OPTIMAL ROW-COLUMN DESIGNS FOR TREATMENT CONTROL COMPARISONS

A THESIS
SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Murat Aksu
August, 1997

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By tarafından bağışlanmıştır

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August, 1997

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

OPTIMAL ROW-COLUMN DESIGNS FOR TREATMENT CONTROL COMPARISONS

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M.S. in Industrial Engineering
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August, 1997

Problem of comparing a set of test treatments with a control or standard treatment arises in many applications. In the literature, there exist a number of alternative design settings available for the situation being considered. Problem is to choose one among several alternatives which is best in some sense. In this thesis, we considered two-way elimination of heterogeneity model with simultaneous confidence coefficient criterion. A procedure for making exact joint confidence statements for multiple comparisons with a control was described and some methods for the construction of the Balanced Treatment Row-Column Designs (BTRCD's) were given. Finally, tables of optimal BTRCD's were provided for practical range of parameter values.

Key words: Balanced Treatment Row-Column Designs, Optimal Designs, Simultaneous Confidence Interval Criterion, Two-Way Elimination of Heterogeneity.

ÖZET

TEST VE KONTROL ÖRNEKLERİNİN KARŞILAŞTIRILMASI İÇİN OPTİMUM SATIR SÜTUN DİZAYNLARI

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Tez Yöneticisi: Doç. Ülkü Gürler
Ağustos, 1997

Test örneklerini bir kontrol örneği ile karşılaştırma problemi ile bir çok uygulamada karşılaşılmaktadır. Göz önünde bulundurulan durum için kullanılabilecek pek çok alternatif dizayn vardır. Problem, alternatifler arasında en iyiyi, belirli bir ölçüte göre seçmektir. Bu çalışmada heterasyonun çift yönü eliminasyonu modeli ve eş anlı güven aralığı ölçütü kullanılmıştır. Eş anlı güven aralığının hesaplanması ve optimum dizaynların bulunması için araç ve metotlar tarif edilmiştir. Son olarak, heterasyonun iki yönlü eliminasyonu ve eş anlı güven aralığı kriterine göre optimum olan satır sütun dizaynlarının belirli parametre değerleri için tabloları verilmiştir.

Anahtar sözcükler: Dengeli Dağılımlı Satır Sütun Dizaynları, Eş Anlı Güven Aralığı Ölçütü, Heterasyonun İki Yönlü Eliminasyonu, Optimum Dizaynlar.

To those who have filled my life with love
My sister Zeliha, and Esra

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Chapter 1

Introduction

Experimental design methods play an important role in process development and process trouble shouting as well as selection of best treatment among several alternatives or comparing new candidates to a control item.

Experiments are performed by investigators in virtually all fields of inquiry, usually to discover something about a particular process or system. Literally, an experiment is a test. A designed experiment is a test or series of tests in which purposeful changes are made to the input variables of a process or system so that we may observe and identify the reasons for changes in the output response.

As an example of an experiment, suppose that a metallurgical engineer is interested in studying the effect of two different hardening processes, oil quenching and salt water quenching, on an aluminum alloy. Here the objective of the experimenter is to determine the quenching solution that produces the maximum hardness for this particular alloy. The engineer decides to subject a number of alloy specimens after quenching. The average hardness of the specimens treated in each quenching solution will be used to determine which solution is the best.

In any experiment, the results and conclusions that can be drawn depend to a large extend on the manner in which the data were collected. To illustrate this point, suppose that the metallurgical engineer in the above experimental setting used specimens from one heat in the oil quench and specimens from a second heat in the saltwater quench. Now, when the mean hardness is compared, the experimenter is unable to say how much of the observed difference is the result of the quenching media and how much is the result of inherit differences between the heats. Indeed, the effects of quenching media and heat were confounded, that is both of these two factors affects the result simultaneously. Thus, the method of data collection has adversely affected the conclusions that can be drawn from the experiment.

According to Montgomery[16], while considering an experimental design like the one described above, number of important points should be answered:

- How many experimental units should be used and in what order should data be collected?
- What difference in average observed values among treatments will be considered important?
- What are the external sources of variability and how could it be dealt with?
- Are there other factors that might affect the results that should be investigated or controlled in the experiments?

Experimental design methods have found broad application in many disciplines. In fact, we may view experimentation as part of the scientific process. Generally, we learn through a series of activities in which we make conjectures about a process, perform experiments to generate data from the process, and then use the information from the experiment to establish new conjectures.

It is obviously the fact that if an experimental design is to be performed most efficiently, then a scientific approach to planning the experiment must be employed.

As a special type of design problem, there are certain experimental conditions where one would like to compare the relative performance of some new treatments with respect to an existing standard or control treatment. Such a problem arises frequently in many biological, industrial and agricultural experiments, for example, in screening experiments or in the beginning of a long term experimental investigation where it is initially desired to determine the relative performance of the new test treatments with respect to the control treatment. If it is possible to employ a completely randomized design this problem can be handled by using the available theory. However, most of the practical situations may require the blocking of experimental units so that the precision of the experiment can be improved and bias is reduced. Consider the following experimental situation: A certain type of alloy is used in the production of specific part of jet engines. The R&D department has developed four new types of alloy that can be used for the same purpose. Each of these four alloys could be easily produced in the existing production facility and can replace the present one if any one of them proved to be stronger. Suppose that we have four testing machines and five operators in order to conduct the experiment. Since the variability between the machines and between the operators are suspected, the experiment must be designed to control such variability.

As a statistical problem, the question of how to compare the test treatments with the control cannot be answered unless it is asked in a more precise manner. To begin with it is needed to postulate a model for the response observed upon application of a treatment, test treatment or control, to an experimental unit. Three possible basic models could be considered: 0-way elimination of heterogeneity model in which all experimental units are homogeneous before application of treatments:

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \tag{1.1}$$

1-way elimination of heterogeneity model in which experimental units can be divided into several homogeneous blocks:

$$y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \tag{1.2}$$

2-way elimination of heterogeneity model in which the experimental units conceptually arranged according to rows and columns:

$$y_{ijh} = \mu + \alpha_i + \beta_j + \tau_h + \varepsilon_{ijh} \tag{1.3}$$

In models 1.1, 1.2 and 1.3 the y's denote observations obtained after applying treatment i to an experimental unit occurring in block j and row h, α_i represents the effect of treatment i, β_j is the effect of column j, τ_h is the effect of row h and ε's are independent random error factors. In other words, α_i, β_j, τ_h could be stated as the effect of new alloy i, machine j, and operator h respectively, in our alloy example.

For the one-way elimination of the heterogeneity model, the information matrix C_d of the differences $\alpha_0 - \alpha_i$ is:

$$C_d = diag\,(r_0, r_1, ..., r_p) - k^{-1} N_d N_d'$$

Here, $N_d = (r_{ij})$ is the incidence matrix of the design, r_i is the number of times treatment i occurs in the whole design and k is the common block size. Also, note that

$$r_i = \sum_{j=1}^{b} r_{ij}$$

where $r_{ij}$is the number of times treatment i is used in block j and the matrix. Let the matrix P

$$P = \begin{bmatrix} 1 & -1 & 0 & & 0 \\ 1 & 0 & -1 & \dots & 0 \\ . & & & & \\ . & & & & \\ . & . & . & & . \\ 1 & 0 & 0 & \dots & -1 \end{bmatrix} \tag{1.4}$$

be a $p \times (p+1)$ matrix. The matrix $PC_d^{-1}P'$ is called the covariance matrix of the vector of the estimators of the contrast $\alpha_0 - \alpha_i, ..., \alpha_0 - \alpha_p$; here C_d^{-1} is a generalized inverse of C_d.

For the 2-way elimination of the heterogeneity model the statistical setup consists of bk experimental units arranged in a $k \times b$ array, and the model of response under design is 1.3. Let

r_{ij} = number of times treatment i occurs in column j.

s_{it} = number of times treatment i occurs in row t.

$r_i = \sum_{j=1}^{b} r_{ij}$,

$N_d = (r_{ij})$, a $(p+1) \times b$ matrix.

$M_d = (s_{it})$, a $(p+1) \times k$ matrix.

P is the $p \times (p+1)$ matrix defined in 1.4

$r_d = (r_0, r_1, .., r_p)'$, then information matrix C_d[7] is calculated as:

$$C_d = diag\,(r_0, r_1, .., r_p) - k^{-1} N_d N_d' - b^{-1} M_d M_d' + (bk)^{-1}\, r_d r_d'$$

Now we may be more precise about what we mean by comparing test treatment with a control. In particular, because the primary goal is to determine which among the test treatments might be better than the control, we would like to estimate the magnitude of each $\alpha_0 - \alpha_i$ with as much precision as possible. More precise comparisons among test treatments found to perform better than the control at this initial stage is generally left to later experimentation. Under the assumptions made above, best linear unbiased estimators (BLUE's) $\hat{\alpha}_0 - \hat{\alpha}_i$ are used to estimate $\alpha_0 - \alpha_i$ under a given design d. In assigning treatments to experimental units, the contrast $\alpha_0 - \alpha_i$ should be made to be estimable. A design satisfying this latter condition is said to be treatment connected and obviously we should restrict our attention to such designs. Clearly there are a number of different designs available for the situation being considered here and we want to choose one of which is best in some sense. For example, one might choose a design that gives the minimal value among all available designs of

$$\sum_{i=1}^{p} var\left(\hat{\alpha}_0 - \hat{\alpha}_i\right) \tag{1.5}$$

or

$$\max_{1 \le i \le p} \ \text{var}\left(\hat{\alpha}_0 - \hat{\alpha}_i\right) \tag{1.6}$$

where $var\left(\hat{\alpha}_0 - \hat{\alpha}_i\right)$ denotes the variance of $\hat{\alpha}_0 - \hat{\alpha}_i$. A design which gives the minimum in 1.5 is called an A-optimal design and one which gives the minimum in 1.6 is called an MV-optimal designs. These minimizations is usually not easy. As in other cases of the exact design theory, it is highly unlikely to obtain one method which is capable of producing A optimal design for arbitrary values of b, k and p.

A-optimality and MV-optimality certainly appear to have very intuitive and appealing statistical interpretations and are the most widely used criteria for treatment-control comparisons type of design settings. Finding an A-optimal design corresponds to minimizing mean square error in inference and finding an MV-optimal design is analogous to finding a minimax procedure.

D-optimality selects a design which minimizes the determinant of the covariance matrix. But for the problem of comparing test treatments with a control, the D-optimality criterion does not seem to be either an intuitively or statistically suitable criterion because the design it selects as being optimal generally do not provide any more information about treatment-control comparisons than they do about comparisons among test treatments.

The structure of optimal designs for treatment-control comparisons seems to depend on the criterion used. Although A and MV-optimal designs are often the same, other criteria yield different designs, usually requiring either fewer or more replications of the control, but otherwise balanced with respect to test treatments. On the other hand, the A and MV-optimality criteria each have a natural and statistically meaningful interpretation as given above.

There are circumstances in which the experimenter is not sure whether to fit a one-way model or a two-way elimination of heterogeneity model to the data. For example, the performance of several technicians are being compared to the control and the days of the week as well as the hours within each day are the possible sources of heterogeneity. In such a situation it would be highly desirable to obtain a design which is A or MV-optimal under each of these models. Hedayat and Majumdar [10] has studied this aspect of the problem and gave some families of model robust designs.

Besides, some other criteria may be relevant in this context. For pilot experiments when the control is taken to be known and the interest lies in testing whether or not the overall effect of the new treatments are appreciable, we may want to contrast the average new treatment effect with the old one and minimize var$(\sum_i \widehat{\alpha}_i/p - \widehat{\alpha}_0)$ $i = 1, .., p$. This criterion seems appropriate to be called as J-optimality because it reduces to minimizing trace$\Big(JPC_d - P'\Big)$, with J the $p \times p$ matrix of all ones. Certain J-optimal designs are also E-optimal, where E-optimality is defined as minimizing the maximum variance of all the estimated contrasts $\sum_i c_i\,(\alpha_0 - \alpha_i)$ with $\sum_i c_i^2 = 1$. Thus although E-optimality does not appear to have a very natural statistical interpretation when there is a control, E-optimal plans may also deserve attention in some cases.

When we handle the design selection problem, we may choose among several optimality criteria. A- and MV-optimality criteria have statistically meaningful interpretation, i.e. both refer to minimizing suitable functions of the variances of the $\widehat{\alpha}_0 - \widehat{\alpha}_i$, they do not take their correlations into account. Thus the optimal designs derived would seem to be appropriate when the results of the experiment are to be reported in terms of the above point estimates accompanied by their estimated standard errors or in terms of separate confidence interval estimates of the $\widehat{\alpha}_0 - \widehat{\alpha}_i$, $i = 1\ , .., p$. However in many practical applications a simultaneous confidence region is much more appropriate than separate confidence intervals. In our alloy example, primary criterion is the parameter $\widehat{\alpha}_0 - \widehat{\alpha}_i$ (test treatments with large values being preferred) but if there also is a secondary criterion such as cost then the precise rules for selection of the test treatments can not be stated in advance. For example, depending on the experimental results the two apparently "best" test treatments (in terms of the $\widehat{\alpha}_0 - \widehat{\alpha}_i$ values) may be selected or even the third or fourth apparently "best" test treatment may be selected. A set of *simultaneous* confidence intervals guarantees a specified confidence coefficient regardless of which test treatments are selected and for which the corresponding confidence interval estimates are reported. Thus, from our viewpoint, the natural optimality criterion for the problem is that of maximizing the coverage probability of simultaneous confidence region.

There are some reasons why simultaneous confidence interval criterion has been less popular in the literature than A-optimality and MV-optimality. First, because there is no closed form for the multivariate t or multivariate normal probability point, it is more difficult to establish that a design is "optimal". Second, because the probability point is function of the yardstick d/σ, it is possible that for different values there will be different "optimal" designs.

In conjunction with the foregoing discussion, it was desirable to provide tables of best available (conjectured optimal) BTRCD's for the 2-way elimination of the heterogeneity model and simultaneous confidence interval criterion. Most difficult part of the research was constructing BTRCD's. In order to solve the construction problem, we have used two main sources:

1. Available BTRCD's in literature, and
2. A method of construction described in the Chapter 4.

The organization of the thesis is as follows: In the next section a literature review will be given. The third chapter is devoted to the preliminaries and problem definition as well as definitions of admissibility and optimality. Derivation of the formulas for variance and correlation figures of treatment-control contrasts are provided in Chapter 4. Last chapter is reserved for conclusion and future directions of research. Tables of parameters of best available designs and building blocks for the construction of those designs will be provided in Appendix A. Finally, for the sake of completeness, derivation of a well known approximation for estimating the multi-variate normal probability integral and the normal equations for the least squares estimates will be given in the Appendix B.

Chapter 2

Literature Review

Comparing various makes or types of a certain item, called test treatments, with a currently used control treatment is a common problem. Examples of such items are different types of apparatus different brands of a drug, different types of fertilizer and different varieties of a plant. A study aimed toward a decision to retain or replace the current item will generally involve an experiment and the use of some statistical methods. Results of these studies, the performance of the items will play an important role in reaching a decision. However, in some cases other factors may play a role as well.

