NON - INTERIOR PIECEWISE - LINEAR PATHWAYS TO ℓ_∞ SOLUTIONS OF OVERDETERMINED LINEAR SYSTEMS

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCE

OF BİLKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Samir Elhedhli

June, 1996

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assist. Prof. Mustafa Ç. Pınar (Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

NON-INTERIOR PIECEWISE-LINEAR PATHWAYS TO ℓ_∞ SOLUTIONS OF OVERDETERMINED LINEAR SYSTEMS

Samir Elhedhli
M.S. in Industrial Engineering
Supervisor: Assist. Prof. Mustafa Ç. Pınar
June, 1996

In this thesis, a new characterization of ℓ_∞ solutions to overdetermined systems of linear equations is described based on a simple quadratic penalty function, which is used to change the problem into an unconstrained one. Piecewise-linear non-interior pathways to the set of optimal solutions are generated from the minimization of the unconstrained function. It is shown that the entire set of ℓ_∞ solutions is obtained from the paths for sufficiently small values of a scalar parameter. As a consequence, a new finite penalty algorithm is given for ℓ_∞ problems. The algorithm is implemented and exhaustively tested using random and function approximation problems. A comparison with the Barrodale-Phillips algorithm is also done. The results indicate that the new algorithm shows promising performance on random (non-function approximation) problems.

Key words: ℓ_∞ Optimization; Overdetermined Linear Systems, Quadratic Penalty Functions, Characterization.

ÖZET

DOĞRUSAL ℓ_∞ PROBLEMİ İÇİN BİR PARÇALI DOĞRUSAL DIŞ NOKTA ALGORİTMASI

Samir Elhedhli
Endüstri Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Yrd. Doç. Mustafa Ç. Pınar
Haziran, 1996

Bu tez çalışmasında, doğrusal ℓ_∞ problemi için yeni bir algoritma önerilmiştir. Algoritma karesel bir ceza fonksiyonunun problemin doğrusal programlama formülasyonuna uygulanması ile elde edilmistir. Karesel ceza fonksiyonunun çözüm kümesi parçalı doğrusal bir yol izleyerek esas problemin (ℓ_∞) çözüm kumesine ulaşır. Algoritmanın sonlu sayıda adımda optimal çözüme ulaştığı gösterilmiştir. Algoritma bilgisayarda programlanmış ve değişik problemler üzerinde denenmiştir. Ayrıca optimizasyon literatüründe en iyi bilinen Barrodale-Phillips simplex algoritması ile karşılaştırılmıştır.

Anahtar sözcükler: ℓ_∞ Problemi, Doğrusal Sistemler, Karesel Ceza Fonksiyonu, Çözüm Kümesi Karakterizasyonu.

To my parents

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Finally, I would like to thank Fatma Gzara for everything.

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Glossary of Symbols and Notations

x	vector of length n.
y	scalar ($\in R$).
(x, y)	pair of vector x and scalar y.
z	$(n+1)$-vector with $z_i = x_i$ for $i = 1, .., n$ and $z_{n+1} = y$.
d_z	$(n+1)$-vector with $d_{z_i} = d_{x_i}$ for $i = 1, .., n$ and $d_{z_{n+1}} = d_y$.
a_j	j-th column of matrix A.
a_i^T	i-th row of matrix A.
e	vector of all 1's of appropriate length, i.e, $e^T = (1, ..., 1)^T$.
e_{n+1}	$(n+1)$-vector with $e_i = 0$ for $i = 1, .., n$ and $e_{n+1} = 1$.
0	vector or matrix of zeroes of appropriate length.
$F(z, t)$	$\equiv F(x, y, t) \equiv F_t(x, y)$.
$\bar{b}$	$2m$-vector of the form $\begin{bmatrix} b \\ b \end{bmatrix}$, where $b \in R^m$.
$\bar{A}$	$(2m) \times (n+1)$-matrix of the form $\begin{bmatrix} A & -e \\ -A & -e \end{bmatrix}$.
$\bar{\Theta}$	$(2m \times 2m)$ matrix of the form $\begin{bmatrix} \bar{\Theta}_1 & \mathbf{0} \\ \mathbf{0} & \bar{\Theta}_2 \end{bmatrix}$ where $\bar{\Theta}_1$ and $\bar{\Theta}_2$ are $m \times m$ matrices.
Q	$(2m) \times (n+1)$-matrix which is equivalent to $\bar{\Theta}\bar{A}$.
P	$(n+1) \times (n+1)$-matrix which is equivalent to $Q^T Q$.
$\bar{P}$	$(n+1) \times (n+1)$-matrix which is equivalent to $\bar{A}^T \bar{\Theta} \bar{A}$.

Chapter 1

Introduction

The ℓ_∞ problem has many applications in a wide range of fields, and for that reason, fast, accurate and stable methods to solve it were all the time sought. The first attempts seem to have been made by statisticians as the problem arises frequently in data fitting analysis. More efficient ways, however, were designed when it was realized that the problem is equivalent to a linear program, and , hence it can be solved via any linear programming method.

Since then, the solution approaches were following developments in linear programming. Barrodale and Phillips [1] designed a simplex-like method in which the special structure of the coefficient matrix is exploited. After Karmarkar's outstanding paper which opened the area of interior-point methods, Ruzinsky and Olsen [14] used the same ideas to design a polynomial algorithm for the ℓ_∞ problem. Later, large developments in the interior-point area led to the solution method of Zhang [16], where an affine scaling approach is used.

Again, in this thesis, the linear programming formulation of the ℓ_∞ problem is used. The constraints are coupled with the objective function through the use of a simple quadratic penalty function and a smoothing parameter. In theory, a solution to the original problem could be obtained from a solution to the unconstrained problem when the parameter tends to zero. It is shown, however, that it is not necessary to let the parameter converge to zero and that there is a threshold value, for which a solution could be detected. This remark is essential both for the efficiency and the numerical stability of the designed algorithm. The approach could be termed as an *exterior point* approach, as a sequence of non-interior iterates is generated that satisfies primal feasibility only upon termination.

The organization of the thesis is as follows. In the next chapter a summary of the most known algorithms for the ℓ_∞ problem is given. Then a new characterization of the solution set is done in the third chapter, followed with a numerical example in the fourth chapter. Chapter five is devoted to the analysis and design of the algorithm. Numerical testing and a comparison with the Barrodale-Phillips algorithm is provided in the fifth chapter. The thesis concludes with some remarks and suggestions for future research.

Chapter 2

Literature Review

2.1 Historical Background

The ℓ_∞ approximation problem is to find $x \in R^n$ that minimizes

$$||Ax - b||_\infty = \max_{i=1..m} |a_i^T x - b_i|$$

where $A \in R^{m \times n}$ with columns a_j and rows a_i^T and $b, y \in R^m$.

The problem is also known as *minimax* or the *Chebychev* approximation problem, and is believed to have been first posed by the French mathematician *Laplace* in 1799 in a study of inconsistent linear systems . However the Russian mathematician *Chebychev* seems to be the first to have studied deeply such class of problems in the 1850's [15].

One of the first systematic methods to solve the ℓ_∞ problem was due to Stiefel (1959), who points out that for a subset J of $n+1$ indices (among the $2m$), either

$$\min_{x \in R^n} \max_{i \in J} |a_i^T x - b_i| \leq \min_{x \in R^n} \max_{i=1..m} |a_i^T x - b_i|$$

or, an other subset J^* of indices can be formed from J by interchanging one index, so that

$$\min_{x \in R^n} \max_{i \in J^*} |a_i^T x - b_i| < \min_{x \in R^n} \max_{i \in J} |a_i^T x - b_i|.$$

The method which he called the *Exchange Method* [15] is similar in principal to the simplex method, as $n+1$ equations among the $2m$, $\begin{bmatrix} A & -e \\ A & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}$ are solved for at a time. Then either $\max_{i \in J} |r_i(x)| \leq \max_{i \in J} |r_i(x)|$ and optimality is reached, or an exchange is made. The method requires that at each step the matrix picked from $\begin{bmatrix} A & -e \\ A & e \end{bmatrix}$ is of full rank.

Among the first to notice that the ℓ_∞ problem could be formulated as a linear program was Zuhovitzki in the 1950's [15], who used the primal formulation.

$$[LINFLP] \qquad \begin{aligned} &\min y \\ \text{s.t} \quad & Ax - ye \leq b \\ & Ax + ye \geq b \end{aligned}$$

to design an algorithm. Kelley (1958), however was the first to use the dual formulation

$$[LINFLD] \qquad \begin{aligned} &\max b^T(v-u) \\ \text{s.t} \quad & A^T(v-u) = 0 \\ & e^T(u+v) = 1 \\ & u, v \geq 0 \end{aligned}$$

which decreases the size of the problem and puts it directly into a standard form. In the late 1960's versions of the simplex method were designed for the problem, the first of which is due to Barrodale and Young (1966) and Bartels and Golub (1968) [15].

2.2 The Algorithm of Barrodale and Phillips (1975)

Barrodale and Phillips [1] use a modified simplex algorithm that exploits the special structure of $[LINFLD]$. Like the simplex algorithm, their algorithm moves from one vertex to a neighboring one that provides a decrease in the objective function. The structure of the problem, however, makes the exchange of u_i and v_i easy to accomplish. Consider the constraint matrix $\begin{bmatrix} A^T & -A^T \\ e^T & e^T \end{bmatrix}$, the columns corresponding to u_i and v_i are $\begin{bmatrix} -a_i \\ 1 \end{bmatrix}$ and $\begin{bmatrix} a_i \\ 1 \end{bmatrix}$ respectively, and suppose that the basis contains the variables $\{x_1, ..., x_n, u_i\}$ and the $(n+1) \times (n+1)$-matrix $B = [\mathbf{w}_1, ..., \mathbf{w}_n, \mathbf{w}_{n+1}]$ where $\mathbf{w}_j's$ $j = 1..., n$ are columns of length

$n+1$ and $\mathbf{w}_{n+1} = \begin{bmatrix} -a_i \\ 1 \end{bmatrix}$ and that v_i will take the place of u_i in the basis, then instead of solving

$$\sum_{j=1}^{n} x_j \mathbf{w}_j + \begin{bmatrix} -a_i \\ 1 \end{bmatrix} u_i = 0,$$

we solve

$$\sum_{j=1}^{n} x_j \mathbf{w}_j + \begin{bmatrix} a_i \\ 1 \end{bmatrix} v_i = 0.$$

In other words,

$$\bar{B}\bar{x} = -v_i \begin{bmatrix} a_i \\ 1 \end{bmatrix}$$

is solved instead of

$$\bar{B}\bar{x} = u_i \begin{bmatrix} a_i \\ 1 \end{bmatrix}.$$

where $\bar{B} = [\mathbf{w}_1, ..., \mathbf{w}_n]$, and thus if $\bar{B}^{-1}$ exists, then it is easy to solve for the second system.

In the first phase of the algorithm, artificial variables are introduced and are taken in the basis, then only variables u_i's are allowed to enter the basis in place of the artificial variables. At the end of the first phase, a basic feasible solution is found where $n+1$ constraints hold with equality. At that point, the ordinary simplex algorithm is applied until optimality is reached.

2.3 The Algorithm of Bartels, Conn and Charalambous (1978)

This algorithm [2] is based upon solving the linear programming formulation [LINFLP] through the use of an exact penalty function. Let [LINFLP] be written as,

$$\min c_0^T w$$
$$c_j^T w \geq \beta_j, \quad j = 1, ..., 2m,$$

where,

$$c_0 = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}.$$

$$c_i = \begin{bmatrix} 1 \\ -a_i \end{bmatrix}, \quad c_{m+i} = \begin{bmatrix} 1 \\ a_i \end{bmatrix}, \quad i = 1, \ldots, m,$$

$$\beta_i = -b_i, \quad \beta_{m+i} = b_i, \quad i = 1, \ldots, m,$$

$$w = \begin{bmatrix} y \\ x \end{bmatrix}.$$

For a fixed parameter $\mu > 0$, they define an unconstrained function,

$$p(w, \nu) = \nu c_0^T w - \sum_{j=1}^{2m} \min(0, c_j^T w - \beta_j), \qquad (2.1)$$

and prove that for decreasing values of ν, a solution to [LINFLP] can be got from a minimizer of 2.1 when ν tends to zero.

The minimization of 2.1 is done through the choice of a descent direction (null space projected gradient) and the computation of a step size (line search). If a minimizer of 2.1 is detected for which primal feasibility is satisfied then that solution is optimal. Otheriwse, if any of the constraints are violated then ν is reduced and the function p is again minimized starting from the previous point. Obviously, the algorithm needs a good starting point, and that was mainly the subject of [3].

2.4 The Algorithm of Ruzinsky and Olsen (1989)

This algorithm is a variant of Karmarkar's algorithm applied to the dual linear programming formulation [LINFLD], which evolves through the interior of the feasible region. It is mainly a rescaling technique coupled with a projected gradient method. Putting [LINFLD] in a standard form, yields,

$$\begin{aligned} &\min \bar{c}^T\bar{x} \\ \text{s.t}\quad &\bar{A}\bar{x} = \bar{b} \\ &\bar{x} \geq 0, \end{aligned}$$

where,

$$\bar{A} \equiv \begin{bmatrix} A^T & -A^T \\ e^T & e^T \end{bmatrix};\ \bar{b} \equiv \begin{bmatrix} \mathbf{0} \\ 1 \end{bmatrix};\ \bar{c} \equiv \begin{bmatrix} -b \\ b \end{bmatrix};\ \bar{x} \equiv \begin{bmatrix} u \\ v \end{bmatrix}.$$

The initial point is chosen as $u_0 = v_0 = \frac{1}{2m}e$. At the kth iteration, Karmarkar would rescale the problem so that e is the center of the feasible region, solve a weighted least squares problem to find a search direction, compute a step size and update the primal variable until the duality gap is closed. This is exactly what Ruzinsky and Olsen [14] did, the only difference is in the stopping criteria. The algorithm is the following.

- **Initialize:** $u = v = \frac{1}{2m}e$, $\epsilon=$ some small positive number.
- **Iterate:**
  - $f = [u_i^2 - v_i^2]$,
  - $D{:=}\text{diag}(u_i^2 + v_i^2)$, $d = -\frac{1}{e^TDe}$ {Rescaling matrix}
  - $x{:=}[\begin{bmatrix} A \\ f^TA \end{bmatrix}^T \begin{bmatrix} D & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} A \\ f^TA \end{bmatrix}]^{-1}[\begin{bmatrix} A \\ f^TA \end{bmatrix}^T \begin{bmatrix} D & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} b \\ f^Tb \end{bmatrix}]$, {Least squares}
  - $r{:=}b - Ax$, {Residual vector}
  - **if** $1 + \frac{df^Te}{\|r\|_\infty} < \epsilon$ **then** stop. {Stopping criteria}
  - **else**
    - $\delta_1{:=}[u_i^2(-df^Tr - r_i)]$; $\delta_2{:=}[v_i^2(-df^Tr + r_i)]$, {Search direction}
    - $\beta{:=}\frac{0.97}{\max(\max_i(-\delta_{1_i}/u_i), \max_i(\delta_{2_i}/v_i))}$, {Step size}
    - $u{:=}u + \beta\delta_1$; $v{:=}v + \beta\delta_2$, {Update}
- **Until** stop.

2.5 The Algorithm of Coleman and Li (1992)

Coleman an Li [4] use a formulation of the ℓ_∞ problem that is based upon the null space of A. Precisely, they consider the $(m-n)\times m$ matrix Z with rank $(m-n)$ such that $AZ = 0$ to formulate the problem as:

$$\begin{aligned} \min \|r\|_\infty &= \phi(r) \\ Zr &= Zb \end{aligned} \qquad \equiv \qquad \begin{aligned} &\min y \\ &Zr = Zb \\ -ye \le\ & r \le ye \end{aligned}$$

as $r = Ax - b$ leads to $Zr = ZAx - Zb$ and $Zr = Zb$. The function $\phi(r)$ is non-differentiable at the points r where more than one residual has maximum magnitude. Assume that this is not the case, and there is only one index j for which $|r_j| = \|r\|_\infty$, then $\phi(r)$ is differentiable in the neighborhood of the current point r and the near-by non-differentiable region is defined by $|r_j| = |r_i|$ for $i \neq j$.
By defining,

$$c_i = s_j r_j - s_i r_i \quad i \neq j \quad \text{and} \quad c_j = |r_j|e - |r| + |r_j|e_j = T^{-1}r,$$

where $s = \text{sign}(r)$ and T is a simple elementary matrix defined by:

$$T = [-s_1e_1, .., -s_{j-1}e_{j-1}, s, -s_{j+1}e_{j+1}, .., -s_me_m]$$

$$\& \quad T^{-1} = [-s_1e_1, .., -s_{j-1}e_{j-1}, s_je, -s_{j+1}e_{j+1}, .., -s_me_m].$$

Then problem becomes,

$$\begin{aligned} &\min \|Tc\|_\infty \\ &ZTc = Zb. \end{aligned}$$

The algorithm then proceeds with finding a descent direction d using a well chosen criteria so that both global convergence and the ability to cross lines of non-differentiability are enhanced,

$$d = -A(A^TT^{-T}D^{-2}T^{-1}A)^{-1}A^Tg,$$

where $g = s_j e_j$. The choice of D is done in a way so that near the solution unit Newton steps are taken, as

$$D = (CD_\theta^{-1})^{\frac{1}{2}},$$

with C=diag(c) and θ is a variable that encapsulates the optimality conditions and tends to zero as the solution is reached.

The next step is to perform a line search along d. In other words, $\phi(r + \alpha d)$ is minimized with respect to α. This is done by considering each breakpoint in turn, adjusting the gradient to reflect a step just beyond the breakpoint and checking if d continues to be a descent direction. Finally r and λ, (the dual variable) are updated as,

$$r^+ = r + \alpha d \quad \text{and} \quad \lambda^+ = g + T^{-T} D^{-2} T^{-1} d.$$

In brief, the algorithm starts by a feasible point r_0 such that $Zr_0 = Zb$, makes a rescaling transformation, computes a step size using a line search and proceeds iteratively until reaching optimality. The main work in the algorithm is done in the solution of the least squares problems.

2.6 Zhang's Algorithm (1993)

As the ℓ_∞ problem is equivalent to a linear program, Zhang [16] uses a primal-dual interior point method to solve for [LINFLP] and [LINFLD]. The Karush-Kuhn-Tucker optimality conditions can be written as:

$$F(p,q,u,v,x,y) = \begin{vmatrix} A^T(u-v) \\ e^T(u+v) - 1 \\ p + Ax - ye - b \\ q - Ax - ye + b \\ Up \\ Uq \end{vmatrix} = 0,$$

where U =diag(u) and V =diag(v) and $(p,q,u,v) \geq 0$.

Starting with an initial point $(p_0, q_0, u_0, v_0, x_0, y_0)$ as:

$$
\begin{aligned}
r_0 &= Ax_0 - b, & u_0 &= \tfrac{1}{2m}\epsilon, & u_0 &= \tfrac{1}{2m}\epsilon, \\
y_0 &= ||r_0||_\infty + \delta, & p_0 &= y_0\epsilon - r_0, & q_0 &= y_0\epsilon + r_0,
\end{aligned}
$$

for an arbitrary x_0 and $\delta > 0$. A Newton search direction is computed at each iteration k, using $F_k' d_k = -F_k$:

$$
\begin{bmatrix}
A^T & -A^T & 0 & 0 & 0 & 0 \\
\epsilon^T & \epsilon^T & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & A & -\epsilon \\
0 & 0 & 0 & I & -A & -\epsilon \\
U & 0 & P & 0 & 0 & 0 \\
0 & V & 0 & Q & 0 & 0
\end{bmatrix}
\begin{bmatrix}
d_{p_k} \\ d_{q_k} \\ d_{u_k} \\ d_{v_k} \\ d_{x_k} \\ d_{y_k}
\end{bmatrix}
=
\begin{bmatrix}
0 \\ 0 \\ 0 \\ 0 \\ r_1 \\ r_2
\end{bmatrix}.
$$

Furthermore, a suitable step size is chosen to satisfy polynomiality and convergenge rate proofs. At each iteration it is shown that, the solution is dual feasible, primal feasible and strictly interior to the feasible region, and the duality gap converges to zero. The algorithm is polynomial in the number of iterations and is shown to have superlinear or quadratic convergence.

Chapter 3

A Quadratic Penalty Function Approach

3.1 Introduction

The purpose of this chapter is to give a new characterization of ℓ_∞ solutions of overdetermined linear systems. The ideas will be based upon the linear programming formulation [LINFLP], and are similar to [13], where general linear programming was investigated. The proofs follow the same lines as in [13], with the necessary modifications. Towards the end of the chapter, a new algorithm is designed and its finite convergence is proved.

Now, let us recall the ℓ_∞ problem,

$[\ell_\infty]$

$$\min_{x \in R^m} ||Ax - b||_\infty = \min_{x \in R^n} \max_{i=1..m} |a_i^T x - b_i|$$

which is known to be equivalent to the following linear program,

$$[LINFLP] \qquad \begin{aligned} &\text{Min } y \\ \text{s.t} \quad & Ax - ye \leq b \\ & Ax + ye \geq b, \end{aligned}$$

and its corresponding dual problem ,

$$[LINFLD] \qquad \begin{aligned} & \text{Max } b^T(v-u) \\ \text{s.t } & A^T(v-u)=0 \\ & e^T(u+v)=1 \\ & u,v \geq 0. \end{aligned}$$

One way to change the above constrained problem into an unconstrained problem is to use a quadratic penalty function. To this end, consider the following quadratic function:

$$F(x,y,t) = ty + \frac{1}{2}r_1^T(x,y)\Theta_1(x,y)r_1(x,y) + \frac{1}{2}r_2^T(x,y)\Theta_2(x,y)r_2(x,y),$$

where $r_1(x,y) = Ax - ye - b$ and $r_2(x,y) = Ax + ye - b$ and $\Theta_1(x,y)$ and $\Theta_2(x,y)$ are $m \times m$ diagonal matrices with,

$$\Theta_{1_{ii}}(x,y) = \begin{cases} 1 & \text{if } a_i^T x - y > b_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\Theta_{2_{ii}}(x,y) = \begin{cases} 1 & \text{if } a_i^T x + y < b_i \\ 0 & \text{otherwise} \end{cases}$$

The first and second derivatives are given by:

$$F_y \equiv \frac{\partial F(x,y,t)}{\partial y} = t + e^T(\Theta_1 + \Theta_2)ey + e^T(\Theta_2 - \Theta_1)Ax + e^T(\Theta_1 - \Theta_2)b.$$

$$F_x \equiv \frac{\partial F(x,y,t)}{\partial x} = A^T(\Theta_2 - \Theta_1)ey + A^T(\Theta_1 + \Theta_2)Ax - A^T(\Theta_1 + \Theta_2)b.$$

$$F_{xy} \equiv \frac{\partial^2 F(x,y,t)}{\partial x \partial y} = A^T(\Theta_2 - \Theta_1)e.$$

$$F_{yx} \equiv \frac{\partial^2 F(x,y,t)}{\partial y \partial x} = e^T(\Theta_2 - \Theta_1)A.$$

$$F_{xx} \equiv \frac{\partial^2 F(x,y,t)}{\partial x^2} = A^T(\Theta_1 + \Theta_2)A.$$

$$F_{yy} \equiv \frac{\partial^2 F(x,y,t)}{\partial y^2} = \epsilon^T(\Theta_1 + \Theta_2)\epsilon.$$

Now, consider the unconstrained minimization problem:

$[LINFCP]$

$$\min_{x\in R^n, y\in R^m} F(x,y,t)$$

For decreasing values of t, let (x_t, y_t) denote a minimizer of $F(x,y,t)$. It can be shown that, (see for example [5] or [6])

Theorem 1

$$\lim_{t\to 0} y_t = y^*$$

$$\lim_{t\to 0} ||Ax_t - b||_\infty = ||Ax^* - b||_\infty$$

where (x^*, y^*) are optimal for [LINFLP] and $||Ax^* - b||_\infty$ is the optimal objective for $[\ell_\infty]$.

3.2 Pathways To ℓ_∞ Solutions

A is assumed to have rank n with no rows or columns identically zero. The following well known theorem shows that the unconstrained minimization of F is well defined.

Theorem 2 *For all positive t, there is a finite pair of points that minimizes F(x,y,t).*

Proof: See for example cite5 or cite6. ∎

Define

$$\theta_{1_i}(x,y) = \begin{cases} 1 & \text{if } a_i^T x - y > b_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\theta_{2_i}(x,y) = \begin{cases} 1 & \text{if } a_i^T x + y < b_i \\ 0 & \text{otherwise} \end{cases}$$

Therefore $\Theta_1 = \text{diag}(\theta_1)$ and $\Theta_2 = \text{diag}(\theta_2)$.

For ease of notation let z be the $m+1$ vector with $z_i = x_i$, $i = 1 \ldots n$ and $z_{n+1} = y$, and denote by X, the set of optimal vectors z to [LINFLP].

3.2.1 The Minimizers of F

In this section, we give a characterization of the set of minimizers of F for fixed $t > 0$. It is obvious that $F(x,y,t)$ is composed of a finite number of quadratic functions. In each domain $\mathcal{D} \subseteq R^{n+1}$ where $\theta_1(x,y), \theta_2(x,y)$ are constants, F is equal to a specific quadratic function. These domains are separated by the union of hyperplanes.

$$\mathcal{B} = \{(x,y) \in R^{n+1};\ \exists i:\ a_i^T x - y - b_i = 0 \ \vee\ a_i^T x + y - b_i = 0\}.$$

So, for a given pair (x,y), the corresponding binary vectors $\theta_1(x,y), \theta_2(x,y)$ are found, and F is represented by $\mathcal{Q}_\theta$ on the subset.

$$\mathcal{C}_\theta = cl\{(\hat{x},\hat{y}) \in R^{n+1}; \theta_1(\hat{x},\hat{y}) = \theta_1 \ \wedge\ \theta_2(\hat{x},\hat{y}) = \theta_2\},$$

where $\mathcal{Q}_\theta$ is defined as follows:

$$\begin{aligned} \mathcal{Q}_\theta(\hat{x},\hat{y},t) &= F(x,y,t) + F_x^T(\hat{x}-x) + F_y^T(\hat{y}-y) \\ &+ \tfrac{1}{2}(\hat{x}-x)^T F_{xx}(\hat{x}-x) + \tfrac{1}{2}(\hat{y}-y)^T F_{yy}(\hat{y}-y) \\ &+ \tfrac{1}{2}(\hat{x}-x)^T F_{xy}(\hat{y}-y) + \tfrac{1}{2}(\hat{y}-y)^T F_{yx}(\hat{x}-x). \end{aligned}$$

Let the set of minimizers of $F(x,y,t)$ be denoted by M_t. The following simple result will be useful later.

Lemma 1 *Let*

$$P = \begin{bmatrix} A^T\Theta_1 A + A^T\Theta_2 A & -A^T\Theta_1 e + A^T\Theta_2 e \\ -e^T\Theta_1 A + e^T\Theta_2 A & e^T\Theta_1 e + e^T\Theta_2 e \end{bmatrix}$$

$$Q = \begin{bmatrix} \Theta_1 A & -\Theta_1 e \\ \Theta_2 A & \Theta_2 e \end{bmatrix}$$

Then

$$Pz = 0 \Rightarrow Qz = 0.$$

Proof: Using the facts that $\Theta_1.\Theta_1 = \Theta_1$. $\Theta_2.\Theta_2 = \Theta_2$, $\Theta_1^T = \Theta_1$ and $\Theta_2^T = \Theta_2$, we have $P = Q^TQ$. Therefore, $Q^TQz = (Qz)^T(Qz) = ||Qz||_2^2$. So $Pz = \mathbf{0} \Rightarrow z^TPz = \mathbf{0} \Rightarrow Qz = \mathbf{0}$. ∎

Lemma 2 $\theta_1(x_t, y_t)$ *and* $\theta_2(x_t, y_t)$ *are constant for* $(x_t, y_t) \in M_t$. *Furthermore* $a_i^T x_t - y_t - b_i$ *is constant for* $\theta_{1_i} = 1$ *and* $a_i^T x_t + y_t - b_i$ *is constant for* $\theta_{2_i} = 1$ $\forall i = 1...m$.

