)$ determines a C^1 extension of g provided that the limiting values of $E(g)(y)$ and $E(g)'(y)$ agree with those of $g(-y)$ and $g'(-y)$ as $y \longrightarrow 0-$; in other words if

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_1 b_1 + a_2 b_2 &= 1 \end{aligned}$$

For distinct $b_1, b_2 < 0$ these equations have a unique solution a_1, a_2.This extension is due to Lichtenstein [13].

Hestenes [11] remarked that the same technique works for any $m < \infty$: a weighted sum of m reflections leads to solving a system of linear equations determined by a Vandermonde matrix.

If $m = \infty$, we can use an infinite sum of reflections [20]:

$$E(g)(y) = \sum_{k=1}^{\infty} a_k \phi(b_k y) g(b_k y), \quad y < 0,$$

where $\{a_k\}, \{b_k\}$ are sequences satisfying

$$\begin{array}{ll} (1) & b_k < 0, b_k \longrightarrow -\infty \quad \text{as} \quad k \longrightarrow \infty; \\ (2) & \sum_{k=1}^{\infty} |a_k||b_k|^n < \infty \quad \text{for all} \quad n \geq 0; \\ (3) & \sum_{k=1}^{\infty} a_k b_k^n = 1 \quad \text{for all} \quad n \geq 0 \end{array}$$

and ϕ is a C^∞ function such that $\phi(y) = 1$ if $0 \leq y \leq 1$ and $\phi(y) = 0$ if $y \geq 2$. In fact condition (1) guarantees that the sum is finite for each $y < 0$. Condition (2) shows that all derivatives converge as $y \longrightarrow 0-$,uniformly in each bounded set, and (3) shows that the limits agree with those of the derivatives of $g(y)$ as $y \longrightarrow 0+$. The continuity of the extension operator also follows from (2).

It is easy to choose sequences $\{a_k\}, \{b_k\}$ satisfying the above conditions. We can take $b_k = -2^k$ and choose a_k using a theorem of Mittag Leffler : there exists an entire function $\sum_{k=1}^{\infty} a_k z^k$ taking arbitrary values (here $(-1)^n$) for a sequence of distinct points (here 2^n) provided that the sequence does not have a finite accumulation point.

It is clear that Seeley's extension operator actually provides a simultaneous extension of all classes of differentiability.

In this article we will be concerned mainly with C^∞ functions. Whitney's theorem on even functions in the C^∞ case is equivalent to the statement that the subalgebra of $\mathcal{E}(\mathbb{R})$ of functions of the form $g(x^2)$ is closed.

1.1 Whitney's Extension theorem

In this section we will examine the classical extension theorem of Whitney [27]. Let U be an open subset of $\mathbb{R}^n$, and X a closed subset of U. Whitney's theorem asserts that a function F^0 defined in X is the restriction of a C^m function in U ($m \in \mathbb{N}$ or $m = +\infty$) provided there exists a sequence $(F^k)_{|k| \le m}$ of functions defined in X which satisfies certain conditions that arise naturally from Taylor's formula.

First we consider $m \in \mathbb{N}$. By a jet of order m on X we mean a set of continuous functions $F = (F^k)_{|k| \le m}$ on X.Here k denotes a multiindex $k = (k_1, ..., k_n) \in \mathbb{N}^n$. Let $J^m(X)$ be the vector space of jets of order m on X. We write

$$|F|_m^K = \sup\{|F^k(x)| : x \in K, |k| \le m\}$$

if K is a compact subset of X, and $F(x) = F^0(x)$.

There is a linear mapping $J^m : \mathcal{E}^m(U) \longrightarrow J^m(X)$ which associates to each $f \in \mathcal{E}^m(U)$ the jet

$$J^m(f) = \left(\frac{\partial^{|k|} f}{\partial x^k} \bigg| X \right)_{|k| \le m}$$

For each k with $|k| \le m$, there is a linear mapping $D^k : J^m(X) \longrightarrow J^{m-|k|}(X)$ defined by $D^k F = (F^{k+l})_{|l| \le m-|k|}$. We also denote by D^k the mapping of $\mathcal{E}^m(U)$ into $\mathcal{E}^{m-|k|}(U)$ given by

$$D^k f = \frac{\partial^{|k|} f}{\partial x^k}$$

This should cause no confusion since

$$D^k \circ J^m = J^{m-|k|} \circ D^k$$

If $a \in X$ and $F \in J^m(X)$, then the *Taylor polynomial (of order m)* of F at a is the polynomial

$$T_a^m F(x) = \sum_{|k| \leq m} \frac{F^k(a)}{k!}(x-a)^k$$

of degree $\leq m$. Here $k! = k_1!...k_n!$. We define $R_a^m F = F - J^m(T_a^m F)$, so that

$$(R_a^m F)^k(x) = F^k(x) - \sum_{|l| \leq m-|k|} \frac{F^{k+l}(a)}{l!}.(x-a)^l$$

if $|k| \leq m$.

Definition 1.4 *A jet $F \in J^m(X)$ is a* Whitney field of class C^m *on X if for each $|k| \leq m$*

$$(R_x^m F)^k(y) = o(|x-y|^{m-|k|}) \tag{1.1}$$

as $|x-y| \longrightarrow 0$, $x, y \in X$.

Let $\mathcal{E}^m(X) \subset J^m(X)$ be the subspace of Whitney fields of class C^m. $\mathcal{E}^m(X)$ is a Frechet space with the seminorms

$$\|F\|_m^K = |F|_m^K + \sup\left\{\frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |k| \leq m\right\},$$

where $K \subset X$ is compact.

There are two more type of norms used to identify the topology in $\mathcal{E}^m(X)$, where one of them is:

$$\|F\|_m^K = |F|_m^K + \sup\left\{\sum_{|k| \leq m} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y\right\},$$

and the other is

$$\|F\|_m^K = max\left\{|F|_m^K, \sup\left\{\frac{|R_x^{m-|k|} F^k(y)|}{|x-y|^{m-|k|}} : x, y \in K, x \neq y, |k| \leq m\right\}\right\}.$$

It is easy to see that topologies given by these system of norms are equivalent.

Remark 1.5 *If $F \in J^m(U)$ and for all $x \in U, |k| \leq m$ we have*

$$\lim_{y \longrightarrow x} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} = 0$$

then there exists $f \in \mathcal{E}^m(U)$ such that $F = J^m(f)$. This simple converse of Taylor's theorem shows that the two spaces we have denoted by $\mathcal{E}^m(U)$ are equivalent. On $\mathcal{E}^m(U)$, the topologies defined by the seminorms $|.|_m^K, \|.\|_m^K$ are equivalent (by the open mapping theorem).

Theorem 1.6 *(Whitney [27]) There is a continuous linear mapping*

$$W : \mathcal{E}^m(X) \longrightarrow \mathcal{E}^m(U)$$

such that $D^k W(F)(x) = F^k(x)$ if $F \in \mathcal{E}^m(X)$, $x \in X$, $|k| \leq m$, and $W(F)\Big|(U - X)$ is C^∞.

Remark 1.7 *The condition (1.1) cannot be weakened to :*

$$\lim_{y \longrightarrow x} \frac{|(R_x^m F)^k(y)|}{|x-y|^{m-|k|}} = 0 \tag{1.2}$$

for all $x \in X$, $|k| \leq m$.

For example let A be the set of points (using one variable) $x = 0, 1/2^s$ and $1/2^s + 1/2^{2s}$ $(s = 1, 2, ...)$. Set $f(x) = 0$ at $x = 0$ and $1/2^s$ and $f(x) = 1/2^{2s}$ at $x = 1/2^s + 1/2^{2s}$.Set $f^1(x) \equiv 0$ in A. The above condition is satisfied but there's no extension of $f(x)$ which has continuous first derivative.

For K a closed subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$. Whitney's extension theorem [27] gives an extension operator (a linear continuous extension operator) from the space $\mathcal{E}^m(K)$ of Whitney jets on K to the space $C^m(\mathbb{R}^n)$. In the case $m = \infty$ such an operator does not exist in general.

Definition 1.8 *For $K \subset \mathbb{R}^n$, K has the* Extension property *if there exists a linear continuous extension operator $L : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^n)$.*

An example for a compact set which does not have the extension property is the set $K = \{0\} \subset \mathbb{R}$. To prove this fact assume that there exists such a continuous extension operator L for $K = \{0\}$. Hence we have

$$\forall p \ \exists q, C \ : \ \|LF\|_p \leq C\|F\|_q \ \forall F \in \mathcal{E}(K).$$

Let $p = 0$, then we have q, C satisfying $\|LF\|_0 \le C\|F\|_q \ \forall F \in \mathcal{E}(K)$.

Let $F = (F_i)_{i=0}^{\infty}$ with $F_{q+1} = 1$ and $F_i = 0$ for all $i \neq q+1$.

It is easy to see that $\|F\|_q = 0$.

But of course $LF \neq 0$ since $LF^{(q+1)}(0) \neq 0$.

Then we get $0 < \|LF\|_0 \le C\|F\|_q = 0$ which is a contradiction.

We can similarly prove that $K = \{0\} \cup [a, b] \subset \mathbb{R} \ \ 0 < a < b$ also does not have the extension property. Generalizing this, it is easy to see that if $K \subset \mathbb{R}^n$ has isolated points then K has no extension property.

1.2 Linear Topological Invariants

In this section we will introduce Fréchet spaces, Köthe spaces and linear topological invariants. We will denote by $\mathbb{K}$ either of the fields $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.9 *A $\mathbb{K}$-vector space F, endowed with a metric, is called* metric linear space, *if in F addition is uniformly continuous and scalar multiplication is continuous.*

A metric linear space F is said to be locally convex *if for each $a \in F$ and each neighborhood V of a there exists a convex neighborhood U of a with $U \subset V$.*

A complete, metric, locally convex space is called a Fréchet space.

Every normed space is a metric linear space and every Banach space is a Fréchet space; however there are Fréchet spaces which are not Banach spaces. The next lemma gives an example of a Fréchet space which is not Banach.The proof can be found in [16] Lemma 5.17.

