FACTORIZATION IN HARDY AND NEVANLINNA
CLASSES

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FACTORORIZATION IN HARDY AND NEVANLINNA CLASSES

A THESIS
Submitted to the Department of Mathematics and the Institute of Engineering and Sciences of Bilkent University in partial fulfillment of the requirements for the degree of Master of Science

By
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August, 1999
I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

FACTORIZATION IN HARDY AND NEVANLINNA CLASSES

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August, 1999

We find conditions under which the factors of a function belonging to Hardy or Nevanlinna class also belong to the corresponding class.

The basis of our method is the theorem on the representation of a function harmonic in the upper half-plane by Poisson integral under much less restrictive conditions than previously known.

Keywords: Poisson integral, Blaschke product, Hardy class, Nevanlinna class.
ÖZET

HARDY VE NEVANLINNA SINIFLARINDA 
FAKTORİZASYON

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Hardy ya da Nevanlinna sınıflarına ait fonksiyonların çarpanlarının da ilgili sınıflara ait olmalarının koşullarını buluyoruz.

Metodumuzun temeli, bilinen teoremlerden daha zayıf koşulları olan, yukarı yarı-düzlemdeki harmonik fonksiyonların Poisson integrali biçiminde gösterilmesi ile ilgili teoremdir.

Anahtar kelimeler: Poisson integralı, Blaschke çarpmı, Hardy sınıfı, Nevanlinna sınıfı.
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Chapter 1

Introduction and statement of results

The classical factorization theorems in Nevanlinna and Hardy classes ([2], Ch.11, [5], Ch.8, [6], Ch.VI) are well-known and have plenty of applications in Complex Analysis and Functional Analysis ([2], [5], [6], [11], [7]). In 1985, I.V. Ostrovskii [9] proved a factorization theorem in the Hardy class $H^\infty(\mathbb{C}_+)$ of quite different kind. This theorem was a basis of his solution [9] to the problem of extension of the Titchmarsh convolution theorem to the measures with unbounded support.

The aim of the presented work is to extend the mentioned theorem of [9] to wider classes than $H^\infty(\mathbb{C}_+)$. The base of such extensions is a new theorem on representation of a function harmonic in a half-plane which may be of interest by its own.

Let us recall the necessary definitions and state the results of this work in detail.

Recall that the Hardy class $H^p(\mathbb{C}_+)$, $0 < p < \infty$, consists of all functions...
$f$ analytic in the upper half-plane $\mathbb{C}_+$ and satisfying the condition

$$\sup_{0 < y < \infty} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty.$$  

The class $H^{\infty}(\mathbb{C}_+)$ consists of all functions analytic and bounded in $\mathbb{C}_+$.

The factorization theorem proved in [9] is the following.

**THEOREM A.** Let $h \neq 0$ belong to $H^{\infty}(\mathbb{C}_+)$. Assume that $h = g_1 g_2$ where $g_1$ and $g_2$ are analytic in $\mathbb{C}_+$ and satisfying the conditions:

1) There is a sequence $\{r_k\} \uparrow \infty$ such that

$$\sup\{|g_1(z)| + |g_2(z)| : |z| < r, \operatorname{Im} z > 0\} \leq \exp\{o(r)\}, \quad r = r_k \uparrow \infty.$$  

1.1)

II) There is an $H > 0$ such that

$$\sup\{|g_1(z)| + |g_2(z)| : 0 < \operatorname{Im} z < H\} < \infty.$$  

Then there are real constants $k_1$, $k_2$ such that $g_j(z)e^{ik_jz} \in H^{\infty}(\mathbb{C}_+), \ j = 1, 2$.

To state our main result, recall that the **Nevanlinna class** is the set of all functions $f$ analytic in $\mathbb{C}_+$ such that $\log|f|$ has positive harmonic majorant in $\mathbb{C}_+$. The connection between the Nevanlinna class and the Hardy classes $H^p(\mathbb{C}_+)$ is the following. Each $H^p(\mathbb{C}_+), 0 < p \leq \infty$, is a subclass of the Nevanlinna class. On the other hand, each function of the Nevanlinna class is a quotient of two functions of $H^{\infty}(\mathbb{C}_+)$.

Our main result is the following.

**THEOREM 1.** Let $h \neq 0$ belong to the Nevanlinna class. Assume that $h = g_1 g_2$ where $g_1$ and $g_2$ are analytic in $\mathbb{C}_+$ and satisfying the conditions:

1) There is a sequence $r_k \uparrow \infty$ such that

$$\int_0^r \log^+ |g_1(re^{i\theta})| \sin \theta d\theta \leq \exp\{o(r)\}, \quad r = r_k \uparrow \infty.$$  

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II) There is an $H > 0$ such that
\[
\sup_{0 < s < H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t + is)|}{1 + t^2} \, dt < \infty, \quad j = 1, 2. \tag{1.2}
\]
Then both $g_1$ and $g_2$ belong to the Nevanlinna class.

The following corollary is immediate.

**COROLLARY 1.** Let $h \notin 0$ belong to $H^p(\mathbb{C}_+)$ for some $p$, $0 < p \leq \infty$. Assume that $h = g_1g_2$ where $g_1$, $g_2$ are analytic in $\mathbb{C}_+$ and satisfying the conditions I) and II) of Theorem 1. Then both $g_1$ and $g_2$ belong to the Nevanlinna class.

The next corollary can be derived from Theorem 1 with the help of Phragmén-Lindelöf principle.

**COROLLARY 2.** Let $h \notin 0$ belong to $H^\infty(\mathbb{C}_+)$. Assume that $h = g_1g_2$ where $g_1$ and $g_2$ are analytic in $\mathbb{C}_+$ and satisfying the conditions:

I) There is a sequence $r_k \uparrow \infty$ such that
\[
\int_0^\pi \log^+ |g_1(re^{i\theta})| \sin \theta \, d\theta \leq \exp(o(r)), \quad r = r_k \uparrow \infty. \tag{1.3}
\]

II) There is an $H > 0$ such that
\[
\sup\{|g_1(z)| + |g_2(z)| : 0 < \text{Im} \, z < H\} < \infty.
\]
Then there are real constants $k_1$, $k_2$ such that $g_j(z)e^{ik_jz} \in H^\infty(\mathbb{C}_+), \ j = 1, 2.$

Evidently, the condition (1.3) is less restrictive than (1.1), and moreover, it relates to only one but not both of functions $g_1, g_2$. That is why Corollary 2 contains Theorem A.

The condition I) in Theorem 1 (in Corollary 2 also) cannot be weakened even by replacing $o(r)$ by $O(r)$. This will be shown by examples which we
consider at the end of chapter 5. Also the condition II) in Theorem 1 cannot be weakened by replacing it with

\[ \exists H > 0, \quad \sup_{0 < s < H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t + is)|}{1 + |t|^\alpha} \, dt < \infty, \quad j = 1, 2, \]  

(1.4)

for some \( \alpha > 2 \). This will also be shown at the end of chapter 5 with other examples related to the sharpness of Theorem 1.

The base of the proof of Theorem 1 is the following theorem on representation of a function harmonic in \( \mathbb{C}_+ \).

**THEOREM 2.** Let \( u \) be a function harmonic in \( \mathbb{C}_+ \) and satisfy the following condition:

There exists \( H > 0 \) such that

\[ \sup_{0 < s < H} \int_{-\infty}^{\infty} \frac{|u(t + is)|}{1 + t^2} \, dt < \infty. \]  

(1.5)

Then \( u \) admits the representation

\[ u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{(x - t)^2 + y^2} + U(z), \quad z = x + iy \in \mathbb{C}_+, \]  

(1.6)

where \( \nu \) is a Borel measure on \( \mathbb{R} \) satisfying

\[ \int_{-\infty}^{\infty} \frac{d|\nu(t)|}{1 + t^2} < \infty, \]

and \( U \) is a function harmonic in the whole plane \( \mathbb{C} \) satisfying \( U(x) = 0 \), \( x \in \mathbb{R} \).

This theorem can be compared with the following well-known result:

Let \( u \) be a function harmonic in \( \mathbb{C}_+ \) and continuous in \( \mathbb{C}_+ \). If

\[ \int_{-\infty}^{\infty} \frac{|u(t)|}{1 + t^2} \, dt < \infty, \]  

(1.7)
then \( u \) admits the representation

\[
    u(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{u(t)dt}{(x-t)^2 + y^2} + U(z)
\]

where \( U \) is a function harmonic in the whole plane \( \mathbb{C} \) satisfying \( U(x) = 0, \ x \in \mathbb{R} \).

Sure, for \( u \) harmonic and continuous in \( \mathbb{C}_+ \), our condition (1.5) is more restrictive than (1.7). Nevertheless, in Theorem 2 we do not assume any kind of continuity in \( \mathbb{C}_+ \). This is important for the application to the proof of Theorem 1. Moreover, Theorem 2 had other applications [3] related to generalizations of the Titchmarsh convolution theorem to collections of linearly dependent measures.

One can expect that, if we weaken the condition (1.2) in Theorem 1 by replacing it with (1.4), then we can show that \( g_1 \) and \( g_2 \) belong to some class wider than the Nevanlinna class. To state our result in this direction we recall some more definitions and known results.

Let \( \{z_n\} \) be a finite or infinite sequence in \( \mathbb{C}_+ \) satisfying the condition

\[
    \sum_n \frac{\text{Im} \ z_n}{1 + |z_n|^2} < \infty.
\]

This condition is called the Blaschke condition. Let us form the finite or infinite product

\[
    B(z) = \left(\frac{z-i}{z+i}\right)^m \prod_{z_n \neq i} \frac{|z_n^2 + 1|}{z_n^2 + 1} \frac{z - z_n}{z - \bar{z}_n}
\]

where \( m \) is the number of \( z_n \)'s equal to \( i \). A product of such kind is called a Blaschke product formed by \( \{z_n\} \). It is known ([6], Ch.VI C ) that for any sequence \( \{z_n\} \) satisfying the Blaschke condition, the corresponding Blaschke product is uniformly convergent on each compact subset of \( \mathbb{C}_+ \) and hence
represents an analytic function in \( \mathbb{C}_+ \) satisfying |\( B(z) \)| < 1, \( z \in \mathbb{C}_+ \). We recall ([7], p.102 ) that zeros of any function of the Nevanlinna class satisfy the Blaschke condition. Moreover ([7], p.104 ) the Nevanlinna class can be described as the class of all functions admitting the representation

\[
f(z) = B(z)e^{i(k_1 z + k_2)} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{1 + tz}{(t - z)(1 + t^2)} d\nu(t) \right\},
\]

where \( B \) is a Blaschke product, \( k_1 \) and \( k_2 \) are real constants, and \( \nu \) is a real-valued Borel measure on \( \mathbb{R} \) satisfying the condition

\[
\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} < \infty.
\]

Our result is the following.

**Theorem 3.** Let a function \( h \neq 0 \) belong to the Nevanlinna class. Suppose that \( h = g_1 g_2 \) where the functions \( g_1 \) and \( g_2 \) are analytic in \( \mathbb{C}_+ \) and satisfying the following conditions:

1) There exists a sequence \( r_k \uparrow \infty \) such that

\[
\int_0^\pi \log^+ |g_1(re^{i\theta})| \sin \theta d\theta \leq \exp\{\nu(r)\}, \quad r = r_k \uparrow \infty.
\]

2) There exist \( \alpha \geq 2 \) and \( H > 0 \) such that

\[
\sup_{0 < x < H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t + is)|}{1 + |t|^\alpha} dt < \infty, \quad j = 1, 2.
\]

Then \( g_j \) admits the following representation

\[
g_j(z) = B_j(z)e^{P_j(z)} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + tz)^q}{(t - z)(1 + t^2)^q} d\nu_j(t) \right\}, \quad q = [\alpha],
\]

where \( B_j \) is the Blaschke product formed by the zeros of \( g_j \), \( P_j \) is a real polynomial whose degree is not greater than \( q + 1 \) and \( \nu_j \) is a real-valued Borel measure satisfying

\[
\int_{-\infty}^{\infty} \frac{d|\nu_j|(t)}{1 + |t|^\alpha} < \infty.
\]
Comparing (1.9) and (1.11), we see that the class of functions of the form (1.11) is wider than the Nevanlinna class. The class of functions (1.11) was firstly considered by R. Nevanlinna [8]. Later on, it was used by N. Govorov [4] in connection with the Riemann boundary problem with infinite index. Note that Theorem 1 is not contained in Theorem 3, because for \( q = 1 \), the condition (1.10) is more restrictive than (1.2).

The base of the proof of Theorem 3 is the following theorem on representation of a function harmonic in \( \mathbb{C}_+ \).

