# PARTIAL DIFFERENTIAL EQUATIONS POSSESSING THE PAINLEVE PROPERTY

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Fehd Jrad
September, 1996

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Uğurhan Muğan(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## PARTIAL DIFFERENTIAL EQUATIONS POSSESSING THE PAINLEVÉ PROPERTY

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M.S. in Mathematics
Advisor: Asst. Prof. Dr. Uğurhan Muğan
September, 1996

In this thesis, applying the Painlevé test developed by Weiss, Tabor and Carnevale (WTC), we investigated the Painlevé property of Burgers' type of equations, KdV type of equations and the KP extensions of the KdV type of equations. We showed that there are infinitely many equations of these types possessing the Painlevé property and thus we classified them with respect to Painlevé property.


*Keywords :* Painlevé property, Singular manifold, Resonances, Compatability conditions.

# ÖZET

## PAİNLEVÉ ÖZELLİĞNE SAHİP KISMİ TÜREVLİ DENKLEMLER

Fahd Jrad
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Danışman: Asst. Prof. Dr. Uğurhan Muğan
Eylül, 1996

Bu tezde Weiss, Tabor ve Carnevale (WTC) tarafından geliştirilen Painlevé testini uygulayarak, Burgers, KdV tipi denklemlerin ve KP genellemelerinin Painlevé özelliği araştırılmıştır. Bu tipte Painlevé özelliğine sahip sonsuz sayıda denklem olduğu gösterilmiş ve bu denklemler Painlevé özelliğine göre sınıflandırılmıştır.

*Anahtar Kelimeler :* Painlevé özelliği, Uyumluluk şartları, Rezonans, Tekil manifold.

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# Chapter 1

# Introduction

The various methods such as symmetries and Painlevé test have been developed to test the integrability of a given partial differential equation (PDE). Between these two approaches, there are intimate relations. Moreover, both methods have been used to classify the integrable PDE 's. In this work, we classify Burgers' type and KdV type of equations which possess the Painlevé property. Furthermore, the KP extensions of the KdV type of equations are considered.

At the turn of this century, Painlevé and Gambier [9] classified the certain type of ordinary differential equations (ODE) that have the Painlevé property; i.e. their solutions are free from the movable critical points.

In 1980, Ablowitz, Ramani and Segur (ARS) [8] stated the conjecture that the system of ODE 's obtained from PDE, which is solvable by inverse scattering, by an exact similarity reduction possesses the Painlevé property. Since the work of Kowalevskaya [10] that was the first connection between the integrability and the Painlevé property, ARS gave an explicit algorithm to determine whether a reduced ODE meets the necessary conditions to possess the Painlevé property. This method is more simple than the $\alpha$ -method used by Painlevé and his school but similar to the method of Kowalevskaya [10]. The method is based on requiring the single valued solution about the movable critical points.

Weiss, Tabor and Carnevale (WTC) [1] extended the algorithm developed by ARS as to be applicable direclly to PDE 's. This extension requires that a given PDE has the Painlevé property if its general solution is free from the noncharacteristic movable branched singularity. As in the case of ODE, WTC test gives the neccessary conditions for a PDE to have the Painlevé property. The test consists of seeking a Laurent series solution of a given PDE in the neighborhood of a noncharacteristic movable singularity manifold. The WTC method has also been used in providing a constructive proof of integral properties of PDE 's. In particular, the truncated Laurent expression can be used to obtain the Lax pairs and the Bäcklund transformations.

The WTC method can be summarized as follows:
Let $F(u, u_t, u_{x_1}, u_{tt}, u_{tx_1}, u_{x_1x_1}, ...) = 0$ be an equation of order $n$ in $m+1$ dimension. If the non characteristic manifold is $\phi(t, x_1, x_2, ..., x_m) = 0$ and $u = u(t, x_1, x_2, ..., x_m)$ is a solution of the PDE, then $u$ is expanded as

$$u = \phi^p \sum_{j=0}^{\infty} u_j \, \phi^j. \tag{1.1}$$

where $\phi(t, x_1, x_2..., x_m)$ and $u_j = u_j(t, x_1, x_2, ..., x_m)$ are analytic functions of $(t, x_1, x_2, ..., x_m)$ in a neighborhood of the manifold and $p$ is a negative integer. [1] The WTC method to check whether a PDE has the Painlevé property contains three basic steps :

**1) The leading order analysis :** Substitute $u = u_0 \phi^p$ where $p$ is a negative integer in the given PDE. For certain values of $p$, two or more terms in the equation may balance and the rest can be ignored. For each such values of $p$ , the terms which can balance are called the leading terms. Requiring that the leading terms do balance determines $u_0$ .

**2) The resonances :** For each $(p, u_0)$ from step1, construct a simplified equation that contains only the leading terms of the original equation. Substitute

$$u = u_0 \, \phi^P + u_j \, \phi^{P+j} \tag{1.2}$$

in the simplified equation. To leading order in $u_j$, this equation reduces to $R_n(j) \, u_j \, \phi^{p+j-n} = 0$ , provided that the given PDE is normalized. The roots of the $n^{\text{th}}$ order polynomial $R_n(j)$ determines the resonances.

(i) One of roots is always $-1$ representing the arbitrariness of the manifold $\phi = 0$ .

(ii) The other $n-1$ roots must be nonnegative integers and distinct.

**3) The compatability conditions :** For a given $p$ from step1, substitute (1.1) in the original equation to get the recursion relations for $u_j$

$$(j+1)(j-r_1).....(j-r_{n-1})u_j = F_j(u_{j-1}, ..., u_0, \phi_{x_i}, ...). \tag{1.3}$$

where $r_i$ 's are the nonnegative integer resonances. If at each $r_i$ the compatability condition $F_i = 0$ is satisfied ( i.e. $u_i$ is arbitrary), we say that the PDE has the Painlevé property.