Lemma 1.10 *Let $(E_n, \|.\|_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. A metric is defined on $E = \prod_{n\in\mathbb{N}} E_n$ by*

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|x_n - y_n\|_n}{1 + \|x_n - y_n\|_n}, \quad x = (x_n)_{n\in\mathbb{N}},\ y = (y_n)_{n\in\mathbb{N}} \ \in E.$$

Then (E, d) is a Fréchet space. (E, d) is not a Banach space if $E_n \neq \{0\}$ for infinitely many $n \in \mathbb{N}$.

Using this lemma it is easy to see that $C(U), C^\infty(U)$ are Fréchet spaces for U an open subset of $\mathbb{R}^n$, and the space of analytic functions on U which we denote by $A(U)$ is a Fréchet space when U is an open subset of $\mathbb{C}$.

$C^\infty(U)$ for U an open subset of $\mathbb{R}^n$,$C^\infty(\overline{U})$-the space of infinitely differentiable functions on an open bounded domain U which are uniformly continuous with all their derivatives, $\mathcal{E}(K)$ for K a compact subset of $\mathbb{R}^n$ and $A(U)$ for U an open domain in $\mathbb{C}^n$ are typical examples of Fréchet spaces.

We now give a simple but useful property of Fréchet spaces by the following proposition:

Proposition 1.11 *For every Fréchet space E and each closed subspace F of E, the spaces F and E/F are Fréchet spaces.*

Definition 1.12 *Let E be a locally convex space. A collection $\mathcal{U}$ of zero neighborhoods in E is called a* fundamental system of zero neighborhoods, *if for every zero neighborhood U there exists a $V \in \mathcal{U}$ and an $\epsilon > 0$ with $\epsilon V \subset U$.*

A family $(\|.\|_\alpha)_{\alpha\in A}$ of continuous seminorms on E is called a fundamental system of seminorms, *if the sets*

$$U_\alpha := \{x \in E : \|x\|_\alpha < 1\}, \quad \alpha \in A,$$

form a fundamental system of zero neighborhoods.

Notation 1.13 *Let E be a locally convex space which has a countable fundamental system of seminorms $(\|.\|_n)_{n\in\mathbb{N}}$. By passing over to $(max_{1\le j\le n}\|\cdot\|_j)_{n\in\mathbb{N}}$ one may assume that*

$$\|x\|_n \le \|x\|_{n+1} \forall x \in E, n \in \mathbb{N}$$

holds. We call $(\|.\|_n)_{n\in\mathbb{N}}$ an increasing fundamental system.

Definition 1.14 *A sequence $(e_j)_{j\in\mathbb{N}}$ in a locally convex space E is called a* Schauder basis *of E, if for each $x \in E$, there is a uniquely determined sequence $(\xi_j(x))_{j\in\mathbb{N}}$ in $\mathbb{K}$, for which $x = \sum_{j=1}^\infty \xi_j(x)e_j$ is true. The maps $\xi_j : E \longrightarrow \mathbb{K}, \quad j \in \mathbb{N}$, are called the* coefficient functionals *of the Schauder basis $(e_j)_{j\in\mathbb{N}}$. They are linear by the uniqueness stipulations.*

A Schauder basis $(e_j)_{j\in\mathbb{N}}$ *of* E *is called an* absolute basis, *if for each continuous seminorm* p *on* E *there is a continuous seminorm* q *on* E *and there is a* $C > 0$ *such that*

$$\sum_{j\in\mathbb{N}} |\xi_j(x)|p(e_j) \leq Cq(x) \quad \forall x \in E.$$

Let $A = (a_{ip})_{i\in I, p\in\mathbb{N}}$ be a matrix of real numbers such that $0 \leq a_{ip} \leq a_{ip+1}$. *Köthe space*, defined by the matrix A, is said to be the locally convex space $K(A)$ of all sequences $\xi = (\xi_i)$ such that

$$|\xi|_p := \sum_{i\in I} a_{ip}|\xi_i| < \infty \quad \forall p \in \mathbb{N}$$

with the topology, generated by the system of seminorms $\{|.|_p, p \in \mathbb{N}\}$. The set of indices I is supposed to be countable, but in general $I \neq \mathbb{N}$. This is convenient for applications, especially when multiple series are considered.

Definition 1.15 *Let* E *and* F *be locally convex spaces ; let us define*

$$L(E, F) := \{A : E \longrightarrow F : A \text{ is linear and continuous } \}$$
$$L(E) := L(E, E) \text{ and } E' := L(E, \mathbb{K})$$

E' *is called the* dual space, *of* E.

A linear map $A : E \longrightarrow F$ *is called an* isomorphism, *if* A *is a homomorphism.* E *and* F *are said to be* isomorphic, *if there exists an isomorphism* A *between* E *and* F. *Then we write* $E \simeq F$.

It is well known that every Fréchet space with absolute basis is isomorphic to some Köthe space. More precisely, If E is a Fréchet space, $\{e_i\}_{i\in I}$ is an absolute basis in E, and $\{\|.\|_p\}_{p\in\mathbb{N}}$ is an increasing sequence of seminorms, generating the topology of E, then E is isomorphic to the Köthe space, defined by the matrix $A = (a_{ip})$, where $a_{ip} = \|e_i\|_p$.

For example the space $C^\infty[-1, 1]$ is isomorphic to the Köthe space $s = K(n^p)$ (see [17]), the space $A(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, is isomorphic to $K(exp(-n/p))$, the space $A(\mathbb{C})$ is isomorphic to $K(exp(pn))$.

It is known ([3],[5],[22],[24],[33]) if the boundary of a domain D is smooth, Lipschitz or even Hölder, then the space $C^\infty(\overline{D})$ is isomorphic to the space s.

To examine whether two given linear topological spaces are isomorphic or not it is useful to deal with some properties of linear topological spaces which are invariant under isomorphisms. More precisely, if Σ is a class of linear topological spaces, Ω is a set with a relation of equivalence $\sim$ and $\Phi : \Sigma \longrightarrow \Omega$ is a mapping, such that

$$X \simeq Y \implies \Phi(X) \sim \Phi(Y)$$

then Φ is called a *Linear Topological Invariant*. We say that the invariant Φ is complete on the class Σ if for any $X, Y \in \Sigma$

$$\Phi(X) \sim \Phi(Y) \Longrightarrow \quad X \simeq Y$$

First linear topological invariants connected with isomorphic classification of Fréchet spaces are due to A.N. Kolmogorov [12] and A. Pelczynski [19]. They introduced linear topological invariants called *approximative dimension* and proved by their help that $A(D)$ is not isomorphic to $A(G)$ if $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$, $m \neq n$ and $A(\mathbb{D}^n)$ is not isomorphic to $A(\mathbb{C}^n)$, where $\mathbb{D}^n$ is the unit polydisc in $\mathbb{C}^n$. Later C. Bessaga, A. Pelczynsky, S. Rolewics [2] and B. Mitiagin [17] considered other linear topological invariants called *diametral dimension*, which turns out to be stronger and more convenient than the approximative dimension. V.Zahariuta [29, 30], introduced some general characteristics as generalizations of Mitiagin's invariants and some new invariants in terms of synthetic neighborhoods [31, 32]. We will give here as an example the invariant β which was used by A. Goncharov and M. Kocatepe [10] based on the Zahariuta's method of synthetic neighborhoods.

Let X be a Fréchet space with a fundamental system of neighborhoods (U_p), and let $t, \tau \in \mathbb{R}_+$. In what follows $t \longrightarrow \infty$ and $\tau = \tau(t) \longrightarrow 0$. Given $0 \leq p < q < r$ we set $\tilde{U} = \tau U_p \cap t U_r$ then

$$\beta(\tau, t : U_p, U_q, U_r) = min\{\dim L : \tilde{U} \subset U_q + L\},$$

where $\min(\Lambda)$ is the minimum of the set Λ. We can see that $\beta(\tau, t) \geq |\{n : d_n(\tilde{U}, U_q) > 1\}|$, where d_n is the Kolmogorov diameter.

Suppose X is a Fréchet space and $(\|.\|_p, p = 1, 2, ...)$ be a system of seminorms generating the topology of X. The following interpolation properties define very important classes of Fréchet spaces. They are invariant under isomorphisms and hence these LTI's are called *Interpolational Invariants*:

$$(DN) \qquad \exists p \forall q \exists r, C : \|x\|_q^2 \le C \|x\|_p \|x\|_r \quad x \in X;$$
$$(\Omega) \qquad \forall p \exists q \forall r \exists \epsilon \exists C : \|x'\|_q^* \le C(\|x'\|_p^*)^\epsilon (\|x'\|_r^*)^{1-\epsilon} \quad x' \in X';$$

Let us note that these notations are due to D.Vogt [16], V. Zahariuta uses the notations D_1, Ω_1 respectively. In this article we will generally use Vogt's notation.

We shall reformulate (DN) in an equivalent way in the following simple propositions.

Proposition 1.16 *A Fréchet space E with an increasing fundamental system $(\|.\|_k)_{k \in \mathbb{N}}$ of seminorms has the property (DN) if and only if the following holds:*

$$\exists p \;\; \forall q \;\; \forall \epsilon > 0 \;\; \exists r, C : \;\; \|x\|_q \le C \|x\|_p^{1-\epsilon} \|x\|_r^\epsilon \tag{1.3}$$

for all $x \in E$.