Proof: Let $z_t \in M_t$ and let $\theta_1 = \theta_1(x_t, y_t)$, $\theta_2 = \theta_2(x_t, y_t)$, i.e, $F(x, y, t) = \mathcal{Q}_\theta(x, y, t)$ for $z \in C_\theta$. Then, if $(x, y) \in C_\theta \cap M_t$ then

$$A^T(\Theta_1 + \Theta_2)A(x - x_t) + A^T(-\Theta_1 + \Theta_2)(y - y_t)e = 0,$$

$$-e^T(\Theta_1 - \Theta_2)A(x - x_t) + e^T(\Theta_1 + \Theta_2)(y - y_t)e = 0,$$

i.e., $P(z - z_t) = \mathbf{0}$. So, using the previous lemma, we get:

$$\Theta_1 A(x - x_t) - \Theta_1 e(y - y_t) = 0$$

$$\Theta_2 A(x - x_t) + \Theta_2 e(y - y_t) = 0$$

Therefore, if $a_i^T x_t - y_t - b_i > 0$, i.e, $\theta_{1_i} = 1$ then $a_i^T(x - x_t) - (y - y_t) = 0 \Rightarrow a_i^T x_t - y_t - b_i = a_i^T x - y - b_i$ and, hence $a_i^T x_t - y - b_i$ is constant for $i = 1...m$.

Next, if $a_i^T x_t - y_t - b_i \leq 0$ then $\theta_{1_i} = 0$. Now, suppose that $a_i^T x - y - b_i > 0$ then by the first part of the proof, $a_i^T x' - y' - b_i$ remains constant for any $z' \in M_t \cap C_\theta$, specially for $a_i^T x_t - y_t - b_i$. However $a_i^T x_t - y_t - b_i \leq 0$, so what was supposed is wrong and $a_i^T x - y - b_i \leq 0$. Therefore, $\theta_{1_i}(x, y) = \theta_{1_i}(x_t, y_t)\ \forall (x, y) \in M_t \cap C_\theta$.

The proof is similar for θ_{2_i} for $i = 1...m$. ∎

Following the lemma, the notation $\theta_1(M_t)$. $\theta_2(M_t)$ is used instead of $\theta_1(x_t, y_t)$, $\theta_2(x_t, y_t)$ for $(x_t, y_t) \in M_t$. The previous lemma supplies the following information about M_t.

Corollary 1 *M_t is a convex set which is contained in one C_θ, where $\theta_1 = \theta_1(M_t)$ and $\theta_2 = \theta_2(M_t)$.*

Corollary 2 *Let $z_t \in M_t$ and $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$. Let $\mathcal{N}_\theta$ be the null space of the matrix P defined previously. Then,*

$$M_t = (z_t + \mathcal{N}_\theta) \cap C_\theta.$$

Proof: To clarify the notation, M_t can be written as:

$$M_t = \{z \in C_\theta : z = z_t + z' \wedge z' \in \mathcal{N}_\theta\}$$

Now, let $z \in M_t$ then $P(z - z_t) = \mathbf{0}$, thus $(z - z_t) \in \mathcal{N}_\theta$. So $z = z_t + z'$, i.e. $x = x_t + x'$ and $y = y_t + y'$ where $z' = (x', y') \in \mathcal{N}_\theta$. Hence $z \in (z_t + \mathcal{N}_\theta) \cap C_\theta$ and $M_t \subseteq (z_t + \mathcal{N}_\theta) \cap C_\theta$.
Similarly, let $z \in (z_t + \mathcal{N}_\theta) \cap C_\theta$, then $\exists\, z' \in \mathcal{N}_\theta$ such that $z = z_t + z'$, which implies that, $P(z - z_t) = \mathbf{0}$. Therefore, by recalling the necessary optimality conditions, $z \in M_t$, and so $(z_t + \mathcal{N}_\theta) \cap C_\theta \subseteq M_t$. ∎

A direct consequence of the previous corollary is the following sufficient condition for the uniqueness of z_t.

Corollary 3 *Let $\theta_1 = \theta_1(M_t)$ and $\theta_2 = \theta_2(M_t)$ then z_t is unique if rank(P)=$n+1$.*

3.2.2 Characterization of ℓ_∞ Solutions

Our purpose, now, is to show how the solution set M_t of F approximates the solution set X of [LINFLP] as t approaches 0. For that, assume $z_t \in M_t$ and let $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$ and N_θ be the null space of P.

Lemma 3 *Let $z_t \in M_t$ and $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$ then the following system is consistent:*

$$Pd_z = e_{n+1}. \tag{3.1}$$

Or, equivalently, $e_{n+1} \in \mathcal{R}(P)$, the column space of P, where

$$P = \begin{bmatrix} A^T\Theta_1 A + A^T\Theta_2 A & -A^T\Theta_1 e + A^T\Theta_2 e \\ -e^T\Theta_1 A + e^T\Theta_2 A & e^T\Theta_1 e + e^T\Theta_2 e \end{bmatrix}, \quad d_z = \begin{bmatrix} d_x \\ d_y \end{bmatrix}$$

and e_{n+1} is the $(n+1)$-dimensional vector with 1 at the $(n+1)$st position and 0 elsewhere.

Proof: Since z_t satisfies the necessary optimality conditions for a minimizer then:

$$t - e^T\Theta_1(Ax_t - b - y_t e) + e^T\Theta_2(Ax_t - b + y_t e) = \mathbf{0} \tag{3.2}$$

$$A^T\Theta_1(Ax_t - b - y_t e) + A^T\Theta_2(Ax_t - b + y_t e) = \mathbf{0} \tag{3.3}$$

which implies that:

$$P\begin{bmatrix} x_t \\ y_t \end{bmatrix} = -\begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} -A^T\Theta_1 & A^T\Theta_2 \\ e^T\Theta_1 & e^T\Theta_2 \end{bmatrix}\begin{bmatrix} b \\ b \end{bmatrix}. \tag{3.4}$$

Hence

$$-\begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} -A^T\Theta_1 & A^T\Theta_2 \\ e^T\Theta_1 & e^T\Theta_2 \end{bmatrix}\begin{bmatrix} b \\ b \end{bmatrix} = -te_{n+1} + Q^T\begin{bmatrix} b \\ b \end{bmatrix}$$

is an element of the row space of P: $\mathcal{R}(P)$. Note also that,

$$P = \begin{bmatrix} -A^T\Theta_1 & A^T\Theta_2 \\ e^T\Theta_1 & e^T\Theta_2 \end{bmatrix}\begin{bmatrix} -\Theta_1 A & \Theta_1 e \\ \Theta_2 A & \Theta_2 e \end{bmatrix} = Q^TQ.$$

and

$$Ph_z = P\begin{bmatrix} h_x \\ h_y \end{bmatrix} = Q^TQh_z = Q^T\begin{bmatrix} b \\ b \end{bmatrix},$$

is always consistent since it corresponds to normal equations of the overdetermined system:

$$Qh_z = \bar{b}. \tag{3.5}$$

where $\bar{b} = \begin{bmatrix} b \\ b \end{bmatrix}$, and so $Q^T\bar{b} \in \mathcal{R}(P)$. Knowing that, $-\begin{bmatrix} 0 \\ t \end{bmatrix} + Q^T\begin{bmatrix} b \\ b \end{bmatrix}$ and $Q^T\begin{bmatrix} b \\ b \end{bmatrix} \in \mathcal{R}(P)$, so any combination of them is in $\mathcal{R}(P)$ (property of vector spaces). Therefore $\frac{1}{t}e_{n+1} - \frac{1}{t}Q^T\bar{b} + \frac{1}{t}Q^T\bar{b} \in \mathcal{R}(P)$, which implies that $e_{n+1} \in \mathcal{R}(P)$. ∎

Let d_z be a solution to $Pd_z = e_{n+1}$, then it is verified that $z_t + td_z$ is the least squares solution to the overdetermined system of linear equations $Qh_z = \bar{b}$. To see that, insert $Pd_z = e_{n+1}$ in $te_{n+1} + Pz_t = Q^T\bar{b}$ to get $tPd_z + Pz_t = Q^T\bar{b}$ which implies that $Q^TQ(z_t + td_z) = Q^T\bar{b}$.

Lemma 4 *Let $z_t \in M_t$ and $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$. If the overdetermined system (3.5) is consistent then*

$$\frac{1}{t}\Theta_1(Ax_t - y_t - b) = -\Theta_1(Ad_x - d_y) \tag{3.6}$$

$$\frac{1}{t}\Theta_2(Ax_t + y_t - b) = -\Theta_2(Ad_x + d_y) \tag{3.7}$$

for any solution $d_z = (d_x, d_y)$ to (3.1).

Proof: We know that $z_t + td_z$ is the least squares solution to the overdetermined system of linear equations (3.5), then . if $Qh_z = \bar{b}$ is consistent, $z + td_z$ solves $Qh_z = \bar{b}$. Therefore, we get

$$Q(z_t + td_z) = \bar{b} \Rightarrow \begin{bmatrix} -\Theta_1 A & \Theta_1 \epsilon \\ \Theta_2 A & \Theta_2 \epsilon \end{bmatrix} \begin{bmatrix} x_t + td_x \\ y_t + td_y \end{bmatrix} = \begin{bmatrix} b \\ b \end{bmatrix}.$$

Premultiplying both sides by $\begin{bmatrix} \Theta_1 & \mathbf{0} \\ \mathbf{0} & \Theta_2 \end{bmatrix}$ and noticing that, $\Theta_1.\Theta_1 = \Theta_1$ and $\Theta_2.\Theta_2 = \Theta_2$, we get,

$$\begin{bmatrix} -\Theta_1 A & \Theta_1 \epsilon \\ \Theta_2 A & \Theta_2 \epsilon \end{bmatrix} \begin{bmatrix} x_t + td_x \\ y_t + td_y \end{bmatrix} = \begin{bmatrix} \Theta_1 b \\ \Theta_2 b \end{bmatrix}.$$

This completes the proof. ∎

Now let d_z solve (3.1) and assume that $\theta_1(x_t + \epsilon d_x, y_t + \epsilon d_y) = \theta_1$ and $\theta_2(x_t + \epsilon d_x, y_t + \epsilon d_y) = \theta_2$, i.e $z_t + \epsilon d_z = (x_t + \epsilon d_x, y_t + \epsilon d_y) \in C_\theta$ for some $\epsilon > 0$. The linearity of the problem implies that $z_t + \delta d_z = (x_t + \delta d_x, y_t + \delta d_y) \in C_\theta$ for $0 \leq \delta \leq \epsilon$. Therefore (3.4) implies that:

$$-t\epsilon_{n+1} + Q^T\bar{b} = Pz_t \Rightarrow tPd_z + Pz_t = Q^T\bar{b} \Rightarrow$$
$$P(\delta d_z + z_t) = P\begin{bmatrix}\delta d_x + x_t \\ \delta d_y + y_t\end{bmatrix} = -(t-\delta)\epsilon_{n+1} + Q^T\bar{b}.$$

Hence, $\delta d_z + z_t = [\delta d_x + x_t, \delta d_y + y_t]^T$ is a minimizer of $F(x, y, t-\delta)$. Using Corollary 2, the following is implied:

Lemma 5 *Let $z_t \in M_t$ and $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$. Let d_z solve (3.1). If $\theta_1(x_t + \epsilon d_x, y_t + \epsilon d_y) = \theta_1$ and $\theta_2(x_t + \epsilon d_x, y_t + \epsilon d_y) = \theta_2$ for $\epsilon > 0$ then $\theta_1(x_t + \delta d_x, y_t + \delta d_y) = \theta_1$ and $\theta_2(x_t + \delta d_x, y_t + \delta d_y) = \theta_2$ and*

$$M_{t-\delta} = (z_t + \delta d_z + \mathcal{N}_\theta) \cap C_\theta,$$

for $0 \leq \delta \leq \epsilon$.

Although t is a continuous parameter, there is only a finite number of binary vectors θ_1 and θ_2. Furthermore, the previous lemma ensures that, whenever there exists t_1, t_2 where $\theta_1(x_{t_1}, y_{t_1}) = \theta_1(x_{t_2}, y_{t_2})$ and $\theta_2(x_{t_1}, y_{t_1}) = \theta_2(x_{t_2}, y_{t_2})$ we have, $\theta_1(x_t, y_t) = \theta_1(x_{t_1}, y_{t_1})$ and $\theta_2(x_t, y_t) = \theta_2(x_{t_1}, y_{t_1})$ for all $t \in [t_1, t_2]$. As a consequence, it can be concluded that,

Corollary 4 *$\theta_1(M_t)$ and $\theta_2(M_t)$ are piecewise constant functions of t.*

It is directly concluded from lemma 5 and corollary 4 that the minimizers of F form a family of piecewise-linear paths as a function of t, and

Theorem 3 *There exists $t_0 > 0$ such that $\theta_1(M_t)$ and $\theta_2(M_t)$ are constants for $0 < t \leq t_0$. Furthermore, If $\theta_1(z_t + \delta d_z) = \theta_1(M_t)$ and $\theta_2(z_t + \delta d_z) = \theta_2(M_t)$ then,*

$$M_{t-\delta} = (z_t + \delta d_z + \mathcal{N}_\theta) \cap C_\theta,$$

for $0 \leq \delta < t \leq t_0$ and where $\theta_1 = \theta_1(M_t), \theta_2 = \theta_2(M_t)$ and d_z solves (3.1).

Proof: As θ_1 and θ_2 are constant functions of t, and can assume only a finite number of values, then as t tends to 0, θ_1 and θ_2 should remain constant in a neighborhood of 0. I.e there exists a point, say t_0, such that θ_1 and θ_2 remain constant for $0 < t \leq t_0$. Using lemma 4 and corollary 4, the theorem is implied. ∎

Now, we propose the following important corollary.

Corollary 5 *Let $0 < t \leq t_0$, where t_0 is given in Theorem 3 and let $\theta_1 = \theta_1(M_t)$, $\theta_2 = \theta_2(M_t)$. Then*

$$\Theta_1 r_1(x_t + t\tilde{d}_x, y_t + t\tilde{d}_y) = 0, \tag{3.8}$$

$$\Theta_2 r_2(x_t + t\tilde{d}_x, y_t + t\tilde{d}_y) = 0, \tag{3.9}$$

where $\tilde{d}_z = \begin{bmatrix} \tilde{d}_x \\ \tilde{d}_y \end{bmatrix}$ is any solution to (3.1). Furthermore,

$$\frac{1}{t}\Theta_1 r_1(x_t, y_t) = -\Theta_1(A\tilde{d}_x - \tilde{d}_y), \tag{3.10}$$

and

$$\frac{1}{t}\Theta_2 r_2(x_t, y_t) = -\Theta_2(A\tilde{d}_x + \tilde{d}_y), \tag{3.11}$$

i.e. $\frac{\Theta_1 r_1(x_t, y_t)}{t}$ and $\frac{\Theta_2 r_2(x_t, y_t)}{t}$ are constants.

Proof: Let $z_{t-\delta} \in M_{t-\delta}$ for $0 \leq \delta < t$. By Theorem 3, $\exists\, d_z$ that solves (3.1) such that $z_{t-\delta} = z_t + \delta d_z$. Hence, there exists d_z^* that solves (3.1) such that $z_t + \delta d_z^* \in M_{t-\delta}$ for all $0 \leq \delta < t$. Now, recall Theorem 1 which implies that,

$$\lim_{t \to 0} \frac{1}{2} r_1^T \Theta_1 r_1 + \frac{1}{2} r_2^T \Theta_2 r_2 = 0.$$

As $(t - \delta) \to 0 \Leftrightarrow \delta \to t$, and $\frac{1}{2} r_1^T \Theta_1 r_1 + \frac{1}{2} r_2^T \Theta_2 r_2 = 0 \Leftrightarrow \Theta_1 r_1 = \Theta_2 r_2 = 0$, we get,

$$\Theta_1 r_1(x_t + t d_x^*, y_t + t d_y^*) = 0,$$

and,

$$\Theta_2 r_2(x_t + t d_x^*, y_t + t d_y^*) = 0.$$

Any solution $\dot{d}_z$ of (3.1) can be expressed as $\dot{d}_z = d_z^* + \eta_z$ where $\eta_z = \left[\begin{smallmatrix}\eta_x\\ \eta_y\end{smallmatrix}\right] \in \mathcal{N}(P)$. However, by lemma 1

$$P\eta_z = \mathbf{0} \Rightarrow Q\eta_z = \mathbf{0} \Rightarrow \begin{bmatrix} -\Theta_1 A & \Theta_1 e \\ \Theta_2 A & \Theta_2 e \end{bmatrix} \begin{bmatrix}\eta_x \\ \eta_y\end{bmatrix} = 0.$$

Inserting that in the above two equations, we get equalities (3.8) and (3.9). Equalities (3.10) and (3.11) follow from Lemma 4, since (3.8) and (3.9) imply that (3.5) is consistent. ∎

For $z_t \in M_t$ and $\theta_1 = \theta_1(M_t)$, $\theta_2 = \theta_2(M_t)$, we define:

$$u_t = \frac{1}{t}\Theta_1 r_1(x_t, y_t). \tag{3.12}$$

and

$$v_t = -\frac{1}{t}\Theta_2 r_2(x_t, y_t). \tag{3.13}$$

By recalling the necessary equations for a minimizer (3.2) and (3.3), we can see that:

$$t - e^T\Theta_1 r_1(x_t, y_t) + e^T\Theta_2 r_2(x_t, y_t) = \mathbf{0} \Rightarrow e^T(u_t + v_t) = 1$$

and

$$A^T\Theta_1 r_1(x_t, y_t) + A^T\Theta_2 r_2(x_t, y_t) = \mathbf{0} \Rightarrow A^T(u_t - v_t) = \mathbf{0},$$

which imply that (u_t, v_t) is feasible for [LINFLD].

Now, we present, without a proof, a classical theorem in linear programming known as the complementarity slackness theorem.

Theorem 4 *Let (x,y) and (u,v) be feasible pairs for [LINFLP] and [LINFLD] respectively, then they are optimal in their respective problems if and only if:*

$$Ax - ye < b \Rightarrow u = 0,$$

$$u > 0 \Rightarrow Ax - ye = b,$$

$$Ax + ye > b \Rightarrow v = 0,$$

$$v > 0 \Rightarrow Ax + ye = b.$$

For the purpose of stating a new characterization for the ℓ_∞ solutions, let $\mathcal{J}_\theta = \{i/\theta_{1_i} = 0 \wedge \theta_{2_i} = 0\}$ and $\mathcal{D}_\theta = \{z = (x, y) \in R^{n+1}/a_i^T x - y \leq b_i \wedge a_i^T x + y \geq b_i \wedge i \in \mathcal{J}_\theta\}$.

Theorem 5 *Let $0 < t \leq t_0$, where t_0 is as given in theorem 3 and let $\theta_1 = \theta_1(M_t)$, $\theta_2 = \theta_2(M_t)$. Let $z_t \in M_t$ and d_z solve (3.1). Then*

$$M_0 \equiv X,$$

where

$$M_0 = (z_t + td_z + \mathcal{N}_\theta) \cap \mathcal{D}_\theta,$$

and

$$u^* = \frac{1}{t}\Theta_1 r_1(x_t, y_t) \quad ; \quad v^* = -\frac{1}{t}\Theta_2 r_2(x_t, y_t)$$

solve [LINFLD].

Proof: Let $z_0 \in M_0$, then there exists a solution d_z^0 to (3.1) and $\eta_z^0 \in \mathcal{N}_\theta$ such that $z_0 = z_t + td_z^0 + \eta_z^0$. Then $\Theta_1 r_1(x_0, y_0) = \Theta_1 r_1(x_t + td_x^0 + \eta_x^0, y_t + td_x^0 + \eta_y^0) = \Theta_1 r_1(x_t + td_x^0, y_t + td_x^0) + \Theta_1(A\eta_x^0 + \eta_y^0)$. The first part equals 0 by Corollary 5 and the second part is 0 by Lemma 1. Similarly for $\Theta_2 r_2(x_0, y_0)$ we get,

$$\Theta_1 r_1(x_0, y_0) = 0 \quad ; \quad \Theta_2 r_2(x_0, y_0) = 0$$

Now, Using the fact that (u^*, v^*) are dual feasible, $e^T(u^* + v^*) = 1$ and $A^T(v^* - u^*) = \mathbf{0}$, we get:

$$\begin{aligned}
y_0 &= y_0 + \mathbf{0}.x_0 \\
&= y_0^T e^T(u^* + v^*) + x_0^T(A^T(v^* - u^*)) \\
&= (-x_0^T A^T + e^T y_0^T + b^T)u^* + (x_0^T A^T + e^T y_0^T - b^T)v^* + b^T(v^* - u^*) \\
&= -\tfrac{1}{t} r_1^T(x_0, y_0)\Theta_1 r_1(x_t, y_t) - \tfrac{1}{t} r_2^T(x_0, y_0)\Theta_2 r_2(x_t, y_t) + b^T(v^* - u^*) \\
&= b^T(v^* - u^*).
\end{aligned}$$

This shows that z_0 and (u^*, v^*) are primal and dual feasible respectively. Furthermore, they are optimal. Since this holds for any $z_0 \in M_0$, $M_0 \subseteq X$.

If z_0 is a singleton, then the proof is complete. Let's assume that it is not the case, and let $z \in X$. Then by feasibility, we have $Ax - ye \leq b$ and $Ax + ye \geq b$, furthermore $y = y_0 = b^T(v^* - u^*)$, for (u^*, v^*) dual feasible. Thus by Theorem 4 we have,

$$\Theta_1 r_1(x, y) = 0 \quad ; \quad \Theta_2 r_2(x, y) = 0$$

$$\Rightarrow P\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A^T\Theta_1 & A^T\Theta_2 \\ -\Theta_1 e & \Theta_2 e \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix}.$$

Furthermore,

$$P\begin{bmatrix} x - x_t \\ y - yt \end{bmatrix} = P\begin{bmatrix} x \\ y \end{bmatrix} - P\begin{bmatrix} x_t \\ yt \end{bmatrix}.$$

Using (3.4), as z_t minimizes $F(x, y, t)$, and the above , this implies:

$$P(z - z_t) = \begin{bmatrix} x - x_t \\ y - yt \end{bmatrix} = \begin{bmatrix} 0 \\ te \end{bmatrix} - \begin{bmatrix} A^T\Theta_1 & A^T\Theta_2 \\ -\Theta_1 e & \Theta_2 e \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix} + \begin{bmatrix} A^T\Theta_1 & A^T\Theta_2 \\ -\Theta_1 e & \Theta_2 e \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ t\epsilon \end{bmatrix}$$

And thus $\frac{z - z_t}{t}$ is a solution to (3.1), and so letting $d_z = \frac{z - z_t}{t}$ and $\eta_z = 0$, we get $z = (z_t + td_z + \eta_z) \in (z_t + td_z) + \mathcal{N}_\theta$. Now observing that $z \in \mathcal{D}_\theta$ by virtue of feasibility, then $z \in M_0$, and thus $X \subseteq M_0$. ▮

Hence, all the optimal solutions to [LINFLP] can be computed from any $z_t \in M_t$ for $t \in (0, t_0]$. This can be performed-at least in theory- by choosing any solution d_z to (3.1) and varying $\eta_z \in \mathcal{N}_\theta$ such that $(z_t + td_z + \eta_z) \in \mathcal{D}_\theta$.

An immediate consequence of the characterization theorem is the following sufficiency condition for the uniqueness of solution in [LINFLP]:

Corollary 6 *X is a singleton if $\mathcal{N}_\theta = \{0\}$ where $\theta_1 = \theta_1(M_t)$ and $\theta_2 = \theta_2(M_t)$ for $t \in (0, t_0]$.*

Proof: Since $\mathcal{N}_\theta = \{0\}$, $z_t \in M_t$ is unique by Corollary 3. Hence $Pd_z = \epsilon_{n+1}$ has a unique solution, d_z^0 say. Therefore, $z_t + td_z^0, +\mathcal{N}_\theta$ is a singleton. So by Theorem 4, X is a singleton. ▮

3.3 Extended Binary Vectors

In analogy to binary vectors, we define the following "extended binary vectors",

$$\bar{\theta}_{1_i}(x,y) = \begin{cases} 1 & \text{if } a_i^T x - y \geq b_i \\ 0 & \text{otherwise} \end{cases}$$

and

$$\bar{\theta}_{2_i}(x,y) = \begin{cases} 1 & \text{if } a_i^T x + y \leq b_i \\ 0 & \text{otherwise} \end{cases}$$

From linear programming, there should exist an optimal solution to [LIN-FLP] where at least $n+1$ equations (among the $2m$) are active (satisfied as equality), and the submatrix formed by picking the corresponding rows of $\left[\begin{smallmatrix} A & | -e \\ A & | \ e \end{smallmatrix}\right]$ has full rank. A similar property exists for the minimizers of $F(x,y,t)$. Now, let's define the following *active* set of indices:

$$\mathcal{A}(z) \equiv \{i/\bar{\theta}_1(x,y) = 1 \vee \bar{\theta}_2(x,y) = 1; i = 1...m\}$$

Theorem 6 *There exists a minimizer z_t of $F(x,y,t)$ for which $\bar{P}(z_t)$ is of rank $n+1$. Where $\bar{P}(z_t)$ is given by:*

$$\begin{bmatrix} A^T\bar{\Theta}_1(x_t,y_t)A + A^T\bar{\Theta}_2(x_t,y_t)A & -A^T\bar{\Theta}_1(x_t,y_t)e + A^T\bar{\Theta}_2(x_t,y_t)e \\ -e^T\bar{\Theta}_1(x_t,y_t)A + e^T\bar{\Theta}_2(x_t,y_t)A & e^T\bar{\Theta}_1(x_t,y_t)e + e^T\bar{\Theta}_2(x_t,y_t)e \end{bmatrix}$$

Proof: Let z be a minimizer of F where rank$(\bar{P}) < n+1$. Therefore, there exists a vector $h_z \in \mathcal{N}(\bar{P})$ with $h_z \neq 0$.
Consider the point $z + \alpha h_z$, $\alpha \in R$. By the fact that $\bar{P}z = \mathbf{0}$ and (3.4), it follows that, if $(z + \alpha h_z) \in C_{\bar{\theta}}$, where $\bar{\theta}_1 = \bar{\theta}_1(x,y)$ and $\bar{\theta}_2 = \bar{\theta}_2(x,y)$, then

$$\bar{P}(z_t + \alpha h_z) = -t e_{n+1} + Q^T\bar{b}.$$

and $z + \alpha h_z$ minimizes $F(x,y,t) = 0$. By lemma 2, if $j \in \mathcal{A}(z)$, then $h_{z_j} = 0$. This implies that $z_j + \alpha h_{z_j} = 0$, and hence

$$\mathcal{A}(z) \subseteq \mathcal{A}(z + \alpha h_z)$$

Recalling that $h_z \neq \mathbf{0}$, so there exists $s \in \{1, ..., n\}/\mathcal{A}(z)$ such that $h_{z_s} \neq 0$ and so $\bar{\theta}_{1_s}(x, y) = 0$ or $\bar{\theta}_{2_s}(x, y) = 0$, say $\bar{\theta}_{1_s}(x, y) = 0$, hence, $r_{1_s}(x, y) < 0$. However, $\alpha \in R$ could be chosen so that, $r_1(x + \alpha h_x, y + \alpha h_y) = 0$, whereas $\theta_{1_s}(x + \alpha h_x, y + \alpha h_y)$ remains 0. Therefore, the active set $\mathcal{A}(z + \alpha h_z)$ has one more index then $\mathcal{A}(z)$.
So far it has been shown that whenever there exists a minimizer for which $\bar{P}(x, y)$ has rank less than $n + 1$, there exists a minimizer for which the corresponding active set has one more element. If the new matrix is also rank deficient, the process could be repeated from the new point until we finally have an active set where the matrix $\bar{P}$ has rank $n + 1$, since when $\Theta_1 = \Theta_2 = I$, the corresponding $\bar{P}$ has rank $n + 1$. ∎

3.3.1 Behavior of the Set of Minimizers near the Feasible Boundary

In this section, we analyze the behavior of the set of extended binary vectors associated with the set of minimizers of $F(x, y, t)$ in the range $(0, t_0]$ where t_0 is as defined in theorem 5. This is important in establishing the finite termination property of the penalty algorithm defined in section 4. Now, let's define the following new concepts and definitions [13].

Let $\sigma_\theta = \{i | \theta_i = 1\}$ for any binary vector θ.

