STABILITY ROBUSTNESS OF LINEAR SYSTEMS: A FIELD OF VALUES APPROACH

A THESIS

SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Karim Saadaoui

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. A. Bülent Özgüler (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. M. Erol Sezer

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Mefharet Kocatepe

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ömer Morgül

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

STABILITY ROBUSTNESS OF LINEAR SYSTEMS: A FIELD OF VALUES APPROACH

Karim Saadaoui
M.S. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. A. Bülent Özgüler
August 1997

One active area of research in stability robustness of linear time invariant systems is concerned with stability of matrix polytopes. Various structured real parametric uncertainties can be modeled by a family of matrices consisting of a convex hull of a finite number of known matrices, the matrix polytope. An interval matrix family consisting of matrices whose entries can assume any values in given intervals are special types of matrix polytopes and it models a commonly encountered parametric uncertainty. Results that allow the inference of the stability of the whole polytope from stability of a finite number of elements of the polytope are of interest. Deriving such results is known to be difficult and few results of sufficient generality exist.

In this thesis, a survey of results pertaining to robust Hurwitz and Schur stability of matrix polytopes and interval matrices are given. A seemingly new tool, the field of values, and its elementary properties are used to recover most results available in the literature and to obtain some new results. Some easily obtained facts through the field of values approach are as follows. Polytopes with normal vertex matrices turn out to be Hurwitz and Schur stable if and only if the vertex matrices are Hurwitz and Schur stable, respectively. If the polytope contains the transpose of each vertex matrix, Hurwitz stability of the symmetric part of the vertices is necessary and sufficient for the Hurwiz stability of the polytope. If the polytope is nonnegative and the symmetric part of each vertex matrix is Schur stable, then the polytope is also stable. For polytopes with spectral vertex matrices, Schur stability of vertices is necessary and sufficient for the Schur stability of the polytope.

Keywords : Robust stability, Structured parametric uncertainties, Matrix polytopes, Field of values.

ÖZET

DOĞRUSAL SİSTEMLERİN GÜRBÜZ KARARLILIĞI: DEĞERLER ALANI YAKLAŞIMI

Karim Saadaoui
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. Dr. A. Bülent Özgüler
Ağustos 1997

Doğrusal sistemlerin gürbüz kararlılığı ile ilgili en aktif araştırma alanlarından birisi de matris politoplarının kararlılığıdır. Çeşitli yapısı belirli gerçel parametrik belirsizlikler, sonlu sayıda matrislerin konveks kombinasyonları yani bir matris politopu olarak modellenebilir. Aralık matris ailesi denilen, her elemanı verilen bir aralıkta herhangi bir değeri alabilen, matrisler kümesi politop matris ailesinin özel bir halidir. Bu matris ailesi belirsiz parametrelerin modellenmesinde sık kullanılırlar. Sonlu sayıda matris elemanının kararlılığından tüm politopun kararlılığını çıkarmaya imkan veren sonuçlar ilgi uyandırmaktadır. Ancak bu türden sonuçlara ulaşmak zordur ve literatürde yeterli genellikte bu türden az sayıda sonuç vardır.

Bu tezde, matris politoplarının ve aralık matris ailelerinin kararlılığına ilişkin literatürde yer alan sonuçları sıraladıktan sonra, yeni bir yöntem olduğuna inandığımız, değer alanları yöntemini kullanarak hem literatürde yer alan bir çok sonucu hem de bazı yeni sonuçları kolayca elde edeceğiz. Değer alanları yöntemi ve bazı basit özellikleri ile elde edilen sonuçlardan bazıları şunlardır. Köşeleri normal matrisler olan politoplar eğer ve ancak köşeleri sırasıyla Hurwitz ve Schur kararlılı ise Hurwitz ve Schur kararlıdırlar. Eğer politop köşe matrislerinin evriklerini de içeriyorsa, o zaman politop eğer ve ancak köşelerinin simetrik kısımları Hurwitz kararlı ise Hurwitz kararlıdır. Elemanları negatif olmayan bir politop, eğer köşelerinin simetrik kısımları Schur kararlı ise Schur kararlıdır. Köşeleri spektral matrisler olan politoplar eğer ve ancak köşeleri Schur kararlılı ise Schur kararlıdırlar.

Anahtar Kelimeler: Gürbüz kararlılık, Yapısı belirli gerçel parametrik belirsizlikler, Matris politopları, Değer alanları.

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To my family.

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Chapter 1

Introduction

A systematic study of any real engineering system requires a mathematical model. The precision of the model is the main factor in the accuracy of any prediction of the future behavior of the real system and in the success of any technique used for affecting a desired behavior pattern for the real system. Since modeling is always done by neglecting some external or even internal factors influencing the real system, the uncertainty in the model parameters is an essential aspect of any type of mathematical model of engineering systems. In addition to model inaccuracies, uncertainties may also arise due to changes in operating conditions, aging, maintenance induced errors, and others. Hence, in analyzing realistic engineering systems, a fixed mathematical model usually leads to limited conclusions on the behavior of the underlying system. This applies even stronger to the mathematical models used for less deterministic real systems such as the models used for economic, biological, or sociological systems.

Stability is one of the fundamental issues in the analysis, design, and performance evaluation of control systems. Hence, it is of great interest to analyze the stability of a system where uncertainties about a nominal (usually linear) model are taken into account. This is known as the stability robustness problem, where robustness of stability is to be ensured for a class of perturbations about the nominal model. Although, a perturbational approach to stability is still a very limited way of handling uncertainties in other fields, in the field of engineering where the model inaccuracies are usually not too gross it is quite an effective way of handling model uncertainties.

Depending on the type of the mathematical nominal model, the techniques used for robust stability analysis vary. If the nominal model is an input-output model, which is a transfer matrix in the case of linear time invariant (LTI) systems, then a frequency domain robust stability technique may be used. If the nominal model is a state space model, which is a linear matrix differential or difference equation in the case of LTI systems, then a time domain robust stability technique may be used.

Among the frequency domain robust stability techniques, in addition to classical method of gain and phase margins studied via Bode or Nyquist plots, one can also mention various methods of analyzing the stability of a family of polynomials. Since the stability of a linear system is studied via the stability of its denominator polynomial, and since uncertainties are reflected as uncertainties on the coefficients of this polynomial, the studies of the stability of a family of polynomials has direct relevance to robust stability. Mainly motivated by the paper of Kharitonov [1], the family of polynomials approach to frequency domain robust stability has received considerable attention in recent years. The time domain robust stability analysis techniques, on the other hand, can be broadly classified under three main approaches divided by assumptions concerning the nature of perturbations. These are unstructured, structured, and parametric perturbation techniques.

In what follows, we give a brief overview of the recent robust stability analysis techniques and their main achievements for LTI systems from a feedback control application viewpoint. We emphasize that the main focus of attention is stability analysis and many important synthesis oriented approaches to model uncertainties such as H_∞-optimization and μ-synthesis techniques are left out as they deserve special attention.

A common uncertain input-output model of a scalar LTI system is an uncertain transfer function

$$\frac{n(s,q)}{d(s,q)}, \tag{1.1}$$

where $n(s,q), d(s,q)$ are real polynomials of the complex variable s with coefficients which are functions of the uncertain parameter vector $q \in \mathbf{R}^k$. The vector q takes values in an uncertainty bounding set Q. The two boundary cases of a certain model and an entirely uncertain model (no knowledge of the real system except that it is linear and time invariant) are represented by Q being a point in $\mathbf{R}^k$ and the whole of $\mathbf{R}^k$, respectively. A usually sufficient model of uncertainty is obtained when each coefficient of the denominator polynomial d depends on at most one component of q and Q is a hyper-rectangle (box) in $\mathbf{R}^k$. In this case, the family of polynomials $d(s,Q) := \{d(s,q);\ q \in Q\}$ has the interval polynomial family representation

$$d(s,Q) = \{d(s,q) = \sum_{i=0}^{l} d_i s^i;\ d_i^- \le d_i \le d_i^+\}, \tag{1.2}$$

for some real numbers $d_i^-, d_i^+; i = 1, ..., l$. Kharitonov showed in this case that Hurwitz stability of four specially constructed extreme polynomials is both necessary and sufficient for Hurwitz stability of $d(s,Q)$ and hence for the stability of all continuous-time systems represented by the uncertain model (1.1).

This result has an immediate application to feedback stability of uncertain systems. Suppose the numerator polynomial $n(s,q)$ also has an uncertainty structure similar to (1.2). A constant output gain $g \in \mathbf{R}$ stabilizes all continuous-time systems represented by (1.1) if and only if all closed-loop denominator polynomials $d(s,q) + g\,n(s,q)$, $q \in Q$ are Hurwitz stable. Kharitonov result applied to this new family of polynomials then yields that stability by the gain g is achieved if and only if g is a common stabilizing gain for four distinguished transfer functions obtained in a similar way to the four distinguished Kharitonov polynomials. The advantage of this result is that the requirement that g stabilizes an infinite number of transfer functions is reduced to the requirement on a finite number of transfer functions. The disadvantage is that the problem of simultaneously stabilizing even two transfer functions by a constant gain is still a very difficult problem (if one desires to state conditions on the transfer functions for the existence of such a common stabilizing gain).

The book [2] contains an extensive discussion on the control oriented applications and various extensions of the Kharitonov's result to other types of uncertainty structures. Among these the powerful results by Rantzer can be singled out as they resolve many issues concerning the extension and application of Kharitonov-like results and, via the concept of convex directions, considers the problem of simultaneous stabilization of two transfer functions by a common gain.

The robust stability techniques for families of polynomials can also be applied to state space uncertainty structures by analyzing the uncertain characteristic polynomial. However, variation in state space parameters often does not give a model whose characteristic polynomial has coefficients varying within a nice uncertainty bounding set, such as a polytope. It is usually more realistic to consider the stability robustness problem through a time domain approach. A common state space model of an uncertain LTI system is in one of the forms

$$\dot{x}(t) = (A + A_u)x(t),\ t \in \mathbf{R};\ x(k+1) = (A + A_u)x(k),\ k \in \mathbf{Z}$$

depending on whether the underlying system is continuous or discrete time. Here, $x(.) \in \mathbf{R}^n$ is the state of the system, $A \in \mathbf{R}^{n\times n}$ is a known nominal system matrix, and $A_u \in \mathbf{R}^{n\times n}$ is an uncertain or perturbation matrix. We now consider the three main approaches to studying robust stability in time domain marked by their assumptions on the perturbation matrix A_u. We consider continuous time systems.

(i) *Unstructured Perturbations*: In this approach, A is assumed Hurwitz stable and no further assumptions on A_u is made. Thus, every entry of A_u can vary independently and the objective is to find a bound on some induced norm of A_u or on its elements that guarantee Hurwitz stability of the overall system. In the literature, one finds almost every result concerning bounds on matrix norms exploited by this approach.

(ii) *Structured Perturbations*: The matrix A is assumed Hurwitz stable and the perturbation model structure is partially known. Bounds on such perturbations are tried to be obtained. Because the structure of perturbations are known, less conservative results are expected. For instance A_u may be of the form $A_u = BKC$

for B,C known matrices and K an uncertain matrix with free entries. Note that for this model of perturbations, stability robustness problem is equivalent to robustness of a stabilizing constant output feedback.

We refer the reader to the thesis [3] for a survey of results obtained for both Hurwitz and Schur stability via various techniques used for (i) and (ii). The stability radius approach of [4] and the μ-analysis approach of [5] have received wide attention in handling structured perturbations.