Proof: For $\epsilon = \frac{1}{2}$ the given condition obviously implies (DN). To prove the converse, let $p \in \mathbb{N}$ be so choosen that $\|.\|_p$ is a dominating norm. If $q \in \mathbb{N}$, $q \ge p$, is given, then we define $r_0 := p$, $r_1 := q$ and iteratively apply (DN) to find $r_{\mu+1} > r_\mu$ and $C_\mu > 0$ such that

$$\|x\|_{r_\mu}^2 \le C_\mu \|x\|_p \|x\|_{r_{\mu+1}} \text{ for all } x \in E.$$

As $\|.\|_p$ is a norm, we have for each $m \in \mathbb{N}$ and all $x \in E$, $x \neq 0$:

$$\left(\frac{\|x\|_q}{\|x\|_p}\right)^m \le \Pi_{\mu=1}^m C_\mu \frac{\|x\|_{r_{\mu+1}}}{\|x\|_{r_\mu}} \le \left(\Pi_{\mu=1}^m C_\mu\right) \frac{\|x\|_{r_{m+1}}}{\|x\|_p}$$

Defining $D_m := \left(\Pi_{\mu=1}^m C_\mu\right)^{1/m}$,it then follows that

$$\|x\|_q \le D_m \|x\|_p^{1-1/m} \|x\|_{r_{m+1}}^{1/m} \text{ for all } x \in E$$

If now $0 < \epsilon < 1$ is given, then we choose $m \in \mathbb{N}$ with $\frac{1}{m} < \epsilon$ and obtain the given condition which holds for $r = r_{m+1}$. If $\epsilon \ge 1$ then the condition trivially holds. □

(1.3) can be stated also as follows :

$$\exists p \;\; \forall q \;\; \forall \epsilon > 0 \;\; \exists r, C : \;\; \|x\|_q^{1+\epsilon} \le C \|x\|_p \|x\|_r^\epsilon \tag{1.4}$$

for all $x \in E$.

Proposition 1.17 *(DN) is equivalent to the following:*

$$\exists p \;\; \forall q \;\; \exists r, C: \;\; \|x\|_q \leq t\|x\|_p + \frac{C}{t}\|x\|_r \quad t > 0 \tag{1.5}$$

Proof: Let (DN) holds. Then we have p as a dominating norm, given $q \in \mathbb{N}$ there exists $r \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_q^2 \leq C\|x\|_p\|x\|_r$$

and by taking the square roots we get

$$\begin{aligned} \|x\|_q &\leq \|x\|_p^{1/2}(C\|x\|_r)^{1/2} = (t\|x\|_p)^{1/2}(\frac{C}{t}\|x\|_r)^{1/2} \quad \forall t > 0 \\ &\leq \frac{1}{2}t\|x\|_p + \frac{1}{2}\frac{C}{t}\|x\|_r \quad \forall t > 0 \\ &\leq t\|x\|_p + \frac{C}{t}\|x\|_r \quad \forall t > 0, \end{aligned}$$

For the proof of the converse take $t^2 = C\frac{\|x\|_r}{\|x\|_p}$, then we get

$$\|x\|_q^2 \leq 4C\|x\|_p\|x\|_r.$$

□

Proposition 1.18 *(1.5) is equivalent to the following:*

$$\exists p \;\; \exists R > 0 \;\; \forall q \;\; \exists r, C: \;\; \|x\|_q \leq t^R\|x\|_p + \frac{C}{t}\|x\|_r \quad t > 0 \tag{1.6}$$

Proof: (1.5) $\Longrightarrow$ (1.6) is trivial. To prove the converse assume we have (1.6) then we have p, R satisfying the condition in (1.6).

Given $q = q_0$, we find $q_{i+1} \geq q_i$ and $C_i > 0$ such that

$$\|x\|_{q_i} \leq t^R\|x\|_p + \frac{C_{i+1}}{t}\|x\|_{q_{i+1}} \quad 0 \leq i \leq R-1$$

Using these R inequalities we get

$$\|x\|_{q_0} \leq (t^R + C_1t^{R-1} + C_1C_2t^{R-2} + ... + C_1...C_{R-1}t)\|x\|_p + \frac{C_1...C_R}{t^R}\|x\|_{q_R}$$

Then there exists $C > C_1...C_R$ such that

$$(t^R+C_1t^{R-1}+C_1C_2t^{R-2}+...+C_1...C_{R-1}t)\|x\|_p+\frac{C_1...C_R}{t^R}\|x\|_{q_R} \leq t^R\|x\|_p+\frac{C}{t^R}\|x\|_{q_R}$$

and hence we have

$$\|x\|_q \leq t\|x\|_p + \frac{C}{t}\|x\|_{q_R} \quad \forall t > 0$$

□

Proposition 1.19 *The following statement is equivalent to DN:*

$$\exists R > 0\ \forall q\ \exists r, C > 0 : |.|_q \ \leq \ t^R |.|_0 + \frac{C}{t} ||.||_r, \ \ t > 0 \tag{1.7}$$

Proof: For the equivalence (1.6) $\Leftrightarrow$ (1.7) see [4] $\square$

1.3 Topological Characterization of Extension Property

Let $(E_i, A_i)_{i\in\mathbb{Z}}$ be a sequence of linear spaces E_i and linear maps $A_i : E_i \longrightarrow E_{i+1}$. The sequence is said to be *exact at the position* E_i in case $R(A_{i-1}) = N(A_i)$. Here R denotes image and N denotes the kernel of the map. The sequence is said to be *exact*, if it is exact at each position. A *short sequence* is a sequence in which at most three successive spaces are different from $\{0\}$. We then write

$$0 \longrightarrow E \longrightarrow^A \ F \longrightarrow^B \ \ G \longrightarrow 0$$

Remark 1.20 *Let F be a Fréchet space and E be a closed subspace of F. Then by Proposition 1.11 , E and F/E are likewise Fréchet spaces. If $j : E \longrightarrow F$ is the inclusion and $q : F \longrightarrow F/E$ is the quotient map, then*

$$0 \longrightarrow E \longrightarrow^j \ F \longrightarrow^q \ \ F/E \longrightarrow 0$$

is a short exact sequence of Fréchet spaces.

Definition 1.21 *A seminorm p on a $\mathbb{K}$-vector space E is called a* Hilbert seminorm, *if there exists a semi-scalar product $\langle .,. \rangle$ on E with $p(x) = \sqrt{\langle x, x \rangle}$ for all $x \in E$.*

A Fréchet-Hilbert space *is a Fréchet space which has a fundamental system of Hilbert seminorms.*

The folowing theorem of D. Vogt from [16] is fundamental in the structure theory of Fréchet spaces.

Theorem 1.22 *(Splitting theorem) Let E, F and G be Fréchet-Hilbert spaces and let*

$$0 \longrightarrow F \longrightarrow^j \ G \longrightarrow^q \ \ E \longrightarrow 0$$

be a short exact sequence with continuous linear maps. If E has the property (DN) and F has the property (Ω), then the sequence splits, ie., q has a continuous linear right inverse and j has a continuous linear left inverse.

M. Tidten used the splitting theorem for the proof of the next theorem which tells that the extension property of K is equivalent to the property (DN) of $\mathcal{E}(K)$.

Theorem 1.23 *[22, Tidten]A compact set K has the extension property iff the space $\mathcal{E}(K)$ has the property (DN).*

Proof: For the proof of the sufficiency part assume that $\mathcal{E}(K)$ has the property (DN) and let L be a cube such that $K \subset L^o$. Now consider the short exact sequence

$$0 \longrightarrow \mathcal{F}(K,L) \longrightarrow^{i} \mathcal{D}(L) \longrightarrow^{q} \mathcal{E}(K) \longrightarrow 0$$

where $\mathcal{D}(L) = C_0^\infty(L)$ is the space of infinitely differentiable functions on L, where the functions and all their derivatives vanish on the boundary of L, and $\mathcal{F}(K,L) = \{f \in \mathcal{D}(L) : f|_K \equiv 0\}$.

By [22] we have that $\mathcal{F}(K,L)$ has property (Ω) $\forall$ compact $K \subset L^o$. Hence we can apply the splitting theorem. This means that there exists an operator ψ, a continuous linear right inverse of q, $\psi : \mathcal{E}(K) \longrightarrow \mathcal{D}(L)$ where obviously $(\psi f)|_K = f$ for $f \in \mathcal{E}(K)$, that is the operator ψ is an extension operator.

On the other hand if there exists an extension operator ψ, then $qo\psi = Id_{\mathcal{E}(K)}$ and ψoq is a continuous projection of $\mathcal{D}(L)$ onto $\mathcal{E}(K)$. We know that $\mathcal{D}(L)$ is isomorphic to s, hence $\mathcal{E}(K)$ is a complemented subspace of s, therefore $\mathcal{E}(K)$ has (DN), since the property (DN) is inherites by subspaces. □

Chapter 2

Review of Previous Results

Whitney's extension theorem provides continuous linear extension operator from the space of C^m Whitney fields ($m < \infty$) on a closed subset X of $\mathbb{R}^n$, to the space of C^m functions on $\mathbb{R}^n$. Though C^∞ Whitney fields on X extend to C^∞ functions on $\mathbb{R}^n$, there does not exist a continuous linear extension operator for every closed subset X. Let $\mathcal{E}(X)$ be the Fréchet space of C^∞ Whitney fields on X. Then $\mathcal{E}(\mathbb{R}^n)$ identifies with the space of C^∞ functions on $\mathbb{R}^n$. The folowing problem arises: Under what conditions on X is there an extension operator $E : \mathcal{E}(X) \longrightarrow \mathcal{E}(\mathbb{R}^n)$? Where we mean by an extension operator, a linear continuous operator such that $E(F)|_X = F$ for all $F \in \mathcal{E}(X)$. Seeley [20] shoved that an extension operator exists if X is a closed half-space $\mathbb{H}^n$. We have described the proof of his theorem in the first chapter.

Mitiagin [17] presented an extension operator for a closed interval in $\mathbb{R}$. Mitiagin in his work proved the fact that the Chebishev Polynomials $T_n(x) = \cos(n\cos^{-1} x)$ form a basis in the space $C^\infty[-1,1]$ ie., for $\Psi(t) \in C^\infty[-1,1]$ and

$$\xi_n = \frac{1}{\pi}\int_{-1}^{1} \frac{\Psi(x)\cos(n\cos^{-1} x)}{\sqrt{1-x^2}}dx$$

we have that

$$\Phi(x) = \sum_{n=0}^{\infty} \xi_n T_n(x) \text{ in } C^\infty[-1,1].$$

It is clear that a linear transformation of the argument sets up an isomorphism between the spaces $C^\infty[-1,1]$ and $C^\infty[a,b]$, $-\infty < a,b < \infty$; therefore the correspondingly transformed Chebishev polynomials form a basis in the space $C^\infty[a,b]$.