Kruskal [2], [3] has simplified the WTC method when introduced the so called reduced ansatz

$$\phi(t, x_1, x_2, ..., x_m) = x_1 + \psi(t, x_2, ..., x_m) \tag{1.4}$$

and so $u_j = u_j(t, x_2, ..., x_m)$ and

$$u(t, x_1, x_2, ..., x_m) = \phi^p \sum_{j=0} u_j(t, x_2, ..., x_m)\phi^j \tag{1.5}$$

In this thesis, classification of Burgers' type of equations and KdV type of equations with their KP extensions which possess the Painlevé property are given .

Burgers' type of equations are given by

$$u_t + P(\tau)\, u_{xx} + Q(\tau)\, u_x = 0 \tag{1.6}$$

while KdV type of equations and their KP extensions are meant to be in the following forms respectively

$$u_t + P(\tau)\, u_{xxx} + Q(\tau)\, u_x = 0 \tag{1.7}$$

$$u_t + P(\tau)\, u_{xxx} + Q(\tau)\, u_x + \sigma u_{yy} = 0 \tag{1.8}$$

where $P$ and $Q$ are polynomials of $\tau$ and $\tau$ is a fuction of the the independent variable $u$ and $\sigma \neq 0$. In this work, $P$ and $Q$ are taken as polynomials in $\tau$ where $\tau = u^{\frac{1}{k}}$, $k \in \mathbb{Z}_+$. In the first chapter, the Burgers' type of equations with time dependent coefficient polynomials P and Q are considered. The second chapter is divided into two sections. The first section is devoted to study the KdV type of equations with constant ceofficient polynomials and the second section to study the equations with time dependent coefficient polynomials P and Q. In the last chapter, the KP extensions of the KdV type of equations considered in the first section of chapter2 are examined.

# Chapter 2

# Burgers' Type Of Equations

In this chapter, we study equations of the following form

$$u_t + P_m(u^{\frac{1}{k}})u_{xx} + Q_n(u^{\frac{1}{k}})u_x = 0, \tag{2.1}$$

where $P_m$ and $Q_n$ are polynomials of $u^{\frac{1}{k}}$ with time-dependent coefficients $a_i(t)$ and $b_j(t)$ and $k$ is a positive integer. These polynomials $P_m$ and $Q_n$ are defined by

$$P_m(u^{\frac{1}{k}}) = \sum_{i=0}^{m} a_i\, u^{\frac{i}{k}}, Q_n(u^{\frac{1}{k}}) = \sum_{i=0}^{n} b_i\, u^{\frac{i}{k}}. \tag{2.2}$$

To examine the Painlevé property of the equations (for all $k$ , $m$ and $n$) in (2.1) first we let $u = q^k$. Under this change of dependent variable Eq.(2.1) becomes

$$\begin{aligned} q\, q_t + P_m(q)\, [(k-1)\, q_x^2 + q\, q_{xx}] \\ + Q_n(q)\, q\, q_x = 0. \end{aligned} \tag{2.3}$$

Eq.(2.3) possess the Painlevé property for all positive integers $k$,$m$ and $n$ provided that the coefficients $a_i$ and $b_j$ ($i = 0, 1, 2, ..., m$ , $j = 0, 1, 2, ..., n$) of the polynomials $P_m$ and $Q_n$ are subject to satisfy some constraints. Around the singular manifold $\phi = x - \psi(t) = 0$ the new dependent variable $q(t, x)$, using the reduced ansatz, is expanded as

$$q(t, x) = \phi^p \sum_{i=0}^{\infty} q_i(t)\, \phi^i. \tag{2.4}$$

If we follow the steps of the WTC method we get the following. In the sequel we call equations with the same $k$ and $n$ as a class $(k, n)$.

**i) Leading order analaysis:** For all $k$ the only possibility for the leading order is $p = -1$ occuring only when $m = n - 1$, which means $a_m = 0$ for all $m > n - 1$

$$q_0 = \frac{a_{n-1}}{b_n}(k+1). \tag{2.5}$$

where $a_{n-1} \neq 0$ and $b_n \neq 0$ .
**ii) Resonances:** For all $n$ the resonances are $r = -1$ , $k + 1$.

**iii) Arbitrary functions:** It turns out that the functions $\psi$ and $q_i$ at the resonances are arbitrary under some compatability conditions among the coefficients $a_i$ and $b_j$ 's of the polynomials $P_{n-1}$ and $Q_n$. The function $q_1$ can be given for all classes $(k, n)$

$$q_1 = \frac{-q_0^2\, \psi_t\, \delta_n^1 - b_{n-1}\, q_0^{n+1} + (k+1)\, a_{n-2}\, q_0^n}{2\, k\, a_{n-1}\, q_0^n}. \tag{2.6}$$

We now give those equations (2.1) passing the Painlevé test for $k = 1, 2, 3$ .
**Case1.** $(k, n) = (1, 1)$ This class is identical to the Burgers equation. Painlevé analysis of this equation with constant coefficient was first given by [1] . Here this equation has the Painlevé propert if $a_0 = c\, b_1$ where $c$ is a constant.