**THEOREM 4.** Let \( u \) be a function harmonic in \( \mathbb{C}_+ \) and satisfying the following condition:

There exists \( \alpha > 0 \) and \( H > 0 \) such that

\[
\sup_{a < s < H} \left( \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+|t|^\alpha} dt \right) < \infty.
\]

Then \( u \) admits the representation

\[
u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} \text{Im} \left\{ \frac{(1+tz)^q}{(t-z)(1+t^2)^\alpha} \right\} d\nu(t) + U(z),
\]

(1.12)

where \( q = [\alpha] \), \( \nu \) is a Borel measure on \( \mathbb{R} \) satisfying

\[
\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1+|t|^\alpha} < \infty,
\]

and \( U \) is a function harmonic in the whole plane \( \mathbb{C} \) satisfying \( U(x) = 0 \), \( x \in \mathbb{R} \).

Comparing (1.6) and (1.12), we see that (1.6) is a particular case of (1.12) with \( q = 1 \). Nevertheless, Theorem 2 is not contained in Theorem 4 by the similar reason as we indicated in connection with Theorem 3.
The presented work is organized in the following way. In the chapter 2 we recall the necessary definitions and state (without proofs) known results which we will use in the sequel. Chapter 3 contains the proof of Theorem 2. In the chapter 4 we prove two auxiliary results needed for the proof of Theorem 1. Chapter 5 contains the proof of Theorem 1 and its corollaries 1 and 2. We also consider there examples related to sharpness of Theorem 1. Finally, the chapters 6 and 7 contain the proof of Theorems 4 and 3 respectively.
Chapter 2

Preliminaries

In this chapter we recall some definitions and results which we will need in the sequel. The material of sections 2.1 and 2.2 is taken from [2], ch.11, [5], ch.8, [6], ch.VI. The material of section 2.3 is taken from [2], ch.11, [6], ch.VI, [7], ch.14. The material of section 2.4 and 2.5 are taken from [7], ch.24, [10], ch.3 and [1], ch.5 respectively.

2.1 Poisson integral

The function

\[ P(x, y) = \frac{1}{\pi} \frac{y}{x^2 + y^2} = \frac{1}{\pi} \text{Im} \left( \frac{-1}{z} \right), \quad z = x + iy \in \mathbb{C}_+ \]

where \( \mathbb{C}_+ := \{ z : \text{Im} z > 0 \} \) is called the Poisson kernel for the upper half-plane.

Here are the main properties of Poisson kernel:

- \( P(x, y) \) is harmonic in \( \mathbb{C}_+ \).

- \( P(x, y) > 0, \quad z = x + iy \in \mathbb{C}_+ \).
\[ \int_{-\infty}^{\infty} P(x - t, y)dt = 1, \quad z = x + iy \in \mathbb{C}_+. \]

\[ \int_{|t-x|\geq\lambda} P(x - t, y)dt \to 0 \quad \text{as} \quad y \to 0+, \quad x \in \mathbb{R} \quad \text{for any} \quad \lambda > 0. \]

\[ P(x - t, y) = \frac{1}{\pi} \text{Im} \frac{1}{t - z} = \frac{y}{\pi} \cdot \frac{1}{|t - z|^2}, \quad t \in \mathbb{R}, \quad z = x + iy \in \mathbb{C}_+. \]

The following theorem is well-known.

**Theorem 2.1.** Let \( \nu \) be a \( \sigma \)-finite (complex-valued) Borel measure on \( \mathbb{R} \) satisfying the condition

\[ \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} dt < \infty. \]

Then the integral

\[ u(z) = \int_{-\infty}^{\infty} P(x - t, y)d\nu(t) \quad (2.1) \]

is convergent (in the Lebesgue sense) for any \( z = x + iy \in \mathbb{C}_+ \) and represents a harmonic function in \( \mathbb{C}_+ \).

The integral (2.1) is called the Poisson integral of the measure \( \nu \).

If the measure \( \nu \) is absolutely continuous with respect to Lebesgue measure and

\[ f(t) = \frac{d\nu}{dt} \]

then the integral (2.1) can be written in the form

\[ u(z) = \int_{-\infty}^{\infty} P(x - t, y)f(t)dt \quad (2.2) \]

and is called the Poisson integral of the function \( f \). The Poisson integral of \( f \) is well-defined for any Lebesgue measurable (complex-valued) function satisfying

\[ \int_{-\infty}^{\infty} \frac{|f(t)|}{1 + t^2} dt < \infty. \quad (2.3) \]
We will need the following theorem.

**THEOREM 2.2.** Let \( f \) be a function continuous on \( \mathbb{R} \) and satisfying (2.3). Then the Poisson integral (2.2) is harmonic in \( \mathbb{C}_+ \) and continuous in \( \overline{\mathbb{C}}_+ \) under agreement \( u(t) = f(t), \ t \in \mathbb{R} \).

We could not find this theorem in the literature therefore we will derive it from the following well-known result.

**THEOREM 2.2'.** If \( f \in L^{\infty}(\mathbb{R}) \), then its Poisson integral represents a harmonic function in \( \mathbb{C}_+ \) and

\[
u(z) \rightarrow f(t_0) \quad \text{as } \ z \rightarrow t_0 \quad (2.4)
\]

for each continuity point \( t_0 \) of \( f \).

**Proof of Theorem 2.2:** The harmonicity of \( u \) immediately follows from Theorem 2.1. To prove (2.4), we choose \( R = 2|t_0| + 1 \) and set

\[
\begin{align*}
f_R(t) &:= f(t) \chi_{[-R, R]}(t), \quad f^R(t) := f(t) - f_R(t). \tag{2.5}
\end{align*}
\]

Then

\[
u(z) = \int_{-\infty}^{\infty} P(x - t, y) f_R(t) dt + \int_{-\infty}^{\infty} P(x - t, y) f^R(t) dt =: I_1(z) + I_2(z).
\]

By Theorem 2.2' we have \( I_1(z) \rightarrow f_R(t_0) = f(t_0) \) as \( z \rightarrow t_0 \). To complete the proof we need to show that \( I_2(z) \rightarrow 0 \) as \( z \rightarrow t_0 \). For \(|z - t_0| < (|t_0| + 1)/2\), \(|t| > R = 2|t_0| + 1\), we have

\[
|t - z| \geq |t - t_0| - |t_0 - z| \geq \frac{|t| + 1}{2} - \frac{|t| + 1}{4} = \frac{|t| + 1}{4} \geq \sqrt{\frac{|t|^2 + 1}{4}} \quad (2.6)
\]

Hence,

\[
|I_2(z)| \leq \frac{y}{\pi} \int_{|t| > R} \left| \frac{f(t)}{|t - z|^2} \right| dt \leq \frac{y}{\pi} \int_{|t| > R} \frac{16|f(t)|}{t^2 + 1} dt \rightarrow 0 \quad \text{as } z \rightarrow t_0. \quad \Box
\]
2.2 Blaschke product

Let \( \{z_n\} \) be a finite or infinite sequence from \( \mathbb{C}_+ \) satisfying the condition

\[
\sum_n \frac{\text{Im} z_n}{1 + |z_n|^2} < \infty.
\]

This condition is called the Blaschke condition. Let us form finite or infinite product

\[
B(z) = \left( \frac{z - i}{z + i} \right)^m \prod_{z_n \neq \pm i} \frac{|z_n|^2 + 1}{z_n^2 + 1} \frac{z - z_n}{z - \overline{z_n}}
\]

where \( m \) is the number of \( z_n \)'s equal to \( i \). A product of such form is called a Blaschke product formed by \( \{z_n\} \). The following theorem is well-known.

**THEOREM 2.3.** For any sequence \( \{z_n\} \) satisfying the Blaschke condition, the Blaschke product formed by \( \{z_n\} \) is uniformly convergent on each compact subset of \( \mathbb{C}_+ \) and, hence represents an analytic function in \( \mathbb{C}_+ \) such that

\[
|B(z)| < 1, \quad z \in \mathbb{C}_+.
\]

Each \( z_n \) is a zero of \( B \), with multiplicity equal to the number of times it occurs in the product, and \( B \) has no other zeros in \( \mathbb{C}_+ \).

2.3 Hardy classes and the Nevanlinna class

The **Hardy class** \( H^p(\mathbb{C}_+) \), \( 0 < p < \infty \), is the class of all functions \( f \) analytic in \( \mathbb{C}_+ \) and satisfying the condition

\[
\sup_{0 < y < \infty} \int_{-\infty}^{\infty} |f(x + iy)|^p dx < \infty.
\]

By \( H^\infty(\mathbb{C}_+) \) is denoted the set of all bounded analytic functions in \( \mathbb{C}_+ \).

The following factorization theorem is a standard tool in the theory of \( H^p \) classes.
THEOREM 2.4. Let $f \neq 0$ be a function belonging to $H^p(\mathbb{C}_+)$ $(0 < p \leq \infty)$. Then the zeros of $f$ satisfy the Blaschke condition and $f$ admits the following factorization

$$f(z) = B(z)g(z), \quad z \in \mathbb{C}_+,$$

where $g$ is a non-vanishing function of $H^p(\mathbb{C}_+)$ and $B$ is the Blaschke product for the upper half-plane formed by the zeros of $f$ in $\mathbb{C}_+$.

A function $f$ analytic in the upper half-plane is said to belong to the Nevanlinna class if $\log |f|$ has a positive harmonic majorant in $\mathbb{C}_+$. It is known that each $H^p(\mathbb{C}_+)$ $0 < p \leq \infty$ is contained in the Nevanlinna class.

This class of functions is closely related to the $H^\infty(\mathbb{C}_+)$ as the following theorem shows.

THEOREM 2.5. Let $f$ be a function analytic in $\mathbb{C}_+$. Then $f$ belong to the Nevanlinna class if and only if $f$ can be written in the form $F_1/E_2$, where $F_j, j = 1, 2,$ belong to $H^\infty(\mathbb{C}_+), |F_j(z)| < 1, z \in \mathbb{C}_+, j = 1, 2$ and $E_2$ does not vanish in $\mathbb{C}_+$.

This theorem and the previous one allow us to conclude the following.

COROLLARY 2.1. Let $f \neq 0$ be a function belonging to the Nevanlinna class. Then the zeros of $f$ satisfy the Blaschke condition and $f$ admits the following factorization

$$f(z) = B(z)F(z), \quad z \in \mathbb{C}_+,$$

where $F$ is a non-vanishing function of the Nevanlinna class and $B$ is the Blaschke product formed by the zeros of $f$ in $\mathbb{C}_+$. 

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The following theorem gives a complete description of the Nevanlinna class.

**Theorem 2.6.** The Nevanlinna class consists of functions representable in the form

$$f(z) = B(z)e^{i(k_1z + k_2)}\exp\left\{\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1 + tz}{(t - z)(1 + t^2)} d\nu(t)\right\},$$

where $B$ is a Blaschke product, $k_1$ and $k_2$ are real constants and $\nu$ is a real-valued Borel measure satisfying the condition

$$\int_{-\infty}^{\infty} \frac{d\nu|(t)}{1 + t^2} < \infty.$$

We will use the following result of [9] several times.

**Theorem 2.7.** If a function $Q \neq 0$ belongs to $H^\infty(\mathbb{C}_+)$, then for any $K > 0$

$$\sup_{0 < s < K} \int_{-\infty}^{\infty} \frac{\log^+ |1/Q(t + is)|}{1 + t^2} dt < \infty.$$

Using Theorem 2.5, we derive the following corollary from Theorem 2.7.

**Corollary 2.2.** If a function $Q \neq 0$ belongs to the Nevanlinna class, then for any $K > 0$

$$\sup_{0 < s < K} \int_{-\infty}^{\infty} \frac{|\log|Q(t + is)||}{1 + t^2} dt < \infty.$$
2.4 Carleman’s and Nevanlinna’s formulas

The following integral formula is called the Carleman’s formula. It connects the modulus and zeros of a function analytic in $\mathbb{C}_+$. This formula has important applications in the theory of entire functions (see, e.g. [7], Ch.24).

**THEOREM 2.8.** Let $F$ be a function analytic in the region $\{z : 0 < \rho \leq |z| \leq R, \ \text{Im} z \geq 0\}$ and $a_k = r_k e^{i\theta_k}$, be its zeros. Then

$$\sum_{\rho < r_k < R} \left(\frac{1}{r_k} - \frac{r_k}{R^2}\right) \sin \theta_k = \frac{1}{2\pi R} \int_0^\pi \log |F(Re^{i\theta})| \sin \theta d\theta +$$

$$+ \int_\rho^R \left(\frac{1}{x^2} - \frac{1}{R^2}\right) \log |F(x)F(-x)|dx + A_\rho(F, R),$$

where

$$A_\rho(F, R) = -\text{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log F(\rho e^{i\theta}) \left( \frac{\rho e^{i\theta}}{R^2} - \frac{e^{-i\theta}}{\rho} \right) d\theta \right\}.$$ 

Remark: If $F$ is analytic for $|z| \geq \rho, \ \text{Im} z \geq 0$, the quantity $A_\rho(F, R)$ is bounded for $R > \rho$ and, as $R \to \infty$, we have the limit

$$A_\rho(F, \infty) = \text{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log F(\rho e^{i\theta}) \frac{e^{-i\theta}}{\rho} d\theta \right\}.$$

We will also use the following formula for a harmonic function in a half-disk which is called the Nevanlinna’s formula.