A "derived-extended-binary-subset" (*debs*) $\mathcal{S}$ of a pair of binary vectors (θ_1, θ_2) is a set of distinct extended binary pairs of vectors $(\bar{\theta}_1, \bar{\theta}_2)$ such that $\sigma_{\theta_1} \subseteq \sigma_{\bar{\theta}_1}$ and $\sigma_{\theta_2} \subseteq \sigma_{\bar{\theta}_2}$ and there exist $(x, y) \in R^{n+1}$ with $\bar{\theta}_1(x, y) = \bar{\theta}_1$, and $\bar{\theta}_2(x, y) = \bar{\theta}_2$.

An "extended-binary-set" (*ebs*) $\mathcal{S}(M_t)$ of a set of minimizers M_t is defined as the set of all distinct extended binary pairs of vectors corresponding to the elements of M_t . In other words, for any $(x_t, y_t) \in M_t$, $(\bar{\theta}_1(x_t, y_t), \bar{\theta}_2(x_t, y_t)) \in \mathcal{S}(M_t)$. Since $(\theta_1(x_t, y_t), \theta_2(x_t, y_t))$ are constants for all $z_t \in M_t$, the *ebs* $\mathcal{S}(M_t)$ of M_t is a *debs* of $(\theta_1(M_t), \theta_2(M_t))$ for any $t > 0$.

Lemma 6 *If $\mathcal{S}(M_{t_1}) = \mathcal{S}(M_{t_2})$ where $0 < t_2 < t_1$ then $\mathcal{S}(M_{t_1}) = \mathcal{S}(M_{t_2}) = \mathcal{S}(M_t)$ for $t_2 \leq t \leq t_2$.*

Proof: Let $(x_{t_1}, y_{t_1}) \in M_{t_1}$ and $(x_{t_2}, y_{t_2}) \in M_{t_2}$ with $\bar{\theta}_1(x_{t_1}, y_{t_1}) = \bar{\theta}_1(x_{t_2}, y_{t_2})$ and $\bar{\theta}_2(x_{t_1}, y_{t_1}) = \bar{\theta}_2(x_{t_2}, y_{t_2})$ and define,

$$x_t = (1-\epsilon)x_{t_2} + \epsilon x_{t_1}$$

where $\epsilon = (t - t_2)/(t_1 - t_2)$. Then obviously, $r_1(x_t, y_t) = (1-\epsilon)r_1(x_{t_2}, y_{t_2}) + \epsilon r_1(x_{t_1}, y_{t_1})$ and $r_2(x_t, y_t) = (1-\epsilon)r_2(x_{t_2}, y_{t_2}) + \epsilon r_2(x_{t_1}, y_{t_1})$. Now, if $\bar{\theta}_1(x_{t_1}, y_{t_1}) = \bar{\theta}_1(x_{t_2}, y_{t_2}) = 0$ then $r_1(x_{t_1}, y_{t_1}) < 0$ and $r_1(x_{t_2}, y_{t_2}) < 0$ and so $r_1(x_t, y_t) < 0$, which implies $\bar{\theta}_1(x_t, y_t) = 0$. Repeating the previous argument for the case when $\bar{\theta}_1(x_{t_1}, y_{t_1}) = \bar{\theta}_1(x_{t_2}, y_{t_2}) = 1$, and for $\bar{\theta}_2(x_t, y_t)$, it follows that :

$$\bar{\theta}_1(x_t, y_t) = \bar{\theta}_1(x_{t_1}, y_{t_1}) = \bar{\theta}_1(x_{t_2}, y_{t_2})$$

and

$$\bar{\theta}_2(x_t, y_t) = \bar{\theta}_2(x_{t_1}, y_{t_1}) = \bar{\theta}_2(x_{t_2}, y_{t_2})$$

Noticing that (x_t, y_t) satisfy the necessary conditions for a minimizer and repeating the above for all $(\bar{\theta}_1, \bar{\theta}_2) \in \mathcal{S}(M_{t_1})$ or $\mathcal{S}(M_{t_2})$, the result follows. ∎

Theorem 7 *There exists $\bar{t}$ such that $\mathcal{S}(M_t)$ is constant for $t \in (0, \bar{t})$ where $0 < \bar{t} \leq t_0$.*

Proof: As $\theta_1(M_t)$ and $\theta_2(M_t)$ remain constants in $(0, t_0]$, and the number of different-derived-extended binary subsets of $(\theta_1(M_t), \theta_2(M_t))$ is finite, the result follows from the previous lemma, as $\mathcal{S}(M_t)$ can not keep changing infinitely as t approaches 0. ∎

Theorem 8 *Let $t \in (0, \bar{t})$ and $(x_t, y_t) \in M_t$ with $\bar{\theta}_1 = \bar{\theta}_1(M_t)$ and $\bar{\theta}_2 = \bar{\theta}_2(M_t)$. Also, let $u^* = \frac{1}{t}\bar{\Theta}_1 r_1(x_t, y_t)$ and $v^* = -\frac{1}{t}\bar{\Theta}_2 r_2(x_t, y_t)$ then,*

$$\bar{\Theta}_1 r_1(x_t + td_x, y_t + td_y) = 0,$$

$$\bar{\Theta}_2 r_2(x_t + td_x, y_t + td_y) = 0.$$

and

$$b^T(v^* - u^*) = y_t + td_y,$$

for any solution $d_z = (d_x, d_y)$ *to (3.1). Furthermore, if* d_z *is unique or* $r_1(x_t + td_x, y_t + td_y) \leq 0$ *and* $r_2(x_t + td_x, y_t + td_y) \geq 0$ *then* $(x_t + td_x, y_t + td_y)$ *solves [LINFLP].*

Proof: Let $t \in (0, \bar{t})$and $(x_t, y_t) \in M_t$ with $\bar{\theta}_1 = \bar{\theta}_1(x_t, y_t)$ and $\bar{\theta}_2 = \bar{\theta}_2(x_t, y_t)$. Consider the system:

$$\bar{P}d_z = e_{n+1}. \tag{3.14}$$

This is a consistent system of linear equations as shown in Lemma 3. By theorem 7 there exists $(x_t, y_t) \in M_t$ such that $\bar{\theta}_1(x_t, y_t) = \bar{\theta}_1$ and $\bar{\theta}_2(x_t, y_t) = \bar{\theta}_2$ for all $t \in (0, \bar{t})$. This implies that there exist d_z that solves (3.14) such that $z_t + \delta d_z \in M_{t-\delta}$ for all $\delta \in (0, t]$ (as in the proof of lemma 5).
Now, recalling theorem 1, which implies that as t approaches 0, z_t solves [LINFLP] and $\bar{\Theta}_1 r_1(x_t, y) = 0$; $\bar{\Theta}_2 r_2(x_t, y_t) = 0$. We can conclude that, $z_t + td_z$ solves [LINFLP] ($t - \delta \to 0 \Rightarrow \delta \to t$), and.

$$\bar{\Theta}_1 r_1(x_t + td_x, y_t + td_y) = 0.$$

$$\bar{\Theta}_2 r_2(x_t + td_x, y_t + td_y) = 0.$$

Since d_z can be replaced by $d_z + \eta_z = (d_x + \eta_x, d_y + \eta_y)$ in the above identity where $\eta_z \in \mathcal{N}(\bar{P})$, it follows that,

$$\bar{\Theta}_1 r_1(x_t + td_x, y_t + td_y) = 0,$$

$$\bar{\Theta}_2 r_2(x_t + td_x, y_t + td_y) = 0.$$

for any solution d_z of (3.14). Clearly, if the solution to (3.14) is unique, $d_z^* = (d_x^*, d_y^*)$ say, then $(x_t + td_x^*, y_t + td_y^*)$ solve [LINFLP].
Let $x_0 = x_t + td_x$ and $y_0 = y_t + td_y$, using the previous two identities and the

fact that (u^*, v^*) are dual feasible, we get:

$$\begin{aligned} y_0 &= y_0 + \mathbf{0}.x_0 \\ &= y_0 e^T(u^* + v^*) + x_0^T(A^T(v^* - u^*)) \\ &= (-x_0^T A^T + y_0 e^T + b^T)u^* + (x_0^T A^T + y_0 e^T - b^T)v^* + b^T(v^* - u^*) \\ &= -\tfrac{1}{t} r_1^T(x_0, y_0)\bar{\Theta}_1 r_1(x_t, y_t) - \tfrac{1}{t} r_2^T(x_0, y_0)\bar{\Theta}_2 r_2(x_t, y_t) + b^T(v^* - u^*) \\ &= b^T(v^* - u^*). \end{aligned}$$

which completes the proof.∎

3.4 The Penalty Algorithm

We are at a position to state the penalty algorithm for solving ℓ_∞ problems.

1. Choose t and compute a minimizer z_t of F.

2. While not STOP

2.1. Reduce t,

2.2. Compute a minimizer $z_t$of F,

where STOP is a binary function that returns TRUE if the duality gap is zero (with in roundoff) and primal feasibility is achieved. For the purpose of proving the finite convergence of the algorithm, we give a summary of the way an unconstrained minimizer is found, and the criteria for reducing the parameter t. A full description, along with implementation details is provided, when the algorithm is fully analyzed in chapter 5.

3.4.1 Computing an Unconstrained Minimizer

The algorithm for computing a minimizer z_t of F is adapted from the *Newton algorithm for robust linear regression using Huber functions*. It is a standard Newton iteration with a simple line search to solve the nonlinear system of equations $F_x(x, y, t) = 0$ and $F_y(x, y, t) = 0$. The idea is to inspect the regions of R^{n+1} to locate the region where the local quadratic $\mathcal{Q}_\theta$ contains its own minimizer . At a given iterate, the Newton step is computed using the quadratic expansion of F. If a unit step in this direction yields a point in the same region, then the global minimizer has been found. That is to say that the quadratic representation of F which contains the global minimizer has been found. Otherwise, the algorithm proceeds with a line search.

A search direction $h_z = (h_x, h_y)$ is computed by minimizing the quadratic $\mathcal{Q}_{\bar\theta}$ where $\bar\theta = \bar\theta(x, y)$ and z is the current iterate. Precisely, we consider the equation

$$\bar{P}h_z = -te_{n+1} - \bar{P}z + \begin{bmatrix} A^T(\bar\Theta_1 + \bar\Theta_2) \\ e^T(\bar\Theta_2 - \bar\Theta_1) \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix}. \tag{3.15}$$

Let's denote the right-hand side vector in equation (3.15) by g, so that we have, $\bar{P}h_z = g$ and let $\mathcal{N}(\bar{P})$ denote the null space of $\bar{P}$. If $\bar{P}$ has full rank then h_z is the solution to (3.15). Otherwise, if the system (3.15) is consistent, a minimum norm solution is computed. If the system is inconsistent, the projection of g onto $\mathcal{N}(\bar{P})$ is computed. These choices are motivated and justified in [7]. After a finite number of iterations (can be shown using the analysis in [7]) we have $z + h_z \in \mathcal{C}_{\bar\theta}$ Therefore, $z + h_z$ is a minimizer of F. The modified Newton Algorithm is summarized below.

repeat

  $\bar\theta_1 = \bar\theta_1(x, y)$; $\bar\theta_2 = \bar\theta_2(x, y)$

  if (3.15) is consistent then

    find h_z from (3.15)

    if $z + h_z \in \mathcal{C}_{\bar\theta}$ then

      $z \leftarrow z + h_z$

      stop = true

else

$z \leftarrow z + \alpha h_z$ (line search)

endif

else

compute h_z = null space projection of g

$z \leftarrow z + \alpha h_z$ (line search)

endif

until stop.

3.4.2 Reducing t

Let z_t be a minimizer of $F(x, y, t)$ for some $t > 0$ and let $\bar{\theta}_1 = \bar{\theta}_1(x_t, y_t)$ and $\bar{\theta}_2 = \bar{\theta}_2(x_t, y_t)$ and consider the system:

$$\bar{P}(z_t)d_z = e_{n+1} \tag{3.16}$$

where

$$\bar{P}(z_t) = \begin{bmatrix} A^T\bar{\Theta}_1 A + A^T\bar{\Theta}_2 A & -A^T\bar{\Theta}_1 + A^T\bar{\Theta}_2 \\ -\bar{\Theta}_1 A + \bar{\Theta}_2 A & \bar{\Theta}_1 + \bar{\Theta}_2 \end{bmatrix}$$

Let d_z be a solution to (3.16). Two cases arise:

Case 1 The duality gap $(y_t + td_y) - b^T(v^* - u^*)$ is zero but $(x_t + td_x, y_t + td_y)$ is infeasible in [LINFLP], i.e., there exists j such that either $r_{1_j}(x_t + td_x, y_t + td_y) > 0$ or $r_{2_j}(x_t + td_x, y_t + td_y) < 0$. Let $\psi_1 \equiv \{\alpha_k, k = 1, 2, ..., m_1\}$ and $\psi_2 \equiv \{\beta_k, k = 1, 2, ..., m_2\}$ be the sets of positive kink points where the components of $r_1(x_t + td_x, y_t + td_y)$ and $r_2(x_t + td_x, y_t + td_y)$ change sign, respectively. I.e., the sets

$$\psi_1 = \{0 < \alpha < 1 | \exists i \in J | r_{1_i}(x_t + t\alpha_i d_x, y_t + t\alpha_i d_y) = 0\},$$

and

$$\psi_2 = \{0 < \beta < 1 | \exists i \in J | r_{2_i}(x_t + t\beta_i d_x, y_t + t\beta_i d_y) = 0\},$$

where

$$J = \{i | 1 \leq i \leq n \wedge (d_{x_i} \neq 0 \vee (d_{y_i} \neq 0)\}.$$

If ψ_1 or ψ_2 is nonempty, we choose,

$$\alpha^* = min(\min_k \alpha_k, \min_k \beta_k),$$

and we let

$$t_{next} \equiv (1 - \alpha^*)t,$$

and

$$x_{t_{next}} \equiv x_t + \alpha^* t d_x; \quad y_{t_{next}} \equiv y_t + \alpha^* t d_y.$$

Otherwise, we let

$$t_{next} \equiv 0.9t,$$

and

$$x_{t_{next}} \equiv x_t + 0.9 t d_x; \quad y_{t_{next}} \equiv y_t + 0.9 t d_y.$$

In both cases, $(x_{t_{next}}, y_{t_{next}})$ is used as the starting point of the modified Newton algorithm, with the reduced value of t.

Case 2 The duality gap is not zero. In this case we reduce t as follows. Let $\Delta((1-\epsilon)t)$ denote the number of changes from $\mathcal{A}(z_t)$ to $\mathcal{A}(z_t + \epsilon t d_z)$, where

$$\mathcal{A}(z) \equiv \{i / \bar{\theta}_1(x,y) = 1 \vee \bar{\theta}_2(x,y) = 1 : i = 1...m\}$$

and d_z solves

$$\bar{P} d_z = e_{n+1}.$$

Then bisection is used to find $\bar{\epsilon}$ s.t $\Delta((1-\bar{\epsilon})t) \approx \frac{1}{2}\Delta(t)$. And,

$$t_{next} = (1 - \bar{\epsilon})t,$$

$$z_{t_{next}} \equiv z_t + \bar{\epsilon} t d_z.$$

3.5 Finite Convergence

In this section, the algorithm is shown to converge finitely. In what follows an iteration of the algorithm means either a modified Newton iteration or an execution of the t-reduction procedure. For that we consider the following lemma.

Lemma 7 *Assume $t \in (0, \bar{t})$. Let $z = (x, y) \in M_t$ with $\bar{\theta}_1 = \bar{\theta}_1(x, y)$ and $\bar{\theta}_2 = \bar{\theta}_2(x, y)$. Let d_z solve (3.16), and $z^+ = (x^+, y^+)$ be generated by one iteration of the algorithm. Then either,*

$$z^+ \equiv z + td_z \in X$$

and the algorithm stops, or

$$z^+ \equiv z + \alpha^* td_z \in M_{t^+},$$

$$t^+ = (1 - \alpha^*)t$$

where α^* is as defined in Case 1 of the reduction procedure, and $\mathcal{A}(z^+)$ is an extension of $\mathcal{A}(z)$.

Proof: Let $u = \frac{1}{t}\bar{\Theta}_1 r_1(x)$ and $v = \frac{-1}{t}\bar{\Theta}_2 r_2(x)$. From Theorem 8, $b^T(u - v) - (y_t + td_y) = 0$. Following, the reduction procedure, either $A(x + td_x) - y_t e \leq b$ and $A(x + td_x) + y_t e \geq b$ and thus $z^+ = z + td_z$ is a solution to [LINFLP] and the algorithm stops, or $\mathcal{A}(z) \subseteq \mathcal{A}(z^+)$. The latter condition follows directly from the choice of α^* in case 1 of the reduction procedure. In addition to that, it guarantees that $z + \alpha^* td_z \in C_{\bar{\theta}}$. Therefore, using the definition of the gradient and the way d_z is calculated, we have

$$z + \alpha^* td_z \in M_{(1-\alpha^*)t} ,$$

which completes the proof. ∎

Let $z \in M_t$ for some $t > 0$. Unless the stopping criteria are met and the algorithm stops with a primal-dual optimal pair, t is reduced by at least a factor of α where $\alpha \in (0\ ,\ 1)$ as discussed in the reduction procedure. Since the modified Newton iteration, is a finite process, t will reach the range $(0, \bar{t})$ where $\bar{t}$ is as defined in Theorem 7 in a finite number of iterations. Now assume $t \in (0, \bar{t})$. From Lemma 7 either the algorithm terminates or the active set $\mathcal{A}$ is expanded. Repeating this argument, in a finite number of iterations the matrix $\bar{P}$ will finally have rank $n + 1$ since A has rank n and $\bar{P} = \begin{bmatrix} A^T A & \mathbf{0} \\ \mathbf{0} & e^T e \end{bmatrix}$. When $\bar{P}$ has full rank the solution d_z to the system (3.16) is unique, and $z^+ = z + td_z$ solves [LINFLP] by Theorem 8. Therefore we have proved the following theorem:

Theorem 9 *The algorithm terminates in a finite number of iterations with a primal-dual optimal pair.*

Chapter 4

Example

Let us consider the following simple example of an ℓ_∞ problem, where $A = \begin{bmatrix}1\\2\end{bmatrix}$ and $b = \begin{bmatrix}3\\5\end{bmatrix}$,

$$\min_{x \in R} \max(|x-3|, |2x-5|).$$

The function is sketched in figure 4.1. Its minimum occurs at the point $\frac{8}{3}$.

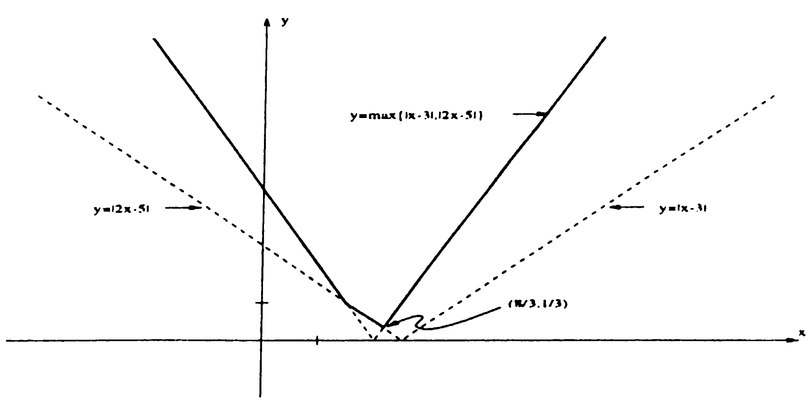

Figure 4.1: Plot of the function $\max(|x-3|, |2x-5|)$.

We use the following linear programming formulation:

$$\begin{aligned} &\text{Min } y \\ \text{s.t}\quad & x - y \le 3 \\ & 2x - y \le 5 \\ & x + y \ge 3 \\ & 2x + y \ge 5, \end{aligned}$$

and the following quadratic function:

$$F(x, y, t) = ty + \frac{1}{2} r_1^T \Theta_1 r_1 + \frac{1}{2} r_2^T \Theta_2 r_2.$$

Where,

$$\Theta_{1_{11}}(x, y) = \begin{cases} 1 & \text{if } x - y > 3 \\ 0 & \text{otherwise} \end{cases} \qquad \Theta_{1_{22}}(x, y) = \begin{cases} 1 & \text{if } 2x - y > 5 \\ 0 & \text{otherwise} \end{cases}$$

$$\Theta_{2_{11}}(x, y) = \begin{cases} 1 & \text{if } x + y < 3 \\ 0 & \text{otherwise} \end{cases} \qquad \Theta_{2_{22}}(x, y) = \begin{cases} 1 & \text{if } 2x + y < 5 \\ 0 & \text{otherwise} \end{cases}$$

$$r_1(x, y) = \begin{bmatrix} \text{x-y-3} \\ \text{2x-y-5} \end{bmatrix}, \qquad r_2(x, y) = \begin{bmatrix} \text{x+y-3} \\ \text{2x+y+5} \end{bmatrix}.$$

Depending on the values of θ_1 and θ_2, R^2 is divided into 11 regions, as shown in figure 4.2 .

	$\Theta_{1_{11}}$	$\Theta_{1_{22}}$	$\Theta_{2_{11}}$	$\Theta_{2_{22}}$
I	1	0	0	0
II	0	0	0	0
III	0	0	0	1
IV	0	0	1	1
V	1	0	1	1
VI	1	1	1	1
VII	1	1	1	0
VIII	1	1	0	0
IX	0	1	1	0
X	0	0	1	0
XI	0	1	1	1

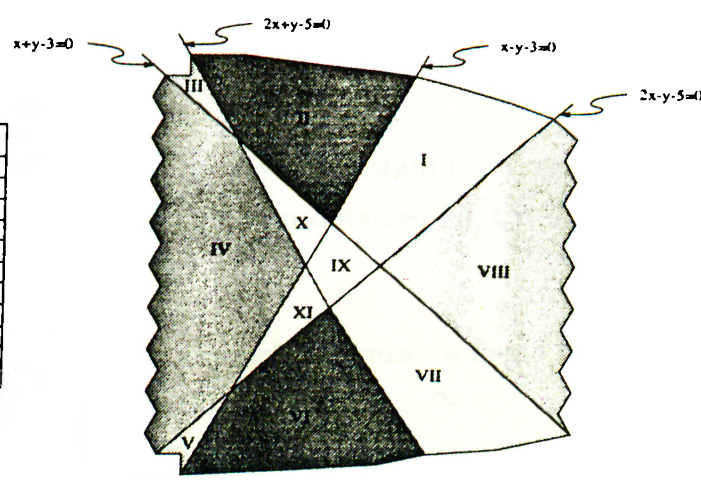

Figure 4.2: The division of R^2 according to the values of Θ_1 and Θ_2.

To find the minimizer of $F(x, y, t)$ for a fixed value of t, we solve

$$\Theta_{1_{11}}(x - y - 3) + 2\Theta_{1_{22}}(2x - y - 5) + \Theta_{2_{11}}(x + y - 3) + 2\Theta_{2_{22}}(2x + y - 5) = 0, \quad (4.1)$$

$$\Theta_{1_{11}}(x-y-3)+\Theta_{1_{22}}(2x-y-5)-\Theta_{2_{11}}(x+y-3)-\Theta_{2_{22}}(2x+y-5)=t. \qquad (4.2)$$

This system however is very difficult to solve. For that purpose, a simple, but exhaustive method was used. For every value of t, all the 11 combinations of Θ_1 and Θ_2 were tried to see if the system can be solved. In that case, the solution is checked whether it falls within the region or not, i.e its Θ vector matches the one of the region or not. If it does, then we have found the solution, otherwise a new region is tried for, and so on. Actually, that was performed through a matlab program. Figure 4.3 displays the region that contains the minimizer for values of t in [0 2]

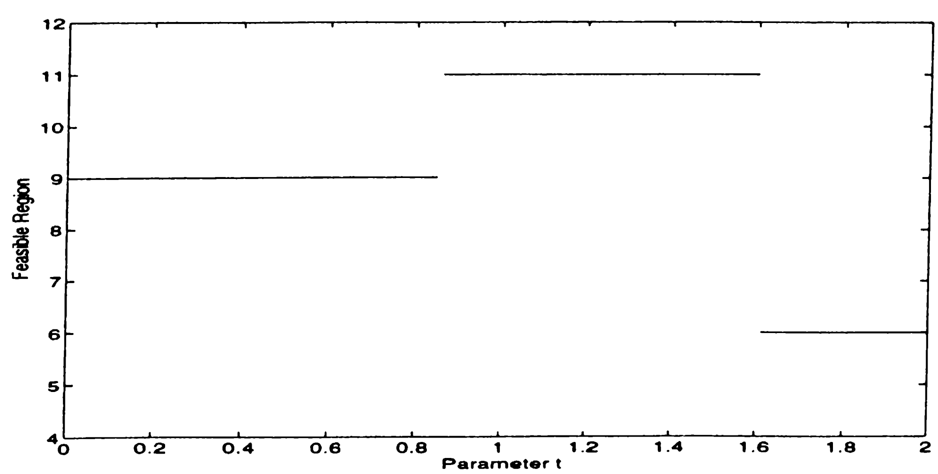

Figure 4.3: The region that contains the minimizer for different values of t.

It is clear that θ_1 and θ_2 are piecewise constant functions of t. In addition to that, a minimizer to the problem can obtained for t in the interval $[0 \quad t_0]$ where $t_0 = 0.85$.

After knowing the region where each minimizer is found for any value of t, it easy to solve for equations (4.1), (4.2).

For $0 \leq t \leq 0.85$, the minimizers are located in region 9, for $0.85 < t \leq 1.6$, they are located in region 11, and for $1.6 < t \leq 2$, they are found in region 6. Inserting the corresponding values of θ_{1_1}, θ_{1_2}, θ_{2_1} and θ_{2_2} we get,

$$x = \frac{8}{3} - \frac{1}{9}t, \qquad y = \frac{1}{3} - \frac{5}{9}t, \qquad \forall\ 0 \le t \le 0.85.$$
$$x = \frac{33}{13} + \frac{1}{26}t, \qquad y = \frac{2}{13} - \frac{9}{26}t, \qquad \forall\ 0.85 < t \le 1.6.4$$
$$x = 2.6, \qquad y = -\frac{1}{4}t, \qquad \forall\ 1.6 \le t \le 2.$$

Obviously, x and y are piecewise linear in terms of t. The graphs of x and y as a function of t are graphed in figures 4.4 and 4.5 respectively.

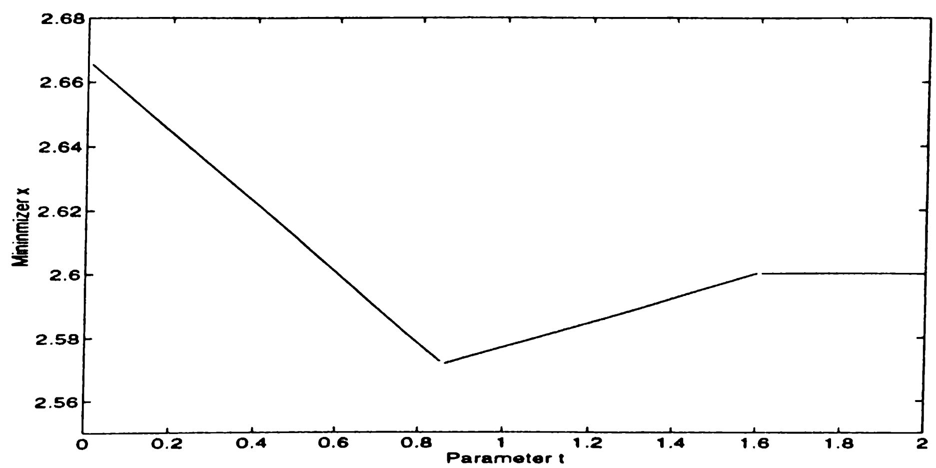

Figure 4.4: The minimizer x as a function of t.