Mitiagin constructs in [17] special extensions $\tilde{T}_n$ for the polynomials $T_n(x)$ and defines the operator $M : C^\infty[-1,1] \longrightarrow C^\infty[-2,2]$ by

$$(M\Phi)(x) = \sum_{n=1}^{\infty} \xi_n(x)(\tilde{T}_n)(x)$$

and by using an infinitely differentiable function $l_0(t)$ on the whole straight line such that

$$l_0(t) \equiv 1 \quad |t| \leq 1 \text{ and } l_0(t) \equiv 0 \quad |t| \geq 1 + \frac{1}{4}$$

he defines the operator $M' : C^\infty[-1,1] \longrightarrow C^\infty(-\infty,\infty)$ by

$$(M'\Phi)(x) = (M\Phi)(x)l_0(x)$$

which is a continuous linear extension operator from $[-1,1]to(-\infty,\infty)$.

Now let us give the definition of Lipschitz domain.

Definition 2.1 *Let $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ be a function which satisfies the* Lipschitz condition *of order γ, $0 < \gamma \leq 1$; ie there is a constant $M > 0$ such that*

$$|\phi(x) - \phi(x')| \leq M|x - x'|^\gamma$$

for all $x, x' \in \mathbb{R}^{n-1}$. We consider points in $\mathbb{R}^n$ as pairs (x,y), $x \in \mathbb{R}^{n-1}$, $y \in \mathbb{R}$. The open subset

$$\{(x,y) \in \mathbb{R}^n : y > \phi(x)\}$$

is called a special Lipschitz domain *of class Lip γ. A rotation around y axis of such a domain will also be called a special Lipschitz domain.*

Let Ω be an open subset of $\mathbb{R}^n$, and $\partial\Omega$ its boundary. We say more generally that Ω is a Lipschitz domain *if for each point a in $\partial\Omega$, there exists an open neighborhood U_a of a in $\mathbb{R}^n$, and a special Lipschitz domain Ω_a such that $\Omega \cap U_a = \Omega_a \cap U_a$. If each Ω_a is of class Lip γ (independent of a), then we say Ω is a Lipschitz domain of class Lip γ.*

The following theorem is due to Stein [21]

Theorem 2.2 *If X is the closure of a Lipschitz domain Ω of class 1, then there exists an extension operator*

$$E : \mathcal{E}(X) \longrightarrow \mathcal{E}(\mathbb{R}^n)$$

Stein's result is extended by Bierstone [3] to the case of a domain with boundary which is Lipschitz of any class, in other words; with boundary of Hölder type. The main result of Bierstone [3], where he used Hironaka's desingularization theorem, is that an extension operator exists if X is a fat closed subanalytic subset of $\mathbb{R}^n$.

The extension property of $K = \bar{\Omega}$ for a domain Ω with boundary of Hölder type was proved also by Tidten [22] using the property (DN) and by Goncharov [5] who proved that in this case $C^\infty(\bar{\Omega})$ is isomorphic to s.

M. Tidten in [23] introduced a geometric property of compact sets in $\mathbb{R}$ which could help to give a geometric characterization for the extension property. Here we define this geometric property.

Definition 2.3 *Let $\alpha \geq 1$. A compact set $K \subset \mathbb{R}$ is said to belong to the class (α) if there exists $\delta_0 > 0$ and $C > 0$ such that, for any point $y \in K$, there is a sequence (x_j) in K with the following properties:*

$$
\begin{array}{ll}
(1) & |y - x_j| \downarrow 0 \\
(2) & |y - x_1| \geq \delta_0 \\
(3) & C|y - x_{j+1}| \geq |y - x_j|^\alpha \text{ for all } j
\end{array}
$$

Tidten proved that

$$K \in (1) \Longrightarrow K \text{ has the extension property } \Longrightarrow K \in (\alpha)$$

and gave an example of $K \notin (1)$ with the extension property. Later Goncharov in [9] shoved that belonging to some class (α) can not be in general a geometric characterization of the extension property for $K \subset \mathbb{R}$.

A. Goncharov and M. Kocatepe in [10] considered compact sets of the following type. For two sequences $(a_n), (b_n)$ such that $0 < \ldots < b_{n+1} < a_n < b_n < \ldots < b_1 < 1$, let $I_n = [a_n, b_n]$ and $K = \{0\} \cup \bigcup_{n=1}^{\infty} I_n$. By ψ_n denote the length of I_n; $h_n = a_n - b_{n+1}$ is the distance between I_n and I_{n+1} and let

$$\psi_n \searrow 0, \quad h_n \searrow 0, \quad \psi_n \leq h_n, \quad n \in \mathbb{N} \tag{2.1}$$

$$\exists Q \in \mathbb{N} : h_n \geq b_{n+1}^Q, \quad n \in \mathbb{N} \tag{2.2}$$

They shoved that $\mathcal{E}(K)$ has property DN if and only if

$$\exists M, \ \forall n, \quad \psi_{n+1} \geq h_n^M$$

It is shown in Chapter 3 that the condition (2.2) can be omitted in the case J_n is bounded, where $J_n = min\{j : b_{n+j} \leq \psi_n\}$.

A. Goncharov in [9] considered Cantor type sets in $\mathbb{R}$ and has given the necessary and sufficient conditions of extension property for those type of compact sets. In Chapter 4 we will see these results and prove that the necessary and sufficient conditions for the extension property of multidimensional cantor type sets is similar to the case one dimensional cantor type sets.

In [1] B. Arslan, A. Goncharov and M. Kocatepe considered generalized Cantor type sets, where the generalized Cantor type sets are produced by removing more than one intervals from all intervals in each step.

Pawlucki and Pleśniak [18] by using the Lagrange interpolational polynomials constructed an extension operator for compact sets satisfying the Markov property. In general Markov property is not equivalent to the Extension property. A Goncharov [6] gave an example of a set with an extension operator but not satisfying the Markov property.

Chapter 3

Some Model Cases

Let $\mathbb{N} = \{1, 2, ...\}$. We will consider compact sets of the following type. For two sequences $(a_n), (b_n)$ such that $0 < ... < b_{n+1} < a_n < b_n < ... < b_1 < 1$, let $I_n = [a_n, b_n]$ and $K = \{0\} \cup \cup_{n=1}^{\infty} I_n$. By ψ_n we denote the length of I_n; $h_n = a_n - b_{n+1}$ is the distance between I_n and I_{n+1}. In what follows we restrict ourselves to the case

$$\psi_n \searrow 0, \quad h_n \searrow 0, \quad \psi_n \leq h_n, \quad n \in \mathbb{N} \tag{3.1}$$

$$\exists Q \in \mathbb{N} : h_n \geq b_{n+1}^Q, \quad n \in \mathbb{N} \tag{3.2}$$

An equivalent form of (3.2) is

$$\exists Q \in \mathbb{N} : h_n \geq b_n^Q, \quad n \in \mathbb{N} \tag{3.3}$$

Let us give some identities about the remainder of the Taylor polynomials that will be used in this chapter. Proofs can be found in [15]:

$$(R_y^q f)^{(i)}(x) = R_y^{q-i} f^{(i)}(x) = f^{(i)}(x) - \sum_{j=i}^{q} \frac{f^{(j)}(y)}{(j-i)!}(x-y)^{j-i} \tag{3.4}$$

If $f \in C^{q+1}[a, b]$ and $x, y \in [a, b]$, then for some $\xi, \eta \in [a, b]$ we have

$$(R_y^q f)^{(i)}(x) = (f^{(q)}(\xi) - f^{(q)}(y))\frac{(x-y)^{q-i}}{(q-i)!} = f^{(q+1)}(\eta)\frac{(x-y)^{q-i+1}}{(q-i+1)!} \tag{3.5}$$

The next two lemmas are from [10].

Lemma 3.1 *Let I be any closed interval in $\mathbb{R}$ with $length(I) \geq \delta_0$ and let $p \leq k \leq r$ be given. Then there exists two constants C_1, C_2 such that*

$$|f^{(k)}(x)| \leq C_1 \delta^{-k+p} |f|_p + C_2 \delta^{r-k} |f|_r \quad \forall f \in C^r(I), \quad \forall \delta \in (0, \delta_0], \quad \forall x \in I$$

Lemma 3.2 *Given positive integers N, p, k such that $k \leq pN$, there is a constant $C(N,p,k)$ with the following properties: For any closed interval $I \subset \mathbb{R}$ with $length(I) = \delta_0$ and for any set of points $a_1, ..., a_N \in I$, let $G(x) = \sqcap_{s=1}^{N} (x - a_s)^p$. Then*

$$|G^{(k)}(x)| \leq C(N,p,k) \delta_0^{pN-k} \quad \forall x \in I$$

For each n, we define $J_n = min\{j : b_{n+j} \leq \psi_n\}$

We have the following result from [10]. When K satisfies both the conditions (3.1) and (3.2) in the cases either (J_n) is bounded or $J_n \longrightarrow \infty$ as $n \longrightarrow \infty$ K has the extension property if and only if

$$\exists M, \quad \forall n, \quad \psi_{n+1} \geq h_n^M$$

In the following theorem arguing as in [10] we see that the same result holds without having the condition (3.2) when (J_n) is bounded.

