EXPLORATIONS IN SUPPLY AND DEMAND FUNCTION EQUILIBRIA

by

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I certify that I have read this thesis and in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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March 1999

In this study, we regard the oligopolistic-oligopsonistic markets within the framework of a "double auction" in which both buyers and sellers make bids. To this end, we introduce games where declarations of supply and demand functions (which need not be true) are treated as strategic variables of producers and consumers, respectively, rather than just as "binding commitments" on the part of these parties. Whether firms produce with positive or zero marginal cost, the number of agents on each side of the market, whether consumers act as a union or not and time structure of the moves lead to different games. Existence of symmetric equilibria of each of these games is established. Most of them are shown to be unique. The equilibrium outcomes of these games are compared with the naked Cournot outcome as well as among themselves regarding the market price, total quantity produced, individual consumer's surplus, individual firm's profit and social welfare they lead to. To allow the consumers to behave strategically along with the producers, naturally makes the former better off and the latter worse off, while the net effect of this on total social welfare turns out to be case-contingent.

Keywords: Demand Function Equlibria, Supply Function Equilibria, Double Auction, Oligopoly, Oligopsony

ÖZET

ARZ VE TALEP FONKSİYONU DENGELERİNE İLİŞKİN İNCELEMELER

Harun Bulut
İktisat Bölümü
Tez Yöneticisi: Prof. Semih Koray
Mart 1999

Bu çalışmada oligopolistik-oligopsonistik piyasaları, hem alıcıların hem de satıcıların teklif verdiği "çift ihale" çerçevesinde düşünüyoruz. Bu yüzden (gerçek olması gerekmeyen) talep ve arz fonksiyonu bildirimlerinin sırasıyla tüketiciler ve üreticiler açından "bağlayıcı taahhütler" olmaktan çok stratejik değişkenler olarak alındığı bazı oyunlar tanımlıyoruz. Firmaların marginal maliyetlerinin pozitif veya sıfır olması, piyasanın her iki tarafındaki aktör sayısı, tüketicilerin birlik olarak hareket edip etmemeleri ve hamlelerin zamanlama yapısı değişik oyunlara yol açmaktadır. Bütün bu oyunların simetrik dengelerinin varlığı ve pek çoğunun da tekliği gösterilmiştir. Ayrıca bu oyunların denge sonuçları, hem kendi aralarında hem de çıplak Cournot denge sonucuyla, yol açtıkları piyasa fiyatı, toplam üretilen miktar, kişisel tüketici artığı, firma başına kar ve toplam sosyal refah temel alınarak karşılaştırılmıştır. Firmaların yanı sıra tüketicilerin de stratejik davranmalarına izin vermek, doğal olarak tüketicileri daha iyi bir duruma getirirken firmaların getirilerini azaltmaktadır. Öte yandan bunun toplam sosyal refaha olan etkisi duruma bağlı olarak değişmektedir.

Anahtar Kelimeler: Talep Fonksiyonu Dengeleri, Arz Fonksiyonu Dengeleri, Çift İhale, Oligopol, Oligopson

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To my family

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

As is well known, in perfectly competitive markets theory consumers and firms are assumed to occur in large numbers. Competitive firms can not affect the market price, nor can the consumers. Market price is determined by the intersection of aggregate demand and aggregate supply. No strategic role is attributed to either consumers or firms, for the impacts of individual agents' actions upon supply and demand are so negligible that they go unnoticed by the market. On the other hand, oligopolistic market theory deals with market interactions of a small number of firms. The literature on the game theoretic analysis of oligopolistic markets mostly attributes a strategic role to firms but not to consumers and justifies this by the asymmetry in the sizes of both parties. Since consumers are assumed to occur in large numbers, each individual consumer remains negligible, and so, their existence can only be traced in the market demand which is regarded as a binding commitment on the part of the whole consumer body, whereas firms, given market demand, enter to competition among each other by utilizing strategic variables which vary from quantity, price, supply function to mark-up over average cost.

However, we also observe that there are markets in which a small number of consumers interact. In fact there are even cases where a monopsonist prevails on the demand side or where consumers are not uniform, but highly differentiated regarding the size of their demands. The soccer transfer market provides a typical example of such markets. In transfer sessions a small number of players and clubs negotiate over contracts. Another example is auctions on government bonds in which a certain number of large banks are allowed to participate. In weapons industry, the government stands as a monopsonist and a small number of firms are awarded contracts. Whenever big firms are demanding a particular good as an intermediate good beyond individuals' consumption demand, they have the power to affect the market price. Energy sector is a typical example of this. Thus, in markets similar to the above it is natural to ascribe a strategic role to the demand side as well. Furthermore, such a consideration allows to analyze the welfare effects of a possible organized behavior on the part of consumers. To this end, here we model games where consumers act as active players by declearing demand functions (which need not not be true, but become a binding commitment once decleared) as strategies along with firms whose strategies are supply functions. The roots of an approach which ascribes to also the consumers a stategic role by allowing them to manipulate demand functions can be traced back in the literature as well and will be discussed in our survey below.

At the initial development of game theoretic analysis of oligopolistic markets Cournot-Nash solution has been mostly used, where firms' strategy is quantity. Bertrand solution also obtained remarkable attention where firms compete with prices. Later Grant and Quiggin[1] study a game where firms'

[1]Grant, S. and J. Quiggin., "Nash Equilibrium with Mark-Up Pricing Oligopolists.",

strategic variable is the mark-up over average cost. Grossman[2] by introducing supply schedules as firms' strategic variable obtains competitive equilibria as a Nash equilibria outcomes of supply functions under some restrictions on supply functions. Therefore, he proves that in a uniform industry with large fixed costs competitive equilibrium outcome can be obtained even if there are few numbers of firms. When competitive equilibrium does not exist due to the integer problem[3], he defines approximate competitive equilibrium and gives supply function strategies yielding this equilibrium. Note that since fixed quantity and fixed price are special cases of supply functions, Cournot competition and Bertrand competition are special cases of supply function competition. Think of a firm commits itself Cournot-Nash equilibrium quantity, i.e. vertical supply function at that quantity, any supply function leading to Cournot-Nash equilibrium price and quantity outcome, including the vertical supply function at the respective quantity is optimal for the other firm. Thus, Cournot-Nash equilibrium is obtained by a Nash equilibrium in supply functions. Similarly, monopoly outcome and Grant-Quiggin mark-up equilibrium outcome can be the Nash equilibrium outcome of supply functions. Though he gives an example in which supply function equilibrium exists but Bertrand equilibrium, both solution concepts are similar in the way that a firm can eliminate its rivals. The multiplicity of equilibrium is one of the

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Economic Letters. 45 (1994), 245-251.

[2]Grossman, S. J., "Nash Equilibrium and the Industrial Organization of Markets with Large Fixed Costs." Econometrica. 49 (1981), 1149-1172.

[3]In Grossman, "Nash"., competitive equilibrium defined as follows: it is a list of a price, a quantity and an integer, (P^c, q^c, n_c) such that $AD(P^c) = n_c q^c$, $MC(q^c) = P^c$, $AC(q^c) = MC(q^c)$. There will not always be an integer satisfying $AD(MC^c) = n_c q^c$. Then integer problem arises.

criticisms raised to Grossman[4]. In his characterization of supply function equilibria, he introduces further restrictions on supply schedules and uses supply correspondences and then every Nash equilibrium in supply functions turns out to be competitive equilibrium. These restrictions are also subject to criticisms. Especially, in an environment where there is no regulation the restrictions to firms on picking supply schedules seem unnatural. Having the same concerns Koray and Sertel[5] and Klemperer and Meyer[6] are two respones with different motivations. The former looks at the problem from the point of view of regulation. Their work is an generalization and extension of Loeb and Magat[7]. In Loeb and Magat's problem given a known industry demand, firms which have private information on their cost structures compete by bidding for a monopoly position. The critical condition is that there must be enough number of contenders. The highest bid comes from the most efficient firm who offers perfectly discriminating monopoly profit. After entry, the winner operates as marginal cost pricer so that it could harvest franchise fee back. At the end it obtains zero profit and consumers surplus is maximized. Moreover, social welfare, the sum of profits and consumers' surplus, is maximized. Note that when there is only one firm, natural monopolist, their procedure does not work. The outcome is standard monopoly

[4] Grossman, "Nash".

[5] Koray, S. and M. R. Sertel., "Socially Optimal Franchise Bidding for an Oligopoly." Unpublished. Bilkent University, Ankara, Turkey and Bogazici University, Istanbul, Turkey, 1989.

[6] Klemperer, P. D. and M. A. Meyer., "Supply Function Equilibria in Oligopoly Under Uncertainty.", Econometrica. 57 (1989), 1243-1277.

[7] Loeb, M. and W. A. Magat., "A Decentralized Method for Utility Regulation.", Journal of Law and Economics. 22:2 (October 1979), 399-404.

outcome. However, it is our observation that when consumers are considered as players in that situation they become better off. For an oligopoly, Koray and Sertel[8] similarly offer a franchise bidding mechanism. In this mechanism, each firm is invited to make a bid including the amount they will produce and a function for monetary compensation the firm asks. Then consumers union to maximize consumers' total welfare picks a group of firms. After defining social welfare as the sum of consumers surplus and profits, under some fairly general conditions on industry demand function and cost function of each firm and under a condition on number of firms providing the competitive behavior, Koray and Sertel show that there is a Nash equilibrium in bids which maximizes social welfare. When industry is uniform, it turns out that every Nash equilibrium in bids leads to the social optimum. Thus, their results give Grossman's supply function equilibrium theorem[9] as corollary. In Koray and Sertel's model consumers are not considered as players. Whether to attribute consumers a strategic role would lead to social optimum remains as open question. Our model is an attempt in this direction. Klemperer and Meyer[10] is the other response to Grossman[11]. Although they neglect fixed costs, they criticize Grossman by demonstrating too many equilibria in supply functions. They solve this problem by introducing uncertainty in industry demand. The justification of firm's commitment to supply functions turns out to be better adaptation to the uncertainty. When industry demand is subject to exogenous random shock, firms set price for each realization of random shock and so they achieve ex-post optimal adjust-

[8] Koray and Sertel, "Franchise"

[9] Grossman, "Nash".

[10] Klemperer and Meyer, "Supply "

[11] Grossman, "Nash"

ment to the shock. This adjustment reduces the set of equilibria in supply functions even to a unique equilibrium under appropriate assumptions. The Klemperer and Meyer's solution concept has been applied to strategic trade policy[12] and in the analysis of electricity spot market[13] with some minor modifications depending on the problem at hand. Green[14] again uses supply function model to analyze increasing competition in British electricity market but takes industry demand as a function of time rather than exogenous random shock. Klemperer and Meyer's solution concept also used in Grant and Quiggin[15]. In their theoretical work they consider two stage game. In the first stage firms make capital commitment and in the second stage firms enter supply function competition. Depending on the technology specification, they show that solution will go from Bertrand to Cournot. For the special case of constant-elasticity demand solution will be equal to mark-up equilibrium solution. Another recent work related to Klemperer and Meyer[16] is Khün[17]. He analyzes a vertically separated duopolistic market in which manufacturers' strategy variable is wholesale price, whereas retailers com-

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[12]See Laussel, D., "Strategic Commercial Policy Revisited: A Supply Function Equilibrium Model.", The American Economic Review. 82:1 (March 1992), 84-99.

[13]See Bolle, F., "Supply Function Equilibria and the Danger of Tacit Collusion. The Case of Spot Markets for Electricity." Energy Economics. (1992), 94-102. and Green, R. J. and D. M. Newbery., "Competition in the British Electricity Spot Market.", Journal of Political Economy. 100: 5 (1992), 929-953.

[14]Green, R. J., "Increasing Competition in the British Electricity Market." The Journal of Industrial Economics. XLIV: 2 (June 1996), 205-216.

[15]Grant, S. and J. Quiggin., "Capital Precommitment and Competition in Supply Schedules.", The Journal of Industrial Economics. XLIV: 4 (December 1996), 427- 441.

[16]Klemperer and Meyer, "Supply"

[17]Kühn, K., "Nonlinear Pricing in Vertically Related Duopolies." RAND Journal of Economics. 28:1 (Spring 1997), 37-62.

pete with quantity. If uncertainty in market demand is additive his model coincides with a model of competition in inverse supply functions. However, under more general forms of shocks to the demand he shows that both models' equilibrium allocations differ. Finally, Bolle[18] is an interesting work in lines with Klemperer and Meyer[19]. It is an extension of Bolle[20]. Important distinguishing feature of his model from the models we cited so far is that some group of buyers are players and their strategies are demand functions. Thus, he deviates from the assumption of large number of uniform buyers. He gives examples of electricity markets such as Norway and New Zealand where demand-side bids are also allowed. He models electricity market in which there are suppliers, big-users and small consumers. Suppliers and big-users behave strategically with their strategies supply function and demand functions respectively and small consumers have an affine autonomous demand function which is subject to additive random shock. Once the supply functions and demand functions are chosen an auctioneer equates excess supply to autonomous demand and obtains equilibrium price as a function of random shock. He defines Bayes-Nash equilibria of the game in which each supplier and big-user maximizes his expected payoff. Then he finds necessary and sufficient conditions for best responses for both demand and supply functions. This leads to system of differential equations and for solving them he suggests power series solution. We define a similar game in the section Nash Game, yet there are important differences. Though Bolle[21] argues that

[18] Bolle, F., "Competition in Supply and Demand Functions." Unpublished. Europa-Universitt Viadrina, Frankfurt, Germany, September 1997.

[19] Klemperer and Meyer, "Supply"

[20] Bolle, "Electricity"

[21] Bolle, "Demand"

deterministic autonomous demand does not make much sense, we do not consider an autonomous demand so in the Bolle's language autonomous demand is zero. In our model all consumers are players and compete with each other and against firms with demand functions. In addition, Bolle assumes a fixed profit rate at each unit of electricity for big users and if equilibrium price is higher than the constant profit rate big users do not demand at all at that price. This assumption can be justified in the context of electricity market, however we consider a more general context in which each consumer has an affine demand function and by "misrepresenting" his demand function he tries to maximize his consumer surplus. With such a set-up we arrive significant results and indicate that deterministic demands matter. We study the both cases where firms are producing with positive marginal cost and with zero marginal cost and observe that zero marginal cost assumption in Bolle[22] is not satisfactory. Furthermore, we analyzed the case where consumers union play on behalf of consumers with aggregate demand against firms both in a Nash game and a Stackelberg game. We investigate how an organized behavior on the side of consumers effect welfare distribution. We answer this question in this context.

Hurwicz[23] is an early reference introducing the idea that consumers can misrepresent their preferences. In an exchange economy with all goods are private if consumers are in finite numbers, he shows that when all other consumers stick to their true preferences and behave as price taker, it can be in

[22] Bolle, "Demand"

[23] Hurwicz, L., "Optimality and Informational Efficiency in Resource Allocation Processes." In Studies in Resource Allocation Processes, eds. L. Hurwicz and K. J. Arrow, 443-457. 1977.

his best interest of the remaining consumer to misrepresent his preferences. Therefore, he concludes that perfect competition may not be individually incentive compatible. Finiteness of consumers is crucial for his conclusion. When consumers are infinitely many, he heuristically argues that perfectly competitive behavior is incentive compatible that is telling the truth is the best response for every consumer when others do so. Firstly, we study the incentive compatibility problem for consumers and firms in an economy where consumption and production take place and having oligopolistic and oligopsonistic features. In the section Nash Game we present a formal proof of the result that when number of consumers goes to the infinity in the limit consumers are telling the truth about their respective private information in a Nash game.

Another early reference in which demand functions are used as strategies is Wilson[24] on share auctions. In Wilson's model finite number of symmetric bidders compete for shares of a single object. They give demand schedules as a function of the price per share. The seller picks the price such that sum of the shares equals to 1. Wilson comes up with the result that buyers are substantially better off in a share auction compared to the unit auction where each bidder names a price for entire object. Note that seller behaves here as if Stackelberg leader and offering 1 unit object for sale is nothing but making a vertical supply function commitment. Thinking of 1 unit of object as an autonomous supply fits better to Wilson's formulation. Though there is no cost of producing the object in Wilson's model, one can attribute a positive cost to the seller. Since object is already produced before demand schedules

[24] Wilson, R., "Auctions of Shares." The Quarterly Journal of Economics. XCIII (1979), 675-89.

are submitted in Wilson's model, when there is positive cost it is better to think of seller as a cost minimizer together with the assumption that good is durable for just 1 period. Buyers are then giving demand schedules and at equilibrium market clears. In the section Stackelberg Game, we present a more general model in which firms are Stackelberg leader and consumers are followers and we analyze equilibrium of demand and supply functions.

Although Grossman[25] assumes a deterministic industry demand and allows only firms to behave strategically, as a remark he mentions about the possible roles of consumers such as behaving monopsonisticly, misrepresenting individual demands in various contexts. In conclusion of his paper as a future research he suggests modeling of buyer choice in finding the correct model of imperfect competition.

Binmore and Swierzbinski[26] is a very recent paper studying the various auction formats in multi-unit auctions. This paper is in their advisory paper series to the Treasury and the Bank of England. They compare uniform and discriminatory auctions by allowing bidders to behave strategically by their demand functions. Bidders true demand functions are derived from a quasi-linear utility function. They criticize Merton Miller and Milton Friedman who advised in favor of uniform auctions and so influenced the USA in starting to experiment uniform auctions. They state that single-unit auction and multi-unit auction are different. For the former, two types of auctions are compared; first price and second price auctions. Though the seller expects the same revenue in both types of auctions, because of transparency it pro-

[25]Grossman, "Nash".

[26]Binmore, K. and J. Swierzbinski., "Uniform or Discriminatory?." Unpublished. August 1998.

vided and its simplicity second-price auctions are advocated by economists and used in practice. On the other hand, the theory of multi-unit auctions is not well developed and one can not guarantee revenue equivalency for the seller for various auction formats. Depending on the information on the seller about buyers demand functions, different formats perform better in terms of revenue. When buyers are allowed to submit any decreasing demand functions, they conclude that it is wrong to consider uniform auction as a generalization of second-price auction to multi-unit case because bidders do not optimize when they give their true demand functions in uniform auction. They find many equilibria in uniform auction so there is strategic uncertainty. However, there is a unique pure strategy equilibrium in discriminatory auction, in which true clearing price is obtained. This in turn contradicts Miller and Friedman's advise. In their paper, seller is not a player. He just commits himself to supply certain number of bonds. Then buyers pick demand functions such that market clears. It can be seen as the extension of Wilson[27] on share auctions to multi-unit case. This is also special case of Stackelberg game that we introduce. Think of seller to make a vertical supply function commitment then buyers play Nash with their demand functions. In our work we do not require buyers true demand functions to come from utility maximization problem, we just depend on their declarations.

When we consider consumers as players with demand functions together with the firms playing with supply functions, we think of oligopolistic markets from the point of view of "double auction" in which sellers make offers and buyers make bids. This approach is very much encouraged in Sonnen-

[27] Wilson, "Share".

schein[28]. He relates oligopoly theory to the auction theory and he himself mentions an example of a simple game in which both parties have strategic role. Furthermore, he emphasizes the need for specification of institutional framework and stresses the importance of it in the development of oligopoly theory. He sees the works of experimentalist economics such as Plot[29] in this direction and gives very much credit. The ideas introduced in Sonnenschein[30] form the basic motivation in our work. We model games distinguished by different institutional assumptions and study the implications of these models.

The plan of this study as follows: We proceed with the section The Model in which we introduce the model in general. Then the section Nash Games follow. There consumers either organized or unorganized play Nash with firms. We cover cases when firms produce with zero cost and positive marginal cost. In each case whether consumers are organized or not leads to different games. Also number of firms and consumers whenever matters leads to different games. Symmetric Nash equilibrium of all these games are given. Most of them are shown to be unique. On the basis of price, total quantity produced, consumer surplus, producer surplus and total social welfare comparisons are made with the outcome of Cournot-Nash game. Limit results are provided. Then, we introduce the Stackelberg Games: There firm is Stackelberg leader and consumers as organized are followers. Whether marginal cost of the firm is positive or zero leads to different games there as well. The unique Stack-

[28]Sonnenschein, H., "Comment.", In Frontiers of Economics, eds. K. J. Arrow and S. Honkapohja, 171-177, 1985.

[29]Plot, C. R., "Industrial Organization Theory and Experimental Economics.", Journal of Economic Literature. XX (December 1982), 1485-1527.

[30]Sonnenschein, "Comment".

elberg equilibrium is given at each game. Finally, we compare outcomes of these games among themselves and with the outcome of Cournot-Nash game. These constitute our four theorems. In the last section we conclude.

CHAPTER 2

THE MODEL

We are in a market for a particular good in the economy. In this market there are n consumers and m firms, where n, m are positive integers. Consumers are identical with their demand functions and firms are identical with their cost functions. Each consumer has an affine demand function, $D(P) = a - bP$, where $a > 0$ and $b > 0$. We assume the slope parameter, b is known, whereas the intercept term, a is private to each consumer. Thus, consumers have an option to manipulate their intercept terms either individually or in an organized manner. If they are not organized, each consumer is picking a positive number, $\gamma > 0$ for his intercept term a and so giving a demand function, by aiming to maximize his consumer surplus. If they are organized, consumer union (CU) plays on behalf of consumers by manipulating the intercept term of the aggregate demand by aiming to maximize total consumers' surplus. Note that true aggregate demand is $AD(P) = na - nbP$ and CU is giving $AD(P) = \Gamma - nbP$, where $\Gamma > 0$. We assume that the contract among consumers is the equal division of aggregate quantity demanded at the resulting equilibrium price. Since consumers are identical, this assumption is the most appropriate one. After division, each consumer can calculate his consumer surplus and compare with the one he obtains when they are not organized. On the other hand, each firm has a quadratic cost function, $C(q) = \alpha q^2$, where $\alpha \geq 0$. Though form of the cost function is known, cost parameter α is private to firms. Thus, firms have an option to misrepresent their cost parameters. Note that quadratic cost function implies linear marginal cost function $MC(q) = 2\alpha q$, which in turn implies a supply function $q(P) = \frac{1}{2\alpha}P$ where slope parameter is private to firms. Thus, each firm by picking a non-negative slope parameter $\beta \geq 0$, in fact by picking a non-negative number for cost parameter, is making a linear supply function commitment, $q(P) = \beta P$ where $\beta \geq 0$. Then, firms' strategies are their supply functions, in particular their slope terms.

Given firms' supply function commitments and consumers' demand function commitments, we define outcome price, P, as the number such that aggregate supply equals to aggregate demand, i.e, P satisfies

$$\sum_{i=1}^{m} q_i(P) = \sum_{i=1}^{n} D_i(P) \tag{2.1}$$

Note that outcome price is a function of supply and demand functions and such a number exists and unique since supply functions are linear and demand functions are affine.

CHAPTER 3

NASH GAME

Consumers either organized or unorganized play Nash with firms. We are only interested in symmetric equilibria.

Definition 1 *Let* $(\gamma^*)_{i=1}^n$ *be a list of strategies of consumers and* $(\beta^*)_{i=1}^m$ *be a list of strategies of firms. We say the list* $((\gamma^*)_{i=1}^n, (\beta^*)_{i=1}^m)$ *forms a Nash equilibrium in intercept terms of consumers and slope terms of firms when other agents stick to their strategies in the list, if for each consumer* γ^* *maximizes*

$$CS_i = \int_0^{\gamma - bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma)) \tag{3.1}$$

with respect to for any positive intercept term,γ*, where* $P(\gamma)$ *is the outcome price and solved from (1) for each* γ *and for each firm* β^* *maximizes*

$$\Pi_i = P(\beta)(\beta P(\beta)) - \alpha(\beta P(\beta))^2 \tag{3.2}$$

with respect to for any non negative slope term, β*, where* $P(\beta)$ *is the outcome price solved from (1).*

When consumers are organized, we define Nash equilibrium as follows:

Definition 2 *Let Γ^* be a particular strategy of consumer union and let $(\beta^*)_{i=1}^m$ be a list of firms' strategies. We say the list $(\Gamma^*, (\beta^*)_{i=1}^m)$ forms a Nash equilibrium in intercept term of aggregate demand and slope parameters of supply functions when other agents stick to their strategies in the list, if Γ^* maximizes total consumer surplus, TCS*

$$TCS = \int_0^{\Gamma - nbP(\Gamma)} (\frac{a}{b} - \frac{1}{bn}t)dt - P(\Gamma)(\Gamma - bnP(\Gamma)) \tag{3.3}$$

with respect to for any non negative Γ, where $P(\Gamma)$ is solved from (1) as $\frac{\Gamma}{nb+m\beta^*}$, and for each firm β^* maximizes

$$\Pi_i = P(\beta)(\beta P(\beta)) - \alpha(\beta P(\beta))^2 \tag{3.4}$$

with respect to for any positive slope term β, where $P(\beta)$ is solved from (1) as $\frac{\Gamma}{nb+(m-1)\beta^*+\beta}$.

Cournot-Nash Game

In Cournot-Nash game, consumers are not players. They submit their true demand functions. Given aggregate demand, firms compete through declearing quantities. Typical firm's problem,

$$\max_{q_j}(q_j(\frac{a}{b} - \frac{1}{bn}(\sum_{i=1}^m q_i)) - \alpha q_j^2) \tag{3.5}$$

Now we proceed case by case and give equilibrium strategies:

Case: $\alpha = 0,\ n \geq 1,\ m \geq 2$

In this case firms produce with zero cost and there are at least one consumer and two firms.

