INVERSE OPTIMAL CONTROL AND POSITIVE REAL SYSTEMS

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL AND ELECTRONICS ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By

YILMAZ ÜNAL

July, 1997

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By tarafından bağışlanmıştır

Yılmaz ÜNAL

July 1997

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Prof. Dr. A. Bülent ÖZGÜLER (Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Erol SEZER

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

INVERSE OPTIMAL CONTROL AND POSITIVE REAL SYSTEMS

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M.S. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. A. Bülent ÖZGÜLER
July 1997

In this thesis an inverse optimal control problem for constant output feedbacks is investigated. Necessary and sufficient conditions for optimality of an output feedback are derived for single-input, single-output systems. The class of systems with members for which any constant positive output feedback is optimal turns out to be precisely the class of positive real systems. It is also shown that for a class of minimum phase systems all "large" positive gains are optimal.

Keywords: linear systems, optimal control, inverse optimal control, positive realness, constant output feedback

ÖZET

EVRİK OPTİMAL DENETİM VE POZİTİF GERÇEL SİSTEMLER

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Bu tezde sabit çıkış geribeslemeleri için evrik optimal denetim problemi incelenmiştir. Tek giriş-tek çıkış sistemlerde, çıkış geribeslemelerinin optimal olması için gerekli ve yeterli koşullar türetilmişir. Elemanları için tüm sabit pozitif çıkış geribeslemeleri optimal olan sistem sınıfının tam tamına pozitif gerçel sistemlerin sınıfına eşit olduğu ortaya konmuştur. Ayrıca minimum fazlı sistemler için bütün "büyük" pozitif kazançların optimal olduğu gösterilmiştir.

Anahtar Kelimeler : Linear sistemler, optimal denetim, evrik optimal denetim, pozitif gerçellik, sabit çıkış geribeslemeleri

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INTRODUCTION

Positive real functions and matrices are the principal objects of study in passive network synthesis. A typical example of a positive real transfer function in electrical network theory is the driving point impedance of passive one-ports, [9].

Many nice properties of positive real systems in control applications have long been known and have been widely exploited. The work of Popov on hyperstability [18] tremendously increased the areas of application for positive real systems. We list below only a few of still active areas of control applications for positive real systems. The references given are the more recent studies among the many that deal with each item.

1. Achieving the *absolute stability or sector stability by nonlinear feedback* hinges on satisfying a positive realness condition (Popov Criterion), [20].
2. In *adaptive output error identification*, the design of positive real transfer functions plays a fundamental role in ensuring the convergence of certain

estimation schemes, [3].

3. There is a direct relation, [4], [17], between the recently popularized concept of *convex directions*, relevant in robust controller synthesis, [19], and positive realness.

4. Robust control of *flexible structures and vibrational systems* heavily rely on the property of positive realness, [10].

This thesis is concerned with the inverse optimal control problem with the purpose of identifying those open-loop systems (plants) for which a prescribed set of output feedbacks are optimal. This objective is worthwhile due to the fact that an optimal feedback has many properties advantageous from a practical viewpoint such as stability, sensitivity reduction, infinite gain margin, large phase margin and others, [2].

Although the problem of determining the exact conditions for the existence of an optimal constant output feedback for a given plant is difficult (see [16] for some partial results), the inverse problem posed here turns out to be relatively easy — at least in the case of scalar plants. It is shown in Theorem 2 below that *the class of systems with members for which any constant positive output feedback is optimal is precisely the class of positive real systems.* This result is comparable to the closed loop stability property of positive real systems, [18], Section 24: The class of systems with the property that the feedback interconnection of any two members gives rise to a Lyapunov-stable closed loop system consists of positive real systems.

In order not to blur the main ideas by generalities, the exposition in this thesis is restricted to linear, time-invariant, continuous-time, strictly proper systems. In Chapter 2, a brief summary of some well-known results on the optimal state-regulator problem and its inverse problem is given. In Chapter 3, we define the optimality of a constant output feedback and give the main results Theorems 1 and 2. Two examples are given in Chapter 4 illustrating possible applications of the main results.

Notation. The set of real and complex numbers are denoted by $\mathbf{R}$ and $\mathbf{C}$, respectively. The imaginary number is denoted by the symbol j. By $\mathbf{C}_-, \mathbf{C}_0, \mathbf{C}_+$, we denote the set of complex numbers with negative, zero, and positive real parts, respectively. Occasionally, we use the combined subscript, like in $\mathbf{C}_{0+}$, to refer to the union of these sets. The magnitude and the real part of $c \in \mathbf{C}$ are denoted by $|c|$ and $Re\,\{c\}$, respectively.

Given a polynomial $p(s) = p_n s^n + p_{n-1} s^{n-1} + \ldots + p_1 s + p_0,\ \ p_n \neq 0$ in the indeterminate s with coefficients p_i in $\mathbf{R}$ or $\mathbf{C}$, the degree of $p(s)$ is $n = deg\,p(s)$. Such a polynomial is called Hurwitz stable if $p(s) = 0$ implies $s \in \mathbf{C}_-$, i.e., if every root has a negative real part. Whenever a rational function in the indeterminate s is written as $\frac{p(s)}{q(s)}$, it is understood that $p(s)$ and $q(s)$ are coprime polynomials, i.e., polynomials with no coinciding roots. Such a rational function is proper (strictly proper) if $deg\,p(s) \leq deg\,q(s)$ $(deg\,p(s) \leq deg\,q(s) - 1)$.

Given a matrix $A \in \mathbf{C}^{k \times l}$, $||A||$ denotes the Euclidean induced norm of A. Note that if $l = 1$, $||A||$ simplifies to the Euclidean norm of the vector A. If $k = l$, $det\,A$ denotes the determinant of A and $\sigma(A)$ denotes the spectrum of A, i.e., the family of eigenvalues of A. If A is real and symmetric, the shortcuts $A > 0$ and $A \geq 0$ are used to indicate that A is positive definite and nonnegative definite, respectively.

Chapter 2

LINEAR OPTIMAL CONTROL PROBLEMS

In this chapter, a brief review of the existing results pertaining to linear optimal control problem by state feedback (the optimal time-invariant infinite-time regulator problem) and its inverse problem are given. Section 1 is devoted to a summary of results in [12], [2] on the optimal regulator problem. In Section 2, the particular inverse optimal control problem of interest is defined and the related results of [1], [5] are summarized. Finally in Section 3, the result of [5] is specialized to single-input situation to recover the return-difference criterion of [13] for optimality of a state feedback.

2.1 Linear time-invariant optimal control problem

Consider a continuous time, linear time-invariant plant

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.1}$$

with the cost functional

$$J = \int_0^\infty (x'(t)Qx(t) + u'(t)u(t))\, dt, \tag{2.2}$$

where x is an n-vector of states and u is an m-vector of piecewise-continuous functions called controls. The matrices A, B, and Q are real matrices of sizes $n \times n$, $n \times m$, and $n \times n$ with Q symmetric nonnegative definite. The optimal control problem is defined as: Find a control law $u^*(t)$ of the form

$$u^*(t) = -K^*x^*(t), \;\; K^* \in \mathbf{R}^{m\times n}, \tag{2.3}$$

which causes the system (2.1) to follow an admissible trajectory x^* that minimizes the cost (2.2). The control u^*, if it exists, is called an optimal control and x^* is called an optimal trajectory. This is the usual optimal regulator problem with control penalty matrix $R = I$, [2], [12]. The optimal solution exists provided (A, B) is stabilizable [2] and it is given by

$$u^*(t) = -B'Px^*(t), \tag{2.4}$$

where P is a nonnegative definite solution of the *algebraic Riccati equation*

$$PA + A'P + Q - PBB'P = 0. \tag{2.5}$$

The optimal value of the cost functional is given by

$$J^* = \frac{1}{2}x_0'Px_0.$$

It is also well known [2] that the optimal state feedback

$$K^* = -B'P$$

is a stabilizing feedback for (2.1), i.e., the nonnegative solution P of (2.5) is such that $\sigma\ (A - BB'P) \subseteq \mathbf{C}_-$ if and only if (H, A) is detectable, where H is an $n \times r$ matrix such that $Q = H'H$ and $r := rank\, Q$.

2.2 Inverse optimal control problem

The inverse of the optimal control problem of section 2.1 is the following: Given a linear constant state feedback control law

$$u(t) = -Kx(t), \tag{2.6}$$

1. find necessary and sufficient conditions on the matrices A, B, and K such that the control law (2.6) minimizes the cost (2.2) for some $Q \geq 0$, and
2. determine all such costs (i.e., all such Q).