For $n > 2$ the classes $(1, n)$ pass the test if the condition

$$a_{n-3}\, a_{n-1}\, b_n + a_{n-2}\, a_{n-1}\, b_{n-1} - a_{n-1}^2\, b_{n-2} - a_{n-2}^2\, b_n = 0.$$

which makes $q_2$ arbitrary, is satisfied.
**Case2.** $(k, n) = (2, 2)$ .The compatability conditions at the resonance $r = 3$ give only $a_0 = 0$ and $a_1 = c\, b_2$ , where $c$ is a constant . The resulting equation passing the Painlevé test is given by

$$u_t + a_1\, u^{\frac{1}{2}}\, u_{xx} + (b_0 + b_1 u^{\frac{1}{2}} + b_2 u)\, u_x = 0. \tag{2.7}$$

The coefficient function $q_3$ is arbitary and $q_i$ for $i = 0, 1$ are given by

$$q_0 = \frac{3\, a_1}{b_2}\ ,\ q_1 = -\frac{3\, b_1}{4\, b_2}. \tag{2.8}$$

For $n > 3$ the class $(2, n)$ pass the test with the following condition that makes $q_3$ arbitrary

$$a_{n-4}\, a_{n-1}^2\, b_n + a_{n-2}\, a_{n-1}\, b_{n-2} + a_{n-2}^3\, b_n + a_{n-3}\, a_{n-1}^2\, b_{n-1} - a_{n-1}^3\, b_{n-3} -$$

$$a_{n-2}^2\, a_{n-1}\, b_{n-1} - 2\, a_{n-3}\, a_{n-2}\, a_{n-1}\, b_n = 0.$$

**Case3.** $(k,n) = (3,2)$ The compatability condition corresponding to the resonance $r = 4$ gives $a_0 = 0$ and $a_1\, b_1 = c\, b_2^2$ ,$c$ is a constant.

The resulting equation passing the Painlevé test is given by

$$u_t + a_1\, u^{\frac{1}{3}}\, u_{xx} + (b_0 + b_1 u^{\frac{1}{3}} + b_3 u^{\frac{2}{3}})\, u_x = 0. \tag{2.9}$$

The coefficient functions $q_i$ for $i = 0, 1$, are given by

$$q_0 = \frac{4\, a_1}{b_2}\ ,\ q_1 = -\frac{2\, b_1}{3\, b_2}. \tag{2.10}$$

**Case4.** $(k,n) = (3,3)$ . The compatability condition at the resonance $r = 4$ implies

$$a_1 = 0 \quad , \quad 4\, a_2 [\frac{a_2}{b_3}]_t + a_0\, a_2\, b_1 - a_0^2\, b_3 = 0.$$

The resulting equation passing the Painlevé test is given by

$$u_t + (a_0 + a_2\, u^{\frac{2}{3}})\, u_{xx} + Q_3(u^{\frac{1}{3}})\, u_x = 0. \tag{2.11}$$

$q_0$ and $q_1$ are given by

$$q_0 = \frac{4\, a_2}{b_3}\ ,\ q_1 = -\frac{2\, b_2}{3\, b_3}. \tag{2.12}$$

For $n > 4$ the class $(3,n)$ have the Painlevé property if the following condition which makes $q_4$ arbitrary is satisfied .

$$\begin{aligned} a_{n-5}\, a_{n-1}^3\, b_n + a_{n-2}\, a_{n-1}^3\, b_{n-3} + a_{n-4}\, a_{n-1}^3\, b_{n-1} + a_{n-3}\, a_{n-1}^3\, b_{n-2} + \\ 3\, a_{n-3}\, a_{n-2}^2\, a_{n-1} b_n + a_{n-2}^2\, a_{n-1}\, b_{n-1} - a_{n-1}^4\, b_{n-4} - 2\, a_{n-4}\, a_{n-2}\, a_{n-1}^2\, b_n - \\ a_{n-2}^2\, a_{n-1}^2\, b_{n-2} - a_{n-3}^2\, a_{n-1}^2\, b_n - 2\, a_{n-3}\, a_{n-2}\, a_{n-1}^2\, b_{n-1} - a_{n-2}^4\, b_n = 0. \end{aligned}$$

When the positive integer $k$ increases resonances become larger. Then it becomes quite difficult to find the compatability conditions. So we stop giving more classes passing the Painlevé test. We conjecture there are infininitely many classes $(k > 3, n)$ possessing the Painlevé property.

# Chapter 3

# Third Order KdV Type Of Equations

In this chapter,we study equations of the form

$$u_t + P_m(u^{\frac{1}{k}})u_{xxx} + Q_n(u^{\frac{1}{k}})u_x = 0 \tag{3.1}$$

where $P_m$ and $Q_n$ are polynomials of $u^{\frac{1}{k}}$ where $k$ is a positive integer. These polynomials are defined by

$$P_m(u^{\frac{1}{k}}) = \sum_{i=0}^{m} a_i u^{\frac{i}{k}}, Q_n(u^{\frac{1}{k}}) = \sum_{i=0}^{n} b_i u^{\frac{i}{k}}. \tag{3.2}$$

To examine the Painlevé property of the equations (for all $k$ , $m$ and $n$) in (3.1) first we let $u = q^k$. Under this change of dependent variable Eq.(3.1) becomes

$$\begin{aligned} &q^2\, q_t + P_m(q)\,[q^2\, q_{xxx} + 3\,(k-1)\, q\, q_x\, q_{xx} \\ &\quad +(k-1)\,(k-2)\, q_x^3] + q^2\, Q_n(q)\, q_x = 0. \end{aligned} \tag{3.3}$$

Eq.(3.3) possess the Painlevé property for all positive integers $k$,$m$ and $n$ provided that the coefficient constants $a_i$ and $b_j$ $(i = 1, 2, ..., m\ ,\ j = 1, 2, ..., n)$ of the polynomials $P_m$ and $Q_n$ are subject to satisfy some constraints. Around the singular manifold $\phi = x - \psi(t) = 0$ the new dependent variable $q(t, x)$, using the reduced ansatz, is expanded as

$$q(t, x) = \phi^p \sum_{i=0}^{\infty} q_i(t)\, \phi^i. \tag{3.4}$$

If we follow the steps of the WTC method, we get the following:

**i) Leading order analaysis:** For all $k$ we have only two possibilities for the leading order , $p = -1$ and $p = -2$ . When $p = -1$ we have $m = n - 2, n > 1$ which means $a_m = 0$ for all $m > n - 2$ and

$$q_0^2 = -\frac{a_{n-2}}{b_n}\,(k+1)\,(k+2)\ ,\quad n > 1. \tag{3.5}$$

where $a_{n-2} \neq 0$ and $b_n \neq 0$.
When $p = -2$ we have $m = n - 1$ which means $a_m = 0$ for all $m > n - 1$ and

$$q_0 = -\frac{2\,a_{n-1}}{b_n}\,(k+1)\,(2k+1) \tag{3.6}$$

where $a_{n-1} \neq 0$ and $b_n \neq 0$.
**ii) Resonances:** When $p = -1$ the resonances are $r = -1$ , $k+2$ and $2\,k+2$ for all $n$ .
On the other hand when the leading order $p = -2$ resonances become $r = -1$ , $2\,k+2$ and $4\,k+2$ for all $n$.