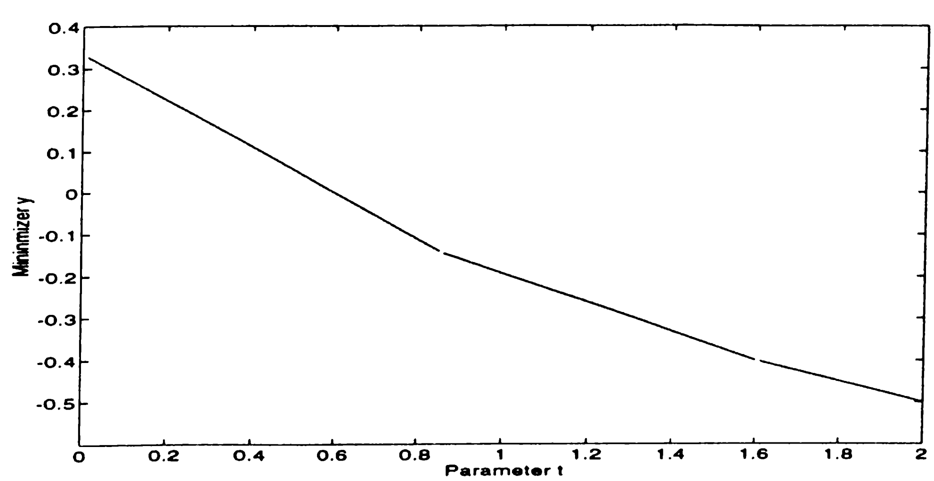

Figure 4.5: The minimizer y as a function of t.

Chapter 5

Numerical Issues and Implementation

In this chapter, the algorithm is fully analyzed. Its different parts are discussed in detail in terms of motivating ideas and implementation.

5.1 The Penalty Algorithm

5.1.1 Description

- Initialize $z_0 = [x_0, y_0]^T$, t_0.
- Compute a minimizer z_{t_0} of $F_{t_0}(x, y)$.
- **Iterate**
  - Reduce t
  - Find a minimizer z_t of $F_t(x, y)$
- **Until STOP.**

Where **STOP** is true if the duality gap. is zero and primal feasibility is obtained, or the parameter t is less than the uncertainty of the residuals, measured in terms of the machine precision.

5.1.2 Computing a minimizer of $F(x, y, t)$

For a certain pair of points $z = (x, y)$ and its corresponding binary vectors $\theta_1(x, y)$ and $\theta_2(x, y)$, the algorithm tries to check whether $z + h_z = (x + h_x, y + h_y)$ falls within the region that contains z. Precisely, the algorithm tries to solve for

$$\bar{A}^T \bar{\Theta} \bar{A}(z + h_z) = -te_{n+1} + \bar{A}^T \bar{\Theta} b;$$

which yields,

$$\bar{A}^T \bar{\Theta} \bar{A} h_z = -te_{n+1} + \bar{A}^T \bar{\Theta} b - \bar{A}^T \bar{\Theta} \bar{A} z. \qquad (5.1)$$

Which is explicitly, equivalent to,

$$\begin{bmatrix} Q_{\bar{\theta}}^{xx} & Q_{\bar{\theta}}^{xy} \\ Q_{\bar{\theta}}^{yx} & Q_{\bar{\theta}}^{yy} \end{bmatrix} \begin{bmatrix} h_x \\ h_y \end{bmatrix} = - \begin{bmatrix} Q_{\bar{\theta}}^{x} \\ Q_{\bar{\theta}}^{y} \end{bmatrix} \qquad (5.2)$$

where:

$$Q_{\bar{\theta}}^{y} = t + e^T(\bar{\Theta}_1 + \bar{\Theta}_2)ye + e^T(\bar{\Theta}_2 - \bar{\Theta}_1)Ax + e^T(\bar{\Theta}_1 - \bar{\Theta}_2)b.$$

$$Q_{\bar{\theta}}^{x} = A^T(\bar{\Theta}_2 - \bar{\Theta}_1)ye + A^T(\bar{\Theta}_1 + \bar{\Theta}_2)Ax - A^T(\bar{\Theta}_1 + \bar{\Theta}_2)b.$$

$$Q_{\bar{\theta}}^{xy} = A^T(\bar{\Theta}_2 - \bar{\Theta}_1)e.$$

$$Q_{\bar{\theta}}^{yx} = e^T(\bar{\Theta}_2 - \bar{\Theta}_1)A.$$

$$Q_{\bar{\theta}}^{xx} = A^T(\bar{\Theta}_1 + \bar{\Theta}_2)e.$$

And thus for a certain point z, we determine $\theta_1(z)$ and $\theta_2(z)$, solve for h_z and then check whether $\theta_1(z) = \theta_1(z + h_z)$ and $\theta_2(z) = \theta_2(z + h_z)$. If that is the case then a minimizer is at hand. The most important step in the algorithm is the solution of the system (5.1) to find a search direction, which must be a descent direction. Depending on the matrix $\bar{A}^T \bar{\Theta} \bar{A}$, three choices of h_z are possible.

Case 1: Suppose that, (5.1) is consistent and $\bar{A}^T\bar{\Theta}\bar{A}$ has full rank, then the solution is unique and

$$h_z = (\bar{A}^T\bar{\Theta}\bar{A})^{-1}(-te_{n+1} - \bar{A}^T\bar{\Theta}\bar{A}z + \bar{A}^T\bar{\Theta}b).$$

Recalling that $\nabla_z F(t) = te_{n+1} + \bar{A}^T\bar{\Theta}\bar{A}z - \bar{A}^T\bar{\Theta}b$, it is clear that $h_z^T\nabla_z F(t) \leq 0$ and so h_z is a descent direction

Case 2: Suppose that, (5.1) is consistent and $\bar{A}^T\bar{\Theta}\bar{A}$ is rank-deficient, then a minimum norm solution is calculated. In this case, system ((5.1)) has many solutions. It is well known that every solution can be written as $h_z = h_z^0 + \gamma\nu$ with h_z^0 a particular solution and ν an element of the null space of $\bar{A}^T\bar{\Theta}\bar{A}$. In addition to that, any vector h_z can be written as $h_z^R + h_z^N$, where h_z^R and h_z^N lie, respectively, in the row space and null space of $\bar{A}^T\bar{\Theta}\bar{A}$. Now by triangular inequality it is clear that, $||h_z^R||_2^2 \leq ||h_z^R||_2^2 + ||h_z^N||_2^2$ and so $||h_z^R||_2$ is a minimum norm solution. Furthermore,

$$h_z^R = P_R\,(-te_{n+1} - \bar{A}^T\bar{\Theta}\bar{A}z + \bar{A}^T\bar{\Theta}b).$$

Where P_R is the projection matrix into the row space of $\bar{A}^T\bar{\Theta}\bar{A}$. Hence it is implied that, $(h_z^R)^T\nabla_z F(t) \leq 0$ and so h_z^R is a descent direction.

Case 3: Suppose that (5.1) is not consistent. in that case h_z is chosen as the projection of $(-te_{n+1} - \bar{A}^T\bar{\Theta}\bar{A}z + \bar{A}^T\bar{\Theta}b)$ into the null space of $\bar{A}^T\bar{\Theta}\bar{A}$. i.e,

$$h_z^N = P_N\,(-te_{n+1} - \bar{A}^T\bar{\Theta}\bar{A}z + \bar{A}^T\bar{\Theta}\bar{A}).$$

Where P_N is the projection matrix into the null space of $\bar{A}^T\bar{\Theta}\bar{A}$. Again, $(h_z^N)^T\nabla_z F(t) \leq 0$ and h_z^N is a descent direction.

In the first and second cases, when a descent direction h_z is determined, the algorithm checks if $z + h_z \in C_\theta$ if so then a minimizer of $F_t(z)$ is found, otherwise it proceeds with a line search. In the third case, however, the algorithm proceeds always with a line search. The modified Newton algorithm is summarized below.

Repeat

$\bar{\theta}_1 = \bar{\theta}_1(x, y)$; $\bar{\theta}_2 = \bar{\theta}_2(x, y)$

If (5.2) consistent then

Find h_z from (5.2)

If $z + h_z \in \mathcal{C}_{\bar{\theta}}$ **then**

$z \leftarrow z + h_z$

Stop= **true**

Else

$z \leftarrow z + \alpha h_z$ (line search)

Endif

Else

Compute h=null space projection of g

$z \leftarrow z + \alpha h_z$ (line search)

Endif

Until Stop

5.1.3 Line Search

Let us consider, $F_t(x + \alpha h_x, y + \alpha h_y) = ty + \frac{1}{2} r_1^T \Theta_1 r_1 + \frac{1}{2} r_2^T \Theta_2 r_2$ where

$$r_1 = r_1(x + \alpha h_x, y + \alpha h_y),$$

$$r_2 = r_2(x + \alpha h_x, y + \alpha h_y),$$

$$\Theta_1 = \Theta_1(x + \alpha h_x, y + \alpha h_y),$$

$$\Theta_2 = \Theta_2(x + \alpha h_x, y + \alpha h_y),$$

which can be written as, $F_t(x+\alpha h_x, y+\alpha h_y) = \frac{1}{2}a_2\alpha^2 + a_1\alpha + a_0$, for

$$\begin{aligned}
a_2 &= (Ah_x - h_y\epsilon)^T\Theta_1(Ah_x - h_y\epsilon) + (Ah_x + h_y\epsilon)^T\Theta_2(Ah_x + h_y\epsilon) \\
a_1 &= th_y + (Ax - ye - b)^T\Theta_1(Ah_x - h_y\epsilon) + (Ax + ye - b)^T\Theta_2(Ah_x + h_y\epsilon) \\
a_0 &= t.y + \tfrac{1}{2}r_1\Theta_1 r_1 + \tfrac{1}{2}r_2\Theta_2 r_2
\end{aligned}$$

It is clear that a_0, a_1 and a_2 are piecewise constant that change at the kink points where one or more of r_1 or r_2 change sign. Now, minimizing $F_t(x+\alpha h_x, y+\alpha h_y)$, is equivalent to finding a zero for its derivative with respect to α which is given as:

$$a_2(\alpha_j)\alpha + a_1(\alpha_j), \quad \alpha_j \le \alpha < \alpha_{j+1}.$$

The subscripts in front of a_2 and a_1 are added for the purpose of indicating its dependence on the kink points α_j's . The strategy used, is to find an interval where,

$$\alpha_j a_2(\alpha_{j-1}) + a_1(\alpha_{j-1}) \le 0 \;\; and \;\; \alpha_j a_2(\alpha_j) + a_1(\alpha_j) \ge 0.$$

The algorithm, then, uses a step size of $\frac{1}{2}(\alpha_j + \alpha_{j+1})$.

5.1.4 Reducing t

Let z_t be a minimizer of $F(x, y, t)$ for some $t > 0$ and let $\bar{\theta}_1 = \bar{\theta}_1(x_t, y_t)$ and $\bar{\theta}_2 = \bar{\theta}_2(x_t, y_t)$ and consider the system:

$$\bar{P}(z_t)d_z = \epsilon_{n+1} \tag{5.3}$$

where

$$\bar{P}(z_t) = \begin{bmatrix} A^T\bar{\Theta}_1 A + A^T\bar{\Theta}_2 A & -A^T\bar{\Theta}_1 + A^T\bar{\Theta}_2 \\ -\bar{\Theta}_1 A + \bar{\Theta}_2 A & \bar{\Theta}_1 + \bar{\Theta}_2 \end{bmatrix}$$

Let d_z be a solution to (5.3). And assume that $\bar{\theta}_1(z_t+\epsilon d_z) = \bar{\theta}_1$ and $\bar{\theta}_2(z_t+\epsilon d_z) = \bar{\theta}_2$ for $\epsilon > 0$. Then, as in lemma 5, $\bar{P}(z_t + \delta d_z) = -(t-\delta)e_{n+1} + \bar{Q}^T\bar{b}$ for $0 \le \delta < \epsilon$, implying that $z_t + \delta d_z$ is a minimizer of $F(x, y, t-\delta)$. Hence,

$$\begin{aligned}
z(t-\delta) &= z_t + \delta d_z, \\
r_1(t-\delta) &= r_1(t) + \delta A d_z,
\end{aligned}$$

$$r_2(t-\delta) = r_2(t) + \delta A d_z.$$

From theorem 8, the corresponding dual variables are,

$$u(t-\delta) = \frac{1}{t-\delta}\bar{\Theta}_1 r_1(t-\delta),$$

$$v(t-\delta) = \frac{-1}{t-\delta}\bar{\Theta}_2 r_2(t-\delta).$$

And, so the duality gap $b^T(u(t-\delta) - v(t-\delta) - y_{t-\delta})$ is given by,

$$b^T(\frac{1}{t-\delta}\bar{\Theta}_1 r_1(t-\delta) + \frac{-1}{t-\delta}\bar{\Theta}_2 r_2(t-\delta)) - y_t + \delta d_y.$$

Now, letting δ increase from 0, if $\bar{\theta}_1(z_t+\delta d_z) = \bar{\theta}_1$ and $\bar{\theta}_2(z_t+\delta d_z) = \bar{\theta}_2$, we check the duality gap. If the latter is zero, then we check for primal feasibility. Otherwise, the choose a value of δ for which one or more of the components of θ_1 or θ_2 change. Precisely, we follow the following heuristic procedure, which was used in [13].

Case 1 The duality gap $(y_t+td_y) - b^T(v^* - u^*)$ is zero but (x_t+td_x, y_t+td_y) is infeasible in [LINFLP], i.e., there exists j such that either $r_{1_j}(x_t+td_x, y_t+td_y) > 0$ or $r_{2_j}(x_t+td_x, y_t+td_y) < 0$. Let $\psi_1 \equiv \{\alpha_k, k = 1, 2, ..., m_1\}$ and $\psi_2 \equiv \{\beta_k, k = 1, 2, ..., m_2\}$ be the sets of positive kink points where the components of $r_1(x_t+td_x, y_t+td_y)$ and $r_2(x_t+td_x, y_t+td_y)$ change sign, respectively. I.e., the set

$$\psi_1 = \{0 < \alpha < 1 | \exists i \in J | r_{1_i}(x_t + t\alpha_i d_x, y_t + t\alpha_i d_y) = 0\},$$

and,

$$\psi_2 = \{0 < \beta < 1 | \exists i \in J | r_{2_i}(x_t + t\beta_i d_x, y_t + t\beta_i d_y) = 0\},$$

where

$$J = \{i | 1 \leq i \leq n \wedge (d_{x_i} \neq 0 \vee (d_{y_i} \neq 0)\}.$$

If ψ is nonempty, we choose

$$\alpha^* = min(\min_k \alpha_k, \min_k \beta_k)$$

and we let
$$t_{next} \equiv (1 - \alpha^*)t,$$
and
$$x_{t_{next}} \equiv x_t + \alpha^* t d_x; \quad y_{t_{next}} \equiv y_t + \alpha^* t d_y$$
Otherwise, we let
$$t_{next} \equiv 0.9t,$$
and
$$x_{t_{next}} \equiv x_t + 0.9td_x; \quad y_{t_{next}} \equiv y_t + 0.9td_y.$$

Case 2 The duality gap is not zero. In this case we reduce t as follows. Let $\Delta((1-\epsilon)t)$ denote the number of changes from $\mathcal{A}(z_t)$ to $\mathcal{A}(z_t + \epsilon t d_z)$, where
$$\mathcal{A}(z) \equiv \{i / \bar{\theta}_1(x,y) = 1 \vee \bar{\theta}_2(x,y) = 1; i = 1...m\}$$
and d_z solves
$$\bar{P} d_z = e_{n+1}.$$
Then bisection is used to find $\bar{\epsilon}$ s.t $\Delta((1-\bar{\epsilon})t) \approx \frac{1}{2}\Delta(t)$. And,
$$t_{next} = (1 - \bar{\epsilon})t,$$
$$x_{t_{next}} \equiv x_t + \bar{\epsilon} t d_x; \quad y_{t_{next}} \equiv y_t + \bar{\epsilon} t d_y.$$

5.2 Implementation and Linear Algebra

The algorithm is implemented in Fortran 77 on a SPARC SUN workstations with 64MB ram. The major work in the Newton algorithm is dominated by the requirement to solve the least squares systems of the form $\bar{A}^T \bar{\Theta} \bar{A}\, x = g$. The AAFAC package of [11] is used to perform this. The solution is obtained via an LDL^T factorization of the matrix $P_k = \bar{A}^T \bar{\Theta} \bar{A}$. Let us recall that $\bar{\Theta}_{ii} = 1$ for the columns of A corresponding to indices in the active set A. Based on this observation, D and L are computed directly from the active columns of $\bar{A}$, i.e., without squaring the condition number as would be the case if P_k was first computed. It is essential for the efficiency of the penalty algorithm to observe that normally only a few entries of the diagonal matrix $\bar{\Theta}$ change between two consecutive iterations. This implies that the factorization of P_k can be obtained by relatively few up- and down-dates of the factorization of P_{k-1}. Occasionally, a refactorization is performed. This consists in the successive updating $LDL^T \leftarrow LDL^T + a_j a_j^T$ for all j in the active set (starting with $L = I, D = 0$). It is considered only when some columns of $\bar{A}$ leave the active set, i.e., when downdating is involved. If many columns are involved in the change from P_{k-1} to P_k it may be cheaper to refactorize P_k. Otherwise, a refactorization is used when a downdating results in a rank decrease and there is an indication that rounding errors have marked influence.

Chapter 6

Numerical Testing

After its implementation in standard Fortran 77, the algorithm is extensively tested on different test problems. This includes randomly generated and function approximation problems. The aim is, first, to test the viability of the approach, the behavior of the algorithm and the numerical accuracy of the method. Second, to experiment with different initialization procedures in order to find the best practical scheme that increases the efficiency of the algorithm. Finally, the potential of the algorithm as a competitor to the well-known algorithms for the ℓ_∞ problem is tested. This involves a comparison with the Barrodale-Phillips algorithm [1], which is among the best known, if not the best, algorithm for solving ℓ_∞ problems.

6.1 The Test Problems

In order to test the behavior of the algorithm, an ℓ_∞ problem generator is designed, which provides problems with a known objective value. The idea is based on linear programming theory. Actually, for given dimensions m and n, appropriate vectors A, x, y, u, v and b are suitably chosen to satisfy the Karush-Kuhn-Tucker optimality conditions for the ℓ_∞ problem:

$$
\begin{array}{ll}
Ax - ye \le b; & \text{Primal Feasibility.} \\
Ax + ye \ge b; & \\
A^T(u - v) = 0; & \text{Dual Feasibility.} \\
e^T(u + v) = 0; & \\
u_i(a_i^T x - b_i - y) = 0 \quad i = 1,..,m; & \text{Complementarity Slackness.} \\
v_i(a_i^T x - b_i + y) = 0 \quad i = 1,..,m. &
\end{array}
$$

The entries of the matrix A, the vector x and the scalar y are chosen from a uniform distribution. The entries of u and v, however, are chosen to satisfy dual feasibility, and b is selected so that complementarity slackness holds. A brief description is provided below.

1. Randomly generate A, x and y.
2. Choose a set of μ components among the $2m$ components of u and v so that for a given index i only u_i or v_i is included. Let it include u_m or v_m. And let $\mathcal{F}$ be the set of the corresponding indices.
3. For $i \in \mathcal{F}$, u_i (or v_i) $= \frac{1}{\nu}$, else zero.
4. Set $a_m = \sum_{i=1..m-1}(u_i - v_i)a_i/(u_m - v_m)$.
5. If $u_i \neq 0$ then $b_i = a_i^T x - y$.
6. If $v_i \neq 0$ then $b_i = a_i^T x + y$.
7. If $u_i = 0$ and $v_i = 0$ then $b_i = a_i^T x$.

Note that at the third and fourth steps, primal feasibility is satisfied, at the fifth and sixth steps, complementarity slackness is satisfied, and at the last step, primal feasibility is satisfied. The parameter ν is used to generate dual degenerate problems with *ddeg* degrees. That is to say, if ν is set to $n - ddeg + 1$, then the generated problem has a solution that has $n - ddeg + 1$ nonzero dual multipliers. The case when $ddeg = 0$ corresponds to a dual nondegenerate problem. Primal degeneracy, on the other hand, is forced through the choice of an additional number of *pdeg* equations to be satisfied as equality at the optimal solution. This is achieved, by changing step 7 of the problem generator, so that *pdeg* b_i's are set to $a_i^T x - y$ or to $a_i^T x + y$. Finally, it is believed that matrices generated with random elements tend to be well conditioned.

The algorithm is also tested through a set of function estimation problems. This is due to the fact that function estimation provides an important source of test problems for ℓ_∞ algorithms. The problem is to estimate a certain function $f(x)$ by a polynomial of degree $n-1$ on a set of m evenly spaced points over an interval $[\xi_1 \ \ \xi_2]$ of length ξ. The estimation is done through the determination of the coefficients of the polynomial. Explicitly, we consider,

$$f(\mu) \ = \ \sum_{j=1}^{n} x_j \mu^{j-1}, \quad \mu = 0, \frac{\xi_2 - \xi_1}{m}, ..., \xi_2 - \xi_1.$$

Obviously,

$$b_i = f(\tfrac{i \times \xi}{m}), \quad \text{for} \quad i = 1, ..., m.$$
$$a_{ij} = (\tfrac{i \times \xi}{m})^{j-1} \quad \text{for} \quad i = 1, ..., m, \text{ and } j = 1, ..., n.$$

6.2 Behavior of the algorithm

In this section, the viability of the approach and the stability of the method are tested. For the problems in this section, the starting solution is the least squares one and the initial threshold value is 0.1. The problem set is chosen to represent problems of different sizes. First, the accuracy of the algorithm is examined. Table 1 gives an insight about the accuracy of the solution provided by the algorithm. The difference between the actual objective value (r^*_{max}) and the one provided by the algorithm ($\bar{r}_{max}$) and the deviation of the actual solution vector x^* from that calculated $\bar{x}$ are given by ,

$$Obj_{err} \ = \ \frac{|r^*_{max} \ - \ \bar{r}_{max}|}{1 + |r^*_{max}|}, \qquad X_{err} = \frac{||x^* \ - \ \bar{x}||_2}{1 + ||x^*||_2}.$$

N	M	X_{err}	Obj_{err}	N	M	X_{err}	Obj_{err}
10	15	3.16D-16	3.67D-15	100	150	1.17D-13	4.53D-14
10	30	1.67D-13	3.16D-16	100	200	3.29D-13	1.14D-13
30	40	2.09D-15	3.59D-14	100	300	7.87D-14	1.33D-14
30	60	8.92D-15	4.66D-15	100	400	6.96D-13	9.60D-15
50	60	3.31D-15	1.19D-13	200	300	1.89D-13	6.54D-12
50	110	3.11D-14	8.41D-14	200	400	5.25D-14	6.94D-12
50	200	2.84D-13	3.92D-15	300	400	1.33D-13	2.41D-12

Table 1: Solution Accuracy of the algorithm.

It can be seen that the algorithm provides a considerably accurate solution, as Obj_{err} and X_{err} are at most 10^{-12}. This actually means that the obtained solution agrees with the actual one in at least 10 digits, as r^*_{max} is generated between 0 and 20 and x^* lies between -5 and 5. This fact shows that the approach is practically accurate and the method is numerically precise.

Next, the algorithm was run on a set of degenerate problems. Tables 2 - a and 2 - b records the CPU times (measured in seconds) and iterations counts for the same set of problems. The solution is chosen to be nondegenerate, primal degenerate with $n + 1 + pdeg$ active equations (satisfied as equality) at the optimal solution, dual degenerate with $n + 1 - ddeg$ nonzero dual multipliers, or primal-dual degenerate with $pdeg$ additional active equations and $ddeg$ less nonzero dual multipliers. In each case, $pdeg$ and $ddeg$ are taken to be the integer parts of $\frac{m-n}{2}$ and $\frac{n}{2}$ respectively.

N	M	Pde	Dde	Ite	CPU	N	M	Pde	Dde	Ite	CPU
10	15	0	0	4	0.036	10	15	2	0	6	0.044
10	30	0	0	3	0.030	10	30	10	0	7	0.053
30	40	0	0	4	0.107	30	40	5	0	7	0.123
30	60	0	0	3	0.103	30	60	15	0	6	0.168
50	60	0	0	4	0.293	50	60	5	0	10	0.383
50	110	0	0	3	0.369	50	110	30	0	6	0.560
50	200	0	0	43	1.685	50	200	75	0	11	1.322
100	200	0	0	5	2.766	100	200	50	0	9	3.239
100	150	0	0	3	1.635	100	150	25	0	9	2.346
100	300	0	0	3	3.060	100	300	100	0	12	5.842
100	400	0	0	41	7.285	100	400	150	0	15	8.418
200	300	0	0	5	17.887	200	300	50	0	10	16.099
200	400	0	0	3	11.741	200	400	50	0	9	15.527
300	400	0	0	5	49.963	300	400	50	0	10	44.515

Table 2-a: Behavior of the algorithm on nondegenerate and primal degenerate problems.

N	M	Pde	Dde	Ite	CPU	N	M	Pde	Dde	Ite	CPU
10	15	0	5	4	0.037	10	15	2	5	4	0.032
10	30	0	5	3	0.031	10	30	10	5	6	0.048
30	40	0	15	5	0.106	30	40	5	15	19	0.213
30	60	0	15	3	0.102	30	60	15	15	6	0.181
50	60	0	25	5	0.263	50	60	5	25	18	0.574
50	110	0	25	3	0.357	50	110	30	25	38	1.260
50	200	0	25	25	1.205	50	200	75	25	13	1.464
100	200	0	50	3	2.143	100	200	50	50	70	7.899
100	150	0	50	3	1.452	100	150	25	50	48	5.513
100	300	0	50	3	2.904	100	300	100	50	63	10.414
100	400	0	50	52	8.722	100	400	150	50	21	9.635
200	300	0	100	5	12.635	200	300	50	100	156	70.647
200	400	0	100	3	10.335	200	400	50	100	156	68.034
300	400	0	150	5	37.558	300	400	50	150	354	304.497

Table 2-b: Behavior of the algorithm on dual degenerate and primal-dual degenerate problems.

In all these runs, the solution agrees in at least eight digits with the actual one. This further supports the fact that the algorithm has a reasonably high accuracy. In addition to that, the numerical stability of the algorithm was not affected which shows its potential to deal with degenerate problems. The speed of the algorithm, however, differed from one case to an other. The CPU times which are graphed in figure 6.1, demonstrate this. The algorithm spends the least time on dual degenerate problems, then nondegenerate problems then primal degenerate problems then primal-dual degenerate problems.

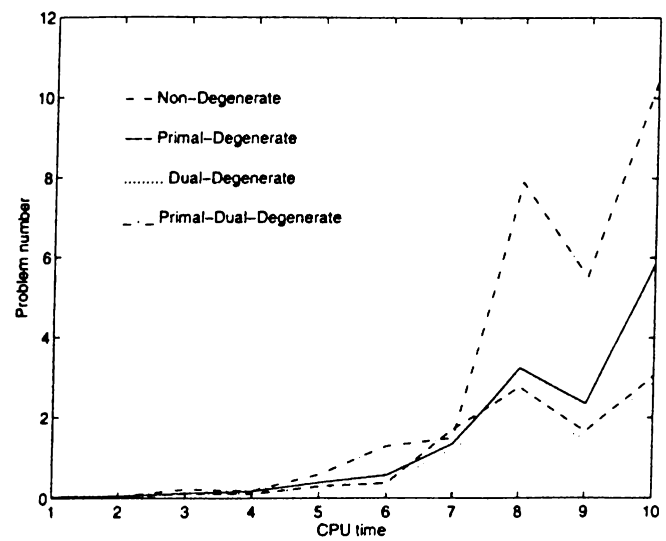

Figure 6.1: CPU time on nondegenerate, primal, dual and primal-dual degenerate problems.

Now, we closely investigate the time spent by the algorithm on the different subroutines. We sum the CPU time and the iteration count for all the previously mentioned random problems. The starting point is chosen to be the least squares solution (to be defined later) and the initial t value is set to 0.1. The statistics are provided in table 3.

	CPU Time	Iteration Count	% CPU
Initialization	26.3018	14	35.986
Threshold reduction	1.5973	25	2.185
Factorization	42.0282	43	57.504
Search direction	2.8503	54	3.900
Kink values	0.0250	14	0.0342
Line search	0.0054	14	0.007
Update Residuals	0.2348	64	0.321
Full step check	0.0443	28	0.060

Table 3 : CPU time and Iteration count spent on different parts of the algorithm.

It is clear that the algorithm spends most of its time in factoring, which accounts for about 50% of all the CPU time, and is the major work done by the algorithm. This, makes the algorithm similar to Interior-point algorithms. Actually, both methods need to solve a least squares system at each iteration. However, Marsten et al. [10] claim that least squares solutions consume about 90% of computational time in Interior-point methods.