Earliest works on this problem was carried out by Dunnet[6]. He also posed the problem of optimally allocating experimental units to control and test treatments so as to maximize the probability associated with the joint confidence statement concerning the many-to-one comparisons between the mean of the control treatment and the means of the test treatments. But this paper and some other early works tacitly assumed that a completely randomized design was used. However, many practical situations may require the blocking of experimental units in order to cut down on bias and improve the precision of the experiment. In traditional design theory the comparisons of all treatment pairs are of equal importance. This leads to the use of such classical designs like balanced incomplete block designs (BIB's).

However, BIB designs apparently are not appropriate for the present setting in which the control plays a distinguished role. Because of the special role played by the control treatment, Cox[5] proposed a design that employs the control treatment an equal number of times (once, twice,etc.) in each block, and the test treatments forming a BIB design in the remaining plots of the blocks; but no analytical details were given for this proposed design. A special case of Cox's design (i.e. the control treatment is employed once in each block) is studied by Pesek[18]; he showed that this design is more efficient than a BIB design for comparisons with a control, but it is less efficient for pairwise comparisons between the test treatments. It is obviously the fact that even the Cox's more general design is quite restrictive.

Bechhofer and Tamhane[3] were the first to study the problem of obtaining optimal block designs for the treatment control comparison problem. But unlike the case of BIB designs, the structure of optimal designs for treatment-control comparisons seems to depend heavily on the criterion used. Bechhofer and Tamhane used the simultaneous confidence coefficient criteria. This criteria chooses the design with maximum joint confidence coefficient. This approach not only requires the equal block sizes which are smaller than or equal to number of treatments and linear model described in 1.2 but also an extra assumption concerning the form of distribution of the random variables involved. The most commonly made distributional assumption is that of normality assumption. Bechhofer and Tamhane proposed a class of designs called balanced treatment incomplete block (BTIB) designs. A BTIB is a design $d \in C(b,k,p)$, where $C(b,k,p)$ is the set of all designs for given b, k, p, if it satisfies the following conditions:

$$\lambda_{01} = \lambda_{02} = \ldots = \lambda_{op} = \lambda_0$$

$$\lambda_{12} = \lambda_{13} = \ldots = \lambda_{21} = .. = \lambda_{p-1,p} = \lambda_1$$

In words, BTIB is a design $d \in C(b,k,p)$ in which each treatment-treatment pair occurs equal number of times in the design and also each treatment-control pair occurs equal number of times. Bechhofer and Tamhane [1] used general linear statistical model for one-way elimination of heterogeneity and they also accepted the normality assumption. Optimal designs within BTIBD class for selected (p,k,b) where $k \leq p$ is the number of plots per block, and b is the total number of blocks available for experimentation were provided. Their first work has a limited scope, optimal designs for $p = 2$,..,6 , $k = 2$ and for $p = 3$, $k = 3$ were provided. Bachhofer and Tamhane provided tables of exact(discrete), optimal designs for these (p,k) values for a range of b's. Also tables of approximate (continuous) optimal designs are given for situations in which very large b-values are required. Following this work Bechhofer and Tamhane [2] improved their results for $p = 4$, $k = 3$ and $p = 5$, $k = 3$. In their later study Bechhofer and Tamhane restricted the search for optimal designs using new definitions of admissibility, and providing strong and combinational admissibility concepts. In the first work for each of the six cases studied it is known that there are only two (non-equivalent) admissible generator designs while for the cases examined in the later study there are many (non-equivalent) admissible generator designs. The full set of generator designs is not known for either of these two cases, so the optimal designs given by Bechhofer and Tamhane are optimal relative to the generator designs which are known. However, they conjecture that they have enumerated all of the admissible generator designs for each of these later cases, and stated that if additional ones do exist the incremental gain that would be achieved by using full set in place of the set used in the study would be very small.

Since the main objective of the experiment is to compare the control treatment with the test treatments, a criterion based on the variances of estimators of the comparisons between the control and each of the test treatments would be a meaningful measure. Majumdar and Notz [15] studied the A-optimality of BTIB designs. They also provided MV-optimal designs as well as A-optimal designs for the 1-way elimination of the heterogeneity model. Main purpose of their article was to investigate further the method for finding A-optimal and A-efficient designs. Hedayat and Majumdar[8] gave an algorithm and a catalog of A- and MV-optimal designs. Hedayat and Majumdar[9] also provided families of A- and MV-optimal designs. Optimal designs for the 2-way elimination of the heterogeneity model was studied by Notz[17]. Majumdar[13] considered the problem of comparing test treatments with more than one controls.

Jacroux [11][12] gave new methods for obtaining MV-optimal designs under 1-way elimination of the heterogeneity model. Hedayat and Majumdar [10] studied designs simultaneously optimal under both 1- and 2-way elimination of heterogeneity models. Cheng, Majumdar, Stufken and Türe [4] gave new families of A- and MV-optimal designs and some approximations for one-way elimination models. Türe[21] derived conditions under which BTRCD's are A-optimal. Later on Türe[20] provided tables of A-optimal and A-efficient BTRCD's. Finally, Majumdar[14] explore connection between maximum joint confidence interval probability and A-optimality criteria for the 1-way elimination of heterogeneity model.

Chapter 3

Problem Definition and Preliminaries

In this chapter we will provide a rationale for choosing a design from a set of competing BTRCD's where $\rho \geq 0$. Also a procedure for calculating the joint confidence interval probability for the case of negative correlation will be offered. Throughout this study we considered one-sided confidence intervals where σ^2 is assumed to be known. For the sake of simplicity, without losing generality, σ^2 is considered as fixed and equal to one. Our purpose is to design an efficient experiment to compare p new treatments with the control. A design is a $k \times b$ array of integers where k is the common block size and b is the total number of blocks. Let the treatments be indexed by $0, 1, ..., p$ with 0 denoting the control treatment and $1, 2, ..., p$, where $p \geq 2$ and $k \leq p$, denoting the test treatments. Thus, $N = kb$ is the total number of blocks available. We will consider the usual additive model:

$$y_{ijh} = \mu + \alpha_i + \beta_j + \tau_h + \varepsilon_{ijh}$$

with $\sum_{i=0}^{p} \alpha_i = \sum_{j=1}^{b} \beta_j = \sum_{h=1}^{k} \tau_h = 0$; the ε_{ijh} are assumed to be i.i.d. $N(0, \sigma^2)$ normal random variables with zero mean and σ^2 variation. It is desired to make an exact joint confidence statement concerning the p differences $\alpha_0 - \alpha_i$ based on their best linear unbiased estimators (BLUE's) $\hat{\alpha}_0 - \hat{\alpha}_i$ $(1 \leq i \leq p)$.

3.1 Choice of Designs

Since the primary aim is to make a confidence statement that applies simultaneously to all of the p differences $\widehat{\alpha}_0 - \widehat{\alpha}_i$, $i = 1, ..., p$, our problem is regarded as being symmetric in these differences. A class of designs for which $var(\widehat{\alpha}_0 - \widehat{\alpha}_i) = \tau^2\sigma^2$ $(1 \leq i \leq p)$ and $corr(\widehat{\alpha}_0 - \widehat{\alpha}_{i_1}, \widehat{\alpha}_0 - \widehat{\alpha}_{i_2}) = \rho\sigma^2$, $(i_1 \neq i_2, 1 \leq i_1, i_2 \leq p)$ will be considered where the parameters τ and ρ depend on the design employed. For the one way elimination problem such designs are balanced treatment incomplete block(BTIB) designs. BTIBD's are balanced with respect to the test treatments. A design is a BTIBD if it satisfies the following conditions:

$$\begin{aligned} \lambda_0 &= \lambda_{01} = \lambda_{02} = ... = \lambda_{op} \\ \lambda_1 &= \lambda_{12} = \lambda_{13} = ... = \lambda_{21} = .. = \lambda_{p-1,p} \end{aligned}$$

where λ_0 , λ_1 are some integers.

In other words, each test treatment must appear with (i.e., in the same block as) the control treatment the same total number of times λ_0 over the design, and each test treatment must appear with every other test treatment the same number of times λ_1 over the design.

For the two way elimination model totally symmetric design class satisfying $var(\widehat{\alpha}_i - \widehat{\alpha}_0) = \tau^2\sigma^2$, $(1 \leq i \leq p)$ and $corr(\widehat{\alpha}_0 - \widehat{\alpha}_{i_1}, \widehat{\alpha}_0 - \widehat{\alpha}_{i_2}) = \rho\sigma^2$, $(i_1 \neq i_2, 1 \leq i_1, i_2 \leq p)$ is the class of equireplicate balanced treatment row column designs (BTRCD's) where all treatments occur same number of times over the whole design. From now on all BTRCD's considered will be equireplicate designs. Note that a design is called as a BTRCD if it satisfies the following conditions:

$$\begin{aligned} \lambda_0 &= \lambda_{01} = \lambda_{02} = ... = \lambda_{op} \\ \lambda_1 &= \lambda_{12} = \lambda_{13} = ... = \lambda_{21} = .. = \lambda_{p-1,p} \\ \mu_0 &= \mu_{01} = \mu_{02} = ... = \mu_{op} \\ \mu_1 &= \mu_{12} = \mu_{13} = ... = \mu_{21} = .. = \mu_{p-1,p} \end{aligned}$$

where μ_0, μ_1, λ_0 and λ_1 are some integers.

In words, each test treatment must appear with (i.e., both in the same block and the same row as) the control treatment the same total number of times λ_0 and μ_0 respectively over the design, and each test treatment must appear with (i.e., both in the same block and the same row as) every other test treatment the same number of times λ_1 and μ_1 over the design.

Remark 1 *Definitions of BTIBD and BTRCD place no restriction on r_i, the number of replications of the i^{th} test treatment used over the whole design. This implies that a design can be BTIB or BTRCD without the r_i $(1 \leq i \leq p)$ being equal. As an example of such a BTIB consider the following design for which $(p, k, b) = (4, 3, 8)$ and $\lambda_0 = 2$, $\lambda_1 = 2$ with $r_1 = r_2 = r_3 = 4$ and $r_4 = 5$:*

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 4 & 2 & 2 & 3 & 3 \\ 1 & 2 & 3 & 4 & 3 & 4 & 4 & 4 \end{bmatrix}$$

But for the BTRCD having an example is not that easy even if we do not restrict ourselves with equireplicate designs. Because a BTRCD should be both columnwise and rowwise symmetric. Additional constraints for the μ_0 and μ_1 force r_i's to be equal in most of the cases.

The specific multiple comparisons with the control (MCC) problem which is considered during our study is that of obtaining joint one-sided confidence intervals of the form

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d \qquad (1 \leq i \leq p)$$

for given values of (p, k, b) when σ^2 is known, and d is a specified yardstick associated with the width of the confidence interval. The probability P associated with this joint confidence statement can be written as

$$\begin{aligned} P &= \Pr\{\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d\,; \quad \{1 \leq i \leq p\}\} \qquad (3.1) \\ &= \Pr\{Z_i \leq d/\tau\sigma\,; \quad \{1 \leq i \leq p\}\} \\ &= \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left[\Phi\left(\frac{d/\tau\sigma + \sqrt{\rho}z}{\sqrt{1-\rho}}\right)\right] \phi(z)\, dz \end{aligned}$$

where $(Z_1, \ ..., Z_p)$ has a $p-variate$ equicorrelated standard normal distribution with common correlation ρ and common variation $\tau^2\sigma^2$ and $\Phi(\cdot)$ denotes the standard univariate normal distribution function. For given p and specified d/τ the probability of 3.1 depends on the BTRCD considered only through ρ and τ. This fact gives us an opportunity to compare BTRCD's among themselves.

3.1.1 Confidence Statement

Joint $100(1-\alpha)$ percent confidence intervals for the $\alpha_0 - \alpha_1 \, (1 \leq i \leq p)$ are given below.

One-sided Confidence Intervals: When σ^2 is unknown, the joint one-sided confidence intervals are defined by Bechhofer and Tamhane[3] as the following:

$$\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - t^{(\alpha)}_{v,p,\rho} \tau s_v \ \ (1 \leq i \leq p) \tag{3.2}$$

In 3.2, $t^{(\alpha)}_{v,p,\rho}$ denotes the upper equicoordinate α point of the p-variate equicorrelated central t distribution with common correlation ρ, and with degree of freedom v.

When σ^2 is known , the joint one-sided confidence intervals are obtained by replacing $t^{(\alpha)}_{v,p,\rho} \tau s_v$ in 3.2 by $z^{(\alpha)}_{p,\rho} \tau \sigma$ where $z^{(\alpha)}_{p,\rho}$ $(= t^{(\alpha)}_{v,p,\rho}$ for $v = \infty)$ denotes the upper equicoordinate α point of the p-variate equicorrelated standard normal distribution with common correlation ρ.

Derivation of the probability statement in 3.1 is provided in Appendix B with a related proof which states that correlation matrix of a BTRCD is positive definite if $\rho \in (-1/(n-1), 1)$.

3.2 Case of Negative Correlation

During our search for the optimal designs we have faced with a few negatively correlated designs. Although there are a negligible number of such BTRCD's, we have to find a method to estimate joint confidence interval probability for those negatively correlated designs. Both in the one-way elimination of heterogeneity and two-way elimination of heterogeneity models the joint probability P could be calculated relatively easily from

$$\begin{aligned} P &= P\left\{Z_i \leq d/\tau\sigma, (1 \leq i \leq p)\right\} \\ &= \int \Phi^p \left[\left(x\sqrt{\rho} + d/\tau\sigma\right)/\sqrt{1-\rho}\right] d\Phi\left(x\right) \end{aligned}$$

where $(Z_1, Z_2, .., Z_p)$ has a *p-variate* equicorrelated standard normal distribution with common correlation ρ and $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Unfortunately, this easy to calculate formula valid for evaluating joint probability integral only for the cases in which $\rho \geq 0$. For the one-way elimination problem all possible designs have positive correlation coefficient, so above formula is valid and sufficient. However, during the search for BTRCD's a few design came-up with negative correlation coefficient especially when $k = p$ and b is close to p value, and hence the above solution for calculating the joint probability coefficient could not be used for these cases. For these rare cases Monte Carlo method may be used to approximate the numerical value of the multinormal integral. To describe a method that applies to the multivariate normal distribution, a sufficient procedure to generate *n-dimensional* normal variates X_1, $X_2, ..., X_n$ with an $N(\mu, \sum)$ distribution should be available. Following well known method is proper to generate *n-dimensional normal variates* with an $N(\mu, \sum)$ distribution

3.2.1 Estimating Joint Probability When Correlation is Negative

The method described below is a well known procedure in literature for generating X_1, $X_2, ..., X_N$ from an $N(\mu, \sum)$ distribution with arbitrary but fixed

μ and $\sum$ (which is positive definite). This method depends on the following result; for every $n \times n$ positive definite matrix $\sum = (\sigma_{ij})$ there exist an $n \times n$ matrix $T = (t_{ij})$ such that $TT' = \sum$. In general, matrix T is not unique. However if we restrict our attention to the subclass of all lower triangular matrices, then it is unique and could be obtained easily.