**iii) Arbitrary functions:** It turns out that the functions $\psi$ and $q_i$ at the resonances are arbitrary under some compatability conditions among the coefficients $a_i$ and $b_j$ 's of the polynomials $P_m$ and $Q_n$. The function $q_1$ can be given for all classes $(k, n)$ with the leading orders $p = -1$ and $p = -2$ respectively

$$q_1 = \frac{b_{n-1}\,q_0^2 + (k+1)\,(k+2)\,a_{n-3}}{2\,(k+1)\,(2k+1)\,a_{n-2}}\quad ,\quad q_1 = 0\quad ,\quad (n > 1). \tag{3.7}$$

where $a_{-i}$ and $b_{-i}$ vanish for all positive integer $i$.

## 3.1 KdV type of equations with constant coefficients

In this section, we give some of the equations (3.3) passing the Painlevé test where $a_i$ 's and $b_i$ 's are constant :

**Case1.** $(k, n) = (1, 1)$ with $p = -2$. This class is identical to the KdV equation. Painlevé analysis of this equation was first given by [1].
**Case2.** $(k, n) = (1, 2)$ with $p = -1$. The compatability conditions at the resonances $r = 3$ and $r = 4$ doesnot bring any conditions among the coefficients $a_0, b_0, b_1$ and $b_2$. The resulting equation passing the Painlevé test is given by

$$u_t + a_0\,u_{xxx} + (b_0 + b_1 u + b_2 u^2)\,u_x = 0. \tag{3.8}$$

The coefficient funcions $q_i$ for $i = 0, 1, 2$ are given by

$$q_0^2 = -\frac{6\, a_0}{b_2} \quad , \quad q_1 = -\frac{b_1}{2b_2} \quad , \quad q_2 = \frac{q_0\,(4\,\psi_t\, b_2 + 4b_0 b_2 - b_1^2)}{24 a_0 b_2}. \tag{3.9}$$

The functions $q_3$ and $q_4$ are arbitrary. Eq.(3.8) is just the superposition of the KdV and mKdV equations.

**Case3.** $(k, n) = (1, 4)$ with $p = -2$. The compatability conditions at $r = 4$ and $r = 6$ respectively give

$$a_1 a_3 b_4 - a_2^2 b_4 + a_2 a_3 b_3 - a_3^2 b_2 = 0,$$
$$a_0\, a_3\, b_4 - a_1\, a_2\, b_4 + a_1\, a_3\, b_3 - a_3^2\, b_1 = 0.$$

The resulting equation passing the Painlevé test is given by

$$u_t + P_3(u)\, u_{xxx} + Q_4(u)\, u_x = 0. \tag{3.10}$$

The coefficient funcions $q_4$ and $q_6$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\frac{12\, a_3}{b_4} \quad , \quad q_1 = 0 \quad , \quad q_2 = \frac{a_2 b_4 - a_3 b_3}{a_3 b_4} \tag{3.11}$$

**Case4.** $(k, n) = (1, 5)$ with $p = -1$. The compatability conditions at $r = 3$ and $r = 4$ respectively give

$$\begin{aligned} -a_0 a_3^2 b_5 + 2a_1 a_2 a_3 b_5 - a_1 a_3^2 b_4 - \\ a_2^3 b_5 + a_2^2 a_3 b_4 - a_2 a_3^2 b_3 + a_3^3 b_2 = 0, \\ a_0\, a_2\, a_3 b_5 - a_0\, a_3^2\, b_4 + a_1^2\, a_3\, b_5 \\ -a_1\, a_2^2\, b_5 + a_1\, a_2\, a_3\, b_4 - a_1\, a_3^2\, b_3 + a_3^3\, b_1 = 0. \end{aligned}$$

The resulting equation passing the Painlevé test is given by

$$u_t + P_3(u)\, u_{xxx} + Q_5(u)\, u_x = 0. \tag{3.12}$$

The coefficient funcions $q_3$ and $q_4$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$\begin{aligned} q_0^2 = -\frac{6\, a_3}{b_5} \quad , \quad q_1 = \frac{a_2\, b_5 - a_3\, b_4}{2\, a_3\, b_5}, \\ q_2 = \frac{4 a_1\, a_3\, b_5^2 - 3\, a_2^2\, b_5^2 + 2\, a_2\, a_3\, b_4\, b_5 - 4\, a_3^2\, b_3\, b_5 + a_3^2\, b_4^2}{4\, a_3^2\, b_5^2\, q_0}. \end{aligned} \tag{3.13}$$

**Case5.** $(k, n) = (1, 5)$ with $p = -2$. The compatability conditions at $r = 4$ and $r = 6$ respectively give

$$a_2\, a_4\, b_5 - a_3^2\, b_5 + a_3\, a_4\, b_4 - a_4^2\, b_3 = 0,$$
$$-a_1\, a_4\, b_5 + a_2\, a_3\, b_5 - a_2\, a_4\, b_4 + a_4^2\, b_2 = 0.$$

The resulting equation passing the Painlevé test is given by

$$u_t + P_4(u)\, u_{xxx} + Q_5(u)\, u_x = 0 \tag{3.14}$$

The coefficient funcions $q_4$ and $q_6$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\frac{12\, a_4}{b_5} \quad , \quad q_1 = 0 \quad , \quad q_2 = \frac{a_3 b_5 - a_4 b_4}{a_4 b_5}. \tag{3.15}$$

For $n > 5$ we have classes $(1, n)$ pass the test both for $p = -1$ and $p = -2$. The compatability conditions satisfied by the constants $a_i$ and $b_i$ where $(i = 1, 2, ..., n)$ become very lengthy.