The second major consumer of CPU time is the initialization procedure. This is the case, because the algorithm makes the first factorization in the initial procedure, and may not need to refactorize in subsequent steps, but only updates or downdates the needed matrix. In addition to that, experience showed that least squares solutions are very good starting points, and the algorithm only performs few steps after that, and thus the difference between the time spent at the initialization and that spent at the factorization is not big. Instead, if we start with the origin, we do not need to make a factorization at the beginning, and the algorithm performs more iterations and refactorizations, making the difference between the initialization time and the factorization time very significant.

6.3 Initialization

The performance of the algorithm highly depends upon the starting point (x_0, y_0) and the initial parameter t_0. Based on the necessary optimality conditions,

$$\begin{bmatrix} -A^T & A^T \\ e^T & e^T \end{bmatrix} \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix} \begin{bmatrix} -A & e \\ A & e \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = -\begin{bmatrix} 0 \\ t \end{bmatrix} + \begin{bmatrix} -A^T & A^T \\ e^T & e^T \end{bmatrix} \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix} \begin{bmatrix} b \\ b \end{bmatrix}, \tag{6.1}$$

we will try to choose a starting point (x, y) and an initial t value that are close to the optimal ones. It can be seen that an initial (x_0, y_0) and t_0 depends on the choice of the region from which the initial point is chosen, this is done through the choice of Θ_1 and Θ_2. Actually, it is very hard to guess a starting region that is close to the final region. And thus our choice will be guided solely by experience.

For the purpose of comparing the different initialization schemes, a set of 14 randomly generated test problems were used with different sizes. These problems are all dense. The compiler and running conditions are similar for all trials.

As a first trial, the origin is chosen to be the initial starting point and t_0 is set to 0.1. The results are displayed in table 5

N	M	Ite	Ref	Red	CPU
10	15	26	5	1	0.152
10	30	18	4	1	0.106
30	40	61	8	1	0.656
30	60	139	14	1	1.554
50	60	115	11	1	2.317
50	110	383	38	1	8.885
50	200	302	29	1	9.168

N	M	Ite	Ref	Red	CPU
100	150	718	66	1	56.375
100	200	617	66	1	55.106
100	300	928	94	1	92.372
100	400	933	87	1	103.240
200	300	1396	158	1	606.160
200	400	1289	161	2	628.259
300	400	3317	343	2	4028.464

Table 5: Statistics of the algorithm for $z_0 = \mathbf{0}$ and $t_0 = 0.1$.

It is clear that the number of iterations and so the number of refactorizations is very high, indicating that the origin is not a good starting point.

Next, a solution to equation (6.1) was sought for previously chosen values of Θ_1 and Θ_2. Table 6 and table 7 display the results for $\Theta_1 = \mathbf{0}$, $\Theta_2 = I$ and $\Theta_1 = I$, $\Theta_2 = \mathbf{0}$ respectively.

N	M	Ite	Ref	Red	CPU
10	15	22	5	1	0.127
10	30	41	9	1	0.251
30	40	92	12	1	0.969
30	60	136	15	1	1.572
50	60	41	7	1	1.006
50	110	208	23	1	5.016
50	200	357	36	1	10.864

N	M	Ite	Ref	Red	CPU
100	150	965	87	1	76.174
100	200	743	73	1	65.492
100	300	757	78	1	77.920
100	400	979	87	1	110.415
200	300	1321	137	1	556.245
200	400	1028	122	1	498.475
300	400	3458	349	3	3972.380

Table 6: Statistics of the algorithm for $\Theta_1 = \mathbf{0}$, $\Theta_2 = I$ and $t_0 = 0.1$.

N	M	Ite	Ref	Red	CPU
10	15	15	5	1	0.095
10	30	16	6	1	0.106
30	40	114	11	1	1.163
30	60	178	18	1	2.032
50	60	239	25	1	4.720
50	110	219	25	1	5.463
50	200	340	34	1	10.58

N	M	Ite	Ref	Red	CPU
100	150	556	49	1	43.500
100	200	994	94	1	82.841
100	300	1134	109	1	110.34
100	400	842	82	1	94.395
200	300	1502	164	1	636.86
200	400	1297	173	1	663.2
300	400	2872	285	1	3245

Table 7: Statistics of the algorithm for $\Theta_1 = I$, $\Theta_2 = \mathbf{0}$ and $t_0 = 0.1$.

Again, the performance of the algorithm was not much improved as the number of iterations is still high, especially for problems of big dimensions. The tables also reveal that the number of factorizations performed is about 10% of the number of iterations. This is a general feature of the algorithm which indicates the importance of the updating and downdating procedure in reducing the refactorization time, and thus the overall computation time.

Finally, the least squares solution was tried, this corresponds to $\Theta_1 = \Theta_2 = I$ and $t = 0$ in (6.1). The resulting system is $A^TAx = A^Tb$ and $e^Tey = 0$. t_0 is again chosen as 0.1. The results are displayed in table 8.

N	M	it.	ref.	red.	CPU
10	15	4	3	1	0.034
10	30	5	2	2	0.045
30	40	4	3	1	0.105
30	60	5	2	2	0.146
50	60	4	3	1	0.293
50	110	5	2	2	0.493
50	200	5	2	2	0.800

N	M	it.	ref.	red.	CPU
100	150	5	3	2	2.472
100	200	5	2	2	2.676
100	300	5	2	2	3.781
100	400	5	2	2	4.811
200	300	5	3	2	17.951
200	400	5	2	2	19.257
300	400	5	3	2	51.206

Table 8: Statistics of the algorithm for $\Theta_1 = I$, $\Theta_2 = I$ and $t_0 = 0.1$.

It is clear from the previous experiments that the least-squares solution provides a good starting point for the algorithm, as the iteration count, the number of refactorizations and reductions and the CPU time are significantly improved. Actually, the number of iterations and refactorizations is nearly constant for all the problems. A preliminary explanation could be based on the fact that the algorithm is an infeasible one which satisfies primal feasibility only upon termination, and the least squares solution, $x_0 = (A^TA)^{-1}A^Tb$, $y_0 = 0$ is the solution that minimizes the primal feasibility gap. Explicitly, $Ax_0 - b - y = Ax_0 - b + y = Ax_0 - b = A(A^TA)^{-1}A^Tb - b$ which satisfies $||Ax_0 - b||_2^2 \le ||Ax - b||_2^2$ for all $x \in R^n$.

In what follows, we will concentrate on effectively using the least squares solution as a starting point. Equation (6.1) reduces to $A^TAx_0 = A^Tb$ and $e^Tey_0 = t_0$ when $\Theta_1 = \Theta_2 = I$. It is believed that t_0 is related to the absolute smallest nonzero residual at the optimal solution. So, the choice of t_0 can be calculated from y_0 as $\beta \times m \times y_0$, where $0 \le \beta \le 1$, and y_0 being derived from the residuals at a certain iteration. Setting y_0 to a value different from zero, leads to a change in the region chosen for the least squares solution. In order to preserve that y_0 should not be too large.

The first experiment is done using y_0 to be the absolute value of smallest entry of the vector $Ax_0 - b$ and $\beta = 1$, the results are displayed in table 9.

N	M	Ite	Ref	Red	CPU
10	15	4	3	1	0.037
10	30	3	1	1	0.035
30	40	4	3	1	0.112
30	60	3	1	1	0.124
50	60	4	3	1	0.322
50	110	3	1	1	0.419
50	200	3	1	1	0.723

N	M	Ite	Ref	Red	CPU
100	150	3	2	1	2.141
100	200	3	1	1	2.293
100	300	3	1	1	3.358
100	400	3	1	1	4.418
200	300	3	2	1	15.684
200	400	3	1	1	17.164
300	400	3	2	1	46.331

Table 9: Statistics for y_0=smallest entry of $Ax_0 - b$ and $\beta = 1$.

In this case, the region is different from that for the least-squares solution. Actually, the first m inequalities are satisfied and the rest are violated. To see that, let us recall that,

$$a_i^T x_0 - b_i \leq 0 \ \ i = 1, ..., m \ \ \wedge \ \ y_0 = \min_{i=1,...m} \{b_i - a_i^T x_0\},$$

then $b_i - a_i^T x_0 \geq -y_0$ for $i = 1, ..., m$ and $b_i - a_i^T x_0 \geq y_0$ for $i = m+1, ..., 2m$ which implies that

$$a_i^T x_0 - b_i - y_0 \leq 0, \quad i = 1, ..., m \quad \wedge \quad a_i T x_0 - b + y_0 \leq 0 \quad i = m+1, ..., 2m.$$

In the second experiment, y_0 was chosen to be the absolute value of the $\frac{m}{2}$-th smallest entry of $Ax_0 - b$, so that by the previous analysis, half of the inequalities is satisfied on both sides of the interval $[-b \ \ b]$ and the other half is violated. Again, β is 0.1. Table 10 reports the results.

N	M	Ite	Ref	Red	CPU
10	15	4	3	1	0.032
10	30	3	2	1	0.026
30	40	4	3	1	0.103
30	60	3	2	1	0.102
50	60	4	3	1	0.294
50	110	3	2	1	0.373
50	200	3	2	1	0.640

N	M	Ite	Ref	Red	CPU
100	150	5	3	2	2.486
100	200	3	2	1	2.096
100	300	3	2	1	3.103
100	400	3	2	1	4.012
200	300	5	3	2	17.610
200	400	3	2	1	15.065
300	400	5	3	2	52.252

Table 10: Statistics for $y_0 = \frac{m}{2}$-th smallest entry of $Ax_0 - b$ and $\beta = 0.1$.

Table 11 and table 12 report the results for y_0 being the absolute value of the m-th smallest entry of $Ax_0 - b$, $\beta = 0.1$ and y_0 being the absolute value of the n-th smallest entry of $Ax_0 - b$, $\beta = 0.1$ respectively. In the first case, the first m equalities are violated and the rest are satisfied. In the second case, all the inequalities are satisfied.

The results reported in tables 9, 10, 11 and 12 are plotted in figure 6.2. The CPU times are normalized so that it is in the range [1 10], for example for the 10 × 15 problem the CPU time was multiplied by 100. The purpose from that is to make the graph look better. The information it contains will not be affected as we are interested in seeing the initialization scheme that outperforms the others.

N	M	Ite	Ref	Red	CPU
10	15	4	3	1	0.037
10	30	3	2	1	0.029
30	40	4	3	1	0.104
30	60	3	2	1	0.103
50	60	4	3	1	0.291
50	110	3	2	1	0.366
50	200	3	2	1	0.634

N	M	Ite	Ref	Red	CPU
100	150	5	3	2	2.435
100	200	3	2	1	2.065
100	300	3	2	1	3.072
100	400	3	2	1	4.043
200	300	5	3	2	17.622
200	400	3	2	1	15.192
300	400	5	3	2	52.144

Table 11: Statistics for $y_0 = m$-th smallest entry of $Ax_0 - b$ and $\beta = 0.1$.

N	M	Ite	Ref	Red	CPU
10	15	4	3	1	0.033
10	30	3	2	1	0.027
30	40	4	3	1	0.105
30	60	3	2	1	0.103
50	60	4	3	1	0.290
50	110	3	2	1	0.373
50	200	43	2	21	1.766

N	M	Ite	Ref	Red	CPU
100	150	5	3	2	2.487
100	200	3	2	1	2.062
100	300	3	2	1	3.078
100	400	11	2	5	4.713
200	300	5	3	2	17.714
200	400	3	2	1	15.198
300	400	5	3	2	52.257

Table 12: Statistics for $y_0 = n$-th smallest entry of $Ax_0 - b$ and $\beta = 0.1$.

Figure 6.2 reveals that the four schemes perform similarly on most of the problems. The only exception is the performance of the fourth approach on the 50×200 random problem. Experience suggests the use of y_0 to be either the $\frac{m}{2}$-th or the n-th smallest nonzero entry of $Ax_0 - b$. For the case when y_0 is set to the smallest nonzero entry of $Ax_0 - b$, the initial t_0 value may be very small, causing the algorithm to crash. This may happen for near-degenerate problems where there is a very small nonzero residual at the solution. Furthermore, if y_0 is set to the m-th smallest residual (biggest), the algorithm may need to do a big number of reductions until it gets to the optimal region. From now on, we choose to use y_0 as the $\frac{m}{2}$-th smallest entry of $Ax_0 - b$ and t_0 to be $0.1 \times m \times y_0$, as this choice seems to slightly outperform the case when $y_0 = n$-th smallest nonzero entry of $Ax_0 - b$.

6.4 Comparison

In this section we compare the performance of the algorithm with the Barrodale and Phillips algorithm (B-P), which is believed to be the most efficient existing algorithm for solving ℓ_∞ problems. The comparison is done in terms of iteration count, CPU time and the ability to deal with highly degenerate problems. As described earlier the B-P algorithm is a modified simplex algorithm that exploits the special structure of the coefficient matrix. The comparison is made on a number of random and function approximation problems. The new algorithm (LINFSOL) uses the least squares solution as a starting point, with y being set to the $\frac{m}{2}$-th nonzero smallest residual and t_0 taken as $0.1 \times m \times y_0$.

The comparison is first made on a set of random problems of different sizes. All the combinations of $n = 10, (10), 390$ and $m = 20, (10), 400$ were performed provided that $n < m$. This resulted in 780 problems . A full report of the results is found in the appendix. Figure 6.3 gives a plot of the ratio of the CPU time of the (B-P) algorithm to the CPU time of the LINFSOL algorithm, for all the problems.

According to the figure, there is no dominance of one algorithm over the other. That is to say, no algorithm outperforms the other on the full set of problems. Among the 780 tests, LINFSOL is faster than B-P on 502 (64.5%) and slower on 278 (35.5%)of the problems. The figure also reveals that there is a certain dependency of the ratio of the CPU times of B-P and LINFSOL to the ratio of m and n. Figure 6.4 clarifies this fact.

It is clear that the B-P algorithm is more efficient when m and n are close to each other, while LINFSOL is better for problems when m is at least the double of n. This fact is related to the approach behind the two algorithms. LINFSOL evolves the exterior of the feasible region and is insensitive to the problem dimensions, whereas B-P is a simplex algorithm that moves from one vertex to an other along the boundary of the feasible region, and is forced to perform more iterations when the number of vertices increases. This is the reason why the LINFSOL's iteration count is nearly constant for all problems and does not exceed 10, while B-P 's varies directly with the dimensions of the problem. Real-life problems are usually very overdetermined, which makes LINFSOL suitable for practical situations.

Next, primal, dual and primal-dual degenerate problems were solved by the two algorithms. The degree of primal degeneracy is $\lfloor \frac{m-n}{2} \rfloor$ and that of dual degeneracy is $\lfloor \frac{n}{2} \rfloor$. The starting point is the least squares and the initial threshold is 0.1. Figures 6.5 , 6.6, 6.7, 6.8, 6.9, 6.10 display the ratios of the CPU times of B-P to LINFSOL and to the ratio of m and n on primal, dual and primal-dual degenerate problems.

On primal degenerate problems, B-P and LINFSOL are again comparable in performance with a slight difference in favor of B-P. Among the 780 test problems, LINFSOL spends less time on 343 (44%) instances and more time on 437 (56.03) instances. However, on primal-dual degenerate problems B-P is more efficient than LINFSOL, especially when m and n either get closer or farther from each other. In fact, B-P outperforms LINFSOL 90% of the time (703 vs 77). On the other hand, for dual degenerate problems, LINFSOL is nearly outperforming B-P on all the problems regardless of the ratio of m and

n. LINFSOL is faster on 711 91% problems among the 780.

Actually, these results are expected because LINFSOL is a primal algorithm, to which primal degeneracy is an obstacle and dual degeneracy is not, and B-P is a dual algorithm that can suffer under dual degeneracy . In Primal-dual degenerate cases, the only explanation may be that, the degree of primal degeneracy is usually greater than that of dual degeneracy. In other words, the particular choice of *pdeg* and *ddeg* makes primal degeneracy more effective than dual degeneracy.

The second set of test problems is function approximation problems. For that purpose we choose to approximate the *exponential, the square-root* and the *sine* functions on the interval [0 1]. Tables 13, 14 and 15 provide the computational results for the exponential, the square root and the sine functions respectively. The starting point is the least-squares and the initial t_0 is set to 10^{-4}.

		B-P		LINFSOL				Rat
N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat
3	50	6	0.00	0.12	10	6	2	0.06
4	50	7	0.01	0.09	13	5	4	0.11
5	50	8	0.01	0.16	20	6	7	0.06
6	50	11	0.01	0.33	35	8	12	0.03
7	50	14	0.02	0.31	31	7	10	0.06
8	50	15	0.03	0.19	13	4	3	0.16
3	100	7	0.02	0.19	13	5	3	0.11
4	100	8	0.02	0.20	14	6	3	0.10
5	100	9	0.03	0.30	23	8	7	0.10
6	100	11	0.04	0.56	38	8	14	0.07
7	100	16	0.05	0.57	28	8	6	0.09
8	100	16	0.06	0.51	20	7	3	0.12
3	150	7	0.03	0.32	17	9	3	0.09
4	150	8	0.03	0.30	17	8	4	0.10
5	150	10	0.04	0.60	32	9	11	0.07
6	150	11	0.06	0.73	37	10	15	0.08
7	150	16	0.06	1.05	35	11	5	0.06
8	150	17	0.08	0.91	23	6	6	0.09

		B-P		LINFSOL				Rat
N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat
3	200	7	0.04	0.41	18	9	3	0 10
4	200	8	0.04	0.56	27	10	7	0.07
5	200	10	0.08	0.85	40	10	13	0 09
6	200	13	0.07	1.33	56	9	24	0.05
7	200	17	0.10	1.65	41	13	4	0.06
8	200	18	0.12	1.28	25	7	4	0.09
3	250	7	0.04	0.59	23	11	3	0.07
4	250	9	0.06	0.75	28	10	7	0.08
5	250	10	0.07	1.14	42	12	12	0.06
6	250	11	0.09	1.20	37	9	14	0.08
7	250	17	0.13	2.16	46	14	4	0.06
8	250	18	0.14	2 02	32	10	4	0.07
3	300	7	0.05	0.73	25	13	3	0.07
4	300	8	0.08	0.95	31	13	6	0.08
5	300	10	0.11	1.27	39	13	10	0.09
6	300	14	0.12	1.45	35	12	11	0.08
7	300	18	0.16	3.10	53	17	3	0.05
8	300	19	0.19	2.63	35	11	3	0.07

Table 13: CPU times for the approximation of the exponential function.

Clearly, B-P outperforms LINFSOL. This is mainly due to three factors. First, the coefficient matrix in function approximation problems tends to be very ill-conditioned. Second, the solution is usually nearly primal degenerate, i.e. the residuals at the solution are in the order of 10^{-7} , requiring a very small threshold value (Actually, in [8], it was shown that t_0 is related to the smallest absolute nonzero residual at the optimal solution). These two factors force LINFSOL to make a lot of iterations, and sometimes it stops because the threshold is very small and not because the duality gap is closed.

Third, the B-P algorithm generates a very good starting point for function approximation problems which is very close to the optimal solution. Actually, Bartels, Conn and Li [3] prove that the high efficiency of the B-P algorithm on function approximation problems is not due the superiority of the approach but rather to the special structure of the function approximation problems. They show that such problems are characterized by a *sign alternating property* which says that at an optimal solution, there are $n+1$ equations whose residuals $|b_i - a_i^T x|$ corresponds to the maximum residual $||Ax-b||_\infty$ and with error terms $(b_i - a_i^T)$ alternating sign as the index moves from 1 to $n+1$. Furthermore, they demonstrate that after the first stage (finding the first vertex) B-P finds a solution that satisfies these properties.

The first two factors weaken the performance of LINFSOL and the third factor increases the efficiency of B-P resulting in a big gap between the computational times of the two algorithms.

		B-P		LINFSOL				
N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat
3	50	5	0.01	0.07	12	6	1	0.14
4	50	7	0 01	0.09	11	6	1	0.11
5	50	8	0.02	0.07	9	5	1	0.29
6	50	12	0.01	0.09	10	6	1	0.11
7	50	14	0.03	0.16	21	8	1	0.19
8	50	15	0.03	0.20	19	6	2	0.15
3	100	7	0.01	0.15	12	6	1	0.07
4	100	9	0.02	0.24	20	7	3	0.08
5	100	9	0.03	0.21	16	6	2	0.14
6	100	12	0.03	0.30	20	8	2	0.10
7	100	18	0.06	0.24	14	7	1	0.25
8	100	14	0.06	0.32	18	7	2	0.19
3	150	7	0.03	0.22	14	9	1	0.14
4	150	8	0.04	0.31	17	7	2	0.13
5	150	11	0.05	0.40	22	10	2	0.12
6	150	13	0.06	0 40	18	10	2	0.15
7	150	18	0.08	0.42	18	7	2	0.19
8	150	16	0.08	0.49	20	10	1	0.16

		B-P		LINFSOL				Rat
N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat
3	200	7	0.03	0.36	18	10	2	0.08
4	200	8	0.06	0.48	24	9	2	0.13
5	200	10	0.07	0.56	21	10	2	0.12
6	200	12	0.08	0.60	23	9	2	0.13
7	200	19	0.11	0.64	23	10	2	0.17
8	200	14	0.09	0.73	25	11	2	0.12
3	250	7	0.05	0.56	25	11	3	0.09
4	250	8	0.06	0.59	22	11	2	0.10
5	250	10	0.08	0.69	23	12	2	0.12
6	250	12	0.08	0.78	24	11	2	0.10
7	250	18	0.15	0.86	22	10	2	0.17
8	250	16	0.13	0.95	24	10	2	0.14
3	300	7	0.07	0.70	26	12	3	0.10
4	300	8	0.07	0.89	30	11	4	0.08
5	300	10	0.09	0.90	26	11	3	0.10
6	300	12	0.10	1.06	29	14	2	0.09
7	300	18	0.17	1 20	25	13	2	0.14
8	300	15	0.16	1.38	31	12	3	0.12

Table 14: CPU times for the approximation of the square root function.

		B-P		LINFSOL							B-P		LINFSOL				Rat
N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat	N	M	Itr	CPU	CPU	Itr	Rf	Rd	Rat
3	50	6	0.01	0.08	10	6	2	0.13	3	200	7	0.04	0.39	19	10	3	0.10
4	50	9	0.01	0.13	14	5	4	0.08	4	200	8	0.05	0.52	24	10	6	0.10
5	50	9	0.02	0.15	17	6	6	0.13	5	200	11	0.06	1.03	46	11	16	0.06
6	50	12	0.02	0.42	50	6	18	0.05	6	200	14	0.10	2.37	94	12	36	0.04
7	50	15	0.02	0.89	98	7	44	0.02	7	200	15	0.10	1.95	50	14	9	0.05
8	50	16	0.02	0.21	14	4	4	0.10	8	200	18	0.13	1.20	25	6	6	0.11
3	100	7	0.01	0.16	14	5	3	0.06	3	250	7	0.04	0.54	22	11	3	0.07
4	100	8	0.02	0.27	21	6	6	0.07	4	250	8	0.06	0.84	32	11	8	0.07
5	100	10	0.03	0.38	28	8	10	0.08	5	250	10	0.08	1.18	42	12	13	0.07
6	100	14	0.05	1.13	82	8	34	0.04	6	250	14	0.11	2.07	61	11	21	0.05
7	100	15	0.04	1.24	79	10	30	0.03	7	250	17	0.14	2.60	56	16	9	0.05
8	100	16	0.06	0.51	20	6	3	0.12	8	250	18	0.13	1.56	27	5	6	0.08
3	150	7	0.04	0.33	16	9	3	0.12	3	300	7	0.05	0.77	27	12	4	0.06
4	150	8	0.04	0.40	25	8	7	0.10	4	300	9	0.08	1.07	33	12	8	0.07
5	150	10	0.05	0.50	28	9	9	0.10	5	300	10	0.09	1.67	49	14	15	0.05
6	150	14	0.06	1.86	96	12	38	0.03	6	300	14	0.13	3.10	88	11	35	0.04
7	150	16	0.07	1.28	47	11	11	0.05	7	300	16	0.15	3.36	59	18	6	0.04
8	150	16	0.07	0.97	26	9	3	0.07	8	300	18	0.20	1.85	25	6	5	0.11

Table 15: CPU times for the approximation of the sine function.

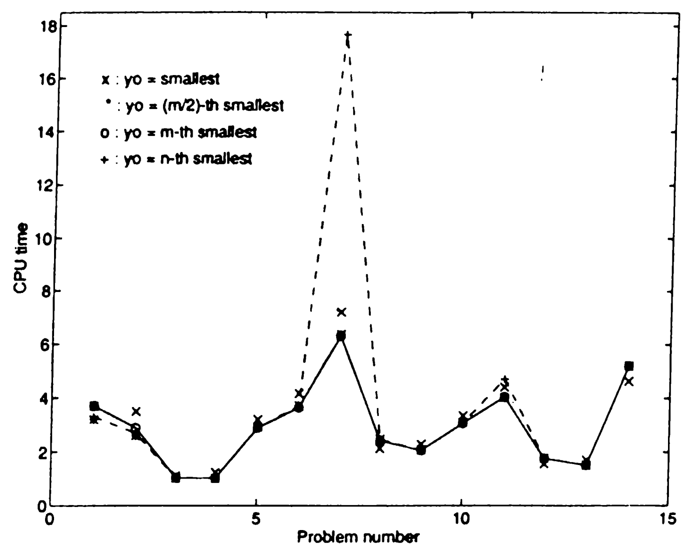

Figure 6.2: Comparison of the different initialization schemes of t_0.

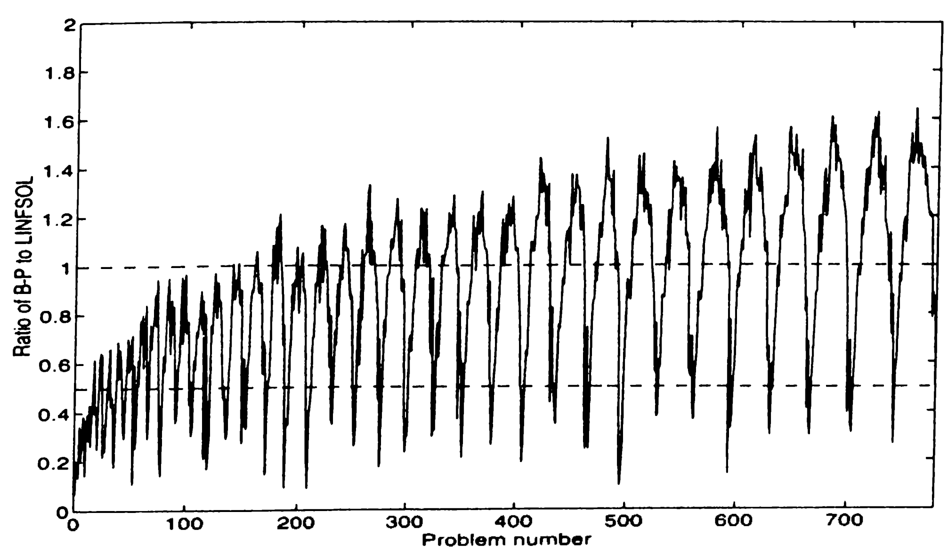

Figure 6.3: CPU time comparison between LINFSOL and B-P for nondegenerate problems.

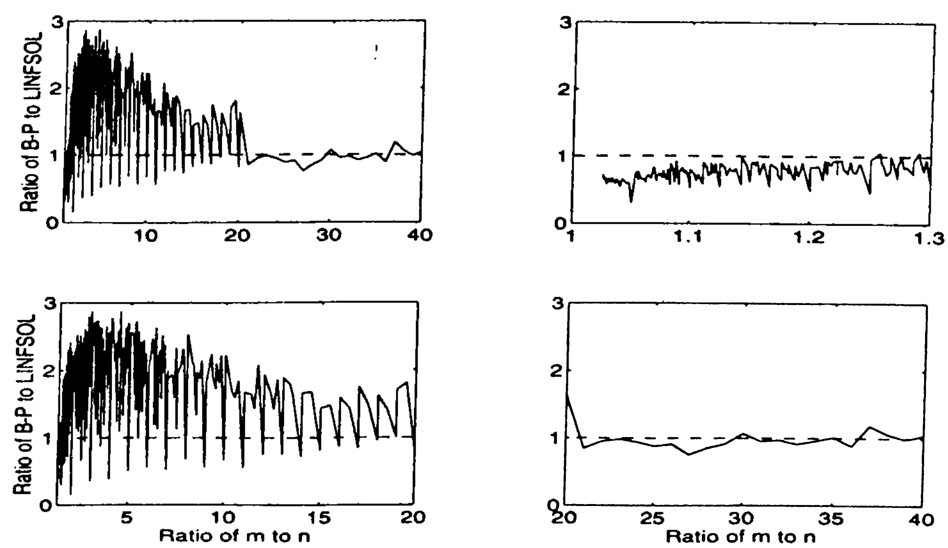

Figure 6.4: CPU time comparison between LINFSOL and B-P for different ratios of m to n on nondegenerate problems.