Proposition 1 *If $\sum = (\sigma_{ij})$ is an $n \times n$ positive definite matrix, then there exists a unique lower triangular matrix $T = (t_{ij})$ satisfying $TT' = \sum$. Furthermore the elements of T are given by:*

$$
\begin{aligned}
t_{ij} &= 0 \text{ for all } 1 \leq i \leq j \leq n, \\
t_{11} &= \sqrt{\sigma_{11}} \\
t_{i1} &= \frac{\sigma_{i1}}{\sqrt{\sigma_{11}}} \text{ for } i = 2, \ldots, n \\
t_{jj} &= \left(\sigma_{jj} - \sum_{k=1}^{j-1} t_{jr}^2 \right)^{1/2} \quad \text{for } j = 2, \ldots, n \\
t_{ij} &= \left(\sigma_{jj} - \sum_{k=1}^{j-1} t_{ir} t_{jr} \right)^{1/2} \quad \text{for } j \leq n \text{ and } i = 2, \ldots, n
\end{aligned}
$$

If Z is $N_n(0, I_n)$ and T is the matrix obtained by applying the above proposition then $X = TZ + \mu$ has an $N_n(\mu, \sum)$ distribution. Consequently to generate independent random variates $X_1, X_2, .., X_n$ according to this distribution, the following algorithm[19] may be used.

1. Compute $T = (t_{ij})$
2. Input R_N, n, $\mu = (\mu_1, .., \mu_n)'$, and $T = (t_{ij})$, where R_N is the number of replications, n denotes the dimension of the multivariate normal distribution and T is the unique lower triangular matrix obtained by using above proposition.

3. Generate $(Z_1, Z_2, .., Z_t)$ (which are (pseudo) independent $N(0,1)$ variates) and apply the transformation $X_{it} = TZ + \mu$

   i.e. compute
$$X_{it} = \sum_{j=1}^{n} t_{ji} Z_j + \mu_i \text{ for } i = 1, .., n$$
   and then form $X_t = (X_{1t}, .., X_{nt})'$

4. Repeat step 3 for $t = 1, .., R_N$.

Above procedure will facilitate the generation of multivariate normal data. After having an efficient method to generate *n-variate* random variable one can follow the following steps to estimate the probability P.

1. Input n, μ, $\sum$, and R_N.
2. Set $c_t = 0$.
3. Repeat the following procedure for $t = 1$ to N: Generate *n-variate normal* and observe
$$c_{t+1} = \begin{cases} c_t & \text{if } X_t \notin A, \\ c_t + 1 & \text{if } X_t \in A \end{cases}$$
4. Compute $\hat{I} = c_n / R_N$ where $\hat{I}$ is the estimator of the probability.

3.3 Admissible and Optimal Designs

In search for the optimal design for given (p, k, b), it is desirable to eliminate from consideration certain designs that are uniformly dominated by other designs and hence cannot be optimal for any d/σ. Following section contains the definition of such inadmissible design as well as optimal and admissible designs

Definition 1 *For given (p, k) and specified d/σ the BTRCD that achieves the highest joint confidence coefficient P with the specified b is said to be optimal for that value of b.*

Definition 2 *If for given (p, k) we have two designs D_1 and D_2 with parameters (b_1, τ_1^2, ρ_1) and (b_2, τ_2^2, ρ_2) with $b_1 < b_2$, and if for every d and σ, D_1 yields a confidence coefficient P at least as large as (larger than) that yielded by D_2 when $b_1 < b_2$ $(b_1 = b_2)$, then D_2 is inadmissible with respect to D_1.*

Definition 3 *If a design is not inadmissible, then it is admissible. If for given (p, k) we have two designs D_1 and D_2 with parameters (b_1, τ_1^2, ρ_1) and (b_2, τ_2^2, ρ_2) and if $b_1 = b_2$, $\tau_1^2 = \tau_2^2$, and $\rho_1 = \rho_2$, then D_1 and D_2 are called as equivalent designs.*

Definition 4 *For given (p, k) consider two BTRC designs D_1 and D_2 with parameters (b_1, τ_1^2, ρ_1) and (b_2, τ_2^2, ρ_2), respectively. Design D_2 is inadmissible with respect to design D_1 if and only if $b_1 \leq b_2$, $\tau_1^2 \leq \tau_2^2$, and $\rho_1 \geq \rho_2$ with at least one inequality is being strict.*

From 3.1 it could be easily seen that as σ decreases for fixed d and ρ, the confidence coefficient P increases. Monotonicity with respect to ρ follows from a special case of Sleepian's inequality which states that if $X = (X_1, ..., X_n)$ has an $N_n(\mu, \sum)$ distribution with positive definite $\sum$ defined in B.1 then

$$P_{\mu\sum}\left[\cap_{i=1}^{n}\{X_i \leq a\}\right]$$

is a strictly increasing function of the correlation coefficient.

Example 1 *For an application of these definitions consider the following two designs for* $(b, p, k) = (10, 6, 6)$*:*

$$D_1 = \left\{ \begin{array}{cccccccccc} 0 & 0 & 0 & 1 & 6 & 5 & 4 & 3 & 2 & 1 \\ 2 & 0 & 0 & 0 & 1 & 6 & 5 & 4 & 3 & 2 \\ 0 & 6 & 0 & 0 & 2 & 1 & 3 & 5 & 4 & 3 \\ 0 & 0 & 1 & 0 & 4 & 3 & 2 & 6 & 5 & 4 \\ 6 & 4 & 3 & 2 & 0 & 0 & 0 & 1 & 6 & 5 \\ 3 & 5 & 4 & 5 & 0 & 0 & 0 & 2 & 1 & 6 \end{array} \right\},$$

$$D_2 = \left\{ \begin{array}{cccccccccc} 0 & 0 & 0 & 0 & 1 & 6 & 2 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 & 3 & 4 & 6 & 1 & 2 & 5 \\ 1 & 0 & 0 & 0 & 0 & 5 & 3 & 2 & 4 & 6 \\ 6 & 0 & 0 & 1 & 4 & 0 & 0 & 3 & 5 & 2 \\ 0 & 5 & 4 & 2 & 0 & 0 & 0 & 6 & 1 & 3 \\ 0 & 3 & 2 & 5 & 0 & 0 & 0 & 4 & 6 & 1 \end{array} \right\},$$

$r_0 = 18$, $r = 7$, $\lambda_0 = 10$, $\mu_0 = 21$, $\lambda_1 = 5$, $\mu_1 = 8$ *and* $\tau_1^2 = 0.22692$, $\rho_1 = 0.328859$ *for* D_1 *and* $r_0 = 24$, $r = 6$, $\lambda_0 = 10$, $\mu_0 = 24$, $\lambda_1 = 4$, $\mu_1 = 6$ *and* $\tau_2^2 = 0.24708$, $\rho_2 = 0.285714$ *for* D_2*. Since* $b_1 = b_2$, $\tau_1^2 < \tau_2^2$, *and* $\rho_1 > \rho_2$ *then we can claim that design* D_2 *is inadmissible with respect to design* D_1*.*

D_1 always yields larger joint confidence coefficient than D_2, so we do not have to consider design D_2 in our search for optimal designs .

Having the optimality and admissibility criteria on hand, we have used the following procedure to determine the optimal designs for specific (p, k, b) and d/σ values. Note that the term *optimal design* refers to the best design among available alternatives. Since the full set of BTRCD's for given (p, k, b) is not known, the optimal designs that we give are optimal relative to the BTRCD's known to us; however, we conjecture that we have enumerated all possible BTRCD's and if additional ones do exist the incremental gain that would be achieved by using the full set in place of our set would be very small.

1. For given values of p and k, start with a value of b.

2. Find all BTRCD's for this combination of (p,k,b) values.

3. Calculate τ^2 and ρ for every design available.

4. Eliminate inadmissible designs from the set of candidate designs for optimality.

5. For d/σ (typically) ranging from 0.1 to 1.0 calculate

$$P = \Pr\{\alpha_0 - \alpha_i \geq \hat{\alpha}_0 - \hat{\alpha}_i - d \quad \{1 \leq i \leq p\}\}$$

   for each admissible BTRCD.

6. The BTRCD with the highest value of P is optimal for that value of (p,k,b) and d/σ.

This procedure requires complete list of BTRCD's for given (p,k,b). This is a much more difficult job than it was for the one-way elimination of the heterogeneity model. In the one-way model if D_1 and D_2 are two BTIBD's with parameters (p,k,b_1), λ_{01}, λ_{11} and (p,k,b_2), λ_{02}, λ_{12} respectively, then $D_1 \cup D_2$ is also a BTIBD with parameters (p,k,b_1+b_2) , $\lambda_0 = \lambda_{01} + \lambda_{02}$ and $\lambda_1 = \lambda_{11} + \lambda_{12}$. Unfortunately, this is no longer the case for the BTRCD's. Union of two BTRCD's does not necessarily yield a new BTRCD. In order to explain this situation consider the following two BTRCD's. Note that all BTRCD's are also BTIB by definition.

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 & 4 \\ 1 & 3 & 4 & 0 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 4 & 2 & 0 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix}$$

D_1 and D_2 are both BTRCD with parameters $(p=4, k=4, b_1=6)$, $\lambda_{01}=6$, $\lambda_{11}=1$, $\mu_{01}=9$, $\mu_{11}=2$ and $(p=4, k=4, b_1=4)$, $\lambda_{01}=3$, $\lambda_{11}=2$, $\mu_{01}=3$, $\mu_{11}=2$ respectively. Let $D_3 = D_1 \cup D_2$

$$D_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 & 4 & 1 & 0 & 3 & 4 \\ 1 & 3 & 4 & 0 & 0 & 0 & 4 & 2 & 0 & 1 \\ 4 & 1 & 2 & 0 & 0 & 0 & 2 & 3 & 4 & 0 \end{bmatrix}$$

D_3 is a BTIB with $(p = 4, k = 4, b_1 = 10)$, $\lambda_{01} = 9$, $\lambda_{11} = 3$, but it is clearly not a BTRCD. Hence the construction of all possible designs remains as a difficult task. Once this is done, a computer program could be used to evaluate the joint confidence interval coefficients and so, optimal design for each (p, k, b) could be determined easily.

Chapter 4

Theoretical Results

4.1 Derivation of Variance and Correlation Expressions

In this section variance and correlation expressions for estimators of treatment control contrasts will be provided for the equireplicate BTRCD's. Let r_0 be the number of units allocated to the control treatment. Define the following quantities

r_{il} = number of replications of treatment i in column l,

s_{ih} = number of replications of treatment i in row h,

$\lambda_{ij} = \sum_l r_{il} r_{jl}$ = number of times treatments i and j are matched in the same column over the whole design,

$\mu_{ij} = \sum_h s_{ih} s_{jh}$ = number of times treatments i and j are matched in the same row over the whole design,

$r_i = \sum_l r_{il} = \sum_h s_{ih} = r$ total number of replications of the test-treatment i.

Now, for design $d \in C(p, b.k)$ the information matrix $M(d) = \{m_{ij}\}$ for estimating all $(\widehat{\alpha}_0 - \widehat{\alpha}_j)$'s given by (see [17][21]).

$$m_{ij} = \begin{cases} r_i - (1/k)\lambda_{ii} - (1/b)\mu_{ii} + (1/bk)r_i^2 & \text{if } i = j \\ -(1/k)\lambda_{ij} - (1/b)\mu_{ij} + (1/bk)r_i r_j & \text{if } i \neq j \end{cases} \tag{4.1}$$

Note that, without loss of generality, we take $\sigma^2 = 1$. From 4.1 , already know that

$$m_{ij} = -(1/k)\lambda_{ij} - (1/b)\mu_{ij} + (1/bk)r_i r_j for i \neq j$$

Now, we consider the expression $r_i - (1/k)\lambda_{ii}$. It can be rewritten as (apart from the divisor of k)

$$(kr_i - \lambda_{ii}) = \sum_{l=1}^{b}(kr_{il} - r_{il}^2) = \sum_{l=1}^{b} r_{il}(k - r_{il})$$

For fixed i, $k - r_{il}$ is the number units in column l which are assigned to all treatments except treatment i, hence it is equal to $r_{0l} + \sum_{j \neq i} r_{jl}$. Then,

$$\begin{aligned} \sum_{l=1}^{b} r_{il}(r_{0l} + \sum_{j\neq i}^{p} r_{jl}) &= \sum_{l=1}^{b} r_{0l}r_{il} + \sum_{j\neq i}^{p}\sum_{l=1}^{b} r_{il}r_{jl} \\ &= \lambda_{0i} + \sum_{j\neq i}^{p} \lambda_{ij} \\ &= (\lambda_0 + (p-1)\lambda_1) \end{aligned}$$

Similarly,

$$r - (1/b)\sum_{i=1}^{p} \mu_{ii} = (\mu_0 + (p-1)\mu_1)/b$$

Hence the diagonal elements of the M matrix is equal to

$$m_{ii} = [b(\lambda_0 + (p-1)\lambda_1) + k(\mu_0 + (p-1)\mu_1) + r^2 - bkr]/bk$$

Since M is completely symmetric it is of the form $aI + cJ$; where I is the identity matrix and J is the matrix of all one's. We can easily see that

$$m_{ii} = a + c \text{ and } m_{ij} = c \text{ if } (i \neq j),$$

and that

$$(M)^{-1} = (1/a)I - [c/a(a + pc)]J,$$

Finally, with using the basic algebra we will get the following:

$$
\begin{aligned}
c &= -(\mu_1/b) - (\lambda_1/k) + (r^2/bk) \\
a &= m_{ii} - m_{ij} \\
&= \left[b(\lambda_0 + (p-1)\lambda_1) + k(\mu_0 + (p-1)\mu_1) - bkr - (-k\mu_1 - b\lambda_1)\right]/bk \\
&= \left[b(\lambda_0 + p\lambda_1) + k(\mu_0 + p\mu_1) - bkr\right]/bk
\end{aligned}
$$

Note that diagonal and off diagonal elements of $(M)^{-1}$ matrix are estimators of *variances* and *covariances* of treatment-control contrasts respectively. Having this information on hand we could easily obtain the formulas for variance and covariance of the estimators of treatment-control contrasts. For the case of equireplicate designs, where $r_i = r$, expressions are as follows:

$$
\begin{aligned}
\rho &= \frac{b\lambda_1 + k\mu_1 - r^2}{\left[b(\lambda_0 + \lambda_1) + k(\mu_0 + \mu_1) + (p-1)r^2 - bkr\right]} \\
\tau^2 &= \frac{bk\left[b(\lambda_0 + \lambda_1) + k(\mu_0 + \mu_1) + (p-1)r^2 - bkr\right]}{\left[b(\lambda_0 + p\lambda_1) + k(\mu_0 + p\mu_1) - bkr\right]\left[b\lambda_0 + k\mu_0 + pr^2 - bkr\right]}
\end{aligned}
$$

Above expressions allows us to use admissibility definition and in this way we could be able to reduce effectively the number of candidate designs for optimality.