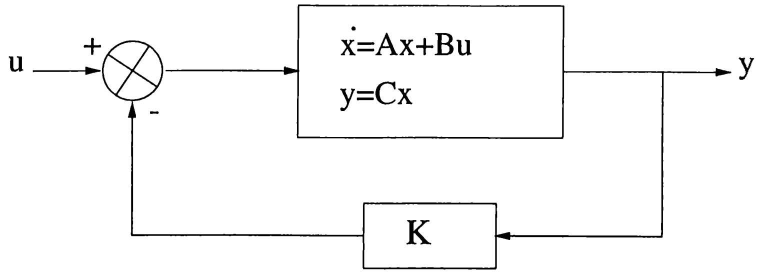

Figure 1.1: Hurwitz stability robustness problem under constant output feedback.

A typical result of these two approaches can be illustrated on the very simple system

$$\dot{x}(t) = (-1 + q)x(t).$$

The method of unstructured or structured perturbations will tell us that this system is Hurwitz stable if $|q| < 1$. The limitation of the approaches is easily seen, the values $q \leq -1$ for which the system is still stable are disregarded. In the parametric perturbations approach this drawback can partially be compensated by treating q as a parameter with a priorily given bounds.

(iii) *Parametric Perturbations*:

$$A_u = \sum_{i=1}^{m} q_i E_i.$$

Here perturbation model structure is known. E_i's are known constant matrices and q_i's are unknown real parameters either free or taking values in given intervals. In the latter case the system matrix $A + A_u$ takes values in a family of matrices. The aim is to either obtain bounds for the parameters that maintain stability or, in the case of a family of matrices, to obtain Kharitonov-like statements, i.e., to identify some (preferably one) distinguished matrices the stability of which imply the stability of the whole class of uncertain systems.

The powerful tools of Lyapunov theory have been widely used in handling all three types of uncertainties in the control literature. See [6] for a recent comprehensive study of structured perturbations via Lyapunov theory and [2] for applications of Lyapunov theory to polytopes of matrices.

The time domain techniques have the following type of application in feedback control. Suppose the matrix A is a stable nominal closed-loop system matrix resulting by the application of a state or output feedback on a nominal open-loop system. The stability robustness bounds obtained for A_u then give confidence regions in which the closed-loop system obtained by this particular feedback (and the particular input matrix) will continue to remain stable in the face of variations in the parameters.

Although, in our brief overview above we found it convenient to classify various robust stability techniques under separate headings, it is clear that there are no clear boundaries, the results obtained by any one of the above techniques find applications in the others.

In this thesis, we give a survey of those results obtained by the above approaches that we consider to be relevant to the robust stability of families of matrices. In chapter 2, we give the main robust Hurwitz stability results obtained for families of polynomials, in particular Kharitonov theorem, Edge theorem, and Rantzer growth condition. Chapter 3 is devoted to a survey of the existing results on the robust stability of polytopes of matrices. We present some approaches used in proving Hurwitz stability of a matrix polytope. In chapter 4, we introduce the field of values concept which proves to be an effective tool for addressing the stability of polytopes of matrices. Through the field of values, we recover most existing results proved using different approaches. We also obtain some new results for both continuous and discrete time systems. Finally, we conclude by some remarks and future research possibilities.

Notation:

The field of real and complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. If $c \in \mathbf{C}$, then $\bar{c}$ denotes the complex conjugate of c, Re(c) the real part, Im(c) the imaginary part, and $|c|$ the magnitude of c. The angle θ of a complex number $c = |c|e^{j\theta}$ is denoted by $\angle c$. Given a matrix $A \in \mathbf{C}^{n\times m}$, A' denotes the transpose of A and A^* denotes the complex conjugate transpose of A. For a square A, $\sigma(A)$ stands for the set of eigenvalues called the spectrum of A. For a real $n \times m$ matrix $A = [a_{ij}]$, $|A|$ denotes the nonnegative matrix $[|a_{ij}|]$. For the notation, terminology, and for various unproved elementary facts concerning vector norms and induced matrix norms used in this thesis, we refer the reader to [7], [8], [9] and [10] .

The set of points in the open left half complex plane and the open unit disk are denoted by $\mathbf{C}_-$ and $\mathbf{D}$, respectively. A polynomial $p(s)$ is said to be Hurwitz (Schur) stable if all its roots lie in $\mathbf{C}_-$ ($\mathbf{D}$). Given a family of polynomials $\mathcal{P} = \{p(.,q);\ q \in Q\}$ with Q some subset of $\mathbf{R}^k$, we say that $\mathcal{P}$ is robustly Hurwitz stable if all the members of $\mathcal{P}$ are Hurwitz stable. If all the polynomials in $\mathcal{P}$ have the same degree we say that $\mathcal{P}$ has invariant degree. A fixed matrix $A \in \mathbf{R}^{n\times n}$ is said to be Hurwitz (Schur) stable if all its eigenvalues lie in $\mathbf{C}_-$ ($\mathbf{D}$). Given a matrix family $\mathcal{A} = \{A(q) : q \in Q\}$ we say that $\mathcal{A}$ is robustly Hurwitz (Schur) stable if all its members are Hurwitz (Schur) stable.

Chapter 2

Families of Polynomials

A general family of polynomials has the description

$$d(s,Q) = \{d(s,q) = \sum_{i=0}^{n} d_i(q)s^i;\ q \in Q\}, \tag{2.1}$$

where $Q \subseteq \mathbf{R}^k$ is an uncertainty bounding set. The degree of an uncertain polynomial $d(s,q)$ is the highest power of s with a nonzero coefficient. The family (2.1) is said to have invariant degree n if all uncertain polynomials in the family have degree n. Clearly, $d(s,Q)$ has invariant degree n if and only if $d_n(q) \neq 0$ for all $q \in Q$.

The uncertainty bounding set is usually taken to be a ball with respect to some norm in $\mathbf{R}^k$. Three usual choices for norms are l^∞, l^1, and l^2 defined by

$$||q||_\infty := \max_i |q_i|,\ ||q||_1 := \sum_{i=1}^{k} |q_i|,\ ||q||_2 = (\sum_{i=1}^{k} q_i^2)^{\frac{1}{2}}.$$

The balls in these norms are referred to as a *box, diamond*, and *sphere*, respectively. For instance, a ball in l^∞ with center q^* is given by $||q - q^*||_\infty \leq 1$ and such a box can be described via componentwise bounds

$$Q = \{q \in \mathbf{R}^k;\ q_i^- \leq q_i \leq q_i^+ \ for\ i = 1, 2, \ldots, k\}$$

where q_i^- and q_i^+ are some lower and upper bounds of q_i. Weighted versions of these norms can also be used in the description of an uncertainty bounding set.

In this chapter, we give a detailed description of available robust stability results obtained for the case where Q is a box referring the reader to [2] for a survey of the results obtained for the other two cases.

The family of polynomials (2.1) is said to have an affine linear uncertainty structure if each coefficient function $d_i(q)$ is an affine linear function of q; i.e, for each $i \in \{0, 1, \ldots, n\}$ there exists a vector $\alpha_i \in \mathbf{R}^k$ and a scalar $\beta_i \in \mathbf{R}$ such that

$$d_i(q) = \alpha_i' q + \beta_i.$$

As an example, $d_i(q) = 2q_1 + 8q_2 - 6q_3 + 1$ is affine linear.

A family of polynomials $\mathcal{P} = \{p(., q);\ q \in Q\}$ is said to be a polytope of polynomials if $p(s, q)$ has an affine linear uncertainty structure and Q is a polytope, i.e., $Q = conv\{q^i\}$ a convex hull of a finite number of points $\{q^i\}$ in $\mathbf{R}^k$. In this case, we call $p(s, q^i)$ the i-th generator of $\mathcal{P}$. For a polytope of polynomials $\mathcal{P}$, its exposed edges are obtained from exposed edges of Q. We call such polynomials edges of $\mathcal{P}$. Note that every polynomial in the family $\mathcal{P} = \{p(., q) : q \in Q\}$ can be expressed as a convex combination of the generators $p(s, q^i)$, i.e, $\mathcal{P} = conv\{p(s, q^i)\}$ which justifies calling $\mathcal{P}$ a polytope of polynomials. For example, the family of polynomials

$$\mathcal{P} = \{p(., q) : q \in Q\} \text{ with } p(s, q) = s^2 + (4q_1 + 3q_2 + 2)s + (2q_1 - q_2 + 5)$$

$|q_1| \leq 1$ and $|q_2| \leq 1$ is a polytope of polynomials. The uncertainty bounding set Q has four extremes $q^1 = (-1, -1)$, $q^2 = (-1, 1)$, $q^3 = (1, -1)$ and $q^4 = (1, 1)$ the four associated generators are given by

$$\begin{aligned} p(s, q^1) &= s^2 - 5s + 4, \\ p(s, q^2) &= s^2 + s + 2, \\ p(s, q^3) &= s^2 + 3s + 8, \\ p(s, q^4) &= s^2 + 9s + 6. \end{aligned}$$

Given $q^* = (0, 0)$ in the uncertainty bounding set, the corresponding polynomial

$p(s, q^*)$ can be expressed as a convex combination of the generators:

$$\begin{aligned} p(s,q^*) &= s^2 + 2s + 5 \\ &= 0.25p(s,q^1) + 0.25p(s,q^2) + 0.25p(s,q^3) + 0.25p(s,q^4) \\ &= \sum_{i=1}^{4} \lambda_i p(s,q^i) \end{aligned}$$

with $\lambda_i \geq 0$, $\sum_{i=1}^{4} \lambda_i = 1$.

2.1 Interval polynomial family and Kharitonov theorem

A special case of affine linear uncertainty structure is an independent linear uncertainty structure. In this case, each component q_i of q enters into only one coefficient. For example, the uncertain polynomial

$$p(s,q) = s^2 + (6 + 3q_1 + q_2)s + (5 + q_3 + 6q_4)$$

has independent uncertainty structure. Usually these uncertainties are lumped to have simply

$$p(s,q) = \sum_{i=0}^{n} q_i s^i.$$

A family of polynomials $\mathcal{P} = \{p(.,q) : q \in Q\}$ is said to be an interval polynomial family if $p(s,q)$ has an independent uncertainty structure, each coefficient depends continuously on q and Q is a box. A convenient notation for an interval polynomial family is

$$p(s,q) = \sum_{i=0}^{n} [q_i^-, q_i^+] s^i. \tag{2.2}$$

It is easy to see that the interval polynomial family is a polytope of polynomials with generators

$$p(s,g) := \sum_{i=0}^{n} g_i s^i,$$

where g_i is equal to one of q_i^- or q_i^+.

Given the interval family (2.2), we extract four distinguished members

$$K_1(s) = q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \dots$$

$$K_2(s) = q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \dots$$

$$K_3(s) = q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \dots$$

$$K_4(s) = q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \dots$$

which are referred to as Kharitonov polynomials.

Theorem 2.1 *[1] Let $\mathcal{P}$ be an interval polynomial family with invariant degree. Then, $\mathcal{P}$ is robustly Hurwitz stable if and only if its associated four Kharitonov polynomials $K_i(s), i = 1, 2, 3, 4$ are Hurwitz stable.*

The power of Kharitonov theorem is derived from the fact that we can determine whether $\mathcal{P}$ is robustly Hurwitz stable by checking the stability of only four fixed polynomials irrespective of the degree of the family of polynomials.

One proof of Kharitonov theorem uses the concept of a value set associated with a family of polynomials and the increasing angle property for Hurwitz polynomials. For alternative proofs, we refer the reader to [11].

The increasing angle (phase) property [12] is the following. *Given any Hurwitz polynomial $p(s)$ of degree n, its angle*

$$\angle p(j\omega)$$

monotonically increases from 0 to $n\pi/2$ as ω increases from 0 to ∞.