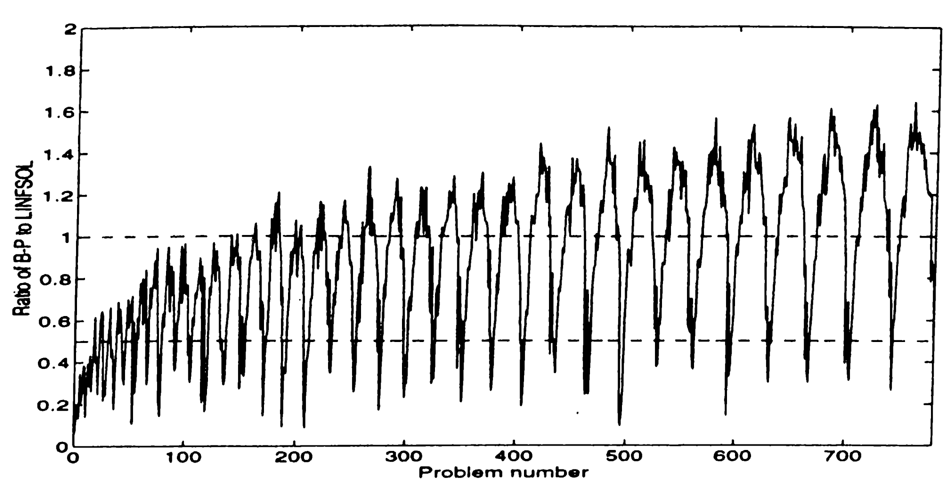

Figure 6.5: CPU time comparison for primal degenerate problems.

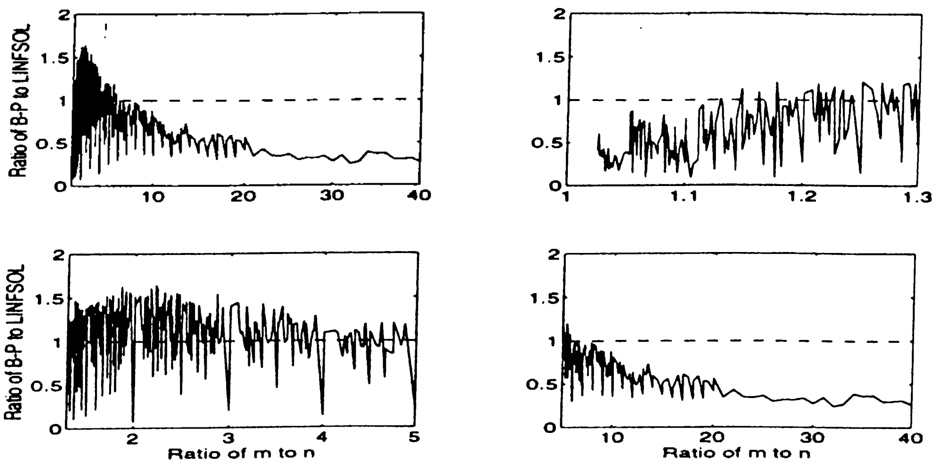

Figure 6.6: CPU time ratio versus m to n ratio for primal degenerate problems.

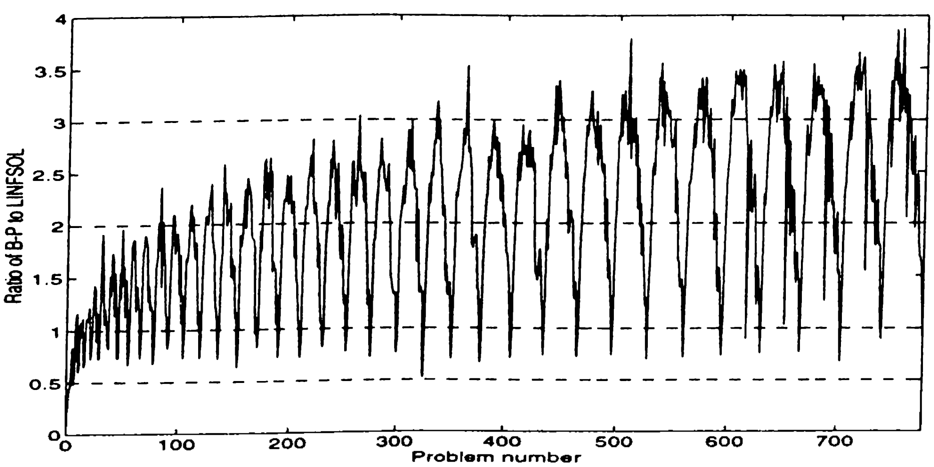

Figure 6.7: CPU time comparison for dual degenerate problems.

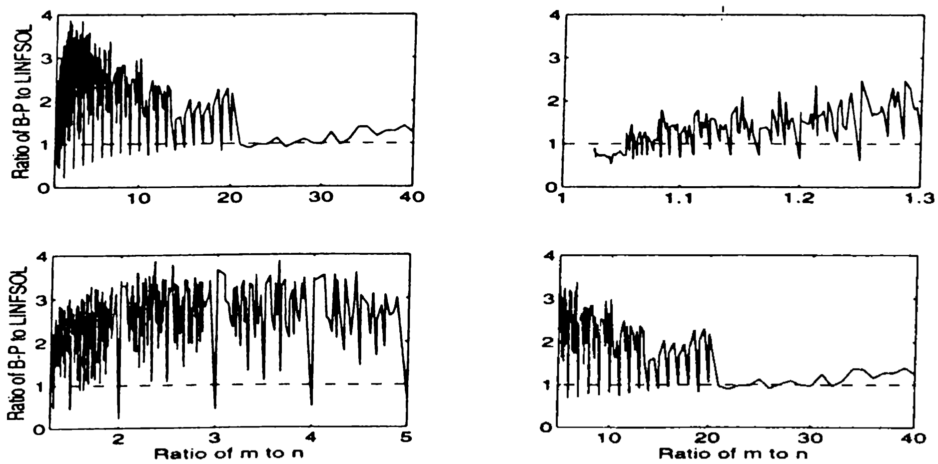

Figure 6.8: CPU time ratio versus m to n ratio for dual degenerate problems.

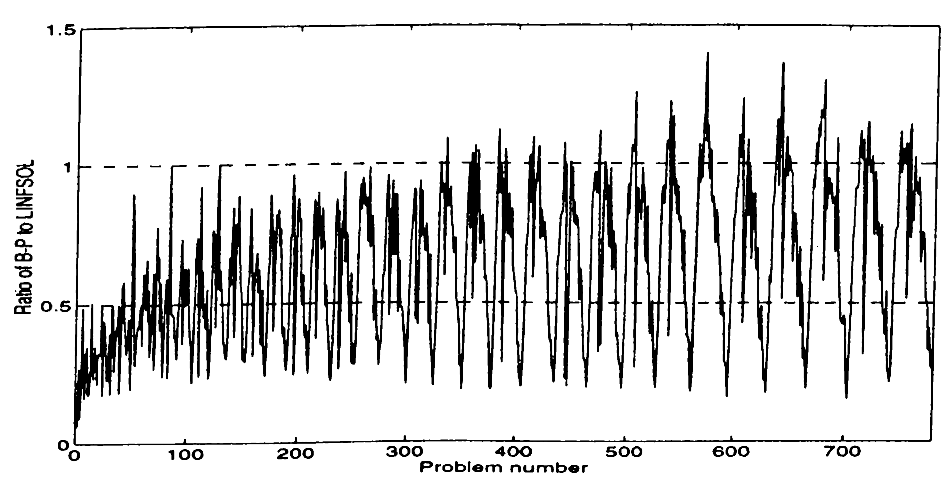

Figure 6.9: CPU time comparison for primal-dual degenerate problems.

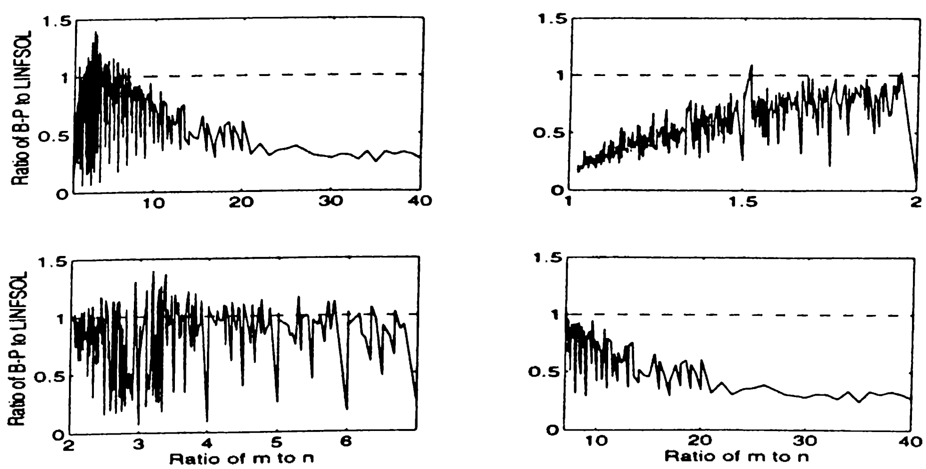

Figure 6.10: CPU time ratio versus m to n ratio for primal-dual degenerate problems.

Chapter 7

Conclusion

In this thesis, new characterizations of ℓ_∞ solutions to overdetermined linear systems was given. The approach was based on the use of a quadratic penalty function and a smoothing parameter to change the problem into an unconstrained one. It is well known that when the parameter tends to zero, a solution to the constrained problem can be got from that of the unconstrained problem. In this work, however, it is shown that the parameter does not need to converge to zero in order to find a solution to the original problem, and there is a threshold value, under which a solution can be detected.

Based on the previous important observation, a penalty algorithm was designed and its finite convergence was proved. In brief, the algorithm follows non-interior piece-wise linear pathways until it reaches the feasible region. The unconstrained solution was found using a modified Newton algorithm.

The Computational results indicate that the algorithm is stable and accurate on different kinds of problems with different levels of degeneracies. In addition to that, it was shown to outperform the Barrodale-Phillips algorithm on a wide range of random problems. On function approximation problems, however, the algorithm was poorly performing.

For future research, we would suggest to increase the performance of the algorithm for function approximation problems, either by choosing a special starting point for such class of problems, or by modifying the problem so that it gets rid of the ill-conditioning and the near-degeneracy obstacles. In addition to that, our approach could be successfully used if linear constraints are added to the problem. This only requires few changes in the algorithm. Finally, it would be interesting to have a sparse implementation of the algorithm.

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Appendix A:

A User's Guide to LINFSOL

1 Purpose

LINFSOL is a Fortran 77 subroutine designed to solve ℓ_∞ problems. The subroutine does not exploit sparsity. Therefore, it is not suitable for large problems with very sparse constraint matrices. The target problem and the description of the algorithm are provided in the main body.

2 Specification of the Subroutine LINFSOL

The heading of subroutine LINFSOL is given below.

```
      SUBROUTINE LINFSOL(M,N,LDA,A,B,START,X,
     &                            E,S,LS,W,LW,ERR)
      INTEGER M,N,LDA,START,ERR,LS,LW
      INTEGER S(LS)
      DOUBLE PRECISION A(LDA,N),B(M), C(N)
      DOUBLE PRECISION X(N), E(21), W(LW)
```

Now, we give a description of the parameters.

3 Description of Parameters

3.1 Input Parameters

`M` INTEGER.
: On entry `M` must specify the total number of constraints m. Unchanged on exit.

`N` INTEGER.
: On entry `N` must specify the number of variables n. Unchanged on exit.

`LDA` INTEGER.
: On entry `LDA` must specify the leading dimension of the constraint matrix array `A` in the program from which LINFSOL is called. Unchanged on exit.

`A DOUBLE PRECISION ARRAY OF DIMENSION (LDA,N).`
: On entry `A` is the matrix that holds the coefficients of the variable in the problem. Unchanged on exit.

`B DOUBLE PRECISION ARRAY OF DIMENSION (M).`
: On entry `B` must contain the elements of the right-hand side b in the problem. Unchanged on exit.

`START` INTEGER.
: On entry, `START` is used to specify whether a "cold" or a "warm" start is requested. If a cold start is requested `START` should be initialized to 1. A warm start is indicated by choosing a positive value greater than 1 for this parameter. See also the description of the input/output parameter S below. Unchanged on exit.

3.2 Workspace Parameters

W DOUBLE PRECISION ARRAY OF DIMENSION AT LEAST (LW).

Used as workspace for LINFSOL.

LW INTEGER.

On entry LW must specify the dimension of W. LW must be at least M (M + 5) + 4 N. Unchanged on exit.

3.3 Input/Output Parameters

S INTEGER ARRAY OF DIMENSION AT LEAST (LS)

On entry, if no prior information about the solution to the problem is available, S need not be specified. This corresponds to a "cold" start (default), and is signaled by initializing START = 1. If information about the solution is available, this could be used to speed up the execution of the algorithm. In this case, S(i) should be initialized to zero if the constraints are believed to be strictly satisfied. To indicate a warm start START should be assigned a value strictly greater than 1; see also section 6.

LS INTEGER.

On entry LS must specify the dimension of S. LS must be at least 3 N + M + 4. Unchanged on exit.

E DOUBLE PRECISION ARRAY OF DIMENSION (21).

E is an input/output parameter that is used to store information about the progress of the algorithm by collecting some statistics. On entry, E(2) should be initialized to machine precision. E(1) must hold an estimate of the accuracy of the elements in A, b, and c.

E(16) and E(17) control the printing level to standard output and to unit 7, respectively. If E(16) and E(17) are initialized to zero, the program will not print any statistics. This option is useful when no intermediate information is needed. If E(16) and E(17) are initialized to 1, the program will print iteration statistics after solving each subproblem successfully, and summary statistics on termination. Specifying E(16)(E(17)) larger than 1 will produce more statistics to be printed at each Newton iteration during the solution of the subproblems. This may lead to massive amount of output for larger problems, and should be used only for debugging and/or research purposes; see also section 4.

E(9) specifies the maximum number of iterations. This number is computed by LINFSOL using the formula 5 max(M,10). If E(9) is initialized to some positive value, this value will be used as the iteration limit.

E(18) is used to control the accuracy requested in the optimal value. It is used as a tolerance in the stopping check where the relative duality gap is checked for zero. The default value of this parameter is 10^{-8}. A different value could be used by initializing E(18) to a number in the interval $(0, 1)$. We summarize the input components of E below:

E	Meaning	Default
E(1)	used for tolerance settings	no default value
E(2)	used for tolerance settings	no default value
E(9)	maximum number of iterations	5 max(10,M)
E(16)	print level on standard output	0
E(17)	print level on unit 7	0
E(18)	zero duality gap tolerance	10^{-8}

On exit, the relevant components of E are as follows:

E	Meaning
E(10)	number of iterations
E(11)	number of refactorizations
E(13)	optimal objective function value
E(19)	number of threshold reductions
E(20)	number of active residuals at the solution
E(21)	rank of the matrix $\bar{A}^T\bar{\Theta}\bar{A}$ on termination

3.4 Output Parameter

X DOUBLE PRECISION ARRAY OF DIMENSION (NS).

On exit with ERR=0, X contains the optimal solution to [LINFLP].

3.5 Diagnostic Parameter

ERR INTEGER.

ERR is a performance indicator. On exit the possible values of ERR and their meaning are as follows:

ERR	Meaning
0	Successful run
-1	E(2)$\leq$0 or E(2) $>$ 0.01
-2	Too little room in work space
-3	Rank(A) $<$ N
-7	Other problems identified by printout on unit 7
$<$ -29	Maximum number of iterations (determined by E(9)) reached

See also section 5 for diagnostic information.

4 Description of Printed Output

The level of output produced by LINFSOL is controlled by the user (see the description of parameter E in section 3.3). When E(17) is initialized to 1 or a larger value, the following intermediate output is printed on unit 7 after the solution of each subproblem:

t is the value of the threshold t in the previous minimization of F.

`D-obj` is the value of the dual objective function at a projected dual point. This is used in the optimality test to monitor the duality gap.

`D-Obj` is the value of the primal objective function at an estimate of primal optimal solution.

`nact` is the number of active residuals at the end of the current minimization of F (i.e., the number of residuals i such that $r_{1_i} \geq 0$ or $r_{2_i} \leq 0$).

`rank` is the rank of the matrix $\bar{A}^T \bar{\Theta} \bar{A}$ corresponding to the current active residuals.

`iters` is the current cumulative number of iterations.

If the printing flag `E(17)` is initialized to 1 or a larger number a summary printout on unit 7 is produced on termination. The summary printout at the end of execution of LINFSOL is described as follows.

`Duality Gap` is an estimate of the relative duality gap on termination.

`Primal infeasibility wrt A x- ye = b` is the ℓ_∞-norm of the residual $Ax - ye - b$ on termination.

`err` is the error indicator (see the description of parameter ERR in section O 3.4).

`refac` is the number of LDL^T refactorizations performed during the execution of the program.

`reductions` is the total number of threshold reductions.

`inf. norm of x` is the ℓ_∞-norm of the optimal solution x.

If `E(17)` is initialized to 1 or a larger value, the value `X(i)` of every variable is also printed to unit 7 on termination.

5 Warnings, Error Indicators and Error Recovery

LINFSOL produces some warning and error messages when it detects a problem during the execution. The purpose of the warning messages is to inform the user about special conditions encountered during the execution of the algorithm. The algorithm may recover from the particular condition and terminate with a solution, or the condition may persist in which case the algorithm stops with an error message. The warning messages produced by LINFSOL along with their meaning and some recommendations for corrective action are described below.

5.1 Warnings

`Requested accuracy not attained.` The program terminated but the accuracy requested (`E(18)`) was not attained. This may be due to numerical difficulties encountered during the execution of the algorithm. The execution may stop with `ERR=-29` or with some other error condition. It is also possible that the execution stops with the warning message `Termination occurred: new t too small`. The meaning of this message is explained below. In any of the cases mentioned above it is recommended to try to run the program with a larger value of `E(18)` using the warm start facility.

`Zero step.` The right-hand side (gradient vector) of the system (??) is zero (within roundoff tolerance). Upon the occurrence of this situation, the iteration is terminated, and the algorithm proceeds to check optimality and reduce the threshold t.

`Step size too small in Newton iteration.` The step size computed in the line search phase of the Newton iteration is smaller than an internal tolerance. In this case, if the active set remained unchanged with respect to the previous iteration, the iteration is terminated, and the program tries to continue with a reduced value of t. This may be an indication that the problem is highly ill-conditioned.

`Inconsistency in threshold reduction.` The system $\bar{A}^T\bar{\Theta}\bar{A}h=\bar{A}^T\bar{\Theta}b$ that corresponds to pseudo-normal equations for the system $\bar{\Theta}\bar{A}h = \bar{b}$ is found numerically inconsistent. In this case, the threshold reduction routine reduces the threshold as $t_{new} := t_{old} * 0.75$. No duality gap is computed, and therefore no statistics are printed.

`Termination occurred: new t too small.` The new threshold t computed by the program is smaller than an internal tolerance. In this case the program terminates.

5.2 Error Messages

In section 3.5 we briefly described the meaning of the error indicator parameter ERR. In this section we give a more detailed overview of error messages and some recommendations for recovery.

`ERR=-1` This indicates some errors in the input as described in section 3.5. The users are advised to check the input carefully.

`ERR=-2` More workspace is needed by LINFSOL. An estimate is printed on unit 7 with an error message.

`ERR=-3` A is rank deficient. There must be dependencies among the columns of A. The users are advised to check their model.

ERR=-7 This error condition is accompanied by a explanatory message which guides the user.

ERR<-29 The maximum number of iterations has been reached. A larger number should be specified and the program should be rerun using a warm start.

6 Warm Starts

The warm start facility is important in the following cases. First the current run may not be successful, and the current approximation to the solution reported by LINFSOL may be used for subsequent runs. Another situation in which a warm start is valuable is when the particular application requires the solution of several similar models successively.

In order to activate the warm start facility, the first step is to initialize the parameter START to a value larger than 1. Furthermore, if the parameter S is available from the previous run, it should be left unchanged. If the problem is to be solved for the first time then use of warm start facility is justified if the user has prior information on the solution of the problem. In this case, S should be initialized as discussed in section 3.3. I.e., If a constraint is believed to be strictly satisfied at an optimal solution S(i) should be assigned the value zero. If it is believed to be satisfied as equality, S(i) should be assigned the value 4 .

7 Example

In this section we will illustrate the use of LINFSOL with A numerical example. This example corresponds to

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 7 \\ 8 \\ 1 \end{bmatrix}.$$

A sample input file used to call LINFSOL to solve the above example is given below.

```
      PROGRAM TEST1
************************************************************************
*  TEST of  LINFSOL WITH EXAMPLE 1
************************************************************************
C SET CONSTANTS
C M..  TOTAL NUMBER OF CONSTRAINTS
C N..  NUMBER OF VARIABLES
C LDA..DECLARED DIMENSION OF A
C LW.. DOUBLE PRECISION WORK SPACE
C LS.. INTEGER WORKSPACE
C START..WARM OR COLD START PARAMETER
C EPSM..MACHINE PRECISION
```

```
C
      INTEGER           ERR,I,J,LS,LW,M,N,START
      DOUBLE PRECISION  E(21),EPSM
      PARAMETER         (EPSM = 2.221D-16,
     &                    M = 2, N = 5, LDA = M,
     &                   LW = M*(M + 4) + 5*N,
     &                   LS = 3*N + M + 4  )
      INTEGER           S(LS)
      DOUBLE PRECISION  A(LDA,N),B(M), X(N), W(LW)
C
C OPEN THE FILE FOR OUTPUT AND ADVISORY MESSAGES
        OPEN(UNIT=7, FILE='TEST1')
C INITIALIZE E(1) AND E(2) TO MACHINE PRECISION EPSM
        E(1) = EPSM
        E(2) = EPSM
C COLD START
        START = 1
C PROBLEM DATA
C.. A    = CONSTRAINT MATRIX
C.. B    = RIGHT-HAND SIDE VECTOR
C
        DO 10 I=1,M
         DO 20 J=1,N
         A(I,J) = 0D0
  20     CONTINUE
        B(I) = 0D0
  10    CONTINUE
        A(1,1) = 1D0
        A(1,2) = 2D0
        A(2,1) = 3D0
        A(2,2) = 4D0
        A(3,1) = 5D0
        A(3,2) = 6D0
        B(1) =  7D0
        B(2) = 8D0
        B(3) = 1D0
C
C
        E(16) = 1D0
        E(17) = 1D0
C SOLVE THE PROBLEM
        CALL LINFSOL(M,N,A,M,B,START,X,E,S,LS,W,LW, ERR)
C
        CLOSE(UNIT = 7)
 999    STOP
      END
```

The output file (unit 7) is as follows.

```
N, M =  2  3

 Iteration Log

-----------------
     t                 Pobj                  Dobj        nact  rank  iters
    2.8643          -25.00000000          -25.00000000   3     2      4


   t          F(x,y,t)          Pobj          Dobj         nact      rank   iter
0.10000000   -1.96250000    2.00000000  -2.00000000       3         3      3
smallest of residual=    -6.9388939039072D-18      tol=  1.0660800000000D-10

   t          F(x,y,t)          Pobj          Dobj         nact      rank   iter
0.10000000  -1.96250000     2.00000000     -2.00000000     3         3      3

 ===== LINFSOL terminal statistics ===

 ===> Duality Gap..   1.0660800000000D-10

 ===> Primal Infeasibility wrt A x-y e= b..  -6.9388939039072D-18


                      LINFSOL :  OBJECTIVE = -2.000000000D+00
                           err, refac, it =   0     2     3
                           reductions =     1
                           norm(Wd) = -1.000000000D+00

 EPS(  1..  5) =  2.221D-16   6.663D-13   3.500D+01   6.857D-01   4.760D-11
 EPS(  6.. 10) =  7.996D-12   5.330D-11   0.000D+00   5.000D+01   3.000D+00
 EPS( 11.. 15) =  2.000D+00   0.000D+00  -2.000D+00  -1.000D+00   1.000D+00
 EPS( 16.. 20) =  0.000D+00   0.000D+00   0.000D+00   1.000D+00   0.000D+00
 EPS( 21.. 22) =  0.000D+00   7.510D-04


INDEX          X
   1      -12.000000000000
   2      10.5000000000000
```

8 System Organization and Installation

LINFSOL is a software system written in standard Fortran 77. It is a collection of several subroutines that implement the algorithm described in the previous section. This section provides information about the organization of the software system. The purpose is to provide enough details for the person who is going to install the software on a computer system.

The source code is distributed in six files that reflect the logical organization of the program. (An additional seventh file contains the BLAS routines that are called by LINFSOL).

LINFSOL: This is the main program that controls the overall flow of the algorithm.

LINFAUX1, LINFAUX2, LINFAUX0, AUXSBR: Contain several service routines. In particular LINFAUX1 contains the subroutines that implement the main tasks of the algorithm.

AAFAC: This file contains routines for the factorization of $\bar{A}^T\bar{\Theta}\bar{A}$, up- and downdates of the factorization, rank and condition estimation, and solver routines. This is a package documented in [11].

BLASSB: This file contains some BLAS routines called by LINFSOL. The use of this file is not necessary if the computer system already contains a callable library of BLAS routines. The users are advised to check with their system administrator before using this file since the use of specially tuned BLAS routines has a significant impact on performance.