Proposition 1 *Let γ^* be equal to a and β^* be equal to ∞. Now, the list $((\gamma^*)_{i=1}^n, (\beta^*)_{i=1}^m)$ forms a Nash equilibrium.*

Proof: Suppose that firm j deviates from the proposed bunch of strategies. Firm j' s problem is to maximize its profit by picking a non-negative number or infinity for its slope term. Now, $AD(P) = na - nbP$ since each consumer tells the true intercept term. Since other firms give infinity for their slope terms aggregate supply is $P = 0$ line, i.e., quantity axes. Firm j can not change aggregate supply by giving non-negative finite number so it will obtain zero profit. Since announcing infinity for its slope term also gives zero profit, it is one of best responses. Return to typical consumer's problem, since firms supply at zero price, when he announces a positive number, γ , his surplus $CS_i(\gamma) = \frac{\gamma}{b}(a - \frac{\gamma}{2})$. When he gives a, $CS_i(a) = \frac{a^2}{2b}$. If $\gamma < a$, then $a = \gamma + \varepsilon$ for some $\varepsilon > 0$. Then $CS_i(a) = \frac{\gamma^2}{2b} + \frac{\gamma\varepsilon}{b} + \frac{\varepsilon^2}{2b}$ and $CS_i(\gamma) = \frac{\gamma^2}{2b} + \frac{\gamma\varepsilon}{b}$. Clearly, the former is bigger. If $\gamma \geq 2a$, then $CS_i(\gamma) \leq 0$, which is less than $CS_i(a)$ since the latter is positive. If $a < \gamma < 2a$, suppose that $CS_i(\gamma) \geq CS_i(a)$. Then $2\gamma a - \gamma\gamma \geq aa$. This leads to $a \geq \gamma$. Contradiction. So, when each of other consumers announces a γ and each firm announces an ∞, announcing a is the best response for the consumer i. In fact whatever the other consumers announce, a maximizes consumer i's surplus as long as a firm announces ∞. Thus, he does not want to deviate. Therefore, the bunch of strategies in which consumers tell their true intercept term and firms tell their true marginal cost functions turns out to be Nash Equilibrium. *QED*

Proposition 2 *Now, the list $((a)_{i=1}^n, (\infty)_{i=1}^m)$ forms a unique symmetric Nash equilibrium.*

Proof: Assume that each consumer announces a $\overline{\gamma} \succ 0$, i.e., there are symmetric strategies on the side of consumers. Then $AD(P) = n\overline{\gamma} - nbP$. Note that $\overline{\gamma}$ need not be equal to γ^*. Also assume that each firm except firm j announces a non-negative finite slope term, $\overline{\beta}$. Consider firm j's problem: Firm j will try to maximize (2) by giving a $\beta_j \geq 0$. Outcome price is solved from (1) as $P(\beta_j) = \frac{n\overline{\gamma}}{nb+(m-1)\overline{\beta}+\beta_j}$. Then firm j' s problem

$$\max_{\beta_j} P(\beta_j)(\beta_j P(\beta_j))$$
$$\text{s. to } \beta_j \geq 0$$

Set the Lagrangian as L $= (\frac{n\overline{\gamma}}{nb+(m-1)\overline{\beta}+\beta_j})^2\beta_j + \mu\beta_j$ and take the first order derivatives

$$\begin{aligned}\frac{\partial L}{\partial \beta_j} &= 2\beta_j \frac{-(n\overline{\gamma})^2}{(nb+(m-1)\overline{\beta}+\beta_j)^3} + (\frac{n\overline{\gamma}}{nb+(m-1)\overline{\beta}+\beta_j})^2 + \mu \\ \frac{\partial L}{\partial \mu} &= \beta_j\end{aligned}$$

Case 1: Assume that $\beta_j > 0$. Then $\mu = 0$ and $\frac{\partial L}{\partial \beta_j} = 0$. From which, obtain $\beta_j = (m-1)\overline{\beta}+nb$. Note that $m > 1$. If m=2, then $\beta_j = \overline{\beta}+nb$. Since $nb > 0$, $\beta_j \neq \overline{\beta}$. Similarly, when $m > 2$, suppose that $\beta_j = \overline{\beta}$. Then $\overline{\beta} = \frac{-nb}{(m-2)} < 0$ Contradiction to $\beta_j \succ 0$. Case 2: Assume that $\beta_j = 0$. Then $\mu \geq 0$. If $\overline{\beta} > 0$, then this is contrary to the symmetricity of strategies of the firms. Consider the case $\overline{\beta} = 0$. Then $\frac{\partial L}{\partial \beta_j} > 0$ since $\overline{\gamma} > 0$ and $\mu \geq 0$. Then maximum can not be at $\beta_j = 0$ for firm j when other firms give zero. Therefore, for this case, i.e. $m \succ 1$ and firms produce with zero cost, when each consumer announces the same $\overline{\gamma} > 0$, there are no symmetric, non-negative, finite equilibrium strategies on the side of firms. This is true for $\overline{\gamma} = a$ in particular. We know from Proposition 1, when each firm annonces ∞, announcing a maximizes each consumer's surplus and so $((a)_{i=1}^n, (\infty)_{i=1}^m)$ is a Nash Equilibrium. So we conlude that it is the unique symmetric one. *QED*

Lemma 1 *Consider the Cournot- Nash game for this case. Now, for* $q^* = \frac{an}{m+1}$ *the list* $(q^*)_{i=1}^m$ *forms a unique symmetric Nash equilibrium.*

Proof: Note that for this case $\alpha = 0$. Typical firm's problem is given in (3.5). First order condition for this problem, $(\frac{a}{b} - \frac{1}{bn}(\sum_{i=1}^m q_i) + q_j(\frac{-1}{bn}) = 0$. From which, $q_j^* = \frac{an}{2} - \frac{1}{2}(\sum_{i=1}^m q_i)$. Since we are interested in symmetric equilibrium strategies, $q^* = \frac{an}{2} - \frac{1}{2}(m-1)q^*$. From which, $q^* = \frac{an}{m+1}$. Note that second order derivative of the objective function is $\frac{-2}{bn}$, which is negative for any non-negative quantity choice. Thus, when other firms submit $q^* = \frac{an}{m+1}$, $q^* = \frac{an}{m+1}$ globally maximizes firm's profit and is the unique symmetric Nash equilibrium. *QED*

We denote market price, total quantity produced, consumer's surplus, firm's profit and total welfare by P, Q, CS, Π and SW respectively and put superscript $C-N$ and CPN to them to indicate that they are outcomes of Cornout-Nash game and the game where consumers play Nash together with the firms, respectively.

Proposition 3 $P^{C-N} > P^{CPN}$, $Q^{C-N} < Q^{CPN}$, $CS^{C-N} < CS^{CPN}$, $\Pi^{C-N} > \Pi^{CPN}$ *and* $SW^{C-N} < SW^{CPN}$.

Proof: We know that $P^{CPN} = 0$. Since each firm produces $\frac{an}{m+1}$, total quantity produced, $Q^{C-N} = m\frac{an}{m+1}$. Then $P^{C-N} = \frac{a}{b} - \frac{1}{bn}(m\frac{an}{m+1}) = \frac{a}{b}(1 - \frac{m}{m+1}) > 0$. So $P^{C-N} > P^{CPN}$. Since in the game where consumers play Nash together with firms total quantity produced is determined by the demand side and each consumer announces a, so $Q^{CPN} = an$. Since $\frac{m}{m+1} < 1$, $Q^{C-N} < Q^{CPN}$. At $P^{C-N} = \frac{a}{b}(1 - \frac{m}{m+1})$, each consumer will consume $a - bP^{C-N} = \frac{am}{m+1}$. Then $CS^{C-N} = \int_0^{\frac{am}{m+1}} (\frac{a}{b} - \frac{1}{b}t)dt - \frac{a}{b}(1 - \frac{m}{m+1})\frac{am}{m+1} = \frac{a^2m^2}{2b(m+1)^2}$. We know from Proposition 1, $CS^{CPN} = \frac{a^2}{2b}$. Clearly, $CS^{C-N} < CS^{CPN}$. Regarding the individual firms profit, $\Pi^{CPN} = 0$ since they sell at zero price and $\Pi^{C-N} = \frac{a}{b}(1 - \frac{m}{m+1})\frac{an}{m+1} = \frac{a^2n}{b(m+1)^2}$. Since the latter is positive, $\Pi^{C-N} > \Pi^{CPN}$. Since we define social welfare as the sum of total consumer surplus and total profit, $SW^{C-N} = nCS^{C-N} + m\Pi^{C-N} = \frac{a^2n(m^2+2m)}{2b(m+1)^2}$ and similarly $SW^{CPN} = \frac{a^2n}{2b}$. Since $\frac{(m^2+2m)}{(m+1)^2} < 1$, $SW^{C-N} < SW^{CPN}$. So we are done. *QED*

Proposition 4

$$\begin{aligned}
\lim_{m\to\infty} P^{C-N} &= P^{CPN} \\
\lim_{m\to\infty} Q^{C-N} &= Q^{CPN} \\
\lim_{m\to\infty} CS^{C-N} &= CS^{CPN} \\
\lim_{m\to\infty} \Pi^{C-N} &= \Pi^{CPN} \\
\lim_{m\to\infty} SW^{C-N} &= SW^{CPN}
\end{aligned}$$

Proof: Straightforward. *QED*

Corollary 1 *The outcome of the game where consumers and firms play Nash is the same with the competitive equilibrium outcome for this case.*

Proof: It follows from the standard result that as the number of firms goes to infinity in the limit the outcome of Cournot-Nash game arrives to the competitive equilibrium outcome and the Proposition 4. *QED*

Case: $\alpha = 0,\ n \geq 1,\ m = 1$

For this case there is natural monopoly producing with zero cost and at least one consumer.

Proposition 5 *Let γ^* be equal to $\frac{a(4n-2)}{(4n-1)}$ and β^* be equal to bn. Now, the list $((\gamma^*)_{i=1}^n, \beta^*)$ forms a Nash equilibrium.*

Proof: Let $\overline{P}$ be the price satisfing $bn\overline{P} = na - nb\overline{P}$. Then $\overline{P} = \frac{a}{2b}$. Note that this is the price when each consumer announces a and the firm announces nb. Now, $\frac{\gamma^*}{b} > \overline{P}$. To see this suppose the contrary. Then $\frac{(4n-2)}{(4n-1)} \leq \frac{1}{2}$. This leads to $(4n-3) \leq 0$. Contradiction. So, $\frac{\gamma^*}{b} > \overline{P}$. Lets check whether consumer i wants to deviate. Assume all other consumers announce γ^* and the firm announces β^*. Consumer i will solve (3.2). Firstly, γ must be smaller than a. To see this suppose the contrary. Then $\gamma > a$. Then $\gamma - bP > a - bP$ for all $P \geq 0$. Let $P(\gamma)$ be the outcome price when consumer i announces γ and $P(a)$ be the outcome price when consumer i announces a. Note that $\gamma^* < a$ since $\frac{(4n-2)}{(4n-1)} < 1$. This will guarantee that when consumer i gives a or any value bigger than a, he will consume positive amount. Since $a - bP < \gamma - bP$ for all $P \geq 0$, aggregate demand, $AD(P)$ will be less at each price when consumer i gives the former. Moreover, since aggregate supply, $AS(P) = \beta^* P = nbP$, which is a linear function so when the consumer i announces a rather than γ, AD and AS will intersect at a lower price, that is, $P(a) < P(\gamma)$. Remember that the firm announces nb and all consumers except consumer i announces γ^*. When consumer i announces a, $P(a)$ is the outcome price and less than $\overline{P}$. Then $\frac{\gamma^*}{b} > P(a)$. This will guarantee that remaining consumers will consume positive amount. Since $P(a) \prec P(\gamma)$, remaining consumers will consume more, whereas consumer i will consume less when he announces a. Now, calculate $CS_i(a) = \int_0^{a-bP(a)} (\frac{a}{b} - \frac{1}{b}t)dt - P(a)(a - bP(a)) = (a - bP(a))(\frac{\frac{a}{b}-P(a)}{2})$. Note that $CS_i(a)$ is also equal to $\int_{P(a)}^{\frac{a}{b}} (a - bt)dt$. Now, since $P(\gamma) > P(a)$ and $a > 0$ and $b > 0$,

$$\int_{P(a)}^{\frac{a}{b}} (a - bt)dt > \int_{P(\gamma)}^{\frac{a}{b}} (a - bt)dt$$

Consumer i's surplus when he announces γ is $CS_i(\gamma) = \int_0^{\gamma-bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma))$. Note that since $\gamma > a$, $\gamma = a + \epsilon$ for some $\epsilon > 0$. Put $a + \epsilon$ instead of γ in $CS_i(\gamma)$ and obtain $CS_i(\gamma)$ as $\frac{a}{b}(a - bP(\gamma)) - \frac{1}{2b}(a - bP(\gamma))^2 - P(\gamma)(a - bP(\gamma)) + \frac{a}{b}\epsilon - \frac{1}{2b}(\epsilon^2 + 2(a - bP(\gamma))\epsilon - P(\gamma)\epsilon$, which is equal to $\int_0^{a-bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(a - bP(\gamma)) + \frac{a}{b}\epsilon - \frac{1}{2b}(\epsilon^2 + 2(a - bP(\gamma))\epsilon) - P(\gamma)\epsilon)$, which in turn equals to $\int_{P(\gamma)}^{\frac{a}{b}} (a - bt)dt + \frac{a}{b}\epsilon - \frac{1}{2b}(\epsilon^2 + 2(a - bP(\gamma))\epsilon) - P(\gamma)\epsilon$. After a little algebra, $CS_i(\gamma) = \int_{P(\gamma)}^{\frac{a}{b}} (a - bt)dt - \frac{1}{2b}\epsilon^2$. Since $\int_{P(a)}^{\frac{a}{b}} (a - bt)dt > \int_{P(\gamma)}^{\frac{a}{b}} (a - bt)dt$ and left hand side of inequality is $CS_i(a)$, $CS_i(a) > CS_i(\gamma)$. Thus, $\gamma \leq a$. Suppose that consumer i announces a $\gamma > 0$ such that at the resulting outcome price he consumes zero, for example very small $\gamma > 0$. Then his surplus is zero. However, if he announced a, he would obtain positive surplus. So γ must be such that at the resulting outcome price he consumes positive amount. Since $\frac{\gamma^*}{b} > \overline{P}$ and $\gamma \leq a$, this implies that all consumers should consume positive amount when consumer is surplus is maximized. Now, define $\underline{P}$ as the price satisfing $(n - 1)\gamma^* - (n - 1)b\underline{P} = nb\underline{P}$. Note that this is the price when we exlude the consumer i. Then $\underline{P} = \frac{(n-1)\gamma*}{(n-1)b+nb}$. Now, if γ is such that $\frac{\gamma}{b} < \underline{P}$,

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

Then $P(\gamma) = \underline{P}$ and it this price consumer i will consume zero so his surplus is zero. However, by giving a he can obtain positive surplus. So, $\forall \gamma < \underline{P}b$: γ can not be the best response of consumer i. Thus, $\underline{P}b \leq \gamma \leq a$, i.e., $\frac{(n-1)\gamma^*}{(2n-1)} \leq \gamma \leq a$. Now, i) Let γ be such that $\frac{\gamma}{b} \in [\underline{P}, \frac{\gamma^*}{b})$. Then

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

Consider the function $G(P) = \gamma + (n-1)\gamma^* - nbP \quad \forall P \geq 0$. Now, $G(P) = AD(P) \quad \forall P \leq \frac{\gamma}{b}$. Now, If $\frac{\gamma}{b} > \underline{P}$, then $\frac{\gamma}{b} > P(\gamma)$. To see this, Assume that $\frac{\gamma^*}{b} > \frac{\gamma}{b} > \underline{P}$. Suppose that $\frac{\gamma}{b} \leq P(\gamma)$. Now, $P(\gamma)$ is the price satisfing $AD(P) = AS(P)$. Since $\frac{\gamma}{b} > \underline{P}$, $AD(P)$

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* \quad nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

Consider the case $\frac{\gamma}{b} = P(\gamma)$. At this price $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$ and $AS(P(\gamma)) = nbP(\gamma)$. Then $P(\gamma) = \underline{P}$. But then $\frac{\gamma}{b} = \underline{P}$. Contradiction. Proceed with the case $\frac{\gamma}{b} < P(\gamma)$. At such an price $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$ and $AS(P(\gamma)) = nb\underline{P}$. But then $\frac{\gamma}{b} = \underline{P}$. Contradiction. So $\frac{\gamma}{b} > P(\gamma)$. Note that if $\frac{\gamma}{b} = \underline{P}$, then $P(\gamma) = \underline{P} = \frac{\gamma}{b}$. Since then

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

By definition of $\underline{P}$, $(n-1)\gamma^* - (n-1)b\underline{P} = nb\underline{P}$. Since AS and AD intersect at a unique point and $\frac{\gamma}{b} = \underline{P}$, it follows that $AS(\underline{P}) = AD(\underline{P})$. Then $\underline{P} = P(\gamma)$ by definition of $P(\gamma)$. Now, ii) Let γ be such that $\frac{\gamma}{b} \in [\frac{\gamma^*}{b}, \frac{a}{b}]$.If $\gamma = \gamma^*$, then $AD(P) = n\gamma^* - nbP$. Then $P(\gamma) = \frac{\gamma^*}{2b}$. This is clearly smaller than $\frac{\gamma^*}{b} = \frac{\gamma}{b}$.

So $\frac{\gamma}{b} > P(\gamma)$. Now,

$$AD(P) = \begin{cases} \gamma - bP & \text{if } P \geq \frac{\gamma^*}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma^*}{b} \end{cases}$$

Note that $\gamma \leq a$ and $\gamma^* < a$. Suppose that $P(\gamma) \geq \frac{\gamma^*}{b}$. Then $AD(P(\gamma)) = \gamma - bP(\gamma)$ and $AS(P(\gamma)) = nbP(\gamma)$. From $AD(P(\gamma)) = AS(P(\gamma))$, $P(\gamma) = \frac{\gamma}{nb+b}$. Then since $\gamma \leq a$, $P(\gamma) = \frac{\gamma}{nb+b} \leq \frac{a}{nb+b} \leq \frac{a}{2b} = \overline{P}$. At the very beginning we showed that $\frac{\gamma^*}{b} > \overline{P}$. Then $\frac{\gamma^*}{b} > \overline{P} \geq P(\gamma)$. Contradiction. So $P(\gamma) < \frac{\gamma^*}{b}$. Then $\forall\gamma \in (\gamma^*, a] : \frac{\gamma}{b} > P(\gamma)$. Combining all the results obtained so far $\forall\gamma \in (\underline{P}b, a] : \frac{\gamma}{b} > P(\gamma)$ and for $\gamma = \underline{P}b$, $\frac{\gamma}{b} = P(\gamma)$. Now, if $\gamma \in (\gamma^*, a]$, then

$$AD(P) = \begin{cases} \gamma - bP & \text{if } P \geq \frac{\gamma^*}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma^*}{b} \end{cases}$$

We showed previously $\gamma^* > P(\gamma)$. Consider the function, $G(P) = \gamma + (n-1)\gamma^* - nbP \quad \forall P \geq 0$. Now, $G(P) = AD(P) \quad \forall P \leq \frac{\gamma^*}{b}$. Since $\gamma \in (\gamma^*, a]$ and so $\frac{\gamma}{b} > \frac{\gamma^*}{b} > P(\gamma)$. So $\forall\gamma \in (\gamma^*, a] : G(P(\gamma) = AD(P(\gamma))$. If $\gamma = \gamma^*$, then $AD(P) = n\gamma^* - nbP \quad \forall P \geq 0$. Clearly, $G(P) = AD(P) \quad \forall P \geq 0$. And so $G(P(\gamma)) = AD(P(\gamma))$.

If $\gamma \in [\underline{P}b, \gamma^*)$, then

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

We showed that $\forall\gamma \in (\underline{P}b, \gamma^*] : P(\gamma) < \frac{\gamma}{b}$. Then $G(P(\gamma)) = AD(P(\gamma))$ and if $\frac{\gamma}{b} = \underline{P}$, $P(\gamma) = \underline{P} = \frac{\gamma}{b}$ and so for $\gamma = \underline{P}b$, $G(P(\gamma)) = AD(P(\gamma))$. We also showed that if $\gamma < \underline{P}b$, then $CS_i(\gamma) = 0$ and if $\gamma > a$, then $CS_i(a) > CS_i(\gamma)$. Now, $CS_i(\gamma)$ is a continuous function and $[\underline{P}b, a]$ is a compact interval so $CS_i(\gamma)$ arrives its maximum in this interval. Then the problem is

$$\max_{\gamma \in [\underline{P}b, a]} CS_i(\gamma) = \int_0^{\gamma - bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma))$$

Since $\forall\gamma \in [\underline{P}b, a] : AD(P(\gamma)) = G(P(\gamma))$ and by definition of $P(\gamma)$, $AS(P(\gamma)) = AD(P(\gamma)) = G(P(\gamma))$. Then $\forall\gamma \in [\underline{P}b, a] : P(\gamma) = \frac{(n-1)\gamma^*+\gamma}{2nb}$. Put it into objective function. F.O.C. for this problem, $-(1-\frac{1}{2n})(-\frac{a}{b}+\frac{2}{2b})(\gamma(1-\frac{1}{2n})-\frac{(n-1)\gamma^*}{2n})+\frac{(n-1)\gamma^*}{2nb}+\frac{\gamma}{2nb})-\frac{1}{2nb}(\gamma(1-\frac{1}{2n})-\frac{(n-1)\gamma^*}{2n})=0$. After arranging terms, one obtains $a2n(2n-1)+\gamma^*(n-1)=\gamma(4n^2-1)$. When we replace γ^* with $\frac{a(4n-2)}{(4n-1)}$, we obtain $\gamma^c = \frac{a(4n-2)}{(4n-1)}$. So γ^c satisfies first order condition. One can verify that $\gamma^c \in (\underline{P}b, a)$. Now second order derivative of the objective function: $-\frac{2}{b}\gamma(1-\frac{1}{2n})-\frac{1}{2nb}(1-\frac{1}{2n})-\frac{1}{2nb}(1-\frac{1}{2nb})$, which is negative for any non-negative γ and so in particular for each element from the interval $[\underline{P}b, a]$. Then γ^c maximizes $CS_i(\gamma)$ over positive real numbers, when every other consumer announce γ^* and the firm announces bn. So γ^c is the best response of consumer i to others proposed strategies and $\gamma^c = \gamma^*$. Since consumers are identical, γ^* is the best response of each consumer when each of remaining consumers stick to γ^* and the firm sticks to bn. Lets check whether the firm wants to deviate. Suppose that consumers announce the same $\overline{\gamma} > 0$, the firm will solve (3.3) by picking a non-negative slope coefficient, β. Note that $P(\beta)$ is solved from (1) as $P(\beta) = \frac{n\overline{\gamma}}{nb+\beta}$. Then the firms problem becomes

$$\max_{\beta} \Pi(\beta) = (\frac{n\overline{\gamma}}{nb+\beta})^2\beta$$

It is clear that objective function is continuos function of β. First order condition for this problem, $2(\frac{n\overline{\gamma}}{nb+\beta})(\frac{-n\overline{\gamma}}{(nb+\beta})^2)\beta+(\frac{n\overline{\gamma}}{nb+\beta})^2=0$. When we arrange the terms we observe that $\overline{\gamma}$ disappears. This means the firms choice is independent of particular value of γ rather it depends on their symmetricity. Then we obtain $\beta^* = nb$. Second order derivative is $\frac{-(n\overline{\gamma})^2((nb+\beta)^3-3(nb+\beta)^2(\beta-nb))}{(nb+\beta)^6}$. At β^* it is negative. One can verify that $\Pi(\beta^*) > \Pi(\beta) \quad \forall\beta > 2nb$. Then we can restrict the domain to a compact interval $[0, 2nb]$. Since profit function is continuos function of β, it arrives its maximum in this restricted compact domain. Note that $\beta^* = bn$ is the only point first order derivaitive vanishes and second order condition is satisfied. we conclude that $\beta^* = bn$ maximizes profit function over non-negative real numbers and so it is the best response of the firm to consumers each of whom sticks to a $\overline{\gamma} > 0$ and so in particular to $(\gamma^*)_{i=1}^n = (\frac{a(4n-2)}{(4n-1)})_{i=1}^n$. Therefore we conclude that the list $((\gamma^*)_{i=1}^n, \beta^*)$ forms a Nash Equilibrium. *QED*

Proposition 6 *Let γ^* and β^* as in Proposition 5. Now, the list $((\gamma^*)_{i=1}^n, \beta^*)$ forms a unique symmetric Nash equilibrium.*

Proof: Suppose there exists another symmetric bunch of strategies, $((\hat{\gamma})_{i=1}^n, \hat{\beta})$ which forms a Nash Equilibrium. Since each consumer announces the same $\hat{\gamma} > 0$, $\hat{\beta}$ must be equal to $\beta^* = bn$ from the firms problem in the proof of Proposition 5. Then $\hat{\gamma}$ must be different than γ^*. Now, a is different than γ^*. Let's check whether $((a)_{i=1}^n, \beta^*)$ forms a symmetric Nash equilibrium or not. Consider consumer i: if he announces a, $P(a) = \frac{a}{2b} = \overline{P}$. At this price he will consume $a - b(\frac{a}{2b}) = \frac{a}{2}$. His surplus $CS_i(a) = \int_{\frac{a}{2b}}^{\frac{a}{b}} (a - bt)dt = \frac{a^2}{8b}$. If he gave $\gamma^* = \frac{a(4n-2)}{(4n-1)}$, then

$$AD(P) = \begin{cases} (n-1)a - (n-1)bP & \text{if } P \geq \frac{\gamma^*}{b} \\ \gamma^* + (n-1)a - nbP & \text{if } P \leq \frac{\gamma^*}{b} \end{cases}$$