The value set at frequency ω_0 of a family of polynomials (2.1) is defined to be the set of all possible values $d(j\omega_0, q)$ assumes as q varies within Q, i.e., the value set of $d(s, Q)$ is given by

$$d(j\omega_0, Q) := \{d(j\omega_0, q);\ q \in Q\}.$$

It is well known that the zeros of a polynomial $d(s, q)$ depend continuously upon its coefficients. If the coefficients are continuous functions of q, then the zeros of

$d(s,q)$ also depend continuously on q. If Q is a "nice" set, then one can easily derive a useful condition in terms of the value set for the stability of the family of polynomials. We state this result for a polytope of polynomials. *Suppose a polytope of polynomials $\mathcal{P}$ has invariant degree and has at least one Hurwitz stable member $p(s,q^*)$. Then, $\mathcal{P}$ is robustly stable if and only if the zero exclusion condition*

$$0 \notin p(j\omega, Q) \; \forall\, \omega \geq 0$$

holds. The value set at any fixed $\omega_0 \in \mathbf{R}$ for the interval polynomial family turns out to be a rectangle with sides parallel to the axes and with its four corners determined by the Kharitonov polynomials, i.e., the four corners are the points $K_i(j\omega_0)$ $i = 1, \cdots, 4$ as shown in Figure 2.1.

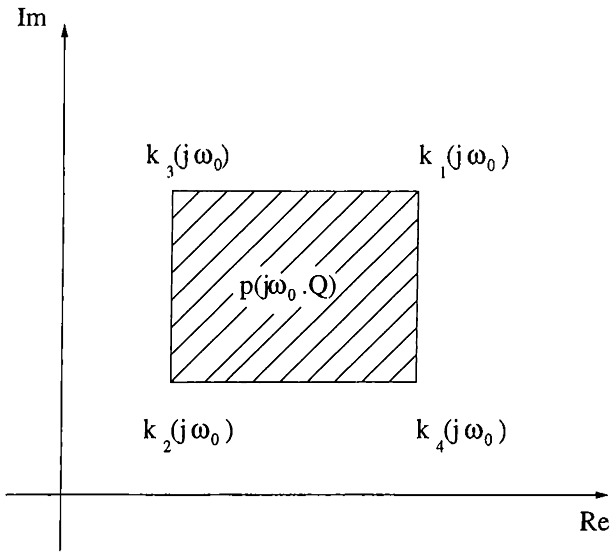

Figure 2.1: A Kharitonov rectangle at a frequency $\omega_0 > 0$.

Evaluating the Kharitonov polynomials at $\omega = 0$, it can be seen that at the zero frequency, the value set degenerates into the interval $[q_0^-, q_0^+]$. If the four Kharitonov polynomials are stable, then the value set at $\omega = 0$ excludes the origin. Suppose now that the interval family contains an unstable polynomial while the Kharitonov polynomials are stable. The zero exclusion condition implies that the rectangle at some ω_1 contains the origin. The continuous motion of the corners with respect to ω gives that, one boundary of the rectangle at some $\omega_2 \in [0, \omega_1]$ includes the origin. Using the increasing angle property for the two stable vertex polynomials of the boundary that includes the origin, it is easy to see that the rectangle at $\omega_2 + \epsilon$ for some $\epsilon > 0$ has no longer sides parallel to the axes, i.e., no longer a value set. This contradiction proves that the stability of the four Kharitonov polynomials imply the stability of the whole interval family.

2.2 Polytopes of polynomials and the edge theorem

The geometric ideas developed for proving Kharitonov theorem carry over to the more general framework of polytopes of polynomials. If $\mathcal{P} = \{p(.,q) : q \in Q\}$ is a polytope of polynomials, then the value set $p(j\omega_0, Q)$ at frequency ω_0 is the polygon on the complex plane with generating set $\{p(j\omega_0, q^i)\}$. The edge theorem tells us the following: When affine linear uncertainty structures are used, Hurwitz stability of the corresponding polytope of polynomials can be ascertained by checking Hurwitz stability of all polynomials associated with edges of Q.

Theorem 2.2 *[13] Let $\mathcal{P}$ be a polytope of polynomials with invariant degree. Then $\mathcal{P}$ is robustly Hurwitz stable if and only if each of the edges of $\mathcal{P}$ are Hurwitz stable.*

Hence, by working with edges, robust Hurwitz stability problem for polytopes of polynomials is reduced to a finite number of one dimensional edge problems which can be solved by classical methods. For example, if q^1 and q^2 are two extreme points of Q, then robust Hurwitz stability test reduces to finding the roots of the polynomial

$$p_{1,2}(s, \lambda) = (1 - \lambda)p(s, q^1) + \lambda p(s, q^2)$$

for $\lambda \in [0, 1]$. Dividing by $\lambda p(s, q^2)$, it becomes clear that the problem reduces to the classical root locus plot of the fictitious plant

$$p_{1,2}(s) = \frac{p(s, q^1)}{p(s, q^2)}$$

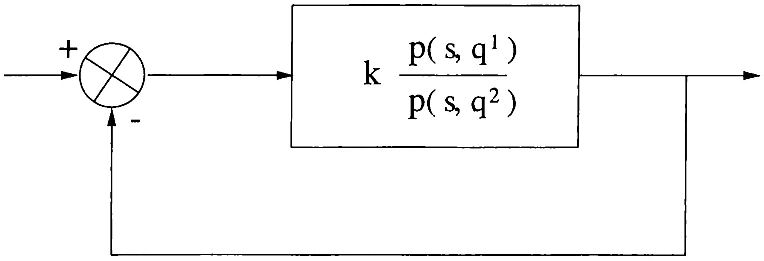

Figure 2.2: Fictitious plant for the solution of an edge problem.

which is compensated via unity feedback.

We close this section by a remark concerning Schur stability of an interval polynomial family. In spite of the above strong result in the case of Hurwitz stability, no similar result for Schur stability exists for a general interval polynomial family. The difficulty is partly explained by a result of Rantzer [2]. For a large class of stability regions $\mathcal{D}$ in the complex plane, Rantzer showed that a Kharitonov-like result exists provided both the stability region and its reciprocal $\mathcal{D}^{-1} := \{z \in \mathcal{C};\ zd = 1,\ d \in \mathcal{D}\}$ are convex. The fact that the open unit disk $\mathbf{D}$ does not have this property is consistent with the lack of Kharitonov-like results for Schur stability.

2.3 Rantzer growth condition

In the preceding section we saw that robust Hurwitz stability of polytopes of polynomials can be ascertained from Hurwitz stability of the edges. In view of this edge type result, we want to have conditions under which Hurwitz stability of the extremes imply Hurwitz stability of the edge. Now we concentrate on one parameter problem. Given $f(s)$ and $g_1(s)$ fixed polynomials, we consider the family $\mathcal{P}$ described by

$$p(s,\lambda) = (1-\lambda)f(s) + \lambda g_1(s)$$

with $\lambda \in [0,1]$. We want to get conditions under which Hurwitz stability of the extremes $p(s,0) = f(s)$, $p(s,1) = g_1(s)$ implies robust Hurwitz stability of the

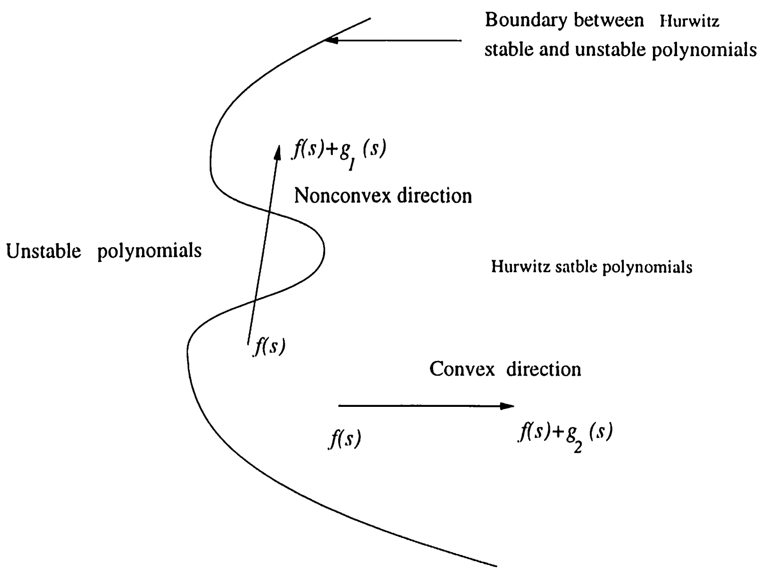

Figure 2.3: Convex directions in the space of polynomials.

family $\mathcal{P}$. Note that the above problem is equivalent to the following one

$$\begin{aligned} p(s,\lambda) &= (1-\lambda)f(s) + \lambda g_1(s) \\ &= f(s) + \lambda(-f(s) + g_1(s)) \\ &= f(s) + \lambda g(s) \end{aligned}$$

where $g(s) = -f(s) + g_1(s)$ so the problem can be restated as, given $f(s)$ and $g(s)$ which define the family

$$p(s,\lambda) = f(s) + \lambda g(s)$$

$\lambda \in [0,1]$, we want to get conditions under which Hurwitz stability of the extremes $p(s,0) = f(s)$, $p(s,1) = f(s)+g(s)$ implies robust Hurwitz stability of the family $\mathcal{P}$. This problem was partially solved by Rantzer [14] using the concept of convex directions. A monic polynomial $g(s)$ is said to be a convex direction (for the space of n^{th} order polynomials) if the following condition is satisfied: *Given any Hurwitz stable n^{th} order polynomial $f(s)$ such that $f(s) + g(s)$ is also Hurwitz stable and $deg(f(s) + \lambda g(s)) = n$ for all $\lambda \in [0,1]$, it follows that $f(s) + \lambda g(s)$ is Hurwitz stable for all $\lambda \in [0,1]$.* The concept of convex direction is depicted graphically in Figure 2.3. From the figure we see that $g_2(s)$ is a convex direction because $f(s) + \lambda g_2(s)$ remains within the stable set for all $\lambda \geq 0$. On the other hand $g_1(s)$ is not a convex direction.

From the above discussion, we see that Hurwitz stability of an edge can be ascertained by Hurwitz stability of the extremes if $g(s)$ is a convex direction. An important paper by Rantzer [14] provides conditions under which $g(s)$ is a convex direction.

Theorem 2.3 *[14] A polynomial $g(s)$ is a convex direction for the space of Hurwitz stable n^{th} order polynomials if and only if*

$$\frac{d}{d\omega} \angle g(j\omega) \leq \left| \frac{\sin 2\angle g(j\omega)}{2\omega} \right|$$

for all $\omega > 0$ such that $g(jw) \neq 0$.

An example of a non-convex direction is the following: Consider $f(s) = 10s^3 + s^2 + 6s + 0.57$ and $g(s) = s^2 + 2s + 1$. It is easy to check that $f(s)$ and $f(s) + g(s)$ are Hurwitz stable, $f(s) + \lambda g(s)$ has a constant degree for all $\lambda \in [0, 1]$ but $f(s) + \lambda g(s)$ for $\lambda = 0.5$ is unstable. The theorem above can be used to obtain classes of polynomials which are convex directions. For instance, it can easily be derived by Theorem 2.3 that all odd polynomials and all even polynomials are convex directions for Hurwitz stable polynomials.

Chapter 3

Polytopes of Matrices

Motivated by strong results obtained for uncertain polynomials, one would like to obtain similar results for uncertain matrices. Actually, such results are needed to address the robust stability of linear systems in state space representation. Consider a continuous-time, unforced system in state space representation

$$\dot{x}(t) = A(q)x(t), \tag{3.1}$$

where q is an uncertainty vector taking values in an uncertainty bounding set Q. The robust Hurwitz stability of this system is achieved if eigenvalues of $A(q)$ lie in $\mathbf{C}_-$ for all values of the uncertainty vector q. One possible approach to study robust stability is to examine the characteristic polynomial

$$det(sI - A(q))$$

of (3.1). However, the uncertainties in the elements of the matrix are reflected in a complicated, nonlinear way to the characteristic polynomial which makes this type of analysis inefficient. Hence, in many cases, it may be advantageous to work directly with $A(q)$.