Appendix B:

Detailed Numerical Results

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
10	20	0	0	0	2.72	15	0.005	0.032	4	2.72	0	2	1	0.15
10	30	0	0	0	2.02	21	0.008	0.023	3	2.02	0	2	1	0.36
20	30	0	0	0	15.53	29	0.016	0.053	4	15.53	0	3	1	0.30
10	40	0	0	0	14.37	20	0.011	0.027	3	14.37	0	2	1	0.39
20	40	0	0	0	8.83	33	0.023	0.041	3	8.83	0	2	1	0.56
30	40	0	0	0	5.92	48	0.041	0.100	4	5.92	0	3	1	0.41
10	50	0	0	0	6.90	24	0.015	0.029	3	6.90	0	2	1	0.52
20	50	0	0	0	18.37	39	0.032	0.048	3	18.37	0	2	1	0.67
30	50	0	0	0	15.06	51	0.055	0.083	3	15.06	0	2	1	0.66
40	50	0	0	0	10.15	62	0.081	0.180	4	10.15	0	3	1	0.45
10	60	0	0	0	11.47	25	0.019	0.033	3	11.47	0	2	1	0.56
20	60	0	0	0	14.64	45	0.044	0.057	3	14.64	0	2	1	0.77
30	60	0	0	0	19.79	45	0.060	0.134	5	19.79	0	2	2	0.45
40	60	0	0	0	13.25	71	0.110	0.143	3	13.25	0	2	1	0.77
50	60	0	0	0	4.08	73	0.142	0.292	4	4.08	0	3	1	0.49
10	70	0	0	0	11.12	21	0.020	0.037	3	11.12	0	2	1	0.53
20	70	0	0	0	18.91	45	0.055	0.063	3	18.91	0	2	1	0.88
30	70	0	0	0	3.35	80	0.117	0.108	3	3.35	0	2	1	1.09
40	70	0	0	0	9.03	98	0.175	0.234	5	9.03	0	2	2	0.75
50	70	0	0	0	6.36	88	0.192	0.243	3	6.36	0	2	1	0.79
60	70	0	0	0	8.65	91	0.230	0.424	4	8.65	0	2	1	0.54
10	80	0	0	0	2.39	27	0.027	0.040	3	2.39	0	2	1	0.68
20	80	0	0	0	1.97	50	0.068	0.071	3	1.97	0	2	1	0.95
30	80	0	0	0	10.29	65	0.114	0.121	3	10.29	0	2	1	0.94
40	80	0	0	0	14.81	104	0.213	0.188	3	14.81	0	2	1	1.13
50	80	0	0	0	4.22	110	0.267	0.428	5	4.22	0	3	2	0.62
60	80	0	0	0	5.82	114	0.325	0.376	3	5.82	0	2	1	0.86
70	80	0	0	0	16.65	112	0.353	0.612	4	16.65	0	2	1	0.58
10	90	0	0	0	17.58	20	0.024	0.043	3	17.58	0	2	1	0.56
20	90	0	0	0	2.45	57	0.083	0.076	3	2.45	0	2	1	1.09
30	90	0	0	0	16.33	71	0.133	0.129	3	16.33	0	2	1	1.03
40	90	0	0	0	10.16	116	0.257	0.202	3	10.16	0	2	1	1.27
50	90	0	0	0	0.19	109	0.288	0.397	5	0.19	0	2	2	0.72
60	90	0	0	0	6.92	138	0.418	0.628	5	6.92	0	3	2	0.67
70	90	0	0	0	18.96	113	0.393	0.534	3	18.96	0	2	1	0.74
80	90	0	0	0	11.65	113	0.443	0.823	4	11.65	0	2	1	0.54
10	100	0	0	0	18.25	24	0.030	0.045	3	18.25	0	2	1	0.67
20	100	0	0	0	1.79	59	0.104	0.085	3	1.79	0	2	1	1.22
30	100	0	0	0	10.78	71	0.150	0.144	3	10.78	0	2	1	1.04
40	100	0	0	0	16.75	92	0.231	0.221	3	16.75	0	2	1	1.04
50	100	0	0	0	3.36	140	0.404	0.321	3	3.36	0	2	1	1.26
60	100	0	0	0	17.08	128	0.450	0.704	5	17.08	0	3	2	0.64
70	100	0	0	0	1.01	149	0.562	0.887	5	1.01	0	3	2	0.63
80	100	0	0	0	3.56	151	0.635	0.752	3	3.56	0	2	1	0.84
90	100	0	0	0	13.95	135	0.639	1.108	4	13.95	0	2	1	0.58
10	110	0	0	0	17.83	20	0.028	0.050	3	17.83	0	2	1	0.56
20	110	0	0	0	16.60	49	0.088	0.089	3	16.60	0	2	1	0.99
30	110	0	0	0	19.92	82	0.184	0.154	3	19.92	0	2	1	1.20
40	110	0	0	0	16.39	99	0.275	0.241	3	16.39	0	2	1	1.14
50	110	0	0	0	14.97	123	0.396	0.349	3	14.97	0	2	1	1.13
60	110	0	0	0	6.29	158	0.574	0.482	3	6.29	0	2	1	1.19
70	110	0	0	0	16.60	143	0.596	0.992	5	16.60	0	3	2	0.60
80	110	0	0	0	6.89	167	0.772	1.216	5	6.89	0	3	2	0.64
90	110	0	0	0	15.94	185	0.941	1.005	3	15.94	0	2	1	0.94
100	110	0	0	0	14.35	150	0.880	1.453	4	14.35	0	2	1	0.61
10	120	0	0	0	15.23	27	0.039	0.051	3	15.23	0	2	1	0.76
20	120	0	0	0	11.71	61	0.117	0.097	3	11.71	0	2	1	1.21
30	120	0	0	0	5.73	88	0.215	0.167	3	5.73	0	2	1	1.29
40	120	0	0	0	15.84	99	0.295	0.256	3	15.84	0	2	1	1.15
50	120	0	0	0	15.60	131	0.452	0.379	3	15.60	0	2	1	1.20
60	120	0	0	0	4.44	146	0.583	0.518	3	4.44	0	2	1	1.13
70	120	0	0	0	13.12	173	0.784	0.682	3	13.12	0	2	1	1.15
80	120	0	0	0	2.64	185	0.956	1.372	5	2.64	0	3	2	0.70
90	120	0	0	0	6.91	226	1.231	1.611	5	6.91	0	3	2	0.76
100	120	0	0	0	0.72	181	1.132	1.339	3	0.72	0	2	1	0.85
110	120	0	0	0	14.13	160	1.056	1.843	4	14.13	0	2	1	0.57
10	130	0	0	0	2.51	26	0.042	0.055	3	2.51	0	2	1	0.76
20	130	0	0	0	2.03	55	0.116	0.103	3	2.03	0	2	1	1.13
30	130	0	0	0	8.21	84	0.222	0.178	3	8.21	0	2	1	1.25
40	130	0	0	0	4.35	115	0.364	0.278	3	4.35	0	2	1	1.31
50	130	0	0	0	0.57	140	0.524	0.408	3	0.57	0	2	1	1.28
60	130	0	0	0	3.12	172	0.732	0.554	3	3.12	0	2	1	1.32
70	130	0	0	0	18.02	217	1.042	0.986	5	18.02	0	2	2	1.06
80	130	0	0	0	10.78	210	1.122	1.478	5	10.78	0	3	2	0.76
90	130	0	0	0	13.49	222	1.371	1.779	5	13.49	0	3	2	0.77
100	130	0	0	0	18.66	187	1.263	2.094	5	18.66	0	3	2	0.60
110	130	0	0	0	6.67	188	1.333	1.699	3	6.67	0	2	1	0.78
120	130	0	0	0	12.75	169	1.310	2.297	4	12.75	0	2	1	0.57
10	140	0	0	0	18.47	22	0.042	0.057	3	18.47	0	2	1	0.73
20	140	0	0	0	8.23	71	0.177	0.117	3	8.23	0	2	1	1.52
30	140	0	0	0	7.35	78	0.267	0.200	3	7.35	0	2	1	1.33
40	140	0	0	0	9.06	132	0.489	0.299	3	9.06	0	2	1	1.63
50	140	0	0	0	19.65	140	0.609	0.431	3	19.65	0	2	1	1.41
60	140	0	0	0	1.16	180	0.916	0.595	3	1.16	0	2	1	1.54
70	140	0	0	0	17.13	198	1.110	0.786	3	17.13	0	2	1	1.41
80	140	0	0	0	13.45	211	1.340	1.024	3	13.45	0	2	1	1.31
90	140	0	0	0	19.83	201	1.388	1.924	5	19.83	0	3	2	0.72
100	140	0	0	0	2.47	226	1.736	2.274	5	2.47	0	3	2	0.76

Table 1: Comparison between LINFSOL and B-P on nondegenerate problems

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
110	140	0	0	0	10.22	211	1.757	2.639	5	10.22	0	3	2	0.67
120	140	0	0	0	5.05	217	1.885	2.136	3	5.05	0	2	1	0.88
130	140	0	0	0	7.10	193	1.882	2.829	4	7.10	0	2	1	0.67
10	150	0	0	0	13.09	25	0.049	0.060	3	13.09	0	2	1	0.82
20	150	0	0	0	18.82	49	0.134	0.118	3	18.82	0	2	1	1.13
30	150	0	0	0	19.67	86	0.289	0.208	3	19.67	0	2	1	1.39
40	150	0	0	0	8.81	150	0.619	0.316	3	8.81	0	2	1	1.96
50	150	0	0	0	4.02	146	0.710	0.461	3	4.02	0	2	1	1.54
60	150	0	0	0	18.55	204	1.072	0.632	3	18.55	0	2	1	1.70
70	150	0	0	0	10.66	245	1.548	0.835	3	10.66	0	2	1	1.85
80	150	0	0	0	17.67	227	1.587	1.077	3	17.67	0	2	1	1.47
90	150	0	0	0	8.69	253	1.913	2.094	5	8.69	0	3	2	0.91
100	150	0	0	0	7.44	218	1.796	2.453	5	7.44	0	3	2	0.73
110	150	0	0	0	3.20	278	2.504	2.885	5	3.20	0	3	2	0.87
120	150	0	0	0	17.05	220	2.094	3.406	5	17.05	0	3	2	0.61
130	150	0	0	0	4.35	242	2.434	2.699	3	4.35	0	2	1	0.90
140	150	0	0	0	15.65	197	2.163	3.460	4	15.65	0	2	1	0.62
10	160	0	0	0	5.12	26	0.056	0.065	3	5.12	0	2	1	0.87
20	160	0	0	0	16.90	68	0.189	0.127	3	16.90	0	2	1	1.49
30	160	0	0	0	10.13	88	0.314	0.221	3	10.13	0	2	1	1.42
40	160	0	0	0	17.77	108	0.501	0.338	3	17.77	0	2	1	1.48
50	160	0	0	0	19.79	127	0.627	0.491	3	19.79	0	2	1	1.28
60	160	0	0	0	19.15	187	1.105	0.674	3	19.15	0	2	1	1.64
70	160	0	0	0	7.04	226	1.489	0.893	3	7.04	0	2	1	1.67
80	160	0	0	0	1.41	271	1.872	1.122	3	1.41	0	2	1	1.67
90	160	0	0	0	4.55	244	1.873	1.397	3	4.55	0	2	1	1.34
100	160	0	0	0	17.37	254	2.315	2.660	5	17.37	0	3	2	0.87
110	160	0	0	0	17.61	295	2.652	3.083	5	17.61	0	3	2	0.86
120	160	0	0	0	10.03	233	2.300	3.538	5	10.03	0	3	2	0.65
130	160	0	0	0	4.50	272	3.061	4.105	5	4.50	0	3	2	0.75
140	160	0	0	0	8.47	218	2.517	3.288	3	8.47	0	2	1	0.77
150	160	0	0	0	12.52	227	2.771	4.146	4	12.52	0	2	1	0.67
10	170	0	0	0	16.54	25	0.057	0.066	3	16.54	0	2	1	0.85
20	170	0	0	0	2.21	61	0.176	0.131	3	2.21	0	2	1	1.34
30	170	0	0	0	2.42	81	0.332	0.239	3	2.42	0	2	1	1.39
40	170	0	0	0	8.38	142	0.622	0.371	3	8.38	0	2	1	1.68
50	170	0	0	0	16.28	171	0.878	0.521	3	16.28	0	2	1	1.69
60	170	0	0	0	1.67	223	1.362	0.714	3	1.67	0	2	1	1.91
70	170	0	0	0	4.09	218	1.416	0.939	3	4.09	0	2	1	1.51
80	170	0	0	0	14.12	234	1.715	1.186	3	14.12	0	2	1	1.45
90	170	0	0	0	15.13	254	2.060	1.467	3	15.13	0	2	1	1.40
100	170	0	0	0	6.19	293	2.777	2.401	5	6.19	0	2	2	1.16
110	170	0	0	0	6.60	280	2.654	3.347	5	6.60	0	3	2	0.79
120	170	0	0	0	18.33	298	3.054	3.779	5	18.33	0	3	2	0.81
130	170	0	0	0	15.96	284	3.164	4.349	5	15.96	0	3	2	0.73
140	170	0	0	0	16.95	293	3.544	4.851	5	16.95	0	3	2	0.73
150	170	0	0	0	3.88	264	3.299	3.997	3	3.88	0	2	1	0.83
160	170	0	0	0	3.45	225	2.999	4.982	4	3.45	0	2	1	0.60
10	180	0	0	0	6.11	29	0.067	0.070	3	6.11	0	2	1	0.96
20	180	0	0	0	0.13	73	0.219	0.139	3	0.13	0	2	1	1.58
30	180	0	0	0	6.12	86	0.366	0.240	3	6.12	0	2	1	1.53
40	180	0	0	0	1.57	130	0.606	0.379	3	1.57	0	2	1	1.60
50	180	0	0	0	12.72	159	0.868	0.546	3	12.72	0	2	1	1.59
60	180	0	0	0	17.46	192	1.222	0.749	3	17.46	0	2	1	1.63
70	180	0	0	0	19.92	233	1.648	0.985	3	19.92	0	2	1	1.67
80	180	0	0	0	8.13	277	2.225	1.249	3	8.13	0	2	1	1.78
90	180	0	0	0	7.90	308	2.702	1.546	3	7.90	0	2	1	1.75
100	180	0	0	0	15.28	295	2.810	1.926	3	15.28	0	2	1	1.46
110	180	0	0	0	10.33	315	3.303	3.548	5	10.33	0	3	2	0.93
120	180	0	0	0	16.77	318	3.515	4.123	5	16.77	0	3	2	0.85
130	180	0	0	0	1.45	327	3.855	4.637	5	1.45	0	3	2	0.83
140	180	0	0	0	10.36	348	4.428	5.269	5	10.36	0	3	2	0.84
150	180	0	0	0	18.78	263	3.535	5.903	5	18.78	0	3	2	0.60
160	180	0	0	0	16.78	257	3.676	4.712	3	16.78	0	2	1	0.78
170	180	0	0	0	16.36	248	3.850	5.948	4	16.36	0	2	1	0.65
10	190	0	0	0	10.71	30	0.074	0.074	3	10.71	0	2	1	1.00
20	190	0	0	0	6.28	74	0.254	0.147	3	6.28	0	2	1	1.73
30	190	0	0	0	13.07	75	0.312	0.261	3	13.07	0	2	1	1.19
40	190	0	0	0	0.37	142	0.706	0.394	3	0.37	0	2	1	1.79
50	190	0	0	0	2.17	163	0.987	0.574	3	2.17	0	2	1	1.72
60	190	0	0	0	10.53	214	1.378	0.812	3	10.53	0	2	1	1.70
70	190	0	0	0	8.61	259	1.878	1.024	3	8.61	0	2	1	1.83
80	190	0	0	0	18.16	269	2.212	1.314	3	18.16	0	2	1	1.68
90	190	0	0	0	3.33	318	2.873	1.624	3	3.33	0	2	1	1.77
100	190	0	0	0	18.88	320	3.253	1.979	3	18.88	0	2	1	1.64
110	190	0	0	0	6.55	299	3.196	2.367	3	6.55	0	2	1	1.35
120	190	0	0	0	18.84	351	4.231	4.330	5	18.84	0	3	2	0.98
130	190	0	0	0	16.39	332	4.205	4.893	5	16.39	0	3	2	0.86
140	190	0	0	0	13.88	361	4.699	5.602	5	13.88	0	3	2	0.84
150	190	0	0	0	5.50	338	4.825	6.194	5	5.50	0	3	2	0.78
160	190	0	0	0	0.20	354	5.329	8.201	19	0.20	0	3	9	0.65
170	190	0	0	0	4.52	263	4.151	5.582	3	4.52	0	2	1	0.74
180	190	0	0	0	8.48	255	4.386	6.800	4	8.48	0	2	1	0.64
10	200	0	0	0	12.39	24	0.065	0.080	3	12.39	0	2	1	0.82
20	200	0	0	0	19.64	59	0.208	0.155	3	19.64	0	2	1	1.35
30	200	0	0	0	12.21	89	0.425	0.268	3	12.21	0	2	1	1.59
40	200	0	0	0	13.58	126	0.654	0.430	3	13.58	0	2	1	1.52
50	200	0	0	0	7.98	176	1.106	0.598	3	7.98	0	2	1	1.85
60	200	0	0	0	18.86	204	1.392	0.814	3	18.86	0	2	1	1.71

Table 2: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
70	200	0	0	0	1.11	246	1.904	1.078	3	1.11	0	2	1	1.77
80	200	0	0	0	0.54	306	2.572	1.390	3	0.54	0	2	1	1.85
90	200	0	0	0	2.16	304	2.935	1.707	3	2.16	0	2	1	1.72
100	200	0	0	0	0.34	354	3.632	2.081	3	0.34	0	2	1	1.75
110	200	0	0	0	4.02	372	4.266	2.466	3	4.02	0	2	1	1.73
120	200	0	0	0	19.16	348	4.203	2.967	3	19.16	0	2	1	1.42
130	200	0	0	0	1.50	369	4.651	3.613	7	1.50	0	2	3	1.29
140	200	0	0	0	19.00	387	5.247	5.982	5	19.00	0	3	2	0.88
150	200	0	0	0	11.72	319	4.762	6.608	5	11.72	0	3	2	0.72
160	200	0	0	0	6.25	320	5.077	7.413	5	6.25	0	3	2	0.68
170	200	0	0	0	10.15	308	5.279	8.154	5	10.15	0	3	2	0.65
180	200	0	0	0	14.89	279	4.854	6.540	3	14.89	0	2	1	0.74
190	200	0	0	0	0.64	308	5.640	8.265	6	0.64	0	2	2	0.68
10	210	0	0	0	13.01	24	0.067	0.078	3	13.01	0	2	1	0.85
20	210	0	0	0	5.28	72	0.261	0.163	3	5.28	0	2	1	1.60
30	210	0	0	0	5.28	103	0.507	0.283	3	5.28	0	2	1	1.79
40	210	0	0	0	6.06	132	0.713	0.431	3	6.06	0	2	1	1.66
50	210	0	0	0	3.17	192	1.275	0.629	3	3.17	0	2	1	2.03
60	210	0	0	0	12.22	204	1.562	0.862	3	12.22	0	2	1	1.81
70	210	0	0	0	19.63	265	2.110	1.143	3	19.63	0	2	1	1.85
80	210	0	0	0	16.07	309	2.767	1.458	3	16.07	0	2	1	1.90
90	210	0	0	0	3.79	266	2.790	1.789	3	3.79	0	2	1	1.56
100	210	0	0	0	15.26	329	3.665	2.172	3	15.26	0	2	1	1.69
110	210	0	0	0	3.19	337	3.945	2.599	3	3.19	0	2	1	1.52
120	210	0	0	0	16.49	378	4.722	3.031	3	16.49	0	2	1	1.56
130	210	0	0	0	9.72	421	5.590	3.558	3	9.72	0	2	1	1.57
140	210	0	0	0	2.48	399	5.715	6.237	5	2.48	0	3	2	0.92
150	210	0	0	0	14.06	411	6.333	6.942	5	14.06	0	3	2	0.91
160	210	0	0	0	12.26	395	6.409	7.767	5	12.26	0	3	2	0.83
170	210	0	0	0	15.91	386	6.587	8.636	5	15.91	0	3	2	0.76
180	210	0	0	0	9.75	363	6.871	8.633	5	9.75	0	2	2	0.80
190	210	0	0	0	12.35	290	5.704	9.025	5	12.35	0	2	2	0.63
200	210	0	0	0	11.55	296	6.141	20.234	4	11.55	0	2	1	0.30
10	220	0	0	0	8.37	27	0.079	0.083	3	8.37	0	2	1	0.95
20	220	0	0	0	13.04	67	0.256	0.170	3	13.04	0	2	1	1.51
30	220	0	0	0	14.99	97	0.471	0.294	3	14.99	0	2	1	1.60
40	220	0	0	0	14.13	128	0.749	0.459	3	14.13	0	2	1	1.63
50	220	0	0	0	4.00	215	1.522	0.675	3	4.00	0	2	1	2.25
60	220	0	0	0	10.30	234	1.766	0.950	3	10.30	0	2	1	1.86
70	220	0	0	0	2.86	268	2.377	1.182	3	2.86	0	2	1	2.01
80	220	0	0	0	16.63	284	2.711	1.500	3	16.63	0	2	1	1.81
90	220	0	0	0	3.30	347	3.545	1.871	3	3.30	0	2	1	1.89
100	220	0	0	0	2.37	416	4.839	2.241	3	2.37	0	2	1	2.16
110	220	0	0	0	0.40	359	4.390	2.728	3	0.40	0	2	1	1.61
120	220	0	0	0	18.27	374	4.843	3.192	3	18.27	0	2	1	1.52
130	220	0	0	0	16.04	449	6.561	3.702	3	16.04	0	2	1	1.77
140	220	0	0	0	16.16	467	6.973	4.272	3	16.16	0	2	1	1.63
150	220	0	0	0	10.46	410	6.494	7.373	5	10.46	0	3	2	0.88
160	220	0	0	0	9.20	414	7.005	8.221	5	9.20	0	3	2	0.85
170	220	0	0	0	13.81	386	7.102	9.089	5	13.81	0	3	2	0.78
180	220	0	0	0	4.61	385	7.309	10.003	5	4.61	0	3	2	0.73
190	220	0	0	0	13.05	355	7.198	10.005	5	13.05	0	2	2	0.72
200	220	0	0	0	6.81	353	7.583	14.493	3	6.81	0	2	1	0.52
210	220	0	0	0	12.14	277	6.310	10.522	4	12.14	0	2	1	0.60
10	230	0	0	0	3.60	28	0.084	0.086	3	3.60	0	2	1	0.99
20	230	0	0	0	18.09	65	0.249	0.174	3	18.09	0	2	1	1.44
30	230	0	0	0	18.40	107	0.640	0.307	3	18.40	0	2	1	2.09
40	230	0	0	0	12.96	144	0.857	0.485	3	12.96	0	2	1	1.77
50	230	0	0	0	18.00	194	1.327	0.700	3	18.00	0	2	1	1.90
60	230	0	0	0	15.60	219	1.728	0.931	3	15.60	0	2	1	1.86
70	230	0	0	0	12.75	264	2.354	1.224	3	12.75	0	2	1	1.92
80	230	0	0	0	19.34	314	3.067	1.565	3	19.34	0	2	1	1.96
90	230	0	0	0	16.71	374	3.962	1.996	3	16.71	0	2	1	1.98
100	230	0	0	0	9.10	382	4.457	2.421	3	9.10	0	2	1	1.84
110	230	0	0	0	5.95	422	5.365	2.807	3	5.95	0	2	1	1.91
120	230	0	0	0	13.66	371	5.121	3.292	3	13.66	0	2	1	1.56
130	230	0	0	0	13.47	504	7.468	3.837	3	13.47	0	2	1	1.95
140	230	0	0	0	6.71	440	6.723	4.515	3	6.71	0	2	1	1.49
150	230	0	0	0	0.70	454	7.496	5.121	3	0.70	0	2	1	1.46
160	230	0	0	0	10.34	444	7.881	8.690	5	10.34	0	3	2	0.91
170	230	0	0	0	19.17	434	8.269	9.597	5	19.17	0	3	2	0.86
180	230	0	0	0	0.70	439	8.645	10.532	5	0.70	0	3	2	0.82
190	230	0	0	0	19.42	359	7.499	11.552	5	19.42	0	3	2	0.65
200	230	0	0	0	9.14	363	7.906	12.907	5	9.14	0	2	2	0.61
210	230	0	0	0	11.05	319	7.526	10.086	3	11.05	0	2	1	0.75
220	230	0	0	0	12.85	289	7.041	12.045	4	12.85	0	2	1	0.58
10	240	0	0	0	8.88	27	0.084	0.089	3	8.88	0	2	1	0.94
20	240	0	0	0	14.57	69	0.276	0.183	3	14.57	0	2	1	1.51
30	240	0	0	0	0.01	112	0.597	0.312	3	0.01	0	2	1	1.91
40	240	0	0	0	6.03	149	0.906	0.486	3	6.03	0	2	1	1.86
50	240	0	0	0	3.93	187	1.435	0.714	3	3.93	0	2	1	2.01
60	240	0	0	0	18.51	264	2.161	1.050	3	18.51	0	2	1	2.06
70	240	0	0	0	1.32	261	2.465	1.285	3	1.32	0	2	1	1.92
80	240	0	0	0	13.35	292	2.959	1.672	3	13.35	0	2	1	1.77
90	240	0	0	0	15.05	357	3.962	2.004	3	15.05	0	2	1	1.98
100	240	0	0	0	17.70	407	4.967	2.444	3	17.70	0	2	1	2.03
110	240	0	0	0	9.91	372	4.956	2.903	3	9.91	0	2	1	1.71
120	240	0	0	0	10.05	400	5.676	3.417	3	10.05	0	2	1	1.66
130	240	0	0	0	14.70	461	7.004	4.057	3	14.70	0	2	1	1.73

Table 3: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
140	240	0	0	0	14.70	431	7.041	4.685	3	14.70	0	2	1	1.50
150	240	0	0	0	13.04	392	6.762	5.382	3	13.04	0	2	1	1.26
160	240	0	0	0	8.56	468	8.404	9.064	5	8.56	0	3	2	0.93
170	240	0	0	0	2.56	484	9.485	10.256	5	2.56	0	3	2	0.92
180	240	0	0	0	8.20	463	9.421	11.150	5	8.20	0	3	2	0.84
190	240	0	0	0	10.67	399	8.774	12.160	5	10.67	0	3	2	0.72
200	240	0	0	0	16.49	433	9.955	13.473	5	16.49	0	3	2	0.74
210	240	0	0	0	16.81	391	9.462	12.940	5	16.81	0	2	2	0.73
220	240	0	0	0	19.29	348	8.698	11.484	3	19.29	0	2	1	0.76
230	240	0	0	0	4.33	324	8.703	13.831	4	4.33	0	2	1	0.63
10	250	0	0	0	19.86	25	0.080	0.091	3	19.86	0	2	1	0.88
20	250	0	0	0	3.87	63	0.266	0.186	3	3.87	0	2	1	1.43
30	250	0	0	0	17.93	113	0.670	0.324	3	17.93	0	2	1	2.07
40	250	0	0	0	2.67	175	1.123	0.510	3	2.67	0	2	1	2.20
50	250	0	0	0	13.25	213	1.557	0.743	3	13.25	0	2	1	2.09
60	250	0	0	0	11.24	261	2.220	1.013	3	11.24	0	2	1	2.19
70	250	0	0	0	15.50	252	2.489	1.322	3	15.50	0	2	1	1.88
80	250	0	0	0	7.31	243	2.542	1.680	3	7.31	0	2	1	1.51
90	250	0	0	0	18.31	357	4.206	2.117	3	18.31	0	2	1	1.99
100	250	0	0	0	18.50	401	5.062	2.519	3	18.50	0	2	1	2.01
110	250	0	0	0	13.60	442	6.030	3.048	3	13.60	0	2	1	1.98
120	250	0	0	0	1.82	414	6.045	3.555	3	1.82	0	2	1	1.70
130	250	0	0	0	4.19	458	7.340	4.141	3	4.19	0	2	1	1.77
140	250	0	0	0	0.89	473	8.103	4.766	3	0.89	0	2	1	1.70
150	250	0	0	0	11.49	468	8.546	5.612	3	11.49	0	2	1	1.52
160	250	0	0	0	10.02	532	10.094	6.306	3	10.02	0	2	1	1.60
170	250	0	0	0	2.64	490	9.892	10.580	5	2.64	0	3	2	0.93
180	250	0	0	0	9.19	475	10.118	11.722	5	9.19	0	3	2	0.86
190	250	0	0	0	1.99	522	11.770	13.909	13	1.99	0	3	6	0.85
200	250	0	0	0	4.92	437	10.605	14.428	5	4.92	0	3	2	0.74
210	250	0	0	0	1.68	432	10.746	15.277	5	1.68	0	3	2	0.70
220	250	0	0	0	13.03	433	11.222	14.578	5	13.03	0	2	2	0.77
230	250	0	0	0	11.41	335	9.097	15.006	5	11.41	0	2	2	0.61
240	250	0	0	0	15.82	355	9.991	15.346	4	15.82	0	2	1	0.65
10	260	0	0	0	17.85	26	0.087	0.095	3	17.85	0	2	1	0.91
20	260	0	0	0	15.31	59	0.265	0.197	3	15.31	0	2	1	1.34
30	260	0	0	0	17.07	111	0.646	0.343	3	17.07	0	2	1	1.88
40	260	0	0	0	3.00	156	1.020	0.529	3	3.00	0	2	1	1.93
50	260	0	0	0	15.78	185	1.508	0.766	3	15.78	0	2	1	1.97
60	260	0	0	0	4.99	251	2.174	1.043	3	4.99	0	2	1	2.08
70	260	0	0	0	3.96	308	3.028	1.407	3	3.96	0	2	1	2.15
80	260	0	0	0	13.57	311	3.358	1.751	3	13.57	0	2	1	1.92
90	260	0	0	0	14.50	376	4.598	2.182	3	14.50	0	2	1	2.11
100	260	0	0	0	3.51	389	5.109	2.625	3	3.51	0	2	1	1.95
110	260	0	0	0	8.35	418	5.957	3.155	3	8.35	0	2	1	1.89
120	260	0	0	0	0.38	472	7.209	3.686	3	0.38	0	2	1	1.96
130	260	0	0	0	2.56	473	7.680	4.281	3	2.56	0	2	1	1.79
140	260	0	0	0	0.40	473	8.423	5.009	3	0.40	0	2	1	1.68
150	260	0	0	0	1.84	554	10.333	5.829	3	1.84	0	2	1	1.77
160	260	0	0	0	8.58	523	10.165	6.407	3	8.58	0	2	1	1.59
170	260	0	0	0	1.35	498	10.370	11.159	5	1.35	0	3	2	0.93
180	260	0	0	0	4.05	479	10.551	12.310	5	4.05	0	3	2	0.86
190	260	0	0	0	0.83	555	12.929	13.543	5	0.83	0	3	2	0.95
200	260	0	0	0	2.02	507	12.596	15.160	5	2.02	0	3	2	0.83
210	260	0	0	0	5.08	457	11.985	15.846	5	5.08	0	3	2	0.76
220	260	0	0	0	19.42	473	12.784	17.247	5	19.42	0	3	2	0.74
230	260	0	0	0	18.05	410	11.702	16.513	5	18.05	0	2	2	0.71
240	260	0	0	0	4.38	432	12.707	15.077	3	4.38	0	2	1	0.84
250	260	0	0	0	1.45	344	10.654	19.756	16	1.45	0	2	7	0.54
10	270	0	0	0	4.40	21	0.074	0.097	3	4.40	0	2	1	0.76
20	270	0	0	0	17.89	73	0.327	0.201	3	17.89	0	2	1	1.62
30	270	0	0	0	17.90	96	0.543	0.351	3	17.90	0	2	1	1.54
40	270	0	0	0	7.76	155	1.108	0.562	3	7.76	0	2	1	1.97
50	270	0	0	0	11.47	173	1.369	0.816	3	11.47	0	2	1	1.68
60	270	0	0	0	4.49	289	2.667	1.080	3	4.49	0	2	1	2.47
70	270	0	0	0	16.00	279	2.877	1.424	3	16.00	0	2	1	2.02
80	270	0	0	0	1.74	320	3.670	1.817	3	1.74	0	2	1	2.02
90	270	0	0	0	8.84	418	5.384	2.300	3	8.84	0	2	1	2.34
100	270	0	0	0	13.29	412	5.531	2.792	3	13.29	0	2	1	1.98
110	270	0	0	0	1.75	425	6.298	3.246	3	1.75	0	2	1	1.94
120	270	0	0	0	15.94	428	6.700	3.828	3	15.94	0	2	1	1.75
130	270	0	0	0	3.20	533	9.138	4.448	3	3.20	0	2	1	2.05
140	270	0	0	0	18.37	587	10.915	5.131	3	18.37	0	2	1	2.13
150	270	0	0	0	13.73	521	10.096	5.914	3	13.73	0	2	1	1.71
160	270	0	0	0	2.80	549	11.413	6.651	3	2.80	0	2	1	1.72
170	270	0	0	0	0.12	565	12.377	11.639	5	0.12	0	3	2	1.06
180	270	0	0	0	8.40	501	11.417	12.722	5	8.40	0	3	2	0.90
190	270	0	0	0	0.30	564	13.676	14.088	5	0.30	0	3	2	0.97
200	270	0	0	0	18.67	507	13.081	15.829	5	18.67	0	3	2	0.83
210	270	0	0	0	13.34	517	13.947	16.667	5	13.34	0	3	2	0.84
220	270	0	0	0	7.26	530	14.741	18.020	5	7.26	0	3	2	0.82
230	270	0	0	0	8.57	453	13.511	19.507	5	8.57	0	3	2	0.69
240	270	0	0	0	9.39	446	13.655	18.611	5	9.39	0	2	2	0.73
250	270	0	0	0	19.63	377	12.078	18.824	4	19.63	0	2	1	0.64
260	270	0	0	0	11.23	374	12.286	19.755	4	11.23	0	2	1	0.62
10	280	0	0	0	9.12	23	0.086	0.101	3	9.12	0	2	1	0.85
20	280	0	0	0	0.96	74	0.343	0.206	3	0.96	0	2	1	1.66
30	280	0	0	0	5.36	110	0.687	0.361	3	5.36	0	2	1	1.90
40	280	0	0	0	16.05	158	1.120	0.561	3	16.05	0	2	1	2.00