**THEOREM 2.9.** Let $u$ be a function harmonic in the half-disk $D_R^+ := \{z : |z| < R, \ \text{Im} z > 0\}$ and continuous in its closure. Then

$$u(z) = \frac{1}{2\pi} \int_0^\pi \left( \frac{(R^2 - r^2)}{|Re^{i\theta} - z|^2} \frac{4R r \sin \theta \sin \varphi}{|Re^{i\theta} - z|^2} \right) u(Re^{i\theta}) d\theta$$

$$+ \frac{r \sin \varphi}{\pi} \int_{-R}^R \left( \frac{1}{|t - z|^2} - \frac{R^2}{|R^2 - tz|^2} \right) u(t) dt, z = re^{i\varphi} \in D_R^+. $$
2.5 Compactness principle for harmonic functions

Recall that a family of analytic or harmonic functions in a region $\Omega$ is said to be normal if every sequence contains a subsequence that converges uniformly on every compact set $E \subset \Omega$.

We have the following well-known compactness principle for analytic functions.

**Theorem 2.10.** A family $F$ of analytic functions in a region $\Omega$ is normal if and only if the functions in $F$ are uniformly bounded on every compact subset $E$ of $\Omega$.

The following analogue theorem for harmonic functions is an immediate corollary of the previous one.

**Theorem 2.11.** Let $\Omega$ be a simply connected region. A family $F$ of harmonic functions in $\Omega$ is normal if and only if the functions in $F$ are uniformly bounded on every compact subset $E$ of $\Omega$. 
Chapter 3

Representation of a function harmonic in the upper half-plane

THEOREM 2. Let \( u \) be a function harmonic in \( \mathbb{C}_+ \) and satisfy the following condition:

There exists an \( H > 0 \) such that

\[
\sup_{0 < s < H} \int_{-\infty}^{\infty} \frac{|u(t + is)|}{1 + t^2} dt < \infty.
\]  

(3.1)

Then \( u \) admits the representation

\[
u(z) = \int_{-\infty}^{\infty} P(x - t, y) d\nu(t) + U(z), \quad z = x + iy \in \mathbb{C}_+,
\]

where \( \nu \) is a Borel measure on \( \mathbb{R} \) satisfying

\[
\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} < \infty
\]

(3.2)

and \( U \) is a function harmonic in \( \mathbb{C} \) satisfying \( U(x) = 0, \ x \in \mathbb{R} \).
**Proof:** Consider the following family of measures on $\mathbb{R}$:

$$\sigma_s(E) = \int_E \frac{u(t+is)}{1+t^2} dt, \quad 0 < s < H,$$

then

$$\|\sigma_s\| := |\sigma_s|((\mathbb{R})) = \int_{-\infty}^{\infty} \frac{|u(t+is)|}{1+t^2} dt \leq M,$$

where $M$ is a constant not depending on $s$.

These measures can also be considered as measures on $\mathbb{R} := \mathbb{R} \cup \{\infty\}$ (one-point compactification of $\mathbb{R}$) assuming $\sigma_s(\{\infty\}) = 0$.

Then the family $\{\sigma_s\}$ belongs to some closed ball of the space of finite measures on $\mathbb{R}$. The space is the dual one for the space $C(\mathbb{R})$ consisting of all functions $f(x)$, continuous on $\mathbb{R}$ and having $\lim_{x \to \infty} f(x) \in \mathbb{R}$.

Since a closed ball of a dual space is weak* compact, from each sequence $\{\sigma_{s_j}\}$ of family $\{\sigma_s\}$, we can extract a subsequence $\{\sigma_{s_{k_j}}\}$ such that

$$\sigma_{s_{k_j}} \rightharpoonup^* \sigma$$

where $\sigma$ is a finite measure on $\mathbb{R}$. That is, for any $f \in C(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f(t)d\sigma_{s_{k_j}}(t) \to \int_{\mathbb{R}} f(t)d\sigma(t) \quad \text{as} \quad j \to \infty.$$

Putting

$$f(t) = (1 + t^2)P(x-t,y)$$

for fixed $x, y \in \mathbb{R}$, $y > 0$, we get

$$\int_{\mathbb{R}} f(t)d\sigma_{s_{k_j}}(t) = \int_{-\infty}^{\infty} P(x-t,y)u(t + is_{k_j})dt \to$$

$$\int_{\mathbb{R}} f(t)d\sigma(t) = \int_{-\infty}^{\infty} P(x-t,y)(1 + t^2)d\sigma(t) + \frac{y}{\pi}\sigma(\{\infty\}) \quad \text{as} \quad j \to \infty.$$
Let us set $dv(t) = (1 + t^2)d\sigma(t)$. We have showed that for any sequence 
\(\{s_k\}, \quad 0 < s_k < H\), there exists a subsequence \(\{s_{k_j}\}\) such that
\[
\int_{-\infty}^{\infty} P(x-t,y)u(t+is_k)dt \to \int_{-\infty}^{\infty} P(x-t,y)dv(t) + \alpha y, \quad (3.3)
\]
where \(\nu\) is a Borel measure on \(\mathbb{R}\) satisfying (3.2) and \(\alpha\) is a constant.

Now let us set for \(0 < s < H\),
\[
U_s(z) := u(z+is) - \int_{-\infty}^{\infty} P(x-t,y)u(t+is)dt, \quad z = x + iy \in \mathbb{C}_+ . \quad (3.4)
\]
By Theorem 2.2, \(U_s\) is harmonic in \(\mathbb{C}_+\) and continuous in \(\mathbb{C}_+\), if we define \(U_s(t) = 0, \quad t \in \mathbb{R}\). By the symmetry principle it can be harmonically extended into \(\mathbb{C}\) and satisfies the following condition:
\[
U_s(z) = -U_s(\bar{z}), \quad z \in \mathbb{C}. \quad (3.5)
\]
Consider the family of harmonic functions \(\{U_s : 0 < s < H/2\}\). Let us show that it is uniformly bounded in the rectangle
\[\Pi_{R,H} = \{z : |\text{Re}z| \leq R, |\text{Im}z| \leq H/4\}\]
for any fixed \(R > 0\).

First we have to prove that \(^1\):
\[
\int_{-\infty}^{\infty} \frac{|U_s(z+iy)|}{1 + x^2} dx \leq C \text{ for } |y| \leq \frac{H}{2}. \quad (3.6)
\]
By (3.5), it suffices to prove (3.6) only for \(0 < y \leq H/2\).

From (3.4) we have
\[
\int_{-\infty}^{\infty} \frac{|U_s(z+iy)|}{1 + x^2} dx \leq \int_{-\infty}^{\infty} \frac{|u(x+iy+is)|}{1 + x^2} dx
\]
\[
+ \int_{-\infty}^{\infty} \left\{\int_{-\infty}^{\infty} P(x-t,y)|u(t+is)|dt\right\} \frac{dx}{1 + x^2} =: I_1 + I_2.
\]

\(^1\)Here and in what follows the letter \(C\) or maybe \(C\) with subscripts denotes the various positive constants
By the condition (3.1), $I_1$ is bounded by some constant which does not depend on $y$ and $s$. Let us show that $I_2$ is also bounded by some constant which does not depend on $y$ and $s$.

By Fubini's theorem we can change the order of integration, and get

$$I_2 = \int_{-\infty}^{\infty} |u(t + is)| \left\{ \int_{-\infty}^{\infty} P(x - t, y) \cdot \frac{dx}{1 + x^2} \right\} dt.$$ 

A standard calculation shows that

$$\int_{-\infty}^{\infty} P(x - t, y) \cdot \frac{dx}{1 + x^2} = \frac{y + 1}{t^2 + (y + 1)^2},$$  \hspace{1cm} (3.7)

and hence

$$I_2 = (y + 1) \int_{-\infty}^{\infty} \frac{|u(t + is)|}{t^2 + (y + 1)^2} dt,$$

$$\leq (H + 1) \int_{-\infty}^{\infty} \frac{|u(t + is)|}{1 + t^2} dt \leq C,$$

where $C$ is a constant not depending on $s$. This implies (3.6).

By the arithmetic mean property of a harmonic function, we have

$$U_s(z) = \frac{1}{\pi \rho^2} \iint_{|\zeta - z| \leq \rho} U_s(\zeta) d\xi d\eta, \quad \zeta = \xi + i\eta.$$ 

Choosing $\rho = H/4$, we get

$$|U_s(z)| \leq \frac{1}{\pi \rho^2} \iint_{|\zeta - z| \leq \rho} (1 + \xi^2) \frac{|U_s(\zeta)|}{1 + \xi^2} d\xi d\eta,$$

$$\leq \frac{1 + (|z| + \rho)^2}{\pi \rho^2} \int_{-H/2}^{H/2} \left\{ \int_{-\infty}^{\infty} \frac{|U_s(\xi + i\eta)|}{1 + \xi^2} d\xi \right\} d\eta \tag{3.8}$$

$$\leq C \cdot (|z| + (H/2))^2,$$

where $C$ is a constant not depending on $s$.

By (3.8), we conclude that the family $\{U_s : 0 < s < H/2\}$ is uniformly bounded in each rectangle $\Pi_{R,H}$. Hence, by the compactness principle for
harmonic functions (Theorem 2.11, p.16), from any sequence \( \{s_k\} \), we can extract a subsequence \( \{s_{k_j}\} \), such that \( \{U_{s_{k_j}}\} \) is uniformly convergent on any compact subsets of \( \{z : |\text{Im } z| < H/4\} \).

Now let \( \{s_k\}, 0 < s_k < H/2 \), be a sequence tending to 0. Let \( U \) be the function harmonic in \( \{z : |\text{Im } z| \leq H/4\} \) to which \( U_{s_k} \) converges uniformly on compact subsets of \( \{z : |\text{Im } z| \leq H/4\} \), for some subsequence \( \{s_{k_j}\} \) of \( \{s_k\} \). Choose a subsequence \( \{s_{k_{n_j}}\} \) of \( \{s_{k_j}\} \), as we did in (3.3), such that

\[
\int_{-\infty}^{\infty} P(x-t,y)u(t + is_{k_{n_j}})dt \to \int_{-\infty}^{\infty} P(x-t,y)d\nu(t) + \alpha y,
\]

where \( \nu \) is a Borel measure on \( \mathbb{R} \) satisfying (3.2) and \( \alpha \) is a constant.

In the formula (3.4) we put \( s_{k_{n_j}} \) instead of \( s \), and let \( l \to \infty \). We get,

\[
U(z) = u(z) - \int_{-\infty}^{\infty} P(x-t,y)d\nu(t) - \alpha y, \quad 0 < y < H/4. \tag{3.9}
\]

Consider the function

\[
V(z) := u(z) - \int_{-\infty}^{\infty} P(x-t,y)d\nu(t) - \alpha y, \quad z = x + iy \in \mathbb{C}_+.
\]

Evidently, \( V \) is harmonic in \( \mathbb{C}_+ \). Since \( U_s(x) = 0, \quad x \in \mathbb{R} \), we have \( U(x) = 0, \quad x \in \mathbb{R} \). Therefore \( V \) is continuous in \( \mathbb{C}_+ \) if we define \( V(x) = 0, \quad x \in \mathbb{R} \). By symmetry principle \( V \) can be extended to the function harmonic in the whole complex plane \( \mathbb{C} \). Together (3.9) this shows that \( V \) is the harmonic extension of \( U \) in \( \mathbb{C} \). With this extension (3.9) is valid for \( z \in \mathbb{C}_+ \). If we redenote \( U(z) + \alpha y \) by \( U \), we get the desired representation for \( u \). \( \square \)
Chapter 4

Estimates for means of Poisson integrals and Blaschke products

**Lemma 1.** Let $B(z)$ be a Blaschke product, then

$$
\int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|} \sin \theta d\theta = O(r), \quad r \to \infty.
$$

**Proof:** Let $\{z_n\}$ be the zeros of $B(z)$. Without loss of generality we can assume $i \not\in \{z_n\}$. We write $B$ in the form $B = B_1 \cdot B_2$ where

$$
B_1(z) = \left( \prod_{|z_n|<1} \frac{|z_n^2 + 1|}{z_n^2 + 1} \cdot \frac{z - z_n}{z - \bar{z}_n} \right) \cdot A
$$

$$
B_2(z) = \left( \prod_{|z_n|\geq1} \frac{|z_n^2 + 1|}{z_n^2 + 1} \cdot \frac{z - z_n}{z - \bar{z}_n} \right) \cdot \frac{1}{A}
$$

and $A$ is chosen to make $B_2(0) = 1$.