A satisfactory solution to problem 1 in the case of single-input plants has been obtained by Kalman [13] in terms of a frequency domain condition. In the general case of multi-input plants a necessary and sufficient condition for optimality is obtained by Anderson in [1]. These conditions have been later improved to obtain a complete solution for the multi-input systems by Fujii and Narazaki in [5]. In [13] and in [6], some partial results on problem 2 can also be found.

In this thesis, we are primarily concerned with problem 1 above. In what follows, we summarize the main results of [1] and [5] on the general multi-input situation.

Given the plant (2.1) and a control law (2.6), consider the *return difference matrix*

$$T(s) := I + K(sI - A)^{-1}B. \tag{2.7}$$

Let

$$\Phi(s) := T(-s)'T(s) - I. \tag{2.8}$$

Note that $\Phi(j\omega)$ is a real and symmetric matrix for each $\omega \in \mathbf{R}$.

Proposition 1. [1] *A stabilizing control law (2.6) is optimal for the plant (2.1) and the cost (2.2)*

(i) only if $\Phi(j\omega) \geq 0 \quad \forall \omega \in \mathbf{R}$,

(ii) if $\Phi(j\omega) > 0 \quad \forall \omega \in \mathbf{R}$.

The sufficient condition (ii) can equivalently be stated as:

$(ii)'$ $\Phi(j\omega) \geq 0$ *and rank* $\Phi(j\omega) = n \quad \forall \omega \in \mathbf{R}$.

The result by Fujii and Narazaki [5] closes the gap between the conditions (i) and (ii). Let χ^0 be the set of states reachable from the origin by control inputs $u(t)$ such that $\Phi(s)U(s) \equiv 0$, where $U(s)$ is the Laplace transform of $u(t)$. Also let $Ker\,(A - sI)$ be the kernel of matrix $(A - sI)$. A triple (H, A, B) is called *right invertible and minimum phase* if

$$rank \begin{bmatrix} A - sI & B \\ H & 0 \end{bmatrix} = n + m \ \ \forall s \in \mathbf{C}_+, \tag{2.9}$$

where $\mathbf{C}_+$ is the open right half complex plane.

Proposition 2. *Suppose (A, B) is controllable. Let K be a control law for the plant (2.1). Then, K is optimal for some $Q = H'H$ such that (H, A) is detectable if and only if*

(i) K stabilizes (2.1) and

(ii.1) $\Phi(j\omega) \geq 0 \quad \forall\, \omega \in \mathbf{R}$, (ii.2) $Ker(A - sI) \cap \chi^0 = \{0\} \ \ \forall \, s \in \mathbf{C}_+$.

Moreover, if K is optimal, a matrix H can be chosen so that (H, A, B) is right invertible and minimum phase.

Proof. The result follows by combining Theorems 4.1 and 4.2 in [5]. □

It is further shown in [5] that, if a control law is stabilizing and if the return difference condition $\Phi(j\omega) \geq 0 \quad \forall \, \omega \in \mathbf{R}$ holds, then the condition (ii.2) of Proposition 2 fails only when there exists an eigenvalue λ of A such that $-\lambda \in \sigma(A - BK)$. Since (A, B) is assumed controllable, this is a condition which fails for almost all K. Hence, the return difference condition (ii.1) of Proposition 2 is *generically* a necessary and sufficient condition for a stabilizing control law to be optimal.

2.3 Single-input inverse problem

Let us now consider the case where (2.1) has only one input, i.e., $m = 1$. A main result of [13] is easily recovered from Proposition 2.

Corollary 1. *Let (2.1) be controllable. A control law K is optimal if and only if*

(i) it is stabilizing and

(ii) the absolute value of the return difference $T(j\omega) = 1 + K(j\omega I - A)^{-1}B$ is at least 1 at all frequencies, or equivalently,

$$\Phi(j\omega) = |T(j\omega)|^2 - 1 \geq 0 \ \ \forall \, \omega \in \mathbf{R}. \tag{2.10}$$

Proof. The "if" part is obvious by Proposition 2. We prove the "only if" part. Since $m = 1$, $T(s)$ and $\Phi(s)$ are both scalars. Suppose (i) and (ii) above hold. Clearly, (i) and (ii.1) of Proposition 2 also hold. If $\Phi(s) \neq 0$ for some s, then $\chi^0 = \{0\}$ so that the condition (ii.2) in Proposition 2 is automatically satisfied. Suppose, on the other hand, that $\Phi(s) \equiv 0$, or equivalently, $K(sI - A)^{-1}B \equiv 0$. By controllability of (A, B), this implies that $K = 0$. By (i) above, $K = 0$ is a stabilizing control law, i.e., $Ker\,(A - sI) = \{0\}$ for every $s \in \mathbf{C}_{0+}$. Consequently, condition (ii.2) in Proposition 2 is again satisfied. We have thus shown that (i) and (ii) above also imply the condition (ii.2) of Proposition 2 completing the proof. □

Let

$$\psi(s) := det\,(sI - A),$$
$$\psi_K(s) := det\,(sI - A + BK).$$

The return difference condition of Corollary 1 can be restated in terms of the polynomials $\psi_K(s)$ and $\psi(s)$. The following result is from [13], the proof is given for the sake of completeness.

Proposition 3. *Let (2.1) be controllable. A control law K is optimal if and only if*

$$(i) \quad \sigma\,(A - BK) \subseteq \mathbf{C}_-, and$$
$$(ii) \quad |\psi_K(j\omega)|^2 - |\psi(j\omega)|^2 \geq 0 \;\; \forall \omega \in \mathbf{R}.$$

Proof. We only need to show the equivalence of the conditions (2.10) and (ii) for a stabilizing control law. Note by a well known determinant identity that

$$\begin{aligned} 1 + K(sI - A)^{-1}B &= det\,(I + (sI - A)^{-1}BK) \\ &= \frac{det\,(sI - A + BK)}{det\,(sI - A)} \\ &= \frac{\psi_K(s)}{\psi(s)}. \end{aligned}$$

It follows that

$$|T(j\omega)|^2 - 1 = \frac{|\psi_K(j\omega)|^2}{|\psi(j\omega)|^2} - 1.$$

Therefore, $|T(j\omega)|^2 \geq 1$ if and only if (ii) holds. □

The following summarizes the procedure in [13] of obtaining a corresponding $Q = H'H$ for an optimal K.

First observe that the condition (ii) of Proposition 3 is equivalent to the existence of a polynomial $\theta(s)$ with all its roots in $\mathbf{C}_{0-}$ such that

$$\psi_K(s)\psi_K(-s) - \psi(s)\psi(-s) = \theta(s)\theta(-s), \tag{2.11}$$

i.e., to the existence of a spectral factorization. In fact, if (2.11) holds, then evaluating at $s = j\omega$ one has (ii). Conversely, if (ii) holds, then

$$\Theta(\omega^2) := |\psi_K(j\omega)|^2 - |\psi(j\omega)|^2$$

is a nonnegative polynomial and has a factorization of the form $\Theta(\omega^2) = \theta(j\omega)\theta(-j\omega)$ for some polynomial $\theta(s)$ as above (see e.g., Section 39 of [18]) and (2.11) follows. Now, by realization theory, there exists $H \in \mathbf{R}^{1\times n}$ such that

$$\frac{\theta(s)}{\psi(s)} = H(sI - A)^{-1}B. \tag{2.12}$$

Here, the pair (H, A) is detectable since, by (2.11) and by the fact that $\psi_K(s)$ is Hurwitz stable, any common root of $\theta(s)$ and $\psi(s)$ is in $\mathbf{C}_-$. The matrix $Q = H'H$ so constructed has the desired property. Note that H obtained by this procedure also satisfies the additional property that the triple (H, A, B) is right invertible and minimum phase.

As shown in [13], although $\theta(s)$ satisfying (2.11) and H satisfying (2.12) are unique, there are other possibilities for determining a suitable Q. The set

$\mathcal{Q}$ of admissible Q's may be described by

$$\mathcal{Q} = \{\bar{H}'\bar{H} : (\bar{H}, A) \textit{ is detectable and } \|\bar{H}(j\omega I - A)^{-1}B\| = |H(j\omega I - A)^{-1}B| \ \ \forall \omega \in \mathbf{R}\}.$$

This is a (partial) solution by [13] to problem 2 of single-input inverse optimal control.