In Section 3.1 below, we first define the particular uncertainty structures to be considered. In Sections 3.2-3.5, various alternative methods of studying the robust stability problem for these uncertainty structures are surveyed.

3.1 Matrix polytopes and interval matrices

We will focus our attention on a class of matrices known as the *matrix polytope* or *polytopic matrix family*:

$$\mathcal{A}_n = \{A \in \mathbf{R}^{n\times n};\ A = \alpha_1 E_1 + \alpha_2 E_2 + \cdots + \alpha_N E_N,\ \alpha_i \geq 0,\ \sum_{i=1}^{N} \alpha_i = 1\}. \quad (3.2)$$

The uncertainty vector is $q = [\alpha_1 ... \alpha_N]'$ and every entry of an uncertain matrix A depends linearly on q. The uncertainty bounding set Q is a box $Q = \{q = [\alpha_1 ... \alpha_N]';\ \alpha_i \in [0,1]\}$. The matrices E_i, $i = 1, ..., N$ are called the *vertex matrices.* Note that

$$\mathcal{A}_n = conv(E_1, ..., E_N).$$

The main motivation for considering matrix polytopes comes from the robust stability studies of matrices with structured perturbations. A common way of representing structured perturbations about a nominal matrix A_0 is to write a perturbed matrix A in the form

$$A = A_0 + q_1 A_1 + \cdots + q_k A_k, \quad (3.3)$$

where the matrices A_k's represent "directions" of perturbations and the parameters $q_1, q_2, \cdots, q_k \in \mathbf{R}$ take their values in a hyper-rectangular region Ω defined by

$$\Omega = \{q \in \mathbf{R}^k;\ \underline{q}_i \leq q_i \leq \bar{q}_i,\ i = 1, 2, \cdots, k\}.$$

It is well known [2] that by the affine transformation (3.3) the region Ω is mapped to a polytope of dimension k in $\mathbf{R}^{n\times n}$ which can be described in terms of some vertex matrices $E_1, E_2, \cdots, E_N (N \leq 2^k)$ as in (3.2). Consequently, stability of matrices with structured perturbations can be studied via robust stability of a matrix polytope.

Another widely studied uncertainty structure is that of an *interval matrix family*:

$$A_I = \{A \in \mathbf{R}^{n\times n};\ L \leq A \leq K,\ L, K \in \mathbf{R}^{n\times n}\}, \tag{3.4}$$

where the inequality sign applies entrywise. More explicitly, if $L = [l_{ij}]$, $K = [k_{ij}]$, then

$$A_I = \{A = [a_{ij}] \in \mathbf{R}^{n\times n};\ l_{ij} \leq a_{ij} \leq k_{ij},\ \forall\, i, j = 1, ..., n\}.$$

Sometimes an interval matrix family is denoted by specifying the intervals that the entries lie in, e.g., a 2×2 interval matrix family is described by

$$A_I = \begin{bmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] \\ [a_{21}^-, a_{21}^+] & [a_{22}^-, a_{22}^+] \end{bmatrix}. \tag{3.5}$$

From the definition we see that the setting for interval matrices is similar to that of interval polynomials. We also note that an element of A_I is a matrix with structured perturbations about a nominal matrix A_0 whose ij-th entry is the midpoint of the ij-th interval. The matrices A_k's are simply the standard basis matrices for $\mathbf{R}^{n\times n}$. Consequently, the interval matrix family is a special matrix polytope. To fix ideas, let us consider the following example which shows how an interval matrix corresponds to a matrix polytope.

Example. Consider for simplicity a 1×2 interval matrix

$$A_I = \begin{bmatrix} [a^-, a^+] & [b^-, b^+] \end{bmatrix}.$$

A typical element of A_I is $A = [a\ b]$, where $a = \alpha_1 a^- + \alpha_2 a^+$ and $b = \alpha_3 b^- + \alpha_4 b^+$ with $\alpha_1 + \alpha_2 = \alpha_3 + \alpha_4 = 1$. In terms of the vertex matrices

$$E_1 = \begin{bmatrix} a^- & b^- \end{bmatrix},\ E_2 = \begin{bmatrix} a^- & b^+ \end{bmatrix},\ E_3 = \begin{bmatrix} a^+ & b^+ \end{bmatrix},\ E_4 = \begin{bmatrix} a^+ & b^- \end{bmatrix},$$

we can write

$$A = \begin{cases} \alpha_3 E_1 + (\alpha_1 - \alpha_3)E_2 + \alpha_2 E_3, & \alpha_1 \geq \alpha_3, \\ \alpha_1 E_1 + (\alpha_3 - \alpha_1)E_4 + \alpha_4 E_3, & \alpha_1 \leq \alpha_3. \end{cases}$$

Since the coefficients are nonnegative and add up to 1 in both cases, we see that $A_I \subseteq conv(E_1, ..., E_4)$. The reverse inclusion is easier to see and we get $A_I = conv(E_1, ..., E_4)$. The situation is illustrated in Figure 3.1. •

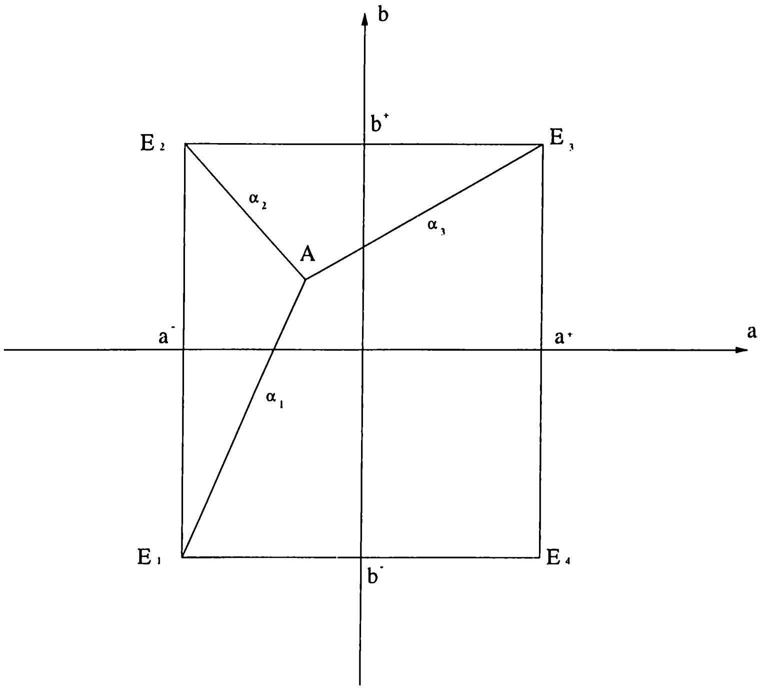

Figure 3.1: An example of a convex combination of vertex matrices, $A = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_3 E_3$ with $\alpha_1 = \frac{1}{3}$, $\alpha_2 = \frac{1}{3}$ and $\alpha_3 = \frac{1}{3}$.

The conclusion of this example can be generalized to $m \times n$ interval matrices. In particular, an interval matrix family (3.4) is equal to a polytopic matrix family (3.2) in which vertex matrices are taken as

$$E_v = [e_{ij}]; \;\; e_{ij} \in \{l_{ij}, k_{ij}\}, \;\; i,j = 1,2,\cdots,n, \;\; v = 1,2,\cdots,2^{n^2}. \tag{3.6}$$

3.2 Hurwitz stability of matrix polytopes

An immediate extension of polynomial results to matrix polytopes is possible. Suppose that $A(q)$ is in companion canonical form

$$A(q) = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \\ \vdots & & & & \\ 0 & 0 & 0 & & 1 \\ q_0 & q_1 & q_2 & \cdots & q_{n-1} \end{bmatrix} \tag{3.7}$$

and the uncertainty bounding set Q is a box as in the case of Kharitonov setup. We are thus considering an interval matrix family whose $n(n-1)$ entries are degenerate intervals, i.e., points. Then, we obtain a result similar to Kharitonov theorem: *$A(q)$ is robustly Hurwitz stable if and only if the four distinguished matrices are Hurwitz stable.* The four distinguished matrices mentioned are obtained from the Kharitonov polynomials associated with the interval polynomial family $\mathcal{P} = \{p(.,q), q \in Q\}$ with $p(s,q) = det(sI - A(q))$. Since the four distinguished matrices are vertex matrices, in the case of interval matrices in companion form, at most four vertex matrices need to be checked irrespective of n.

More generally, if q enters affine linearly into a single row or column of $A(q)$, then the characteristic polynomial $p(s,q)$ of $A(q)$ turns out to have an affine linear uncertainty structure and we can use many results, like edge theorem, obtained for polytopes of polynomials. Although there have been studies of a class of uncertain matrices having characteristic polynomials with affine linear uncertainty, see e.g. El Ghaoui [15], it is clear that such classes of matrices are rather special.

As pointed out by Wang [16], a matrix polytope $\mathcal{A}_n$ with *upper triangular vertex matrices* is Hurwitz stable if and only if the vertex matrices are Hurwitz stable. This is easily seen as the polytope will consist only of upper triangular matrices with eigenvalues the diagonal entries. As these eigenvalues are convex combinations of the diagonal element of the vertex matrices, the result follows.

Such results unfortunately are exceptions rather than the rule. In fact, for a general matrix polytope (3.2), one can easily construct a counterexample to the effect that extreme point results do not exist for matrix polytopes even at the level of 2×2 matrices. Consider a 2×2 matrix polytope $\mathcal{A}_2$ given by (3.2). Let $\mathcal{A}_2$ be a polytope of matrices with two vertices

$$E^1 = \begin{bmatrix} -1 & 1.5 \\ 0.6 & -1 \end{bmatrix}$$

$$E^2 = \begin{bmatrix} -1 & 0.5 \\ 4 & -4 \end{bmatrix}$$

which are both Hurwitz stable. However, the matrix

$$A = 0.7361E^1 + 0.2639E^2 = \begin{bmatrix} -1 & 1.236 \\ 1.497 & -1.791 \end{bmatrix}$$

which is a member of the polytope is not Hurwitz stable as it has an eigenvalue 0.021.

Actually, according to Cobb and DeMarco [17], while the stability of all *faces* of dimension $2n-4$ is sufficient to conclude the robust stability of $\mathcal{A}_n$ with $n \geq 3$, there are examples of unstable polytopes $\mathcal{A}_n$ for which all faces of dimension $2n-5$ are stable.

Also in the special case of interval matrices, the extreme point results cease to exist even for two vertex matrices. Historically, the first attempts for obtaining necessary and sufficient results for Hurwitz stability of a matrix polytope were due to Bialas [18]. He considered the special case of independent uncertainty structures, and tried to extend the results of Hurwitz stability of interval polynomials to Hurwitz stability of interval matrices. Bialas [18] claimed that the interval matrix family A_I is Hurwitz stable if and only if the vertex matrices are Hurwitz stable. However, Barmish and Hollot [19] have shown via a counterexample that Bialas condition is not sufficient. In fact, consider the set of 3×3

interval matrices A_I with

$$K = \begin{bmatrix} -0.5 & -12.06 & -0.06 \\ -0.25 & 0 & 1 \\ 0.25 & -4 & -1 \end{bmatrix},$$

$$L = \begin{bmatrix} -1.5 & -12.06 & -0.06 \\ -0.25 & 0 & 1 \\ 0.25 & -4 & -1 \end{bmatrix}.$$

It is easily verified that L and K are stable. According to the conjecture, the interval matrix A_I should be Hurwitz stable. A_I can be written as

$$A_I = \begin{bmatrix} -0.5 - r & -12.06 & -0.06 \\ -0.25 & 0 & 1 \\ 0.25 & -4 & -1 \end{bmatrix}$$

for any $r \in [0,1]$. Considering $r \in (0.5 - \sqrt{0.06}, 0.5 + \sqrt{0.06})$, the matrices obtained belong to A_I but they can be easily verified to be unstable. Hence, the conjecture of Bialas fails. Note that the uncertainty occurs only at one entry of the interval matrix.