First consider $B_1(z)$. Evidently, $B_1(z)$ is analytic in $\{z : |z| > 1\}$ and

$$\lim_{z \to \infty} |B_1(z)| = A.$$
Hence, we get
\[ \int_0^\pi \log^+ \frac{1}{|B_1(re^{i\theta})|} \sin \theta d\theta = O(1), \quad r \to \infty. \tag{4.1} \]

Now put for \(0 < h \leq 1/2\)
\[ B_2^{(h)}(z) := B_2(z + ih). \]

Evidently, \(B_2^{(h)}\) is analytic in \(\{z : \text{Im}z \geq 0\}\). Applying Carleman's formula (Theorem 2.8, p.15) to \(B_2^{(h)}\) in the region \(\{z : h \leq |z| \leq r, \text{Im}z \geq 0\}\), we have
\[
\sum_{h < |a_{k,h}| < r} \left( \frac{1}{|a_{k,h}|} - \frac{|a_{k,h}|}{r^2} \right) \sin \theta_{k,h} = \frac{1}{\pi r} \int_0^\pi \log |B_2^{(h)}(re^{i\theta})| \sin \theta d\theta \\
+ \frac{1}{2\pi} \int_h^r \left( \frac{1}{t^2} - \frac{1}{r^2} \right) \log |B_2^{(h)}(t)B_2^{(h)}(-t)| dt \\
+ A_h(B_2^{(h)}, r),
\]
where \(a_{k,h} = |a_{k,h}|e^{i\theta_{k,h}}\) are the zeros of \(B_2^{(h)}(z)\) in \(\mathbb{C}_+\) and
\[
A_h(B_2^{(h)}, r) = -\text{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log B_2^{(h)}(he^{i\theta}) \left( \frac{he^{i\theta}}{r^2} - \frac{e^{-i\theta}}{h} \right) d\theta \right\} \\
= -\text{Im} \left\{ \frac{1}{2\pi} \int_0^\pi \log B_2(h(e^{i\theta} + ih) \left( \frac{he^{i\theta}}{r^2} - \frac{e^{-i\theta}}{h} \right) d\theta \right\}. (4.2)
\]

Since the term in the left hand side is nonnegative and the second term in the right hand side is non-positive, we have
\[
\frac{1}{\pi r} \int_0^\pi \log^+ \frac{1}{|B_2^{(h)}(re^{i\theta})|} \sin \theta d\theta = \frac{1}{\pi r} \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta} + ih)|} \sin \theta d\theta \\
\leq A_h(B_2^{(h)}, r). \tag{4.3}
\]

Since \(B_2\) is analytic in \(|z| \leq 1/2\) and \(B_2(0) = 1\), we have the power series expansion:
\[ B_2(z) = 1 + cz + O(|z|^2) \quad \text{for} \quad |z| \leq 1/2. \]
Hence, for $h$ is small enough, we have

$$B_2(he^{i\theta} + ih) = 1 + che^{i\theta} + cih + O(h^2), \quad h \to 0.$$  

Using the power series expansion for $\log(1 + z) = z - \frac{z^2}{2} + \cdots$, we obtain

$$\log B_2(he^{i\theta} + ih) = che^{i\theta} + cih + O(h^2), \quad h \to 0. \quad (4.4)$$

If we substitute (4.4) into (4.2), we get

$$A_h(B^{(k)}_2, r) = -\Im \left\{ \frac{1}{2\pi i} \oint_0^{2\pi} \left\{ -c - ice^{-i\theta} + O(h) \right\} d\theta \right\} = -\frac{\Im c}{2} + O(h), \quad h \to 0,$$

which implies

$$\lim_{h \to 0} A_h(B^{(k)}_2, r) = -\frac{\Im c}{2} = -\frac{\Im(B'_2(0))}{2} \leq \frac{|B'_2(0)|}{2}. \quad (4.5)$$

Now, applying Fatou's lemma, we obtain

$$\frac{1}{\pi r} \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|} \sin \theta d\theta \leq \liminf_{h \to 0} \frac{1}{\pi r} \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta} + ih)|} \sin \theta d\theta. \quad (4.6)$$

Putting (4.3), (4.5) and (4.6) together, we get

$$\frac{1}{\pi r} \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|} \sin \theta d\theta \leq \frac{|B'_2(0)|}{2},$$

and, hence

$$\int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|} \sin \theta d\theta = O(r), \quad r \to \infty. \quad (4.7)$$

So,

$$\int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|} \sin \theta d\theta = \int_0^\pi \log^+ \frac{1}{|B_1(re^{i\theta})B_2(re^{i\theta})|} \sin \theta d\theta \leq \int_0^\pi \log^+ \frac{1}{|B_1(re^{i\theta})|} \sin \theta d\theta \quad \text{and} \quad + \int_0^\pi \log^+ \frac{1}{|B_2(re^{i\theta})|} \sin \theta d\theta.$$
Using (4.1), and (4.7), we get,

$$
\int_0^\pi \log^+ \frac{1}{|B(re^{i\theta})|} \sin \theta d\theta = O(r), \quad r \to \infty. \quad \square
$$

**Lemma 2.** Let $f$ be a Poisson integral of measure, i.e.

$$
f(z) := \int_{-\infty}^{\infty} P(x - t, y) d\nu(t), \quad z = x + iy \in \mathbb{C}_+,$$

where $\nu$ is a Borel measure on $\mathbb{R}$ satisfying (3.2).

Then,

$$
\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta = O(r), \quad r \to \infty.
$$

**Proof:** We have by the definition of $f$ that:

$$
f(re^{i\theta}) = \frac{r \sin \theta}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{r^2 + t^2 - 2rt \cos \theta}.
$$

Hence

$$
|f(re^{i\theta})| \leq \frac{r \sin \theta}{\pi} \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{r^2 + t^2 - 2rt \cos \theta},
$$

and

$$
\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \leq \int_0^\pi \frac{r \sin^2 \theta}{\pi} \left\{ \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{r^2 + t^2 - 2rt \cos \theta} \right\} d\theta.
$$

By Fubini's theorem, we can change the order of integration and get:

$$
\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \leq \frac{r}{\pi} \int_{-\infty}^{\infty} \left\{ \int_0^\pi \frac{\sin^2 \theta}{r^2 + t^2 - 2rt \cos \theta} d\theta \right\} d|\nu|(t). \quad (4.8)
$$

A standard calculation shows that

$$
\int_0^\pi \frac{\sin^2 \theta d\theta}{r^2 + t^2 - 2rt \cos \theta} = \begin{cases} \frac{\pi}{2r^2}, & \text{for } |t| \leq r \\ \frac{\pi}{2r^2}, & \text{for } |t| \geq r \end{cases} \quad (4.9)
$$
Substituting this into (4.8), we get (assuming $r > 1$):

$$\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \leq r \int_{|t| \leq r} \frac{d|\nu|(t)}{2r^2} + r \int_{|t| > r} \frac{d|\nu|(t)}{2t^2}$$

$$\leq r \int_{|t| \leq r} \frac{d|\nu|(t)}{1 + t^2} + r \int_{|t| > r} \frac{d|\nu|(t)}{1 + t^2},$$

and hence

$$\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \leq r \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + t^2} = O(r), \quad r \to \infty. \quad \square$$
Chapter 5

Factorization in Hardy and Nevanlinna classes

THEOREM 1. Let a function $h \neq 0$ belong to the Nevanlinna class. Suppose that $h = g_1 g_2$ where the functions $g_1$ and $g_2$ are analytic in $\mathbb{C}_+$ and satisfy the following conditions:

I) There exists a sequence $\{R_k\} \uparrow \infty$ such that

$$\int_0^\pi \log^+ |g_1(R_k e^{i\theta})| \sin \theta d\theta \leq \exp(o(R_k)).$$

II) There exists an $H > 0$ such that

$$\sup_{0 < s < H} \int_{-\infty}^{\infty} \log^+ \frac{|g_j(t + is)|}{1 + t^2} dt < \infty, \quad j = 1, 2.$$

Then $g_j$ belongs to the Nevanlinna class, $j = 1, 2$.

Since $H^p(\mathbb{C}_+)$ is contained in the Nevanlinna class for each $p$, $0 < p \leq \infty$, we get the following corollary immediately.

COROLLARY 1. Let a function $h \neq 0$ belong to $H^p(\mathbb{C}_+)$ for some $p$,
0 < p \leq \infty. Suppose that h = g_1g_2 where the functions g_1 and g_2 satisfy the conditions of Theorem 1. Then g_j belongs to the Nevanlinna class, j = 1, 2.

**COROLLARY 2.** Let h \neq 0 belongs to \( H^\infty(\mathbb{C}_+) \). Assume that h = g_1g_2 where g_1 and g_2 are analytic in \( \mathbb{C}_+ \) and satisfying the conditions:

1) There is a sequence \( r_k \uparrow \infty \) such that

\[
\int_0^r \log^+ |g_1(re^{i\theta})| \sin \theta d\theta \leq \exp\{o(r)\}, \quad r = r_k \uparrow \infty.
\]

II') There is an \( H > 0 \) such that

\[
\sup\{|g_1(z)| + |g_2(z)| : 0 < \text{Im } z < H\} < \infty.
\]

Then there are real constants \( k_j \) such that \( g_j(z)e^{ik_jz} \in H^\infty(\mathbb{C}_+) \), j = 1, 2.

We will derive this corollary after proving Theorem 1.

**Proof of Theorem 1:** By Corollary 2.1 of Theorem 2.5, p.13 the zeros of h satisfy the Blaschke condition. Therefore the zeros of \( g_1 \) and \( g_2 \) also satisfy this condition.

Denote by \( B_1 \) the Blaschke product corresponding to the zeros of \( g_1 \). The function

\[
f := \frac{g_1}{B_1}
\]

is analytic and non-vanishing in \( \mathbb{C}_+ \). So \( \log |f| \) is harmonic in \( \mathbb{C}_+ \).

Let us show that this function satisfies the conditions of Theorem 2, that is

\[
\sup_{0 < \alpha < H} \int_{-\infty}^{\infty} \frac{\log |f(t + is)|}{1 + t^2} dt < \infty.
\]

We have

\[
\int_{-\infty}^{\infty} \frac{\log |f(t + is)|}{1 + t^2} dt \leq \int_{-\infty}^{\infty} \frac{\log |g_1(t + is)|}{1 + t^2} dt + \int_{-\infty}^{\infty} \frac{\log |B_1(t + is)|}{1 + t^2} dt.
\]
Further,
\[
\int_{-\infty}^{\infty} \frac{\log |g_1(t + is)|}{1 + t^2} dt = \int_{-\infty}^{\infty} \frac{\log^+ [g_1(t + is)]}{1 + t^2} dt + \int_{-\infty}^{\infty} \frac{\log^+ [1/g_1(t + is)]}{1 + t^2} dt,
\]
and
\[
\int_{-\infty}^{\infty} \frac{\log^+ |1/g_1(t + is)|}{1 + t^2} dt = \int_{-\infty}^{\infty} \frac{\log^+ [g_2(t + is)/h(t + is)]}{1 + t^2} dt
\leq \int_{-\infty}^{\infty} \frac{\log^+ [g_2(t + is)]}{1 + t^2} dt + \int_{-\infty}^{\infty} \frac{\log^+ [1/h(t + is)]}{1 + t^2} dt.
\]

So, combining all these together, we get
\[
\int_{-\infty}^{\infty} \frac{\log |f(t + is)|}{1 + t^2} dt \leq \int_{-\infty}^{\infty} \frac{\log^+ [g_1(t + is)]}{1 + t^2} dt + \int_{-\infty}^{\infty} \frac{\log^+ [g_2(t + is)]}{1 + t^2} dt
+ \int_{-\infty}^{\infty} \frac{\log^+ [1/h(t + is)]}{1 + t^2} dt + \int_{-\infty}^{\infty} \frac{\log |B_1(t + is)|}{1 + t^2} dt.
\]

Now, by condition II) and Corollary 2.2 (see, p.14) we get (5.2).

Applying Theorem 2 to the function \( u = \log |f| \), we get the following representation:
\[
\log |f(z)| = \int_{-\infty}^{\infty} P(x - t, y) d\nu(t) + U(z), \quad z = x + iy \in \mathbb{C}_+,
\]
where \( U(z) \) is harmonic in \( \mathbb{C} \) and \( U(x) = 0, \ x \in \mathbb{R} \), and \( \nu \) is a Borel measure satisfying (3.2). Note that both \( U \) and \( \nu \) are real-valued in (5.3) because \( u = \log |f| \) is real-valued.