Since $\frac{\gamma^*}{b} > \overline{P}$ and $\overline{P} > P(\gamma^*)$, $P(\gamma^*) = \frac{\gamma^* + (n-1)a}{2nb}$. And he will consume at this price as $\gamma^* - bP(\gamma^*) = \gamma^* - b\frac{\gamma^* + (n-1)a}{2nb}$. His surplus can be calculated from, $CS_i(\gamma^*) = \frac{a}{b}(\gamma^* - bP(\gamma^*)) - \frac{1}{2b}(\gamma^* - bP(\gamma^*))^2 - P(\gamma^*)(\gamma^* - bP(\gamma^*)) = \frac{a^2}{8b} \frac{4n(4n^2-3n+1)(4n-1) - (4n^2-3n+1)^2 - 2(4n^2-n+1)(4n^2-3n+1))}{n^2(4n-1)^2} = \frac{a^2}{8b}\theta$, where

θ is the corresponding coefficient. Note that $CS_i(a) = \frac{a^2}{8b}$. Suppose that $CS_i(\gamma^*) \leq CS_i(a)$. Then $\theta \leq 1$. Then $(4n^2 - 3n + 1)(4n(4n - 1) - 4n^2 - 3n + 1 - 2(4n^2 - n + 1)) \leq n^2(4n - 1)^2$. This leads to $(4n^2 - 2n + 1) \leq 0$. Contradiction since $n \geq 2$. So $CS_i(\gamma^*) > CS_i(a)$. Then when the firm announces $\beta^* = bn$ and each of other consumers announces $\hat{\gamma} = a$, announcing $\hat{\gamma} = a$ is not the best response of consumer i. Thus, $((a)_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Now, let $\hat{\gamma}$ be such that $\gamma^* < \hat{\gamma} < a$. Let's check whether $((\hat{\gamma})_{i=1}^n, \beta^*)$ forms a Nash equilibrium. Note that $\hat{\gamma} = t\gamma^* + (1-t)a$ for some $t \in (0,1)$. Then $\hat{\gamma} = t\frac{a(4n-2)}{(4n-1)} + (1-t)a = a\frac{(4n-1-t)}{(4n-1)}$ for some $t \in (0,1)$. Let's check whether consumer i wants to deviate. if he announces $\hat{\gamma}$, then his surplus $CS_i(\hat{\gamma}) = \frac{a}{b}(\hat{\gamma} - bP(\hat{\gamma})) - \frac{1}{2b}(\hat{\gamma} - bP(\hat{\gamma}))^2 - P(\hat{\gamma})(\hat{\gamma} - bP(\hat{\gamma}))$, where $P(\hat{\gamma}) = \frac{n\hat{\gamma}}{2nb}$ from (1) using $AD(P) = n(\hat{\gamma} - bP)$ and $AS(P) = \beta^* P = bnP$. After a little algebra, $CS_i(\hat{\gamma}) = \frac{4a\hat{\gamma} - 3\hat{\gamma}^2}{8b}$. Replace $\hat{\gamma}$ with $a\frac{(4n-1-t)}{(4n-1)}$. Then consumer i's surplus is calculated as $CS_i(\hat{\gamma}) = \frac{a^2}{8b}(\frac{(4n-1)^2 + 2t(4n-1) - 3t^2}{(4n-1)^2})$. If he announces γ^* while other consumers announce $\hat{\gamma}$ and the firm announces β^*, then $P(\gamma^*) = \frac{(n-1)\hat{\gamma} + \gamma^*}{2nb}$ since $\frac{\hat{\gamma}}{b} > \frac{\gamma^*}{b} > \overline{P} > P(\hat{\gamma}) > P(\gamma^*)$. Now, his surplus can be calculated by replacing $P(\gamma^*)$ with $\frac{(n-1)\hat{\gamma}+\gamma^*}{2nb}$, $\hat{\gamma}$ with $a\frac{(4n-1-t)}{(4n-1)}$ from $CS_i(\gamma^*) = \frac{a}{b}(\gamma^* - bP(\gamma^*)) - \frac{1}{2b}(\gamma^* - bP(\gamma^*))^2 - P(\gamma^*)(\gamma^* - bP(\gamma^*))$. After a bit massy algebra, $CS_i(\gamma^*) = \frac{a^2}{8b}\frac{(4n^2-3n+1+t(n-1))(12n^2-5n+3+3t(n-1))}{n^2(4n-1)^2}$. Suppose that $CS_i(\hat{\gamma}) \geq CSi(\gamma^*)$. Then $\frac{(4n^2-3n+1+t(n-1))(12n^2-5n+3+3t(n-1))}{n^2(4n-1)^2} \leq (\frac{(4n-1)^2+2t(4n-1)-3t^2}{(4n-1)^2})$ After a little algebra, this reduces to $32n^4 - 48n^3 + 38n^2 - 14n + 12t(n-1)n^2 - 5nt(n-1) + 10tn^2(4n-1) - 9nt(4n-1) + 3 + 3t(4n-1) + 3t(n-1) + 3t^2(4n-1)(n-1) + 3tn^2 \leq 0$, where $n \geq 2$. Contradiction. Note that left hand side is positive since $32n^4 - 48n^3 > 0$, $38n^2 - 14n > 0$, $12t(n-1)n^2 - 5nt(n-1) > 0$ and $10tn^2(4n-1) - 9nt(4n-1) > 0$ and other terms are positive. Thus, $CS_i(\gamma^*) > CS_i(\hat{\gamma})$. So $\hat{\gamma}$ is not the best response of consumer i to the other players' strategy profile in which each consumer announces $\hat{\gamma}$ and the firm announces β^*. Thus, $((\hat{\gamma} : \gamma^* < \hat{\gamma} < a,\ ((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Combining both results, $\forall \hat{\gamma} : \gamma^* < \hat{\gamma} \leq a$, $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Now, focus on the values less than γ^*. Let $\hat{\gamma}$ be such that $\hat{\gamma} < \gamma^*$. Then $\hat{\gamma} = t\gamma^* = t\frac{a(4n-2)}{(4n-1)}$ for some $t \in (0,1)$. Let's check $((\hat{\gamma})_{i=1}^n, \beta^*)$ is a Nash equilibrium or not. Now, all consumers except consumer i sticks $\hat{\gamma}$ and the firm sticks to β^*. If consumer i announces also $\hat{\gamma}$, then the outcome price $P(\hat{\gamma}) = \frac{n\hat{\gamma}}{2nb}$ as in above. Then his surplus $CS_i(\hat{\gamma}) = \frac{a}{b}(\hat{\gamma} - bP(\hat{\gamma})) - \frac{1}{2b}(\hat{\gamma} - bP(\hat{\gamma}))^2 - P(\hat{\gamma})(\hat{\gamma} - bP(\hat{\gamma}))$ is calculated by replacing $P(\hat{\gamma})$ with $\frac{n\hat{\gamma}}{2nb}$ and $\hat{\gamma}$ with $t\frac{a(4n-2)}{(4n-1)}$. After a little algebra, $CS_i(\hat{\gamma}) = \frac{a^2}{8b}(\frac{4t(4n-2)(4n-1)-3t^2(4n-2)^2}{(4n-1)^2})$. If consumer announces γ^* instead, then there are two cases to be considered. Define $\underline{P^*}$ as the price satisfying $\gamma^* - b\underline{P^*} = nb\underline{P^*}$. Then $\underline{P^*} = \frac{\gamma^*}{(n+1)b} = \frac{t\gamma^*}{b} = \frac{\hat{\gamma}}{b}$, where $t = \frac{1}{n+1}$. Note that if $t \leq \frac{1}{n+1}$, then $P(\gamma^*) = \underline{P^*}$ and only consumer i consumes positive amount at $\underline{P^*}$. If $t > \frac{1}{n+1}$, then $P(\gamma^*) = \frac{(n-1)\hat{\gamma}+\gamma*}{2nb} = \frac{((n-1)t+1)\gamma^*}{2nb}$. To see this suppose the contrary. Since

$$AD(P) = \begin{cases} \gamma^* - bP & \text{if } P \geq \frac{\hat{\gamma}}{b} \\ \gamma^* + (n-1)\hat{\gamma} - nbP & \text{if } P \leq \frac{\hat{\gamma}}{b} \end{cases}$$

, $P(\gamma^*) > \frac{\hat{\gamma}}{b}$. But then $P(\gamma^*) = \underline{P^*} = \frac{1}{n+1}\frac{\gamma^*}{b} > \frac{\hat{\gamma}}{b} = \frac{t\gamma^*}{b} > \frac{1}{n+1}\frac{\gamma^*}{b}$. Contradiction. So $P(\gamma^*) = \frac{(n-1)\hat{\gamma}+\gamma*}{2nb} = \frac{((n-1)t+1)\gamma^*}{2nb}$. Note that each consumer consumes a positive amount at $P(\gamma^*)$. Proceed with the case $\hat{\gamma} : \hat{\gamma} = t\gamma^*$ for some $\frac{1}{(n+1)} < t < 1$. Now, if consumer i announces γ^* while other players stick to their strategies in the proposed bunch strategies, his surplus can be calculated by replacing $P(\gamma^*)$ with $\frac{((n-1)t+1)\gamma^*}{2nb}$ and γ^* with $(\frac{a(4n-2)}{(4n-1)})$ from

$CS_i(\gamma^*) = \frac{a}{b}(\gamma^* - bP(\gamma^*)) - \frac{1}{2b}(\gamma^* - bP(\gamma^*))^2 - P(\gamma^*)(\gamma^* - bP(\gamma^*))$. After a little algebra, $CS_i(\gamma^*)$ equals to $\frac{a^2(4n-2)}{8b(4n-1)^2}\frac{(2n-1-(n-1)t)}{n^2}(4n(4n-1) - (4n-2)(2n-1-(n-1)t) - 2((n-1)t+1)(4n-2))$. Note that when consumer i announces $\hat{\gamma}$ instead, then his surplus as above is $\frac{a^2}{8b}\frac{4t(4n-2)(4n-1)-3t^2(4n-2)^2}{(4n-1)^2}$. Suppose that $CS_i(\hat{\gamma}) \geq CS_i(\gamma^*)$. Then $n^2(4t(4n-1) - 3t^2(4n-2)) \geq (2n-1-(n-1)t)(16n^2 - 4n - (4n-2)(2n-1-(n-1)t) - 2((n-1)t+1)(4n-2))$. Open up brackets in each side, make cancellations, collect the terms in one side and obtain $0 \geq 16n^3 - 16n^2 - 32tn^3 + 32tn^2 - 16tn + 8n + 4t - 2 + 16t^2n^3 - 16t^2n^2 + 8t^2n - 2t^2$. Arrange the terms and obtain $0 \geq n^3(16-32t+16t^2)+n^2(-16+32t-16t^2)+n(-16t+8+8t^2)+4t-2-2t^2$, which in turn can be written as $0 \geq n^2[(n-1)(16-32t+16t^2)] + n(-16t+8+8t^2) + 4t - 2 - 2t^2$. Note that $t \mapsto 1 \Leftrightarrow \hat{\gamma} \mapsto \gamma^* \Leftrightarrow CS_i(\hat{\gamma}) \mapsto CS_i(\gamma^*)$. This can be seen in the preceding inequality since as $t \mapsto 1$, right hand side goes to zero. Arrange the terms in the preceding inequality a bit more $2(1-2t+t^2) \geq n^2(n-1)(16-32t-16t^2) + n(-16t+8+8t)$. Then $\frac{2}{n}(1-2t+t^2) \geq (n(n-1)16(1-2t+t^2)+(16t+8+8t)) > n(n-1)16(1-2t+t^2)$. But then $\frac{2}{n} > n(n-1)16$. Contradiction since $n \geq 2$. Thus, $\forall\hat{\gamma}$: for some $\frac{1}{n+1} < t < 1$, $CS_i(\gamma^*) > CS_i(\hat{\gamma})$ and so $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Now, we proceed with the case $t \leq \frac{1}{n+1}$. Let $\hat{\gamma}$ be such that $\hat{\gamma} = t\gamma^*$ for some $0 < t \leq \frac{1}{n+1}$. Now, all consumers except consumer i sticks to $\hat{\gamma}$ and the firm sticks to β^*. If consumer i announces $\hat{\gamma}$, his surplus as in the previous case $CS_i(\hat{\gamma}) = \frac{a^2}{2b}(4n-2)(\frac{4t(4n-1)-3t^2(4n-2)}{4(4n-1)^2})$. If he deviates to γ^*, his surplus again calculated from , $CS_i(\gamma^*) = \frac{a}{b}(\gamma^* - bP(\gamma^*)) - \frac{1}{2b}(\gamma^* - bP(\gamma^*))^2 - P(\gamma^*)(\gamma^* - bP(\gamma^*))$, where $P(\gamma^*) = \underline{P^*} = \frac{\gamma^*}{(n+1)b}$ since $\frac{\hat{\gamma}}{b} \leq \underline{P^*}$, i.e, only consumer i consumes positive amount at the outcome price. $CS_i(\gamma^*)$ is calculated by replacing $P(\gamma^*) = \frac{\gamma^*}{(n+1)b}$ and γ^* with $\frac{a(4n-2)}{(4n-1)}$ as $\frac{a^2}{2b}(4n-2)(\frac{2(4n-1)n(n+1)-2n(4n-2)-(4n-2)n^2}{(n+1)^2(4n-1)^2})$. Suppose $CS_i(\hat{\gamma}) \geq CS_i(\gamma^*)$. Then $(\frac{2(4n-1)n(n+1)-2n(4n-2)-(4n-2)n^2}{(n+1)^2(4n-1)^2}) \leq (\frac{4t(4n-1)-3t^2(4n-2)}{4(4n-1)^2})$. After eliminating the denominators, making cancellations and collecting the terms in one side, one obtains the following inequality $0 \leq n^3(-16+16t+28t) - 12t^2n^3 - n^2 18t^2 + n(8t-8) + t(6t-4)$. Note that maxumum value for t can be $\frac{1}{3}$. Then $(-16+16t+28t)$ and other terms are negative. Contradiction. So $CS_i(\gamma^*) > CS_i(\hat{\gamma})$. Thus, $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Summing up the results so far we obtained, while each of other consumers sticks to $\hat{\gamma}$ and firm sticks to β^*, $\forall \hat{\gamma} \in (0,a] \setminus \{\gamma^*\} : ((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Let $\hat{\gamma} > a$. Then $\hat{\gamma} = a + \epsilon$ for some $\epsilon > 0$. Let $\underline{\hat{P}}$ be the price satisfing $(n-1)\hat{\gamma} - (n-1)b\underline{\hat{P}} = nb\underline{\hat{P}}$. Then $\underline{\hat{P}} = \frac{(n-1)\hat{\gamma}}{(2n-1)b}$. For $\underline{\hat{P}}$ to be equal to $\frac{a}{b}$, ϵ must be equal to $\frac{an}{(n-1)}$, where $n \geq 2$. Let $\hat{\gamma}$ be such that $\hat{\gamma} = a + \epsilon$ for $\epsilon \geq \frac{an}{(n-1)}$. Assume that each of consumers except consumer i sticks to $\hat{\gamma}$ and the firm sticks to β^*. If consumer i announces a then he will consume zero since $P(a) = \underline{\hat{P}} \geq \frac{a}{b}$. Then $CS_i(a) = 0$. If consumer announces $\hat{\gamma}$ instead, then $P(\hat{\gamma}) = \frac{n\hat{\gamma}}{2nb}$ from (1) by using $AD(P) = n(\hat{\gamma} - bP)$ and $AS(P) = bnP$. By using $\hat{\gamma} = a + \epsilon \geq (a + \frac{an}{(n-1)})$, we conclude that $P(\hat{\gamma}) > \frac{a}{b}$. But then $CS_i(\hat{\gamma}) < 0 = CS_i(a)$. Thus, $\forall \hat{\gamma} : \hat{\gamma} = a + \epsilon$ for $\epsilon \geq \frac{an}{(n-1)}$, $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Consider the case $\hat{\gamma} = a + \epsilon$ for $\epsilon < \frac{an}{(n-1)}$. Assume that that each of consumers except consumer i sticks to $\hat{\gamma}$ and the firm sticks to β^*. If consumer i announces $\hat{\gamma}$, as in above $P(\hat{\gamma}) = \frac{n\hat{\gamma}}{2nb}$. And $CS_i(\hat{\gamma})$ is calculated from $CS_i(\hat{\gamma}) = \frac{a}{b}(\hat{\gamma} - bP(\hat{\gamma})) - \frac{1}{2b}(\hat{\gamma} - bP(\hat{\gamma}))^2 - P(\hat{\gamma})(\hat{\gamma} - bP(\hat{\gamma}))$ as $CS_i(\hat{\gamma}) = \frac{4a\hat{\gamma} - 3\hat{\gamma}^2}{8b} = \frac{(4a-3(a+\epsilon)(a+\epsilon)}{8b} = \frac{a^2+a\epsilon-3\epsilon a-3\epsilon^2}{8b} = \frac{a^2n^2+a\epsilon n^2-3\epsilon an^2-3\epsilon^2n^2}{8bn^2}$. If consumer i announces a instead, then $P(a) = \frac{(n-1)\hat{\gamma}+a}{2nb} = \frac{(n-1)\epsilon+na}{2nb}$. To see this suppose contrary and verify that it leads to contradiction. Then from $CS_i(a) = \frac{a}{b}(a - bP(a)) - \frac{1}{2b}(a - bP(a))^2 - P(a)(a - bP(a))$ by replacing $P(a)$ with $\frac{(n-1)\epsilon + na}{2nb}$ and doing a little algebra, $CS_i(a) = \frac{a^2n^2 + a\epsilon n^2 - 3\epsilon an^2 + (2an\epsilon + \epsilon^2 n^2 - 2n\epsilon^2 + \epsilon^2)}{8bn^2}$. Now, $CS_i(a) > CS_i(\hat{\gamma})$ since $\epsilon^2 n^2 - 2n\epsilon^2 \geq 0$, where $n \geq 2$. Thus, $\forall \hat{\gamma} : \hat{\gamma} = a + \epsilon$ for $\epsilon < \frac{an}{(n-1)}$, $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash equilibrium. Combining all, $\forall \hat{\gamma} : \hat{\gamma} \in (0, \infty) \setminus \{\gamma^*\}$, $((\hat{\gamma})_{i=1}^n, \beta^*)$ is not a Nash Equilibrium. Since we know from Proposition 5 $((\gamma^*)_{i=1}^n, \beta^*)$ is a Nash equilibrium, it turns out that it is the unique symmetric one. *QED*

Proposition 7 *Assume that consumers are organized. Let Γ^* be equal to $\frac{2an}{3}$ and β^* be equal to bn. Now, the list (Γ^*, β^*) forms a unique Nash equilibrium.*

Proof: Let β^* be chosen by the firm. Consumer Union (CU), will solve (3.4). Now, total consumers' surplus (TCS) is $TCS(\Gamma) = \frac{a}{b}(\Gamma - bnP(\Gamma)) - (\Gamma - bnP(\Gamma))^2 - P(\Gamma)(\Gamma - bnP(\Gamma))$, where $P(\Gamma)$ is $\frac{\Gamma}{nb+\beta^*}$. F.O.C. for this problem: $(\frac{2\Gamma}{nb+\beta^*})(\frac{-\Gamma}{(nb+\beta^*)^2}) + (\frac{\Gamma}{nb+\beta^*})^2 = 0$ Replace β^* with bn and solve for Γ. Then $\Gamma^c = \frac{2an}{3} = \Gamma^*$. Find the second order derivative (S.O.D.) of the objective function as $\frac{-1}{4bn} - \frac{1}{2bn} < 0 \quad \forall \Gamma \geq 0$. Moreover, since objective function is continuos and first order derivative vanishes at Γ^* , we conclude that TCS arrives its maximum at Γ over $(0, \infty)$. Now, let a $\Gamma > 0$ be chosen by CU. The firms problem is given in (3.5) for $\alpha = 0$ and $m = 1$, of course. Then the firm's problem

$$\max_{\beta} \Pi(\beta) = (\frac{\Gamma}{nb + \beta})^2 \beta$$

F.O.C. for this problem: $(\frac{2\Gamma}{nb+\beta})(\frac{-\Gamma}{(nb+\beta)^2} + \frac{\Gamma^2}{(nb+\beta)^2}) = 0$. One can verify that Γ cancels out and β^c is solved as $bn = \beta^*$. S.O.D. of the profit function is calculated as $\frac{-4\Gamma^2 bn + 2\Gamma^2 \beta}{(nb+\beta)^4}$, which is negative for $\beta^* = bn$. One can verify that

$\Pi(\beta^*) > \Pi(\beta) \quad \forall\beta > 2bn$. So we can restrict domain to a compact interval, $[0, 2bn]$. Since profit function is continuous, it arrives to its maximum on a compact domain. Since β^* is the only point satisfying both F.O.C. and S.O.C., it maximizes profit of the firm over $[0, \infty)$. Thus, it is the best response of the firm for any $\Gamma > 0$ chosen by CU, in particular to $\Gamma^* = \frac{2an}{3}$. Since CU's unique best response to $\beta^* = bn$ is $\Gamma^* = \frac{2an}{3}$, the list (Γ^*, β^*) is the unique Nash equilibrium. *QED*

Lemma 2 *Consider the Cournot-Nash Game. Now,* $Q^M = \frac{an}{2}$ *is the monopolist's profit maximizing level of output.*

Proof: Since m=1, Cournot Nash Game reduces to natural monopolist's problem. Inverse industry demand is $P = \frac{a}{b} - \frac{1}{bn}$ and total revenue (TR) equals to $Q(\frac{a}{b} - \frac{1}{bn}Q)$. Marginal Revenue (MR) is $\frac{a}{b} - 2bnQ$. Since F.O.C. is necessary and sufficient for this problem, monopolist's profit maximizing level of output solved from $0 = MC = MR$ as $Q^M = \frac{an}{2}$. *QED*

As a new notation the variables with superscript M belong to monopoly structure and those with superscript $CUPN$ belong to the game in which consumer union and firms play Nash.

Proposition 8 *The following is true :*

$$\begin{aligned}
P^M \succ\ & P^{CPN} \geq P^{CUPN} \\
Q^M \succ\ & Q^{CPN} \geq Q^{CUPN} \\
CS^{CUPN} \geq\ & CS^{CPN} \succ CS^M \\
\Pi^M \succ\ & \Pi^{CPN} \geq \Pi^{CUPN} \\
SW^M \succ\ & SW^{CPN} \geq SW^{CUPN}
\end{aligned}$$

(when $n \geq 2$, inequalities are strict.)