**Case6.** $(k, n) = (2, 1)$ with $p = -2$. The compatability conditions at the resonances do not bring any condition on the coefficients $a_i$ and $b_i$. The resulting equation passing the Painlevé test is given by

$$u_t + a_0\, u_{xxx} + (b_0 + b_1\, u^{\frac{1}{2}})\, u_x = 0. \tag{3.16}$$

The coefficient funcions $q_6$ and $q_{10}$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\frac{30\, a_0}{b_1} \quad , \quad q_1 = 0 \quad , \quad q_2 = -\frac{5\psi_t + b_0}{8 b_1}$$

Here we can take, without loss of generality, $a_0 = 1$, $b_0 = 0$ and $b_1 = 1$. Hence (3.16) reduces to

$$u_t + u_{xxx} + u^{\frac{1}{2}}\, u_x = 0. \tag{3.17}$$

Painlevé analysis of this equation was first studied by Xiao [4].

**Case7.** $(k, n) = (2, 2)$ with $p = -1$. The compatability conditions at the resonances $r = 4, 6$ give only $b_1 = 0$. This equation is nothing but the KdV equation. Painlevé analysis of this equation was first given by [1].

**Case8.** $(k, n) = (2, 2)$ with $p = -2$. The compatability conditions at the resonances $r = 6, 10$ give only $a_0 = 0$. The resulting equation passing the Painlevé test is given by

$$u_t + a_1\, u^{\frac{1}{2}}\, u_{xxx} + (b_0 + b_1 u^{\frac{1}{2}} + b_2 u)\, u_x = 0. \tag{3.18}$$

The coefficient funcions $q_6$ and $q_{10}$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\frac{30\, a_1}{b_2} \quad , \quad q_1 = 0 \quad , \quad q_2 = -\frac{5 b_1}{8 b_2}. \tag{3.19}$$

**Case9.** $(k, n) = (2, 3)$ with $p = -1$. The compatability conditions at the resonances $r = 4$ and $r = 6$ imply only $a_0 = 0$ . The resulting equation passing the Painlevé test is given by

$$u_t + a_1\, u^{\frac{1}{2}}\, u_{xxx} + Q_3(u^{\frac{1}{2}})\, u_x = 0. \tag{3.20}$$

The coefficient funcions $q_4$ and $q_6$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0^2 = \frac{-12\, a_1}{b_3} \quad , \quad q_1 = \frac{-2\, b_2}{5\, b_3} \quad , \quad q_2 = \frac{q_0\, (25\, b_1\, b_3 - 8\, b_2^2)}{600\, a_1\, b_3}. \tag{3.21}$$

**Case10.** $(k, n) = (2, 4)$ with $p = -2$. The compatability conditions at the resonances $r = 6, 10$ imply $a_2 = 0$ and

$$a_0\, b_4 + a_1\, b_3 - a_3\, b_1 = 0,$$
$$a_0\, (a_1\, b_4 - a_3\, b_2) = 0.$$

The resulting equation passing the Painlevé test is given by

$$u_t + (a_0 + a_1\, u^{\frac{1}{2}} + a_3 u^{\frac{3}{2}})\, u_{xxx} + Q_4(u^{\frac{1}{2}})\, u_x = 0. \tag{3.22}$$

The coefficient funcions $q_6$ and $q_{10}$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\frac{30\, a_3}{b_4} \quad , \quad q_1 = 0 \quad , \quad q_2 = -\frac{5\, b_3}{8\, b_4}. \tag{3.23}$$

**Case11.** $(k, n) = (2, 5)$ with $p = -1$. The compatability conditions at $r = 4, 6$ give $a_2 = 0$ and

$$a_0\, (a_3\, b_2 - a_1\, b_4 - a_0\, b_5) = 0,$$
$$a_0\, a_3\, b_4 - a_1^2\, b_5 + a_1\, a_3\, b_3 - a_3^2\, b_1 = 0.$$

The resulting equation passing the Painlevé test is given by

$$u_t + (a_0 + a_1\, u^{\frac{1}{2}} + a_3 u^{\frac{3}{2}})\, u_{xxx} + Q_5(u^{\frac{1}{2}})\, u_x = 0. \tag{3.24}$$

The coefficient funcions $q_4$ and $q_6$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$\begin{aligned} {}^{2}_{0} &= -\frac{12\, a_3}{b_5} \quad , \quad q_1 = \frac{-2\, b_4}{5\, b_5}, \\ q_2 &= -\frac{q_0\, (25\, a_1\, b_5^2 - 25\, a_3\, b_3\, b_5 + 8\, a_3\, b_4^2)}{600\, a_3^2\, b_5}. \end{aligned} \tag{3.25}$$

The class $(2,6)$ passes the test only for $p=-2$. The classes $(2,n)$ for $n>6$ pass the test both for $p=-1$ and $p=-2$. For all these cases the compatability conditions are quite lengthy.