Now put
\[
\psi(z) := \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \left\{ \frac{1}{t - z} - \frac{t}{1 + t^2} \right\} d\nu(t) \right\}, \quad \text{Im} z > 0.
\]

Evidently, \( \psi \) is a function analytic in \( \mathbb{C}_+ \). Moreover
\[
\log |\psi(z)| = \int_{-\infty}^{\infty} P(x - t, y) d\nu(t)
\]
\[
\leq \int_{-\infty}^{\infty} P(x - t, y) d\nu^+(t)
\]
This implies that log \(|\psi(z)|\) has a positive harmonic majorant in \(\mathbb{C}_+\).

By (5.3), we have for \(z \in \mathbb{C}_+\),
\[
U(z) = \log \left| \frac{f(z)}{\psi(z)} \right| = \log \left| \frac{g_1(z)}{B_1(z)\psi(z)} \right|.
\]

Put
\[
G(z) := i \log \left\{ \frac{g_1(z)}{B_1(z)\psi(z)} \right\}, \quad \text{Im } z > 0. \quad (5.5)
\]

Evidently, \(G\) is a function analytic in \(\mathbb{C}_+\) and \(\text{Im } G(z) = U(z), \ z \in \mathbb{C}_+\).

Since \(\text{Im } G\) can be extended to the harmonic function \(U\) in the whole complex plane \(\mathbb{C}\), \(G\) can be analytically extended to the entire function, which we shall also denote by \(G\).

Let us show that there exists a sequence \(\{r_k\} \uparrow \infty\) such that
\[
|G(z)| \leq \exp \{o(r_k)\}, \quad |z| = r_k \uparrow \infty. \quad (5.6)
\]

For \(z \in \mathbb{C}_+\),
\[
\text{Im } G(z) = \log \left| \frac{g_1(z)}{B_1(z)\psi(z)} \right| \\
\leq \log^+ |g_1(z)| + \log^+ \frac{1}{|B_1(z)|} + |\log |\psi(z)||,
\]
which implies
\[
(\text{Im } G(z))^+ \leq \log^+ |g_1(z)| + \log^+ \frac{1}{|B_1(z)|} + |\log |\psi(z)||, \quad z \in \mathbb{C}_+. \quad (5.7)
\]

Let us apply Nevanlinna’s formula (Theorem 2.9, p.15) to \(u = \text{Im } G\), \(z = i\), \(R = R_k > 2\) where \(R_k\) is taken from the condition 1) of Theorem 1.

Taking into account that \(\text{Im } G(t) = 0\) for \(t \in \mathbb{R}\), we have
\[
\text{Im } G(i) = \frac{1}{2\pi} \int_0^\pi \text{Im } G(R_k e^{i\theta}) \frac{(R_k^2 - 1)4R_k \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} d\theta.
\]
We write \( \text{Im} G \) as \( (\text{Im} G)^+ - (\text{Im} G)^- \) and get

\[
\frac{1}{2\pi} \int_0^\pi (\text{Im} G(R_k e^{i\theta}))^- \frac{(R_k^2 - 1)4R_k \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} d\theta
\]

\[
= -\text{Im} G(i) + \frac{1}{2\pi} \int_0^\pi (\text{Im} G(R_k e^{i\theta}))^+ \frac{(R_k^2 - 1)4R_k \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} d\theta.
\]

Therefore multiplying the both sides of the above equation by \( R_k \), we have

\[
\frac{1}{2\pi} \int_0^\pi (\text{Im} G(R_k e^{i\theta}))^- \frac{(R_k^2 - 1)4R_k^2 \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} d\theta
\]

\[
= -\text{Im} G(i) \cdot R_k + \frac{1}{2\pi} \int_0^\pi (\text{Im} G(R_k e^{i\theta}))^+ \frac{(R_k^2 - 1)4R_k^2 \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} d\theta.
\]

Since we assumed \( R_k > 2 \), the following inequalities hold

\[
\frac{3R_k^2}{4} < R_k^2 - 1 < R_k^2 \quad \text{and} \quad \frac{R_k^2}{4} \leq |R_k e^{i\phi} \pm i|^2 \leq 4R_k^2,
\]

and so

\[
\frac{3}{16} \sin \theta \leq \frac{(R_k^2 - 1)4R_k^2 \sin \theta}{|R_k e^{i\theta} - i|^2|R_k e^{i\theta} + i|^2} \leq 64 \sin \theta.
\]

Substituting the previous inequalities into (5.8), we get

\[
\frac{3}{32\pi} \int_0^\pi (\text{Im} G(R_k e^{i\phi}))^- \sin \theta d\theta \leq -\text{Im} G(i) \cdot R_k + \frac{32}{\pi} \int_0^\pi (\text{Im} G(R_k e^{i\phi}))^+ \sin \theta d\theta,
\]

or, in other words,

\[
\int_0^\pi (\text{Im} G(R_k e^{i\phi}))^- \sin \theta d\theta \leq C_1 \cdot R_k + C_2 \int_0^\pi (\text{Im} G(R_k e^{i\phi}))^+ \sin \theta d\theta,
\]

(5.9)

where \( C_1 \) and \( C_2 \) are positive constants not depending on \( k \).

Let us estimate

\[
\int_0^\pi (\text{Im} G(R_k e^{i\phi}))^+ \sin \theta d\theta.
\]
By (5.7) we have

\[
\int_0^\pi (\text{Im} \, G(R_k e^{i\phi}))^+ \sin \theta d\theta \\
\leq \int_0^\pi \log^+ |g_1(R_k e^{i\phi})| \sin \theta d\theta + \\
+ \int_0^\pi \log^+ \frac{1}{|B_1(R_k e^{i\phi})|} \sin \theta d\theta + \\
+ \int_0^\pi |\log |\psi(R_k e^{i\phi})|| \sin \theta d\theta. \\
(5.10)
\]

Using the condition 1) and the results of Lemma 1 and Lemma 2 of chapter 4 in (5.10), we get

\[
\int_0^\pi (\text{Im} \, G(R_k e^{i\phi}))^+ \sin \theta d\theta \leq \exp(o(R_k)). \\
(5.11)
\]

Putting (5.9) and (5.11) together, we get

\[
\int_0^\pi (\text{Im} \, G(R_k e^{i\phi}))^- \sin \theta d\theta \leq \exp(o(R_k))
\]

and hence

\[
\int_0^\pi |\text{Im} \, G(R_k e^{i\phi})| \sin \theta d\theta \leq \exp(o(R_k)). \\
(5.12)
\]

Applying Nevanlinna’s formula (Theorem 2.9, p.15) once more, with

\[u = \text{Im} \, G, \quad R = R_k > 2 \quad \text{and} \quad z = R_k e^{i\varphi}/2 \in D_{R_k}^+\]

we have

\[
\text{Im} \, G(z) = \frac{1}{2\pi} \int_0^\pi \text{Im} \, G(R_k e^{i\phi}) \frac{(R_k^2 - (\frac{R_k}{2})^2)4R_k(R_k e^{i\phi}) \sin \theta \sin \varphi}{|R_k e^{i\phi} - (\frac{R_k}{2} e^{i\varphi}/2|)R_k e^{i\phi} - (\frac{R_k}{2} e^{-i\varphi}/2|)} d\theta
\]

and so

\[
|\text{Im} \, G(z)| \leq \frac{1}{2\pi} \int_0^\pi |\text{Im} \, G(R_k e^{i\phi})| \frac{3R_k^2 \sin \theta}{R_k^2 R_k^2} d\theta \\
= \frac{12}{\pi} \int_0^\pi |\text{Im} \, G(R_k e^{i\phi})| \sin \theta d\theta.
\]

By (5.12) and symmetry, we obtain

\[
|\text{Im} \, G(z)| \leq \exp(o(R_k)), \quad \text{for} \quad |z| = R_k/2.
\]
Applying the Schwarz formula

\[ G(z) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{Im} G(Re^{i\theta}) \frac{Re^{i\theta} + z}{Re^{i\theta} - z} d\theta + \text{Re} G(0) \]

with \( R = R_k/2, \ |z| = R_k/4, \) we have

\[ |G(z)| \leq \frac{R + |z|}{R - |z|} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\text{Im} G(Re^{i\theta})| d\theta + |\text{Re} G(0)| \]

whence we make sure that (5.6) is fulfilled for \( r_k = R_k/4. \)

Let us show that

\[ \sup_{-H < s < H} \int_{-\infty}^{\infty} \frac{|\text{Im} G(t + is)|}{1 + t^2} dt < \infty \]  \quad (5.13)

where \( H > 0 \) is taken from the condition II).

We have for \( 0 < \text{Im} \ z < H \)

\[ \text{Im} G(z) = \log \left| \frac{g_1(z)}{B_1(z)\psi(z)} \right| \]
\[ \leq \log^+ |g_1(z)| + \log^+ \frac{1}{|B_1(z)|} + \log^+ \frac{1}{|\psi(z)|} \]

and

\[ -\text{Im} G(z) = \log \left| \frac{B_1(z)\psi(z)}{g_1(z)} \right| = \log \left| \frac{B_1(z)\psi(z)g_2(z)}{h(z)} \right| \]
\[ \leq \log^+ |\psi(z)| + \log^+ |g_2(z)| + \log^+ \frac{1}{|h(z)|}. \]

Hence

\[ |\text{Im} G(z)| \leq \log^+ |g_1(z)| + \log^+ |g_2(z)| + \log^+ \frac{1}{|B_1(z)|} \]
\[ + \log^+ \frac{1}{|h(z)|} + |\log |\psi(z)||. \]

The inequality (5.13) is now an immediate consequence of condition II) and Corollary 2.2 of Theorem 2.7, p.14.
Let us prove that

\[ |G(z)| = o(|z|^2), \quad |z| \to \infty, \quad |\text{Im}z| \leq H/4. \quad (5.14) \]

By harmonicity of the function \( \text{Im} G \), we have for \( |\text{Im} z| \leq 3H/4 \), \( \rho = H/4 \) that,

\[
|\text{Im} G(z)| \leq \frac{1}{\pi \rho^2} \int_{|t+i\sigma-z| \leq \rho} |\text{Im} G(t+is)| dt ds \\
\leq \frac{1 + (|z| + \rho)^2}{\pi \rho^2} \int_{|t+i\sigma-z| \leq \rho} \frac{|\text{Im} G(t+is)|}{1 + t^2} dt ds \\
\leq \frac{1 + (|z| + \rho)^2}{\pi \rho^2} \int_{-H}^{H} \left( \int_{\text{Re}z-\rho}^{\text{Re}z+\rho} \frac{|\text{Im} G(t+is)|}{1 + t^2} dt \right) ds.
\]

The last integral tends to 0 as \( \text{Re} z \to \infty \) because from (5.13) it follows that

\[
\int_{-H}^{H} \left( \int_{-\infty}^{\infty} \frac{|\text{Im} G(t+is)|}{1 + t^2} dt \right) ds < \infty.
\]

Hence

\[ |\text{Im} G(z)| = o(|z|^2), \quad |z| \to \infty, \quad |\text{Im} z| \leq 3H/4. \quad (5.15) \]

By the Schwarz formula,

\[
G(z + \zeta) = \frac{i}{2\pi} \int_{-\pi}^{\pi} \text{Im} G(z + \rho e^{i\theta}) \frac{\rho e^{i\theta} + \zeta}{\rho e^{i\theta} - \zeta} d\theta + \text{Re} G(z), \quad |\zeta| < \rho.
\]

Differentiating this equality with respect to \( \zeta \) and putting \( \zeta = 0 \), we obtain

\[
G'(z) = \frac{i}{\pi} \int_{-\pi}^{\pi} \text{Im} G(z + \rho e^{i\theta}) \frac{e^{-i\theta}}{\rho} d\theta. \quad (5.16)
\]

Using (5.15), we obtain \( |G'(z)| = o(|z|^2), \quad z \to \infty, \quad |\text{Im} z| \leq H/2. \) Integrating it with respect to \( z \), we get (5.14).

Let us complete the proof of the theorem.

From (5.6) and (5.14) we conclude by virtue of the Phragmén-Lindelöf principle and Liouville theorem that the function \( G \) is a polynomial of a
degree not higher than 2. Since \( \text{Im} \, G(t) = 0 \) for \( t \in \mathbb{R} \), the coefficients of the polynomial are real. Thus, \( G(z) = az^2 + bz + c \) where \( a, b, c \in \mathbb{R} \), whence \( \text{Im} \, G(t + is) = 2ats + bs \). It follows from (5.13) that \( a = 0 \). We see from (5.5) that

\[
g_1(z) = B_1(z) \psi(z) e^{-i(bz+c)}, \quad z \in \mathbb{C}_+,
\]

and

\[
\log |g_1| = \log |B_1| + \log |\psi| + by \\
\leq \log |\psi| + b_+ y
\]

where \( b_+ = \max(b,0) \).