4.2 Results Concerning Variance and Correlation

During our search for optimal designs we have observed that symmetric designs usually perform better than the highly asymmetric cases. To be more specific, consider two BTRCD's D_1 and D_2 for a given (p, k, b). If three of the following four parameters λ_0, λ_1, μ_0, μ_1 are equal to or very close to each other and the remaining one differs significantly, then the design with significantly larger fourth parameter performs better than the other one. In this section we will try to give some results to verify our observations. For a BTRCD $\in C(p, k, b)$, define the following quantities for the sake of simplification.

$$A = b\lambda_1 + k\mu_1 - r^2$$

$$B = b\left(\lambda_0 + k\mu_0 + pr^2 - bkr\right) \tag{4.2}$$

Lemma 2 *For fixed λ_0, μ_0, r if λ_1 or μ_1 (or both) increases then ρ increases.*

Proof.

$$\begin{aligned}\frac{\partial \rho}{\partial \lambda_1} &= \frac{b(A+B) - bA}{(A+B)^2} \\ &= b\frac{B}{(A+B)^2}\end{aligned}$$

If B > 0 then the proof is complete. Obviously,

$$p\lambda_0 = \sum_{i=1}^{b} (k - r_{0i})\, r_{0i} \tag{4.3}$$

and $\lambda_0 = \frac{\sum_{i=1}^{b}(k-r_{0i})r_{0i}}{p}$
Similarly,

$$\mu_0 = \frac{\sum_{j=1}^{k} (b - s_{0j})\, s_{0j}}{p} \tag{4.4}$$

, also $r = \frac{bk - r_0}{p}$, and

$$\begin{aligned}pr^2 - bkr &= r(pr - bk) \\ &= -rr_0\end{aligned}$$

From 4.2, 4.3 and 4.4 we can immediately write:

$$\begin{aligned}B &= \frac{b\sum_{i=1}^{b} (k - r_{0i})\, r_{0i}}{p} + \frac{\sum_{j=1}^{k} (b - s_{0j})\, s_{0j}}{p} - rr_0 \\ &= \frac{bkr_0 - \sum_{i=1}^{b} r_{0i}^2 + bkr_0 - \sum_{j=1}^{k} s_{0j}^2}{p} - rr_0 \\ &\geq \frac{bkr_0 - r_0^2 + bkr_0 - r_0^2}{p} - rr_0 = \frac{2r_0\left(\overbrace{bk - r_0}^{pr}\right)}{p} - rr_0 \geq rr_0 \geq 0\end{aligned}$$

That implies $\partial\rho/\partial\lambda_1 \geq 0$ or ρ is an increasing function of λ_1. ∎

Lemma 3 *τ^2 is a decreasing function of λ_0 (μ_0).*

Proof. Since $\tau^2 = A + B \ / \ B(pA + B)$, derivative of τ^2 with respect to λ_0 is equal to:

$$\begin{aligned}
\frac{\partial \tau^2}{\partial \lambda_0} &= \frac{bB(pA+B) - b(A+B)(pA+2B)}{B^2(pA+B)^2} \\
&= -b\frac{pA^2 + 2AB + B^2}{B^2(pA+B)^2} \\
&= \frac{-b\left((p-1)A^2 + (A+B)^2\right)}{B^2(pA+B)^2} < 0
\end{aligned}$$

■

Lemma 4 *For fixed λ_1, μ_1, r if λ_0 or μ_0 (or both) increases then B increases, hence ρ decreases, provided that A is positive(which is not guaranteed).*

Proof. Since $\rho = A/(A+B)$ derivative of ρ is as follows: $\frac{\partial \rho}{\partial \lambda_1} = \frac{-Ab}{(A+B)^2}$, since B is always positive ρ decreases (provided that A is positive) with increasing λ_0 (μ_0). ■

Lemma 5 *τ^2 is a decreasing function of λ_1 (μ_1).*

Proof.

$$\begin{aligned}
\frac{\partial \tau^2}{\partial \lambda_0} &= \frac{bB(pA+B) - bpB(A+B)}{B^2(pA+B)^2} \\
&= \frac{(1-p)bB}{B^2(pA+B)^2}
\end{aligned}$$

Hence τ^2 (the variance) decreases when λ_1 increases. ■

Above lemmas state that, if everything is fixed larger λ_0 (λ_1) implies larger correlation coefficient and smaller variance, hence better design. By the perfect symmetry of the problem the same argument applies to μ_0 (μ_1).

4.3 Construction of BTRCD's

METHOD 1: If there exists BTRCD for given values of (p, k, b) then a new design with $b_{new} = b + 1$ $(k_{new} = k + 1)$ could be easily constructed by adding a block (row) of zeros $[0, ..., 0]'$ to the available BTRCD. Resulting design will be a BTRCD with parameters $\lambda_{0_{new}} = \lambda_0$ $(\lambda_{0_{new}} = \lambda_0 + r)$, $\lambda_{1_{new}} = \lambda_1$, $\mu_{0_{new}} = \mu_0 + r$ $(\mu_{0_{new}} = \mu_0)$, $\mu_{1_{new}} = \mu_1$ and $r_{0_{new}} = r_0 + k$ $(r_{0_{new}} = r_0 + b)$.

Example 2 *Consider two designs D_1 and D_2 with parameters $b_{d1} = 6$, $p_{d1} = 3$, $k_{d1} = 3$, $r = 5$, $r_{0_{d1}} = 3$ $\lambda_{0_{d1}} = 2$, $\lambda_{1_{d1}} = 4$, $\mu_{o_{d1}} = 3$, $\mu_{1_{d1}} = 9$ and $b_{d1} = 7$, $p_{d1} = 3$, $k_{d1} = 3$, $r = 5$, $r_{0_{d1}} = 6$ $\lambda_{0_{d1}} = 2$, $\lambda_{1_{d1}} = 4$, $\mu_{o_{d1}} = 8$, $\mu_{1_{d1}} = 9$ respectively. D_2 is produced simply by adding a column of zeros to the design D_1.*

$$D_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 2 & 1 & 3 & 2 & 3 & 1 \\ 3 & 2 & 1 & 3 & 1 & 2 \end{bmatrix}$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 3 & 2 & 3 & 1 \\ 0 & 3 & 2 & 1 & 3 & 1 & 2 \end{bmatrix}$$

METHOD 2: For the $k = p$ and $b = p$ there always exist a BTRCD such that all diagonal elements of the design matrix are 0's. New design D_2 could be produced easily by adding a column containing treatments $p = 1, ..., p$.

$$D_1 = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 3 & 0 & 4 & 1 \\ 4 & 1 & 0 & 2 \\ 1 & 3 & 2 & 0 \end{bmatrix}$$

Define $T = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}'$ as being a column of test treatments, new design D_2 is:

$$D_2 = \begin{bmatrix} 0 & 2 & 3 & 4 & 1 \\ 3 & 0 & 4 & 1 & 2 \\ 4 & 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 & 4 \end{bmatrix}$$

If p of T is added to the basic design D_1 then the resulting design will also be a BTRCD.

$$D_3 = \begin{bmatrix} 0 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\ 3 & 0 & 4 & 1 & 2 & 3 & 4 & 1 \\ 4 & 1 & 0 & 2 & 3 & 4 & 1 & 2 \\ 1 & 3 & 2 & 0 & 4 & 1 & 2 & 3 \end{bmatrix}$$

METHOD 3: It seems appropriate to provide some definitions before considering the third method which is our computer algorithm. Also note that we are searching for equireplicate BTRCD's with $k \leq p$. Let us define the quantities,

N_c : Maximum number of treatment-treatment pairs in the design over blocks.

$$N_c = \binom{p}{2}\lambda_1 = \sum_{i=1}^{b} \binom{k - r_{0i}}{2} \tag{4.5}$$

N_r : Maximum number of treatment-treatment pairs in the design over rows.

$$N_r = \binom{p}{2}\mu_1 = \sum_{h=1}^{k} \binom{b - s_{0h}}{2} \tag{4.6}$$

M_{ci} : Maximum number of treatment-treatment pairs in column i.

$$M_{ci} = \binom{k - ri}{2}$$

M_{rh} : Maximum number of treatment-treatment pairs in row i.

$$M_{rh} = \binom{b - s_{0h}}{2}$$

S_c : Total number of treatment-control pairs in the design over blocks.

$$S_c = p\lambda_0 = \sum_{i=1}^{b} (k - r_{0i})\, r_{0i} \tag{4.7}$$

S_r : Total number of treatment-control pairs in the design over rows.

$$S_r = p\mu_0 = \sum_{h=1}^{k} (b - s_{0h})\, s_{0h} \tag{4.8}$$

c_t : Total number of blocks satisfying the condition of $(k - r_{0i}) = t$ for $i = 1, .., b$

d_t : Total number of rows satisfying the condition of $(b - s_{0h}) = t$ for $h = 1, .., k$

$excess_c$: $N_c \bmod \binom{p}{2}$

$excess_r$: $N_r \bmod \binom{p}{0}$

Following constraints come from definition of BTRCD's and our earlier assumptions.

- Since we are searching for the equireplicate designs, $r_i = r \; (i = 1, .., p)$ and
$$[(k \times b) - r_0] \bmod p = 0$$
- r_0 should be obviously greater than zero. Otherwise, treatment-control comparison could not be evaluated.

Example 3 *Consider the design D_1 below with parameters $b = 4$, $p = 4$, $k = 4$, $r = 4$, $r_0 = 0$ $\lambda_0 = 0$, $\lambda_1 = 4$, $\mu_o = 0$, $\mu_1 = 4$*

$$D_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

Although D_1 is a BTRCD, it could not provide any information about treatment-control differences.

- $\lambda_{01} =, .., = \lambda_{0p} = \lambda_0$ should be an integer. From 4.7 it is known that
$$\lambda_0 = \frac{S_c}{p} = \frac{\sum_{i=1}^{b} (k - r_{0i}) r_{0i}}{p}$$
so, $S_c \bmod p = 0$ should be.

- Similarly $\mu_{01} =, .., = \mu_{0p} = \mu_0$ should be an integer. From 4.8

$$\mu_0 = \frac{S_r}{p} = \frac{\sum_{h=1}^{k} (b - s_{0h})\, s_{0h}}{p}$$

and above formula implies the $S_r \bmod p = 0$

Utilizing the prior information and adding some extra conditions we end up with the following algorithm which decides if there exist a BTRCD for the given values of (p, k, b) and r_0.

Algorithm 4.1

1. *For the specific values of* (p, k, b) *set* $r_0 = 0$
2. *If* $r_0 = kb$ *then stop else* $r_0 = r_0 + 1$
3. *Allocate control treatments over blocks and rows. If all possible combinations of* r_0 *controls are enumerated then go to step2 else continue.*
4. *Set* $c_t = 0$ $(t = 1, .., k)$ *and* $d_t = (t = 1, .., b)$
5. *If* $[(kb) - r_0] \bmod p = 0$ *then continue else go to step3*
6. *If* $S_c \bmod p \neq 0$ *then go to step3*
7. *If* $S_c \bmod p = 0$ *then check the following condition for each* $c_t > 0$.

   if $c_t > 0$ *and* $t < k$ *then continue if* $(t \times c_t) \bmod p = 0$

   else go to step3
8. *If* $S_r \bmod p \neq 0$ *then go to step3*
9. *If* $S_r \bmod p = 0$ *then check the following condition for each* $d_t > 0$.

   if $d_t > 0$ *and* $t < b$ *then continue if* $(t \times d_t) \bmod p = 0$

   else go to step3

10. If $N_c \bmod \binom{p}{2} = 0$ *then*

consider the following if statement

if $c_t \bmod \binom{p}{t} = 0$ *for all* $c_t > 0$ $(t = 1, .., b)$ *then continue else go to step3*

else begin

if $excess_c = p$ *and* $c_2 = p$ *then go to step8 else go to step3*

if $excess_c = 3p$ *and* $c_3 = 3p$ *then go to step8 else go to step3*

end else

11. If $N_r \bmod \binom{p}{2} = 0$ *then*

consider the following if statement

if $d_t \bmod \binom{p}{t} = 0$ *for all* $d_t > 0$ $(t = 1, .., k)$ *then continue else go to step3*

else begin

if $excess_r = p$ *and* $d_2 = p$ *then go to step9 else go to step3*

if $excess_r = 3p$ *and* $d_3 = 3p$ *then go to step9 else go to step3*

end else

12. Save the information of r_0 *and allocation of controls over rows and columns. There is a BTRCD for this value of* (p, k, b) *and* r_0.

13. If the search is not finished, turn back to step3.

The algorithm described here makes it possible to construct probably the all equireplicate BTRCD's for given (p, k, b). Another main source of BTRCD's is the tables of A or MV-optimal BTRCD's provided by several authors. This tables allow us to check out our construction methodology with serving as a valuable source of designs.

Chapter 5

Conclusion

There are two widely studied criteria to arrive at optimal designs for comparing a set of test treatments with a control. First one, offered by Kiefer for the first time, chooses a design that gives the best estimators for treatment-control contrast; the most widely accepted criterion of this type is A-optimality which minimizes the sum of the variances of the estimators. Second one, introduced by Bechhofer and Tamhane[3], is to find a design that maximizes the coverage probability of a simultaneous confidence intervals for treatment control contrasts.

For the one-way elimination of the heterogeneity model, optimal BTIBD's for several (p, k, b) combinations were provided in the literature. The problem of determining the optimal set of BTRCD's for maximum joint confidence interval criteria was an open and unanswered problem. But, this later problem is quite a difficult one. In our study, we have focused on this open problem of finding optimal design for the two-way elimination of the heterogeneity model with a special emphasis on treatment-control comparisons. We attempted to generate all possible BTRCD's for specific values of (p, k, b). The admissibility criteria offered by Bechhofer and Tamhane[3] was found to be useful for the two-way elimination problem and it is used to eliminate inadmissible design from the set of all candidate BTRCD's. This elimination allows us to consider a much smaller set of candidate designs for the optimality. Variance and correlation figures are calculated for all admissible designs and joint confidence probabilities are calculated for each d/σ typically ranging from 0.1 to 1.0. Also some methods to construct BTRCD's were given. Finally tables of optimal designs for several p, k, b and d/σ values were provided.

We will conclude by noting some of the open problem areas that need further research. First problem is that of finding efficient or optimal designs for comparing test treatments with several controls for the two-way elimination of the heterogeneity model and simultaneous confidence coefficient criterion.

The next problem is that of establishing a relationship between different optimality criteria. Although an optimal design for a criterion can mostly be optimal or near optimal for the other criterion, much remains to be done in this area. Majumdar's paper[14] was an important improvement for the one-way elimination model. He derived several inadmissibility and an admissibility results for BTIB designs. Such considerations should be taken into account for the two-way elimination model also.

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Appendix A

Tables of Optimal Designs

In the following section of Appendix A interested reader can find tables of optimal designs. All parameter values which are needed to distinguish a specific BTRCD are given in these tables. For the sake of completeness, in the last column of each table basic building blocks to construct the given design are provided. For example, the layout of optimal design for $k = 4$, $p = 4$ and $b = 5$ is given as $A + T$ in the table $K4P4$. We can easily find the layout of A and T in the last section of Appendix A. Let us write these designs

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 4 & 2 & 0 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix} \qquad T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Obviously

$$A + T = \begin{bmatrix} 0 & 1 & 2 & 3 & 1 \\ 1 & 0 & 3 & 4 & 2 \\ 4 & 2 & 0 & 1 & 3 \\ 2 & 3 & 4 & 0 & 4 \end{bmatrix}$$

is the design that we are looking for. But sometimes construction problem could not be handled easily. As an example, let us try to construct design $A + 5T$. If we simply add up five blocks of T to A, then the resulting design will not be a BTRCD. Consider the design given below

$$A + 5T = \begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 4 & 2 & 2 & 2 & 2 & 2 \\ 4 & 2 & 0 & 1 & 3 & 3 & 3 & 3 & 3 \\ 2 & 3 & 4 & 0 & 4 & 4 & 4 & 4 & 4 \end{bmatrix}$$

At the first glance it seems as if it is not a BTRCD. Indeed it is true, but if we rearrange the rows of some T's then we could get the following design which is the right one.

$$A + 5T = \begin{bmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 1 \\ 1 & 0 & 3 & 4 & 2 & 3 & 4 & 1 & 2 \\ 4 & 2 & 0 & 1 & 3 & 4 & 1 & 2 & 3 \\ 2 & 3 & 4 & 0 & 4 & 1 & 2 & 3 & 4 \end{bmatrix}$$

Since it is practically not feasible to write out the all possible building blocks we choose to give the basic blocks. Although most of the time small rearrangements are enough to obtain the desired design, in some cases construction of the BTRCD's may require trial and error. Related to this problem, things that the reader should be careful about could be summarized as follows:

- Be aware of the fact that layout information in the tables are provided just to help the construction of the exact design. So, construction might require trial and error.
- If you need to rearrange the building blocks, do never increase or decrease the number of treatments and keep fixed the zeros in their original places.
- Finally, checkout the parameters of the design constructed with the parameter values of the optimal design given in the table. Obviously, two sets of values should be equal and if you end up with different parameter values try to reconstruct the design.