**Case12.** $(k,n)=(3,3)$ with $p=-1$. The compatablity conditions corresponding to the resonances $r=5$ and $r=8$ are respectively $a_0=0$ and $b_2=0$. The resulting equation passing the Painlevé test is given by

$$u_t+a_1\,u^{\frac{1}{3}}\,u_{xxx}+(b_0+b_1u^{\frac{1}{3}}+b_3u)\,u_x=0. \tag{3.26}$$

The coefficient funcions $q_5$ and $q_8$ are arbitrary and $q_i$ for $i=0,1,2$ are given by

$$q_0^2=-\frac{20\,a_1}{b_3}\quad,\quad q_1=\frac{-5\,b_2}{14\,b_3}\quad,\quad q_2=-\frac{1960b_1}{5292b_3\,q_0}. \tag{3.27}$$

**Case13.** $(k,n)=(3,3)$ with $p=-2$. The compatablity conditions at the resonances $r=8$ and $r=14$ imply

$$a_0=a_1=0. \tag{3.28}$$

The resulting equation passing the Painlevé test is given by

$$u_t+a_2\,u^{\frac{2}{3}}\,u_{xxx}+Q_3(u^{\frac{1}{3}})\,u_x=0. \tag{3.29}$$

The coefficient funcions $q_8$ and $q_{14}$ are arbitrary and $q_i$ for $i=0,1,2$ are given by

$$q_0=-\frac{56\,a_2}{b_3}\quad,\quad q_1=0\quad,\quad q_2=-\frac{1568b_2}{3042b_3}\quad,\quad q_3=0. \tag{3.30}$$

We stop here giving more examples of the classes $(k,n)$ for each leading order $p=-1$ and $p=-2$. When the positive integer $k$ increases resonances become larger. Then it becomes quite difficult to find the compatability conditions.We conjecture that there are infinitely many classes $(k>3,n)$ possess the Painlevé property.

## 3.2 KdV type of equations with time-dependent coefficients

We have also considered Eq (3.3) when the coefficients of $P_m$ and $Q_n$ , given in (3.2), are time dependent i.e. $a_i=a_i(t)$ and $b_i=b_i(t)$ where $k=1$ . It turns out , except for the cases when $(k,n,p)=(1,1,-2)$ and $(k,n,p)=(1,2,-1)$ , the classes $(k,n)=(1,n)$ are of Painlevé type with the same compatability conditions and the same data mentioned in the first section . The exceptional cases are

**Case1.** $(k,n) = (1,1)$ with $p = -2$ . The compatability condition at the resonance $r = 4$ is identically satisfied while at $r = 6$ it is given by

$$\frac{d(\frac{a_0}{b_1})}{dt} = cb_1 \tag{3.31}$$

where c is a constant. When $a_0$ is a constant this class is identical to the cylindrical KdV equation. [3]

$$u_t + 6\,u\,u_x + .u_{xxx} + \frac{1}{2t}u = 0. \tag{3.32}$$

**Case2.** $(k,n) = (1,2)$ with $p = -1$ . The compatability conditiontions at $r = 3$ and $r = 4$ give $\frac{a_o}{b_2} = c$ and $\frac{b_1}{b_2} = d$ where $c$ and $d$ are constant. The resulting equation having the Painlevé property is

$$u_t + a_0 u_{xxx} + (b_0 + b_1 u + b_2 u^2) u_x = 0. \tag{3.33}$$

The functions $q_3$ and $q_4$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0^2 = -6\frac{a_0}{b_2} \quad , \quad q_1 = -\frac{b_1}{2b_2} \quad , \quad q_2 = \frac{q_0(4\psi_t b_2 4 b_0 + 4 b_0 b_2 - b_1^2)}{24 a_0 b_2}. \tag{3.34}$$

Eq.(3.33) is nothing but the superposition of the KdV and mKdV equations.

# Chapter 4

# KP Type Of Equations

In this chapter we shall consider The KP extensions of the classes studied in chapter2. These extensions are given as follows:

$$[u_t + P_m(u^{\frac{1}{k}})u_{xxx} + Q_n(u^{\frac{1}{k}})u_x]_x + \sigma u_{yy} = 0. \tag{4.1}$$

where $\sigma \neq 0$ is a constant and $k$ is a positive integer. Here we shall consider that the coefficients of the polynomials $P_m$ and $Q_n$ are constants. The polynomials $P_m$ and $Q_n$ are given in chapter1 and chapter2. If we let $u = q^k$ around the singularity manifold, by using the reduced anzats, $\phi = x - \psi(y,t) = 0$ the function $q(x,y,t)$ has the expansion

$$q(x,y,t) = \phi^p \sum_{i=1}^{\infty} q_i(y,t)\phi^i. \tag{4.2}$$

If we follow the WTC method we get the following:

**i) Leading order analysis:** For all $k$ we have two possibilities for the leading order , $p = -1$ and $p = -2$ . When $p = -1$ we have $m = n-2, n > 1$ which means $a_m = 0$ for all $m > n-2$ and

$$q_0^2 = -\frac{a_{n-2}}{b_n}(k+1)(k+2)\ , \quad n > 1. \tag{4.3}$$

where $a_{n-2} \neq 0$ and $b_n \neq 0$ . When $p = -2$ we have $m = n-1$ which means $a_m = 0$ for all $m > n-1$ and

$$q_0 = -\frac{2\,a_{n-1}}{b_n}(k+1)(2k+1). \tag{4.4}$$

where $a_{n-1} \neq 0$ and $b_n \neq 0$ .

**ii) Resonances:** When $p = -1$ the resonances are $r = -1\ ,\ k+2\ ,\ 2k+2$ and $k+n+1$ . While, when $p = -2$ resonances become $r = -1\ ,\ 2k+2\ ,$

$4k+2$ and $2k+2n+1$ .
We note that we must exclude the cases when we have a resonance of multiplicity 2 as the equations which produce such resonances do not have the Painlevé property. When $p=-2$ we do not have such a trouble. On the contrary, when $p=-1$, the classes $(k,1)$ and $(k,k+1)$ have double resonances and thus do not have the Painlevé property.