In spite of such negative results, several authors have revealed that the result of Bialas will be correct if some assumptions are made on matrix polytopes.

Xin [20] considered interval matrices A_I defined in (3.4) with L and K such that

$$k_{ii} < 0 \quad i = 1, 2, \cdots, n; \quad l_{ij} \geq 0 \quad i \neq j, \quad i, j = 1, 2, \cdots, n.$$

He showed that A_I is Hurwitz stable if and only if the matrix K is Hurwitz stable.

Shi and Gao [21] have shown that if the set of interval matrices is restricted to be symmetric, then Hurwitz stability of the vertex matrices is necessary and sufficient to guarantee Hurwitz stability of the set of interval matrices.

Jiang [22] considered Hurwitz stability of the interval matrix A_I defined by (3.4). He showed that Hurwitz stability of the symmetric parts of vertex matrices is sufficient to conclude Hurwitz stability of the interval matrix A_I.

Soh [23] considered a polytope of symmetric interval matrices (3.2), i.e, E_i, $i = 1, \cdots, N$ are symmetric. He showed that this polytope is Hurwitz stable if and only if the vertex matrices are Hurwitz stable, using the fact that positively weighted sums of negative definite matrices are still negative definite.

Çevik [24], gave a rectangular bounding region in the complex plane for the eigenvalues of a matrix, see Figure 4.1, and have shown that a matrix polytope is Hurwitz stable provided the symmetric parts of the vertex matrices E_i, $i = 1, 2, \cdots, N$ are Hurwitz stable.

3.3 Gershgorin's theorem applied

Hurwitz stability of interval matrices can also be addressed using Gershgorin's theorem and its extensions, as they are useful in estimating eigenvalue locations of matrices. Chen [25], used this fact to establish a number of sufficient conditions given below for Hurwitz stability of interval matrices A_I of (3.4).

Gershgorin's theorem tells us that for an $n \times n$ matrix A, every eigenvalue λ must be in at least one of the circles described by

$$|\lambda - a_{ii}| \leq \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad i = 1, 2, \cdots, n. \tag{3.8}$$

For Hurwitz stability, we are interested in the real parts of eigenvalues. By (3.8), we can write

$$Re(\lambda) \leq a_{ii} + \sum_{j=1, j\neq i}^{n} |a_{ij}| \quad i = 1, 2, \cdots, n. \tag{3.9}$$

As eigenvalues are invariant under similarity transformation, inequalities in (3.9) can be tightened by using matrix scalings. Define

$$\mathcal{R} = \{diag\{r_1, r_2, \cdots, r_n\} : r_i > 0, i = 1, 2, \cdots, n\}. \tag{3.10}$$

Therefore, $\forall R \in \mathcal{R}$, $Re(\lambda)$ must satisfy

$$Re(\lambda) \leq a_{ii} + \sum_{j=1, j\neq i}^{n} \frac{r_j}{r_i}|a_{ij}| \quad i = 1, 2, \cdots, n. \tag{3.11}$$

Now, we consider the interval matrix family A_I of (3.4). Suppose that $k_{ii} < 0 \;\; \forall i$. Using (3.9), we get that A_I is Hurwitz stable if

$$k_{ii} + \sum_{j=1, j\neq i}^{n} \max\{|l_{ij}|, |k_{ij}|\} < 0, \;\; i = 1, \cdots, n. \tag{3.12}$$

A tighter condition can be obtained using (3.11): A_I is Hurwitz stable if there exists $R \in \mathcal{R}$ such that

$$k_{ii} + \sum_{j=1, j\neq i}^{n} \frac{r_j}{r_i} \max\{|l_{ij}|, |k_{ij}|\} < 0, \;\; i = 1, \cdots, n. \tag{3.13}$$

Upon defining a new matrix W_h by

$$W_h = [w_{ij}], w_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{\max\{|l_{ij}|, |k_{ij}|\}}{|k_{ii}|} & \text{if } i \neq j \end{cases} \tag{3.14}$$

and assuming that $k_{ii} < 0 \;\; \forall i$, the above two results respectively reduce to
(i) A_I is Hurwitz stable if $||W_h||_\infty < 1$.
(ii) A_I is Hurwitz stable if there exists an $R \in \mathcal{R}$ such that $||R^{-1}W_hR||_\infty < 1$.

As these conditions are obtained using Gershgorin's theorem, they suffer from an inherent shortcoming that all the endpoints k_{ii} are required to be negative $k_{ii} < 0$. An extension of Gershgorin's theorem allows us to overcome this limitation. Any interval matrix given by (3.4) can be written as

$$A_I = A_0 + E_I, \; A_0 := \frac{K+L}{2}, \; E_I := [-D, D], \; D := \frac{K-L}{2}. \tag{3.15}$$

Since $K \geq L$, D is a nonnegative matrix. Let T be the transformation such that $T^{-1}A_0T = \Lambda + U$. With $J = \Lambda + U$ the Jordan form of A_0, $\Lambda = diag[\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n]$, $\hat{\lambda}_i$ $\forall i$ being an eigenvalue of A_0. Given $A \in A_I$, it can be written as $A = A_0 + E$ with $E \in E_I$. Under similarity transformation

T and T^{-1}, $T^{-1}AT = \Lambda + U + T^{-1}ET$. Let $F = U + |T^{-1}|D|T| = [f_{ij}]$, it follows that every eigenvalue λ of a matrix A must be in one of the circles described by

$$|\lambda - \hat{\lambda}_i| \leq \sum_{j=1}^{n} f_{ij}, \; i = 1, 2, \cdots, n.$$

So the real part of each eigenvalue must satisfy one of the conditions

$$Re(\lambda) \leq Re(\hat{\lambda}_i) + \sum_{j=1}^{n} f_{ij}$$

for some $i = 1, 2, \cdots, n$. With matrix scaling by $R = diag\,\{r_1, ..., r_n\}$, we also get

$$Re(\lambda) \leq Re(\hat{\lambda}_i) + \sum_{j=1}^{n} \frac{r_j}{r_i} f_{ij}, \; i = 1, 2, \cdots, n.$$

Hence, $\forall A \in A_I$ written as in (3.15), we obtain the following results. Suppose that $Re(\hat{\lambda}_i) < 0 \;\; \forall i, \;\; A_I$ is Hurwitz stable if either of the following conditions hold:

$$\begin{array}{ll} \text{(i)} & Re(\hat{\lambda}_i) + \sum_{j=1}^{n} f_{ij} < 0, \quad i = 1, 2, \cdots, n, \\ \text{(ii)} & Re(\hat{\lambda}_i) + \sum_{j=1}^{n} \frac{r_j}{r_i} f_{ij} < 0, \quad i = 1, 2, \cdots, n. \end{array} \tag{3.16}$$

Alternatively, these results can be stated in a more compact form. Assuming $Re(\hat{\lambda}_i) < 0 \;\; \forall i$, define the matrix

$$\Gamma_h = [\gamma_{ij}], \; \gamma_{ij} := \frac{f_{ij}}{|Re(\hat{\lambda}_i)|}.$$

Then, A_I is Hurwitz stable if either of the following hold:
(i) $||\Gamma_h||_\infty < 1$.
(ii) there exists a diagonal nonsingular $R \in \mathbf{R}^{n \times n}$ such that $||R^{-1}\Gamma_h R||_\infty < 1$.

In some cases it is possible to conclude the Schur stability of the whole interval matrix family (3.4) from the stability of only one test matrix. Such a result requires rather strong assumptions on the family. A result of Sezer and Siljak [26] uses Gershgorin's theorem in obtaining the following result. The main assumption on the interval family is that it is "almost nonnegative".

A matrix $A \in \mathbf{R}^{n \times n}$ is called a Morishima matrix if there exists $S = diag\{s_1, ..., s_n\}$, $s_i = \pm 1$ such that $SAS = |A|$. An extreme vertex $\hat{E} = [\hat{e}_{ij}]$ of (3.4) is defined as a vertex matrix with entries satisfying

$$|\hat{e}_{ij}| = \max\{|l_{ij}|, |k_{ij}|\}.$$

Note that in general an extreme vertex may not belong to the family (3.4). However, in the case of $L \geq 0$ corresponding to the case of the whole family being nonnegative, for instance, the unique extreme vertex is $\hat{E} = K$. In [26], it has been shown that *if there exists an extreme vertex $\hat{E}$ which is a Morishima matrix, then the interval matrix family is stable if and only if $|\hat{E}|$ is stable.* If $L \geq 0$, this result implies that the interval matrix family is stable if and only if K is stable. A similar result for Hurwitz stability of (3.4) is also given in [26].

3.4 Lyapunov approach to interval matrices

The robust stability of the interval matrix family defined by (3.15) can be examined using the tools of Lyapunov stability theory. The main tool used is the following theorem of Lyapunov, see e.g. [27]: *A matrix $A \in \mathbf{R}^{n \times n}$ is Hurwitz stable if for some and only if for all symmetric positive definite $Q \in \mathbf{R}^{n \times n}$ a symmetric positive definite $P \in \mathbf{R}^{n \times n}$ exists satisfying the Lyapunov equation*

$$A'P + PA = -Q.$$

In Wang [28], the interval matrix family defined by (3.15) is considered. The following main result is obtained. *Let $A_0 := \frac{L+K}{2}$ be Hurwitz stable and let a positive definite matrix P determined by $PA_0 + A_0'P = -I$. If*

$$\max\{\|K - L\|_1, \|K - L\|_\infty\} < \frac{1}{||P||_\infty} \tag{3.17}$$

then the interval family A_I is Hurwitz stable.

The proof consists of showing that $\forall A \in A_I$ the trivial solution $x = 0$ of $\dot{x} = Ax$ is asymptotically stable. For this purpose, let $\Delta A := A - A_0 = A - \frac{L+K}{2}$.

Then,

$$|\Delta a_{ij}| = |a_{ij} - \frac{1}{2}(l_{ij} + k_{ij})| \leq \frac{1}{2}(k_{ij} - l_{ij})$$

which implies

$$||\Delta A||_1 \leq \frac{1}{2}||K - L||_1, \ ||\Delta A||_\infty \leq \frac{1}{2}||K - L||_\infty.$$

By (3.17), it follows that

$$\max\{||\Delta A||_1, ||\Delta A||_\infty\} < \frac{1}{2||P||_\infty}. \tag{3.18}$$

Now, for any $A \in A_I$, we have

$$PA + A'P = -I + P\Delta A + (\Delta A)'P$$

and in order to show that $PA + A'P$ is negative definite and hence by Lyapunov theorem A is Hurwitz stable, it suffices to show that the spectral radius $\rho(P\Delta A + (\Delta A)'P)$ is less than unity. This however follows by (3.18) and the fact that any eigenvalue $\lambda(A)$ of a matrix A satisfies $|\lambda(A)| \leq ||A||_\infty$, i.e.,

$$|\lambda(P\Delta A + (\Delta A)'P)| \leq ||P\Delta A + (\Delta A)'P||_\infty \leq 2||\Delta A||_\infty ||P||_\infty < 1.$$

We note that the proof hinges on finding a common positive definite matrix P that works for every element in the interval matrix family. This is a common feature of all Lyapunov approaches to robust stability of families of matrices.