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	3	2	2	2	2	1	2	1	2	A+T
0.1 ..0.4	4	2	2	2	3	1	2	2	5	C+2T
0.5 .. 1	4	2	2	2	3	1	3	2	4	A+T
0.1 .. 0.2	5	2	2	2	4	1	3	3	8	C+3T
0.3 .. 0.7	5	2	2	2	4	1	4	3	8	A+2T
0.8 .. 1	5	2	2	4	3	2	6	1	4	A+B+T
0.1 .. 0.2	6	2	2	2	5	1	4	4	13	C+4T
0.3 .. 0.4	6	2	2	2	5	1	5	4	12	A+4T
0.5 .. 1	6	2	2	4	4	2	8	2	8	A+B+2T
0.1 .. 0.2	7	2	2	2	6	1	5	5	18	C+5T
0.3	7	2	2	2	6	1	6	5	18	A+5T
0.4 .. 0.7	7	2	2	4	5	2	10	3	12	A+B+2T
0.8 .. 1	7	2	2	6	4	3	12	1	8	2A+B+T
0.1 .. 0.2	8	2	2	2	7	1	6	6	25	C+6T
0.3	8	2	2	2	7	1	7	6	24	A+6T
0.4 .. 0.5	8	2	2	4	6	2	12	4	18	A+B+2T
0.6 .. 1	8	2	2	6	5	3	15	2	12	2A+B+2T
0.1	9	2	2	2	8	1	7	7	32	C+7T
0.2	9	2	2	2	8	1	8	7	32	A+7T
0.3	9	2	2	4	7	2	14	5	24	A+B+5T
0.4 .. 1	9	2	2	6	6	3	18	3	18	2A+B+3T
0.1 .. 0.2	10	2	2	2	9	1	8	8	41	C+8T
0.3	10	2	2	4	8	2	16	6	32	A+B+6T
0.4 .. 0.6	10	2	2	6	7	3	21	4	24	2A+B+4T
0.7 .. 1	10	2	2	8	6	4	24	2	18	2A+2B+2T
0.1	11	2	2	2	10	1	9	9	50	C+9T
0.2	11	2	2	2	10	1	10	9	50	A+9T
0.3	11	2	2	4	9	2	18	7	40	A+B+7T
0.4 .. 0.5	11	2	2	6	8	3	24	5	32	2A+B+5T
0.6 .. 1	11	2	2	8	7	4	28	3	24	2A+2B+3T
0.1	12	2	2	2	11	1	10	10	61	C+10T
0.2	12	2	2	2	11	1	11	10	60	A+10T
0.3	12	2	2	6	9	3	27	6	40	2A+B+6T
0.4 .. 0.7	12	2	2	8	8	4	32	4	32	2A+2B+4T
0.8 .. 1	12	2	2	10	7	5	35	2	24	3A+2B+2T
0.1	13	2	2	2	12	1	11	11	72	C+11T
0.2	13	2	2	2	12	1	12	11	72	A+11T
0.3	13	2	2	6	10	3	30	7	50	2A+B+7T
0.4 .. 0.5	13	2	2	8	9	4	36	5	40	2A+2B+5T
0.6 .. 1	13	2	2	10	8	5	40	3	32	3A+2B+3T
0.1	14	2	2	2	13	1	12	12	85	C+12T
0.2	14	2	2	4	12	2	24	10	72	A+B+10T
0.3	14	2	2	6	11	3	33	8	60	2A+B+8T
0.4	14	2	2	8	10	4	40	6	50	2A+2B+6T
0.5 .. 0.7	14	2	2	10	9	5	45	4	40	3A+2B+4T
0.8 .. 1	14	2	2	12	8	6	48	2	32	3A+3B+2T
0.1	15	2	2	2	14	1	13	13	98	C+13T
0.2	15	2	2	4	13	2	26	11	84	A+B+11T
0.3	15	2	2	8	11	4	44	7	60	2A+2B+7T
0.4 .. 0.6	15	2	2	10	10	5	50	5	50	3A+2B+5T

Table A.1: Table K2P2

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.7 .. 1	15	2	2	12	9	6	54	3	40	3A+3B+3T
0.1	16	2	2	2	15	1	14	14	113	C+14T
0.2	16	2	2	4	14	2	28	12	98	A+B+12T
0.3	16	2	2	8	12	4	48	8	72	2A+2B+8T
0.4	16	2	2	10	11	5	55	6	60	3A+2B+6T
0.5 .. 0.9	16	2	2	12	10	6	60	4	50	3A+3B+4T
1	16	2	2	14	9	7	63	2	40	4A+3B+2T
0.1	17	2	2	2	16	1	15	15	128	C+15T
0.2	17	2	2	6	14	3	42	11	98	2A+B+11T
0.3 .. 0.4	17	2	2	10	12	5	60	7	72	3A+2B+7T
0.5 .. 0.6	17	2	2	12	11	6	66	5	60	3A+3B+5T
0.7 .. 1	17	2	2	14	10	7	70	3	50	4A+3B+3T
0.1	18	2	2	2	17	1	16	16	145	C+16T
0.2	18	2	2	6	15	3	45	12	112	2A+B+12T
0.3	18	2	2	10	13	5	65	8	84	3A+2B+8T
0.4 .. 0.5	18	2	2	12	12	6	72	6	72	3A+3B+6T
0.6 .. 1	18	2	2	14	11	7	77	4	60	4A+3B+4T
0.1	19	2	2	2	18	1	17	17	162	C+17T
0.2	19	2	2	6	16	3	48	13	128	2A+B+13T
0.3	19	2	2	10	14	5	70	9	98	3A+2B+9T
0.4	19	2	2	12	13	6	78	7	84	3A+3B+7T
0.5 .. 0.7	19	2	2	14	12	7	84	5	72	4A+3B+5T
0.8 .. 1	19	2	2	16	11	8	88	3	60	4A+4B+3T
0.1	20	2	2	2	19	1	18	18	181	C+18T
0.2	20	2	2	8	16	4	64	12	128	2A+2B+12T
0.3	20	2	2	12	14	6	84	8	98	3A+3B+8T
0.4 .. 0.5	20	2	2	14	13	7	91	6	84	4A+3B+6T
0.6 .. 1	20	2	2	16	12	8	96	4	72	4A+4B+4T
0.1	21	2	2	2	20	1	19	19	200	C+18T
0.2	21	2	2	8	17	4	68	13	144	2A+2B+13T
0.3	21	2	2	12	15	6	90	9	112	3A+3B+9T
0.4	21	2	2	14	14	7	98	7	98	4A+3B+7T
0.5 .. 0.7	21	2	2	16	13	8	104	5	84	4A+4B+5T
0.8 .. 1	21	2	2	18	12	9	108	3	72	5A+4B+3T
0.1	22	2	2	2	21	1	20	20	221	C+20T
0.2	22	2	2	8	18	4	72	14	162	2A+2B+14T
0.3	22	2	2	12	16	6	96	10	128	3A+3B+10T
0.4 .. 0.5	22	2	2	16	14	8	112	6	98	4A+4B+6T
0.6 .. 1	22	2	2	18	13	9	117	4	84	5A+4B+4T
0.1	23	2	2	2	22	1	21	21	242	C+21T
0.2	23	2	2	10	18	5	90	13	162	3A+2B+13T
0.3	23	2	2	14	16	7	112	9	128	4A+3B+9T
0.4	23	2	2	16	15	8	120	7	112	4A+4B+7T
0.5 .. 0.9	23	2	2	18	14	9	126	5	98	5A+4B+5T
1	23	2	2	20	13	10	130	3	84	5A+5B+3T
0.2	24	2	2	2	23	1	22	22	265	C+22T
0.2	24	2	2	10	19	5	95	14	180	3A+2B+14T
0.3	24	2	2	14	17	7	118	10	145	C+3A+3B+10T
0.4	24	2	2	16	16	8	128	8	128	4A+4B+8T
0.5	24	2	2	18	15	9	135	6	112	5A+4B+6T

Table A.2: Table K2P2

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.6 .. 1	24	2	2	20	14	10	140	4	98	5A+5B+4T
0.1	25	2	2	2	24	1	23	23	288	C+23T
0.2	25	2	2	10	20	5	100	15	200	3A+2B+15T
0.3	25	2	2	14	18	7	126	11	162	C+3A+3B+11T
0.4 .. 1	25	2	2	18	16	9	144	7	128	5A+4B+7T

Table A.3: Table K2P2

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	6	3	2	3	3	1	3	1	5	A+T
0.1 .. 0.6	9	3	2	3	5	1	6	2	13	A+2T
0.7 .. 1	9	3	2	6	4	2	12	1	8	A+B+T
0.1 .. 0.4	12	3	2	3	7	1	9	3	25	A+3T
0.5 .. 1	12	3	2	6	6	2	18	2	18	A+B+2T
0.1 .. 0.4	15	3	2	3	9	1	12	4	41	A+4T
0.5 .. 0.6	15	3	2	6	8	2	24	3	32	A+B+3T
0.7 .. 0.8	15	3	2	9	7	3	30	2	25	2A+B+2T
0.9 .. 1	15	3	2	12	6	4	36	1	18	2A+2B+T
0.1 .. 0.3	18	3	2	3	11	1	15	5	61	A+5T
0.4	18	3	2	6	10	2	30	4	50	A+B+4T
0.5 .. 0.6	18	3	2	9	9	3	39	3	41	2A+B+3T
0.7 .. 1	18	3	2	12	8	4	48	2	32	2A+2B+2T
0.1 .. 0.3	21	3	2	3	13	1	18	6	85	A+6T
0.4	21	3	2	6	12	2	36	5	72	A+B+5T
0.5 .. 0.8	21	3	2	12	10	4	60	3	50	2A+2B+3T
0.9 .. 1	21	3	2	15	9	5	66	2	41	3A+2B+2T
0.1 .. 0.2	24	3	2	3	15	1	21	7	113	A+7T
0.3	24	3	2	6	14	2	42	6	98	A+B+6T
0.4	24	3	2	9	13	3	57	5	85	2A+B+5T
0.5	24	3	2	12	12	4	72	4	72	2A+2B+4T
0.6 .. 0.7	24	3	2	15	11	5	81	3	61	3A+2B+3T
0.8 .. 1	24	3	2	18	10	6	90	2	50	3A+3B+2T
0.1 .. 0.2	27	3	2	3	17	1	24	8	145	A+8T
0.3	27	3	2	6	16	2	48	7	128	A+B+7T
0.4	27	3	2	12	14	4	84	5	98	2A+2B+5T
0.5	27	3	2	15	13	5	96	4	85	3A+2B+4T
0.6 .. 1	27	3	2	18	12	6	108	3	72	3A+3B+3T
0.1 .. 0.2	30	3	2	3	19	1	27	9	181	A+9T
0.3	30	3	2	9	17	3	75	7	145	2A+B+7T
0.4	30	3	2	12	16	4	96	6	128	2A+2B+6T
0.5 .. 0.6	30	3	2	18	14	6	126	4	98	3A+3B+4T
0.7 .. 0.9	30	3	2	21	13	7	135	3	85	4A+3B+3T
1	30	3	2	24	12	8	144	2	72	4A+4B+2T
0.1 .. 0.2	33	3	2	3	21	1	30	10	221	A+10T
0.3	33	3	2	12	18	4	108	7	162	2A+2B+7T
0.4 .. 0.5	33	3	2	18	16	6	144	5	128	3A+3B+5T
0.6 .	33	3	2	21	15	7	156	4	113	4A+3B+4T
0.7 .. 1	33	3	2	24	14	8	168	3	98	4A+4B+3T
0.1 .. 0.2	36	3	2	3	23	1	33	11	265	A+11T
0.3	36	3	2	12	20	4	120	8	200	2A+2B+8T
0.4	36	3	2	18	18	6	162	6	162	3A+3B+6T
0.5 .. 0.8	36	3	2	24	16	8	192	4	128	4A+4B+4T
0.9 .. 1	36	3	2	27	15	9	201	3	113	5A+4B+3T
0.1 .. 0.2	39	3	2	3	25	1	36	12	313	A+12T
0.3	39	3	2	15	21	5	156	8	221	3A+2B+8T
0.4	39	3	2	21	19	7	198	6	181	4A+3B+6T
0.5	39	3	2	24	18	8	216	5	162	4A+4B+5T
0.6 .. 0.7	39	3	2	27	17	9	228	4	145	5A+4B+4T
0.8 .. 1	39	3	2	30	16	10	240	3	128	5A+5B+3T

Table A.4: Table K2P3

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 0.2	42	3	2	3	27	1	39	13	365	A+13T
0.3	42	3	2	18	22	6	198	8	242	3A+3B+8T
0.4	42	3	2	24	20	8	240	6	200	4A+4B+6T
0.5	42	3	2	27	19	9	255	5	181	5A+4B+5T
0.6 .. 1	42	3	2	30	18	10	270	4	162	5A+5B+4T