Chapter 3

OPTIMALITY OF A CLASS OF OUTPUT FEEDBACKS

Consider a single-input, single-output, linear, time-invariant plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) \end{aligned} \tag{3.1}$$

with an output feedback

$$u(t) = -\alpha y(t), \tag{3.2}$$

where $\alpha \in \mathbf{R}$.

We assume throughout this section that (3.1) is controllable and observable.

Definition. *We call α or the feedback (3.2) an* **optimal output feedback** *if the corresponding state feedback $K := \alpha C$ is optimal with respect to the cost (2.2).*

In this chapter, we first obtain, by a direct application of Proposition 3, conditions for optimality of an output feedback. We then identify those plants for which all positive (or negative) output feedbacks are optimal. Such plants turn out to be characterized by positive realness of their transfer functions, a result that may be expected from similar results on hyperstable systems, [18].

3.1 Optimality of an output feedback

By Proposition 3, $K = \alpha C$ is optimal if and only if

$$\begin{array}{ll} (i) & \sigma(A - B\alpha C) \subseteq \mathbf{C}_-, \, and \\ (ii) & |\psi_{\alpha C}(j\omega)|^2 - |\psi(j\omega)|^2 \geq 0 \;\; \forall \omega \in \mathbf{R}. \end{array}$$

Let the plant transfer function be written as

$$C(sI - A)^{-1}B = \frac{p(s)}{q(s)},$$

for coprime polynomials $p(s)$ and $q(s)$ with $q(s)$ monic, i.e., has leading coefficient equal to 1. By the assumption of controllability and observability of (3.1), we have $\psi(s) = det\,(sI - A) = q(s)$. Moreover,

$$\begin{aligned} det\,(sI - A + B\alpha C) &= q(s)\, det\,(1 + \alpha C(sI - A)^{-1}B) \\ &= q(s)\, det\,(1 + \alpha \tfrac{p(s)}{q(s)}) \\ &= q(s) + \alpha p(s) \end{aligned}$$

so that $\psi_{\alpha C}(s) = q(s) + \alpha p(s)$. It follows that the conditions (i) and (ii) above are equivalent to

$$\begin{array}{ll} (1) & q(s) + \alpha p(s) \; is \; Hurwitz \; stable, \; and \\ (2) & |q(j\omega) + \alpha p(j\omega)|^2 - |q(j\omega)|^2 \geq 0 \;\; \forall \;\; \omega \in \mathbf{R}. \end{array}$$

The following first main result is thus obtained.

Theorem 1. *An output feedback $u(t) = -\alpha y(t)$, $\alpha \in \mathbf{R}$, is optimal for (3.1) and (2.2) if and only if*

(*i*) $q(s) + \alpha p(s)$ *is Hurwitz stable, and*

$$(ii) \quad \left\{ \begin{array}{lll} \alpha > 0 & \Rightarrow & Re\{\frac{q(j\omega)}{p(j\omega)}\} \geq -\frac{\alpha}{2} \\ \alpha < 0 & \Rightarrow & Re\{\frac{q(j\omega)}{p(j\omega)}\} \leq -\frac{\alpha}{2} \end{array} \right\} \quad \forall \omega \in \mathbf{R} \text{ such that } p(j\omega) \neq 0.$$

Proof. If $\alpha = 0$, then the condition (2) prior to theorem statement is satisfied and $\alpha = 0$ is optimal if and only if (i) holds, i.e., $q(s)$ is Hurwitz stable. Suppose $\alpha \neq 0$. The condition (2) is

$$\begin{aligned} |q(j\omega) + \alpha p(j\omega)|^2 - |q(j\omega)|^2 &= 2\alpha Re\{p(-j\omega)q(j\omega)\} + \alpha^2|p(j\omega)|^2 \\ &\geq 0 \end{aligned}$$

which holds for all ω such that $p(j\omega) \neq 0$ if and only if

$$\alpha(2Re\{\frac{q(j\omega)}{p(j\omega)}\} + \alpha) \geq 0$$

holds. Considering the cases $\alpha > 0, \alpha < 0$, it follows that, this inequality is satisfied if and only if (ii) in the theorem statement holds. □

If a given $\alpha \in \mathbf{R}$ satisfies the conditions of Theorem 1, then a corresponding Q_α of the cost (2.2) can be determined by setting $K = \alpha C$, performing the spectral factorization (2.11), and following the procedure outlined at the end of Chapter 2. Alternatively, a symmetric $Q_\alpha \geq 0$ can be determined such that for some symmetric $P_\alpha > 0$, the following algebraic relations are satisfied

$$\begin{aligned} &B'P_\alpha = \alpha C, \\ &P_\alpha A + A'P_\alpha + Q_\alpha - P_\alpha BB'P_\alpha = 0. \end{aligned}$$

It is interesting to observe that if one pair $P_\kappa > 0, Q_\kappa \geq 0$ satisfies these relations for some $\kappa > 0$, then the pair

$$\begin{aligned} P_\alpha &:= \tfrac{\alpha}{\kappa} P_\kappa \\ Q_\alpha &:= \tfrac{\alpha}{\kappa} Q_\kappa + \alpha(\alpha - \kappa)C'C \end{aligned} \tag{3.3}$$

also satisfies these relations for any $\alpha \in \mathbf{R}$ as can be verified by direct substitution. It follows that if a cost matrix $Q_\kappa \geq 0$ for one $\kappa > 0$ (respectively, $\kappa < 0$) is determined, then Q_α defined in (3.3) is a cost matrix for every $\alpha > \kappa$ (respectively, $\alpha < \kappa$).

Suppose $p(s)$ has no $j\omega$-axis roots. Then, the condition (ii) of Theorem 1 has a simple interpretation in terms of the *inverse Nyquist plot* of $p(s)/q(s)$ (or the polar plot of $q(s)/p(s)$). Suppose $\alpha > 0$. Then, the condition (ii) of Theorem 1 holds if and only if the inverse Nyquist plot of $p(s)/q(s)$ is contained in the right half plane $Re\,\{q(j\omega)/p(j\omega)\} \geq -\alpha/2$. Note that if (ii) holds, then the inverse Nyquist plot does not encircle the point $(-\alpha, j0)$. Applying the inverse Nyquist criterion, in order for condition (i) of Theorem 1 also to hold, it is necessary and sufficient that the polynomial $p(s)$ has no roots in $\mathbf{C}_+$, i.e., in the strict right half plane. We thus arrive at the following geometric restatement of Theorem 1. *Let $p(j\omega) \neq 0$ for all $\omega \in \mathbf{R}$. An output feedback with $\alpha \geq 0$ is optimal if and only if $p(s)$ has no roots in the right half complex plane and the inverse Nyquist plot of $p(s)/q(s)$ is contained in the closed half plane $Re\,\{\frac{q(j\omega)}{p(j\omega)}\} \geq -\alpha/2$.* The inverse Nyquist diagram of a typical system for which $\alpha > 0$ is optimal is shown in Figure 3.1.

There is yet another equivalent restatement of Theorem 1 in terms of the Nyquist plot of $p(s)/q(s)$. We first note that condition (ii) of Theorem 1 for $\alpha > 0$ can be written as

$$Re\,\{\frac{p(j\omega)}{q(j\omega)}\} \geq -\frac{\alpha}{2}\,\frac{|p(j\omega)|^2}{|q(j\omega)|^2} \;\; \forall\omega \in \mathbf{R} \text{ for which } q(j\omega) \neq 0 \qquad (3.4)$$

using the identity

$$Re\,\{\frac{q(j\omega)}{p(j\omega)}\}p(j\omega)p(-j\omega) = Re\,\{\frac{p(j\omega)}{q(j\omega)}\}q(j\omega)q(-j\omega). \qquad (3.5)$$

The inequality (3.4) means that, in the $\frac{p(j\omega)}{q(j\omega)}$-plane, the Nyquist plot of $\frac{p(j\omega)}{q(j\omega)}$ lies