Since \( \log |\psi| \) has a positive harmonic majorant in \( \mathbb{C}_+ \) we conclude that \( \log |g_1| \) has positive harmonic majorant, that is \( g_1 \) belongs to the Nevanlinna class.

Since \( h \) and \( g_1 \) are functions of the Nevanlinna class, by Theorem 2.5 ( p.13) and its Corollary 2.1, we can write \( h \) and \( g_1 \) in the form

\[
h = B \cdot \frac{H_1}{H_2} \quad g_1 = B_1 \frac{G_1}{G_2}
\]

where \( B \) and \( B_1 \) are Blaschke products formed by zeros of \( h \) and \( g_1 \) respectively, \( H_j, G_j \) are analytic and non-vanishing in \( \mathbb{C}_+ \) such that \( |H_j| < 1 \), \( |G_j| < 1 \). Then \( g_2 \) is in the form

\[
g_2 = \frac{h}{g_1} = \frac{B}{B_1} \cdot \frac{H_1 G_2}{H_2 G_1}.
\]

Note that \( B_2 := B/B_1 \) is the Blaschke product formed by zeros of \( g_2 \) and therefore \( B_2 \) is analytic in \( \mathbb{C}_+ \) and \( |B_2| < 1 \). Putting \( F_1 := B_2 H_1 G_2 \) and \( F_2 := H_2 G_1 \), we write \( g_2 = F_1/F_2 \) where \( F_j \)'s are analytic in \( \mathbb{C}_+ \) and
$|F_j| < 1$ for $j = 1, 2$, $F_2 \neq 0$. Hence by Theorem 2.5 (see p.13) $g_2$ belongs to the Nevanlinna class. □

**Proof of Corollary 2:** Clearly $h$, $g_1$ and $g_2$ satisfy the conditions of Theorem 1. Then according to the Theorem 1, $g_1$ and $g_2$ belong to the Nevanlinna class. By Theorem 2.6 (see p.14) $g_j$ has the following representation

$$g_j(z) = B_j(z)e^{i(a_j z + b_j)} \exp \left\{ \int_{-\infty}^{\infty} \frac{1 + \tau z}{(t - z)(1 + t^2)} \, d\nu_j(t) \right\}, \quad (5.17)$$

where $B_j$ is a Blaschke product, $a_j$ and $b_j$ are real constants and $\nu_j$ is a real-valued Borel measure satisfying the condition

$$\int_{-\infty}^{\infty} \frac{d\nu_j(t)}{1 + t^2} < \infty, \quad j = 1, 2.$$

I claim that $g_j(z)e^{-i(a_j - \epsilon)z} \in H^\infty(\mathbb{C}_+)$ for any fixed $\epsilon > 0$. Evidently, this function is bounded in $\{z : 0 < \text{Im} \, z < H\}$ by the condition $2'$. By (5.17) for $y \geq H$ and any fixed $N > 1$ we have

$$|g_j(z)e^{-i a_j z}| \leq \exp \left\{ \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{d\nu_j^+(t)}{(x - t)^2 + y^2} \right\} \leq \exp \left\{ \frac{1}{\pi y} \int_{-N}^{N} \frac{d\nu_j^+(t)}{x-N(t^2+y^2)} + \frac{2t^2}{\pi y} \int_{|t| > N} \frac{d\nu_j^+(t)}{1 + t^2} \right\} \leq \exp \left\{ \frac{1}{\pi H} \int_{-N}^{N} d\nu_j^+(t) + \frac{2(x^2 + y^2)}{\pi y} \int_{|t| > N} \frac{d\nu_j^+(t)}{1 + t^2} \right\}, \quad z = x + iy.$$

Since $N$ can be taken arbitrarily large, we get

$$|g_j(z)e^{-i a_j z}| = e^{o(|z|^2)}, \quad |z| \to \infty, \text{ Im } z \geq H.$$

and

$$|g_j(z)e^{-i a_j z}| = e^{o(|z|)}, \quad |z| \to \infty, \quad |\pi/2 - \arg z| \leq \pi/4. \quad (5.18)$$

Evidently, $|g_j(z)e^{-i(a_j - \epsilon)z}| = e^{o(|z|^2)}, \quad |z| \to \infty, \text{ Im } z \geq H$. By (5.18) and condition $2'$ it is bounded on the boundary of the regions $\{z : \text{Re } z >$
Applying the Phragmén-Lindelöf principle to the function $g_i(z)e^{-(i\alpha - r)z}$ in these regions we conclude that it is bounded in \( \{ z : \text{Im } z \geq H \} \) and therefore it is bounded in \( C_+ \). □

Now let us consider some examples related to the sharpness of Theorem 1.

**Example 1:** Consider the functions $g_1(z) = \exp(\cos z)$, $g_2(z) = \exp(-\cos z)$. Evidently, these functions satisfy condition I) with $O(r)$ instead of $o(r)$, satisfy II) and satisfy $g_1g_2 = 1 \in H^\infty(C_+)$. Nevertheless, neither $g_1$ nor $g_2$ belongs to the Nevanlinna class. Indeed, if $f$ is a function belonging to the Nevanlinna class, then by representation (5.17) and similar estimation as in the proof of Corollary 2 (cf. (5.18)), we have

$$
\log^+ |f(z)| \leq ky + o(|z|), \quad |z| \to \infty, |\pi/2 - \arg z| \leq \pi/4.
$$

But, in our case $\log^+ |g_1(iy)| = (e^y + e^{-y})/2$ does not satisfy the above inequality. This shows that $g_1$ does not belong to the Nevanlinna class. Hence $g_2$ does not belong to the Nevanlinna class. This example shows that the condition I) cannot be weakened replacing $o(r)$ by $O(r)$.

The following example shows that the condition II) in Theorem 1 cannot be weakened by replacing it with

$$
\exists H > 0, \quad \sup_{0 < r < H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t + is)|}{1 + |t|^\alpha} dt < \infty, \quad j = 1, 2,
$$

for some $\alpha > 2$.

**Example 2:** Consider the functions $g_1(z) = \exp(iz^2)$, $g_2(z) = \exp(-iz^2)$. Since $\log |g_1(t + is)| = -2ts$, $\log |g_2(t + is)| = 2ts$, they satisfy the above condition. The condition I) of Theorem 1 and $g_1g_2 = 1 \in H^\infty(C_+)$ are also satisfied. Nevertheless, neither $g_1$ nor $g_2$ belongs to the Nevanlinna class by the same reason as in the Example 1, now it is enough to look the growth of
log⁺ |g₁| and log⁺ |g₂| on the rays \( \{ z : \arg z = 3\pi/4 \} \) and \( \{ z : \arg z = \pi/4 \} \) respectively.

The condition I) of Theorem 1 touches only one of functions \( g₁, g₂ \). But it is impossible to change the condition II) in a similar way as the following example shows.

**Example 3:** Consider the functions \( g₁(z) = \exp(z^2) \), \( g₂(z) = \exp(-z^2) \).

Clearly \( g₁ \) and \( g₂ \) satisfy the condition I) and satisfy \( g₁g₂ = 1 \in H∞(C₊) \) and \( g₂ \) satisfy condition II) but

\[
\sup_{0<s<\infty} \int_0^\infty \frac{\log⁺ |g₁(t + is)|}{1 + t^2} dt = \infty.
\]

In this case, we see by the same reason as in the previous examples that neither \( g₁ \) nor \( g₂ \) belong to the Nevanlinna class.
Chapter 6

Generalized representation of a function harmonic in the upper half-plane

THEOREM 4. Let $h$ be a function harmonic in $\mathbb{C}_+$ and satisfy the following condition:

There exist $\alpha > 0$ and $H > 0$ such that

$$\sup_{0<s<H} \int_{-\infty}^{\infty} \frac{|h(t + is)|}{1 + |t|^\alpha} dt < \infty.$$

(6.1)

Then $h$ admits the representation

$$h(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z, t) d\nu(t) + U(z), \quad z = x + iy \in \mathbb{C}_+,$$

where

$$P_q(z, t) = \text{Im} \left\{ \frac{(1 + tz)^q}{(1 + t^2)^q(t - z)} \right\}, q = [\alpha],$$

(6.2)

$\nu$ is a Borel measure on $\mathbb{R}$ satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + |t|^\alpha} < \infty$$

(6.3)
and $U$ is a function harmonic in $\mathbb{C}$ satisfying $U(x) = 0$, $x \in \mathbb{R}$.

**Proof:** The proof is based on the same ideas as the proof of Theorem 2, but some new things appear because the kernel $P_\eta(z, t)$ is more complicated than the Poisson kernel.

Consider the following family of measures on $\mathbb{R}$:

$$\sigma_s(E) = \int_E \frac{h(t + is)}{1 + |t|^\alpha} dt, \ 0 < s < H.$$  

Then

$$\|\sigma_s\| := |\sigma_s|((\mathbb{R}) = \int_-\infty^\infty \frac{|h(t + is)|}{1 + |t|^\alpha} dt \leq M,$$

where $M$ is a constant not depending on $s$.

The family $\{\sigma_s\}$ belong to some closed ball of the space of finite measures on $\mathbb{R}$. The space is the dual one for the space $C_0(\mathbb{R})$ consisting of all functions $f(x)$, continuous on $\mathbb{R}$ and having $\lim_{x \to -\infty} f(x) = 0$.

Since the closed balls of dual spaces are weak$^*$ compact, from each sequence $\{\sigma_{s_j}\}$ of family $\{\sigma_s\}$ we can extract a subsequence $\{\sigma_{s_{j_k}}\}$ such that

$$\sigma_{s_{j_k}} \xrightarrow{w^*} \sigma,$$

where $\sigma$ is a finite measure on $\mathbb{R}$. That is for any $f \in C_0(\mathbb{R})$, we have

$$\int_-\infty^\infty f(t) d\sigma_{s_{j_k}}(t) \rightarrow \int_-\infty^\infty f(t) d\sigma(t) \text{ as } j \to \infty.$$

Set $f(t) = (1 + |t|^\alpha)P_\eta(z, t)$, for fixed $z$. Since, for $|t|$ large enough

$$|f(t)| \leq (1 + |t|^\alpha) \frac{|1 + tz|^q}{(1 + t^2)^q|t - z|} \leq C_{q, z} \frac{|t|^{\alpha + q}}{|t|^{2q + 1}} \leq \frac{C_{q, z}}{|t|^{\eta + 1 - \alpha}} \to 0 \text{ as } |t| \to \infty,$$
we have \( f \in C_0(\mathbb{R}) \) and
\[
\int_{-\infty}^{\infty} f(t) d\sigma_{s_k}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} h(t + is_k) P_q(z, t) dt \rightarrow \int_{-\infty}^{\infty} f(t) d\sigma(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} (1 + |t|^\alpha) P_q(z, t) dt \sigma(t) \text{ as } j \to \infty.
\]
(6.4)

Let us set \( d\nu(t) = (1+|t|^\alpha) d\sigma(t) \). We have showed that: For each sequence \( \{s_k\}, 0 < s < H \), there exists a subsequence \( \{s_{k_j}\} \) such that
\[
\frac{1}{\pi} \int_{-\infty}^{\infty} h(t + is_{k_j}) P_q(z, t) dt \rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z, t) d\nu(t), \quad j \to \infty,
\]
where \( \nu \) is a Borel measure satisfying (6.3).

Now let us set
\[
U_s(z) := h(z + is) - \frac{1}{\pi} \int_{-\infty}^{\infty} h(t + is) P_q(z, t) dt.
\]
(6.6)

By the following lemma, whose proof we postpone to the end of this chapter, \( U_s \) is harmonic in \( \mathbb{C}_+ \) and continuous in \( \overline{\mathbb{C}_+} \) if we define \( U_s(x) = 0, x \in \mathbb{R} \).

**LEMMA 3.** Let \( f \) be a function continuous on \( \mathbb{R} \) and satisfying the condition:
\[
\exists q \in \mathbb{N}, \quad \int_{-\infty}^{\infty} \frac{|f(t)|}{1 + |t|^q} dt < \infty.
\]
(6.7)

Then the function
\[
u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) P_q(z, t) dt, \quad z = x + iy \in \mathbb{C}_+
\]
is harmonic in \( \mathbb{C}_+ \) and continuous in \( \overline{\mathbb{C}_+} \) if we put \( u(t) = f(t), t \in \mathbb{R} \).