Table A.5: Table K2P3

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	10	4	2	4	4	1	8	1	8	B+C
0.1 .. 1	12	4	2	12	3	3	16	0	5	2A+D
0.1 .. 1	14	4	2	8	5	2	18	1	13	A+B+C
0.1 .. 0.8	16	4	2	4	7	1	12	2	25	A+C+F
0.9 .. 1	16	4	2	12	5	2	28	1	13	2O+A+B+C
0.1 .. 0.6	18	4	2	8	7	1	26	2	25	2O+A+C+F
0.7 .. 1	18	4	2	12	6	3	36	1	18	A+B+D+C
0.1 .. 0.3	20	4	2	4	9	1	16	2	41	A+C+F+E
0.4	20	4	2	8	8	2	24	2	34	2A+C+F
0.5 .. 1	20	4	2	8	8	2	32	2	32	A+D+C+F
0.1 .. 0.6	22	4	2	4	10	1	20	3	50	B+2C+F
0.7 .. 1	22	4	2	16	7	4	54	1	25	2A+D+B+C
0.1 .. 0.4	24	4	2	12	9	3	36	2	45	3A+C+F
0.5 .. 1	24	4	2	12	9	3	52	2	41	2A+D+C+F
0.1 .. 0.6	26	4	2	8	11	2	42	3	61	A+B+2C+F
0.7 .. 1	26	4	2	20	8	5	80	1	32	2A+2D+B+C
0.1 .. 0.4	28	4	2	4	13	1	24	4	85	A+2C+2F
0.5 .. 1	28	4	2	16	10	4	80	2	50	2A+2D+C+F
0.1 .. 0.3	30	4	2	8	13	1	50	4	85	2O+A+2C+2F
0.4 .. 0.8	30	4	2	12	12	3	72	3	72	A+D+B+2C+F
0.9 .. 1	30	4	2	24	9	6	106	1	41	3A+2D+B+C
0.1 .. 0.3	32	4	2	4	15	1	28	4	113	A+E+2C+2F
0.4 .. 0.5	32	4	2	8	14	2	56	4	98	A+D+2C+2F
0.6 .. 1	32	4	2	20	11	5	108	2	61	3A+2D+C+F
0.1 .. 0.4	34	4	2	4	16	1	32	5	128	B+3C+2F
0.5 .. 0.8	34	4	2	16	13	4	102	3	85	2A+B+D+2C+F
0.9 .. 1	34	4	2	28	10	7	140	1	50	3A+3D+B+C
0.1 .. 0.2	36	4	2	8	16	2	56	4	130	2A+E+2C+2F
0.3	36	4	2	12	15	3	72	4	117	3A+2C+2F
0.4 .. 0.5	36	4	2	12	15	3	88	4	113	2A+D+2C+2F
0.6 .. 1	36	4	2	24	12	6	144	2	72	3A+3D+C+F
0.1 .. 0.4	38	4	2	8	17	2	66	5	145	A+B+3C+2F
0.5 .. 1	38	4	2	20	14	5	140	3	98	2A+2D+B+2C+F
0.1 .. 0.3	40	4	2	4	19	1	36	6	181	A+3C+3F
0.4 .. 0.6	40	4	2	16	16	4	128	4	128	2A+2D+2C+2F
0.7 .. 1	40	4	2	28	13	7	180	2	85	4A+3D+C+F

Table A.6: Table K2P4

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	15	5	2	5	5	1	10	1	13	A+T
0.1 .. 0.2	20	5	2	5	7	1	15	1	25	A+B+T
0.3 .. 0.6	20	5	2	10	6	2	20	1	20	2A+T
0.7 .. 1	20	5	2	10	6	2	30	1	18	A+D+T
0.1 .. 0.6	25	5	2	5	9	1	20	2	41	A+2T
0.7 .. 1	25	5	2	15	7	3	50	1	25	2A+D+T
0.1 .. 0.3	30	5	2	5	11	1	25	2	61	A+B+2T
0.4	30	5	2	10	10	2	40	2	52	2A+2T
0.5 .. 0.8	30	5	2	10	10	2	50	2	50	A+D+2T
0.9 .. 1	30	5	2	20	8	4	80	1	32	2A+2D+T
0.1 .. 0.5	35	5	2	5	13	1	30	3	85	A+3T
0.6 .. 0.9	35	5	2	15	11	3	80	2	61	2A+D+2T
1	35	5	2	25	9	5	110	1	41	3A+2D+T
0.1 .. 0.3	40	5		5	15	1	35	3	113	A+B+3T
0.4 .. 0.5	40	5	2	10	14	2	70	3	98	A+D+3T
0.6 .. 1	40	5	2	20	12	4	120	2	72	2A+2D+2T
0.1 .. 0.4	45	5	2	5	17	1	40	4	145	A+4T
0.5	45	5	2	15	15	3	110	3	113	2A+D+3T
0.6 .. 1	45	5	2	25	13	5	160	2	85	3A+2D+2T
0.1 .. 0.3	50	5	2	5	19	1	45	4	181	A+B+4T
0.4	50	5	2	10	18	2	90	4	162	A+D+4T
0.5 .. 0.6	50	5	2	20	16	4	160	3	128	2A+2D+3T
0.7 .. 1	50	5	2	30	14	6	210	2	98	3A+3D+2T

Table A.7: Table K2P5

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	3	3	3	3	2	2	2	1	1	A
0.1 .. 1	4	3	3	3	3	2	3	2	3	A+T
0.1 .. 1	5	3	3	3	4	2	4	3	5	A+2T
0.1 .. 0.5	6	3	3	3	5	2	3	4	9	B+3T
0.6 .. 1	6	3	3	6	4	4	8	2	5	2A
0.1 .. 0.6	7	3	3	3	6	2	6	5	12	A+4T
0.6 .. 1	7	3	3	6	5	4	8	3	9	A+B+T
0.1 .. 0.4	8	3	3	3	7	2	7	6	16	A+5T
0.5 .. 1	8	3	3	6	6	4	12	4	12	2A+2T
0.1 .. 0.3	9	3	3	3	8	2	6	7	22	B+6T
0.4 .. 0.5	9	3	3	6	7	4	14	5	16	2A+3T
0.6 .. 1	9	3	3	9	6	6	18	3	12	3A
0.1 .. 0.3	10	3	3	3	9	2	9	8	27	A+7T
0.4	10	3	3	6	8	4	14	6	22	A+B+4T
0.5 .. 1	10	3	3	9	7	6	21	4	16	3A+T
0.1 .. 0.3	11	3	3	3	10	2	10	9	33	A+8T
0.4 .. 0.5	11	3	3	6	9	4	18	7	27	2A+5T
0.6 .. 1	11	3	3	9	8	6	22	5	22	2A+B+2T
0.1 .. 0.3	12	3	3	3	11	2	9	10	41	B+9T
0.4	12	3	3	6	10	4	20	8	33	2A+6T
0.5 .. 0.6	12	3	3	9	9	6	27	6	27	3A+3T
0.7 .. 1	12	3	3	12	8	8	30	4	22	4A
0.1 .. 0.3	13	3	3	3	12	2	12	11	48	A+10T
0.4	13	3	3	9	10	6	30	7	33	3A+4T
0.5 .. 1	13	3	3	12	9	8	36	5	27	4A+T
0.1 .. 0.2	14	3	3	3	13	2	13	12	56	A+11T
0.3	14	3	3	6	12	4	24	10	48	2A+8T
0.4	14	3	3	9	11	6	31	8	41	2A+B+5T
0.5 .. 1	14	3	3	12	10	8	40	6	33	4A+2T
0.1 .. 0.2	15	3	3	3	14	2	12	13	66	B+12T
0.3	15	3	3	6	13	4	26	11	56	2A+9T
0.4	15	3	3	9	12	6	36	9	48	3A+6T
0.5 .. 1	15	3	3	15	10	10	50	5	33	5A
0.1 .. 0.2	16	3	3	3	15	2	15	14	75	A+13T
0.3	16	3	3	9	13	6	39	10	56	3A+7T
0.4 .. 0.5	16	3	3	12	12	8	48	8	48	4A+4T
0.6 .. 1	16	3	3	15	11	10	53	6	41	4A+B+T
0.1 .. 0.2	17	3	3	3	16	2	16	15	85	A+14T
0.3	17	3	3	9	14	6	40	11	66	2A+B+8T
0.4	17	3	3	12	13	8	52	9	56	4A+5T
0.5 .. 1	17	3	3	15	12	10	60	7	48	5A+2T
0.1 .. 0.2	18	3	3	3	17	2	15	16	97	B+15T
0.3	18	3	3	9	15	6	45	12	75	3A+9T
0.4	18	3	3	15	13	10	65	8	56	5A+3T
0.5 .. 1	18	3	3	18	12	12	72	6	48	6A
0.1 .. 0.2	19	3	3	3	18	2	18	17	108	A+16T
0.3	19	3	3	12	15	8	60	11	75	4A+7T
0.4	19	3	3	15	14	10	68	9	66	4A+B+4T
0.5 .. 1	19	3	3	18	13	12	78	7	56	6A+T
0.1 .. 0.2	20	3	3	3	19	2	19	18	120	A+17T

Table A.8: Table K3P3

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.3	20	3	3	12	16	8	64	12	85	4A+8T
0.4	20	3	3	15	15	10	75	10	75	5A+5T
0.5 .. 1	20	3	3	18	14	12	82	8	66	5A+B+2T
0.1 .. 0.2	21	3	3	3	20	2	18	19	134	B+18T
0.3	21	3	3	12	17	8	66	13	97	3A+B+9T
0.4 .. 0.5	21	3	3	18	15	12	90	9	75	6A+3T
0.6 .. 1	21	3	3	21	14	14	96	7	66	6A+B
0.1 .. 0.2	22	3	3	3	21	2	21	20	147	A+19T
0.3	22	3	3	12	18	8	72	14	108	4A+10T
0.4	22	3	3	18	16	12	96	10	85	6A+4T
0.5 .. 1	22	3	3	21	15	14	105	8	75	7A+T
0.1 .. 0.2	23	3	3	3	22	2	22	21	161	A+20T
0.3	23	3	3	15	18	10	90	13	108	5A+8T
0.4 .. 1	23	3	3	21	16	14	112	9	85	7A+2T
0.1	24	3	3	3	23	2	21	22	177	A+21T
0.2	24	3	3	6	22	4	44	20	161	2A+18T
0.3	24	3	3	15	19	10	95	14	120	5A+9T
0.4	24	3	3	21	17	14	117	10	97	6A+B+3T
0.5 .. 1	24	3	3	24	16	16	128	8	85	8A
0.1	25	3	3	3	24	2	24	23	192	A+22T
0.2	25	3	3	6	23	4	44	21	177	A+B+19T
0.3	25	3	3	18	19	12	114	13	120	6A+7T
0.4	25	3	3	21	18	14	126	11	108	7A+4T
0.5 .. 1	25	3	3	24	17	16	134	9	97	7A+B+T

Table A.9: Table K3P3

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	6	4	3	6	3	3	6	1	3	A
0.1 .. 1	8	4	3	4	5	2	4	2	9	B+T
0.1 .. 1	10	4	3	6	6	3	12	3	12	A+T
0.1 .. 0.5	12	4	3	4	8	2	8	4	22	B+2T
0.6 .. 1	12	4	3	12	6	6	24	2	12	2A
0.1 .. 1	14	4	3	6	9	3	18	5	27	A+2T
0.1 .. 0.4	16	4	3	4	11	2	12	6	41	B+3T
0.5 .. 1	16	4	3	12	9	6	36	4	27	2A+T
0.1 .. 0.5	18	4	3	6	12	3	24	7	48	A+3T
0.6 .. 1	18	4	3	18	9	9	54	3	27	3A
0.1 .. 0.3	20	4	3	4	14	2	16	8	66	B+4T
0.4 .. 0.9	20	4	3	12	12	6	48	6	48	2A+2T
1	20	4	3	16	11	8	56	4	41	2A+B+T
0.1 .. 0.3	22	4	3	6	15	3	30	9	75	A+4T
0.4 .. 1	22	4	3	18	12	9	72	5	48	3A+T
0.1 .. 0.3	24	4	3	4	17	2	20	10	97	B+5T
0.4	24	4	3	12	15	6	60	8	75	2A+3T
0.5 .. 1	24	4	3	24	12	12	96	4	48	4A
0.1 .. 0.3	26	4	3	6	18	3	36	11	108	A+5T
0.4 .. 1	26	4	3	18	15	9	90	7	75	3A+2T
0.1 .. 0.2	28	4	3	4	20	2	24	12	134	B+6T
0.3	28	4	3	12	18	6	72	10	108	2A+4T
0.4 .. 1	28	4	3	24	15	12	120	6	75	4A+T
0.1 .. 0.3	30	4	3	6	21	3	42	13	148	A+6T
0.4	30	4	3	18	18	9	108	9	108	3A+3T
0.5 .. 1	30	4	3	30	15	15	150	5	75	5A
0.1 .. 0.2	32	4	3	4	23	2	28	14	177	B+7T
0.3	32	4	3	12	21	6	84	12	147	2A+5T
0.4 .. 1	32	4	3	24	18	12	144	8	108	4A+2T
0.1 .. 0.2	34	4	3	6	24	3	48	15	192	A+7T
0.3	34	4	3	18	21	9	126	11	147	3A+4T
0.4 .. 1	34	4	3	30	18	15	180	7	108	5A+T
0.1 .. 0.2	36	4	3	4	26	2	32	16	226	B+8T
0.3	36	4	3	12	24	6	96	14	192	2A+6T
0.4	36	4	3	24	21	12	168	10	147	4A+3T
0.5 .. 1	36	4	3	36	18	18	216	6	108	6A
0.1 .. 0.2	38	4	3	6	27	3	54	17	243	A+8T
0.3	38	4	3	18	24	9	144	13	192	3A+5T
0.4 .. 1	38	4	3	30	21	15	210	9	147	5A+2T
0.1 .. 0.2	40	4	3	4	29	2	36	18	281	B+9T
0.3	40	4	3	24	24	12	192	12	192	4A+4T
0.4 .. 1	40	4	3	36	21	18	252	8	147	6A+T

Table A.10: Table K3P4

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 0.8	10	5	3	5	5	2	5	2	9	A
0.9 .. 1	10	5	3	10	4	4	10	1	6	B
0.1 .. 0.6	15	5	3	5	8	2	10	3	22	D+C
0.7 .. 1	15	5	3	15	6	4	30	2	12	E+C
0.1 .. 0.4	20	5	3	5	11	2	15	5	41	A+C
0.5	20	5	3	10	10	4	30	4	34	B+C
0.6 .. 1	20	5	3	15	9	6	45	3	27	B+F
0.1 .. 0.4	25	5	3	5	14	2	20	6	66	D+2C
0.5	25	5	3	15	12	4	60	5	48	C+E+F
0.6 .. 0.7	25	5	3	15	12	6	60	4	48	B+C+G
0.8 .. 0.9	25	5	3	25	10	8	80	3	34	B+E+F
1	25	5	3	30	9	10	90	2	27	B+E+H
0.1 .. 0.3	30	5	3	5	17	2	25	8	97	A+2C
0.4	30	5	3	15	15	6	75	6	75	B+C+F
0.5	30	5	3	20	14	8	90	5	66	A+B+F
0.6 .. 0.7	30	5	3	25	13	10	105	4	57	2B+F
0.8 .. 1	30	5	3	30	12	12	120	3	48	2B+H
0.1 .. 0.3	35	5	3	5	20	2	30	9	134	D+3C
0.4	35	5	3	15	18	6	90	7	108	B+2C+G
0.5 .. 0.6	35	5	3	30	15	10	150	5	75	B+C+E+H
0.7 .. 1	35	5	3	30	15	12	150	4	75	2B+G+F
0.1 .. 0.2	40	5	3	5	23	2	35	11	177	A+3C
0.3	40	5	3	10	22	4	70	10	162	B+3C
0.4	40	5	3	25	19	10	155	7	121	2B+C+FF
0.5 .. 0.6	40	5	3	30	18	12	180	6	108	2B+C+H
0.7 .. 0.8	40	5	3	35	17	14	195	5	97	2B+A+H
0.9 .. 1	40	5	3	40	16	16	210	4	86	3B+H
0.1 .. 0.2	45	5	3	5	26	2	40	12	226	D+4C
0.3	45	5	3	15	24	6	120	10	192	B+G+3C
0.4	45	5	3	30	21	10	210	8	147	E+B+H+2C
0.5	45	5	3	30	21	12	210	7	147	2B+C+E+G
0.6 .. 1	45	5	3	45	18	16	270	5	108	E+2B+H+C
0.1 .. 0.2	50	5	3	5	29	2	45	14	281	A+4C
0.3	50	5	3	20	26	8	170	11	226	A+B+F+2C
0.4	50	5	3	30	24	12	240	9	192	2B+H+2C
0.5	50	5	3	40	22	16	290	7	162	3B+H+C
0.6 .. 1	50	5	3	45	21	18	315	6	147	3B+F+H