**iii) Arbitrary functions:** It turns out that the functions $\psi$ and $q_i$ at the resonances are arbitrary functions of $y$ and $t$ under some compatability conditions among the coefficients $a_i$ and $b_j$ 's of the polynomials $P_m$ and $Q_n$. The function $q_1$ can be given for all classes $(k,n)$ with the leading orders $p=-1$ and $p=-2$ respectively

$$q_1=\frac{b_{n-1}q_0^2+(k+1)(k+2)a_{n-3}}{2(k+1)(2k+1)a_{n-2}} \quad , \quad q_1=0 \quad , \quad (n>1). \qquad (4.5)$$

where $a_{-i}$ and $b_{-i}$ vanish for all positive integer $i$ .
We now give some equations (4.1) passing the Painlevé test for $k=1,2$ .
**Case1.** $(k,n)=(1,1)$ with $p=-2$ . The compatability conditions at the resonances $r=4$ , $r=5$ and $r=6$ are identically satisfied. Painlevé analysis of this class with constant coefficients was first done by [1].
**Case2.** $(k,n)=(1,4)$ with $p=-2$. The compatability conditions at $r=4$ and $r=6$ respectively give

$$a_1a_3b_4-a_2^2b_4+a_2a_3b_3-a_3^2b_2=0,$$
$$a_0\,a_3\,b_4-a_1\,a_2\,b_4+a_1\,a_3\,b_3-a_3^2\,b_1=0.$$

The compatability condition at $r=11$ is identically satisfied. The resulting equation passing the Painlevé test is given by

$$[u_t+P_3(u)u_{xxx}+Q_4(u)u_x]_x+\sigma u_{yy}=0. \qquad (4.6)$$

The coefficient functions $q_4$ , $q_6$ and $q_{11}$ are arbitrary and $q_i$ for $i=0,1,2$ are given by

$$q_0=-\tfrac{12a_3}{b_4} \quad , \quad q_1=0 \quad , \quad q_2=\tfrac{a_2b_4-a_3b_3}{a_3b_4}. \qquad (4.7)$$

**Case3.** $(k,n)=(1,5)$ with $p=-1$. The compatability conditions at $r=3$ and $r=4$ respectively give

$$-a_0a_3^2b_5+2a_1a_2a_3b_5-a_1a_3^2b_4-$$
$$a_2^3b_5+a_2^2a_3b_4-a_2a_3^2b_3+a_3^3b_2=0,$$
$$a_0\,a_2\,a_3b_5-a_0\,a_3^2\,b_4+a_1^2\,a_3\,b_5$$
$$-a_1\,a_2^2\,b_5+a_1\,a_2\,a_3\,b_4-a_1\,a_3^2\,b_3+a_3^3\,b_1=0.$$

while that at $r = 7$ is identically satisfied. The resulting equation passing the Painlevé test is given by

$$[u_t + P_3(u)u_{xxx} + Q_5(u)u_x]_x + \sigma u_{yy} = 0. \tag{4.8}$$

The coefficient functions $q_3$ , $q_4$ and $q_7$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$\begin{gathered} {}^{2}_{0} = -\tfrac{6\,a_3}{b_5} \quad , \quad q_1 = \tfrac{a_2 b_5 - a_3 b_4}{2 a_3 b_5}, \\ q_2 = \tfrac{4 a_1 a_3 b_5^2 - 3 a_2^2 b_5^2 + 2 a_2 a_3 b_4 b_5 - 4 a_3^2 b_3 b_5 + a_3^2 b_4^2}{4 a_3^2 b_5^2 q_0}. \end{gathered} \tag{4.9}$$

**Case4.** $(k, n) = (1, 5)$ with $p = -2.$. The compatability conditions at $r = 4$ and $r = 6$ respectively give

$$\begin{gathered} a_2\, a_4\, b_5 - a_3^2\, b_5 + a_3\, a_4\, b_4 - a_4^2\, b_3 = 0, \\ -a_1\, a_4\, b_5 + a_2\, a_3\, b_5 - a_2\, a_4\, b_4 + a_4^2\, b_2 = 0. \end{gathered}$$

while the compatability condition at $r = 13$ is identically satisfied. The resulting equation passing the Painlevé test is given by

$$[u_t + P_4(u)u_{xxx} + Q_5(u)u_x]_x + \sigma u_{yy} = 0 \tag{4.10}$$

The coefficient functions $q_4$ , $q_6$ and $q_{13}$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$q_0 = -\tfrac{12\,a_4}{b_5} \quad , \quad q_1 = 0 \quad , \quad q_2 = \tfrac{a_3 b_5 - a_4 b_4}{a_4 b_5}. \tag{4.11}$$

**Case5.** $(k, n = (1, 6)$ with $p = -1$. The compatability conditions at $r = 3$ and $r = 4$ respectively give

$$\begin{aligned} -a_1\, a_4^2\, b_6 + 2\, a_2\, a_3\, a_4\, b_6 - a_2\, a_4^2\, b_5 \\ -a_3^3\, b_6 + a_3^2\, a_4\, b_5 - a_3\, a_4^2\, b_4 + a_4^3\, b_3 = 0, \\ a_0\, a_4^2\, b_6 - a_1\, a_3\, a_4\, b_6 + a_1\, a_4^2\, b_5 - a_2^2\, a_4\, b_6 + a_2\, a_3^2\, b_6 \\ -a_2\, a_3\, a_4\, b_5 + a_2\, a_4^2\, b_4 - a_4^3\, b_2 = 0. \end{aligned}$$

whereas the compatability condition at $r = 8$ is identically satisfied. The resulting equation passing the Painlevé test is given by

$$[u_t + P_4(u)u_{xxx} + Q_6(u)u_x]_x + \sigma u_{yy} = 0. \tag{4.12}$$

The coefficient functions $q_3$ , $q_4$ and $q_8$ are arbitrary and $q_i$ for $i = 0, 1, 2$ are given by

$$\begin{gathered} q_0^2 = -\tfrac{6\,a_4}{b_6} \quad , \quad q_1 = \tfrac{a_3\, b_6 - a_4\, b_5}{2\, a_4\, b_6}, \\ q_2 = \tfrac{1}{24\, a_4^3\, b_6}[-4\, a_2\, a_4\, b_6^2\, c_0 + q_0\, (3\, a_3^2\, b_6^2 - 2\, a_3 a_4 b_5\, b_6 + 4\, a_4^2\, b_4\, b_6 - a_4^2\, b_4^2)]. \end{gathered} \tag{4.13}$$