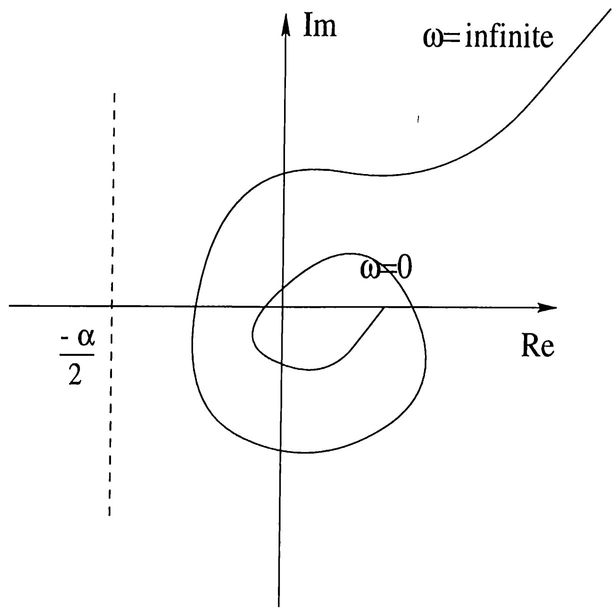

Figure 3.1: Inverse Nyquist plot for an optimal α.

outside the open unit disk of radius $1/\alpha$ and of center $(-1/\alpha, j0)$. Using the Nyquist criterion in interpreting the condition (i) of Theorem 1, we arrive at the following geometric criterion for optimality. It is understood that whenever $q(s)$ has roots on $\mathbf{C}_0$, the Nyquist plot is obtained by introducing small semi-circles at the Nyquist contour so as to avoid these roots.

Corollary 2. *An output feedback with $\alpha > 0$ is optimal if and only if the Nyquist plot of $p(s)/q(s)$ avoids the open disk of radius $1/\alpha$ centered at $(-1/\alpha, j0)$ and encircles the disk counterclockwise as many times as the number of $\mathbf{C}_+$-roots of $q(s)$.*

The Nyquist diagram of a typical system for which $\alpha > 0$ is optimal is shown in Figure 3.2.

Example 1. Consider

$$\frac{p(s)}{q(s)} = \frac{s}{s^2 - s + 1}.$$

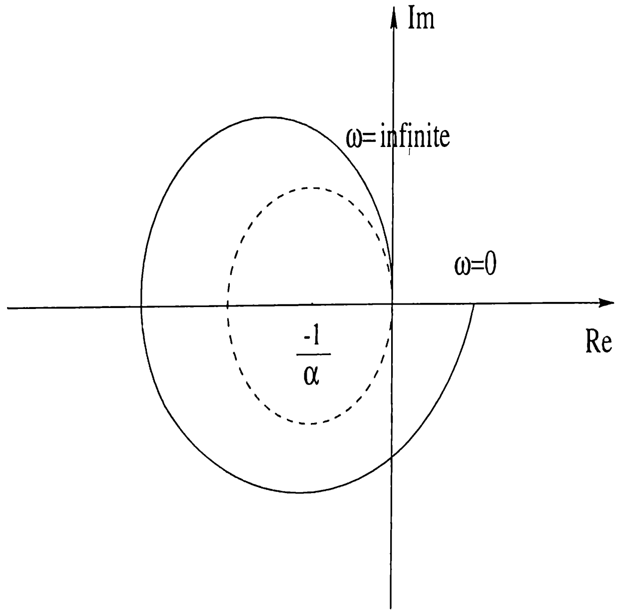

Figure 3.2: Nyquist plot for an optimal α with two $\mathbf{C}_+$-roots of $q(s)$.

We have

$$Re\,\{\frac{q(j\omega)}{p(j\omega)}\} = Re\,\{\frac{-\omega^2 - j\omega + 1}{j\omega}\} = -1$$

so that all $\alpha \geq 2$ are optimal for this system. On the other hand, for

$$\frac{p(s)}{q(s)} = \frac{s^2+1}{(s^2+2)(s+3)}.$$

no feedback with $\alpha > 0$ is optimal since

$$Re\,\{\frac{q(j\omega)}{p(j\omega)}\} = \frac{3(2-\omega^2)}{1-\omega^2}$$

goes to $-\infty$ as $\omega \to 1^+$. Note that condition (i) of Theorem 1 is satisfied for all $\alpha > 0$ since $q(s) + \alpha p(s)$ is Hurwitz stable for all $\alpha > 0$ by a simple application of Routh-Hurwitz criterion. •

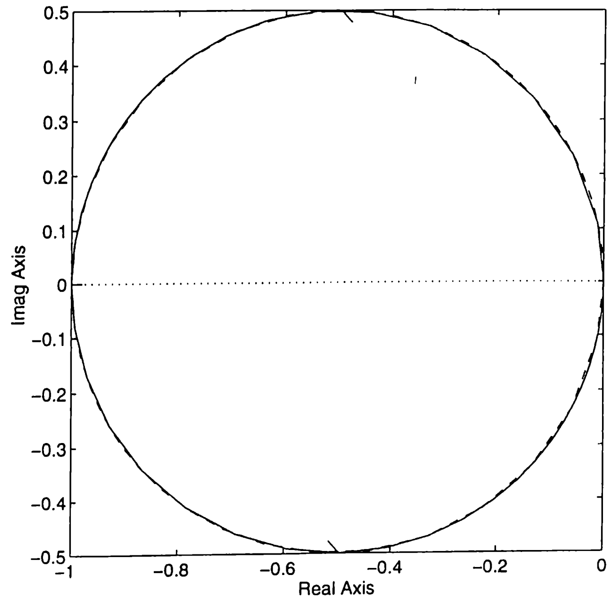

Figure 3.3: Nyquist plot of $\frac{p(s)}{q(s)} = \frac{s}{s^2-s+1}$.

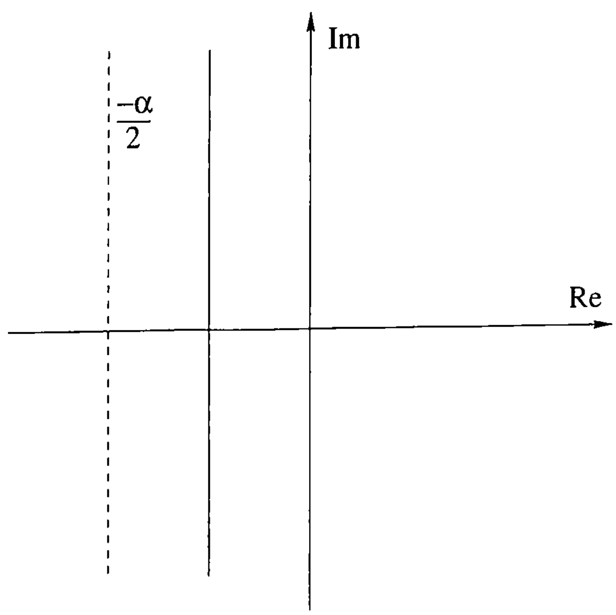

Figure 3.4: Inverse Nyquist plot of $\frac{p(s)}{q(s)} = \frac{s}{s^2-s+1}$.

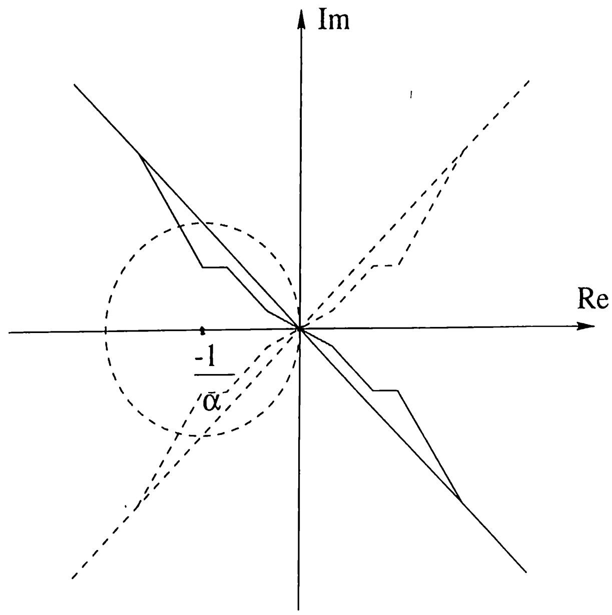

Figure 3.5: Nyquist plot of $\frac{p(s)}{q(s)} = \frac{s^2+1}{(s^2+2)(s+3)}$.