Mansour [29] gave a simple proof of the result of Jiang [22] using Lyapunov theory and the fact that any member of the interval matrix can be written as a convex combination of vertex matrices (3.2). If the symmetric part $H(E_i)$ of each vertex matrix E_i is Hurwitz stable, then it is also negative definite. The symmetric part $\frac{A+A'}{2}$ of any A in the interval matrix family, being a convex combination of negative definite matrices, is also negative definite. Since $A + A'$ is negative definite, the Lyapunov equation is satisfied by $P = I$ and we easily conclude that A is Hurwitz stable. This yields a simple proof of the fact that if $H(E_i)$ are Hurwitz stable, then the interval matrix family $conv\{E_1, ..., E_n\}$ is robustly Hurwitz stable.

Mansour [30] further simplified this result by considering only a part of the extreme matrices. Consider a subset of the vertex matrices $\{E_\nu\}$ defined by

$$\tilde{E}_v = [e_{ij}],\ e_{ii} := k_{ii},\ e_{ij} := k_{ij} \text{ or } l_{ij}$$

for $i,j = 1,2,\cdots,n$ and $v = 1,2,\cdots,\frac{n(n-1)}{2}$. It is obvious that $\{E_v\} \supset \{\tilde{E}_v\}$. The interval matrix family A_I of (3.4) can be shown to be Hurwitz stable if the symmetric parts of $\tilde{E}_v, v = 1,2,\cdots,\frac{n(n-1)}{2}$ are Hurwitz stable. Thus, instead of checking Hurwitz stability of 2^{n^2} symmetric matrices, one needs to check only $2^{\frac{n(n-1)}{2}}$ symmetric matrices.

3.5 Copositive matrices

Using the fact that for a 2×2 matrix, Hurwitz stability is equivalent to positivity of the coefficients of the associated second order characteristic polynomial, we can obtain necessary and sufficient conditions for Hurwitz stability of a 2×2 matrix polytope in terms of the copositivity of an auxiliary matrix.

Let us consider (3.2) for $n = 2$. Let $\mathcal{A}_2$ have N vertex matrices and denote its k-th vertex matrix by

$$E^k = \begin{bmatrix} e_{11}^k & e_{12}^k \\ e_{21}^k & e_{22}^k \end{bmatrix}.$$

We construct a new $N \times N$ symmetric matrix $\hat{A} = [\hat{a}_{kl}]$ by

$$\hat{a}_{kl} := \frac{1}{2}[(e_{11}^k e_{22}^l - e_{21}^k e_{12}^l) + (e_{11}^l e_{22}^k - e_{21}^l e_{12}^k)].$$

For a polytope $\mathcal{A}_2$ with two vertex matrices E^1 and E^2, this matrix is

$$\hat{A} = \begin{bmatrix} det(E^1) & \frac{1}{2}\left(det \begin{bmatrix} e_{11}^1 & e_{12}^2 \\ e_{21}^1 & e_{22}^2 \end{bmatrix} + det \begin{bmatrix} e_{11}^2 & e_{12}^1 \\ e_{21}^2 & e_{22}^1 \end{bmatrix}\right) \\ \frac{1}{2}\left(det \begin{bmatrix} e_{11}^2 & e_{12}^1 \\ e_{21}^2 & e_{22}^1 \end{bmatrix} + det \begin{bmatrix} e_{11}^1 & e_{12}^2 \\ e_{21}^1 & e_{22}^2 \end{bmatrix}\right) & det(E^2) \end{bmatrix}$$

Definition. [31] A matrix $Q \in \mathbf{R}^{n \times n}$ is said to be strictly copositive if $\alpha' Q \alpha > 0$ for all nonzero $\alpha \in \mathbf{R}^n$ such that $\alpha_k \geq 0,\ k = 1, 2, \cdots, n$.

Fact 3.1 *A polytope of matrices $\mathcal{A}_2$ is robustly Hurwitz stable if and only if*
(i) trace $E^k < 0,\ k = 1, 2, \cdots, N$,
(ii) $\hat{A}$ is a strictly copositive matrix.

Proof. [Only if] Suppose that the polytope is Hurwitz stable. Then, $traceE^k < 0$ since $E^k,\ k = 1, 2, \cdots, N$ are members of the polytope and since the trace of a matrix is the sum of its eigenvalues. Moreover, each element $A = \sum_{k=1}^{N} \alpha_k E^k$ of $\mathcal{A}_2$, which is Hurwitz stable by hypothesis, must have its determinant strictly greater than zero. With

$$\alpha := \begin{bmatrix} \alpha_1 \\ \\ \alpha_N \end{bmatrix}$$

we have

$$\begin{aligned} det(A) &= (\sum_{k=1}^{N} \alpha_k e_{11}^k)(\sum_{k=1}^{N} \alpha_k e_{22}^k) - (\sum_{k=1}^{N} \alpha_k e_{21}^k)(\sum_{k=1}^{N} \alpha_k e_{12}^k) \\ &= [\alpha_1 \cdots \alpha_N] \hat{A} \begin{bmatrix} \alpha_1 \\ \\ \alpha_N \end{bmatrix} > 0 \end{aligned} \tag{3.19}$$

where $\sum_{k=1}^{N} \alpha_k = 1,\ \alpha_k \geq 0$. Now, if (**ii**) fails and $\beta' \hat{A} \beta \leq 0$ for some nonzero and nonnegative $\beta \in \mathbf{R}^N$, then $\alpha := \beta / \|\beta\|_1$ is such that $\sum_{k=1}^{N} \alpha_k = 1,\ \alpha_k \geq 0$ and the inequality in (3.19) also fails. Hence, conditions (**i**) and (**ii**) hold.
[If] Suppose that trace $E^k < 0,\ k = 1, 2, \cdots, n$ and $\hat{A}$ is a strictly copositive matrix. Given any $A \in \mathcal{A}_2$, it is of the form

$$A = \begin{bmatrix} \sum_{k=1}^{N} \alpha_k e_{11}^k & \sum_{k=1}^{N} \alpha_k e_{12}^k \\ \sum_{k=1}^{N} \alpha_k e_{21}^k & \sum_{k=1}^{N} \alpha_k e_{22}^k \end{bmatrix},$$

for some $\alpha_k \geq 0, \sum_{k=1}^{N} \alpha_k = 1$. Hence,

$$traceA = \sum_{k=1}^{N} \alpha_k (e_{11}^k + e_{22}^k) < 0.$$

Moreover, we have

$$det(A) \quad = \quad \alpha' \hat{A} \alpha > 0, \tag{3.20}$$

where the inequality is by strict copositiveness of $\hat{A}$. Therefore, the characteristic polynomial $s^2 - s\ traceA + det(A)$ of A has positive coefficients and A is Hurwitz stable. □

This result has an immediate application to a 2×2 interval matrix family recovering a result by [32].

Fact 3.2 *[32] A 2×2 interval matrix family A_I given by (3.5) is Hurwitz stable if and only if all the vertex matrices are Hurwitz stable.*

Proof. [Only if] This part is obvious as the vertex matrices are elements of A_I.

[If] If the vertex matrices are stable, then

$$traceE^k < 0, \;\; k = 1, \cdots, N$$

or, condition (**i**) of Fact 3.1 is satisfied. In addition, diagonal elements of $\hat{A}$ are determinants of vertex matrices and therefore positive. The crucial observation is that in the case of interval matrices, an off diagonal element

$$\hat{a}_{kl} := \frac{1}{2}[(e_{11}^k e_{22}^l - e_{21}^k e_{12}^l) + (e_{11}^l e_{22}^k - e_{21}^l e_{12}^k)], \; k \neq l$$

of the matrix $\hat{A}$ is the mean value of the determinants of two vertex matrices, i.e., the determinant

$$e_{11}^k e_{22}^l - e_{21}^k e_{12}^l = det \begin{bmatrix} e_{11}^k & e_{12}^l \\ e_{21}^k & e_{22}^l \end{bmatrix}$$

is the determinant of a vertex matrix and similarly for the other determinant that appears in the expression for $\hat{a}_{kl}$. It follows that

$$\hat{a}_{kl} > 0, \ \forall\, k, l = 1, ..., N$$

and hence $\hat{A}$ is a positive matrix. It immediately follows that $\hat{A}$ is strictly copositive. By Fact 3.1, A_I is robustly Hurwitz stable. □

For an $n \times n$ matrix polytope $\mathcal{A}_n$, Hurwitz stability of a matrix $A \in \mathcal{A}_n$ requires much more than positivity of the coefficients of the associated characteristic polynomial. A result by Qian and DeMarco [33] shows that, after transferring the robust stability problem into a robust nonsingularity problem, an extension of the concept of copositivity still yields a necessary and sufficient condition for the robust stability of a polytope of matrices.

Let us first review the main steps of transferring the stability problem to a nonsingularity problem through the use of Kronecker sums.

Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{p\times p}$. The Kronecker product of A and B [12], denoted by $A \otimes B$, is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & & a_{1n}B \\ \vdots & & \\ a_{n1}B & \cdots & a_{nn}B \end{bmatrix} \in \mathbf{R}^{np\times np}.$$

The Kronecker sum of A and B, denoted by $A \oplus B$, is defined as

$$A \oplus B = A \otimes I + I \otimes B \in \mathbf{R}^{np\times np}.$$

Let $\lambda_i(A) \in \sigma(A)$ denote an eigenvalue of A. Then,

$$\sigma(A \oplus B) = \{\lambda_i(A) + \lambda_j(B),\ i = 1, \cdots, n,\ j = 1, \cdots, p\}. \qquad (3.21)$$

Using (3.21), we are able to transform Hurwitz stability problem for the general polytope

$$\mathcal{A}_n = \{A \in \mathbf{R}^{n\times n};\ A = \alpha_1 E_1 + \alpha_2 E_2 + \cdots + \alpha_N E_N,\ \alpha_i \geq 0,\ \sum_{i=1}^{N} \alpha_i = 1\}$$

into an equivalent nonsingularity problem. Suppose the polytope contains at least one Hurwitz stable element A_s. If the polytope is nevertheless unstable, then there exists $A_{us} \in \mathcal{A}_n$ which has some roots in the closed right half complex plane. By the continuity of the eigenvalues of

$$A = \alpha_1 E_1 + \alpha_2 E_2 + \cdots + \alpha_N E_N$$

with respect to $\alpha = [\alpha_1 \dots \alpha_N]'$, it follows that there also exists an element in $\mathcal{A}_n$ with at least one imaginary eigenvalue. We have thus shown that, given a Hurwitz stable element in $\mathcal{A}_n$, a necessary and sufficient condition for $\mathcal{A}_n$ to be robustly Hurwitz stable is that $\forall A \in \mathcal{A}_n$, A has no imaginary eigenvalues. Suppose then that $\mathcal{A}_n$ has at least one stable element A_s, and define

$$\tilde{\mathcal{A}}_{n^2} = \{\tilde{A} \in \mathbf{R}^{n^2 \times n^2},\ \tilde{A} = A \oplus A,\ \text{where } A \in \mathcal{A}_n\} \tag{3.22}$$

Using (3.21), it easily follows that $\mathcal{A}_n$ is robustly Hurwitz stable if and only if $\tilde{\mathcal{A}}_{n^2}$ is robustly nonsingular. The following simple fact is also easy to prove.

Fact 3.3 *Suppose that $\tilde{A}_{ns} \in \tilde{\mathcal{A}}_{n^2}$ is nonsingular. Then, $\tilde{\mathcal{A}}_{n^2}$ is robustly nonsingular if and only if*

$$\gamma det\tilde{A} > 0 \textit{ for all } \ \tilde{A} \in \tilde{\mathcal{A}}_{n^2}$$

where $\gamma = sign(det\tilde{A}_{ns})$.