Proof: Firstly, let's find the outcomes of three structures regarding these variables: we know that $Q^M = \frac{an}{2}$. Put it into inverse industry demand and find $P^M = \frac{a}{2b}$. At this price each consumer will consume $\frac{a}{2}$. Then individual consumer surplus, $CS^M = \int_0^{\frac{a}{2}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a}{2}\frac{a}{2} = \frac{a^2}{8b}$. Monopolists profit ,$\Pi^M = P^M Q^M = \frac{a^2 n}{4b}$. Then the total social welfare under monopoly, $SW^M = n\frac{a^2}{8b} + \frac{a^2 n}{4b} = \frac{3a^2 n}{8b}$. Return to the game where consumers and firms play Nash: We know that for $\gamma^* = \frac{a(4n-2)}{(4n-1)}$ and $\beta^* = bn$, $((\gamma^*)_{i=1}^n, \beta^*)$ forms a unique Nash equilibrium. From, $P^{CPN} = \frac{n\gamma^*}{2nb}$, $P^{CPN} = \frac{a(4n-2)}{2b(4n-1)}$. Each consumer at this price will consume $\gamma^* - bP^{CPN} = \frac{a(4n-2)}{2(4n-1)}$. Then total quantity demanded (total quantity produced), $AD(P) = n\frac{a(4n-2)}{2(4n-1)}$. Surplus for each consumer can be calculated from $CS^{CPN} = \int_0^{\frac{a(4n-2)}{2(4n-1)}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(4n-2)}{2b(4n-1)}\frac{a(4n-2)}{2(4n-1)} = \frac{a^2(4n^2-1)}{2b(4n-1)^2}$. And the firms profit is $\Pi^{CPN} = P^{CPN}(\beta^* P^{CPN}) = (\frac{a(4n-2)}{2b(4n-1)})^2 bn = \frac{a^2 n(4n-2)^2}{4b(4n-1)^2}$. Then total social welfare calculated as $SW^{CPN} = n\frac{a^2(4n^2-1)}{2b(4n-1)^2} + \frac{a^2 n(4n-2)^2}{4b(4n-1)^2} = \frac{a^2 n[(4n-2)^2 + 2(4n^2-1)]}{4b(4n-1)^2}$. Now, return to the game where CU plays Nash with the firms: We know that for $\Gamma^* = \frac{2an}{3}$ and $\beta^* = bn$, (Γ^*, β^*) forms a unique Nash equilibrium. Now, from $P^{CUPN} = \frac{\Gamma^*}{2nb}$, $P^{CUPN} = \frac{a}{3b}$. Total quantity demanded (total quantity produced) at this price, $Q^{CUPN} = \Gamma^* - nbP^{CUPN} = \frac{an}{3}$. Since consumers equally share this amount, each consumer takes $\frac{a}{3}$. Then his surplus is calculated from $CS^{CUPN} = \int_0^{\frac{a}{3}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a}{3b}\frac{a}{3} = \frac{a^2}{6b}$. And the profit of the firm is $\Pi^{CUPN} = P^{CUPN}(\beta^* P^{CUPN}) = \frac{a^2 n}{9b}$. Then total social welfare is calculated as $SW^{CUPN} = nCS^{CUPN} + \Pi^{CUPN} = \frac{5a^2 n}{18b}$. Now, we have all the results to make comparisons: We know that $P^M = \frac{a}{2b}$, $P^{CUPN} = \frac{a}{3b}$ and $P^{CPN} = \frac{a(4n-2)}{2b(4n-1)}$. If $n = 1$, then $P^{CUPN} = P^{CPN}$. Since $\frac{a}{2b} > \frac{a}{3}$, $P^M \succ P^{CPN} \geq P^{CUPN}$ holds for $n = 1$. If $n \geq 2$, then

$\frac{(4n-2)}{(4n-1)} \succ \frac{1}{3}$. Suppose not. Then $\frac{(4n-2)}{(4n-1)} \leq \frac{1}{3}$. This leads to $4n-4 \leq 0$. Since $n \geq 2$, contradiction. Since $\frac{(4n-2)}{(4n-1)} < 1$, $\frac{a}{2b} > \frac{a(4n-2)}{2b(4n-1)}$. Combining both, $P^M \succ P^{CPN} \succ P^{CUPN}$ for $n \geq 2$. We know that $Q^M = \frac{an}{2}$, $Q^{CUPN} = \frac{an}{3}$ and $Q^{CPN} = n\frac{a(4n-2)}{2(4n-1)}$. By a similar way in the prices, $Q^M \succ Q^{CPN} \succ Q^{CUPN}$ for $n \geq 2$ and $Q^M \succ Q^{CPN} = Q^{CUPN}$ for $n = 1$. About consumer's surplus, we know that $CS^M = \frac{an}{2}$, $CS^{CPN} = \frac{a^2(4n^2-1)}{2b(4n-1)^2}$ and $CS^{CUPN} = \frac{a^2}{6b}$. Clearly, $CS^{CUPN} \succ CS^M$ for $n \geq 1$. When $n = 1$, $\frac{(4n^2-1)}{(4n-1)^2} = \frac{1}{3}$ and so $CS^{CPN} = CS^{CUPN}$. Then, $CS^{CPN} = CS^{CUPN} \succ CS^M$ for $n = 1$. If $n \geq 2$, $\frac{(4n^2-1)}{2(4n-1)^2} < \frac{1}{6}$. Suppose not. Then $\frac{(4n^2-1)}{2(4n-1)^2} \geq \frac{1}{6}$. This leads to $0 \geq 8n^2 - 16n + 8$. Since $n \geq 2$, contradiction. Then, $CS^{CUPN} > CS^{CPN}$ for $n \geq 2$. Suppose that $CS^{CPN} \leq CS^M$, i.e, $\frac{(4n^2-1)}{2(4n-1)^2} \leq \frac{1}{8}$.This leads to $0 \leq -16n + 10$. Since $n \geq 2$, contradiction. So $CS^{CPN} > CS^M$. So, $CS^{CUPN} \succ CS^{CPN} \succ CS^M$ for $n \geq 2$. Regarding the profits, we know that $\Pi^M = \frac{a^2n}{4b}$, $\Pi^{CPN} = \frac{a^2n(4n-2)^2}{4b(4n-1)^2}$ and $\Pi^{CUPN} = \frac{a^2n}{9b}$ .Clearly, $\Pi^M \succ \Pi^{CUPN}$ for $n \geq 1$. When n=1, $\Pi^{CPN} = \frac{a^2}{9b}$ and so $\Pi^{CUPN} = \Pi^{CPN}$. So when $n = 1$, $\Pi^M \succ \Pi^{CPN} = \Pi^{CUPN}$ holds. Suppose that $\Pi^{CUPN} \geq \Pi^{CPN}$ for $n \geq 2$. Then $\frac{(4n-2)^2}{4(4n-1)^2} \leq \frac{a^2}{9}$. This leads to $80n^2 - 112n + 32n \leq 0$. Since $n \geq 2$, contradiction. Then $\Pi^{CPN} > \Pi^{CUPN}$ when $n \geq 2$. Clearly, $\Pi^M > \Pi^{CUPN}$ for $n \geq 2$. So, $\Pi^M \succ \Pi^{CPN} \succ \Pi^{CUPN}$ for $n \geq 2$. About total social welfares, we know that $SW^M = \frac{3a^2n}{8b}$, $SW^{CPN} = \frac{a^2n[(4n-2)^2+2(4n^2-1)]}{4b(4n-1)^2}$ and $SW^{CUPN} = \frac{5a^2n}{18b}$. When $n = 1$, $SW^{CPN} = \frac{a^25}{18b}$ and so $SW^{CPN} = SW^{CUPN}$. Since $\frac{5}{18} \prec \frac{3}{8}$, $SW^M \succ SW^{CPN} = SW^{CPN}$ holds for $n = 1$. Now, $SW^{CPN} = SW^{CUPN}\frac{[(4n-2)^2+2(4n^2-1)]9}{10(4n-1)^2}$. Suppose $\frac{[(4n-2)^2+2(4n^2-1)]9}{10(4n-1)^2} \leq 1$. This leads to $0 \geq 56n^2 - 64n + 8$. Since $n \geq 2$, contradiction. Then, $SW^{CPN} \succ SW^{CUPN}$ for $n \geq 2$.Now, $SW^{CPN} = SW^M\frac{[(4n-2)^2+2(4n^2-1)]2}{3(4n-1)^2}$.

Suppose $1 \leq \frac{[(4n-2)^2+2(4n^2-1)]2}{3(4n-1)^2}$. This leads to $0 \geq 8n-1$. Since $n \geq 2$, contradiction. So, $SW^M \succ SW^{CPN}$ for $n \geq 2$. Combining both, $SW^M \succ SW^{CPN} \succ SW^{CUPN}$ for $n \geq 2$. So we are done. *QED*

As a new notation, any varible with a superscript C belongs to the competitive equilibrium outcome.[31]

Proposition 9 *The following is true:*

$$\begin{aligned}
\lim_{n\to\infty} \gamma^* &= a \\
\lim_{n\to\infty} P^{CPN} &= \lim_{n\to\infty} P^M = \frac{a}{2b} \succ \lim_{n\to\infty} P^{CUPN} = \frac{a}{3b} \succ \lim_{n\to\infty} P^C = 0 \\
\lim_{n\to\infty} Q^C &= \lim_{n\to\infty} Q^{CPN} = \lim_{n\to\infty} Q^M = \lim_{n\to\infty} Q^{CUPN} = \infty \\
\lim_{n\to\infty} CS^{CPN} &= \lim_{n\to\infty} CS^M = \frac{a^2}{8b} \prec \lim_{n\to\infty} CS^{CUPN} = \frac{a^2}{6b} \prec \lim_{n\to\infty} CS^C = \frac{a^2}{2b} \\
\lim_{n\to\infty} \Pi^C = 0 &\prec \lim_{n\to\infty} \Pi^{CPN} = \lim_{n\to\infty} \Pi^M = \lim_{n\to\infty} \Pi^{CUPN} = \infty \\
\lim_{n\to\infty} SW^C &= \lim_{n\to\infty} SW^{CPN} = \lim_{n\to\infty} SW^M = \lim_{n\to\infty} SW^{CUPN} = \infty
\end{aligned}$$

Proof: Note that since there is only one firm, industry marginal cost (IMC) function equals to MC function which is zero for all non-negative levels of production. Then P^C=0. Then $Q^C = na$. Each individual consumer will consume a at zero price. His surplus will be $CS^C = \frac{a^2}{2b}$. The firm obtains zero profit since it sells at zero price. Then total social welfare will be equal to $\frac{na^2}{2b}$. Combining them with the results given in Proposition 8, limits can be easily verified. *QED*

[31]When we say competitive equilibrium outcome, we mean socially optimal outcome. Consider a single-product industry. Let (P^s, q^s) be such that $AD(P^s) = IMC(P^s)$ and $AD(P^s) = q^s$, where $IMC(q)$ is the industry marginal cost function. Moreover, if IMC function and AD intersect more than one point, then at (P^s, q^s) social welfare is maximized.

Proposition 9 tells that as number of consumer gets very large, they tell the truth. In fact, as their number gets large, unless they are organized, they lose their ability to manipulate their intercept terms. The outcomes of Nash game we define and monopoly structure coincide in the limit. When consumers are organized, in the limit individual consumer is still better off compared to the situation where they are not organized and monopoly structure .

Case: $\alpha \succ 0$, $n \geq 1$, $m \geq 2$

In this case firms are producing with positive marginal cost and there is at least one consumer and more than one firm.

Proposition 10 *Let* γ^* *be equal to* $\frac{\frac{a}{b}\theta}{\frac{1}{b}\theta^2 + \frac{2}{nb+m\beta^*}\theta - \frac{(n-1)b}{(nb+m\beta^*)^2}}$ *where* $\theta = (1 - \frac{b}{nb+m\beta^*})$ *and* β^* *be equal to* $\frac{-(m-2-2\alpha nb)-\sqrt{\Delta}}{-4\alpha(m-1)}$ *where* $\Delta = (m-2-2\alpha nb)^2 + 8\alpha nb(m-1)$. *Now, the list* $((\gamma^*)_{i=1}^{n}, (\beta^*)_{i=1}^{m})$ *forms a Nash equilibrium.*

Proof: Let a $\gamma \succ 0$ be chosen by each consumer and β^* be chosen by all firms other than firm j. Now, firm j's problem is given in (5), where $P(\beta_j)$ is obtained from (1) as $P(\beta_j) = \frac{n\gamma}{(m-1)\beta^*+\beta_j+nb}$ by using $AD(P) = n(\gamma - bP)$ and $AS(P) = (m-1)\beta^* P + \beta_j P$. Set the Lagrangian as $L = (\frac{n\gamma}{(m-1)\beta^*+\beta_j+nb})^2(\beta_j - \alpha\beta_j^2)$ F.O.C. for this problem: $\frac{-2(n\gamma)^2}{((m-1)\beta^*+\beta_j+nb)^3}(\beta_j - \alpha\beta_j^2) + (\frac{n\gamma}{(m-1)\beta^*+\beta_j+nb})^2(1-2\alpha\beta^*) = 0$. Put the proposed value above for β^* and solve for β_j. During this a bit massy algebra one can notice that γ cancels out and β_j is found as equal to β^*. About the S.O.D., we used Mathematica software to take the derivative of F.O.D. with respect to β_j and then to impose $\beta_j = \beta^*$. It supplied the following expression $\frac{6(\beta^* - \alpha(\beta^*)^2\gamma^2 n^2 - 4(1-2\alpha\beta^*)\gamma^2 n^2(m\beta^*+nb) - 2\alpha\gamma^2 n^2(m\beta^*+nb)^2}{(\beta^* m+nb)^4}$. Focus on the numerator, open up the parentheses and make the cancellations and obtain the numerator as $6\gamma^2 n^2\beta^* - 6\gamma^2 n^2\alpha(\beta^*)^2 - 4\gamma^2 n^2\beta^* m - 4\gamma^2 n^3 b + 8\gamma^2 n^3 b\alpha\beta^* - 2\gamma^2 n^2\alpha(\beta^*)^2 m^2 - 4\alpha\gamma^2 n^3 b\beta^* m - 2\gamma^2 n^4\alpha b^2$. Observe that $6\gamma^2 n^2\beta^* - 4\gamma^2 n^2\beta^* m \prec 0$ and $8\gamma^2 n^3 b\alpha\beta^* - 4\alpha\gamma^2 n^3 b\beta^* m \prec 0$ since $m \geq 2$ and other terms are negative. Thus the whole expression turns out to be negative for $\beta_j = \beta^*$. Thus, S.O.C is satisfied for $\beta_j = \beta^*$. Note that if $0 \leq \beta_j \leq \frac{1}{\alpha}$, profit of the firm is non-negative and if $\beta_j \succ \frac{1}{\alpha}$, then the profit of the firm is negative. So we can restrict the domain to a compact interval $[0, \frac{1}{\alpha}]$. Since profit function is continuous function of β_j, it arrives its maximum on $[0, \frac{1}{\alpha}]$. Since $\beta_j = \beta^*$ is the unique point satisfying F.O.C. and S.O.C., it maximizes firm j's profit on $[0, \infty)$. Thus, it is the best response of the firm j to others' strategy profile in which each firm sticks to β^* and each consumer sticks to the same $\gamma \succ 0$. Implying $\beta_j = \beta^*$ is the best response of the firm j to the others'strategy profile in which each firm sticks to β^* and each consumer sticks to γ^*, in particular. Since firms are identical this is true for each firm. Let's turn to typical consumer's problem: Assume that each firm sticks to β^* and each consumer other than consumer i sticks to γ^*. Then consumer i's problem is as given in (2). Now, as in the proof of Proposition 5, let $\overline{P}$ be the price satisfying $m\beta^*\overline{P} = n(a - b\overline{P})$. Then, $\overline{P} = \frac{na}{m\beta^* + nb}$. Let's a bit manipulate $\frac{\gamma^*}{b} = \frac{1}{b}\frac{\frac{a}{b}\theta}{\frac{1}{b}\theta^2 + \frac{2}{nb+m\beta^*}\theta - \frac{(n-1)b}{(nb+m\beta^*)^2}}$ where $\theta = (1 - \frac{b}{nb+m\beta^*})$ and obtain $\frac{a(nb+\beta^* m)}{((n-1)b+\beta^* m)b + 2b^2 - \frac{(n-1)b^3}{((n-1)b+\beta^* m)}}$. Suppose that $\overline{P} \geq \frac{\gamma^*}{b}$. Then $\frac{(nb+\beta^* m)}{((n-1)b+\beta^* m)b + 2b^2 - \frac{(n-1)b^3}{((n-1)b+\beta^* m)}} \leq \frac{1}{\frac{m\beta^* + nb}{n}}$. Then, $b^2((n - 1) + 2) + m\beta^* b - \frac{(n-1)b^3}{((n-1)b+\beta^* m)} \geq \frac{(\beta^*)^2 m^2 + 2\beta^* mnb + n^2 b^2}{n}$. After a little algebra one can obtain $0 \geq b^2(\frac{(n-1)b}{(n-1)+\beta^* m} - 1) + \beta^* mb + \frac{(\beta^*)^2 m^2}{n^2}$. This leads to $0 \geq \beta^* mb(\frac{-1}{(n-1) + \frac{m\beta^*}{b}} + 1) + \frac{(\beta^*)^2 m^2}{n^2}$. Now, when $n \geq 2$, $(n-1) \geq 1$. Moreover, $\frac{m^2\beta^*}{b} \succ 0$. So right hand side is positive. Contradiction . So, $\frac{\gamma^*}{b} \succ \overline{P}$. Note that when $n = 1$, there is no problem in terms of outcome price. So whether $\frac{\gamma^*}{b}$ is bigger than $\overline{P}$ or not is not interesting. Let's also prove that $\gamma^* \prec a$. Now, $\theta = (1 - \frac{b}{nb+m\beta^*}) \prec 1$ and positive since $n \geq 1$ and $\beta^* \succ 0$ and $m > 1$. Suppose that $\gamma^* \geq a$. Then $\frac{\theta}{b} \geq \frac{\theta^2}{b} + \frac{2\theta}{nb+m\beta^2} - \frac{(n-1)b}{(nb+m\beta^*)^2}$. One can verify that $\frac{2\theta}{nb+m\beta} - \frac{(n-1)b}{(nb+m\beta^*)^2} = \frac{(n-1)b+m\beta^*}{(nb+m\beta^*)^2} + \frac{m\beta^*}{(nb+m\beta^*)^2}$. And $(\frac{\theta}{b} - \frac{\theta^2}{b}) = \theta(\frac{(n-1)b+m\beta^*}{nb+m\beta^*})$. Then $\theta(\frac{(n-1)b+m\beta^*}{nb+m\beta^*}) \geq \frac{(n-1)b+m\beta^*}{(nb+m\beta^*)^2} + \frac{m\beta^*}{(nb+m\beta^*)^2}$. Since $\theta \prec 1$, Contradiction. So $\gamma^* \prec a$. Now, we have all the tools to claim that in consumer i's problem (5), consumer i will not give values bigger than a. To prove this, follow the same steps in Proposition 5. So $\gamma \leq a$. Let $\underline{P}$ be the price satisfying $(n-1)\gamma^* - (n-1)b\underline{P} = mb\underline{P}$. Then $\underline{P} = \frac{(n-1)\gamma^*}{(n-1)b+m\beta^*}$. If consumer i announces a $\gamma \succ 0$ such that $\frac{\gamma}{b} \prec \underline{P}$, then he will consume zero so his surplus will be zero. Then $AD(P)$ will be

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

And by definition of $\underline{P}$,$P(\gamma) = \underline{P}$. So $P(\gamma) > \frac{\gamma}{b}$. So consumer i will consume zero. if he announced a instead, he would consume positive amount and obtain positive surplus. So, any γ less than $\underline{P}b$ can not be the best response of consumer i. Thus, $\underline{P}b \leq \gamma \leq a$. Note that $\underline{P} \prec \frac{\gamma^*}{b}$ since $\underline{P} = \frac{\gamma^*}{b+\frac{m\beta^*}{(n-1)}}$ and $n \geq 2$. Note that if $n = 1$, there is no need to define $\underline{P}$ since there is no problem in terms of outcome price. Consider the case $\frac{\gamma}{b} \in [\underline{P}, \frac{\gamma^*}{b}]$. For all such γ's,

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

Consider the function, $G(P) = \gamma + (n-1)\gamma^* - nbP \quad \forall P \geq 0$. Now, $G(P) = AD(P)\ \forall P \leq \frac{\gamma}{b}$. We observed above that if $\frac{\gamma}{b} = \underline{P}$, then $P(\gamma) = \underline{P}$. Then

$G(\underline{P}) = AD(\underline{P})$ and $G(P(\gamma)) = AD(P(\gamma))$. We need to prove the following statement $\forall \frac{\gamma}{b} \in (\underline{P}, \frac{\gamma^*}{b}] : \frac{\gamma}{b} \succ P(\gamma)$. Let $\frac{\gamma}{b} \in (\underline{P}, \frac{\gamma^*}{b}]$. Suppose the contrary, so $\frac{\gamma}{b} \leq P(\gamma)$. Now,$P(\gamma)$ is the price satisfying $AD(P(\gamma)) = AS(P(\gamma))$. Proceed with the case: $\frac{\gamma}{b} = P(\gamma)$. At this price $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$ and $AS(P(\gamma)) = m\beta^* P(\gamma)$. Then $P(\gamma) = \underline{P}$. Then $\frac{\gamma}{b} = \underline{P}$. Contradiction. Left with the case $\frac{\gamma}{b} \prec P(\gamma)$. At $P(\gamma)$, $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$ and $AS(P(\gamma)) = m\beta^* P(\gamma)$ and $AD(P(\gamma)) = AS(P(\gamma))$. From which, $P(\gamma) = \underline{P}$. Then $\frac{\gamma}{b} \prec P(\gamma) = \underline{P}$. Contradiction. So $\forall \frac{\gamma}{b} \in (\underline{P}, \frac{\gamma^*}{b}]$, $\frac{\gamma}{b} \succ P(\gamma)$. Since $G(P) = AD(P) \quad \forall P \leq \frac{\gamma}{b}$, $G(P(\gamma)) = AD(P(\gamma)) \quad \forall \gamma \in [\underline{P}b, \gamma^*]$. (Note that we proved when $\frac{\gamma}{b} = \underline{P}$, $G(P(\gamma)) = AD(P(\gamma))$). Let's look at the set $(\gamma^*, a]$. For all elements in this set,

$$AD(P) = \begin{cases} \gamma - bP & \text{if } P \geq \frac{\gamma^*}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma^*}{b} \end{cases}$$

Note that $\gamma \leq a$ and $\gamma^* \prec a$. Suppose that $P(\gamma) \geq \frac{\gamma^*}{b}$. Then, $AD(P(\gamma)) = \gamma - bP(\gamma) = AS(P(\gamma)) = m\beta^* P(\gamma)$. Note that $\gamma \leq a$ and $\gamma^* \prec a$. Then $P(\gamma) = \frac{\gamma}{b+\beta^* m} \leq \frac{a}{b+\beta^* m} \prec \frac{a}{b+\frac{\beta^* m}{n}} = \frac{na}{bn+\beta^* m} = \overline{P}$ since $n \geq 2$ and so $\frac{\beta^* m}{n} \prec m\beta^*$. We know that $\frac{\gamma^*}{b} \succ \overline{P}$. But then $\frac{\gamma^*}{b} \succ P(\gamma)$. Contradiction to the supposition.So $P(\gamma) \prec \frac{\gamma^*}{b}$. Then, $\frac{\gamma}{b} \succ \frac{\gamma^*}{b} \succ P(\gamma)$. So $\forall \gamma \in (\gamma^*, a] : \frac{\gamma}{b} \succ P(\gamma)$. Thus, $\forall \gamma \in (\gamma^*, a] : G(P(\gamma)) = AD(P(\gamma))$.Combining all the results we obtained so far, $\forall \gamma \in [\underline{P}b, a] : G(P(\gamma)) = AD(P(\gamma))$. We know that if $\gamma \prec \underline{P}b$, then $CS_i(\gamma) = 0 \prec CS_i(a)$ and if $\frac{\gamma}{b} \succ \frac{a}{b}$, then $CS_i(a) \succ CS_i(\gamma)$. So maximum of CS_i, a continuous function of γ, lies in the compact interval $[\underline{P}b, a]$. So consumer i's problem

$$\max_{\gamma \in [\underline{P}b, a]} CS_i = \int_0^{\gamma - bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma))$$

Since $\forall \gamma \in [\underline{P}b, a] : G(P(\gamma)) = AD(P(\gamma))$, $P(\gamma) = \frac{(n-1)\gamma^* + \gamma}{nb+\beta^* m}$. After a little algebra from F.O.C. of this problem, γ^c is solved as $\frac{\frac{b(n-1)\gamma^*}{(nb+\beta^* m)^2}+\frac{a}{b}\theta}{\frac{\theta^2}{b}+\frac{2\theta}{nb+\beta^* m}}$ where θ is as defined before. = where as before. Putting the corresponding values for β^* and γ^* and doing a little algebra, one obtains γ^c as equal to γ^*. S.O.D. of the objective function with respect to γ is $\frac{-1}{b}(1-\frac{b}{nb+/beta^* m})^2$ which is negative for any choice of γ. When each of remaining consumers stick to γ^* and each firm sticks to β^*, $\gamma^c = \gamma^*$ is the unique point satisfying F.O.C. and S.O.D. is negative for any $\gamma \in [\underline{P}b, a]$. We conclude that γ^* maximizes consumer i's surplus.Thus, it is the best response of consumer i to the other' strategy profile in which each cinsumer announces γ^* and each firm announces β^*. This is true for any consumer since they are identical. By adding the results of typical firm's problem, we conclude that the list $((\gamma^*)_{i=1}^n, (\beta^*)_{i=1}^m)$ forms a Nash equlibrium. *QED*

The following lemma provides some limit results related to the Nash game in which consumers play Nash with the firms.

Lemma 3

$$
\begin{aligned}
\gamma^* &\prec a \\
\lim_{n\to\infty} \gamma^* &= a \\
\lim_{n\to\infty} \beta^* &= \frac{1}{2\alpha} \\
\lim_{n\to\infty} P^{CPN} &= \frac{a}{b} \\
\lim_{n\to\infty} (\gamma^* - bP^{CPN}) &= 0 \\
\lim_{n\to\infty} AD(P^{CPN}) &= \frac{ma}{2\alpha b} \\
\lim_{m\to\infty} \gamma^* &= a \\
\lim_{m\to\infty} \beta^* &= \frac{1}{2\alpha}
\end{aligned}
$$

$$\begin{aligned}
\lim_{m\to\infty} P^{CPN} &= 0 \\
\lim_{m\to\infty} (\gamma^* - bP^{CPN}) &= a \\
\lim_{m\to\infty} AD(P^{CPN}) &= na
\end{aligned}$$

Proof: In the proof of Proposition 10 we showed that $\gamma^* \prec a$. Start with one of the limits on slope parameters of firms. Now,

$$\begin{aligned}
\lim_{m\to\infty} \beta^* &= \lim_{m\to\infty} \frac{-(m-2-2\alpha nb) - \sqrt{(m-2-2\alpha nb)^2 + 8\alpha nb(m-1)}}{-4\alpha(m-1)} \\
&= \lim_{m\to\infty} \frac{(1-\frac{1}{m}-\frac{2\alpha nb}{m}) + \sqrt{(1-\frac{2}{m}-\frac{2\alpha nb}{m})^2 + \frac{8\alpha nb}{m} - \frac{1}{m}}}{(4\alpha - \frac{1}{m})} = \frac{1}{2\alpha}
\end{aligned}$$

Note that $\lim_{m\to\infty} m\beta^* = \infty$ since as β^* goes to a positive finite number m explodes and so $\lim_{m\to\infty} \theta = \lim_{m\to\infty}(1 - \frac{b}{nb+\beta^* m}) = 1$. By using this

$$\begin{aligned}
\lim_{m\to\infty} \gamma^* &= \lim_{m\to\infty} \frac{\frac{a}{b}\theta}{\frac{1}{b}\theta^2 + \frac{2}{nb+m\beta^*}\theta - \frac{(n-1)b}{(nb+m\beta^*)^2}} = a \\
\lim_{m\to\infty} P^{CPN} &= \lim_{m\to\infty} \frac{n\gamma^*}{nb+\beta^* m} = 0
\end{aligned}$$

since the numerator goes to na, a poitive fixed number, whereas denominator explodes. This implies that in the limit each firm produces zero amount since $\lim_{m\to\infty} \beta^* P^{CPN} = 0$. Then $\lim_{m\to\infty}(\gamma^* - bP^{CPN}) = a$. Then, $\lim_{m\to\infty} n(\gamma^* - bP^{CPN}) = na$. Now,

$$\begin{aligned}
\lim_{n\to\infty} \beta^* &= \lim_{n\to\infty} \frac{-(m-2-2\alpha nb) - \sqrt{(m-2-2\alpha nb)^2 + 8\alpha nb(m-1)}}{-4\alpha(m-1)} \\
&= \lim_{n\to\infty} \frac{-2b}{(\frac{m}{n}-\frac{2}{n}-2\alpha b) - \sqrt{(\frac{m}{n}-\frac{2}{n}-2\alpha b)^2 + \frac{8\alpha nb(m-1)}{n}}} \\
&= \frac{1}{2\alpha}
\end{aligned}$$

Note that $\lim_{n\to\infty}\theta = \lim_{n\to\infty}(1 - \frac{b}{nb+m\beta^*}) = 1$. Then

$$\begin{aligned}
\lim_{n\to\infty}\gamma^* &= \lim_{n\to\infty}\frac{\frac{a}{b}\theta}{\frac{1}{b}\theta^2 + \frac{2}{nb+m\beta^*}\theta - \frac{(n-1)b}{(nb+m\beta^*)^2}} \\
&= \lim_{n\to\infty}\frac{\frac{a}{b}\theta}{\frac{1}{b}\theta^2 + \frac{2}{nb+m\beta^*}\theta - \frac{-b}{n(b+\frac{m\beta^*}{n})^2} + \frac{b}{(bn+m\beta^*)^2}} \\
&= a
\end{aligned}$$

Now, $\lim_{n\to\infty} P^{CPN} = \lim_{n\to\infty}\frac{n\gamma^*}{nb+m\beta^*} = \lim_{n\to\infty}\frac{\gamma^*}{b+\frac{m\beta^*}{n}} = \frac{a}{b}$ Then each firm produces in the limit $\lim_{n\to\infty}\beta^* P^{CPN} = \frac{a}{2\alpha b}$. So in the limit total quantity produced (total quantity demanded) becomes $\lim_{n\to\infty} m\beta^* P^{CPN} = \frac{ma}{2\alpha b}$. So we are done. *QED*

Lemma 4 *Consider the Cornout-Nash game. Let q^* be equal to $\frac{an}{2\alpha bn+m+1}$. Now, the list $(q^*)_{i=1}^m$ forms a unique symmetric Nash equilibrium.*

Proof: Note that there are more than one firms in the market and each of which produces with positive cost. Typical firm's problem is given in (6). F.O.C. for its problem:$(\frac{a}{b} - \frac{1}{bn}(\sum_{i=1}^m q_i) - q_j\frac{1}{bn} - 2\alpha q_j = 0$. From which q_j is solved as $q_j = \frac{an-\sum_{i\neq j}^m q_i}{2+2\alpha bn}$. Since we are interested in symmetric equilibria strategies, we impose $q_j = q^* \quad \forall j$. Then, q^* is solved as $q^* = \frac{an}{2\alpha bn+m+1}$. If we check the S.O.D., it is $\frac{-2}{bn} - 2\alpha$. Clearly, it is negative for any $q_j \geq 0$. Then, F.O.C. is sufficient as well and when each of other firms sticks to q^*, q^* satisfies F.O.C. of the problem firm j. Thus, maximizes firm j's profit and so it is the best response of firm j. Since firms are identical, this is true for any firm. Thus, $(q^*)_{i=1}^m$ is a Nash equilibrium.It follows from the first order condition of firm j's problem that this Nash equilibrium is the only symmetric one. *QED*

The following lemma indicates the fact that in the limit outcomes of Cornout-Nash game and the Nash game in which consumers are players along with firms coincide.