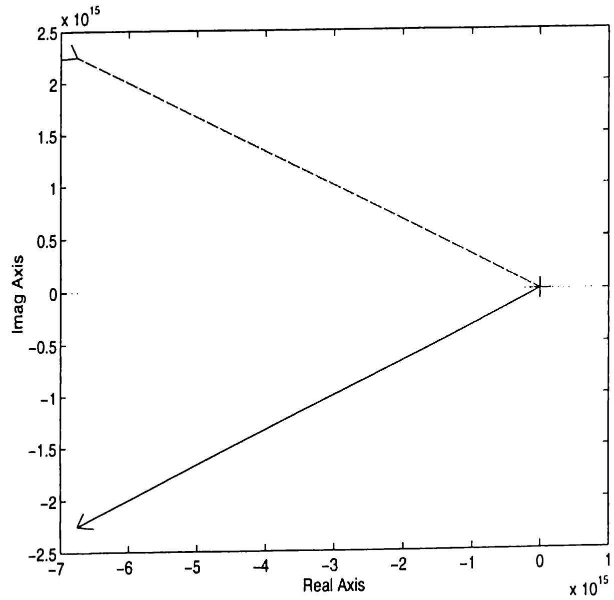

Figure 3.6: Inverse Nyquist plot of $\frac{p(s)}{q(s)} = \frac{s^2+1}{(s^2+2)(s+3)}$.

3.2 Optimality for second and third order systems

In this section, we give coefficient conditions on open loop transfer functions of second and third order systems for a given $\alpha > 0$ to be optimal.

Consider a second order strictly proper transfer function

$$\frac{p(s)}{q(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}, \; b_0^2 - b_0 b_1 a_1 + b_1^2 a_0 \neq 0.$$

By Theorem 1, a given $\alpha > 0$ turns out to be optimal if and only if

$$\begin{aligned} a_1 + \alpha b_1 &> 0 \\ a_0 + \alpha b_0 &> 0 \\ a_1 b_1 - b_0 + \tfrac{\alpha}{2} b_1^2 &\geq 0 \\ a_0 b_0 + \tfrac{\alpha}{2} b_0^2 &\geq 0, \end{aligned}$$

where the first two conditions ensure that $q(s) + \alpha p(s)$ is Hurwitz stable.

Consider a third order strictly proper open loop transfer function

$$\frac{p(s)}{q(s)} = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0},$$

where the resultant matrix

$$\begin{bmatrix} b_2 & b_1 & b_0 & & \\ & b_2 & b_1 & b_0 & \\ & & b_2 & b_1 & b_0 \\ & 1 & a_2 & a_1 & a_0 \\ 1 & a_2 & a_1 & a_0 & \end{bmatrix}$$

is assumed nonsingular. Again by Theorem 1, a given $\alpha > 0$ is optimal if and only if the following inequalities hold:

$$a_0 + \alpha b_0 > 0$$

$$
\begin{aligned}
a_2 + \alpha b_2 &> 0 \\
(a_1 + \alpha b_1)(a_2 + \alpha b_2) - (a_0 + \alpha b_0) &> 0 \\
a_0 b_0 + \frac{\alpha}{2} b_0^2 &\geq 0 \\
a_2 b_2 - b_1 + \frac{\alpha}{2} b_2^2 &\geq 0
\end{aligned}
$$

Either

$$
a_1 b_1 + \frac{\alpha}{2} b_1^2 - a_0 b_2 - b_0 a_2 - \alpha b_0 b_2 \geq 0
$$

or

$$
a_1 b_1 + \frac{\alpha}{2} b_1^2 - a_0 b_2 - b_0 a_2 - \alpha b_0 b_2 \leq -2\sqrt{(a_0 b_0 + \frac{\alpha}{2} b_0^2)(a_2 b_2 - b_1 + \frac{\alpha}{2} b_2^2)}
$$

Following [13], the last "either-or" condition can be replaced by

$$
a_1 b_1 + \frac{\alpha}{2} b_1^2 - a_0 b_2 - b_0 a_2 - \alpha b_0 b_2 = -\sqrt{(a_0 b_0 + \frac{\alpha}{2} b_0^2)(a_2 b_2 - b_1 + \frac{\alpha}{2} b_2^2)}.
$$

3.3 Plants for which all positive feedbacks are optimal

We show in this section that the set of open loop systems for which all feedbacks (3.2) with $\alpha > 0$ are optimal, are systems having positive real transfer functions. Among the many equivalent characterizations possible for positive realness of (rational) transfer functions, see [7] and Appendix C of [18], we use the following as a definition. The definition of strict positive real transfer functions is attributed to [11].

Definition. *A transfer function* $\frac{p(s)}{q(s)}$ *is* **positive real** *if it has no poles in* $\mathbf{C}_+$*, the poles with zero real parts (if any) are simple (i.e., have multiplicities* *equal to 1) with real and positive residues, and*

$$Re\,\{\frac{p(j\omega)}{q(j\omega)}\} \geq 0 \;\; \forall \omega \in \mathbf{R} \text{ for which } q(j\omega) \neq 0. \tag{3.6}$$

A transfer function $\frac{p(s)}{q(s)}$ *is* **strict positive real** *if* $q(s)$ *is Hurwitz stable and*

$$Re\,\{\frac{p(j\omega)}{q(j\omega)}\} > 0 \;\; \forall \omega \in \mathbf{R}.$$

It is well known, [18], [11], that $\frac{p(s)}{q(s)}$ is (strict) positive real if and only if $\frac{q(s)}{p(s)}$ is.

The following lemma simplifies the test for positive realness.

Lemma 1. *Given a transfer function* $\frac{p(s)}{q(s)}$, *suppose there exists* $\alpha > 0$ *such that* $q(s) + \alpha p(s)$ *is Hurwitz stable. Then,* $\frac{p(s)}{q(s)}$ *is positive real if and only if (3.6) holds.*

Proof. Let (3.1) be a reachable and observable realization of the transfer function $\frac{p(s)}{q(s)}$. By Proposition 2 of [18], Section 15, if (3.1) is *minimally stable*, then the second statement of the lemma is valid. The system (3.1) is defined to be minimally stable if for every initial state $x(0)$, there exists an input u such that the solution $x(t)$ of (3.1) satisfies

$$\lim_{t \to \infty} \|x(t)\| = 0 \tag{3.7}$$

and the integral constraint

$$\int_0^{t_1} y(t)u(t)dt$$

is nonpositive for all $t_1 \geq 0$. We show that (3.1) is minimally stable under the hypothesis of the lemma. In fact if there exists $\alpha > 0$ such that $q(s) + \alpha p(s)$ is Hurwitz stable, consider the input $u(t) := -\alpha y(t)$. By Hurwitz stability of $q(s) + \alpha p(s)$ and by the reachability and observability of the plant, the closed loop system consisting of the plant (3.1) and the feedback $-\alpha$ is internally stable. It follows that the solution $x(t)$ of (3.1) is asymptotically stable and hence satisfies (3.7). Moreover, the integral becomes

$$-\alpha \int_0^{t_1} y(t)^2 dt$$

which is nonpositive for all $t_1 \geq 0$. Hence (3.1) is minimally stable and the proof is complete. □

Let us recall a fundamental property of plants with positive real transfer functions. This well-known result follows by Sections 23 and 24 of [18]. We give a simple Nyquist plot argument for the second statement.

Lemma 2. *A plant with a positive real transfer function is stabilized by all output feedbacks $u(t) = -\alpha y(t)$ with $\alpha > 0$. A plant with a strict positive real transfer function is stabilized by all output feedbacks $u(t) = -\alpha y(t)$ with $\alpha \geq 0$.*

Proof. By Section 24 of [18], it follows that any negative feedback connection of two positive real transfer functions gives rise to a "Lyapunov-stable" closed loop system. From the discussion on asymptotic stability in Section 23 of [18] (see condition 1° of Theorem 1), it follows that if one of the systems is strict positive real, then the closed loop system is asymptotically stable. This proves both statements. In order to fix ideas, we give the following simple Nyquist plot argument for the proof of the second statement.

If the transfer function $p(s)/q(s)$ is strict positive real, then $q(s)$ is Hurwitz stable so that at $\alpha = 0$ the claim is true. Moreover, the Nyquist plot of $p(s)/q(s)$ is contained in the open right half $p(j\omega)/q(j\omega)$-plane. Hence, for any $\alpha > 0$, the point $(-1/\alpha, j0)$ can not be enclosed by the Nyquist plot. By Hurwitz stability of $q(s)$ and by the Nyquist criterion, it follows that $q(s) + \alpha p(s)$ is Hurwitz stable for all $\alpha \geq 0$. □

The following is the main result of this section.