Theorem 3.3 *Let $J_N < J$ for each n. K is a compact set as it is described in this chapter satisfying condition (3.1). Then $\mathcal{E}(K)$ has property DN if and only if*

$$\exists M, \quad \forall n, \quad \psi_{n+1} \geq h_n^M$$

Proof: *(Necessity)* We have p from DN. We let $q = (2J+1)(p+1)$ and find r, C according to DN. We fix n and define

$$f = f_n = \begin{cases} (x \sqcap_{s=n}^{n+2J-1} (x - a_s))^{p+1} & x \leq b_n \\ 0 & x \geq a_{n-1} \end{cases}$$

Since $b_{n+J_n} \leq \psi_n$ we have $b_{n+J} \leq \psi_n$ for all n. Because f is a polynomial of degree q on $[0, b_n]$ we have $\|f\|_q \geq |f|_q \geq |f^{(q)}|_0 = q!$ Now let us find upper bounds for $\|f\|_p$ and $\|f\|_r$

To find the upper bound for $\|f\|_p$ let $x \leq b_{n+J}$. Then $f(x) = x^{p+1} G(x)$ where $G(x)$ is the product of the other terms. For $k \leq p$,

$$|f^{(k)}(x)| \leq \lambda_n b_{n+J}^{p+1-k} \qquad \lambda_n = C_p b_n^{(p+1)J} \tag{3.6}$$

If $x \leq b_{n+2J}$ then

$$|f^{(k)}(x)| \leq \lambda_n b_{n+2J}^{p+1-k} \tag{3.7}$$

If $x \in I_l \quad n \leq l \leq n+2J-1$ then

$$|f^{(k)}(x)| \leq \lambda_n \psi_l^{p+1-k} \tag{3.8}$$

We therefore have $|f^{(k)}(x)| \leq \lambda_n \psi_n$ if $x \leq b_{n+2J}$ or $x \in I_l \quad n \leq l \leq n+2J-1$.

Next consider $A_p = \frac{|(R_y^p f)^{(i)}(x)|}{|x-y|^{p-i}} \quad x, y \in K \quad x \neq y \quad i \leq p$

If $x, y \leq b_{n+2J}$ or $x, y \in I_l$ $(n \leq l \leq n+2J-1)$ then by (3.5) we have

$$A_p \leq 2\lambda_n \psi_n$$

If $x \in I_l$ and $y \in I_m$ $(n \leq l, m \leq n+2J-1)$ then

$$|x-y| \geq max\{h_l, h_m\} \geq max\{\psi_l, \psi_m\}$$

and from (3.8) we see that

$$A_p \leq 4\lambda_n \psi_n$$

Clearly the same estimate holds if $l \geq n, m \leq n-1$.

If $x \leq b_{n+2J}$ and $y \in I_m \quad n \leq m \leq n+J-1$ then $|x-y| \geq h_n + J - 1 \geq b_{n+2J}$ and so (3.7) implies

$$\frac{|f^{(i)}(x)|}{|x-y|^{p-i}} \leq \lambda_n \frac{b_{n+2J}^{p+1-i}}{b_{n+2J}^{p-i}} = \lambda_n b_{n+2J} \leq \lambda_n \psi_n$$

Clearly the estimate holds if $x \leq b_{n+2J}$ and $y \in I_m \quad m \leq n$

Now there is only one remaining case to consider which is $x \leq b_{n+2J}$ and $y \in I_m \quad n+J \leq m \leq n+2J-1$

But then $x, y \leq b_{n+J}$ and then by (3.5) we have

$$R_y^p f^{(i)}(x) = (f^{(p)}(\xi) - f^{(p)}(y))\frac{(x-y)^{p-i}}{(p-i)!}$$

where $0 < \xi < b_{n+J}$ and therefore

$$A_p \leq 2\lambda_n \psi_n$$

Hence we have that $\|f\|_p \leq 5\lambda_n \psi_n \leq \psi_n$ for $n \geq n_p$ since $\lambda_n \to 0$

Upper bound for $\|f\|_r$: by Lemma 3.2 $|f^{(k)}(x)| \leq C(2J+1,p,k)b_n^{q-k}$ for $k \leq q$ and 0 otherwise.Thus

$$|f|_r \leq \max_{k \leq q} C(J+1,p,k) = C_q$$

Clearly $R_y^r f(x) \equiv 0$ when $x, y \leq b_n$. If either $x \geq a_{n-1}$ or $y \geq a_{n-1}$ then since $|x-y| \geq h_{n-1}$ by (3.4) we have

$$\frac{|(R_y^r f)^{(i)}(x)|}{|x-y|^{r-i}} \leq |f|_r(1+\sum_{j=i}^{r}\frac{1}{(j-i)!})\frac{1}{|x-y|^r} \leq 4C_q h_{n-1}^{-r}$$

Thus $\|f\|_r \leq 5C_q h_{n-1}^{-r}$

Now replacing f by f_n in DN , we obtain

$$q! \leq t\psi_n + \frac{C}{t}5C_q h_{n-1}^{-r} \leq t\psi_n + \frac{1}{th_{n-1}^{r+1}}$$

for large enough n and arbitrary t. Let $t = h_{n-1}^{-r-1}$. Since $q \geq 2$ we obtain $h_{n-1}^M \leq \psi_n$ for n large enough.$M \geq r+1$,increasing the value of M if necessary we get $h_{n-1}^M \leq \psi_n \forall n$

(Sufficiency) Let $p = 0$ $R = 7M+3$ for given $q \geq 1$. Let $r = 3q$. It is enough to prove the implication

$$\left.\begin{array}{l} \|f\|_0 \leq \tau = t^{-Rq} \\ \|f\|_r \leq t^q \end{array}\right\} \Longrightarrow \|f\|_q \leq 1$$

For any t s.t. $t^2 > \frac{1}{h_1}$. Find n s.t. $h_{n+1} \leq t^{-2} < h_n$

Let us first estimate $B = |f^{(k)}(z)|t^{2(q-k)}$ $\quad z \in K \quad k \leq 3q$ If $z \geq a_{n+1}$ apply Lemma 3.1,

$$\begin{aligned} B &\leq (C_1\psi_{n+1}^{-k}|f|_0 + C_2\psi_{n+1}^{r-k}|f|_r)t^{2(q-k)} \\ &\leq (C_1 t^{2Mk} t^{-Rq} + C_2 t^{-2(R-k)} t^q)t^{2(q-k)} \\ &\leq C_1 t^{-Mq-q} + C_2 t^{-q} \leq C_3 t^{-q} \end{aligned}$$

If $z = b_{n+2}$ then consider taylor expansion of f^k at the point $a = a_{n+1}$

$$f^{(k)}(z) = \sum_{i=k}^{3q} f^{(i)}(a)\frac{(z-a)^{i-k}}{(i-k)!} + (R_a^{3q}f)^{(k)}(z)$$

Therefore for $B_k = |f^{(k)}(z)|t^{2(q-k)}$ $k \leq 2q$ we have

$$\begin{aligned} B_k &\leq eC_3t^{-q} + \|f\|_{3q}t^{-2(3q-k)} \\ &\leq eC_3t^{-q} + t^{-q} \leq (eC_3+1)t^{-q} = C_4t^{-q} \end{aligned}$$

And for $2q \leq k \leq 3q$ we have

$$B_k = |f^{(k)}(z)|t^{2(q-k)} \leq t^{q+2q-2k} \leq t^{3q-4q} = t^{-q}$$

Hence for $z = b_{n+2}$ we have $B_k(z) \leq C_4t^{-q}$ $0 \leq k \leq 3q$

If $z = a_{n+2}$ then consider taylor expansion of f^k at the point $a = b_{n+2}$

$$f^{(k)}(z) = \sum_{i=k}^{3q} f^{(i)}(a)\frac{(z-a)^{i-k}}{(i-k)!} + (R_a^{3q}f)^{(k)}(z)$$

Therefore for $B_k = |f^{(k)}(z)|t^{2(q-k)}$ $k \leq 2q$ we have

$$\begin{aligned} B_k &\leq eC_4t^{-q} + \|f\|_{3q}t^{-2(3q-k)} \\ &\leq eC_4t^{-q} + t^{-q} \leq (eC_4+1)t^{-q} = C_5t^{-q} \end{aligned}$$

And for $2q \leq k \leq 3q$ we have

$$B_k = |f^{(k)}(z)|t^{2(q-k)} \leq t^{q+2q-2k} \leq t^{3q-4q} = t^{-q}$$

Hence for $z = b_{n+2}$ we have $B_k(z) \leq C_5t^{-q}$ $0 \leq k \leq 3q$ Now it is easy to see that we can find an inequality for $B_k(z)$ for $z \in \{b_{n+2}, a_{n+2}, b_{n+3}, a_{n+3}, ..., b_{n+J}\}$ for every element in the sequence using the inequality for the previous element.

$$\begin{aligned} B_k(b_{n+m}) &\leq C_{2m}t^{-q} \ \ 2 \leq m \leq J \\ B_k(a_{n+m}) &\leq C_{2m+1}t^{-q} \ \ 2 \leq m \leq J-1 \end{aligned}$$

Where C_m has the recurrence relation $C_m = eC_{m-1} + 1$ Using this recurrence relation we get $C_m = e^{m-3}C_3 + e^{m-4} + ... + e + 1$. It is easy to see that (C_m) is increasing.

If $z \in [a_{n+m}, b_{n+m}]$ $2 \leq m \leq J-1$ then by considering the Taylor expansion of f^k at $a = b_{n+m}$ we obtain

$$B_k(z) \leq eC_{2m}t^{-q} + t^{-q} = C_{2m+1}t^{-q} \leq C_{2J+1}t^{-q}$$

If $z \leq b_{n+J}$ then consider taylor expansion of f^k at the point $a = b_{n+J}$

$$f^{(k)}(z) = \sum_{i=k}^{3q} f^{(i)}(a)\frac{(z-a)^{i-k}}{(i-k)!} + (R_a^{3q}f)^{(k)}(z)$$

and since $|z-a| \leq b_{n+J} \leq h_{n+1} \leq t^{-2}$ we have

$$B_k(z) \leq eC_{2J}t^{-q} + t^{-q} = C_{2J+1}t^{-q}$$

Hence we have proved that

$$\begin{aligned} B_k(z) &\leq C_{2J+1}t^{-q} \ \forall z \in K \ \ k \leq 3q \\ and|f|_q &\leq C_{2J+1}t^{-q} \end{aligned} \tag{3.9}$$

Next we estimate $A_q = \frac{|(R_y^q f)^{(i)}(x)|}{|x-y|^{q-i}}$ $\quad x, y \in K \ \ x \neq y \ \ i \leq p$ If $|x-y| > t^{-2}$,then by (3.4) and (3.9) we have

$$\begin{aligned} A_q &\leq |f^{(i)}(x)||x-y|^{i-q} + \sum_{k=i}^{q} |f^{(k)}(y)|\frac{|x-y|^{k-q}}{(k-i)!} \\ &\leq |f^{(i)}(x)|t^{2(q-i)} + \sum_{k=i}^{q} |f^{(k)}(y)|\frac{t^{2(q-i)}}{(k-i)!} \leq \frac{C_5(e+1)}{t^q} \end{aligned}$$