Lemma 5

$$\begin{aligned}
\lim_{m\to\infty} CS^{CPN} &= \lim_{m\to\infty} CS^{C-N} = \frac{a^2}{2b} \\
\lim_{m\to\infty} \Pi^{CPN} &= \lim_{m\to\infty} \Pi^{C-N} = 0 \\
\lim_{m\to\infty} SW^{CPN} &= \lim_{m\to\infty} SW^{CPN} = \frac{na^2}{2b} \\
\lim_{n\to\infty} CS^{CPN} &= \lim_{n\to\infty} CS^{C-N} = 0 \\
\lim_{n\to\infty} \Pi^{CPN} &= \lim_{n\to\infty} \Pi^{C-N} = \frac{a^2}{4\alpha b} \\
\lim_{n\to\infty} SW^{CPN} &= \lim_{n\to\infty} SW^{CPN} = \frac{ma^2}{4\alpha b}
\end{aligned}$$

Proof: Firstly, we know that for $q^* = \frac{an}{2\alpha bn+m+1}$, $(q^*)_{i=1}^m$ forms a unique symmetric Nash equilibrium. Then total quantity produced is $Q^{C-N} = \frac{anm}{2\alpha bn+m+1}$. Putting Q^{C-N} into inverse industry demand gives $P^{C-N} = \frac{a}{b} - \frac{1}{bn}Q^{C-N} = \frac{2\alpha abn+a}{b(2\alpha bn+m+1)}$. At this price typical consumer will consume $a - b\frac{2\alpha abn+a}{b(2\alpha bn+m+1)} = \frac{am}{2\alpha bn+m+1}$ and his surplus $CS^{C-N} = \int_0^{\frac{am}{2\alpha bn+m+1}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{2\alpha abn+a}{b(2\alpha bn+m+1)}\frac{am}{2\alpha bn+m+1} = \frac{a^2m^2}{2b(2\alpha bn+m+1)^2}$. Each firm's profit is $\Pi^{C-N} = P^{C-N}q^* - \alpha(q^*)^2 = \frac{\alpha bn^2a^2+a^2n}{b(2\alpha bn+m+1)^2}$. Total social welfare $SW^{C-N} = m\Pi^{C-N} + nCS^{C-N} = \frac{2m\alpha bn^2a^2+a^2nm^2+2mna^2}{2b(2\alpha bn+m+1)^2}$. Now, we are ready to take limits. $\lim_{m\to\infty} CS^{C-N} = \lim_{m\to\infty} \frac{a^2m^2}{2b(2\alpha bn+m+1)^2} = \lim_{m\to\infty} \frac{a^2}{2b(\frac{(2\alpha bn)^2}{m^2} + \frac{2(2\alpha bn+1)^2}{m}+1)} = \frac{a2}{2b}$, $\lim_{m\to\infty} CS^{CPN} = \lim_{m\to\infty}(\frac{a}{b}(\gamma^* - bP^{CPN}) - \frac{1}{2b}(\gamma^* - bP^{CPN})^2 - P^{CPN}(\gamma^* - bP^{CPN}) = \frac{a2}{2b}$ by using the results $\lim_{m\to\infty}(\gamma^* - bP^{CPN}) = a$ and $\lim_{m\to\infty}(P^{CPN}) = 0$ obtained in Lemma 4. $\lim_{m\to\infty} \Pi^{CPN} = \lim_{m\to\infty}(P^{CPN})^2(\beta^* - \alpha(\beta^*)^2)) = 0$ since $\lim_{m\to\infty}(P^{CPN}) = 0$ and $\lim_{m\to\infty} \beta^* = \frac{1}{2\alpha}$. Clearly, $\lim_{m\to\infty} \Pi^{C-N} =$

$\lim_{m\to\infty} \frac{\alpha bn^2a^2+a^2n}{b(2\alpha bn+m+1)^2} = 0$. $\lim_{m\to\infty} SW^{C-N} = \lim_{m\to\infty} \frac{2m\alpha bn^2a^2+a^2nm^2+2mna^2}{2b(2\alpha bn+m+1)^2} = \lim_{m\to\infty} \frac{\frac{2m\alpha bn^2a^2}{m}+a^2n+\frac{2na^2}{m}}{2b(\frac{(2\alpha bn+1)^2}{m^2}+1+\frac{(2\alpha bn+1)}{m})^2} = \frac{na^2}{2b}$. Observe that $\lim_{m\to\infty} \Pi^{CPN} = 0$, $\lim_{m\to\infty} nCS^{CPN} = \frac{na^2}{2b}$ and then $\lim_{m\to\infty} SW^{CPN} = \lim_{m\to\infty} m\Pi^{CPN} + nCS^{CPN} = \frac{na^2}{2b}$. Clearly, $\lim_{n\to\infty} CS^{C-N} = \lim_{n\to\infty} \frac{a^2m^2}{2b(2\alpha bn+m+1)^2} = 0$, $\lim_{n\to\infty} CS^{CPN} = \lim_{n\to\infty}(\frac{a}{b}(\gamma^* - bP^{CPN}) - \frac{1}{2b}(\gamma^* - bP^{CPN})^2 - P^{CPN}(\gamma^* - bP^{CPN}) = 0$ by using the results $\lim_{n\to\infty}(\gamma^*-bP^{CPN}) = 0$ and $\lim_{n\to\infty} P^{CPN} = 0$ obtained in Lemma 4. Easily, $\lim_{n\to\infty} \Pi^{C-N} = \lim_{m\to\infty} \frac{\alpha bn^2a^2+a^2n}{b(2\alpha bn+m+1)^2} = \frac{a^2}{4\alpha b}$. $\lim_{n\to\infty} \Pi^{CPN} = \lim_{n\to\infty}(P^{CPN})^2(\beta^*-\alpha(\beta^*)^2)) = \frac{a^2}{4\alpha nb}$ since $\lim_{n\to\infty}(P^{CPN}) = \frac{a}{b}$ and $\lim_{n\to\infty} \beta^* = \frac{1}{2\alpha}$. $\lim_{n\to\infty} SW^{C-N} = \lim_{n\to\infty} \frac{2m\alpha bn^2a^2+a^2nm^2+2mna^2}{2b(2\alpha bn+m+1)^2} = \lim_{n\to\infty} \frac{2m\alpha ba^2+\frac{a^2m^2}{n}+\frac{2ma^2}{n}}{2b(2b((2\alpha b)^2+\frac{(m+1)^2}{n^2}+\frac{2\alpha(m+1)}{n})} = \frac{ma^2}{4\alpha b^2}$. Now, $\lim_{n\to\infty} m\Pi^{CPN} = \frac{ma^2}{4\alpha b^2}$. We conjecture that $\lim_{n\to\infty} nCS^{CPN} = 0$. In that case $\lim_{n\to\infty} SW^{CPN} = \lim_{n\to\infty} m\Pi^{CPN} + \lim_{n\to\infty} nCS^{CPN} = \frac{ma^2}{4\alpha b^2}$. *QED*

Example 1 : For $a = 1$, $b = 1$, $\alpha = 1$, $n = 1$, $m = 2$, the following is true: $P^{C-N} \succ P^{CPN}$, $Q^{C-N} \succ Q^{CPN}$, $CS^{CPN} \succ CS^{C-N}$, $\Pi^{C-N} \succ \Pi^{CPN}$ and $SW^{C-N} \succ SW^{CPN}$. One can put the values into the corresponding formulas given in Lemma 5 and verify the results.

The following example is to check if there exist more than one consumer whether relations above remains the same or not. It turns out that for this example relations stay the same.

Example 2 For $a = 1$, $b = 1$, $\alpha = 1$, $n = 2$, $m = 2$, the following is true: $P^{C-N} \succ P^{CPN}$, $Q^{C-N} \succ Q^{CPN}$, $CS^{CPN} \succ CS^{C-N}$, $\Pi^{C-N} \succ \Pi^{CPN}$ and $SW^{C-N} \succ SW^{CPN}$. The same procedure applies.

Case: $\alpha \succ 0,\ n \geq 2,\ m = 1$

For this case there are one firm producing with positive marginal cost and at least two consumers.

Proposition 11 *Let* γ^* *be equal to* $\frac{-2a(1+\alpha bn)(-1+2n-2\alpha bn+2\alpha bn^2)}{(1-4n+4\alpha bn-8\alpha bn^2+4\alpha^2b^2n^2-4\alpha^2b^2n^3)}$ *and* β^* *be equal to* $\frac{nb}{1+2\alpha bn}$ *. Now the list* $((\gamma*)_{i=1}^n, \beta^*)$ *forms a Nash Equilibrium.*

Proof: The proof follows similar lines to the one for Proposition 5. Define $\overline{P}$ as the price satisfying $n(a - b\overline{P}) = m\beta^*\overline{P} = \frac{nb}{1+2\alpha bn}\overline{P} = \frac{a(1+2\alpha bn)}{b(2+2\alpha bn)}$. Now, open up the parentheses in the numerator of γ^* and obtain $\frac{\gamma^*}{b} = \frac{a(-2+4n-6\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}{b(1-4n+4\alpha bn-8\alpha bn^2+4\alpha^2b^2n^2-4\alpha^2b^2n^3)}$. Suppose that $\frac{\gamma^*}{b} \leq \overline{P}$. Then

$$\frac{(-2+4n-6\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \leq \frac{(1+2\alpha bn)}{(2+2\alpha bn)}$$

After a little algebra, this leads to $-3+4n-10\alpha bn+8\alpha n^2b-8\alpha^2n^2b^2+4n^3\alpha^2b^2 \leq 0$. Observe that when $n \geq 2$, left hand side is positive. Contradiction. So $\frac{\gamma^*}{b} \succ \overline{P}$. Now, without pencil work one can observe that for $n \geq 1$ $0 \prec \frac{(-2+4n-6\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \prec 1$ So, $\gamma^* \prec a$. So we have the tools that $\frac{\gamma^*}{b} \succ \overline{P}$ and $\gamma^* \prec a$. Now, return to the consumer i's problem: Assume that the firm announces β^* and each of remaining consumers announces γ^*. Consumer i's problem is as given in (2): he will maximize his surplus by giving $a \succ 0$. Now, we claim that $CS_i(a) \succ CS_i(\gamma) \quad \forall\gamma \succ a$. This is proved by a similar reasoning in Proposition 5 by using the results $\frac{\gamma^*}{b} \succ \overline{P}$ and $\gamma^* \prec a$. Note that $CS_i(a) \succ 0$ since $\frac{a}{b} \succ \frac{\gamma^*}{b} \succ \overline{P}$. By definition of $\overline{P}$ it must be true that $P(a) \prec \overline{P}$. Then $\frac{a}{b} \succ \overline{P}$. So $CS_i(a) \succ 0$. Again define $\underline{P}$ as the price satisfying $(n-1)\gamma^* + (n-1)b\underline{P} = \beta^*\underline{P}$. So $\underline{P} = \frac{(n-1)\gamma^*}{(n-1)b+\beta^*}$. If consumer i gives a $\gamma \succ 0$ such that $\gamma \prec \underline{P}b$, then he will consume zero amount so his surplus is zero. This is due to $P(\gamma) = \underline{P} \quad \forall\gamma \prec \underline{P}b$. Note that when consumer i announces such a γ, then aggregate demand will be

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

and by defination of $\underline{P}$ it follows that $P(\gamma) = \underline{P}$. So $\forall\gamma \prec \underline{P}b \quad CS_i(a) = 0$. But since $CS_i(a) \succ 0$, such γs can not be the best response of consumer i. Clearly, $\frac{\gamma^*}{b} \succ \underline{P}$. Now, $\forall\gamma \in [\underline{P}b, \gamma^*]$, aggregate demand will be

$$AD(P) = \begin{cases} (n-1)\gamma^* - (n-1)bP & \text{if } P \geq \frac{\gamma}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma}{b} \end{cases}$$

Consider the function $G(P) = \gamma + (n-1)\gamma^* - nbP \quad \forall P \succ 0$. Now, $G(P) = AD(P) \quad \forall P \leq \frac{\gamma}{b}$. Suppose that $\frac{\gamma}{b} = \underline{P}$. By definition of $\underline{P}$, $P(\gamma) = \underline{P}$. Moreover, $G(\underline{P}) = AD(\underline{P})$ since $\frac{\gamma}{b} = \underline{P}$. Since $P(\gamma) = \underline{P}$, $G(P(\gamma)) = AD(P(\gamma))$. Now, we will prove the following statement $\forall\frac{\gamma}{b} \in (\underline{P}, \frac{\gamma}{b}] : \frac{\gamma}{b} \succ P(\gamma)$. Let $\frac{\gamma}{b} \in (\underline{P}, \frac{\gamma}{b}]$. Suppose that $\frac{\gamma}{b} \leq P(\gamma)$. Analyze the equality case. Then $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$. By definition of $P(\gamma)$, $(n-1)\gamma^* - (n-1)bP(\gamma) = P(\gamma)$. Then $P(\gamma) = \underline{P}$. Contradiction since $\frac{\gamma}{b} \succ \underline{P}$. Left with the case $\frac{\gamma}{b} \prec P(\gamma)$.Then $AD(P(\gamma)) = (n-1)\gamma^* - (n-1)bP(\gamma)$. By definition of $P(\gamma)$, $(n-1)\gamma^* - (n-1)bP(\gamma) = \beta^* P(\gamma)$. But then $P(\gamma) = \underline{P}$. Then $\frac{\gamma}{b} \prec P(\gamma) = \underline{P}$. Contradiction to the fact that $\frac{\gamma}{b}$ comes form $(\underline{P}, \frac{\gamma^*}{b}]$. So $\forall\frac{\gamma}{b} \in (\underline{P}, \frac{\gamma^*}{b}] : \frac{\gamma}{b} \succ P(\gamma)$. Since $\forall P \leq \frac{\gamma}{b} : G(P) = AD(P)$, $\forall P \leq \frac{\gamma}{b} \in (\underline{P}, \frac{\gamma^*}{b}] : G(P(\gamma)) = AD(P(\gamma))$. Adding the previous result,$\forall P \leq \frac{\gamma}{b} \in [\underline{P}, \frac{\gamma^*}{b}] : G(P(\gamma)) = AD(P(\gamma))$. Now, let's look at the case $\gamma \in (\gamma^*, a]$. For all such γs

$$AD(P) = \begin{cases} \gamma - bP & \text{if } P \geq \frac{\gamma^*}{b} \\ \gamma + (n-1)\gamma^* - nbP & \text{if } P \leq \frac{\gamma^*}{b} \end{cases}$$

Note that $\gamma \leq a$ and $n \geq 2$. Suppose that $P(\gamma) \geq \frac{\gamma}{b}$ for some $\gamma \in (\gamma^*, a]$. Then $AD(P(\gamma)) = \gamma - bP(\gamma)$. By definition of $P(\gamma)$, $\gamma - bP(\gamma) = \beta^* P(\gamma)$. Then $P(\gamma) = \frac{\gamma}{b+\beta^*}$. Since $\gamma \leq a$ and $n \geq 2$, $P(\gamma) = \frac{\gamma}{b+\beta^*} \leq \frac{a}{b+\beta^*} \prec \frac{a}{b+\frac{\beta^*}{n}} = \frac{na}{bn+\beta^*} = \overline{P}$. We know that $\frac{\gamma^*}{b} \succ \overline{P}$. But then $\frac{\gamma^*}{b} \succ \overline{P} \succ P(\gamma)$. Then $\frac{\gamma}{b} \succ \frac{\gamma^*}{b} \succ \overline{P} \succ P(\gamma)$.Contradiction to the supposition. Then $\forall \gamma \in (\gamma^*, a] : \frac{\gamma}{b} \succ P(\gamma)$. Thus, $\forall \frac{\gamma}{b} \in (\frac{\gamma^*}{b}, \frac{a}{b}] : \frac{\gamma}{b} \succ \gamma^* b \succ \overline{P} \succ P(\gamma)$. Since $G(P) = AD(P) \quad P \leq \frac{\gamma^*}{b}$, $G(P(\gamma)) = AD(P(\gamma)) \quad \forall \gamma \in (\gamma^*, a]$. Combining with the previous result, $\forall \gamma \in [\underline{P}b, a] : G(P(\gamma)) = AD(P(\gamma))$. We also showed that if $\gamma \prec \underline{P}b$, then $CS_i(\gamma) = 0 \prec CS_i(a)$ and if $\gamma \succ a$, $CS_i(a) \succ CS_i(\gamma)$. Thus, we can restrict the whole domain to $[\underline{P}b, a]$. Now, $CS_i(\gamma)$ is a continuous function and $[\underline{P}b, a]$ is a compact interval so $CS_i(\gamma)$ arrives its maximum in this interval. Then the problem is

$$\max_{\gamma \in [\underline{P}b, a]} CS_i(\gamma) = \int_0^{\gamma - bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma))$$

Since $\forall \gamma \in [\underline{P}b, a] : AD(P(\gamma)) = G(P(\gamma))$ and by definition of $P(\gamma)$, $AS(P(\gamma)) = AD(P(\gamma)) = G(P(\gamma))$. Then $\gamma \in [\underline{P}b, a] : P(\gamma) = \frac{(n-1)\gamma^* + \gamma}{nb+\beta^*}$. Put the corresponding value for $P(\gamma)$ into objective function. F.O.C. for this problem, $\frac{a}{b}(1 - \frac{b}{nb+\beta^*}) - \frac{\gamma}{b}((1 - \frac{b}{nb+\beta^*}) - \frac{2b}{nb+\beta^*})(1 - \frac{b}{nb+\beta^*}) + \frac{b(n-1)\gamma^*}{(nb+\beta^*)^2} = 0$ From which one obtains $\gamma^c = \frac{\frac{a}{b}(1 - \frac{b}{nb+\beta^*}) + \frac{b(n-1)\gamma^*}{(nb+\beta^*)^2}}{\frac{1}{b}(1 - \frac{b}{nb+\beta^*})^2 + \frac{2}{(nb+\beta^*)}(1 - \frac{b}{nb+\beta^*})}$. After replacing γ^* and β^* with corresponding expressions, one can verify that $\gamma^c = \gamma^*$. Now, S.O.D. of the objective function with respect to γ is $\frac{-1}{b}(1 - \frac{b}{nb+\beta^*})^2$ and it is negative $\forall \gamma > 0$. Thus, F.O.C. turns out to be necessary and sufficient. Thus, when each of other consumers announces γ^* and the firm announce β^* , announcing γ^* is the best response of consumer i. Now, let's turn to firm's problem: Assume that each of the consumers announces the same $\gamma \succ 0$, that is ,there are symmetric set of strategies on the side of consumers.

Firm's problem is given in (3). Note that the outcome price is solved from (1) as $P(\beta) = \frac{n\gamma}{\beta+nb}$ using $AD(P) = n(\gamma - bP)$ and $AS(P) = \beta P$. So firm's problem

$$\max_{\beta} \Pi(\beta) = (\frac{n\gamma}{nb+\beta})^2(\beta - \alpha\beta^2)$$

F.O.C. for this problem $2(\frac{n\gamma}{nb+\beta})(\frac{-n\gamma}{(nb+\beta)^2})(\beta - \alpha\beta^2) + (\frac{n\gamma}{nb+\beta})^2(1 - 2\alpha\beta) = 0$ After a little algebra, β is solved as proposed so $\beta = \frac{nb}{1+2\alpha nb} = \beta^*$. Note that it is independent of the particular value of $\gamma \succ 0$ rather it depends on their symmetricity. Now, S.O.D. of the profit function with respect to β (after simplifying in Mathematica) $\frac{2\gamma^2 n^2(\beta - 2bn + 2\alpha b\beta n - \alpha b^2 n^2)}{(\beta+bn)^4}$. Clearly, denominator is positive. suppose that the numerator is nonnegative. Put $\beta = \beta^*$. This leads to $1 \geq (2 + \alpha bn)$. Contradiction. So S.O.D. is negative at β^*. So β^* satisfies S.O.C. Note that if $\beta \succ \frac{1}{\alpha}$, profit is negative whereas if $\beta \leq \frac{1}{\alpha}$, then profit is non-negative. So we can restrict the domain to a compact interval, $[0, \frac{1}{\alpha}]$. Since profit function is continuous function of β it arrives its maximum in this interval. Since F.O.D. of the objective function vanishes only at β^* and S.O.C. is satisfied at β^*, we conclude that β^* maximizes it over $[0, \infty)$. So β^* is the firm's best response to the consumers each of whom announce the same value. Thus, it is the best response of firm to the strategy profile in which each consumer announces γ^*, in particular. By adding the results from typical consumer's problem, we conclude that $((\gamma^*)_{i=1}^n, \beta^*)$ forms a Nash equilibrium. *QED*

Proposition 12 *Assume that consumers are organized. Let* Γ^* *be equal to* $\frac{an(2+2\alpha nb)}{(3+4\alpha nb)}$ *and* β^* *be equal to* $\frac{nb}{1+2\alpha nb}$. *Now, the list* (Γ^*, β^*) *forms a unique symmetric Nash equilibrium.*

Proof: Let β^* be chosen by the firm. Then CU's problem is to maximize total consumers surplus as in (3.4). Note that the outcome price is solved from (2.1) as $P(\Gamma) = \frac{\Gamma}{nb+\beta^*}$ by using $AD(P) = \Gamma - bnP$ and $AS(P) = \beta^* P$. Put the corresponding expression for $P(\Gamma)$ in the objective function. F.O.C. for this problem $\frac{a}{b}(1-\frac{bn}{nb+\beta^*}) - \frac{\Gamma}{bn}(1-\frac{bn}{nb+\beta^*})^2 - \frac{2\Gamma}{nb+\beta^*}(1-\frac{bn}{nb+\beta^*}) = 0$. From which $\Gamma^c = \frac{an(nb+\beta^*)}{\beta^*+2nb}$. After putting $\beta^* = \frac{nb}{1+2\alpha nb}$, one obtains $\Gamma^c = \frac{an(2+2\alpha nb)}{(3+4\alpha nb)} = \Gamma^*$. S.O.D. of the objective function $\frac{-1}{bn}(1 - \frac{bn}{nb+\beta^*})^2 - \frac{2}{nb+\beta^*}$ and it is negative for any $\Gamma \succ 0$. Thus, Γ^* maixmizes total consumers surplus when the firm announces β^* and so it is the best respone of CU. Let's return to the firm's problem. Assume that CU announces a $\Gamma \succ 0$. Now, the firm's problem is given in (3.5) and the outcome price is obtained from (2.1) as $P(\beta) = \frac{\Gamma}{nb+\beta}$. So firm's problem

$$\max_{\beta} \Pi(\beta) = (\frac{\Gamma}{nb+\beta})^2(\beta - \alpha\beta^2)$$

F.O.C. for this problem $2(\frac{\Gamma}{nb+\beta})(\frac{-\Gamma}{(nb+\beta)^2})(\beta - \alpha\beta^2) - (\frac{\Gamma}{nb+\beta})^2(1 - 2\alpha\beta) = 0$. From which $\beta^c = \frac{nb}{1+2\alpha nb} = \beta^*$. Note that it is independent of a particular value of $\Gamma \succ 0$. S.O.D. of the objective function

$$\frac{-2\Gamma^2 nb + 4\alpha\beta\Gamma^2 nb - 2\Gamma^2\alpha n^2 b^2 - \Gamma^2 2nb + 2\Gamma^2\beta}{(nb+\beta)^4}.$$

Arrange the terms in the numerator and focus on it since we are sure that denominator is positive. Suppose numer ator is non-negative then evaluate it at β^*. This leads to $0 \succ 1 + \alpha nb + 2\alpha^2 n^2 b^2$. Contradiction. So S.O.D. is negative at β^*. Note that if firm announces a β which is greater than $\frac{1}{\alpha}$, then its profit will be negative. If it announces between a β which is nonnegative and less than $\frac{1}{\alpha}$, then its profit will bw nonnegative. So we can restrict the domain into a compact interval, which is $[0, \frac{1}{\alpha}]$. Since profit function is continuous function of β, it arrives its maximum in this interval. Since β^* is the only point satisfying both F.O.C. and S.O.C., we conclude that it maximizes firm's profit over $(0, \infty)$. Thus, firm's best response to any $\Gamma \succ 0$ and in particular to Γ^*. Combining this result with the one in CU's problem, we conclude that the list (Γ^*, β^*) is a Nash equilibrium. About uniqueness, β^* is the best response of the firm to any $\Gamma \succ 0$ and it follows from the problem of CU that Γ^* is the best response of CU to β^*. Thus, the list (Γ^*, β^*) is the unique Nash equilibrium. *QED*

Lemma 6 *Consider the monopoly structure. Let q^* be equal to $\frac{an}{2(1+\alpha bn)}$. Now, q^* is the monopolist profit maximizing level of output.*

Proof: The monopolist with a cost function, $C(q) = \alpha q^2$ will solve (6). Note that $MR = (\frac{a}{b} - \frac{2}{bn}q)$ and $MC = 2\alpha q$. From F.0.C., MR=MC, q^c is obtained as proposed. Note that S.O.D. is negative for any non-negative level of production. So q^* is the profit maximizing level of output. *QED*

Proposition 13 *The following is true:*

$$\begin{array}{rcl} P^M \succ & P^{CPN} \succ & P^{CUPN} \\ Q^M \succ & Q^{CPN} \succ & Q^{CUPN} \\ CS^{CUPN} \succ & CS^{CPN} \succ & CS^M \\ \Pi^M \succ & \Pi^{CPN} \succ & \Pi^{CUPN} \\ SW^{CPN} \succ & SW^M \succ & SW^{CUPN} \end{array}$$

Proof: Firstly, return to the game in which consumers play Nash with the firm. We know that $P^{CPN} = \frac{n\gamma^*}{nb+\beta^*}$. After putting the corresponding expressions for γ^* and β^*, we obtain $P^{CPN} = \frac{a(-1+2n-2\alpha bn+2\alpha bn^2)(1+2\alpha bn)}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}$.