Theorem 2. *All output feedbacks $\alpha > 0$ are optimal for the plant (3.1) and the cost (2.2) if and only if the transfer function $\frac{p(s)}{q(s)}$ is positive real.*

Proof. [If] By Lemma 2, if $\frac{p(s)}{q(s)}$ is positive real, then $q(s)+\alpha p(s)$ is Hurwitz stable for all $\alpha > 0$. Moreover, by the identity (3.5), it also follows that $Re\{\frac{q(j\omega)}{p(j\omega)}\} \geq 0$ for all ω for which $p(j\omega) \neq 0$. Therefore, both conditions of Theorem 1 hold for every $\alpha > 0$ and all such feedbacks are optimal.

[Only if] By Theorem 1, if all $\alpha > 0$ are optimal, then

$$(a) \quad q + \alpha p \text{ is Hurwitz stable for all } \alpha > 0, \text{ and}$$

$$(b) \quad Re\{\frac{q(j\omega)}{p(j\omega)}\} \geq 0 \quad \forall \omega \in \mathbf{R} \text{ for which } p(j\omega) \neq 0.$$

By condition (a) and Lemma 1, the transfer function is positive real provided $Re\{\frac{p(j\omega)}{q(j\omega)}\} \geq 0$ for all ω for which $q(j\omega) \neq 0$. The latter follows by condition (b) and the identity (3.5). □

Remarks.(1) We observe from Theorem 1 that $\alpha < 0$ is optimal for the plant transfer function $\frac{p(s)}{q(s)}$ if and only if $-\alpha > 0$ is optimal for $-\frac{p(s)}{q(s)}$. For this reason, the discussion in the rest of the thesis is restricted to feedbacks (3.2) with $\alpha > 0$.

(2) It has been noted in [8] in the multivariable situation that, any output feedback $u(t) = -R^{-1}y(t)$ with R^{-1} positive definite is optimal with respect to (3.1) and the cost

$$\int_0^\infty [x(t)'(C'R^{-1}C + Q)x(t) + u(t)'Ru(t)]dt$$

provided the plant is positive real. Although this is a different inverse problem of optimal control as the input penalization matrix R is assumed unknown, the result still points out to the optimality property of positive real systems. Our result above applies to scalar systems and to fixed input penalization $R = 1$. It also provides a converse to the result of [8], namely, if all positive $\alpha > 0$ are optimal for a plant, then the plant must be positive real.

Corollary 3. *Let $p(j\omega) \neq 0$ for all $\omega \in \mathbf{R}$. Then, all output feedbacks $\alpha \geq 0$ are optimal for (3.1) and (2.2) if and only if the transfer function $\frac{p(s)}{q(s)}$ of (3.1) is strict positive real.*

Proof. If the transfer function $p(s)/q(s)$ is strict positive real, then, by Lemma 2, $q(s) + \alpha p(s)$ is Hurwitz stable and, by (3.5), $Re\,\{\frac{q(j\omega)}{p(j\omega)}\} > 0 \geq -\alpha/2$ for any $\alpha \geq 0$. By Theorem 1, we have that any $\alpha \geq 0$ is optimal for the plant (3.1) and the cost (2.2). Conversely, if all $\alpha \geq 0$ are optimal, then, by Theorem 2 considering $\alpha > 0$, the transfer function is positive real and, by the hypothesis, $p(s)$ is Hurwitz stable. By Theorem 1 considering $\alpha = 0$, the polynomial $q(s)$ is also Hurwitz stable. By continuity of $\frac{p(j\omega)}{q(j\omega)}$ with respect to ω, it follows that $Re\,\{\frac{p(j\omega)}{q(j\omega)}\} > 0$ for all $\omega \in \mathbf{R}$ and $p(s)/q(s)$ is strict positive real. □

3.4 Minimum phase systems

In this section, a class of feedbacks optimal for a class of minimum phase systems are determined as an application of Corollary 3. We first show that under a preliminary (usually high gain) feedback, a minimum phase system of the type defined below can be made strict positive real. Any feedback $u(t) = -\alpha y(t)$ with α larger than two times the value of this preliminary feedback is then shown to be optimal for the original minimum phase plant.

We first adopt a "non-standard" definition.

Definition. *A strictly proper plant of transfer function* $\frac{p(s)}{q(s)}$ *will be called* **strict minimum phase** *if* $p(s)$ *is Hurwitz stable,* $deg\, p(s) = deg\, q(s) - 1$*, and the highest coefficients of* $p(s)$ *and* $q(s)$ *are of the same sign.*

Lemma 3. *Given a strict minimum phase* $\frac{p(s)}{q(s)}$*, there exists* $k > 0$ *such that*

$$\frac{p(s)}{q(s) + \delta p(s)}$$

is strict positive real for all $\delta \geq k$.

Proof. Let us assume without loss of generality that the highest coefficients of $p(s)$ and $q(s)$ are both positive. Since $p(s)$ is Hurwitz stable and $deg\, p = deg\, q - 1$, by root-locus considerations it is easy to see that there exists $l > 0$ such that $r(s) := q(s) + \beta p(s)$ is Hurwitz stable for all $\beta \geq l$. In what follows, we show that there exists $k > 0$ such that $p(s)/[r(s) + \gamma p(s)]$ is strict positive real for all $\gamma \geq k$. From this it easily follows that $p(s)/[q(s) + \alpha p(s)]$ is strict positive real for all $\alpha \geq k + l$. The transfer function $p(s)/[r(s) + \gamma p(s)]$ is strict positive real if and only if

$$Re\,\{r(j\omega)p(-j\omega)\} + \gamma p(j\omega)p(-j\omega) > 0 \;\; \forall\, \omega \in \mathbf{R}. \tag{3.8}$$

Since both $r(s)$ and $p(s)$ are Hurwitz stable, all coefficients are nonzero and of the same sign (see e.g., [7]). By our assumption that the highest coefficients of $q(s)$ and $p(s)$ are positive and by $deg\, p(s) = deg\, q(s) - 1$, $deg\, q(s) = deg\, r(s)$, it follows that all coefficients of $r(s)$ and $p(s)$ are positive. Hence the following equality holds for the degrees in ω:

$$deg_\omega Re\,\{r(j\omega)p(-j\omega)\} = deg_\omega p(j\omega)p(-j\omega).$$

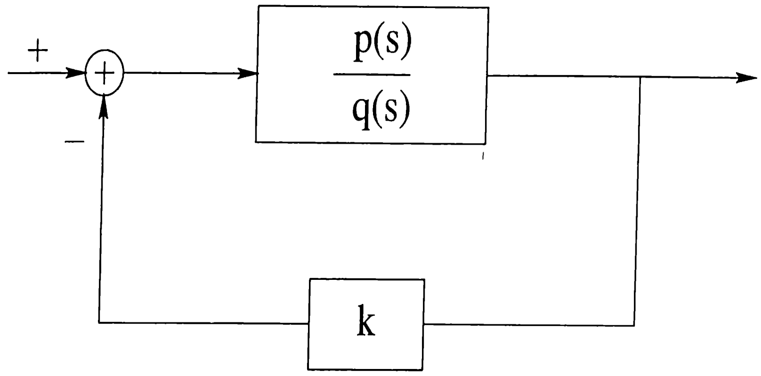

Figure 3.7: Feedback system for minimum phase plants.