If $|x-y| \leq t^{-2}$, then

$$(R_y^q f)^{(i)}(x) = f^{(q+1)}(y)\frac{(x-y)^{q+1-i}}{(q+1-i)!} + \ldots + f^{(q+q)}(y)\frac{(x-y)^{2q-i}}{(2q-i)!} + (R_y^{2q}f)^{(i)}(x)$$

and using this last equation and (last) we get

$$\begin{aligned} A_q &\leq C_{2J+1}(t^{-q+2}\frac{t^{-2}}{(q+1-i)!} + \ldots + t^{-q+2q}\frac{t^{-2q}}{(2q-i)!}) + \|f\|_{2q}t^{-2q} \\ &\leq C_{2J+1}et^{-q} + t^{-q} = C_{2J+2}t^{-q} \end{aligned}$$

Therefore for large enough t we obtain $\|f\|_q \leq 1$ $\square$

Now we will consider compact sets $K \subset \mathbb{R}^2$ of the following type.For two sequences $(a_n), (b_n)$ such that $0 < \ldots < b_{n+1} < a_n < b_n < \ldots < b_1 < 1$ let $c_n = \frac{1}{2}(a_n + b_n)$,let D_n be the closed disc with center $(c_n, 0)$ and radius $r_n = \frac{1}{2}(b_n - a_n)$ then $K = \{0\} \cup_{n=1}^{\infty} D_n$. By $\psi_n = 2r_n$ we denote the diameter of D_n; $h_n = a_n - b_{n+1}$ is the distance between D_n and D_{n+1} . We restrict ourselves to the case where (3.1) and (3.2) hold.

$\mathcal{E}(K)$ is equipped with the topology defined by the sequence of norms

$$\|f\|_q = |f|_q + \sup\left\{\frac{|(R_y^q f)^{(k)}(x)|}{|x-y|^{q-|k|}} : x, y \in K, x \neq y, |k| \leq q\right\},$$

$|k| = k_1 + k_2$
$q = 0, 1, ...,$ where $|f|_q = sup\{|f^{(k)}(x)| : x \in K, |k| \leq q\}$ and

$$R_y^q f(x) = f(x) - T_y^q f(x) = f(x) - \sum_{|k|\leq q} \frac{f^{(k)}(y)}{k_1!k_2!}(x_1 - y_1)^{k_1}(x_2 - y_2)^{k_2}$$

is the Taylor remainder.

Let Ω be a bounded domain in $\mathbb{R}^2$, $\delta > 0$. For a point $x \in \Omega$ we denote $x \in Q(\delta)$ if x represents a point of a square, situated in Ω, with the side of the length δ. The next lemma is from [8].

Lemma 3.4 *Let $f \in C^\infty(\overline{\Omega})$, $k \in \mathbb{Z}_+^2$, $p \leq |k| \leq s$, $x \in Q(\delta)$. Then*

$$|f^{(k)}(x)| \leq C_3.\delta^{-|k|+p}|f|_p + C_4.\delta^{s-|k|}|f|_s$$

Theorem 3.5 *Let the compact set $K \subset \mathbb{R}^2$ be as it is described. Then $\mathcal{E}(K)$ has (DN) if and only if*

$$\exists M > 0 \ : \ \psi_n \geq h_{n-1}^M \tag{3.10}$$

Proof: *(Necessity)* It is easy to see that under conditoin (3.2) the statement (3.10) is equivalent to the following:

$$\exists M > 0 \ : \ \psi_n \geq h_n^M$$

We have p from (DN). Let $q = p + 1$, and let

$$f(x_1, x_2) = f_n(x_1, x_2) = \begin{cases} (x_1 - a_n)^q/q! & \text{if } x \in D_n \\ 0 & \text{otherwise} \end{cases}$$

Clearly $\|f\|_q \geq 1$. We shall estimate $\|f\|_p$ and $\|f\|_r$ from above. We have

$$\|f\|_p = |f|_p + \sup \frac{|(R_y^p f)^{(i)}(x)|}{|x-y|^{p-|i|}}$$

$(R^p_y f)^{(i)}(x) = 0$ for $i_2 > 0$ so let $i_2 = 0$.

For $x, y \in D_n$ we have

$$\begin{aligned}(R^p_y f)^{(i_1,0)}(x) &= f^{(i_1,0)}(x) - \sum_{j \geq i, |j| \leq p} \frac{f^{(j)}(y)}{(j-i)!}(x_1 - y_1)^{j_1 - i_1}(x_2 - y_2)^{j_2 - i_2} \\ &= f^{(i_1,0)}(x) - \sum_{i_1 \leq j_1 \leq p} \frac{f^{(j_1,0)}(y)}{(j_1 - i_1)!}(x_1 - y_1)^{j_1 - i_1} \\ &= f^{(p+1,0)}(\eta)\frac{(x_1 - y_1)^{p+1-i}}{(p+1-i)!} \text{ for some } \eta \in D_n \text{ using (3.5)}\end{aligned}$$

Then we have

$$A_{p,i} = \frac{|(R^p_y f)^{(i)}(x)|}{|x-y|^{p-|i|}} \leq \frac{|x_1 - y_1|}{(p+1-i)!} \leq \psi_n$$

For $x \in D_n, y \notin D_n$ we have

$$(R^p_y f)^{(i_1,0)}(x) = f^{(i_1,0)}(x) = (x_1 - a_n)^{q-i_1}/(q-i_1)!$$

Hence $A_{p,i} \leq \psi_n$

For $y \in D_n, x \notin D_n$ we have

$$(R^p_y f)^{(i_1,0)}(x) = -\sum_{i_1 \leq j_1 \leq p} \frac{f^{(j_1,0)}(y)}{(j_1 - i_1)!}(x_1 - y_1)^{j_1 - i_1} = \sum \frac{(y_1 - a_n)^{q-j_1}(x_1 - y_1)^{j_1 - i_1}}{(q-j_1)!(j_1 - i_1)!}$$

So we have $A_{p,i} \leq e\psi_n$ in this case, it is clear that $|f|_p \leq \psi_n$. Hence we have $\|f\|_p \leq 4\psi_n$.

By doing a similar work we see that $\|f\|_r \leq 4h_n^{-r}$.

Combining all these estimations in (DN) for $t = 8Ch_n^{-r}$ we obtain $1 \leq 64Ch_n^{-r}\psi_n$ hence there exists $M > 0$ such that $\psi_n \geq h_n^M$.

(Sufficiency) Let $p = 0$ and $R = 2MQ + 1$ where for a given $q \geq 1$ let $r = 3q$ and $m = Mq + 1$. It is enough to prove the implication

$$\|f\|_0 \leq \tau, \quad \|f\|_r \leq t \Longrightarrow \|f\|_q \leq 1$$

where $\tau = \frac{1}{t^{Rq}}$

For any t such that $t^2 > \frac{1}{b_1}$ find n such that $b_{n+1} \leq t^{-2} < b_n$. Then

$$h_n \geq b_n^Q > \frac{1}{t^{2Q}}$$

and by the hypothesis, we have

$$\psi_{n+1} \geq \delta := \frac{1}{t^{2MQ}}$$

It is clear that $\delta t^2 \leq 1$ and $\frac{\tau}{\delta^q} < \frac{1}{t}$

Let us first estimate

$$B_k(z) := |f^{(k)}(z)| t^{2(q-|k|)} \quad z = (z_1, z_2) \in K \quad |k| \leq 3q$$

If $z_1 \geq a_{n+1}$ then one can apply Lemma 3.4 for $|k| \leq q$

$$\begin{aligned} B_k(z) &\leq \left(C_1 \delta^{-|k|} |f|_0 + C_2 \delta^{r-|k|} |f|_r \right) t^{2(q-|k|)} \\ &\leq \left(C_1 \delta^{-|k|} \tau + C_2 \delta^{r-|k|} t \right) t^{2(q-|k|)} = C_1 (\delta t^2)^{q-|k|} \delta^q \tau + C_2 t^{2(q-r)+1} \\ &\leq C_1 t^{-1} + C_2 t^{1-2q} \leq C_3 t^{-1} \text{ for some } C_3 \geq 1 \end{aligned}$$

The same estimation already holds for $q \leq k \leq 3q$

If $z_1 \leq b_{n+2}$ then we consider the Taylor expansion of $f^{(k)}$ at the point $a = (a_{n+1}, 0)$

$$f^{(k)}(z) = \sum_{i \geq k, |i| \leq 3q} f^{(i)}(a) \frac{(z_1 - a_{n+1})^{i_1 - k_1} (z_2 - 0)^{i_2 - k_2}}{(i_1 - k_1)!(i_2 - k_2)!} + (R_a^{3q} f)^{(k)}(z)$$

We apply Lemma 3.4 to the terms $f^{(i)}(a)$. Since $|z_1 - a_{n+1}| \leq a_{n+1} \leq b_{n+1} < t^{-2}$ and $|z_2 - 0| < \psi_{n+2} < b_{n+2} < t^{-2}$ we have

$$\begin{aligned} B_k(z) &\leq \sum_{i \geq k, |i| \leq 3q} B_i(a) t^{2(|i|-|k|)} \frac{t^{-2(|i|-|k|)}}{(i_1 - k_1)!(i_2 - k_2)!} + \|f\|_{3q} t^{-2(3q-|k|)} \\ &\leq \sum \frac{B_i(a)}{(i_1 - k_1)!(i_2 - k_2)!} + t^{-q} \text{ for } |k| \leq 2q \\ &\leq e^2 C_3 t^{-1} + t^{-q} \text{ for } |k| \leq 2q \\ &\leq C_4 t^{-1} \text{ for some } C_4 \geq 1 \end{aligned}$$