Total quantity demanded (total quantity produced), $Q^{CPN} = n(\gamma^* - bP^{CPN})$. Put the corresponding expressions for γ^* and P^{CPN} obtain

$$Q^{CPN} = \frac{na(-1 + 2n - 2\alpha bn + 2\alpha bn^2)}{b(-1 + 4n - 4\alpha bn + 8\alpha bn^2 - 4\alpha^2 b^2 n^2 + 4\alpha^2 b^2 n^3)}$$

Each consumer will consume as much as $(\gamma^* - bP^{CPN})$ and utilize $CS^{CPN} = \frac{a}{b}(\gamma^* - bP^{CPN}) - \frac{a}{2b}(\gamma^* - bP^{CPN})^2 - P^{CPN}(\gamma^* - bP^{CPN})$, which is calculated as (by using Mathematica) $CS^{CPN} = \frac{a^2(-1-4\alpha bn+4n^2-4\alpha^2b^2n^2+8\alpha bn^3+4\alpha^2b^2n^4)}{2b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2}$. Regarding the firm's profit, from $\Pi^{CPN} = (P^{CPN})^2(\beta^* - \alpha\beta^2)$, $\Pi^{CPN} = \frac{a^2n(-1+2n-2\alpha bn+2\alpha bn^2)^2(1+\alpha bn)}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2}$. Then total social welfare can be calculated from $SW^{CPN} = nCS^{CPN} + m\Pi^{CPN}$ by putting corresponding expressions for CS^{CPN} and Π^{CPN}. Now, return to the game in which CU plays Nash with the firm: We know that for $\Gamma^* = \frac{an(2+2\alpha nb)}{(3+4\alpha nb)}$ and $\beta^* = \frac{bn}{1+2\alpha nb}$, (Γ^*, β^*) is the unique Nash equilibrium. Now, the outcome price is calculated from $P^{CUPN} = \frac{\Gamma^*}{nb+\beta^*}$ as $P^{CUPN} = \frac{a(1+2\alpha bn)}{b(3+4\alpha bn)}$ Total quantity demanded (total quantity produced) at this price $Q^{CUPN} = \Gamma^* - bnP^{CUPN} = \frac{an}{(3+4\alpha bn)}$ Each consumer will consume $\frac{a}{(3+4\alpha bn)}$ since they are sharing total amount equally. Then typical consumer's surplus $CS^{CUPN} = \frac{a}{b}\frac{a}{(3+4\alpha bn)} - \frac{1}{2b}(\frac{a}{(3+4\alpha bn)})^2 - \frac{a(1+2\alpha bn)}{b(3+4\alpha bn)}\frac{a}{(3+4\alpha bn)} = \frac{3a^2+4b\alpha a^2n}{2b(3+4\alpha bn)^2}$. One can verify that $TCS = nCS^{CUPN}$. Profit of the firm can be calculated from $\Pi^{CUPN} = (P^{CUPN})^2(\beta^* - \alpha\beta^2)$ as $\Pi^{CUPN} = \frac{a^2n(1+\alpha bn}{b(3+4\alpha bn)^2}$. Total social welfare calculated from $SW^{CUPN} = nCS^{CUPN} + m\Pi^{CUPN}$ as $SW^{CUPN} = \frac{5a^2n+6b\alpha n^2a^2}{2b(3+4\alpha bn)^2}$. Left with the monopoly structure, we know that total quantity produced, $Q^M = q^* = \frac{an}{2(1+\alpha bn)}$. The outcome price calculated from $P^M = \frac{a}{b} - \frac{1}{bn}Q^M = \frac{a(1+2\alpha bn)}{2b(1+\alpha bn)}$. Each consumer will consume $\frac{a}{2(1+2\alpha bn)}$ since industry demand is n times individual demand. Then $CS^M = \int_0^{\frac{a}{2(1+\alpha bn)}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(1+2\alpha bn)}{2b(1+\alpha bn)}\frac{a}{2(1+\alpha bn)} = \frac{a^2}{8b(1+\alpha bn)^2}$. Now, monopolist's profit is calculated from $\Pi^M = P^M Q^M - \alpha(Q^M)^2$ as

$\Pi^M = \frac{a^2n(1+\alpha bn)}{4b(1+\alpha bn)^2}$. Using them, $SW^M = nCS^M + m\Pi^M = \frac{3a^2n+2\alpha ba^2n^2}{8b(1+\alpha bn)^2}$. Now, we are ready to make comparisons. Suppose that $P^{CUPN} \geq P^{CPN}$, i.e., $\frac{a(1+2\alpha bn)}{b(3+4\alpha bn)} \geq \frac{a(-1+2n-2\alpha bn+2\alpha bn^2)(1+2\alpha bn)}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}$. After a little algebra this leads to $-2+2n-6\alpha bn+6\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3 \leq 0$. Since $n \geq 2$, left hand side is positive. Contradiction. So $P^{CPN} \succ P^{CUPN}$. Now, suppose that $P^{CPN} \geq P^M$, i.e., $\frac{a(-1+2n-2\alpha bn+2\alpha bn^2)(1+2\alpha bn)}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \geq \frac{a(1+2\alpha bn)}{2b(1+\alpha bn)}$. This leads to $-1-2\alpha bn \geq 0$. Contradiction. So $P^M \succ P^{CPN}$. Combining both results, $P^M \succ P^{CPN} \succ P^{CUPN}$. About total quantities produced, $Q^{CUPN} \prec Q^M$ since $3+4\alpha bn \succ 2+2\alpha bn$. Now, suppose that $Q^{CPN} \leq Q^M$. Then $\frac{(-1+2n-2\alpha bn+2\alpha bn^2}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \leq \frac{1}{(3+4\alpha bn)}$. This leads to $-2+2n-6\alpha bn+6\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3 \leq 0$. Since $n \geq 2$, left hand side is positive. Contradiction. So $Q^{CPN} \succ Q^{CUPN}$. Suppose that $Q^{CPN} \geq Q^{CUPN}$. Then $\frac{(-1+2n-2\alpha bn+2\alpha bn^2}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \geq \frac{1}{2(1+\alpha bn)}$. This leads to $-1-2\alpha bn \geq 0$. Contradiction. So $Q^M \succ Q^{CPN}$. Combining both results, $Q^M \succ Q^{CPN} \succ Q^{CUPN}$. About individual consumer's surplus, suppose that $CS^{CPN} \geq CS^{CUPN}$. Then, $\frac{(-1-4\alpha bn+4n^2-4\alpha^2b^2n^2+8\alpha bn^3+4\alpha^2b^2n^4)}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2} \geq \frac{1}{(3+4\alpha bn)}$. After a bit massy algebra, this leads to $0 \geq 4+4n^2-8n+24\alpha bn+40\alpha bn^3-64\alpha bn^2+20\alpha^2b^2n^2+84\alpha^2b^2n^4-72\alpha^2b^2n^3+16\alpha^3b^3n^3(2-5n+3n^2)+16\alpha^4b^4n^4+16\alpha^4b^4n^6-32\alpha^4b^4n^5$ Since $n \geq 2$, right hand side is positive. Contradiction. So $CS^{CPN} \prec CS^{CUPN}$. Now, suppose that $CS^{CPN} \leq CS^M$.Then $1 \geq \frac{n(4+8\alpha bn+4\alpha^2b^2n^2)(-1-4\alpha bn+4n^2-4\alpha^2b^2n^2+8\alpha bn^3+4\alpha^2b^2n^4)}{n(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2}$. Call this inequality (*). Since $n \geq 2$, the denominator and numerator are positive. Define $Expr1$ as $Expr1 = \frac{(4n+8\alpha bn^2+4\alpha^2b^2n^3)}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)}$. And $Expr2$ as $Expr2 = \frac{(-1-4\alpha bn+4n^2-4\alpha^2b^2n^2+8\alpha bn^3+4\alpha^2b^2n^4)}{(-n+4n^2-4\alpha bn^2+8\alpha bn^3-4\alpha^2b^2n^3+4\alpha^2b^2n^4)}$. Note that right hand side of (*) equals to $Expr1$ times $Expr2$. Focus on $Expr1$. Observe that its denominator is positive and less than the numerator. So $Expr1$ is bigger than 1. Now, both denominator and numerator of $Expr2$ are positive and $|n(-1-4\alpha bn)| \succ |-1-4\alpha bn|$ and $|-4\alpha^2b^2n^3| \succ |-4\alpha^2b^2n^2|$ and other terms of denominator and the numerator are the same. Then denominator is less than the numerator. So $Expr2$ is bigger than 1. So the right hand side of (*) is bigger than 1. Contradiction. So $CS^{CPN} \succ CS^M$. Combining both results, we have $CS^{CUPN} \succ CS^{CPN} \succ CS^M$. Regarding profits, $\Pi^M \succ \Pi^{CUPN}$ since $\frac{1}{(3+4\alpha bn)^2} \prec \frac{1}{(2+2\alpha bn)^2}$. Now, suppose that $\Pi^{CUPN} \geq \Pi^{CPN}$. Then $\frac{1}{(3+4\alpha bn)^2} \geq \frac{(-1+2n-2\alpha bn+2\alpha bn^2)^2}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2}$. This leads to $0 \geq -2+2n-6\alpha bn+6\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3$. Contradiction since right hand side is positive. So $\Pi^{CPN} \succ \Pi^{CUPN}$. Now, suppose that $\Pi^{CPN} \geq \Pi^M$. Then $\frac{(-1+2n-2\alpha bn+2\alpha bn^2)^2}{(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)^2} \geq \frac{1}{(2+2\alpha bn)^2}$ After a little algebra, this leads to $-1-2\alpha bn \geq 0$. Contradiction. So $\Pi^M \succ \Pi^{CPN}$. Combining both results, we conclude that $\Pi^M \succ \Pi^{CPN} \succ \Pi^{CUPN}$. Finally total social welfares, suppose that $SW^M \leq SW^{CUPN}$. Then $\frac{3+2\alpha bn}{8b(1+\alpha bn)^2} \leq \frac{5+6b\alpha n}{2b(3+4\alpha bn)^2}$. This leads to $0 \geq 7+2n+26\alpha bn+16\alpha^3b^3n^3+28\alpha^2b^2n^2$. Contradiction. So $SW^M \succ SW^{CUPN}$. Now, suppose that $SW^M \geq SW^{CPN}$. We know Π^M, Π^{CPN}, CS^{CPN} and CS^M. By supposition, $n(CS^{CPN}-CS^M)-(\Pi^M-\Pi^{CPN}) \leq 0$. Define $\Lambda = (-1+4n^2-4\alpha bn+8\alpha bn^3-4\alpha^2b^2n^2+4\alpha^2b^2n^4)$ and $\Theta = (-1+4n-4\alpha bn+8\alpha bn^3-4\alpha^2b^2n^2+4\alpha b^2n^3)^2$. Check that $(-1+2n-2\alpha bn+2\alpha bn^2)^2 = \Lambda+\tau$, where $\tau = 2+8\alpha bn-12\alpha bn^2+8\alpha^2b^2n^2-8\alpha^2b^2n^3$. Now, $n(CS^{CPN}-CS^M)-(\Pi^M-\Pi^{CPN}) \leq 0$ is can be written as $\frac{a^2n}{2b}[(\frac{\Lambda}{\Theta}-\frac{1}{4(1+\alpha bn)^2})-(\frac{2(1+\alpha bn)}{4(1+\alpha bn)^2}-\frac{2(1+\alpha bn)(\Lambda+\tau)}{\Theta})] \leq 0$. Then $\Lambda(3+2\alpha bn)(1+\alpha bn)^2 4+8\tau(1+\alpha bn)^3-\Lambda(3+2\alpha bn)\Theta \leq 0$. Mathematica simplified left hand side into $-15+24n-42\alpha bn+64\alpha bn^2-36(\alpha bn)^2+56(\alpha b)^2n^3+$

$8(\alpha bn)^3 + 16(\alpha bn)^3$ which is positive. Contradiction. So $SW^M \prec SW^{CPN}$. Combining both, we have $SW^{CPN} \succ SW^M \succ SW^{CUPN}$. So we are done.
QED

Proposition 14

$$\begin{aligned}
\lim_{n\to\infty} \gamma^* &= a \\
\lim_{n\to\infty} \beta^* &= \frac{1}{2\alpha} \\
\lim_{n\to\infty} P^{CPN} &= \lim_{n\to\infty} P^M = \lim_{n\to\infty} P^C = \frac{a}{b} \succ \lim_{n\to\infty} P^{CUPN} = \frac{a}{2b} \\
\lim_{n\to\infty} Q^{CPN} &= \lim_{n\to\infty} Q^M = \lim_{n\to\infty} Q^C = \frac{a}{2\alpha b} \succ \lim_{n\to\infty} Q^{CUPN} = \frac{a}{4\alpha b} \\
\lim_{n\to\infty} CS^{CPN} &= \lim_{n\to\infty} CS^M = \lim_{n\to\infty} CS^C = \lim_{n\to\infty} CS^{CUPN} = 0 \\
\lim_{n\to\infty} TCS^{CUPN} &= \frac{a^2}{8\alpha b^2} \\
\lim_{n\to\infty} \Pi^{CPN} &= \lim_{n\to\infty} \Pi^M = \lim_{n\to\infty} \Pi^C = \frac{a^2}{4\alpha b^2} \succ \lim_{n\to\infty} \Pi^{CUPN} = \frac{a^2}{16\alpha b^2} \\
\lim_{n\to\infty} SW^{CPN} &= \lim_{n\to\infty} SW^M = \lim_{n\to\infty} SW^C = \frac{a^2}{4\alpha b^2} \\
\lim_{n\to\infty} SW^{CUPN} &= \frac{3a^2}{16\alpha b^2}
\end{aligned}$$

Proof: Firstly, since there is one firm, IMC function becomes equal to the MC function of the firm. By using $D(P) = n(a - bP)$ and $MC(q) = 2\alpha q$, we obtain $P^C = \frac{na}{(\frac{1}{2\alpha}+nb)}$. Then Q^C is obtained as $Q^C = \frac{a}{\frac{1}{n}+2\alpha b}$. Each individual consumer will utilize $\int_{P^C}^{\frac{a}{b}}(a - bt)dt = \frac{a^2}{2b(1+4\alpha nb+4\alpha^2b^2n^2)}$ as a surplus. One can easily verify that $\lim_{n\to\infty} P^C = \frac{a}{b}$ and $\lim_{n\to\infty} Q^C = \frac{a}{2\alpha b}$. By using them $\lim_{n\to\infty} \Pi^C = \lim_{n\to\infty}(P^C Q^C - \alpha(Q^C)^2) = \frac{a^2}{4\alpha b^2}$ It is also straightforward that $\lim_{n\to\infty} CS^C = 0$ and $\lim_{n\to\infty} nCS^C = 0$. Then $\lim_{n\to\infty} SW^C = \frac{a^2}{4\alpha b^2}$. Note that By using corresponding formula from Proposition 13 for each economic variable, one can verify other limits. *QED*

An interesting observation from this proposition, the existence of an institution like CU leads to the suboptimality as the number of consumers goes to infinity. Although total consumer surplus is a positive finite amount in the limit, when it is distributed among very large number of consumers, individual consumer's share goes to the nill. The loser here is the firm.

Case: $\alpha \succ 0$, $n = 1$, $m = 1$

In this case there is one consumer and one firm. The firm produces with positive cost.

Proposition 15 *Let γ^* be equal to $\frac{a(2+2\alpha b)}{3+4\alpha b}$ and β^* be equal to $\frac{b}{(1+2\alpha b)}$. Now, the list (γ^*, β^*) forms a unique Nash equilibrium.*

Proof: Let β^* be chosen by the firm. Consumer's problem is given in (3.2). The outcome price is solved easily from (2.1) as $P(\gamma) = \frac{\gamma}{b+\beta^*}$. F.O.C. for his problem $\frac{a}{b}(1 - \frac{b}{b+\beta^*}) - \frac{\gamma}{b}(1 - \frac{b}{b+\beta^*})^2 - \frac{2\gamma}{(b+\beta^*)}(1 - \frac{b}{b+\beta^*})$. From which, $\gamma^c = \frac{a(b+\beta^*)}{(2b+\beta^*)}$. Put $\frac{b}{(1+2\alpha b)}$ instead of β^* and verify that $\gamma^c = \gamma^*$. S.O.D. of the objective function is $\frac{-1}{b}(1 - \frac{b}{b+\beta^*})^2 - \frac{2}{(b+\beta^*)}(1 - \frac{b}{b+\beta^*}) \prec 0$ for any $\gamma \succ 0$. Note that consumer's surplus function is continuous function of γ and F.O.D. vanishes only at γ^*. Thus, γ^* maximizes his surplus over $(0, \infty)$ and so it is the best response of consumer to β^*. Let's return to the firm's problem. Let consumer announce any $\gamma \succ 0$. Firm's problem is given in (3.3). Note that the outcome price is solved from (2.1) as $P(\beta) = \frac{\gamma}{b+\beta}$. F.O.C. for this problem, $2\frac{\gamma}{b+\beta}(\frac{-\gamma}{(b+\beta^*)^2})(\beta - \alpha\beta^2) + (\frac{\gamma}{b+\beta})^2(1 - 2\alpha\beta) = 0$ From which β^c is solved as $\frac{b}{(1+2\alpha b)} = \beta^*$. Note that it is independent of the particular value of γ. Now, S.O.D. of the objective function after a bit manipulating is

$\frac{-2\gamma^2}{(\beta+b)^4}[-\beta+2b-2\alpha\beta b+\alpha b^2]$. Concentrate on $[-\beta+2b-2\alpha\beta b+\alpha b^2]$. Suppose that it is non-positive. Evaluate it at β^*. Then this leads to $1 \geq 2+\alpha b$. Contradiction. So S.O.D. of the profit function is negative at β^*. So it satisfies S.O.C..One can verify the for $\beta \succ \frac{1}{\alpha}$ profit is negative otherwise it is non-negative. So we can restrict the domain to a compact interval $[0, \frac{1}{\alpha}]$. Since profit function is continuos function of β it arrives its maximum in this interval. Since β^* is the unique point satisfying F.O.C and S.O.C., we conclude that β^* maximizes firm's profit. So it is the best response of the firm to any $\gamma \succ 0$ and so it is the dominant strategy to the firm. Since β^* is the best response of firm to γ^*, in particular, we conclude that the list (γ^*, β^*) forms a Nash equilibrium. Uniqueness follows from the fact that β^* is the dominant strategy of the firm and γ^* is the best response of consumer to β^*. *QED*

CHAPTER 4

STACKELBERG GAME

We can consider two games. In one of the games a firm is Stackelberg leader, whereas in the other consumers as organized are leaders. The latter reduces to Nash game because we observe that in all cases we cover so far, as long as there is symmetric bunch of strategies on the side of consumers, typical firm's problem is independent from the particular values of intercept terms. Thus, we proceed with the game in which the firm is Stackelberg leader and consumers are either organized or unorganized followers. Data and strategy sets are as defined in the Model section and we search for symmetric equilibria. The firm is leader so it moves first and announces a number for its slope term, which induces a game for consumers. Given the firm's strategy, consumers play Nash among themselves.

Definition 3 *Let $\gamma^* \succ 0$ and $\beta^* \geq 0$ and $(\gamma^*)_{i=1}^n$ and (β^*) be a symmetric strategy profile of the consumers and the firm, respectively. We say the list $((\gamma^*)_{i=1}^n, (\beta^*))$ forms a Stackelberg equilibrium if $(\gamma^*)_{i=1}^n$ is the Nash equilibrium of the game induced by (β^*) and $((\gamma^*)_{i=1}^n, (\beta^*))$ is a Nash equilibrium of* *the whole game.*

Now, we proceed case by case and give equilibrium strategies.

Case: $\alpha = 0,\ n = 1,\ m = 1$

In this case there is one consumer and one firm. The firm produces with zero cost.

Proposition 16 *Assume that the firm is Stackelberg leader. Let γ^* be equal to $\frac{3a}{4}$ and β^* be equal to $2b$. Now, the list (γ^*, β^*) is the unique Stackelberg equilibrium.*

Proof: Now, the firm is Stackelberg leader and the consumer is follower. Consumer's problem for a given $\beta \geq 0$ to maximize his surplus by giving a $\gamma \succ 0$ as

$$\max_{\gamma} CS(\gamma) = \int_0^{\gamma - bP(\gamma)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma)(\gamma - bP(\gamma))$$

where $P(\gamma)$ is the outcome price and solved from (1) as $P(\gamma) = \frac{\gamma}{b+\beta}$. F.O.C. for this problem $\frac{a}{b}\frac{\beta}{b+\beta} - \frac{\gamma}{b}(\frac{\beta}{b+\beta})^2 - \frac{2\gamma\beta}{(b+\beta)^2} = 0$ From which γ^c is solved as a function of β, $\gamma^c = \frac{a(b+\beta)}{(2b+\beta)}$. Note that S.O.C is satisfied for any $\gamma \succ 0$. Now, the firm knows that consumer is rational and recognize how consumer will respond when it announces a $\beta \geq 0$. Then firm's problem

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta))$$

where $P(\beta)$ is the outcome price and solved from (2.1) as $P(\beta) = \frac{\gamma^c}{b+\beta}$. Put $\frac{a(b+\beta)}{(2b+\beta}$ for γ^c. Then firm's problem,

$$\max_{\beta} \Pi(\beta) = (\frac{a}{\beta + 2b})^2 \beta$$

Then F.O.C. for this problem, $2\frac{a}{\beta+2b}\frac{-a}{(\beta+2b)^2}\beta+(\frac{a}{\beta+2b})^2=0$ From which β^c is solved as $2b=\beta^*$. S.O.D. for the profit function $\frac{2a^2\beta-8a^2b}{(2b+\beta)^4}$. Put $\beta^*=2b$ and verify that it is negative. One can verify that $\Pi(\beta^*)\succ\Pi(\beta)\quad\forall\beta\succ 4b$. So we can restrict the domain into an compact interval since profit function is continuous function of β it arrives its maximum in this interval.Since F.O.D. vanishes at β^* and S.O.C. is satisfied at β^*, it maximizes firm's profit. Now, one can verify that $\gamma^c=\gamma^*$ for β^*. Since consumer's surplus function is continuous function of γ and F.O.D. vanishes at γ^* for given β^*. Since S.O.C. satisfied for any $\gamma\succ 0$, γ^* maximizes consumer's surplus for given β^*. Therefore, (γ^*,β^*) forms a Stackelberg equilibrium. Uniqueness is straightforward.

QED

Before giving our first theorem, any variable with superscript FSL belongs to the game in which firm is Stackelberg leader consumer is follower.