By the symmetry principle \( U_s \) can be harmonically extended into \( \mathbb{C} \) and satisfies the following condition:
\[
U_s(z) = -U_s(z), \quad z \in \mathbb{C}.
\]
(6.8)
Now we consider the family of harmonic functions \( \{ U_s : 0 < s < H/2 \} \).

Let us show that it is uniformly bounded in the rectangle

\[ \Pi_{R,H} := \{ z : |\text{Re}z| \leq R, |\text{Im}z| \leq H/4 \} \quad (6.9) \]

for any fixed \( R > 0 \).

First we have to prove that

\[ \int_{-\infty}^{\infty} \frac{|U_s(x + iy)|}{1 + |x|^{q+2}} \, dx \leq C_{q,H}, \quad \text{for } |y| \leq \frac{H}{2}. \quad (6.10) \]

By (6.8), it suffices to prove (6.10) only for \( 0 < y < H/2 \).

From (6.6) we have \( (z = x + iy) \)

\[ \int_{-\infty}^{\infty} \frac{|U_s(x + iy)|}{1 + |x|^{q+2}} \, dx \leq \int_{-\infty}^{\infty} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx + \]

\[ + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + |x|^{q+2}} \left\{ \int_{-\infty}^{\infty} |h(t + is)| |P_q(z, t)| \, dt \right\} \, dx. \quad (6.11) \]

Further,

\[ \int_{-\infty}^{\infty} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx = \left\{ \int_{|x| < 1} + \int_{|x| \geq 1} \right\} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx. \]

Since

\[ \int_{|x| < 1} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx \leq \int_{|x| < 1} |h(x + i(y + s))| \, dx \leq 2 \int_{|x| < 1} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx, \]

we have

\[ \int_{-\infty}^{\infty} \frac{|h(x + i(y + s))|}{1 + |x|^{q+2}} \, dx \leq 2 \int_{-\infty}^{\infty} \frac{|h(x + i(y + s))|}{1 + |t|^{q+2}} \, dt. \]

Since \( y + s < H \) by the condition (6.1), we conclude that the first integral on the right hand side of (6.11) is bounded by some constant which does not depend on \( y \) and \( s \).
To show that the second integral on the right hand side of (6.11) is also bounded by some constant which does not depend on \( y \) and \( s \), we need the following inequality:

\[
|P_q(z, t)| \leq C \frac{y(1 + |z|)^q}{|t - z|^q(1 + |t|)^{q-1}}.
\]  

(6.12)

where \( C \) is a positive constant depending only on \( q \).

We have

\[
P_q(z, t) = \text{Im} \left\{ \frac{(1 + tz)^q}{(1 + t^2)^q(t - z)} \right\} = \frac{\text{Im}\{(t - \bar{z})(1 + tz)^q\}}{(t^2 + 1)^q|t - z|^2}
\]

and, for \( z = |z|e^{i\theta} \),

\[
\text{Im}\{(t - \bar{z})(1 + tz)^q\} = \text{Im} \left\{ \sum_{k=0}^{q} \binom{q}{k} t^{q-k+1} z^k - \sum_{k=0}^{q} \binom{q}{k} t^k |z|^{2k} z^{k-1} \right\}
\]

\[
= \sum_{k=1}^{q} \binom{q}{k} t^k |z|^k \{ t \sin k\theta - |z| \sin(k-1)\theta \}.
\]

Using the inequality \(|\sin k\theta| \leq k \sin \theta, 0 \leq \theta \leq \pi, k \in \mathbb{N}\), we have

\[
|\text{Im}\{(t - \bar{z})(1 + tz)^q\}| \leq \sum_{k=1}^{q} \binom{q}{k} |t|^k |z|^k (|t|k + |z||k - 1|) \sin \theta \leq 2q^2 q! (1 + |t|)^{q+1} (1 + |z|)^q y.
\]

Therefore

\[
|P_q(z, t)| = \frac{|\text{Im}\{(t - \bar{z})(1 + tz)^q\}|}{(t^2 + 1)^q|t - z|^2} \leq 2q^2 q! (1 + |t|)^{q+1} (1 + |z|)^q y \frac{1}{(t^2 + 1)^q|t - z|^2}
\]

\[
\leq C q \frac{(1 + |z|)^q y}{|t - z|^2(1 + |t|)^{q-1}},
\]

(we utilized the evident inequality \(1 + t^2 \geq (1 + |t|)^2/2\) and (6.12) is shown.
Now let us estimate the second integral on the right hand side of (6.11), i.e.

\[
I := \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + |x|^{q+2}} \left\{ \int_{-\infty}^{\infty} |h(t + is)||P_q(z, t)| dt \right\} dx.
\]

By Fubini’s theorem

\[
I = \int_{-\infty}^{\infty} |h(t + is)| \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|P_q(z, t)|}{1 + |x|^{q+2}} dx \right\} dt.
\]

Without loss of generality we can assume that \( H < 1 \). Then, using inequality (6.12), we have

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} |P_q(z, t)| \frac{dx}{1 + |x|^{q+2}} \leq \frac{C}{(1 + |t|)^{q+1}} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{1 + |x|^{q+2}} \frac{(2 + |x|)^q}{1 + |x|^{q+2}} dx \\
\leq \frac{C}{(1 + |t|)^{q+1}} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y}{(x - t)^2 + y^2} \cdot \frac{1}{1 + x^2} dx.
\]

Using equality (3.7), we obtain

\[
\int_{-\infty}^{\infty} |P_q(z, t)| \frac{dx}{1 + |x|^{q+2}} \leq \frac{C}{1 + |t|^{q+1}}.
\]

Hence

\[
I \leq C \int_{-\infty}^{\infty} |h(t + is)| \cdot \frac{1}{(1 + |t|)^{q+1}} dt.
\]

By condition (6.1), we conclude that \( I \) is bounded by a constant not depending on \( y \) and \( s \). This proves that the second integral in the right hand side of (6.11) is bounded by some constant which does not depend on \( y \) and \( s \) and hence (6.10) is true.

Now suppose \( |\text{Im} z| \leq H/4 \) and let \( \rho = H/4 \). Since the function \( U_s \) is harmonic, we have

\[
U_s(z) = \frac{1}{\pi \rho^2} \int_{|\zeta - z| \leq \rho} U_s(\zeta) d\xi d\eta, \quad \zeta = \xi + i\eta.
\]
Hence

\[ |U_s(z)| \leq \frac{1}{\pi \rho^2} \iint_{|\zeta - z| \leq \rho} |U_s(\zeta)| \, d\xi \, d\eta \]

\[ = \frac{1}{\pi \rho^2} \iint_{|\zeta - z| \leq \rho} \left(1 + |\zeta|^{q+2}\right) \frac{|U_s(\zeta)|}{1 + |\zeta|^{q+2}} \, d\xi \, d\eta \]

\[ \leq \frac{1 + (|z| + \rho)^{q+2}}{\pi \rho^2} \int_{-\frac{H}{2}}^{\frac{H}{2}} \left\{ \int_{-\infty}^{\infty} \frac{|U_s(\zeta + i\eta)|}{1 + |\zeta|^{q+2}} \, d\zeta \right\} \, d\eta \]

\[ \leq C_{q,H} \cdot (|z| + 1)^{q+2}. \]

Hence

\[ |U_s(z)| \leq C_{q,H} \cdot (R + 1)^{q+2}, \text{ for } |\text{Im } z| \leq H/4, |\text{Re } z| \leq R, \quad 0 < s < H/2, \]

and this proves that the family \( \{U_s : 0 < s < H/2\} \) is uniformly bounded in (6.9) for any \( R > 0 \). By the compactness principle for harmonic functions (Theorem 2.11, p.16), we conclude that there exists a subsequence \( \{U_{s_k}\} \) \((s_k \text{ tends to } 0 \text{ as } k \to \infty)\) uniformly convergent in each compact set in the strip \( \{z : |\text{Im } z| < H/4\} \) to a function \( U \) harmonic in the strip.

We can extract a subsequence \( \{s_{k_j}\} \) from \( \{s_k\} \) such that

\[ \frac{1}{\pi} \int_{-\infty}^{\infty} h(t + is_{k_j})P_q(z,t) \, dt \rightarrow \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z,t) \, d\nu(t), \quad j \to \infty, \]

for some Borel measure \( \nu \) on \( \mathbb{R} \) satisfying (6.3).

Consider the difference

\[ U_{s_{k_j}}(z) = h(z + is_{k_j}) - \frac{1}{\pi} \int_{-\infty}^{\infty} h(t + is_{k_j})P_q(z,t) \, dt. \]

If we let \( j \to \infty \) we get

\[ U(z) = h(z) - \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z,t) \, d\nu(t), \quad 0 < \text{Im } z < H/4. \]
By the same reasoning as at the end of the proof of Theorem 2 (see, p.21), it follows that \( h \) has the desired representation. □

**Proof of Lemma 3:** Consider the following functions

\[
 u_N(z) := \frac{1}{\pi} \int_{-N}^{N} f(t) P_q(z, t) dt, \quad z = x + iy \in \mathbb{C}_+, \; N > 0.
\]

Each \( u_N \) is harmonic in \( \mathbb{C}_+ \). Using inequality (6.12), and the condition (6.7) one can easily show that \( u_N \) converges to \( u \) uniformly on compact subsets of \( \mathbb{C}_+ \) as \( N \to \infty \), which implies that \( u \) is harmonic in \( \mathbb{C}_+ \).

Now let us show that \( u \) is continuous in \( \overline{\mathbb{C}_+} \), that is

\[
 u(z) \to f(t_0) \quad \text{as} \quad z \to t_0, \; \text{Im} \; z > 0, \; t_0 \in \mathbb{R}.
\]

The reasonings are similar to those of the proof of Theorem 2.2 (see, p.11).

Choosing \( R = 2|t_0| + 1 \), define \( f_R \) and \( f^R \) by (2.5). Then we have

\[
 u(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f_R(t) P_q(z, t) dt + \frac{1}{\pi} \int_{-\infty}^{\infty} f^R(t) P_q(z, t) dt =: I_1(z) + I_2(z).
\]

First let us show that

\[
 I_2(z) \to 0 \text{ as } z \to t_0.
\]

By inequality (6.12), we have

\[
 |I_2(z)| \leq C \int_{|t| > R} \frac{(1 + |z|)^y}{|t - z|^2(1 + |t|)^{y-1}} |f(t)| dt.
\]

Since for \( |z - t_0| < (|t_0| + 1)/2 \), \( |t| > R = 2|t_0| + 1 \), we have (2.6), we get

\[
 |I_2(z)| \leq C \cdot y \int_{|t| > R} \frac{|f(t)|}{(1 + |t|)^{y+1}} dt.
\]

Using (6.7), we see that \( I_2(z) \to 0 \) as \( z \to t_0 \).
To consider the integral $I_1(z)$, we represent it in the form

$$I_1(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} f_R(t) \text{Im} \left\{ \frac{1}{l - z} - \frac{1}{(l^2 + 1)^q} \left[ \frac{(t^2 + 1)^q - (tz + 1)^q}{l - z} \right] \right\} dt$$

$$= \int_{-\infty}^{\infty} P(x - t, y) f_R(t) dt - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f_R(t)}{(t^2 + 1)^q} \text{Im} \left\{ \frac{(t^2 + 1)^q - (tz + 1)^q}{l - z} \right\} dt$$

$$=: I_1^1(z) - I_1^2(z)$$

The integral $I_1^1(z)$ is the Poisson integral of the bounded function $f_R$ and converges to $f_R(t_0) = f(t_0)$ as $z \to t_0$ according to Theorem 2.2' (p.11).

To conclude that $u(z)$ is continuous on $\overline{C_+}$, it only remains to show $I_1^2(z)$ tend to 0 as $z \to t_0$.