Now we note that (3.22) can be written as

$$\tilde{\mathcal{A}}_{n^2} = \{\tilde{A} \in \mathbf{R}^{n^2 \times n^2},\ \tilde{A} = \sum_{i=1}^{N} \alpha_i \tilde{E}_i,\ \text{where } \tilde{E}_i = E_i \oplus E_i, \alpha \in \Gamma\},$$

where

$$\Gamma := \{\alpha \in \mathbf{R}^N;\ \alpha_i \geq 0,\ \text{and } \sum_{i=1}^{N} \alpha_i = 1\},$$

i.e., the set of nonnegative vectors of unit l_1-norm. Define for every $\alpha \in \Gamma$ the polynomial

$$\begin{aligned} p(\alpha) &:= \gamma det\tilde{A} \\ &= \sum_{i_1+\cdots+i_N=l} a_{i_1 i_2 \cdots i_N} \alpha_1^{i_1} \alpha_2^{i_2} \cdots \alpha_N^{i_N}. \end{aligned}$$

Such a polynomial is called a homogeneous multivariable polynomial of order l (size of $\tilde{A}$) and dimension N. The homogeneous polynomial is said to be strictly copositive if

$$p(\alpha) > 0 \text{ for all } \alpha \in \Gamma.$$

Fact 3.4 *The matrix polytope $\mathcal{A}_n$ is robustly Hurwitz stable if and only if $\gamma det(\tilde{A})$ is strictly copositive for all $\tilde{A} \in \tilde{\mathcal{A}}_{n^2}$.*

It is easy to see that for $\mathcal{A}_2$, Fact 3.4 reduces to the conditions given in Fact 3.1, where strict copositivity of the matrix $\hat{A}$ has to be checked. The above result of [33] can thus be considered as a generalization of the 2×2 result described. As expected, checking copositivity or giving a simple characterization of copositive matrices or polynomials is not an easy task. Some useful necessary or sufficient conditions however still result from this approach, [33], [31].

Chapter 4

Stability of Matrix Polytopes Through the Field of Values Approach

We have seen that simple necessary and sufficient conditions for the stability of matrix polytopes are difficult to obtain. Such results have been obtained only for special polytopes of matrices, namely for polytopes with symmetric, normal, upper-lower triangular vertex matrices. In this approach, Hurwitz stability of the entire family of matrices is proved by establishing Hurwitz stability of test matrices which are usually the vertex matrices of the polytope or matrices generated from them such as their symmetric parts.

In this chapter, we employ the concept of the field of values associated with a matrix to obtain conditions for the Hurwitz and Schur stability of matrix polytopes. The reader is referred to the book [34] for an excellent exposure to various properties of the field of values and their applications. In Section 4.1, we give a summary of those properties relevant to the stability of matrix polytopes. In

Section 4.2, the field of values of the matrix polytope under consideration is examined. Sections 4.3 and 4.4 are devoted to the application of the concept of field of values to Hurwitz and Schur stability of matrix polytopes, respectively.

4.1 Some properties of the field of values

This section contains the definition and a summary of the properties of field of values. For a more in-depth discussion and for the proofs [34] can be consulted.

Definition 4.1 *The* **field of values** *of* $A \in \mathbf{R}^{n\times n}$ *is*

$$F(A) = \{x^*Ax;\ x \in \mathbf{C}^n,\ x^*x = 1\}.$$

Thus, $F(A)$ is a set of complex numbers associated with a given matrix $A \in \mathbf{R}^{n\times n}$. Alternatively, $F(.)$ can be viewed as a function from $\mathbf{R}^{n\times n}$ to the complex plane like the spectrum $\sigma(.)$, the set of eigenvalues of A.

By considering the unit eigenvectors associated with each eigenvalue of A, it immediately follows that

$$\sigma(A) \subseteq F(A). \tag{4.1}$$

A fundamental property of $F(A)$, known as the Toeplitz-Hausdorff theorem, is that it is a (compact and) convex subset of the complex plane. It follows that any information on the location and the shape of this convex set can be used to bound the eigenvalues. For matrices of size 2, the field of values is always an ellipse (possibly degenerate) with eigenvalues at the foci. When the size of the matrix is larger than 2 however, a variety of shapes are possible in general.

A useful measure of the size of $F(A)$ is the radius of the smallest disc centered at the origin of the complex plane that contains $F(A)$.

Definition 4.2 *The* **numerical radius** *of* $A \in \mathbf{R}^{n\times n}$ *is*

$$r(A) = \max\{|z| :\ z \in F(A)\}.$$

Recall that the **spectral radius** of $A \in \mathbf{R}^{n\times n}$ is the nonnegative real number

$$\rho(A) = \max\{|\lambda| : \ \lambda \in \sigma(A)\},$$

which is the radius of the smallest disc centered at the origin in the complex plane that includes all eigenvalues of A. In view of (4.1), we have

$$\rho(A) \leq r(A)$$

for any $A \in \mathbf{R}^{n\times n}$.

Given $A \in \mathbf{R}^{n\times n}$, let $H(A)$ and $S(A)$ denote the symmetric and the skew-symmetric part of A, i.e.,

$$H(A) := \frac{A+A'}{2}, \ \ S(A) := \frac{A-A'}{2}.$$

For any $x \in \mathbf{C}^n$ such that $x^*x = 1$, we have

$$\begin{aligned} x^*H(A)x &= \frac{1}{2}(x^*Ax + x^*A'x) \\ &= \frac{1}{2}(x^*Ax + (x^*Ax)^*) \\ &= \frac{1}{2}(x^*Ax + \overline{x^*Ax}) \\ &= Re(x^*Ax). \end{aligned}$$

Moreover,

$$\begin{aligned} x^*S(A)x &= \frac{1}{2}(x^*Ax - x^*A'x) \\ &= \frac{1}{2}(x^*Ax - (x^*Ax)^*) \\ &= \frac{1}{2}(x^*Ax - \overline{x^*Ax}) \\ &= j\, Im(x^*Ax). \end{aligned}$$

We thus obtain the following property.

Property 4.1 (Projection) *For $A \in \mathbf{R}^{n\times n}$ with symmetric part $H(A)$ and skew-symmetric part $S(A)$*

$$F(H(A)) = Re(F(A)) := \{Re(z); \ z \in F(A)\},$$

$$F(S(A)) = j\, Im(F(A)) := \{j\, Im(z); \ z \in F(A)\}.$$

It is easy to see that for any symmetric A, $F(A)$ is the closed interval on the real axis with end points $\lambda_{min}(A)$ and $\lambda_{max}(A)$. Similarly, for any skew-symmetric A, $F(A)$ is the closed interval on the imaginary axis with endpoints $j\,\lambda_{min}(A)$ and $j\,\lambda_{max}(A)$. It follows that

$$F(H(A)) = [\lambda_{min}(H(A)), \lambda_{max}(H(A))],$$

$$F(S(A)) = [j\,\lambda_{min}(S(A)), j\,\lambda_{max}(S(A))].$$

The property 4.1 thus gives another bound for the location of the field of values: *For any matrix A, $F(A)$ is contained in a rectangle in the complex plane with vertical sides going through the smallest and the largest eigenvalues of $H(A)$ and with horizontal sides going through the smallest and the largest eigenvalues of $-j\,S(A)$.*

The two regions, one circular and one rectangular, in which $F(A)$ is inscribed are shown in Figure 4.1.

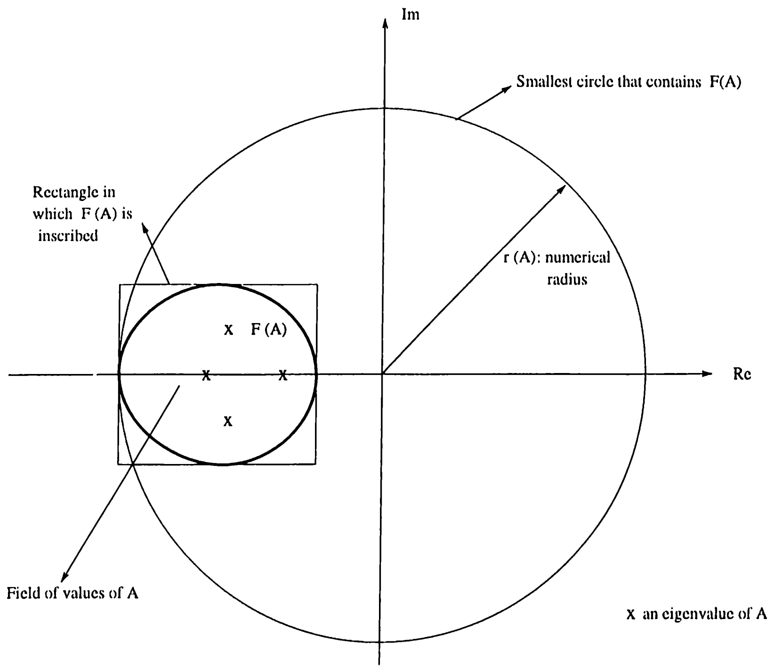

Figure 4.1: The field of values of a matrix A.

Another property of the field of values is its invariance under unitary similarity transformations, obtained as an easy consequence of its definition.

Property 4.2 (Unitary similarity invariance) *For all $A \in \mathbf{R}^{n\times n}$, $U \in \mathbf{C}^{n\times n}$ with U unitary*

$$F(U^*AU) = F(A).$$

The unitary similarity invariance property allows us to get a good description of the field of values of a normal matrix. Recall that if A is a normal matrix, then it is unitarily congruent to a diagonal matrix having its eigenvalues as diagonal entries. The field of values of a diagonal matrix, on the other hand, can easily be seen to be a polygon in the complex plane having the diagonal elements at its vertices. This yields the following property.

Property 4.3 (Normality) *If $A \in \mathbf{R}^{n\times n}$ is normal, then*

$$F(A) = conv(\sigma(A)) := \{\sum_{i=1}^{n} \alpha_i\lambda_i;\ \alpha_i \geq 0,\ \sum_{i=1}^{n} \alpha_i = 1,\ \lambda_i \in \sigma(A)\}.$$

A simple bound on the numerical radius is easily obtained on noting that

$$r(A) = \max_{||x||_2=1} |x^*Ax| \leq \max_{||x||_2=1} ||Ax||_2||x||_2 = ||A||_2.$$

Hence for any $A \in \mathbf{R}^{n\times n}$

$$r(A) \leq ||A||_2. \tag{4.2}$$

In the case of l_1 and l_∞ induced norms, a similar inequality to (4.2) is not possible. However, it can be shown that (see Corollary 1.5.4 in [34])

$$r(A) \leq \frac{1}{2}(||A||_1 + ||A||_\infty). \tag{4.3}$$

For nonnegative matrices, better bounds on the numerical radius are possible. A real matrix A is called **nonnegative** if every entry of A is nonnegative. Recall that if A is nonnegative, then the spectral radius $\rho(A)$ is an eigenvalue of A. If A is nonnegative, then so is $H(A)$. By the fact that $F(H(A))$ is an interval on the real axis with $\lambda_{max}(H(A))$ being the rightmost endpoint, it follows that $r(H(A)) = \rho(H(A))$. On the other hand, for any $x \in \mathbf{C}^n$ and nonnegative $A = [a_{ij}]$, we have

$$|x^*Ax| = |\sum_i \sum_j a_{ij}\bar{x}_i x_j| \leq \sum_i \sum_j a_{ij}|x_i|\,|x_j|$$

so that $r(A) \leq \max\{x'Ax;\ x \in \mathbf{R}^n,\ x_i \geq 0,\ x'x = 1\} = \max\{x'H(A)x;\ x \in \mathbf{R}^n,\ x_i \geq 0,\ x'x = 1\} = \rho(H(A))$. Moreover, by property 4.1, it is easily seen that $r(H(A)) \leq r(A)$. We thus arrive at the following property of the field of values of nonnegative matrices.