Theorem 1 *The following is true:*

$$\begin{array}{rll} P^M\succ & P^{CPN}\succ & P^{FSL} \\ Q^M= & Q^{FSL}\succ & Q^{CPN} \\ CS^{FSL}\succ & CS^{CPN}\succ & CS^M \\ \Pi^M\succ & \Pi^{FSL}\succ & \Pi^{CPN} \\ SW^M= & SW^{FSL}\succ & SW^{CPN} \end{array}$$

Proof: Firstly, return to the Stackelberg game in which for $\gamma^*=\frac{3a}{4}$ and $\beta^*=2b$, the list (γ^*,β^*) forms a unique Nash equilibrium. Then $P^{FSL}=\frac{\gamma^*}{b+\beta^*}=\frac{a}{4b}$. Using this, $Q^{FSL}=\gamma^*-bP^{FSL}=\frac{a}{2}$. Now, consumer's surplus $CS^{FSL}=\frac{a}{b}\frac{a}{2}-\frac{1}{2b}(\frac{a}{2})^2-\frac{a}{4b}\frac{a}{2}=\frac{a^2}{4b}$. Firm's profit is $\Pi^{FSL}=(\frac{a}{4b})^2 2b=\frac{a^2}{8b}$.

Then total social welfare is $SW^{FSL} = \frac{a^2}{4b} + \frac{a^2}{8b} = \frac{3a^2}{8b}$. Now, in the proof of Proposition 8, we gave the formulas regarding these variables for the game in which consumers play Nash with the firm and monopoly structure. By putting $n = 1$ there calculate $P^{CPN} = \frac{a}{3b}$, $Q^{CPN} = \frac{a}{3}$, $CS^{CPN} = \frac{a^2}{6b}$, $\Pi^{CPN} = \frac{a^2}{6b}$ and $SW^{CPN} = \frac{3a^2}{8b}$ for the game in which consumer and firm play Nash and $P^M = \frac{a}{2b}$, $Q^M = \frac{a}{2}$, $CS^M = \frac{a^2}{8b}$, $\Pi^M = \frac{a^2}{4b}$ and $SW^M = \frac{3a^2}{8b}$. Then, one can easily verify the proposed relations. *QED*

Case: $\alpha = 0$, $n \geq 2$, $m = 1$

Only difference with the previous case is that there are more than one consumers. We require consumers to form a union, otherwise things are getting really complicated. As a new notation $FSL + CUF$ is the abbreviation for "Firm Stackelberg leader and CU is follower" and any variable takes it as superscript belongs to this game.

Proposition 17 *Assume that consumers are organized and the firm is Stackelberg leader. Let Γ^* be equal to $\frac{3an}{4}$ and β^* be equal to $2bn$. Now, the list (Γ^*, β^*) forms a unique Stackelberg equilibrium.*

Proof: The proof is very similar to the proof of Proposition 16. Firm is Stackelberg leader and CU is follower. The firm knows that CU is a rational player and for any $\beta \geq 0$ it will solve CU's problem which is to maximize total consumer surplus by giving a $\Gamma \succ 0$ as

$$\max_{\Gamma} TCS(\Gamma) = \int_0^{\Gamma - bnP(\Gamma)} (\frac{a}{b} - \frac{1}{bn}t)dt - P(\Gamma)(\Gamma - bnP(\Gamma))$$

where $P(\Gamma)$ is the outcome price and solved from (1) as $P(\Gamma) = \frac{\Gamma}{bn+\beta}$. Then F.O.C. for this problem $\frac{a}{b}\frac{\beta}{bn+\beta} - \frac{\Gamma}{bn}(\frac{\beta}{bn+\beta})^2 - \frac{2\Gamma\beta}{(nb+\beta)^2} = 0$ From which Γ^c is solved as a function of β, $\Gamma^c = \frac{an(bn+\beta)}{(2bn+\beta)}$. Note that S.O.C is satisfied for any $\Gamma \succ 0$. Now, the firm by recognizing how CU will respond when it announces a $\beta \geq 0$ will solve its own problem, which is

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta))$$

where $P(\beta)$ is the outcome price and solved from (1) as $P(\beta) = \frac{\Gamma^c}{bn+\beta}$. Then firm's problem

$$\max_{\beta} \Pi(\beta) = (\frac{an}{\beta + 2bn})^2 \beta$$

F.O.C. for this problem, $2\frac{an}{2bn+\beta} - an(2bn+\beta)^2\beta + (\frac{an}{2bn+\beta})^2 = 0$ From which β^c is solved as $2bn = \beta^*$. One can verify that S.O.C. is satisfied at β^* and $\Pi(\beta^*) \succ \Pi(\beta) \quad \forall \beta \succ 4bn$. So we can restrict the domain into a compact interval $[0, 4bn]$. Since profit function is continuous function of β, it arrives its maximum in this interval and F.O.D. vanishes only at β^* and S.O.C. is satisfied at β^*, it maximizes firm's profit. Now, one can verify that $\Gamma^c = \Gamma^*$ for β^*. Since total consumer's surplus function is continuous function of Γ and F.O.D. vanishes at Γ for given β^* and S.O.C. is satisfied for any $\Gamma \succ 0$, Γ^* maximizes consumer's surplus for given β^*. Therefore, (Γ^*, β^*) forms a Stackelberg equilibrium. Uniqueness is straightforward. *QED*

Theorem 2 *The following is true:*

$$\begin{aligned}
P^M \succ \quad & P^{CPN} \succ \quad && P^{CUPN} \succ P^{FSL+CUF} \\
Q^{FSL+CUF} = \quad & Q^M \succ \quad && Q^{CPN} \succ Q^{CUPN} \\
CS^{FSL+CUF} \succ \quad & CS^{CUPN} \succ \quad && CS^{CPN} \succ CS^M \\
\Pi^M \succ \quad & \Pi^{CPN} \succ \quad && \Pi^{FSL+CUF} \succ \Pi^{CUPN} \\
SW^M = \quad & SW^{FSL+CUF} \succ \quad && SW^{CPN} \succ SW^{CUPN}
\end{aligned}$$

Proof: Firstly, return to the Stackelberg game in which firm is Stackelberg leader and CU is follower. We know that for $\Gamma^* = \frac{3an}{4}$ and $\beta^* = 2bn$. Now, the list (Γ^*, β^*) forms a unique Stackelberg equilibrium. First, equilibrium price from $P^{FSL+CUF} = \frac{\Gamma^*}{bn+\beta^*}$, $P^{FSL+CUF} = \frac{a}{4b}$. Total quantity produced equals to total quantity demanded at equilibrium so $Q^{FSL+CUF} = \Gamma^* - bnP^{FSL+CUF} = \frac{an}{2}$. Since consumers equally divide this amount among themselves, each consumer will consume $\frac{a}{2}$. Then individual consumer's surplus $CS^{FSL+CUF} = \int_0^{\frac{a}{2}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a}{2}\frac{a}{4b} = \frac{a^2}{4b}$ One can verify that $TCS^{FSL+CUF} = nCS^{FSL+CUF}$. Now, firm's profit $\Pi^{FSL+CUF} = (P^{FSL+CUF})^2\beta^* = \frac{a^2n}{8b}$. Then total social welfare $SW^{FSL+CUF} = nCS^{FSL+CUF} + \Pi^{FSL+CUF} = \frac{3a^2n}{8b}$. Now, we know from Proposition 8 (when $n \geq 2$) the following relations

$$\begin{array}{ccccc} P^M & \succ & P^{CPN} & \succ & P^{CUPN} \\ Q^M & \succ & Q^{CPN} & \succ & Q^{CUPN} \\ CS^{CUPN} & \succ & CS^{CPN} & \succ & CS^M \\ \Pi^M & \succ & \Pi^{CPN} & \succ & \Pi^{CUPN} \\ SW^M & \succ & SW^{CPN} & \succ & SW^{CUPN} \end{array}$$

and their magnitudes. Now, $P^{CUPN} = \frac{a}{3b} \succ \frac{a}{4b} = P^{FSL+CUF}$. Then it follows that $P^M \succ P^{CPN} \succ P^{CUPN} \succ P^{FSL+CUF}$. Since $Q^{FSL+CUF} = Q^M = \frac{an}{2}$, it follows that $Q^{FSL+CUF} = Q^M \succ Q^{CPN} \succ Q^{CUPN}$. Now, $CS^{FSL+CUF} = \frac{a^2}{4b} \succ \frac{a^2}{6b} = CS^{CUPN}$. Then it follows that $CS^{FSL+CUF} \succ CS^{CUPN} \succ CS^{CPN} \succ CS^M$. Suppose that $\Pi^{FSL+CUF} \geq \Pi^{CPN}$. Then $\frac{a^2n}{8b} \geq \frac{a^2n(4n-2)^2}{4b(4n-1)^2}$. This leads to $16n^2 - 24n + 7 \leq 0$. Since $n \geq 2$, left hand side is positive. Then contradiction. So $\Pi^{CPN} \succ \Pi^{FSL+CUF}$. Now, since $\frac{a^2n}{4b} \succ \frac{a^2n}{8b} \succ \frac{a^2n}{9b}$, $\Pi^M \succ \Pi^{FSL+CUF} \succ \Pi^{CUPN}$. Clearly, $\Pi^M \succ \Pi^{CPN}$. Then it follows that $\Pi^M \succ \Pi^{CPN} \succ \Pi^{FSL+CUF} \succ \Pi^{CUPN}$. Since $SW^{FSL+CUF} =$

SW^M, it follows that $SW^M = SW^{FSL+CUF} \succ SW^{CPN} \succ SW^{CUPN}$. So we are done. *QED*

Case: $\alpha \succ 0$, $n = 1$, $m = 1$

For this case there is one consumer and one firm. The firm produces with positive marginal cost.

Proposition 18 *Assume that firm is Stackelberg leader. Let γ^* be equal to $\frac{a(3+4\alpha b)}{4+8\alpha b}$ and β^* be equal to $\frac{2b}{1+4\alpha b}$. Now, the list (γ^*, β^*) is a unique Stackelberg equilibrium.*

Proof: The consumer's problem is as in the proof of proposition 16 and solved in the same way and $\gamma^c = \frac{a(3+4\alpha b)}{4+8\alpha b}$ is obtained from the F.O.C. of his problem and S.O.D. is negative for any positive choice of $\gamma \succ 0$. Now, firm' s problem only differs with positive cost parameter. So

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta)) - \alpha(\beta P(\beta))^2$$

where $P(\beta)$ is the outcome price and solved from (1) as $P(\beta) = \frac{\gamma^c}{b+\beta}$. F.O.C. for this problem $2\frac{a}{2b+\beta}\frac{-a}{(2b+\beta)^2}(\beta - \alpha\beta^2) + (\frac{a}{2b+\beta})^2(1 - 2\alpha\beta) = 0$. From which we obtain $\beta^c = \frac{2b}{1+4\alpha b} = \beta^*$. Now, S.O.D. is $\frac{-2a\alpha}{(2b+\beta)^2} - \frac{4a^2(1-2\alpha\beta)}{(2b+\beta)^3} + \frac{ba^2(\beta-\alpha\beta^2)}{(2b+\beta)^4}$. Mathematica simplified this into $\frac{2a^2(-4b-4\alpha b^2+\beta+4\alpha\beta)}{(2b+\beta)^4}$ Suppose that the numerator is non-negative and evaluate it at β^*. This leads to $1 \geq 2 + 2\alpha b$. Contradiction. So S.O.D. is negative at β^*. So S.O.C. is satisfied at β^*. Note that we can restrict the domain into a compact interval, $[0, \frac{1}{\alpha}]$ since for all βs greater than $\frac{1}{\alpha}$ profit is negative and nonnegative otherwise. Since profit function is continuous it arrives its maximum in this interval. So β^* maximizes firm's profit. Now, if we put β^* at γ^c, we obtain γ^*. By the same reasoning in the proof of Proposition 16, we conclude that the list is the unique Stackelberg equilibrium. *QED*

Theorem 3 *The following is true:*

$$\begin{array}{ccccc} P^M \succ & P^{CPN} \succ & P^{FSL} \\ Q^M \succ & Q^{FSL} \succ & Q^{CPN} \\ CS^{FSL} \succ & CS^{CPN} \succ & CS^M \\ \Pi^M \succ & \Pi^{FSL} \succ & \Pi^{CPN} \\ SW^M \succ & SW^{FSL} \succ & SW^{CPN} \end{array}$$

Proof: Firstly, return to the Stackelberg Game. We know that for $\gamma^* = \frac{a(3+4\alpha b)}{4+8\alpha b)}$ and $\beta^* = \frac{2b}{1+4\alpha b}$, (γ^*, β^*) is the unique Stackelberg equilibrium. From $P^{FSL} = \frac{\gamma^*}{b+\beta^*}$, the outcome price solved as $P^{FSL} = \frac{a(1+4\alpha b)}{b(4+8\alpha b)}$. Total quantity produced equals to total quantity demanded so $Q^{FSL} = \gamma^* - bP^{FSL} = \frac{a}{(2+4\alpha b)}$. So the consumer consumes $\frac{a}{(2+4\alpha b)}$ and his surplus $CS^{FSL} = \int_0^{\frac{a}{(2+4\alpha b)}} (\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(1+4\alpha b)}{b(4+8\alpha b)} \frac{a}{(2+4\alpha b)} = \frac{a^2}{b(4+8\alpha b)}$. And firm's profit $\Pi^{FSL} = P^{FSL}(\beta^* P^{FSL}) - \alpha(\beta^* P^{FSL})^2 = \frac{a^2}{8b(1+2\alpha b)}$. Using them total social welfare $SW^{FSL} = CS^{FSL} + \Pi^{FSL} = \frac{3a^2}{2b(4+8\alpha b)}$ Now, the outcome of Nash game is calculated as follows: We know from Proposition 14 that for $\gamma^* = \frac{a(2+2\alpha)}{(3+4\alpha b)}$ and $\beta^* = \frac{b}{(1+2\alpha b)}$, the list (γ^*, β^*) forms a unique Nash equilibrium. Then the outcome price is $P^{CPN} = \frac{\gamma^*}{b+\beta^*} = \frac{a(1+2\alpha b)}{b(3+4\alpha b)}$. At this price the consumer consumes $\gamma^* - bP^{CPN} = \frac{a}{(3+4\alpha b)}$. This is also total quantity produced, Q^{CPN} at equilibrium price. The consumer's surplus is $CS^{CPN} = \int_0^{\frac{a}{(3+4\alpha b)}} (\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(1+2\alpha b)}{b(3+4\alpha b)} \frac{a}{(3+4\alpha b)} = \frac{a^2}{2b(3+4\alpha b)}$. And firm's profit is $\Pi^{CPN} = P^{CPN}(\beta^* P^{CPN}) - \alpha(\beta^* P^{CPN})^2 = \frac{a^2(1+\alpha b)}{b(3+4\alpha b)^2}$. Then total social welfare $SW^{CPN} = CS^{CPN} + \Pi^{CPN} = \frac{5a^2+6a^2\alpha b}{2b((3+4\alpha b)^2}$. Now, the monopoly structure is left. From Lemma 6 by putting $n = 1$ obtain $Q^M =$

$\frac{a}{2(1+\alpha b)}$ is the monopolist's profit maximizing level of output. Put it into inverse industry demand (individual consumer's demand here) and obtain $P^M = \frac{a}{b} - \frac{1}{b}Q^M = \frac{a(1+2\alpha b)}{2b(1+\alpha b)}$. The consumer's surplus is $CS^M = \int_0^{\frac{a}{2(1+\alpha b)}}(\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(1+2\alpha b)}{2b(1+\alpha b)}\frac{a}{2(1+\alpha b)}$. And the monopolist's profit $\Pi^M = P^M(\beta^* P^M) - \alpha(\beta^* P^M)^2 = \frac{a^2}{4b(1+\alpha b)}$. Finally, total social welfare is calculated from $SW^M = CS^M + \Pi^M = \frac{3a^2+2a^2\alpha b}{8b(1+\alpha b)^2}$. Now, we have all the necessary information to make the comparisons. Start with the prices. Suppose that $P^{CPN} \leq P^{FSL}$. Then $\frac{(1+2\alpha b)}{(3+4\alpha b)} \leq \frac{(1+4\alpha b)}{(4+8\alpha b)}$. This leads to $1 \geq 0$. Contradiction. So $P^{CPN} \succ P^{FSL}$. Clearly, $P^{CPN} \prec P^M$. So $P^M \succ P^{CPN} \succ P^{FSL}$ Since $2+2\alpha b \prec 2+4\alpha b \prec 3+4\alpha b$, it follows that $Q^M \succ Q^{FSL} \succ Q^{CPN}$. The relation $CS^{FSL} \succ CS^{CPN} \succ CS^M$ is straightforward. Regarding the profits suppose that $\Pi^{CPN} \geq \Pi^{FSL}$. Then $\frac{(1+\alpha b)}{(3+4\alpha b)^2} \geq \frac{1}{8(1+2\alpha b)}$. This leads to $0 \geq 1$. Contradiction. So $\Pi^{CPN} \prec \Pi^{FSL}$. Clearly, $\Pi^{FSL} \prec \Pi^M$. Then $\Pi^M \succ \Pi^{FSL} \succ \Pi^{CPN}$. Finally, total social welfares: Suppose that $SW^{CPN} \geq SW^{FSL}$. Then $\frac{5+6\alpha b}{(3+4\alpha b)^2} \geq \frac{3}{(4+8\alpha b)}$. This leads to $0 \geq 7 + 8\alpha b$. Contradiction. So $SW^{CPN} \prec SW^{FSL}$. Suppose that $SW^{FSL} \geq SW^M$. Then $\frac{3}{(1+2\alpha b)} \geq \frac{3+2\alpha b}{(1+\alpha b)^2}$. This leads to $0 \geq 2\alpha b + (\alpha b)^2$. Contradiction. So $SW^M \succ SW^{FSL}$. Then $SW^M \succ SW^{FSL} \succ SW^{CPN}$. So we are done. *QED*

Case: $\alpha \succ 0$, $n \geq 2$, $m = 1$

In this case there are more than two consumers and one firm again. The firm produces with positive marginal cost.

Proposition 19 *Assume that consumers are organized. Let Γ^* be equal to $\frac{an(3+4\alpha bn)}{(4+8\alpha bn)}$ and β^* be equal to $\frac{2bn}{1+4\alpha bn}$. Now, the list (Γ^*, β^*) forms a unique Stackelberg equilibrium.*

Proof: The CU's problem is as in the proof of Proposition 17 and it is solved in the same way. So one obtains $\Gamma^c = \frac{an(3+4\alpha bn)}{(4+8\alpha bn)}$. Now, the firm's problem only differs with positive cost parameter. So its problem

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta)) - \alpha(\beta P(\beta))^2$$

where $P(\beta)$ is the outcome price and solved from (1) as $P(\beta) = \frac{\Gamma^c}{bn+\beta}$. F.O.C. for this problem $(\frac{an}{\beta+2bn})(\frac{-an}{(\beta+2bn})^2)(\beta - \alpha\dot{\beta}^2) + (\frac{an}{\beta+2bn})^2(1-2\alpha) = 0$ From which one obtains $\beta^c = \frac{2bn}{1+4\alpha bn}\frac{2bn}{1+4\alpha bn} = \beta^*$. One can check that S.O.C. satisfied at β^*. Again it is possible to restrict the domain into a compact interval $[0, \frac{1}{\alpha}]$. When one puts β^* in Γ^c, one obtains Γ^*. By the same reasoning in Proposition 18, we conclude that the list (Γ^*, β^*) forms a unique Stackelberg equilibrium. *QED*

Note that $FSL{+}CUF$ is the abbreviation of the statement "Firm Stackelberg Leader and CU is follower". As a new notation, any variable with superscript $FSL + CUF$ reveals that it belongs to that game.

Theorem 4 *The following is true:*

$$\begin{array}{rcccc}
 & P^M \succ & P^{CPN} \succ & P^{CPN} \succ P^{FSL} \\
\textit{if } \alpha b \in \left(0, \frac{\sqrt{2}-1}{2n}\right): & Q^M \succ & Q^{FSL+CUF} \succ & Q^{CPN} \succ Q^{CUPN} \\
\textit{if } \alpha b \in \left(\frac{\sqrt{2}-1}{2n}, \infty\right): & Q^M \succ & Q^{CPN} \succ & Q^{FSL+CUF} \succ Q^{CUPN} \\
\textit{if } \alpha b = \frac{\sqrt{2}-1}{2n}: & Q^M \succ & Q^{CPN} = & Q^{FSL+CUF} \succ Q^{CUPN} \\
 & CS^{FSL+CUF} \succ & CS^{CUPN} \succ & CS^{CPN} \succ CS^M \\
 & \Pi^M \succ & \Pi^{CPN} \succ & \Pi^{FSL+CUF} \succ \Pi^{CUPN} \\
 & SW^{CPN} \succ & SW^M \succ & SW^{FSL+CUF} \succ SW^{CUPN}
\end{array}$$

Proof: Firstly, return to the Stackelberg game. We proved in Proposition 18 that for $\Gamma^* = \frac{an(3+4\alpha bn)}{4+8\alpha bn}$ and $\beta^* = \frac{2bn}{1+4\alpha bn}$, (Γ^*, β^*) forms the unique Stackelberg equilibrium. The outcome price is calculated from $P^{FSL+CUF} = \frac{\Gamma^*}{bn+\beta^*}$ as $P^{FSL+CUF} = \frac{a(1+4\alpha bn)}{b(4+8\alpha bn)}$. Total quantity produced equals to total quantity demanded so $Q^{FSL+CUF} = \Gamma^* - bP^{FSL+CUF} = \frac{an}{2+4\alpha bn}$. Since the contract among the consumers is the equal division of total quantity produced at equilibrium, each consumer will consume $\frac{a}{2+4\alpha bn}$. Then each consumer will obtain the surplus $CS^{FSL+CUF} = \int_0^{\frac{a}{2+4\alpha bn}} (\frac{a}{b} - \frac{1}{b}t)dt - \frac{a(1+4\alpha bn)}{b(4+8\alpha bn)} \frac{a}{2+4\alpha bn} = \frac{a^2}{2b(1+2\alpha bn)}$. One can verify that $TCS^{FSL+CUF} = nCS^{FSL+CUF}$ so the contract among the consumers equally divides the total consumers surplus. Now, firm's profit $\Pi^{FSL+CUF} = (P^{FSL+CUF})^2(\beta^* - \alpha(\beta^*)^2)) = \frac{2a^2n}{16b(1+2\alpha bn)}$ Then total social welfare $SW^{FSL+CUF} = nCS^{FSL+CUF} + \Pi^{FSL+CUF} = \frac{3a^2n}{8b(1+2\alpha bn)}$ From Proposition 13 we know the following relation

$$\begin{array}{ccc}
P^M \succ & P^{CPN} \succ & P^{CUPN} \\
Q^M \succ & Q^{CPN} \succ & Q^{CUPN} \\
CS^{CUPN} \succ & CS^{CPN} \succ & CS^M \\
\Pi^M \succ & \Pi^{CPN} \succ & \Pi^{CUPN} \\
SW^{CPN} \succ & SW^M \succ & SW^{CUPN}
\end{array}$$

and their magnitudes. So, we have all the necessary information to carry out the comparisons. Start with the prices. Suppose that $P^{FSL+CUF} \geq P^{CUPN}$. Then $\frac{(1+4\alpha bn)}{(4+8\alpha bn)} \geq \frac{(1+2\alpha bn)}{(3+4\alpha bn)}$. This leads to $0 \geq 1$. Contradiction. So $P^{CUPN} \succ P^{FSL+CUF}$. Combining with the above relation, $P^M \succ P^{CPN} \succ P^{CUPN} \succ P^{FSL}$. Since $3+4\alpha bn \succ 2+4\alpha$, $Q^{CUPN} \prec Q^{FSL+CUF}$. Suppose that $Q^{CPN} \leq Q^{FSL+CUF}$. Then $\frac{(-1+2n-2\alpha bn+2\alpha bn^2}{b(-1+4n-4\alpha bn+8\alpha bn^2-4\alpha^2b^2n^2+4\alpha^2b^2n^3)} \leq \frac{1}{(2+4\alpha bn)}$. This leads to $n(4\alpha bn - 4\alpha b - 4\alpha^2b^2n^2 + 4\alpha^2b^2n^2) \leq 1$. This implies $\frac{1}{4\alpha bn(1+\alpha bn)} + 1 \geq n \geq 2$.