Hence we have reached to the result

$$|f^{(k)}(z)| t^{2(q-|k|)} \leq C_4 t^{-1} \quad z \in K, \quad |k| \leq q \tag{3.11}$$

hence $|f|_q \leq C_4 t^{-1}$

Next we estimate

$$A_q = \frac{(R_y^q f)^{(i)}(x)}{|x-y|^{q-|i|}} \qquad x, y \in K,\ x \neq y,\ |i| \leq q$$

If $|x-y| \leq t^{-2}$, then

$$\begin{aligned}
(R_y^q f)^{(i)}(x) &= (R_y^{q+1} f)^{(i)}(x) + \sum_{|k|=q+1, k\geq i} f^{(k)}(y) \frac{(x_1-y_1)^{k_1-i_1}(x_2-y_2)^{k_2-i_2}}{(k_1-i_1)!(k_2-i_2)!} \\
&\leq (R_y^{q+1} f)^{(i)}(x) + |f|_{q+1}.|x-y|^{q+1-|i|} \sum_{|k|=q+1, k\geq i} \frac{1}{(k_1-i_1)!(k_2-i_2)!} \\
&\leq (R_y^{q+1} f)^{(i)}(x) + e^2 |f|_{q+1}.|x-y|^{q+1-|i|}
\end{aligned}$$

and it follows that

$$A_q \leq (\|f\|_{q+1} + e^2|f|_{q+1})|x-y| \leq \frac{10}{t}$$

If $|x-y| > t^{-2}$ then we will use the identity

$$(R_y^q f)^{(i)}(x) = f^{(i)}(x) - \sum_{j\geq i, |j|\leq q} \frac{f^{(j)}(y)}{(j_1-i_1)!(j_2-i_2)!}(x_1-y_1)^{j_1-i_1}(x_2-y_2)^{j_2-i_2}$$

and (3.10), then we have

$$\begin{aligned}
A_q &\leq |f^{(i)}(x)||x-y|^{|i|-q} + \sum_{j\geq i, |j|\leq q} |f^{(j)}(y)| \frac{|x-y|^{|j|-q}}{(j_1-i_1)!(j_2-i_2)!} \\
&\leq |f^{(i)}(x)| t^{2(q-|i|)} + \sum |f^{(j)}(y)| \frac{t^{2(q-|j|)}}{(j_1-i_1)!(j_2-i_2)!} \\
&\leq \frac{C_4}{t}\left(1 + \sum \frac{1}{(j_1-i_1)!(j_2-i_2)!}\right) \leq \frac{C_4}{t}(1+e^2) \leq 10\frac{C_4}{t}
\end{aligned}$$

Therefore for large enough t we obtain $\|f\|_q \leq 1$ $\square$

Chapter 4

Multidimensional Cantor type sets

We concider a problem of the existence of a linear continuous extension operator for the space of Whitney functions given on a generalized multidimensional Cantor set.

4.1 Introduction

In what follows we will consider only C^∞-determining compact sets. A compact set $K \in \mathbb{R}^n$ is called *C^∞-determining* if for each $f \in C^\infty(\mathbb{R}^n), f|_K = 0$ implies $f^{(k)}|_K = 0$ for all $k \in \mathbb{N}^n$. Therefore we can consider not jets but functions.

Let $(l_n)_{n=0}^{\infty}$ be a sequence such that $l_0 = 1, 0 < 2l_{n+1} < l_n, n \in N$.Let K be the Cantor set associated with the sequence (l_n) that is $K = \cap_{n=0}^{\infty} K_n$, where $K_0 = I_{0,1} = [0,1], K_n$ is a union of 2^n closed intervals $I_{n,k}$ of length l_n and K_{n+1} is obtained by deleting the open concentric subinterval of length $l_n - 2l_{n+1}$ from each $I_{n,k}, k = 1, 2, ..., 2^n$.

Fix $\alpha > 1$ and $l_1 < 1/2$ with $2l_1^{\alpha-1} < 1$.We will denote by $K^{(\alpha)}$ the Cantor set associated with the sequence (l_n),where $l_0 = 1, l_{n+1} = l_n^\alpha = ... = l_1^{\alpha^n}, n \geq 0$.

Theorem 4.1 *[9, Goncharov]If $\alpha > 2$ then $K^{(\alpha)}$ does not have the extension property.*

Theorem 4.2 *[9, Goncharov] If $1 < \alpha < 2$ then $K^{(\alpha)}$ has the extension property.*

4.2 Cantor type sets in $\mathbb{R}^n$ and the extension property

We see that the critical point for the one dimensional Cantor sets is $\alpha = 2$. We want to find the critical point for the set $K^{(\alpha_1)} \times K^{(\alpha_2)} \times ... \times K^{(\alpha_n)}$. Let for $i \leq n$ $K^{[\alpha_1,...,\alpha_i]}$ denote the set $K^{(\alpha_1)} \times K^{(\alpha_2)} \times ... \times K^{(\alpha_i)}$. For simplicity we will use the folowing notation:

Notation 4.3 $\|f\|_q^{(i)}$ *denotes the q^{th} norm of $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_i]})$ $\forall i \in \mathbb{N}$.*

For $x = (x_1, ..., x_n) \in K^{[\alpha_1,...,\alpha_n]}$ and $k = (k_1, ..., k_n) \in \mathbb{N}^n$ let

$$\begin{aligned}
\vec{x} &= (x_1, ..., x_n) \\
\vec{x}_i &= (x_i, ..., x_n) \\
\bar{x}_i &= (x_1, ..., x_i) \\
k! &= k_1!...k_n! \\
x^k &= x_1^{k_1}...x_n^{k_n} \\
\vec{x} \geq \vec{y} &\Leftrightarrow x_i \geq y_i \;\; \forall i \leq n \\
\vec{x} = \vec{y} &\Leftrightarrow x_i = y_i \;\; \forall i \leq n \\
\vec{x} > \vec{y} &\Leftrightarrow \vec{x} \geq \vec{y} \text{ and } \vec{x} \neq \vec{y}
\end{aligned}$$

Lemma 4.4 *Let $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_n]})$. For $n \geq 2$ fix $c \in K^{[\alpha_2,...,\alpha_n]}$ and let $f_c(x) = f(x, c), x \in K^{(\alpha_1)}$ then $\|f\|_q^{(n)} \geq \|f_c\|_q^{(1)}$*

Proof:

$$\begin{aligned}
|f|_q^{(n)} &= \sup_{x,\vec{j}}\{|f^{(\vec{j})}(x)|\} = \sup\{|f^{(j_1,...j_n)}(x_1,...,x_n)| : x_i \in K^{(\alpha_i)}, |\vec{j}| \leq q\} \\
&\geq \sup_{x_1,j_1}\{|f^{(j_1,\vec{0})}(x_1, c)| : x_1 \in K^{(\alpha_1)}, c \in K^{[\alpha_2,...,\alpha_n]}, j_1 \leq q\} \\
&= \sup\{|f_c^{(j_1)}(x_1)| : x_1 \in \mathbb{R}, j_1 \leq q\} = |f_c|_q^1
\end{aligned}$$

On the other hand

$$\begin{aligned}
S_q^n(f) &= \sup_{x,y,i}\left\{|\frac{(R_y^q f)^{(i)}(x)}{|x-y|^{q-|i|}}| : x, y \in K^{[\alpha_1,...,\alpha_n]}, x \neq y, |i| \leq q\right\} \\
&= \sup\left\{\frac{|f^{(i)}(x) - \sum \frac{f^{(j)}(y)}{(j_1-i_1)!...(j_n-i_n)!}(x_1-y_1)^{j_1-i_1}...(x_n-y_n)^{j_n-i_n}|}{|x-y|^{q-|i|}}\right\}
\end{aligned}$$

$$\geq \sup_{\substack{i_1 \leq q \\ x_1, y_1 \in K^{(\alpha_1)}}} \left\{ \frac{|f^{(i_1,\bar{0})}(x_1,c) - \sum \frac{f^{(j_1,\bar{0})}(y_1,c)}{(j_1-i_1)!}(x_1-y_1)^{j_1-i_1}|}{|x_1-y_1|^{q-i_1}} : x_1 \neq y_1 \right\}$$

$$= S_q^1(f_c) \text{ for } c \in K^{[\alpha_2,\ldots,\alpha_n]}$$

hence $\|f\|_q^{(n)} = |f|_q^{(n)} + S_q^n(f) \geq |f_c|_q^{(1)} + S_q^1(f_c) = \|f_c\|_q^{(1)}$

□

Lemma 4.5 *Let $f \in \mathcal{E}(K^{[\alpha_1,\ldots,\alpha_n]})$. For $n \geq 2$ fix $c \in K^{(\alpha_n)}$ and let $f_c^{(i)}(y) = \frac{\partial^i}{\partial x_n^i} f(y,c), y \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}$ then $\|f\|_q^{(n)} \geq \|f_c^{(i)}\|_{q-i}^{(n-1)}$*

Proof

For the proof of this inequality we will use a strategy similar to the one in the proof of the previous lemma.

$$\begin{aligned} |f|_q^{(n)} &= \sup_{x,j}\{|f^{(j)}(x)| : x \in K^{[\alpha_1,\ldots,\alpha_n]}, |j| \leq q\} = \sup\{|f^{(j_1,\ldots,j_n)}(x_1,\ldots,x_n)|\} \\ &\geq \sup\{|f^{(\bar{j}_{n-1},j_n)}(y,c)| : c \in K^{(\alpha_n)}, y \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}\} \\ &= |f_c^{(j_n)}|_{q-j_n}^{(n-1)} \end{aligned}$$