**Case6.** $(k,n) = (1,6)$ with $p = -2$. The compatability conditions at $r = 4$ and $r = 6$ respectively give

$$a_3\,a_5\,b_6 - a_4^2\,b_6 + a_4\,a_5\,b_5 - a_5^2\,b_4 = 0$$
$$-a_2\,a_5\,b_6 + a_3\,a_4\,b_6 - a_3\,a_5\,b_5 + a_5^2\,b_3 = 0.$$

while the compatability condition at $r = 15$ is identically satisfied. The resulting equation passing the Painlevé test is given by

$$[u_t + P_5(u)u_{xxx} + Q_6(u)u_x]_x + \sigma u_{yy} = 0. \tag{4.14}$$

The coefficient functions $q_4$ , $q_6$ and $q_{15}$ are arbitrary and $q_i$ for $i = 0,1,2$ are given by

$$q_0 = -\frac{12\,a_5}{b_6} \quad , \quad q_1 = 0 \quad , \quad q_2 = \frac{a_4\,b_6 - a_5\,b_5}{a_5\,b_6}. \tag{4.15}$$

For $n > 6$ we have classes $(1,n)$ pass the test both for $p = -1$ and $p = -2$. The compatability conditions satisfied by the constants $a_i$ and $b_i$ where $i = 1,2,...,n$ for $n > 6$ become very lengthy. It is interesting to note that the compatability conditions obtained here for the classes $(1,n)$ are the same conditions obtained for the corresponding KdV type of equations with constant coefficients (see sec3.1).

**Case7.** $(k,n) = (2,2)$ with $p = -1$. The compatability condition at $r = 4$ gives only $b_1 = 0$ ,while those at the resonances $r = 5$ and $r = 6$ are identically satisfied. The resulting equation having the Painlevé property is given by

$$[u_t + a_0 u_{xxx} + Q_2(u^{\frac{1}{2}})u_x]_x + \sigma u_{yy} = 0. \tag{4.16}$$

The coefficient functions $q_4$ , $q_5$ and $q_6$ are arbitrary and $q_i$ for $i = 0,1,2$ are given by

$$\begin{aligned} q_0^2 &= -\frac{12\,a_0}{b_2} \quad , \quad q_1 = -\frac{2\,b_1}{5\,b_2}, \\ q_2 &= \frac{-25\,\psi_t\,b_2 - 25\,\psi_y^2\,b_2\,\sigma - 25\,b_0\,b_2 + 8\,b_1^2}{50\,q_0\,b_2^2}. \end{aligned} \tag{4.17}$$

**Case8.** $(k,n) = (2,4)$ with $p = -2$. The compatability conditions at the resonances $r = 6$ , $r = 10$ and $r = 13$ gives respectively $a_0 b_4 + a_1 b_3 - a_3 b_1 = 0$ , $a_2 = 0$ and $a_0(a_3 b_2 - a_1 b_4) = 0$ . The equation passing the Painlevé property is given by

$$[u_t + (a_0 + a_1 u^{\frac{1}{2}} + a_3 u^{\frac{2}{3}})u_{xxx} + Q_4(u^{\frac{1}{2}})]_x + \sigma u_{yy} = 0. \tag{4.18}$$

$q_6$ , $q_{10}$ and $q_{13}$ are arbitrary and for $i = 0,1,2$ , $q_i$ are given by

$$q_0 = -\frac{30\,a_3}{b_4} \quad , \quad q_1 = 0 \quad , \quad q_2 = \frac{5\,(a_2\,b_4 - a_3\,a_3 b_3)}{8\,a_3\,b_4}. \tag{4.19}$$

When $k$ and $n$ increase, it gets so difficult to obtain the compatability conditions which become so lengthy as the resonances get bigger.

We conjecture that for $k > 2$ , there are infinitely many classes (k,n) having the Painlevé property.

# Chapter 5

# Conclusion

In this work, we have applied the Painlevé test of the WTC method to classes of evolution equations of the forms

$$u_t + p(\tau)\, u_{xx} + q(\tau)\, u_x = 0. \tag{5.1}$$

$$u_t + p(\tau)\, u_{xxx} + q(\tau)\, u_x = 0. \tag{5.2}$$

where $p$ and $q$ are functions of $\tau$ which is in terms a function of $u$.

For the sake of applicacibility of Painlevé test, we considered $\tau = u^{\frac{1}{k}}$, where k is a positive integer , and the functions $p$ and $q$ as polynomials of $\tau$ of orders $m$ and $n$ respectively. Finding infinitely many PDE 's belonging to the mentioned-above classes such that they possess the Painlevé property, it has made sense to consider KP extensions of (5.2) in the form

$$u_t + p(\tau)\, u_{xxx} + q(\tau)\, u_x + \sigma u_{yy} = 0. \tag{5.3}$$

where $p, q$ and $\tau$ are as mentioned above. Again, we have found infinitely many equations of the form (5.3) that pass the Painlevé test.

Hence, we have classified three types of nonlinear PDE 's with respect to the Painlevé approach given by WTC and we believe this classification is not of less importance than any other classification done so far.

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