So if $4\alpha bn(1+\alpha bn)+1$, is less than or equal to 1, then it is consistent with the supposition otherwise it contradicts the supposition. So focus on the equation $4\alpha bn+4\alpha^2b^2n^2-1=0$. Set $x=\alpha b$ and find the roots of $4n^2x^2+4xn-1=0$. It turns out that $x_1=\frac{-1-\sqrt{2}}{2n}$ and $x_2=\frac{-1+\sqrt{2}}{2n}$ are the roots. Since $x=\alpha b\succ 0$, we use x_2. Now, $\forall\alpha b\in(0,\frac{-1+\sqrt{2}}{2n}):4x^2n^2+4xn-1\prec 0$ and this is consistent with the negativity side of the supposition, $Q^{CPN}\prec Q^{FSL+CUF}$. If $x=\alpha b=\frac{-1+\sqrt{2}}{2n}$, then $4x^2n^2+4xn-1=0$ and this is consistent with the equality side of the supposition, $Q^{CPN}=Q^{FSL+CUF}$. Finally, $\forall\alpha b\in(\frac{-1+\sqrt{2}}{2n},\infty)$: $4x^2n^2+4xn-1\succ 0$ and and this contradicts the supposition so for this case its negation is true. Thus, $\forall\alpha b\in(\frac{-1+\sqrt{2}}{2n},\infty):Q^{CPN}\succ Q^{FSL+CUF}$. Clearly, $Q^M\succ Q^{FSL+CUF}$. Combining all,

$$\begin{aligned}
\text{if } \alpha b\in\left(0,\tfrac{\sqrt{2}-1}{2n}\right)&: Q^M\succ \; Q^{FSL+CUF}\succ \; Q^{CPN}\succ Q^{CUPN}\\
\text{if } \alpha b\in\left(\tfrac{\sqrt{2}-1}{2n},\infty\right)&: Q^M\succ \; Q^{CPN}\succ \; Q^{FSL+CUF}\succ Q^{CUPN}\\
\text{if } \alpha b=\tfrac{\sqrt{2}-1}{2n}&: Q^M\succ \; Q^{CPN}= \; Q^{FSL+CUF}\succ Q^{CUPN}
\end{aligned}$$

Now, we come to the consumer's surplus. Clearly, $CS^{FSL+CUF}\succ CS^{CUPN}$, so combining with the relation above $CS^{FSL+CUF}\succ CS^{CUPN}\succ CS^{CPN}\succ CS^M$. Regarding total social welfares: Suppose that $SW^M\leq SW^{FSL+CUF}$ Then $\frac{(3+2\alpha bn)}{(1+\alpha bn)^2}\leq\frac{3}{(1+2\alpha bn)}$. This leads to $2\alpha bn+\alpha b^2n^2\leq 0$. Contradiction. So $SW^M\succ SW^{FSL+CUF}$. Suppose that $SW^{CUPN}\geq SW^{FSL+CUF}$. Then $\frac{(5+6\alpha bn)}{(3+4\alpha bn)^2}\geq\frac{3}{4(1+2\alpha bn)}$. This leads to $7+8\alpha bn\leq 0$. Contradiction. So $SW^{FSL+CUF}\succ SW^{CUPN}$. Combining both results with the above relations, we conclude that $SW^{CPN}\succ SW^M\succ SW^{FSL+CUF}\succ SW^{CUPN}$. Now, let's return to the profits. Suppose that $\Pi^{CUPN}\geq\Pi^{FSL+CUF}$. Then $\frac{(1+\alpha bn)}{(3+4\alpha bn)^2}\geq\frac{1}{8(1+2\alpha bn)}$. This leads to $0\geq 1$. Contradiction. So $\Pi^{FSL+CUF}\succ$

Π^{CUPN}. We know that $SW^{CPN} \succ SW^{FSL+CUF}$ and $CS^{FSL+CUF} \succ CS^{CPN}$. This implies $\Pi^{M} \succ \Pi^{FSL+CUF}$ Adding the relations above, we conclude that $\Pi^{M} \succ \Pi^{CPN} \succ \Pi^{FSL+CUF} \succ \Pi^{CUPN}$. So we are done. *QED*

CHAPTER 5

EXTENSION

Remark 1 Consider the case in which there are one consumer and one firm. The firm produces with zero cost. Now, let's extend the consumer's strategy set so that the consumer can also announce a slope parameter. So the firm announces a $\beta \in \mathbb{R}_+ \cup \{\infty\}$ for the slope parameter of its linear supply function and the consumer announces a $\gamma \in \mathbb{R}_{++} \cup \{\infty\}$ for the intercept term and a $\xi \in \mathbb{R}_{++} \cup \{\infty\}$ for the slope parameter of its affine demand function. Assume that a (γ, ξ) is already announced by the consumer. Let's analyze the firm's problem, which is

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta))$$

by giving a non-negative number. Note that $P(\beta)$ is the outcome price and solved from (2.1) by using $AD(P) = \gamma - \xi P$ and $AS(P) = \beta P$ as $\frac{\gamma}{\beta+\xi}$. F.O.C. for this problem $2(\frac{\gamma}{\beta+\xi})\beta\frac{-\gamma}{(\beta+\xi)^2} + (\frac{\gamma}{\beta+\xi})^2 = 0$. From which one obtains $\beta^* = \xi$. Now, S.O.D. of the objective function $\frac{2\gamma^2\beta - 4\gamma^2\xi}{(\beta+\xi)^4}$ One can verify that at β^* S.O.D is negative. Moreover, one can also check that $\Pi(\beta^*) \succ \Pi(\beta) \quad \forall \beta \succ 2\xi$ and so we can restrict the domain to $[0, 2\xi]$ which is

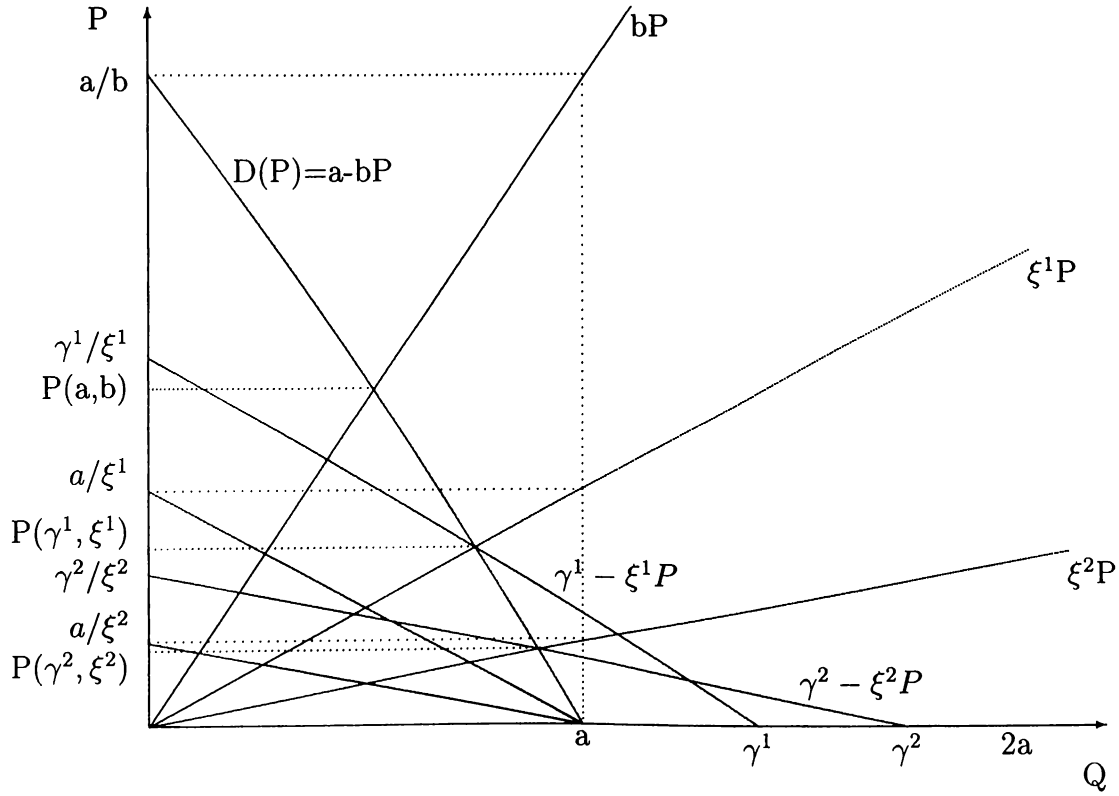

Figure 5.1:

compact and profit function is continuous on non-negative numbers. Thus, profit function arrives its maximum in this interval. Note that β^* is the only point satisfying F.O.C and S.O.C.. So profit function has the maximum at β^*. Therefore, the firm's best response to any positive (γ, ξ) is $\beta^* = \xi$. Let's analyze the consumer's problem through Figure 5.1 above. Note that since $\beta^* \succ 0$ and $MC(q) = 0$, then the firm does not announce its MC function as a best response. Remember that $D(P) = a - bP$ is the consumer's true demand function. Since $MC(q) = 0$, consumer surplus is maximized when $P = 0$. Also note that when $P = 0$, total social welfare is maximized. For $P^c = 0$ and $q^c = a$, (P^c, q^c) is the competitive equilibrium outcome. The consumer's problem is solved algebraicly as follows:

$$CS(\gamma,\xi) = \int_0^{\gamma-\xi P(\gamma,\xi)} (\frac{a}{b} - \frac{1}{b}t)dt - P(\gamma,\xi)\gamma - \xi P(\gamma,\xi)$$

, where $P(\gamma,\xi) = \frac{\gamma}{\xi+\beta^*}$. F.O.Ds.

$$\begin{aligned} \frac{\partial CS}{\partial \gamma} &= \frac{a}{2b} - \frac{1}{4b}\gamma - \frac{\gamma}{2\xi} \\ \frac{\partial CS}{\partial \xi} &= \frac{\gamma}{4\xi^2} \end{aligned}$$

Note that any finite number for consumer's slope parameter do not maximize his surplus since $\frac{\partial CS}{\partial \xi} \succ 0$ and it becomes zero only on the limit as $\xi \to \infty$. From $\frac{\partial CS}{\partial \gamma} = 0$, $\gamma^c = \frac{2a}{1+\frac{1}{\xi}}$. So as $\xi \to \infty$, $\gamma^c \to 2a$. One can also check that as $\xi \to \infty$, S.O.D. of the objective with respect to γ goes to zero. Now, we return to the graphical analysis: If consumer tells the true parameters, firm will respond by b. This will lead to the price $P(a,b)$. However, consumer can do better by announcing (γ^1,ξ^1) so that he can obtain the price $P(\gamma^1,\xi^1)$. To follow this, let him announce a sufficiently large ξ^1 so that $\frac{a}{\xi^1} \prec P(a,b)$ as he keeps the intercept parameter in a. Demand function will rotate as in the figure and supply function will shift downward. How much will be that shift? Imagine a rectangle whose three vertices are origin, a, $\frac{a}{\xi^1}$. Now, from the remaining vertice supply function should pass. This will guarantee that supply function's new slope equals to ξ^1. Now, let consumer fix ξ^1 and increase intercept term to obtain the price, we call $P(\gamma^1,\xi^1)$, at which true demand function and the new supply function intersect. Then his demand curve, $\gamma - bP$, will move on $\xi^1 P$ from $a - \xi^1 P$ to $\gamma^1 - \xi^1 P$. Clearly, by giving (γ^1,ξ^1) consumer is better off compared to by giving (a,b) since he consumes more at a lower price. Both outcomes are on the true demand function so we can compare them directly. Now, consumer still can do better. He can give a sufficiently large ξ^2 so that $\frac{a}{\xi^2} \prec P(\gamma^1, \xi^1)$. Take the demand $a - \xi^2 P$ as reference. Firm responds to this slope change by announcing ξ^2 so supply function will shift downward. It will pass from the remaining vertice of the rectangle whose vertices are origin , a, and $\frac{a}{\xi^2}$. To determine γ^2, fix ξ^2 and increase the intercept term of $a - \xi^2 P$. That is move $\gamma - \xi^2 P$ on $\xi^2 P$ from $a - \xi^2 P$ to obtain the price at which ξ^2 and true demand function intersect. We call this price $P(\gamma^2, \xi^2)$ which is less than $P(\gamma^1, \xi^1)$ and consumer consumes more in the former. So consumer is better off. Note that $a \prec \gamma^1 \prec \gamma^2$ and $b \prec \xi^1 \prec \xi^2$. Consumer can still do better he can announce sufficiently large ξ^3 such that $\frac{a}{\xi^3} \prec P(\gamma^2, \xi^2)$. Then supply function will shift downward appropriately. Again consumer by moving $a - \xi^3 P$ on $\xi^3 P$ by increasing the intercept term to γ^3, he can obtain $P(\gamma^3, \xi^3)$ and $\gamma^3 - \xi^3 P(\gamma^3, \xi^3) = a - bP(\gamma^3, \xi^3)$. Clearly, $P(\gamma^3, \xi^3) \prec P(\gamma^2, \xi^2)$ and so $a - bP(\gamma^3, \xi^3) \succ a - bP(\gamma^2, \xi^2)$. So this is better for consumer. Importing thing is that $\gamma^3 \succ \gamma^2$ and $\xi^3 \succ \xi^2$. The process is clear. The consumer will have sequences in his mind as $(a, \gamma^1, \gamma^3, \gamma^3, ...)$ and $(b, \xi^1, \xi^2, \xi^3, ...)$ such that $\lim_{i \to \infty} \xi^i = \infty$, $\lim_{i \to \infty} \gamma^i = 2a$ and so $\lim_{i \to \infty} P(\gamma^i, \xi^i) = 0$ and $\lim_{i \to \infty}(\gamma^i - \xi^i P(\gamma^i, \xi^i)) = \lim_{i \to \infty}(a - bP(\gamma^i, \xi^i)) = a$. So in the limit competitive outcome is obtained. Thus, the consumer would target the socially optimizing outcome. In other words, interest of consumer and society coincide here. Similarly, CU, if there were more than one consumers and they formed union, would target the socially optimizing outcome since it is as if one consumer and plays with aggregate demand on behalf of consumers. Therefore, if there are one firm producing with zero cost, whose strategy slope parameter of its linear supply function and CU representing consumers, whose strategies intercept term and slope parameter of the aggregate demand and if they play Nash, the socially optimizing outcome is the limit outcome.

Remark 2 Consider the case in which there are one consumer and one firm. The firm has a cost function $C(q) = \alpha q^2$. Implying that it has a $MC(q) = 2\alpha$. Assume that for this case strategy set of the consumer is extended so that the consumer can also announce a slope parameter. Thus, the firm announces a $\beta \in \mathbb{R}_+ \cup \{\infty\}$ for the slope parameter of its linear supply function and the consumer announces a $\gamma \in \mathbb{R}_{++} \cup \{\infty\}$ for the intercept term and a $\xi \in \mathbb{R}_{++} \cup \{\infty\}$ for the slope parameter of its affine demand function. Assume that a (γ, ξ) is chosen by the consumer. Now, the firm's problem is

$$\max_{\beta} \Pi(\beta) = P(\beta)(\beta P(\beta)) - \alpha(\beta P(\beta))^2$$

by giving a non-negative number. Note that the outcome price, $P(\beta) = \frac{\gamma}{\beta+\xi}$ by using $AD(P) = \gamma - \xi P$ and $AS(P) = \beta P$ from market clearing. F.O.C. for this problem $2(\frac{\gamma}{\beta+\xi})\beta(\frac{-\gamma}{(\beta+\xi)^2})(\beta - \alpha\beta^2) + (\frac{\gamma}{\beta+\xi})^2(1 - 2\alpha\beta) = 0$ From which $\beta^* = \frac{\xi}{1+2\alpha\xi} = \frac{1}{\frac{1}{\xi}+2\alpha} \prec \frac{1}{2\alpha}$ One can verify that S.O.D is negative at β^*. Note that profit of the firm is negative if β is greater than $\frac{1}{\alpha}$ and so we can restrict the domain into a compact interval, $[0, \frac{1}{\alpha}]$. Since the firm's profit function is continuous, it arrives its maximum in this interval. One can verify that S.O.D is negative at β^* and it is the only point satisfying F.O.C. as well. We conclude that β^* maximizes the firm's profit. Now, let's try to solve the consumer's problem algebraicly. It is the same with the one in Remark 1 but now, the firm announces $\beta^* = \frac{\xi}{1+2\alpha\xi}$. From F.O.C. of his problem $\gamma^* = \frac{\frac{a}{b}}{\frac{1}{b}(1-\frac{1}{(1+\frac{1}{1+2\alpha\xi})})+\frac{2}{\xi(1+\frac{1}{1+2\alpha\xi})}}$. Note that $\lim_{\xi\to\infty} \gamma^* = \infty$. For critical value of ξ Mathematica supplies one page long expression, which is useless. So we proceed with the graphical analysis. Observe that β^* is a function of slope parameter of consumer, ξ and $\lim_{\xi \to \infty} \beta^* = \frac{1}{2\alpha}$. Thus, as the value of consumer's slope parameter gets larger and larger, i.e., as demand function becomes perfectly elastic, the firm becomes marginal cost pricer. In other words, consumer has the power to make the firm marginal cost pricer at least arbitrarily close to that. Consider the following outcome. Let (P^s, q^s) be such that $AD(P^s) = IMC(P^s)$ and $AD(P^s) = q^s$, where $IMC(q)$ is the industry marginal cost function. Note that since there is one firm here, $IMC(q) = MC(q)$. One can prove that (P^s, q^s) is the socially optimum outcome. That is, at this outcome total social welfare, sum of total consumers' surplus and total profits, is maximized.[32] Using $AD(MC(q^s)) = a - b2\alpha q^s = q^s$, $q^s = \frac{a}{1+2\alpha b}$ and $P^s = MC(q^s) = \frac{2\alpha a}{1+2\alpha b}$. We depict this outcome in the Figure 5.2 in the next page. Now, our concern whether the consumer would target this point or not. We know that as consumer makes his demand function perfectly elastic, the firm becomes marginal cost pricer. Suppose that the firm marginal cost pricer, that is, it submits $\overline{\beta} = \frac{1}{2\alpha}$. Now, formulate the consumer's problem as picking a positive number for its intercept term to maximize his surplus. Consumer's problem is the same with the one in proof of Proposition 14. From there we know that when firm submits a non-negative β, from the F.O.C. of the consumer's problem $\gamma^c = \frac{a(b+\beta)}{(2b+\beta)}$. So for $\overline{\beta} = \frac{1}{2\alpha}$, $\gamma^c = \frac{a(1+2\alpha\beta)}{(1+4\alpha\beta)}$. Since S.O.D is negative for any positive choice of intercept term, consumer's surplus function is concave so F.O.C. turns out to be necessary and sufficient. So when firm announces $\overline{\beta} = \frac{1}{2\alpha}$, announcing $\overline{\gamma} = \frac{a(1+2\alpha\beta)}{(1+4\alpha\beta)}$ maximizes consumer's surplus. Note that $\overline{\gamma} \prec a$. Thus, as the consumer forces firm to become marginal cost pricer by

[32] According to the definition in Grossman, "Nash"., competitive equilibrium does not exist for this case since $MC(q) \succ AC(q) \quad \forall q \geq 0$

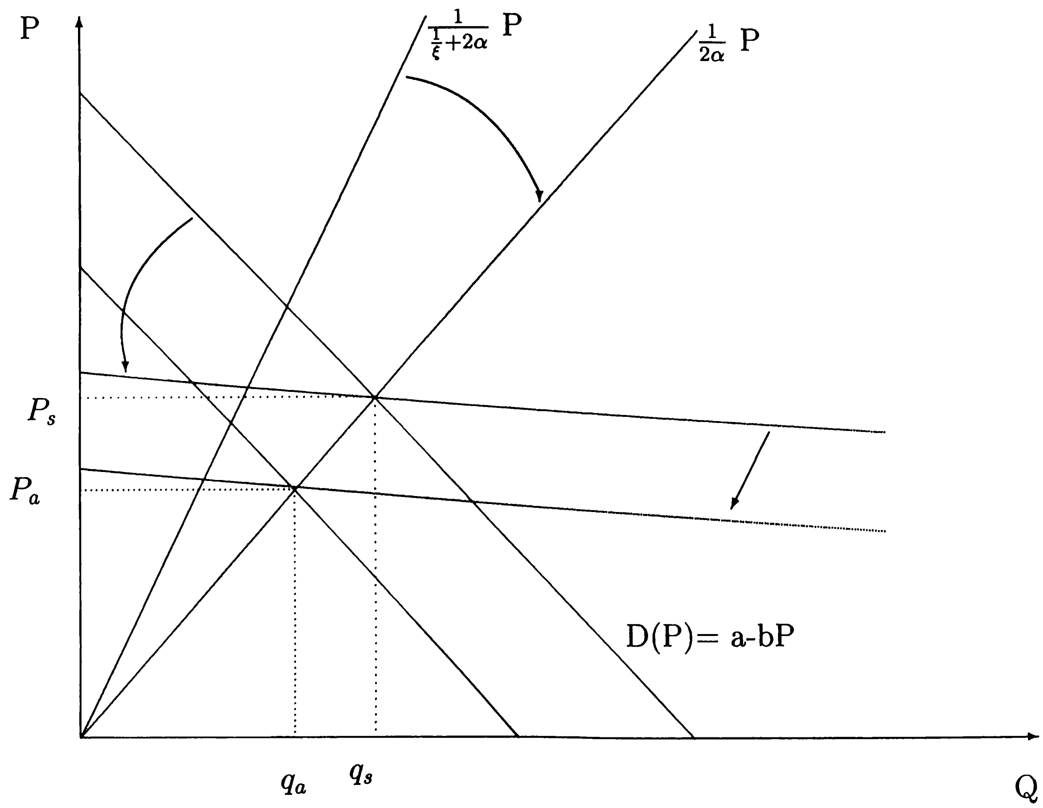

Figure 5.2:

making his demand infinitely elastic, he would not target (P^s, q^s) at which true demand function and firm's marginal cost intersects but rather it would target the point at which $\overline{\gamma} - bP$ and marginal cost of the firm intersects. We depict this outcome in Figure 5.2 as (P_a, q_a). Since at the latter outcome consumer's surplus is maximized when firm announces its marginal cost which is almost the case when consumer makes its demand function almost perfectly elastic. Therefore, consumer would not target socially optimum outcome. That is, the interest of consumer and society does not coincide here. Similarly, CU on behalf consumers would not target the socially optimal outcome since it is as if one consumer and plays with the aggregate demand function. Therefore, we do not expect the socially optimum outcome when CU whose strategies are intercept and slope parameter of the aggregate demand function and one firm whose strategy is the slope parameter of its linear supply function play Nash.

CHAPTER 6

CONCLUSION

We have considered consumers as players along with firms in oligopolistic markets by defining games in which identical firms with quadratic cost functions restricted to announce linear supply functions and identical consumers with affine true demand functions are allowed to manipulate their intercept terms either in organized or unorganized manner. We have established the symmetric equilibria of both the Nash game and the Stackelberg game. In the latter a firm is assumed to be leader. When the consumers as organized are Stackelberg leaders, the game reduces to the Nash game. This is due to the assumption that consumers can only manipulate their intercept terms. When consumers' strategy sets are extended to include their slope parameters, this will not be the case. We know from our remarks, firm's best response is a function of consumers' slope terms so when they are manipulated firm's strategy will change accordingly. When consumers are followers they are assumed to be organized. The case in which they are unorganized followers leads to the complications so we do not study here. For the Nash game, when there are at least two firms producing with positive marginal cost, we only make comparisons for particular set of values of parameters since outcome expressions are cumbersome. We support this peace of information with the limit results. Nevertheless, we cover fairly large class of cases to get an extensive idea on the possible effects of considering consumers as players along with firms and our findings are as follows:

Only for the case in which there are at least two firms producing with zero cost, telling the truth is the best response of each consumer to the others' strategy profile in which each of remaining consumers tells the truth and each firm announces its MC function. This Nash equilibrium is the unique symmetric one as we prove in Proposition 2. Moreover, the outcome of this equilibrium leads to the social optimum. In all remaining cases that we cover, consumers misrepresent their intercept terms downwards as their best responses so declared demand function lies to the left of their true demand function at each price and firms announce a slope parameter which is less than the value in the supply function obtained from MC function as their best responses so declared supply function lies above of their MC functions at each quantity. We show that if either the number of consumers or firms gets larger and larger (whenever the case allows), they tell the truth in the limit.

One finding regarding the case of natural monopolist is that whenever forming union is meaningful, that is there are at least two consumers, it is strength for consumers so they increase their utility. Furthermore, if they form union in the Stackelberg game, they are better off compared to the Nash Game in which they form union. When there is one consumer, he or she is better off in Stackelberg game and the firm loses less compared to the Nash game. So Stackelberg equilibrium is an improvement for both parties in this case. When number of consumers more than one, this is not true anymore. Though consumers obtain the most compared to the other structures, the firm loses more compared to the Nash game in which consumers and the firm are players in this case. Thus, Stackelberg game outcome is an improvement for the consumers over the Nash game outcome.

The case of one firm is interesting for another reason. Since there is not any other firm providing the competitive behavior, franchise bidding procedure for monopoly due to Loeb and Magat [33] is inapplicable. Only remaining option is to tax the monopolist. This policy might have deficiency since the cost parameter of the firm is private to itself. However, our results show that allowing consumers to manipulate their demand functions is equivalent to taxing the monopolist. Under these alternative institutional assumptions consumers are better off and the firm is loser while the society gains or lose depending on the case. For example, for the case in which there are at least two consumers and one firm producing with positive cost there is welfare gain in the game all agents are players compared to the monopoly structure. Then total gain of consumers must be higher than the loss of the firm. This means consumers may even compensate the firm's loss and can be still better off than the initial monopoly structure while firm is made indifferent between two structures. However, there is much to gain for consumers in this case. It is shown in Theorem 4 consumers obtain the most benefit under the Stackelberg game when CU is the follower. Thus, the designer can bargain to get the amount of tax that is equal to increase in total consumers' surplus when agents switch from monopoly structure to Stackelberg game. The firm does

[33] Loeb and Magat, "Regulation".

not lose the most profit in this game rather it is in the worst situation when it plays Nash with CU. From the relation about social welfare outcomes of both games we observe that the total loss of the firm is less than the total gain of consumers so there is profit arising from the design of the market. To sum up there are alternative structures and all of them transfer income from the monopolist to the consumers, which provides the designer of the market a bargaining power against the monopolist. When the firm produces with zero or positive cost and there is one consumer or when the firm produces with zero cost and there are at least two consumers, the same reasoning is valid.

In all cases, the distribution of total social welfare leads to the result that consumers become better off and so firms are losers compared to the Cournot-Nash game. Regarding the society's gain, we find that when there are at least two firms producing with zero cost, the Nash equilibrium in intercept terms of demand functions and slope parameters of supply functions yields the socially optimum outcome. For the positive marginal cost however, total social welfare outcome of the game where consumers and firms play Nash turns out to be even less than the total social welfare obtained from the Cournot-Nash game. We note that as the number of consumers goes to infinity, the outcomes of Cournot-Nash game and the Nash game we define coincide in the limit. Although for the case in which there is one firm producing with positive cost, there is some gain in total social welfare compared to the Cournot-Nash game, since when they are finite in numbers equilibrium do not occur on true demand function and IMC function, socially optimizing outcome is not obtained. Restricting the consumer to manipulate only his intercept term seems to be the reason. However, we observe in the Remark 2, although consumers are allowed to manipulate their slope parameters as well, CU does not target the socially optimal outcome. What would happen if consumers were not organized in the same situation? This is an interesting case to study but it is tough to solve. We note that in the case of one firm producing with zero cost and finite number of consumers represented by CU which is allowed to manipulate both intercept and slope parameters, then social optimal outcome is obtained in the limit of strategies.

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