On the other hand

$$\begin{aligned} S_q^n(f) &= \sup\left\{ \left|\frac{(R_y^q f)^{(i)}(x)}{|x-y|^{q-|i|}}\right| : x,y \in K^{[\alpha_1,\ldots,\alpha_n]}, x \neq y, |i| \leq q \right\} \qquad (4.1) \\ &= \sup\left\{ \frac{|f^{(i)}(x) - \sum \frac{f^{(j)}(y)}{(j_1-i_1)!\ldots(j_n-i_n)!}(x_1-y_1)^{j_1-i_1}\ldots(x_n-y_n)^{j_n-i_n}|}{|x-y|^{q-|i|}} \right\} \end{aligned}$$

$$\geq \sup\left\{ \frac{|f^{(\bar{i}_{n-1},i_n)}(\bar{x}_{n-1},c) - \sum_{\bar{j}_{n-1}\geq \bar{i}_{n-1}, |\bar{j}_{n-1}| \leq q-i_n} \frac{f^{(\bar{j}_{n-1},i_n)}(\bar{y}_{n-1},c)}{(j_1-i_1)!\ldots(j_{n-1}-i_{n-1})!}(\bar{x}_{n-1}-\bar{y}_{n-1})^{\bar{j}_{n-1}-\bar{i}_{n-1}}|}{|\bar{x}_{n-1}-\bar{y}_{n-1}|^{q-i_n-|\bar{i}_{n-1}|}} \right.$$

$$\left. : \bar{x}_{n-1}, \bar{y}_{n-1} \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}, \bar{x}_{n-1} \neq \bar{y}_{n-1}, |\bar{i}_{n-1}| \leq q - i_n \right\} \qquad \text{for fixed } i_n$$

$$= S_{q-i_n}^{n-1}(f_c^{(i_n)})$$

hence $\|f\|_q^{(n)} = |f|_q^{(n)} + S_q^n(f) \geq |f_c^{(i)}|_{q-i}^{(n-1)} + S_{q-i}^{n-1}(f_c^{(i)}) = \|f_c^{(i)}\|_{q-i}^{(n-1)}$

□

Theorem 4.6 $K^{[\alpha_1,\ldots,\alpha_n]}$ *has the extension property for* $1 < \alpha_i < 2, i = 1, \ldots, n$.

Proof

We will prove by induction on n. We know the statement is true for $k = 1$. Now suppose the statement is true for $k \leq n-1$. Then take

$$z_0 = (x_0, y_0) \in K^{[\alpha_1,\ldots,\alpha_n]}$$

where $x_0 \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}$ and $y_0 \in K^{(\alpha_n)}$

fix $f \in \mathcal{E}(K^{[\alpha_1,\ldots,\alpha_n]})$ fix q.Given $R > 0$ Now fix $k_2 \leq q$

Let $g_1(x) := f^{(\vec{0},k_2)}(x, y_0)$. Then $g_1(x) \in \mathcal{E}(K^{[\alpha_1,\ldots,\alpha_{n-1}]})$

Therfore by proposition 1.19 and by our induction assumption

$$\exists r, C > 0 : |g_1|_q^{(n-1)} \leq t^R |g_1|_0^{(n-1)} + \frac{C}{t}||g_1||_r^{(n-1)}, \quad t > 0$$

So $\forall \vec{k_1} \in \mathbb{N}^{n-1}$ s.t. $|\vec{k_1}| \leq q - k_2$ we have

$$|f^{(\vec{k_1},k_2)}(z_0)| \leq t^R \sup_{x \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}} |f^{(\vec{0},k_2)}(x, y_0)| + \frac{C}{t}||g_1||_r^{(n-1)}, \quad t > 0 \tag{4.2}$$

Now let $g_2(y) := f(x, y)$ then $g_2(y) \in \mathcal{E}(K^{(\alpha_n)})$ using our assumption again, if we fix x we will have

$$|f^{(\vec{0},k_2)}(x, y_0)| \leq d^R \sup_{y \in K^{(\alpha_n)}} |f(x, y)| + \frac{C}{d}||g_2||_r^{(1)}, \quad d > 0$$

then

$$\begin{aligned} \sup_{x \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}} |f^{(\vec{0},k_2)}(x, y_0)| &\leq \sup_{x \in K^{[\alpha_1,\ldots,\alpha_{n-1}]}} (d^R \sup_{y \in K^{(\alpha_n)}} |f(x, y)| + \frac{C}{d}||g_2||_r^{(1)}) \\ &\leq d^R \sup_{(x,y)} |f(x, y)| + \frac{C}{d} \sup_x ||g_2||_r^{(1)} \quad \forall d > 0 \end{aligned}$$

By Lemma 4.4

$$||g_2||_r^{(1)} \leq ||f||_r^{(n)}$$

and by Lemma 4.5

$$||g_1||_r^{(n-1)} \leq ||f||_{r+k_2}^{(n)} \leq ||f||_{2r}^{(n)}$$

then

$$|f^{(\vec{k_1},k_2)}(z_0)| \leq t^R d^R |f|_0 + t^R \frac{C}{d} ||f||_{2r} + \frac{C}{t} ||f||_{2r}$$

Now let $d = t^{R+1}$ then

$$|f^{(\vec{k_1},k_2)}(z_0)| \leq t^{R^2+2R} |f|_0 + \frac{2C}{t} ||f||_{2r} \forall t > 0$$

□

Lemma 4.7 *Let $f \in \mathcal{E}(K^{[\alpha_1,...,\alpha_n]})$ s.t. $f(x) = f(x_1,...,x_n) = F(x_1), F(x_1) \in \mathcal{E}(K^{(\alpha_1)})$ that is, f depends only on the first variable.Then $||f||_q^{(n)} = ||F||_q^{(1)}$*

Proof: Since $F^{(k_1,\vec{k}_2)}(x_1) = 0$ for $\vec{k}_2 > 0$ we have

$$\begin{aligned} |f|_q^{(n)} &= sup_{x_1,\vec{x}_2,k_1,\vec{k}_2}\{|f^{(k_1,\vec{k}_2)}(x_1,\vec{x}_2)| : k_1 + |\vec{k}_2| \leq q, x_1 \in K^{(\alpha_1)}, \vec{x}_2 \in K^{[\alpha_2,...,\alpha_n]}\} \\ &= sup_{x_1,k_1,\vec{k}_2}\{|F^{(k_1,\vec{k}_2)}(x_1)| : k_1 + |\vec{k}_2| \leq q, x_1 \in K^{(\alpha_1)}\} \\ &= sup_{x_1,k_1}\{|F^{(k_1)}(x_1)| : k_1 \leq q, x_1 \in K^{(\alpha_1)}\} \\ &= |F|_q^{(1)} \end{aligned}$$

On the other hand we have

$F^{(i_1,\vec{i}_2)}(x_1) - \sum_{j\geq i|j|\leq q} \frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j_1-i_1)!...(j_n-i_n)!}(x_1-y_1)^{j_1-i_1}...(x_n-y_n)^{j_n-i_n} = 0$ for $\vec{i}_2 > 0$ and $F^{(j_1,\vec{j}_2)}(x_1) = 0$ for $\vec{j}_2 > 0$ therefore

$$S_q^n(f) = \sup_{x,y,i}\left\{|\frac{(R_y^q f)^{(i)}(x)}{|x-y|^{q-|i|}}| : x,y \in K^{[\alpha_1,...,\alpha_n]}, x \neq y, |i| \leq q\right\}$$

$$= \sup\left\{\frac{|f^{(i)}(x) - \sum_{j\geq i,|j|\leq q} \frac{f^{(j)}(y)}{(j-i)!}(x-y)^{j-i}|}{|x-y|^{q-|i|}} : x \neq y, |i| \leq q\right\}$$

$$= \sup\left\{\frac{|F^{(i_1,\vec{i}_2)}(x_1) - \sum_{j\geq i,|j|\leq q} \frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j-i)!}(x-y)^{j-i}|}{|x-y|^{q-|i|}}\right\}$$

$$= \sup_{x,y,i_1}\left\{\frac{|F^{(i_1)}(x_1) - \sum_{j_1\geq i_1,\vec{j}_2\geq\vec{0},|j|\leq q} \frac{F^{(j_1,\vec{j}_2)}(y_1)}{(j_1-i_1)!\vec{j}_2!}(x_1-y_1)^{j_1-i_1}(\vec{x}_2-\vec{y}_2)^{\vec{j}_2}|}{|x-y|^{q-i_1}}\right\}$$

for $i_1 \leq q$

$$= \sup\left\{\frac{|F^{(i_1)}(x_1) - \sum \frac{F^{(j_1)}(y_1)}{(j_1-i_1)!}(x_1-y_1)^{j_1-i_1}|}{(\sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2})^{q-i_1}} : x \neq y, i_1 \leq q\right\}$$

$$
\begin{aligned}
&= \sup\left\{\frac{|F^{(i_1)}(x_1) - \sum \frac{F^{(j_1)}(y_1)}{(j_1-i_1)!}(x_1-y_1)^{j_1-i_1}|}{|x_1-y_1|^{q-i_1}} : x_1, y_1 \in \mathbb{R}, x_1 \neq y_1, i_1 \leq q\right\} \\
&= S_q^1(F)
\end{aligned}
$$

Hence we get $\|f\|_q^{(n)} = \|F\|_q^{(1)}$ □

Theorem 4.8 *$K^{[\alpha_1,\ldots,\alpha_n]}$ does not have the extension property if at least one of the α_i's is greater than 2.*

Proof: Suppose wlog $\alpha_1 > 2$ .By the proof of Theorem 2 in [9] we have

$$\forall p \ \exists \epsilon \ \exists q \ \forall r > q \ \exists (f_m) \subset \mathcal{E}(K^{(\alpha_1)}) : \frac{\|f_m\|_p^{(1)} \|f_m\|_r^{(1)\epsilon}}{\|f_m\|_q^{(1)1+\epsilon}} \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty$$

Now define $g_m(x_1, \ldots, x_n) = f_m(x_1)$ By Lemma 4.7 $\|g_m\|_q^{(n)} = \|f_m\|_q^{(1)}$

Hence we have

$$\forall p \ \exists \epsilon \ \exists q \ \forall r > q \ \exists (g_m) \subset \mathcal{E}(K^{[\alpha_1,\ldots,\alpha_n]}) : \frac{\|g_m\|_p^{(n)} \|g_m\|_r^{(n)\epsilon}}{\|g_m\|_q^{(n)1+\epsilon}} \longrightarrow 0 \ \ as \ \ n \longrightarrow \infty$$

which shows the negation of (1.4)

□

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