Since

$$\frac{(t^2 + 1)^q - (tz + 1)^q}{l - z} = \sum_{k=0}^{q} \binom{q}{k} t^k (t^k - z^k)$$

$$= \sum_{k=0}^{q} \binom{q}{k} t^k \left( \sum_{i=0}^{k-1} t^{(k-1)-i} |z|^i \right)$$

we have

$$\left| \text{Im} \left\{ \frac{(t^2 + 1)^q - (tz + 1)^q}{l - z} \right\} \right| \leq \sum_{k=0}^{q} \binom{q}{k} |t|^k \left( \sum_{i=0}^{k-1} |t|^{(k-1)-i} |z|^{i+1} \right)$$

$$\leq C_{q,t_0} \cdot y.$$ 

Therefore

$$|I_1^2(z)| \leq y \cdot C_{q,t_0} \int_{-R}^{R} \frac{|f(t)|}{(t^2 + 1)^q} dt \to 0 \text{ as } y \to 0. \quad \square$$
Chapter 7

Generalized factorization in Hardy and Nevanlinna classes

**THEOREM 3.** Let a function $h \neq 0$ belong to the Nevanlinna class. Suppose that $h = g_1 g_2$ where the functions $g_1$ and $g_2$ are analytic in $\mathbb{C}_+$ and satisfying the following conditions:

I) There exists a sequence $\{R_k\} \uparrow \infty$ such that

$$
\int_0^\pi \log^+ |g_1(R_k e^{i\theta})| \sin \theta d\theta \leq \exp(o(R_k)).
$$

II) There exist $a \geq 2$, $H > 0$ such that

$$
\sup_{0 < s < H} \int_{-\infty}^{\infty} \frac{\log^+ |g_j(t + is)|}{1 + |t|^a} dt < \infty, \quad j = 1, 2.
$$

Then $g_j$ admits the following representation

$$
g_j(z) = B_j(z) \cdot e^{P_j(z)} \cdot \exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + tz)^q}{(t - z)(1 + t^2)^q} dv_j(t) \right), \quad q = [\alpha] (7.1)
$$

where $B_j$ is the Blaschke product formed by the zeros of $g_j$, $P_j$ is a real polynomial whose degree is not greater than $q + 1$, and $v_j$ is a real-valued
Borel measure satisfying
\[ \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + |t|^\alpha} < \infty. \] (7.2)

**COROLLARY.** Let a function \( h \neq 0 \) belongs to \( H^p(\mathbb{C}_+) \) for some \( p \), \( 0 < p \leq \infty \). Suppose that \( h = g_1 g_2 \) where the functions \( g_1 \) and \( g_2 \) satisfy the conditions of Theorem 3. Then \( g_j \) admits the representation (7.1), \( j = 1, 2 \).

**Proof of Theorem 3:** The proof is similar to that of Theorem 1. As in that proof, we note that the zeros of \( h, g_1, g_2 \) satisfy the Blaschke condition and consider the function \( f \) defined by (5.1).

Using condition II) and Corollary 2.2 of Theorem 2.7 (p.14), it can be shown in the same manner, as we did in the proof of Theorem 2 (see, p.28-29), that \( \log |f| \) satisfies the conditions of Theorem 4.

Applying Theorem 4 to the function \( \log |f| \), we get the following representation
\[
\log |f(z)| = \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z, t) d\nu(t) + U(z), \quad z = x + iy \in \mathbb{C}_+ \tag{7.3}
\]
where
\[
P_q(z, t) = \text{Im} \left\{ \frac{(1 + tz)^q}{(1 + t^2)^{q(t - z)}} \right\}, \quad q = [\alpha],
\]
\( \nu \) is a real-valued Borel measure on \( \mathbb{R} \) satisfying (6.3), and \( U \) is a function harmonic in \( \mathbb{C} \) satisfying \( U(x) = 0, x \in \mathbb{R} \).

Now put
\[
\psi(z) := \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + tz)^q}{(1 + t^2)^{q(t - z)}} d\nu(t) \right\}, \quad \text{Im} \, z > 0. \tag{7.4}
\]
Evidently, $\psi$ is analytic in $\mathbb{C}_+$. By (7.3) we have for $z \in \mathbb{C}_+$,

$$U(z) = \log \left| \frac{f(z)}{\psi(z)} \right| = \log \left| \frac{g_1(z)}{B_1(z)\psi(z)} \right|.$$ 

Put

$$G(z) = i \log \left\{ \frac{g_1(z)}{B_1(z)\psi(z)} \right\}, \quad \text{Im} \, z > 0. \quad (7.5)$$

Evidently $G$ is a function analytic in $\mathbb{C}_+$ and $\text{Im} \, G(z) = U(z), \ z \in \mathbb{C}_+$.

Since $\text{Im} \, G$ can be extended to the harmonic function $U$ in the whole complex plane $\mathbb{C}$, $G$ can be analytically extended to the entire function, which we shall also denote by $G$.

Further we will need the following two lemmas whose proofs we postpone to the end of this chapter.

**Lemma 4.** Let $f$ be a function of the form

$$f(z) = \frac{1}{\pi} \int_{-\infty}^{\infty} P_q(z,t)d\nu(t), \quad z = x + iy \in \mathbb{C}_+, \quad (7.6)$$

where

$$P_q(z,t) = \text{Im} \left\{ \frac{(1 + tz)^q}{(t^2 + 1)^q(t - z)} \right\}$$

for some $q \in \mathbb{N}$, and $\nu$ is a real-valued Borel measure satisfying

$$\int_{-\infty}^{\infty} \frac{d|\nu|(t)}{1 + |t|^{q+1}} < \infty.$$ 

Then

$$\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta = O(r^{q+1}), \quad r \to \infty. \quad (7.7)$$

**Lemma 5.** Let $f$ be a function of the form (7.6) where $\nu$ and $P_q(z,t)$ as in the Lemma 4.
Then for any \( K > 0 \)

\[
\sup_{0 < y < K} \int_{-\infty}^{\infty} \frac{|f(x + iy)|}{1 + |x|^{q+2}} \, dx < \infty.
\]

Now we prove that there exists a sequence \( \{r_k\} \uparrow \infty \) such that

\[
|G(z)| \leq \exp(o(r_k)), \quad |z| = r_k \uparrow \infty.
\] (7.8)

in the same manner as we did (5.6) in the proof of Theorem 1 (see p.30-33). The only difference is that now we use Lemma 4 instead of Lemma 2 of Chapter 4.

Then we prove that

\[
\sup_{-H < s < H} \int_{-\infty}^{\infty} \frac{|\text{Im} G(t + is)|}{1 + |t|^{q+2}} \, dt < \infty
\] (7.9)

where \( H > 0 \) is taken from the condition II), in the same manner as we did (5.13) in the proof of Theorem 1 (see p.33). The only difference is that now we use Lemma 5 instead of Corollary 2.2 of Theorem 2.7, p.14.

Let us prove that

\[
|G(z)| = o(|z|^{q+5}), \quad |z| \to \infty, \quad |\text{Im} z| \leq H/2.
\] (7.10)

Because of harmonicity of the function \( \text{Im} G \) we have for \( |\text{Im} z| \leq 3H/4, \rho = H/4 \) that

\[
|\text{Im} G(z)| \leq \frac{1}{\pi \rho^2} \int_{|t+is-z| \leq \rho} |\text{Im} G(t + is)| \, dt \, ds
\]

\[
\leq \frac{1 + (|z| + \rho)^{q+2}}{\pi \rho^2} \int_{|t+is-z| \leq \rho} \frac{|\text{Im} G(t + is)|}{1 + |t|^{q+2}} \, dt \, ds
\]

\[
\leq \frac{1 + (|z| + \rho)^{q+2}}{\pi \rho^2} \int_{-H}^H \left( \int_{\text{Re} z - \rho}^{\text{Re} z + \rho} \frac{|\text{Im} G(t + is)|}{1 + |t|^{q+2}} \, dt \right) \, ds.
\]
From (7.9) it follows that
\[ \int_{-H}^{H} \left( \int_{-\infty}^{\infty} \frac{|\text{Im} G(t + i\zeta)|}{1 + |t|^{q+2}} dt \right) d\zeta < \infty, \]

hence
\[ |\text{Im} G(z)| = o(|z|^{q+2}), \quad |z| \to \infty, \quad |\text{Im} z| \leq 3H/4. \tag{7.11} \]

Using the formula (5.16) and (7.11), we obtain that
\[ |G'(z)| = o(|z|^{q+2}), \quad |z| \to \infty, \quad |\text{Im} z| \leq H/2. \]

Integrating it with respect to \( z \), we get (7.10).

From (7.8) and (7.10) we conclude by virtue of Phragmén-Lindelöf principle and Liouville theorem that the function \( G \) is a polynomial of a degree not higher than \( q + 2 \). Since \( \text{Im} G(t) = 0, \ t \in \mathbb{R} \), the coefficients of the polynomial are real. Thus \( G(z) = a_{q+2}z^{q+2} + \cdots + a_0, \ a_k \in \mathbb{R}, \ 0 \leq k \leq q+2 \).

It follows from (7.9) that \( a_{q+2} = 0 \), that is \( G(z) = a_{q+1}z^{q+1} + \cdots + a_0, \ a_k \in \mathbb{R} \).

By (7.5) and (7.4) we have the following representation for \( g_1 \)
\[
g_1(z) = B_1(z)e^{-iG(z)}\psi(z) = B_1(z)e^{-iG(z)}\exp\left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + t^2)^q}{(t - z)(1 + t^2)^q} d\nu(t) \right).
\]

Thus we have proved the theorem for \( g_1 \).

Now consider the following factorization of \( h \):
\[
h = h \cdot 1
\]

instead of \( g_1 \) we put \( h \) itself and instead of \( g_2, 1 \) only. Clearly \( h \) and 1 satisfy the conditions I) and II). Applying the same procedure to this factorization we conclude that \( h \) also admits the same representation, that is:
\[
h(z) = B(z)e^{iQ(z)}\exp \left( \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1 + t^2)^q}{(t - z)(1 + t^2)^q} d\mu(t) \right)
\]
where \( \mu \) is a real-valued Borel measure satisfying (7.2), \( Q \) is a polynomial with real coefficients whose degree is not greater than \( q+1 \), and \( B \) is the Blaschke product formed by the zeros of \( h \).

Since \( g_2 = h/g_1 \) we have

\[
g_2(z) = B_2(z) e^{i(Q(z)-P_1(z))} \exp \left\{ \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{(1+tz)^q}{(t-z)(1+t^2)^q} d(\mu - \nu_1)(t) \right\},
\]

where \( B_2 := B/B_1 \) is the Blaschke product formed by the zeros of \( g_2 \). This implies that \( g_2 \) has also the desired representation. □

**Proof of Lemma 4:** We have

\[
|f(z)| \leq \frac{1}{\pi} \int_{-\infty}^{\infty} |P_\phi(z,t)| d|\nu|(t).
\]

Without loss of generality we can assume \( r > 1 \). By inequality (6.12) we have

\[
|P_\phi(re^{i\theta})| \leq C_q \frac{\sin \theta}{(t^2 + r^2 - 2tr \cos \theta)(1 + |t|)q-1}.
\]

Hence

\[
\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \\
\leq C_q r^{q+1} \int_0^\pi \sin^2 \theta \left\{ \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{(t^2 + r^2 - 2tr \cos \theta)(1 + |t|)q-1} \right\} d\theta \\
= C_q r^{q+1} \int_{-\infty}^{\infty} \frac{1}{(1 + |t|)q-1} \left\{ \int_0^\pi \frac{\sin^2 \theta}{r^2 + t^2 - 2rt \cos \theta} d\theta \right\} d|\nu|(t).
\]

By equality (4.9) we get

\[
\int_0^\pi |f(re^{i\theta})| \sin \theta d\theta \\
\leq C_q r^{q+1} \int_{|t|\leq r} \frac{1}{(1 + |t|)q-1 t^2} d|\nu|(t) + \\
+ C_q r^{q+1} \int_{|t|> r} \frac{1}{(1 + |t|)q-1 t^2} d|\nu|(t) \\
\leq C_q r^{q+1} \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{(1 + |t|)q+1}.
\]
which implies (7.7). □

**Proof of Lemma 5:** We have

\[
\int_{-\infty}^{\infty} \frac{|f(x + iy)|}{1 + |x|^{q+2}} \, dx \leq \int_{-\infty}^{\infty} \frac{1}{1 + |x|^{q+2}} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} |P_q(z, t)| \, d|\nu|(t) \right\} \, dx
\]

\[
= \int_{-\infty}^{\infty} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} |P_q(z, t)| \, d|\nu|(t) \right\} \, dx
\]

By inequality (6.12), we have for \(0 < y < K\)

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|P_q(z, t)|}{1 + |x|^{q+2}} \, dx \leq \frac{C_q}{(1 + |t|)^{q-1}} \cdot \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(|x| + K)^q}{1 + |x|^{q+2}} \cdot \frac{y}{(x - t)^2 + y^2} \, dx
\]

\[
\leq \frac{C_{q,K}}{(1 + |t|)^{q-1}} \left\{ \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1 + x^2} \cdot \frac{y}{(x - t)^2 + y^2} \, dx \right\}
\]

Using equality (3.7), we get

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|P_q(z, t)|}{1 + |x|^{q+2}} \, dx \leq \frac{C_{q,K}}{(1 + |t|)^{q+1}}.
\]

Hence

\[
\int_{-\infty}^{\infty} \frac{|f(x + iy)|}{1 + |x|^{q+2}} \, dx \leq C_{q,K} \int_{-\infty}^{\infty} \frac{d|\nu|(t)}{(1 + |t|)^{q+1}} \leq C_{q,K}. \quad \square
\]
Bibliography