Property 4.4 *If $A \in \mathbf{R}^{n\times n}$ is nonnegative, then*

$$r(A) = r(H(A)) = \rho(H(A)).$$

Let

$$|A| := [|a_{ij}|],$$

i.e., $|A|$ is the matrix whose elements are the absolute values of the elements of A. Clearly, for any $x \in \mathbf{C}^n$ and any $A = [a_{ij}]$, we have

$$|x^*Ax| = |\sum_i \sum_j a_{ij}\bar{x}_i x_j| \leq \sum_i \sum_j |a_{ij}|\,|x_i|\,|x_j|$$

so that $r(A) \leq r(|A|)$. By property 4.4, we get the following bound of the numerical radius of any matrix $A \in \mathbf{R}^{n\times n}$

$$\rho(A) \leq r(A) \leq r(|A|) = \rho(H(|A|)). \tag{4.4}$$

4.2 The field of values of matrix polytopes

We now turn to our main objective of examining the stability of

$$\mathcal{A}_n = \{A = \sum_{i=1}^{N} \alpha_i E_i \in \mathbf{R}^{n\times n};\ \alpha_i \geq 0,\ \sum_{i=1}^{N} \alpha_i = 1\}. \tag{4.5}$$

Using the definition of $F(A)$, we easily obtain the inclusion

$$F(A) \subseteq conv(F(E_1) \cup \ldots \cup F(E_N)) \tag{4.6}$$

for any $A \in \mathcal{A}_n$. The reverse inclusion holds only under very special circumstances such as $n = 1$ or $E_i = e_i I$ for $e_i \in \mathbf{R}$, $i = 1, \ldots, N$. An immediate consequence of

(4.6) is the following inequality for the numerical radii:

$$r(A) \leq \sum_{i=1}^{N} \alpha_i r(E_i), \quad \forall\, A \in \mathcal{A}_n. \tag{4.7}$$

Similarly, by (4.6) and by 4.1, we have that

$$\max\{Re(z);\ z \in F(A)\} \leq \max_i\{\beta;\ \beta \in F(H(E_i))\}. \tag{4.8}$$

4.3 Hurwitz stability of matrix polytopes

Property 4.1 and its consequence (4.8) explain why the Hurwitz stability of symmetric parts of vertex matrices is sufficient to conclude the Hurwitz stability of a matrix polytope. In fact, if the symmetric parts of E_i, $i = 1, \cdots, N$ are Hurwitz stable, then $F(H(E_i))$ is contained in the negative real axis for every $i = 1, ..., N$. It follows by (4.8) that $\max\{Re(z);\ z \in F(A)\} < 0$ for every $A \in \mathcal{A}_n$. Therefore, the matrix polytope is Hurwitz stable. Hence, using the projection property of the field of values, we recover the results of Jiang [22], Soh [23], Shi and Gao [21] and Çevik [24].

If the symmetric part of a matrix is Hurwitz stable, then the matrix is Hurwitz stable by property 4.1. The converse of this statement is not generally true but turns out to be true for normal matrices.

Fact 4.1 *Given a normal matrix $A \in \mathbf{R}^{n\times n}$ with symmetric part $H(A)$, A is Hurwitz stable if and only if $H(A)$ is Hurwitz stable.*

Proof. By property 4.3, the field of values of a normal matrix is a polygon with vertices determined by the eigenvalues of the matrix. It follows by the projection property 4.1 that the field of values of the symmetric part is contained in the negative real axis if and only if all eigenvalues are contained in $\mathbf{C}_-$. $\square$

The following fact recovers the result by Wang [16].

Fact 4.2 *The polytope of matrices $\mathcal{A}_n$ defined by (4.5) with normal vertex matrices is Hurwitz stable if and only if the vertex matrices are Hurwitz stable.*

Proof. Since the vertices are in the polytope, the [only if] part is trivial. To see the [if] part, suppose the vertices are stable. Since they are normal, by Fact 4.1, their symmetric parts are also stable. The result now follows by (4.8) or by the discussion at the beginning of this section. □

The result concerning symmetric parts of the vertex matrices can be stated in the form of a necessary and sufficient condition for Hurwitz stability of the matrix polytope.

Fact 4.3 *Suppose $\mathcal{A}_n$ has the property that $E_i' \in \mathcal{A}_n$, $i = 1, \cdots, N$. Then, $\mathcal{A}_n$ is Hurwitz stable if and only if $H(E_i)$, $i = 1, \cdots, N$ are Hurwitz stable.*

Proof. [Only if] Since $E_i, E_i' \in \mathcal{A}_n$, $H(E_i) = \frac{E_i + E_i'}{2} \in \mathcal{A}_n$.
If $\mathcal{A}_n$ is Hurwitz stable then $H(E_i)$ is Hurwitz stable for $i = 1, \cdots, N$.

[If] This follows easily by property 4.1 and its consequence (4.8) as pointed out at the beginning of this section. □

Fact 4.3 is applicable to interval matrices. A particular case in which the above assumption holds for interval matrices is the following.

Fact 4.4 *The interval matrix family*

$$A_I = [[a_{ij}^-, a_{ij}^+]]$$

with the additional property

$$a_{ij}^- = a_{ji}^-, \;\; a_{ij}^+ = a_{ji}^+, \;\; i, j = 1, \cdots, n. \tag{4.9}$$

is Hurwitz stable if and only if the symmetric parts of the vertex matrices are Hurwitz stable.

Proof. With the above property (4.9), a matrix E_i is a vertex matrix if and only if E_i' is a vertex matrix. Consequently, Fact 4.3 applies. □

An example of interval matrix for which the result holds is

$$A_I = \begin{bmatrix} [a_{11}^-, a_{11}^+] & [a_{12}^-, a_{12}^+] & [a_{13}^-, a_{13}^+] \\ [a_{12}^-, a_{12}^+] & [a_{22}^-, a_{22}^+] & [a_{23}^-, a_{23}^+] \\ [a_{13}^-, a_{13}^+] & [a_{23}^-, a_{23}^+] & [a_{33}^-, a_{33}^+] \end{bmatrix}.$$

A similar type of interval matrix was considered before by Shi and Gao [21] and they proved a similar result. However in their development, they assumed that $a_{ij} = a_{ji},\ i,j = 1,\cdots,N$ resulting in a symmetric polytope with symmetric vertex matrices and therefore a result identical to that of Soh [23] was obtained. By relaxing their assumption in Fact 4.4, we are no longer restricted to symmetric polytopes and yet the same result holds.

4.4 Schur stability of matrix polytopes

When discrete time systems are considered, stability is no more a requirement on the eigenvalues to be in the open left half plane. Instead, they are required to be in the open unit disk **D** of the complex plane. Consequently, in general robust stability results obtained in the continuous time case are not expected to hold in discrete time context. However, conditions obtained by Wang [16] and Soh [23] hold in both cases. In what follows, we will try to explain why these two results hold for Schur stability using field of values arguments.

The unitary similarity invariance property 4.2, allows us to determine the numerical radius of a normal matrix.

Fact 4.5 *If $A \in \mathbf{R}^{n\times n}$ is normal, then*

$$r(A) = \rho(A).$$

Proof. Since $F(A) = conv(\sigma(A))$ by property 4.3, the result is immediate from the geometry of the field of values. $\square$

Now consider the polytope of matrices $\mathcal{A}_n$ given in (4.5), with the assumption that E_i's are normal. Note that this includes the case of E_i's being symmetric.

Fact 4.6 *Let $\mathcal{A}_n$ be such that every vertex matrix E_i is normal. Then, $\mathcal{A}_n$ is Schur stable if and only if the vertex matrices are Schur stable.*

Proof. As the [Only if] part is obvious, let us prove the [If] part using the numerical radius. Since $E_i,\ i = 1, \cdots, N$ are normal and Schur stable, using Fact 4.5, we get $r(E_i) < 1,\ i = 1, \cdots, N$. The inequality (4.7) now gives for any $A \in \mathcal{A}_n$ that $r(A) < 1$. Hence the polytope $\mathcal{A}_n$ is Schur stable. $\square$

The previous vertex result can be generalized. A matrix $A \in \mathbf{R}^{n\times n}$ is called **spectral** if $\rho(A) = r(A)$. In view of Fact 4.5, normal matrices are spectral. However, the converse is true only in the case $n = 2$. The following result is immediate.

Fact 4.7 *Let $\mathcal{A}_n$ be such that every vertex matrix E_i is spectral. Then, $\mathcal{A}_n$ is Schur stable if and only if the vertex matrices are Schur stable.*

The norm bounds (4.2) and (4.3) on the numerical radius give the following result.

Fact 4.8 *Given a polytope $\mathcal{A}_n$ in (4.5), suppose that*

$$\min\{\|E_i\|_2, \frac{1}{2}(\|E_i\|_1 + \|E_i\|_\infty)\} < 1, \tag{4.10}$$

for every $i = 1, \cdots, N$. Then, $\mathcal{A}_n$ is Schur stable if and only if the vertex matrices are Schur stable.

Proof. If the vertex matrices satisfy (4.10), then by (4.2) or (4.3), we have that $r(E_i) < 1$ for every vertex matrix. By (4.7), it follows that for every $A \in \mathcal{A}_n$, the inequality $r(A) < 1$ holds and $F(A) \subset \mathbf{D}$. $\square$

The part of this result for l_2 induced norm was obtained by Mori and Kokame [35] for the special case of interval matrix families.

Let $A \in \mathbf{R}^{n\times n}$ be a nonnegative matrix. Upon using property 4.4 we obtain the following sufficient condition for Schur stability of a nonnegative matrix polytope.

Fact 4.9 *Given a nonnegative matrix polytope $\mathcal{A}_n$ defined by (4.5), if $H(E_i)$, $i = 1, \cdots, N$ are Schur stable then $\mathcal{A}_n$ is Schur stable.*

Proof. If $H(E_i)$, $i = 1, \cdots, N$ are Schur stable then $r(E_i) < 1$, $i = 1, \cdots, N$ by property 4.4. The result follows by (4.7). □

The inequality (4.4) gives a slight generalization of this fact.

Fact 4.10 *Given a matrix polytope $\mathcal{A}_n$ defined by (4.5), if $H(|E_i|)$, $i = 1, \cdots, N$ are Schur stable, then $\mathcal{A}_n$ is Schur stable.*

Proof. The result follows by (4.4) and (4.7). □

We remark before closing this chapter that all the results obtained in this chapter apply to complex matrix polytopes as well.

Chapter 5

Conclusions

We have examined the robust stability of matrix polytopes. For the polytope $\mathcal{A}_2$, in Fact 3.1, we gave a necessary and sufficient condition in terms of the copositivity of an auxiliary symmetric matrix via a brute-force approach to the problem. For higher dimensional polytopes, we demonstrated that the elementary properties of the field of values directly yield many existing results in the literature and some others such as Facts 4.3, 4.8, 4.9, 4.10 that may be new.

We have not yet fully exploited all properties of the field of values that have applications in robust stability of matrix polytopes and more general results through this approach may be obtained. A limitation of the approach is clear. Like the Gershgorin circles, the field of values also yield regions in the complex plane where the eigenvalues lie in. The field of values like Gershgorin circles can not capture a full information on the spectrum. Unlike the Gershgorin's theorem or its extensions, however, there are stronger links between the type of the matrix and the field of values as witnessed by the properties listed in Chapter 4.

The polytope of matrices with normal vertex matrices seems to be the most general family for which Hurwitz stability of vertex matrices is both necessary and sufficient for stability of the polytope. On the other hand, for an interval matrix in companion canonical form (3.7), we have a vertex result through the application of Kharitonov theorem although this matrix is not in the above mentioned family. A closer investigation is hence necessary to enlarge the class of polytopes for which a vertex result is possible.

Finally, as pointed out in [2], construction of parametric Lyapunov functions for matrix polytopes is a research direction not yet fully exploited. A field of values approach to parametric Lyapunov functions seems also possible in view of some results in [34].

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