Since

$$p(j\omega)p(-j\omega) > 0 \;\forall \omega \in \mathbf{R},$$
$$\lim_{\omega\to\pm\infty} p(j\omega)p(-j\omega) = +\infty,$$

there exists a suitable $k > 0$ such that for all $\gamma \geq k$ the inequality (3.8) is satisfied. Therefore, the transfer function

$$\frac{p(s)}{r(s) + \gamma p(s)}$$

is strict positive real for all $\gamma \geq k$ since it has a strict positive real part at $s = j\omega$ and $p(s)$ (and $r(s) + \gamma p(s)$) is Hurwitz stable. □

Theorem 3. *Given a strict minimum phase plant (3.1), let $k > 0$ be such that the closed loop system of Figure 3.7 is strict positive real for all $\delta \geq k$. Then, every $\alpha \geq 2k$ is optimal for (3.1) and (2.2).*

Proof. A preliminary feedback $k > 0$ such that

$$\frac{p(s)}{q(s) + \delta p(s)}$$

is strictly positive real for all $\delta \geq k$ exists by Lemma 3. By definition of strict positive realness, the denominator polynomial $q(s) + \delta p(s)$ is Hurwitz stable for all $\delta \geq k$, and also using (3.5), we have

$$Re\,\{\frac{q(j\omega) + \delta p(j\omega)}{p(j\omega)}\} > 0$$

for all $\delta \geq k$. This yields

$$Re\{\frac{q(j\omega)}{p(j\omega)}\} > -k \geq -\frac{\alpha}{2}$$

and $q(s) + \alpha p(s)$ is Hurwitz stable. Therefore, by Theorem 1, α is optimal for $\frac{p(s)}{q(s)}$. □

Remark. Although the discussion above has been restricted to strictly proper plants, all the results above have appropriate extensions to the more general case of a single-input, single-output plant

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{3.9}$$

with an output feedback

$$u(t) = -\alpha y(t), \tag{3.10}$$

where $\alpha \in \mathbf{R}$. We call a feedback (3.10) optimal for the plant (3.9), and the cost (2.2) if (i) $1 + \alpha D \neq 0$ and (ii) the corresponding state feedback $u(t) = -Kx(t)$ with

$$K := \frac{\alpha}{1 + \alpha D} C$$

is optimal for (3.9) and the cost (2.2). The condition (i) is included for the closed loop system to be well-defined. Since the statement of the results for this more general case are slightly more involved (mainly due to the nonlinear dependence of K on α), we have restricted our discussion to strictly proper plants (D=0).

Chapter 4

APPLICATIONS

Here, we examine one mechanical and one electro-mechanical system with the application of the results of Chapter 3 in mind. The second example, stepping motor, is a particularly difficult case since its transfer function is far from being positive real. We examine the possibility of altering the transfer function by a feedforward compensation so that it becomes positive real for some values of the motor constants.

4.1 Automobile suspension system

Figure (4.1) shows a schematic diagram of an automobile suspension system. The linearized equation of motion for this standard textbook example, [15], is obtained as follows.

As the car moves along the road, the vertical displacement of the tires act as the motion excitation to the automobile suspension system. The motion of this system consists of a translational motion of the center of mass, indicated in the figure as m. The equation of motion for the system is:

$$m\ddot{x}_o + b(\dot{x}_o - \dot{x}_i) + k(x_o - x_i) = 0 \quad or \tag{4.1}$$

$$m\ddot{x}_o + b\dot{x}_o + kx_o = b\dot{x}_i + kx_i \tag{4.2}$$

Assuming that the motion x_i at point P, which is the center of mass of the tires, is the input to the system and the vertical motion x_o of the body is the output, considering the motion of the body m only in the vertical direction, we obtain the transfer function as:

$$\frac{X_o(s)}{X_i(s)} = \frac{bs + k}{ms^2 + bs + k}. \tag{4.3}$$

For this second order transfer function, the coefficient condition in Section 3.2 can be applied verbatim on setting

$$b_1 = a_1 = \frac{b}{m},\ b_0 = a_0 = \frac{k}{m}.$$

Furthermore, all feedbacks with $\alpha > 0$ are optimal for this transfer function, i.e., the transfer function is positive real (Section 3.2.1) if and only if

$$b^2 \geq mk. \tag{4.4}$$

4.2 Stepping motors

The following description and the linear model of a stepping motor is adapted from [14]. DC motors are devices which converts an electrical input into a mechanical motion. Stepping motors can perform the same or similar functions with the following significant advantages:

- No feedback is usually required for either position or speed control,

- positional error is noncumulative, and

- stepping motors are compatible with modern digital equipment.

Here we examine the Two Phase Variable Reluctance (VR) voltage driven stepping motor. The transfer functions of other types of stepping motors have the same structure. Consider the model of a VR motor shown in Figure 4.2, where λ is the pitch angle and θ_i is an equilibrium position. An equilibrium position is a magnetic null obtained by circulating nominal steady state currents through the windings of the stator. It was found that a very small displacement of this magnetic null around the stepping position can be obtained by applying differential currents to the same windings of the stator. Let us now define the desired position as the magnetic null where the load is to be fine positioned by the motor. Magnetic null is obtained by interaction of the rotor motion in order to reduce the reluctance and the magnetic fields of the excited windings. The equation of motion of the rotor is

$$J_m \frac{d^2\theta}{dt^2} + D_{air}\frac{d\theta}{dt} + T_a + T_b = 0, \tag{4.5}$$

where J_m is the inertia of the rotor, θ is the actual position of the rotor, D_{air} is the viscous damping coefficients of the air and friction, T_a is the torque due to current in phase A and T_b is the torque due to current in phase B. The expressions for T_a and T_b are:

$$T_a = p\Phi_m i_b sin(p\theta) \tag{4.6}$$

$$T_b = p\Phi_m i_a sin(p(\theta - \lambda)), \tag{4.7}$$

where p is the number of pairs of magnetic poles, Φ_m is the flux, λ is the pitch angle, i_a is the current in phase A and i_b is the current in phase B. The electrical equations neglecting the mutual inductance are:

$$v_a = ri_a + L\frac{di_a}{dt} + \frac{d}{dt}(n\Phi_m cos(N_r\theta)) \qquad (4.8)$$

$$v_b = ri_b + L\frac{di_b}{dt} + \frac{d}{dt}(n\Phi_m sin(N_r\theta)), \qquad (4.9)$$

where r is the winding resistance and N_r is equal to p. For proper positioning of the rotor, magnetic damping and mechanical damping are used. If we connect a mechanical damper to the rotor, the equation of motion is changed as:

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + T_a + T_b + T_d = 0, \qquad (4.10)$$

where T_d is the viscous torque exerted on the damper housing, J is the equivalent inertia of the rotor and the damper housing and D is the viscous damping coefficients of the air, friction and damper. The expression for T_d is obtained from

$$D_d(\frac{d\theta}{dt} - \frac{d\theta_{do}}{dt}) = \tau_d = J_{do}\frac{d^2\theta_{do}}{dt^2}, \qquad (4.11)$$

where D_d is the viscous damping coefficient of the damper, θ_{do} is the position of the inertial flywheel, and J_{do} is the inertia of the damper housing. Using the last four equations, after linearizing, we obtain the following transfer function

$$\overline{H}(s) = \frac{\Theta(s)}{V_b(s) - V_a(s)} = \frac{b_1 s + b_0}{a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}, \qquad (4.12)$$

where $V_a(s)$ and $V_b(s)$ are the Laplace transforms of the applied voltages at phases A and B, respectively, and where

$$\begin{aligned}
a_4 &= J_m J_{do} L + J_{di} J_{do} L \\
a_3 &= J_m J_{do} r + J_m D_d L + J_{do} D L + J_{di} J_{do} r + J_{di} D_d L + J_{do} D_d L \\
a_2 &= J_m D_d r + J_{do} D_d r + D D_d L + J_{di} D_d r + c_1 J_{do} L + J_{do} D_d r + \\
&\quad p\Phi_m^2 n^2 sin(\frac{N_r\lambda}{2})(sin(\frac{N_r\lambda}{2}) - cos(\frac{N_r\lambda}{2}))N_r J_{do}
\end{aligned}$$

$$
\begin{aligned}
a_1 &= DD_d r + c_1 J_{do} r + c_1 D_d L + p\Phi_m^2 n^2 sin(\frac{N_r\lambda}{2})(sin(\frac{N_r\lambda}{2}) - cos(\frac{N_r\lambda}{2}))N_r D_d \\
a_0 &= c_1 D_d r \\
b_1 &= p\Phi_m n sin(\frac{N_r\lambda}{2}) J_{do} \\
b_0 &= p\Phi_m n sin(\frac{N_r\lambda}{2}) D_d \\
c_1 &= 2p^2\Phi_m n I_o cos(\frac{N_r\lambda}{2}).
\end{aligned}
$$

A positive α is optimal for $\overline{H}(s)$ provided the following conditions are satisfied by the coefficients:

$$
\begin{aligned}
0 &< a_4 \\
0 &< a_3 \\
0 &< a_3 a_2 - a_4(a_1 + \alpha b_1) \\
0 &< [a_3 a_2 - a_4(a_1 + \alpha b_1)](a_1 + \alpha b_1) - a_3^2(a_0 + \alpha b_0) \\
0 &< a_0 + \alpha b_0 \\
0 &\leq (a_4 b_0 - a_3 b_1) \\
0 &\leq 2a_1 b_1 + \alpha b_1 - 2a_2 b_0 \\
0 &\leq 2a_0 b_0 + \alpha b_0^2
\end{aligned}
$$

The last three conditions are sufficient to satisfy the condition (ii) of Theorem 1. The first two conditions automatically hold.