Table A.11: Table K3P5

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	7	6	3	3	3	1	3	1	3	A
0.1 .. 1	13	6	3	15	4	3	16	1	6	A+B
0.1 .. 1	14	6	3	6	6	2	12	2	12	2A
0.1 .. 1	15	6	3	15	5	5	21	1	9	C
0.1 .. 1	17	6	3	21	5	5	31	1	9	D+E
0.1 .. 0.5	18	6	3	12	7	4	12	2	19	F+G+H
0.6 .. 1	18	6	3	12	7	4	24	2	17	F+I+H
0.1 .. 1	20	6	3	12	8	4	28	2	22	2A+J
0.1 .. 1	21	6	3	9	9	3	27	3	27	3A
0.1 .. 0.2	22	6	3	12	9	4	36	2	27	K+L+M+N
0.3 .. 1	22	6	3	18	8	6	44	2	22	A+C
0.1 .. 0.6	24	6	3	6	11	2	18	4	41	2A+N
0.7 .. 1	24	6	3	18	9	6	54	2	27	I+J+H+P
0.1 .. 0.4	25	6	3	15	10	5	34	3	36	A+F+G+H
0.5 .. 1	25	6	3	15	10	5	46	3	34	A+F+I+H
0.1 .. 0.5	26	6	3	6	12	2	24	4	48	U+2N
0.6 .. 1	26	6	3	24	9	8	72	2	27	R+E
0.1 .. 0.8	28	6	3	12	12	4	48	4	48	4A
0.9 .. 1	28	6	3	24	10	8	76	2	34	A+C+H
0.1 .. 1	29	6	3	21	11	7	73	3	41	E+F+H+V
0.1 .. 0.4	30	6	3	12	13	4	36	4	59	J+2O+N
0.5	30	6	3	12	13	4	48	4	57	H+2O+N
0.6 .. 1	30	6	3	30	10	10	96	2	34	2T
0.1 .. 1	31	6	3	9	14	3	38	5	66	A+2O+N

Table A.12: Table K3P6

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	14	7	3	14	4	2	14	1	6	A+C
0.1 .. 0.7	21	7	3	7	8	2	14	2	22	B+2C
0.8 .. 1	21	7	3	21	6	6	42	1	12	D
0.1 .. 0.5	28	7	3	7	11	2	21	3	41	B+3C
0.6 .. 1	28	7	3	21	9	6	63	2	27	D+C
0.1 .. 0.4	35	7	3	7	14	2	28	4	66	B+4C
0.5 .. 1	35	7	3	21	12	6	84	3	48	D+2C
0.1 .. 0.4	42	7	3	7	17	2	35	5	97	B+5C
0.5 .. 0.7	42	7	3	21	15	6	105	4	75	D+3C
0.8 .. 1	42	7	3	42	12	12	168	2	48	2B
0.1 .. 0.3	49	7	3	7	20	2	42	6	134	B+6C
0.4 .. 0.5	49	7	3	21	18	6	126	5	108	D+4C
0.6 .. 1	49	7	3	42	15	12	210	3	75	2B+C
0.1 .. 0.3	56	7	3	7	23	2	49	7	177	B+7C
0.4	56	7	3	21	21	6	147	6	147	D+5C
0.5 .. 1	56	7	3	42	18	12	252	4	108	2B+2C

Table A.13: Table K3P7

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	4	4	4	4	3	3	3	2	2	A
0.1 .. 1	5	4	4	4	4	3	4	3	4	A+T
0.1 .. 1	6	4	4	4	5	3	5	4	6	A+2T
0.1 .. 0.6	7	4	4	8	5	4	10	3	6	E+3T
0.7 .. 1	7	4	4	12	4	6	12	2	4	B+T
0.1 .. 0.5	8	4	4	4	7	3	4	6	13	C+4T
0.6 .. 0.8	8	4	4	8	6	6	8	4	10	C+D
0.9 .. 1	8	4	4	12	5	6	15	3	6	B+2T
0.1 .. 0.5	9	4	4	4	8	3	8	7	16	A+5T
0.6 .. 1	9	4	4	8	7	6	11	5	13	A+C+T
0.1 .. 0.4	10	4	4	4	9	3	9	8	20	A+6T
0.5 .. 1	10	4	4	8	8	6	16	6	16	2A+2T
0.1 .. 0.2	11	4	4	8	9	4	18	7	20	E+7T
0.3 .. 1	11	4	4	8	9	6	18	7	20	2A+3T
0.1 .. 0.3	12	4	4	4	11	3	8	10	31	C+8T
0.4	12	4	4	8	10	6	16	8	26	C+D+4T
0.5 .. 1	12	4	4	12	9	9	27	6	20	4A
0.1	13	4	4	8	11	6	10	9	33	2C+5T
0.2 .. 0.3	13	4	4	4	12	3	12	11	36	A+9T
0.4	13	4	4	8	11	6	19	9	31	A+C+5T
0.5 .. 0.8	13	4	4	12	10	9	26	7	26	A+C+D+T
0.9 .. 1	13	4	4	16	9	9	36	6	20	F+G+3T
0.1 .. 0.3	14	4	4	4	13	3	13	12	42	A+10T
0.4	14	4	4	8	12	6	24	10	36	2A+6T
0.5 .. 0.6	14	4	4	12	11	9	30	8	31	2A+C+2T
0.7 .. 1	14	4	4	20	9	12	45	5	20	H+G
0.1 .. 0.2	15	4	4	8	13	4	26	11	42	E+11T
0.3	15	4	4	8	13	6	26	11	42	2A+7T
0.4 .. 1	15	4	4	12	12	9	36	9	36	3A+3T
0.1 .. 0.3	16	4	4	4	15	3	12	14	57	C+12T
0.4	16	4	4	12	13	9	39	10	42	3A+8T
0.5 .. 1	16	4	4	16	12	12	48	8	36	4A
0.1 .. 0.3	17	4	4	4	16	3	16	15	64	A+13T
0.4 .. 1	17	4	4	16	13	12	52	9	42	4A+T
0.1 .. 0.2	18	4	4	4	17	3	17	16	72	A+14T
0.3	18	4	4	8	16	6	32	14	64	2A+10T
0.4 .. 0.5	18	4	4	16	14	12	52	10	50	2A+D+C+2T
0.6 .. 1	18	4	4	24	12	15	72	7	36	I+J+G
0.1 .. 0.2	19	4	4	8	17	4	34	15	72	E+15T
0.3	19	4	4	8	17	6	34	15	72	2A+11T
0.4 .. 0.5	19	4	4	16	15	12	57	11	57	3A+C+3T
0.6 .. 1	19	4	4	24	13	15	78	8	42	I+J+G+T
0.1 .. 0.2	20	4	4	4	19	3	16	18	91	C+16T
0.3 .. 1	20	4	4	12	17	9	51	14	72	3A+8T
0.1 .. 0.2	22	4	4	12	19	9	45	16	93	A+2C+10T
0.3	22	4	4	12	19	9	54	16	91	2A+C+10T
0.4 .. 0.8	22	4	4	20	17	15	85	12	72	5A+2T
0.9 .. 1	22	4	4	28	15	18	102	9	57	I+A+J+G
0.1 .. 0.2	23	4	4	8	21	4	42	19	110	E
0.3	23	4	4	12	20	9	60	17	100	4A+11T

Table A.14: Table K4P4

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.4	23	4	4	20	18	15	86	13	82	4A+C+D+3T
0.5 .. 1	23	4	4	28	16	18	112	10	64	A+I+J+H+T
0.1 .. 0.2	24	4	4	4	23	3	20	22	133	C+20T
0.3	24	4	4	16	20	12	80	16	100	4A+8T
0.4 .. 1	24	4	4	24	18	18	104	12	82	4A+C+D
0.1 .. 0.2	25	4	4	4	24	3	24	23	144	A+21T
0.3	25	4	4	16	21	12	84	17	110	4A+9T
0.4 .. 1	25	4	4	24	19	18	111	13	91	4A+C+T

Table A.15: Table K4P4

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1 .. 1	5	5	5	5	4	4	4	3	3	A
0.1 .. 1	6	5	5	5	5	4	5	4	5	A+T
0.1 .. 1	7	5	5	5	6	4	6	5	7	A+2T
0.1 .. 1	8	5	5	15	5	6	15	3	5	F+3T
0.1 .. 1	9	5	5	15	6	6	18	4	7	F+4T
0.1 .. 0.4	10	5	5	5	9	4	5	8	17	B+5T
0.5 .. 0.7	10	5	5	10	8	8	10	6	14	B+C
0.8 .. 1	10	5	5	20	6	12	24	3	7	G
0.1 .. 0.5	11	5	5	5	10	4	10	9	20	A+6T
0.6 .. 0.9	11	5	5	10	9	8	14	7	17	A+B+T
1	11	5	5	20	7	12	22	4	11	H
0.1 .. 0.4	12	5	5	5	11	4	11	10	24	A+7T
0.5 .. 1	12	5	5	10	10	8	20	8	20	2A+2T
0.1	13	5	5	15	10	6	30	8	20	F+8T
0.2 .. 0.8	13	5	5	10	11	8	22	9	24	2A+3T
0.9 .. 1	13	5	5	20	9	12	32	6	17	I
0.1 .. 0.4	14	5	5	15	11	6	33	9	24	F+9T
0.5 .. 1	14	5	5	20	10	12	40	7	20	G+4T
0.1 .. 0.3	15	5	5	5	14	4	10	13	40	B+10T
0.4	15	5	5	10	13	8	20	11	35	B+C+5T
0.5 .. 0.6	15	5	5	15	12	12	30	9	30	B+C+D
0.7 .. 1	15	5	5	25	10	16	50	6	20	E+J+K
0.1 .. 0.3	16	5	5	5	15	4	15	14	45	A+11T
0.4	16	5	5	10	14	8	24	12	40	A+B+6T
0.5	16	5	5	15	13	12	33	10	35	A+B+C+T
0.6 .. 1	16	5	5	25	11	16	55	7	24	E+J+K
0.1 .. 0.3	17	5	5	5	16	4	16	15	51	A+12T
0.4	17	5	5	10	15	8	30	13	45	2A+7T
0.5 .. 0.6	17	5	5	15	14	12	38	11	40	2A+B+2T
0.7 .. 1	17	5	5	25	12	16	54	8	30	L+2T
0.1	18	5	5	15	15	6	45	13	45	F+13T
0.2 .. 0.3	18	5	5	10	16	8	32	14	51	2A+8T
0.4 .. 0.8	18	5	5	15	15	12	45	12	45	3A+3T
0.9 .. 1	18	5	5	25	13	16	59	9	35	M+4T
0.1 .. 0.2	19	5	5	15	16	6	48	14	51	F+14T
0.3 .. 0.6	19	5	5	15	16	12	48	13	51	3A+4T
0.7 .. 1	19	5	5	25	14	16	66	10	40	N
0.1 .. 0.2	20	5	5	5	19	4	15	18	73	B+15T
0.3	20	5	5	10	18	8	30	16	66	B+C+10T
0.4 .. 0.9	20	5	5	20	16	16	64	12	51	4A
1	20	5	5	30	14	20	80	9	40	D+E+J+V
0.1 .. 0.2	21	5	5	5	20	4	20	19	80	A+16T
0.3	21	5	5	10	19	8	34	17	73	A+B+11T
0.4	21	5	5	20	17	16	62	13	59	A+B+C+D+T
0.5 .. 1	21	5	5	30	15	20	90	10	45	E+P+T
0.1 .. 0.2	22	5	5	5	21	4	21	20	88	A+17T
0.3	22	5	5	10	20	8	40	18	80	2A+12T
0.4	22	5	5	20	18	16	66	14	66	2A+B+C+2T
0.5 .. 1	22	5	5	30	16	20	96	11	51	E+P+2T

Table A.16: Table K5P5

d/σ	b	p	k	r_0	r	λ_0	μ_0	λ_1	μ_1	The layout
0.1	23	5	5	15	20	6	60	18	80	F+18T
0.2	23	5	5	10	21	8	42	19	88	2A+13T
0.3	23	5	5	15	20	12	60	17	80	3A+8T
0.4 .. 0.5	23	5	5	20	19	16	72	15	73	3A+B+3T
0.6	23	5	5	30	17	20	96	12	59	R+3T
0.7 .. 1	23	5	5	40	15	24	120	9	45	U+3T
0.1 .. 0.2	24	5	5	15	21	6	63	19	88	F+19T
0.3	24	5	5	15	21	12	63	18	88	3A+9T
0.4 .. 0.5	24	5	5	20	20	16	80	16	80	4A+4T
0.6 .. 1	24	5	5	40	16	24	128	10	51	V+J+S+2G
0.1 .. 0.2	25	5	5	5	24	4	20	23	116	B+20T
0.3	25	5	5	20	21	16	84	17	88	4A+5T
0.4 .. 1	25	5	5	25	20	20	100	15	80	5A

Table A.17: Table K5P5

A.1 Building Blocks For Designs

A.1.1 Table K2P2

$$A = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} T = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

A.1.2 Table K2P3

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$$

A.1.3 Table K2P4

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix} B = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 1 & 2 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 1 & 2 & 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 & 3 & 4 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} E = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} F = \begin{bmatrix} 1 & 2 & 3 & 4 & 3 & 4 \\ 2 & 3 & 4 & 1 & 1 & 2 \end{bmatrix}$$

A.1.4 Table K2P5

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} B = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

A.1.5 Table K3P3

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} B = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix} T = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

A.1.6 Table K3P4

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 0 & 0 & 4 & 1 \\ 1 & 2 & 3 & 4 & 0 & 0 \end{bmatrix} \; B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix} \; T = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 4 & 1 & 2 & 3 \end{bmatrix}$$

A.1.7 Table K3P5

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 3 & 4 & 5 & 1 & 2 \\ 2 & 3 & 4 & 5 & 1 & 5 & 1 & 2 & 3 & 4 \end{bmatrix} \; B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 1 & 2 & 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 & 2 & 3 & 4 & 5 & 1 \\ 5 & 1 & 2 & 3 & 4 & 3 & 4 & 5 & 1 & 2 \end{bmatrix} \; D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix} \; F = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 & 3 & 4 & 5 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \; H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 & 5 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A.1.8 Table K3P6

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 3 & 4 & 5 & 6 & 1 \\ 4 & 5 & 0 & 6 & 1 & 2 & 3 \end{bmatrix} \; B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 0 & 0 & 0 & 1 & 2 & 3 \\ 2 & 3 & 1 & 3 & 1 & 2 & 6 & 4 & 5 & 6 & 4 & 5 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 1 & 2 & 3 & 4 & 5 & 0 \end{bmatrix} \; E = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 & 3 & 5 & 6 & 1 & 4 \\ 2 & 4 & 4 & 3 & 1 & 5 & 6 & 3 & 1 & 2 & 5 \\ 6 & 1 & 2 & 4 & 5 & 0 & 0 & 0 & 0 & 3 & 6 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 3 & 1 & 2 \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 6 & 4 & 5 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{bmatrix} \quad I = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 4 & 5 & 3 & 1 & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix} \quad K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 5 \end{bmatrix} \quad L = \begin{bmatrix} 5 & 2 & 4 & 6 \\ 0 & 0 & 0 & 0 \\ 5 & 2 & 4 & 6 \end{bmatrix}$$

$$M = \begin{bmatrix} 4 & 6 & 1 & 3 \\ 4 & 6 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 & 5 \\ 2 & 3 & 4 & 5 & 6 & 1 & 2 & 4 & 5 & 6 \\ 5 & 6 & 1 & 2 & 3 & 4 & 3 & 6 & 4 & 1 \end{bmatrix}$$

$$O = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 & 5 \\ 1 & 2 & 3 & 2 & 4 & 5 & 6 \\ 4 & 5 & 6 & 3 & 6 & 4 & 1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 6 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0 & 0 & 0 & 2 & 3 & 4 & 5 & 6 & 1 & 4 & 3 & 5 & 6 & 2 & 1 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 6 & 6 \\ 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 & 1 & 6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 & 3 & 4 & 5 & 6 & 1 & 2 & 0 & 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 0 & 0 & 3 & 4 & 5 & 6 \\ 1 & 2 & 0 & 0 & 5 & 6 \\ 1 & 2 & 3 & 4 & 0 & 0 \end{bmatrix}$$

A.1.9 Table K3P7

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 & 4 & 5 & 6 & 7 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A.1.10 Table K4P4

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 3 & 4 \\ 4 & 2 & 0 & 1 \\ 2 & 3 & 4 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 & 3 & 4 \\ 1 & 3 & 4 & 0 & 0 & 0 \\ 4 & 1 & 2 & 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \end{bmatrix} \quad E = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 & 0 \\ 3 & 4 & 1 & 2 & 2 & 3 \end{bmatrix}$$

$$G = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 4 & 1 & 2 & 3 & 4 \\ 3 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 2 \\ 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 & 1 & 2 & 0 & 0 & 3 & 4 \\ 2 & 3 & 4 & 1 & 3 & 4 & 1 & 2 & 0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 4 & 1 & 2 \end{bmatrix} \quad T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

A.1.11 Table K5P5

$$A = \begin{bmatrix} 0 & 2 & 3 & 4 & 5 \\ 1 & 0 & 4 & 5 & 3 \\ 4 & 5 & 0 & 1 & 2 \\ 2 & 3 & 5 & 0 & 1 \\ 3 & 4 & 1 & 2 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \\ 4 & 5 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad F = \begin{bmatrix} 0 & 0 & 0 & 4 & 5 \\ 1 & 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 4 & 0 \\ 0 & 2 & 3 & 0 & 0 \\ 0 & 2 & 3 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 1 \\ 2 & 4 & 0 & 0 & 0 & 0 & 5 & 1 & 2 & 3 \\ 0 & 0 & 3 & 5 & 0 & 0 & 1 & 2 & 3 & 4 \\ 3 & 5 & 4 & 1 & 2 & 4 & 0 & 0 & 0 & 0 \\ 4 & 1 & 5 & 2 & 3 & 5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 4 & 5 & 0 & 0 & 0 & 2 \\ 5 & 1 & 4 & 3 & 5 & 1 & 2 & 3 & 4 & 0 \\ 3 & 5 & 2 & 4 & 1 & 2 & 3 & 4 & 5 & 0 \end{bmatrix}$$

$$I = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 4 & 2 & 3 & 5 & 2 & 4 & 1 & 5 & 3 \\ 5 & 3 & 1 & 0 & 0 & 0 & 4 & 2 & 3 & 5 & 2 & 1 & 4 \\ 1 & 4 & 2 & 5 & 1 & 4 & 0 & 3 & 0 & 0 & 5 & 3 & 2 \\ 3 & 5 & 4 & 2 & 5 & 3 & 1 & 0 & 0 & 0 & 4 & 2 & 1 \end{bmatrix} \quad J = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

$$K = \begin{bmatrix} 2 & 4 & 5 & 1 & 3 \\ 3 & 5 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 4 & 1 & 2 & 3 & 5 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 5 & 4 & 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 5 & 1 \\ 0 & 0 & 0 & 1 & 4 & 5 & 3 & 0 & 0 & 0 & 0 & 4 & 5 & 2 & 3 \\ 3 & 4 & 5 & 2 & 1 & 2 & 4 & 5 & 1 & 3 & 3 & 0 & 0 & 1 & 2 \\ 4 & 5 & 1 & 3 & 2 & 1 & 2 & 3 & 4 & 5 & 4 & 5 & 1 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0&0&0&0&0&0&0&0&4&1&1&2&3&4&5\\ 1&0&0&0&0&0&0&0&0&5&2&3&4&5&1\\ 0&2&3&4&5&1&2&3&0&0&3&4&5&1&2\\ 2&3&4&5&1&3&4&5&2&0&0&0&1&2&3\\ 3&4&5&1&2&4&5&2&1&3&4&5&0&0&0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0&0&0&0&0&0&0&0&0&1&2&3&4&1&5\\ 0&0&0&0&3&5&4&2&1&2&3&5&1&2&4\\ 4&3&4&5&0&0&0&1&2&0&4&1&2&5&3\\ 3&4&5&1&2&3&5&0&0&0&0&2&3&4&1\\ 1&2&3&4&5&1&2&3&4&5&5&0&0&0&0 \end{bmatrix}$$

$$P = \begin{bmatrix} 0&0&0&0&0&0&1&3&4&5&4&4&5&1&3\\ 1&2&0&0&0&0&0&0&3&4&3&5&1&2&2\\ 0&0&1&2&4&5&0&0&0&0&1&2&3&4&5\\ 2&3&5&4&1&3&2&5&0&0&0&0&0&0&1\\ 3&4&2&5&3&1&4&2&5&1&2&3&4&5&0 \end{bmatrix}$$

$$R = \begin{bmatrix} 0&0&0&0&0&0&0&0&5&4&1&2&3&4&5&1&2&3&4&5\\ 2&3&0&0&0&0&0&0&0&0&2&3&4&5&1&2&3&4&5&1\\ 0&0&1&2&3&4&5&1&0&0&0&0&0&0&2&3&4&5&1&2\\ 5&4&4&5&1&2&3&2&3&1&3&4&5&1&0&0&0&1&2&3\\ 1&2&3&4&5&1&2&3&4&5&4&5&1&2&3&4&5&0&0&0 \end{bmatrix}$$

$$S = \begin{bmatrix} 1&2&3&4&5\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 2&3&4&5&1\\ 3&4&5&1&2 \end{bmatrix} \quad T = \begin{bmatrix} 1\\ 2\\ 3\\ 4\\ 5 \end{bmatrix}$$

$$U = \begin{bmatrix} 0&0&0&0&0&0&0&0&5&4&1&2&3&4&5&1&2&3&4&5\\ 2&3&0&0&0&0&0&0&0&0&0&0&4&5&1&3&4&5&1&2\\ 0&0&1&2&3&4&5&1&0&0&0&0&5&1&2&4&5&1&2&3\\ 5&4&4&5&1&2&3&2&3&1&2&3&0&0&0&0&0&0&0&0\\ 1&2&3&4&5&1&2&3&4&5&3&4&0&0&0&0&0&0&0&0 \end{bmatrix}$$

$$V = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 5 & 1 \\ 3 & 4 & 5 & 1 & 2 \end{bmatrix}$$

Appendix B

Derivation of the Integral Expression

BTRCD's have totally symmetric covariance matrix. This fact facilitates the derivation of an easy to calculate formula for the p-variate equicorrelated multinormal integral of simultaneous confidence region. Expression in 3.1 is valid if the correlation matrix of the design is positive definite and $\rho \geq 0$. Following lemma states that correlation matrix of BTRCD's are positive definite if $\rho \in (-1/(n-1), 1)$.

Lemma 6 *Let $\sum = (\sigma_{ij})$ be an $n \times n$ matrix satisfying*

$$\sigma_{ij} = \begin{cases} \sigma^2 & for\ i = j \\ \rho\sigma^2 & for\ i \neq j \end{cases}$$

then: $\sum$ is a positive definite matrix if $\rho \in (-1/(n-1), 1)$.

Proof. Consider $a = (a_1, ..., a_n) \neq 0$ is a given real vector. Then

$$\begin{aligned} a' \sum a &= \sigma^{2n} \left[\rho \sum_{i \neq j} a_i a_j + \sum_{i=1}^{n} a_i^2 \right] \\ &= \sigma^{2n} \left[\rho \left(\sum_{i=1}^{n} a_i \right)^2 + (1-\rho) \sum_{i=1}^{n} a_i^2 \right] \end{aligned}$$

$\rho\left(\sum_{i=1}^{n} a_i\right)^2$ is always positive while ρ is positive. If $\rho \in (-1/(n-1), 0)$ it is obvious that

$$\text{if } \sum_{i \neq j} a_i a_j \leq 0 \text{ then } \rho \sum_{i \neq j} a_i a_j + \sum_{i=1}^{n} a_i^2 > 0$$

and, if $\sum_{i \neq j} a_i a_j > 0$ and $\rho \in (-1/(n-1), 0)$ we have

$$\begin{aligned} \rho \sum_{i \neq j} a_i a_j + \sum_{i=1}^{n} a_i^2 &> \left[\sum_{i=1}^{n} a_i^2 - \frac{1}{(n-1)} \sum_{i \neq j} a_i a_j\right] \\ &= \frac{1}{(n-1)} \left[\sum_{i<j} (a_i - a_j)^2\right] \geq 0 \end{aligned}$$

So $\sum$ is positive definite for $\rho \in (-1/(n-1), 1)$. ■

Single integral formula of 3.1 could be derived easily from the direct result of the following well known theorem.

Theorem 7 *The following statements are equivalent provided that $\lambda_i \in [-1, 1]$, $\sum$ is positive definite and $\sum = (\sigma_{ij})$, where*

$$\sigma_{ij} = \begin{cases} \sigma_i^2 & for\ i = j \\ \lambda_i \lambda_j \sigma_i \sigma_j & for\ i \neq j \end{cases} \tag{B.1}$$

1. *$X = (X_1, \ldots, X_n)'$ has an $N_n(\mu, \sum)$ distribution such that $X_1, \ldots, X_n$ are $N(\mu, \sum)$ single variate normal variables.*

2. *X and*

$$\left(\sigma_1\left(\sqrt{1-\lambda_1^2} Z_1 + \lambda_1 Z_0\right) + \mu_1, \ldots, \sigma_n\left(\sqrt{1-\lambda_n^2} Z_n + \lambda_n Z_0\right) + \mu_n\right)$$

are identically distributed where $Z_0, \ldots, Z_n$ are i.i.d. standard normal variables $N(0,1)$, $\mu \in \Re$, $\sigma > 0$, and $\rho_{ij} = \lambda_i \lambda_j$, $\lambda_i \in [-1, 1]$.

Above theorem (refer to Tong[19]) states that, if X is distributed according to $N_n(\mu, \sum)$ and conditions of the theorem are satisfied, X and

$$\left(\sigma_1\left(\sqrt{1-\lambda_1^2} Z_1 + \lambda_1 Z_0\right) + \mu_1, \ldots, \sigma_n\left(\sqrt{1-\lambda_n^2} Z_n + \lambda_n Z_0\right) + \mu_n\right)$$

are identically distributed, where Z_0, Z_1,..., Z_n are i.i.d. $N(\mu, \sigma^2)$ normal variables. Thus if A is an n-dimensional region given by

$$A = \{x : x \in \Re^n, b_i \le x_i \le a_i, i = 1, .., n\},$$

where $-\infty \le b_i \le a_i \le \infty \quad (i = 1, .., n)$, then the probability $P[X \in A]$ can be expressed as

$$\begin{aligned} & P[X \in A] \\ = & \int_{-\infty}^{\infty} \prod_{i=1}^{n} P\left[b_i \le \sigma_1 \left(\sqrt{1-\lambda_1^2} Z_1 + \lambda_1 Z_0\right) + \mu_1 \le a_i\right] \phi(z)\, dz \qquad \text{(B.2)} \\ = & \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left[\Phi\left(\frac{(a_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1-\lambda_i^2}}\right) - \Phi\left(\frac{(b_i - \mu_i)/\sigma_i + \lambda_i z}{\sqrt{1-\lambda_i^2}}\right)\right] \phi(z)\, dz \end{aligned}$$

where Φ and ϕ are standard normal distribution function and the density function, respectively. The right hand side of the B.3 is a *single* integral instead of a multiple integral over $\Re^n$. A special problem of interest, like the one we are dealing with during our research, concerns the covariance matrix of an n-dimensional normal variable with a common mean $\mu = 0$, a common variance $\tau^2\sigma^2$ and a common correlation coefficient $\rho \ge 0$. In this special case B.3 reduces to

$$P[X \in A] = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \left[\Phi\left(\frac{(a_i)/\sigma + \sqrt{\rho} z}{\sqrt{1-\rho}}\right) - \Phi\left(\frac{(b_i)/\sigma + \sqrt{\rho} z}{\sqrt{1-\rho}}\right)\right] \phi(z)\, dz \qquad \text{(B.3)}$$

The expressions in B.3 could be used to evaluate multivariate normal probability integral when $\rho \ge 0$. Since it involves the integral of only one variable, it is obviously easier to evaluate numerically on a computer.

Appendix C

Least Squares Normal Equations

In this section, we will give the normal equations for the least squares estimates (BLUE) for interested readers. Let TT_i denote the sum of all observations obtained with the ith treatment $(0 \leq i \leq p)$, BT_j denote the sum of all observations in the jth $(1 \leq j \leq b)$ block and RT_t is the sum of all observations in the row t $(1 \leq t \leq k)$. $GT = \sum_{j=1}^{b} BT_j = \sum_{t=1}^{k} RT_t$. The least squares normal equations are:

$$N \hat{\mu} + \sum_{i=0}^{p} r_i \hat{\alpha}_i + k \sum_{j=1}^{b} \hat{\beta}_j + b \sum_{t=1}^{k} \hat{\tau}_t = GT \tag{C.1}$$

$$r_i \hat{\mu} + r_i \hat{\alpha}_i + \sum_{j=1}^{b} r_{ij} \hat{\beta}_j + \sum_{t=1}^{k} s_{it} \hat{\tau}_t = TT_i \tag{C.2}$$

$$k \hat{\mu} + \sum_{n=0}^{p} r_{nj} \hat{\alpha}_n + k \hat{\beta}_j + \sum_{t=1}^{k} \hat{\tau}_t = BT_j \tag{C.3}$$

$$b \hat{\mu} + \sum_{h=0}^{p} s_{ht} \hat{\alpha}_h + b \hat{\tau}_t + \sum_{j=1}^{b} \hat{\beta}_j = RT_t \tag{C.4}$$

where $\hat{\alpha}_i$ is the estimator of the treatment effect, $\hat{\beta}_j$ is the estimator of the block effect, $\hat{\tau}_t$ is the estimator of the row effect and $\hat{\mu}$ is the estimator of the common mean and $\sum_{i=0}^{p} \hat{\alpha}_i = \sum_{j=1}^{b} \hat{\beta}_j = \sum_{t=1}^{k} \hat{\tau}_t = 0$ by definition.

Vitae

Murat Aksu was born in 1972. He studied high school at İstanbul Atatürk Fen Lisesi. He graduated from Department of Industrial Engineering, Boğaziçi University in 1994. In October 1994, he joined to the Department of Industrial Engineering at Bilkent University as a graduate student. From that time to the present, he worked with Prof. T. E. Türe for his graduate study at the same department.