Since $\overline{H}(s)$ has relative degree 3, it is not positive real irrespective of the values of its coefficients. Let us now suppose that a precompensator is added in the feedforward path as shown in the block diagram at Figure 4.3. We assume that a precompensator of derivative type is used. More specifically, linearized precompensator is assumed to have a transfer function of the form

$$H_1(s) = as^2 + b. \tag{4.13}$$

So the overall transfer function of the precompensator and the motor becomes

$$H(s) = \frac{\Theta(s)}{E(s)} = \frac{ab_1s^3 + ab_0s^2 + bb_1s + bb_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}, \tag{4.14}$$

A positive α is optimal for $H(s)$ provided the following conditions hold.

$$\begin{aligned}
0 &\leq -2aa_4b_0 + 2aa_3b_1 + \alpha a^2b_1^2 \\
0 &\leq 2a_4bb_0 + 2aa_2b_0 - 2a_3bb_1 - 2aa_1b_1 + \alpha a^2b_0^2 - 2\alpha abb_1^2 \\
0 &\leq -aa_0b_0 - a_2bb_0 + a_1bb_1 + \alpha b^2b_1^2 - 2\alpha abb_0^2 \\
0 &\leq 2a_0bb_0 + \alpha b^2b_1^2 \\
0 &< a_3 + \alpha ab_1 \\
0 &< a_0 + \alpha bb_0 \\
0 &< (a_3 + \alpha ab_1)(a_2 + \alpha ab_0) - a_4(a_1 + \alpha bb_1) \\
0 &< [(a_3 + \alpha ab_1)(a_2 + \alpha ab_0) - a_4(a_1 + \alpha bb_1)](a_1 + \alpha bb_1) \\
&\quad -(a_3 + \alpha ab_1)(a_0 + \alpha bb_0)
\end{aligned}$$

Furthermore, the transfer function $H(s)$ is positive real and therefore any $\alpha > 0$ is optimal provided a and b are so chosen that:

$$\begin{aligned}
0 &< a_3a_2 - a_4a_1 \\
0 &< a_1(a_3a_2 - a_4a_1) - a_0a_3^2 \\
0 &< aa_3b_1 - aa_4b_0 \\
0 &< aa_2b_0 + a_4bb_0 - aa_1b_1 - a_3bb_1 \\
0 &< a_1bb_1 - aa_0b_0 - a_2bb_0 \\
0 &< a_0bb_0
\end{aligned}$$

The fact that stepping motors hardly need feedback compensation for position control indicates that the denominator polynomial of $H(s)$ has stable poles for realistic values of the parameters a_i. Hence, it may be possible to satisfy the above set of inequalities for a wide range of stepping motors by a suitable choice of a and b.

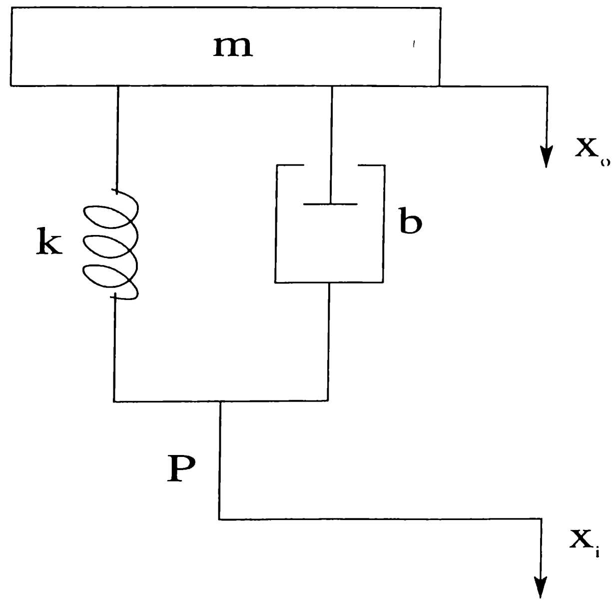

Figure 4.1: Automobile suspension system.

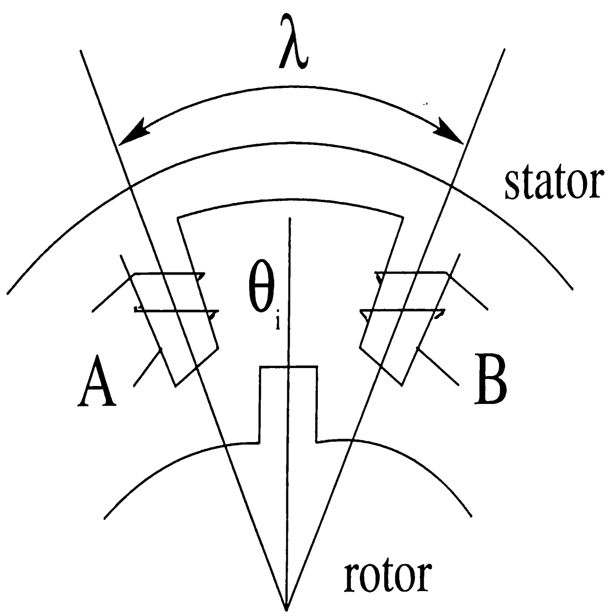

Figure 4.2: Model for VR stepping motor.

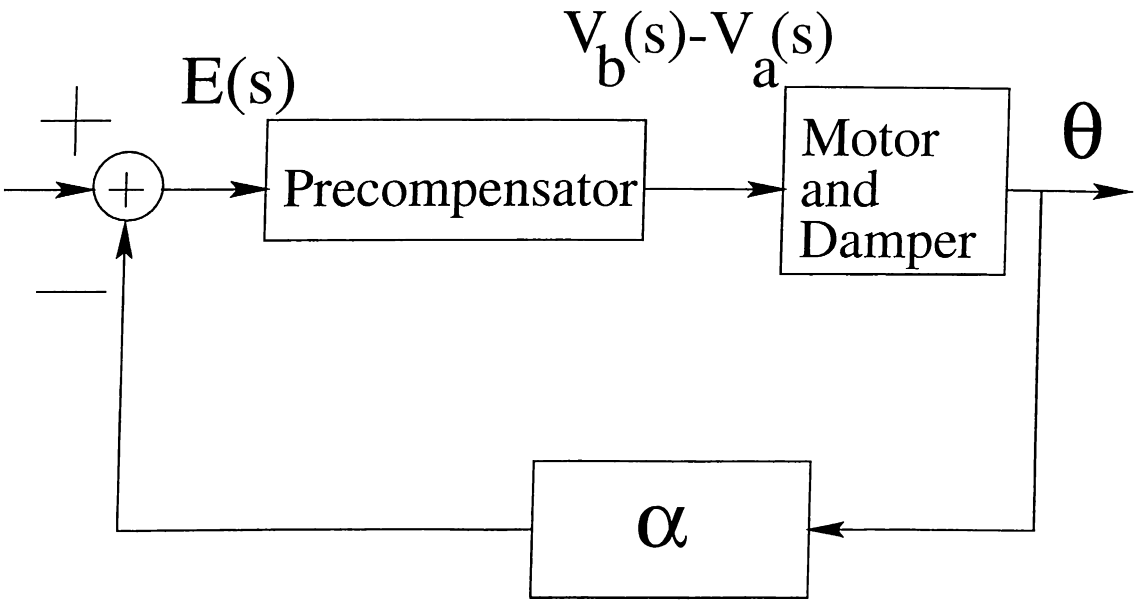

Figure 4.3: Feedback configuration for the precompensated motor.

Chapter 5

CONCLUSIONS

From our investigation of systems for which some or all positive constant output feedbacks are optimal emanates the positive realness conditions in Theorems 1 and 2. As expected, generally high gain feedbacks turn out to be optimal for minimum phase systems and small gain feedbacks may not be optimal or even stabilizing. This follows from Theorem 3 and Theorem 1.

The fields of optimal control and positive real systems both being very old and well investigated, all these results are probably known in some form or other. What may be new is the focus of attention at the optimality of an output feedback. The connection between optimal constant output feedbacks and the property of positive realness seems to be either newly being noticed or being rediscovered, [8].

We have only considered scalar, continuous-time, strictly proper systems leaving the pursuit of the extensions to more general situations for future work.

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