Table 4: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
50	280	0	0	0	8.09	212	1.767	0.821	3	8.09	0	2	1	2.15
60	280	0	0	0	17.01	248	2.391	1.115	3	17.01	0	2	1	2.14
70	280	0	0	0	2.16	286	3.006	1.494	3	2.16	0	2	1	2.01
80	280	0	0	0	8.35	312	3.635	1.904	3	8.35	0	2	1	1.91
90	280	0	0	0	2.70	344	4.549	2.325	3	2.70	0	2	1	1.96
100	280	0	0	0	19.74	442	6.283	2.865	3	19.74	0	2	1	2.19
110	280	0	0	0	0.44	478	7.398	3.378	3	0.44	0	2	1	2.19
120	280	0	0	0	7.71	461	7.645	3.948	3	7.71	0	2	1	1.94
130	280	0	0	0	9.63	485	8.393	4.620	3	9.63	0	2	1	1.82
140	280	0	0	0	1.24	513	9.794	5.313	3	1.24	0	2	1	1.84
150	280	0	0	0	18.02	529	10.608	6.050	3	18.02	0	2	1	1.75
160	280	0	0	0	4.68	546	11.701	6.970	3	4.68	0	2	1	1.68
170	280	0	0	0	12.21	612	13.679	7.748	3	12.21	0	2	1	1.77
180	280	0	0	0	13.78	628	15.031	13.318	5	13.78	0	3	2	1.13
190	280	0	0	0	6.89	611	15.287	14.553	5	6.89	0	3	2	1.05
200	280	0	0	0	4.80	580	15.492	16.362	5	4.80	0	3	2	0.95
210	280	0	0	0	6.73	616	17.077	17.636	5	6.73	0	3	2	0.97
220	280	0	0	0	13.96	552	16.075	18.738	5	13.96	0	3	2	0.86
230	280	0	0	0	3.74	586	17.617	20.360	5	3.74	0	3	2	0.87
240	280	0	0	0	9.00	551	17.292	19.917	5	9.00	0	2	2	0.87
250	280	0	0	0	12.10	458	15.047	20.497	5	12.10	0	2	2	0.73
260	280	0	0	0	13.34	424	14.649	18.721	3	13.34	0	2	1	0.78
270	280	0	0	0	6.94	398	13.993	21.560	4	6.94	0	2	1	0.65
10	290	0	0	0	14.29	25	0.095	0.104	3	14.29	0	2	1	0.92
20	290	0	0	0	6.47	75	0.362	0.215	3	6.47	0	2	1	1.68
30	290	0	0	0	14.50	111	0.680	0.373	3	14.50	0	2	1	1.82
40	290	0	0	0	10.93	154	1.113	0.578	3	10.93	0	2	1	1.92
50	290	0	0	0	2.38	222	1.898	0.865	3	2.38	0	2	1	2.20
60	290	0	0	0	5.45	256	2.544	1.155	3	5.45	0	2	1	2.20
70	290	0	0	0	15.62	303	3.288	1.531	3	15.62	0	2	1	2.15
80	290	0	0	0	6.72	322	4.119	1.937	3	6.72	0	2	1	2.13
90	290	0	0	0	19.69	410	5.646	2.391	3	19.69	0	2	1	2.36
100	290	0	0	0	8.02	419	6.092	2.891	3	8.02	0	2	1	2.11
110	290	0	0	0	16.19	436	7.038	3.499	3	16.19	0	2	1	2.01
120	290	0	0	0	7.22	493	8.356	4.072	3	7.22	0	2	1	2.05
130	290	0	0	0	13.62	487	9.035	4.803	3	13.62	0	2	1	1.88
140	290	0	0	0	6.27	579	11.238	5.508	3	6.27	0	2	1	2.04
150	290	0	0	0	7.22	556	11.435	6.446	3	7.22	0	2	1	1.77
160	290	0	0	0	14.62	516	11.395	7.153	3	14.62	0	2	1	1.59
170	290	0	0	0	16.57	547	12.707	7.919	3	16.57	0	2	1	1.60
180	290	0	0	0	8.64	569	14.020	8.946	3	8.64	0	2	1	1.57
190	290	0	0	0	18.06	552	14.389	15.176	5	18.06	0	3	2	0.95
200	290	0	0	0	8.63	604	16.386	16.925	5	8.63	0	3	2	0.97
210	290	0	0	0	5.85	625	17.821	18.100	5	5.85	0	3	2	0.98
220	290	0	0	0	3.51	585	17.438	19.680	5	3.51	0	3	2	0.89
230	290	0	0	0	18.35	596	18.302	21.104	5	18.35	0	3	2	0.87
240	290	0	0	0	8.02	539	17.745	22.822	5	8.02	0	3	2	0.78
250	290	0	0	0	17.41	552	18.517	22.162	5	17.41	0	2	2	0.84
260	290	0	0	0	7.73	463	16.422	23.199	5	7.73	0	2	2	0.71
270	290	0	0	0	18.53	421	15.285	20.600	3	18.53	0	2	1	0.74
280	290	0	0	0	18.73	364	13.997	23.878	4	18.73	0	2	1	0.59
10	300	0	0	0	10.16	30	0.114	0.107	3	10.16	0	2	1	1.07
20	300	0	0	0	14.39	63	0.318	0.220	3	14.39	0	2	1	1.45
30	300	0	0	0	0.32	134	0.866	0.394	3	0.32	0	2	1	2.20
40	300	0	0	0	16.46	159	1.188	0.610	3	16.46	0	2	1	1.95
50	300	0	0	0	14.19	186	1.713	0.890	3	14.19	0	2	1	1.92
60	300	0	0	0	1.79	241	2.385	1.214	3	1.79	0	2	1	1.96
70	300	0	0	0	4.51	308	3.452	1.570	3	4.51	0	2	1	2.20
80	300	0	0	0	18.70	392	5.046	2.023	3	18.70	0	2	1	2.49
90	300	0	0	0	10.66	386	5.383	2.550	3	10.66	0	2	1	2.11
100	300	0	0	0	10.65	428	6.369	3.013	3	10.65	0	2	1	2.11
110	300	0	0	0	18.62	563	9.176	3.587	3	18.62	0	2	1	2.56
120	300	0	0	0	15.66	532	9.317	4.272	3	15.66	0	2	1	2.18
130	300	0	0	0	8.42	574	11.104	4.893	3	8.42	0	2	1	2.27
140	300	0	0	0	8.76	574	11.589	5.667	3	8.76	0	2	1	2.04
150	300	0	0	0	17.17	579	12.361	6.536	3	17.17	0	2	1	1.89
160	300	0	0	0	15.66	652	14.770	7.365	3	15.66	0	2	1	2.01
170	300	0	0	0	0.85	656	15.826	8.266	3	0.85	0	2	1	1.91
180	300	0	0	0	3.50	670	16.821	9.216	3	3.50	0	2	1	1.83
190	300	0	0	0	11.22	682	18.437	10.271	3	11.22	0	2	1	1.80
200	300	0	0	0	15.49	654	18.273	11.634	3	15.49	0	2	1	1.57
210	300	0	0	0	1.30	652	19.134	21.187	17	1.30	0	3	8	0.90
220	300	0	0	0	13.91	618	19.226	20.421	5	13.91	0	3	2	0.94
230	300	0	0	0	15.41	564	18.097	22.024	5	15.41	0	3	2	0.82
240	300	0	0	0	19.73	588	19.756	23.833	5	19.73	0	3	2	0.83
250	300	0	0	0	11.20	561	20.299	25.377	5	11.20	0	3	2	0.80
260	300	0	0	0	4.08	585	21.384	24.893	5	4.08	0	2	2	0.86
270	300	0	0	0	17.39	479	18.178	25.345	5	17.39	0	2	2	0.72
280	300	0	0	0	13.57	443	17.536	25.949	5	13.57	0	2	2	0.68
290	300	0	0	0	8.65	410	16.818	26.510	4	8.65	0	2	1	0.63
10	310	0	0	0	4.16	26	0.104	0.109	3	4.16	0	2	1	0.96
20	310	0	0	0	12.95	66	0.339	0.229	3	12.95	0	2	1	1.48
30	310	0	0	0	2.79	104	0.674	0.400	3	2.79	0	2	1	1.68
40	310	0	0	0	1.43	184	1.432	0.621	3	1.43	0	2	1	2.31
50	310	0	0	0	3.18	230	2.128	0.906	3	3.18	0	2	1	2.35
60	310	0	0	0	15.41	283	2.971	1.239	3	15.41	0	2	1	2.40
70	310	0	0	0	19.50	309	3.573	1.632	3	19.50	0	2	1	2.19
80	310	0	0	0	9.14	353	4.733	2.057	3	9.14	0	2	1	2.30
90	310	0	0	0	17.24	367	5.214	2.609	3	17.24	0	2	1	2.00

Table 5: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
100	310	0	0	0	18.43	489	7.770	3.129	3	18.43	0	2	1	2.48
110	310	0	0	0	6.29	549	9.304	3.755	3	6.29	0	2	1	2.48
120	310	0	0	0	19.12	525	9.662	4.342	3	19.12	0	2	1	2.23
130	310	0	0	0	7.99	557	10.846	5.106	3	7.99	0	2	1	2.12
140	310	0	0	0	16.79	502	10.467	5.826	3	16.79	0	2	1	1.80
150	310	0	0	0	17.91	541	11.970	6.614	3	17.91	0	2	1	1.81
160	310	0	0	0	13.82	653	15.436	7.701	3	13.82	0	2	1	2.00
170	310	0	0	0	19.82	598	14.961	8.432	3	19.82	0	2	1	1.77
180	310	0	0	0	3.05	659	17.605	9.504	3	3.05	0	2	1	1.85
190	310	0	0	0	18.82	665	18.396	10.643	3	18.82	0	2	1	1.73
200	310	0	0	0	4.36	713	20.624	11.782	3	4.36	0	2	1	1.75
210	310	0	0	0	13.81	651	20.035	19.755	5	13.81	0	3	2	1.01
220	310	0	0	0	15.13	655	21.004	21.223	5	15.13	0	3	2	0.99
230	310	0	0	0	8.55	642	21.501	23.083	5	8.55	0	3	2	0.93
240	310	0	0	0	6.58	654	23.068	25.015	5	6.58	0	3	2	0.92
250	310	0	0	0	11.94	640	23.300	26.562	5	11.94	0	3	2	0.88
260	310	0	0	0	13.41	602	22.759	28.404	5	13.41	0	3	2	0.80
270	310	0	0	0	18.82	559	21.796	27.273	5	18.82	0	2	2	0.80
280	310	0	0	0	8.33	497	20.372	28.011	5	8.33	0	2	2	0.73
290	310	0	0	0	1.48	491	21.181	25.405	3	1.48	0	2	1	0.83
300	310	0	0	0	5.92	433	19.420	28.913	4	5.92	0	2	1	0.67
10	320	0	0	0	9.75	26	0.109	0.112	3	9.75	0	2	1	0.97
20	320	0	0	0	7.95	69	0.374	0.235	3	7.95	0	2	1	1.59
30	320	0	0	0	1.82	112	0.767	0.416	3	1.82	0	2	1	1.84
40	320	0	0	0	2.26	159	1.345	0.650	3	2.26	0	2	1	2.07
50	320	0	0	0	14.28	197	1.906	0.941	3	14.28	0	2	1	2.02
60	320	0	0	0	14.14	288	3.017	1.285	3	14.14	0	2	1	2.35
70	320	0	0	0	11.29	300	3.596	1.678	3	11.29	0	2	1	2.14
80	320	0	0	0	9.56	350	4.725	2.144	3	9.56	0	2	1	2.20
90	320	0	0	0	3.58	390	5.654	2.654	3	3.58	0	2	1	2.13
100	320	0	0	0	13.97	450	7.281	3.194	3	13.97	0	2	1	2.28
110	320	0	0	0	18.43	470	8.405	3.789	3	18.43	0	2	1	2.22
120	320	0	0	0	7.40	505	9.698	4.504	3	7.40	0	2	1	2.15
130	320	0	0	0	10.29	578	11.624	5.156	3	10.29	0	2	1	2.25
140	320	0	0	0	15.76	557	11.931	6.019	3	15.76	0	2	1	1.98
150	320	0	0	0	8.59	580	13.562	6.866	3	8.59	0	2	1	1.98
160	320	0	0	0	14.34	695	16.929	7.723	3	14.34	0	2	1	2.19
170	320	0	0	0	12.99	657	16.889	8.731	3	12.99	0	2	1	1.93
180	320	0	0	0	13.22	647	17.709	9.848	3	13.22	0	2	1	1.80
190	320	0	0	0	5.95	705	20.353	10.909	3	5.95	0	2	1	1.87
200	320	0	0	0	13.96	716	21.974	12.165	3	13.96	0	2	1	1.81
210	320	0	0	0	15.21	656	21.356	13.635	3	15.21	0	2	1	1.57
220	320	0	0	0	4.84	778	25.648	22.296	5	4.84	0	3	2	1.15
230	320	0	0	0	15.32	755	26.092	23.888	5	15.32	0	3	2	1.09
240	320	0	0	0	4.78	744	26.733	25.776	5	4.78	0	3	2	1.04
250	320	0	0	0	7.10	631	23.414	27.616	5	7.10	0	3	2	0.85
260	320	0	0	0	18.49	560	21.857	29.545	5	18.49	0	3	2	0.74
270	320	0	0	0	15.40	559	22.695	31.775	5	15.40	0	3	2	0.71
280	320	0	0	0	5.64	515	21.926	29.688	5	5.64	0	2	2	0.74
290	320	0	0	0	7.38	513	22.975	30.972	5	7.38	0	2	2	0.74
300	320	0	0	0	15.37	461	20.888	31.695	5	15.37	0	2	2	0.66
310	320	0	0	0	1.94	406	19.411	31.932	4	1.94	0	2	1	0.61
10	330	0	0	0	15.48	27	0.114	0.125	3	15.48	0	2	1	0.91
20	330	0	0	0	15.54	64	0.360	0.249	3	15.54	0	2	1	1.45
30	330	0	0	0	6.88	94	0.708	0.422	3	6.88	0	2	1	1.68
40	330	0	0	0	2.42	170	1.410	0.662	3	2.42	0	2	1	2.13
50	330	0	0	0	14.74	211	2.199	0.959	3	14.74	0	2	1	2.29
60	330	0	0	0	4.63	254	2.820	1.322	3	4.63	0	2	1	2.13
70	330	0	0	0	17.58	341	4.302	1.728	3	17.58	0	2	1	2.49
80	330	0	0	0	6.41	357	4.885	2.183	3	6.41	0	2	1	2.24
90	330	0	0	0	17.44	411	6.261	2.717	3	17.44	0	2	1	2.30
100	330	0	0	0	9.68	458	7.685	3.356	3	9.68	0	2	1	2.29
110	330	0	0	0	18.31	498	9.031	3.896	3	18.31	0	2	1	2.32
120	330	0	0	0	15.38	518	9.941	4.630	3	15.38	0	2	1	2.15
130	330	0	0	0	19.77	588	12.139	5.339	3	19.77	0	2	1	2.27
140	330	0	0	0	15.03	606	13.411	6.207	3	15.03	0	2	1	2.16
150	330	0	0	0	17.23	608	14.369	7.032	3	17.23	0	2	1	2.04
160	330	0	0	0	10.05	671	16.599	8.001	3	10.05	0	2	1	2.07
170	330	0	0	0	9.87	668	17.620	9.225	3	9.87	0	2	1	1.91
180	330	0	0	0	16.95	622	17.358	10.062	3	16.95	0	2	1	1.73
190	330	0	0	0	5.06	797	23.568	11.199	3	5.06	0	2	1	2.10
200	330	0	0	0	8.45	731	22.608	12.442	3	8.45	0	2	1	1.82
210	330	0	0	0	13.40	764	24.985	13.792	3	13.40	0	2	1	1.81
220	330	0	0	0	0.36	823	28.134	15.042	3	0.36	0	2	1	1.87
230	330	0	0	0	17.06	769	27.482	24.780	5	17.06	0	3	2	1.11
240	330	0	0	0	2.55	848	31.809	26.944	5	2.55	0	3	2	1.18
250	330	0	0	0	17.83	705	27.443	28.706	5	17.83	0	3	2	0.96
260	330	0	0	0	6.69	696	28.104	31.367	7	6.69	0	3	3	0.90
270	330	0	0	0	11.77	721	30.541	33.004	5	11.77	0	3	2	0.93
280	330	0	0	0	15.56	654	28.632	34.712	5	15.56	0	3	2	0.82
290	330	0	0	0	15.59	597	27.430	33.303	5	15.59	0	2	2	0.82
300	330	0	0	0	2.86	577	27.525	34.229	5	2.86	0	2	2	0.80
310	330	0	0	0	15.16	507	25.078	34.995	5	15.16	0	2	2	0.72
320	330	0	0	0	2.60	416	21.526	35.026	4	2.60	0	2	1	0.61
10	340	0	0	0	10.92	26	0.114	0.119	3	10.92	0	2	1	0.96
20	340	0	0	0	16.77	75	0.439	0.250	3	16.77	0	2	1	1.75
30	340	0	0	0	10.22	100	0.723	0.438	3	10.22	0	2	1	1.65
40	340	0	0	0	12.74	157	1.342	0.680	3	12.74	0	2	1	1.97
50	340	0	0	0	3.59	223	2.376	1.000	3	3.59	0	2	1	2.37

Table 6: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
60	340	0	0	0	17.40	269	3.025	1.350	3	17.40	0	2	1	2.24
70	340	0	0	0	9.81	358	4.627	1.774	3	9.81	0	2	1	2.61
80	340	0	0	0	17.57	384	5.484	2.260	3	17.57	0	2	1	2.43
90	340	0	0	0	7.68	417	6.512	2.819	3	7.68	0	2	1	2.31
100	340	0	0	0	7.13	465	7.986	3.396	3	7.13	0	2	1	2.35
110	340	0	0	0	16.53	509	9.759	4.295	3	16.53	0	2	1	2.27
120	340	0	0	0	3.87	547	10.776	4.749	3	3.87	0	2	1	2.27
130	340	0	0	0	19.64	649	13.810	5.538	3	19.64	0	2	1	2.49
140	340	0	0	0	4.85	617	14.088	6.377	3	4.85	0	2	1	2.21
150	340	0	0	0	19.07	624	14.982	7.227	3	19.07	0	2	1	2.07
160	340	0	0	0	2.89	728	18.647	8.209	3	2.89	0	2	1	2.27
170	340	0	0	0	11.72	637	17.463	9.340	3	11.72	0	2	1	1.87
180	340	0	0	0	2.94	735	20.833	10.430	3	2.94	0	2	1	2.00
190	340	0	0	0	10.82	672	20.468	11.522	3	10.82	0	2	1	1.78
200	340	0	0	0	11.62	889	28.282	13.011	3	11.62	0	2	1	2.17
210	340	0	0	0	16.52	747	24.733	14.167	3	16.52	0	2	1	1.75
220	340	0	0	0	12.09	850	29.746	15.539	3	12.09	0	2	1	1.91
230	340	0	0	0	3.19	884	32.367	25.674	5	3.19	0	3	2	1.26
240	340	0	0	0	7.89	862	33.093	27.788	5	7.89	0	3	2	1.19
250	340	0	0	0	15.42	738	29.898	29.599	5	15.42	0	3	2	1.01
260	340	0	0	0	2.54	782	32.892	31.767	5	2.54	0	3	2	1.04
270	340	0	0	0	14.06	682	30.313	34.161	5	14.06	0	3	2	0.89
280	340	0	0	0	2.51	725	33.212	36.257	5	2.51	0	3	2	0.92
290	340	0	0	0	3.68	662	31.570	38.586	5	3.68	0	3	2	0.82
300	340	0	0	0	4.42	601	29.682	37.240	5	4.42	0	2	2	0.80
310	340	0	0	0	6.24	570	29.418	37.344	5	6.24	0	2	2	0.79
320	340	0	0	0	10.66	492	26.599	38.538	5	10.66	0	2	2	0.69
330	340	0	0	0	14.83	458	25.678	38.671	4	14.83	0	2	1	0.66
10	350	0	0	0	2.18	28	0.123	0.122	3	2.18	0	2	1	1.01
20	350	0	0	0	19.76	63	0.393	0.266	3	19.76	0	2	1	1.48
30	350	0	0	0	19.37	115	0.922	0.444	3	19.37	0	2	1	2.08
40	350	0	0	0	3.13	170	1.491	0.696	3	3.13	0	2	1	2.14
50	350	0	0	0	10.12	187	1.914	1.021	3	10.12	0	2	1	1.87
60	350	0	0	0	4.57	269	3.137	1.389	3	4.57	0	2	1	2.26
70	350	0	0	0	16.77	307	4.102	1.828	3	16.77	0	2	1	2.24
80	350	0	0	0	13.01	367	5.400	2.343	3	13.01	0	2	1	2.30
90	350	0	0	0	5.87	414	6.688	2.880	3	5.87	0	2	1	2.32
100	350	0	0	0	2.15	492	8.584	3.480	3	2.15	0	2	1	2.47
110	350	0	0	0	9.47	526	10.016	4.118	3	9.47	0	2	1	2.43
120	350	0	0	0	11.34	544	11.510	4.916	3	11.34	0	2	1	2.34
130	350	0	0	0	19.16	586	13.115	5.650	3	19.16	0	2	1	2.32
140	350	0	0	0	19.52	589	14.011	6.490	3	19.52	0	2	1	2.16
150	350	0	0	0	10.03	710	17.689	7.450	3	10.03	0	2	1	2.37
160	350	0	0	0	16.86	722	19.130	8.479	3	16.86	0	2	1	2.26
170	350	0	0	0	8.19	781	21.680	9.466	3	8.19	0	2	1	2.29
180	350	0	0	0	17.80	685	20.603	10.608	3	17.80	0	2	1	1.94
190	350	0	0	0	14.07	793	25.215	12.358	3	14.07	0	2	1	2.04
200	350	0	0	0	13.56	802	27.265	13.282	3	13.56	0	2	1	2.05
210	350	0	0	0	6.76	807	28.271	14.547	3	6.76	0	2	1	1.94
220	350	0	0	0	3.95	834	30.125	16.155	3	3.95	0	2	1	1.86
230	350	0	0	0	12.14	821	30.728	17.343	3	12.14	0	2	1	1.77
240	350	0	0	0	8.56	840	32.892	28.607	5	8.56	0	3	2	1.15
250	350	0	0	0	7.11	756	31.347	30.798	5	7.11	0	3	2	1.02
260	350	0	0	0	19.77	877	38.272	33.188	5	19.77	0	3	2	1.15
270	350	0	0	0	2.28	792	36.282	38.096	15	2.28	0	3	7	0.95
280	350	0	0	0	16.33	749	35.383	37.422	5	16.33	0	3	2	0.95
290	350	0	0	0	15.46	685	34.382	40.416	5	15.46	0	3	2	0.85
300	350	0	0	0	4.72	673	34.871	39.334	5	4.72	0	2	2	0.89
310	350	0	0	0	14.05	619	33.614	40.305	5	14.05	0	2	2	0.83
320	350	0	0	0	10.37	606	34.452	41.489	5	10.37	0	2	2	0.83
330	350	0	0	0	1.54	517	30.520	46.445	15	1.54	0	2	7	0.66
340	350	0	0	0	17.51	431	26.130	42.853	4	17.51	0	2	1	0.61
10	360	0	0	0	12.85	24	0.113	0.128	3	12.85	0	2	1	0.89
20	360	0	0	0	18.90	72	0.429	0.264	3	18.90	0	2	1	1.63
30	360	0	0	0	3.20	114	0.884	0.458	3	3.20	0	2	1	1.93
40	360	0	0	0	6.60	158	1.439	0.783	3	6.60	0	2	1	1.84
50	360	0	0	0	1.44	190	2.020	1.046	3	1.44	0	2	1	1.93
60	360	0	0	0	9.41	244	3.004	1.446	3	9.41	0	2	1	2.08
70	360	0	0	0	17.31	339	4.784	1.878	3	17.31	0	2	1	2.55
80	360	0	0	0	12.96	377	5.860	2.435	3	12.96	0	2	1	2.41
90	360	0	0	0	16.65	429	7.432	3.004	3	16.65	0	2	1	2.47
100	360	0	0	0	11.34	444	8.267	3.660	3	11.34	0	2	1	2.26
110	360	0	0	0	1.44	522	10.171	4.265	3	1.44	0	2	1	2.38
120	360	0	0	0	15.10	585	12.463	5.035	3	15.10	0	2	1	2.48
130	360	0	0	0	7.97	706	16.449	5.878	3	7.97	0	2	1	2.80
140	360	0	0	0	18.24	717	17.579	6.733	3	18.24	0	2	1	2.61
150	360	0	0	0	7.69	606	15.962	7.657	3	7.69	0	2	1	2.08
160	360	0	0	0	8.36	728	20.282	8.642	3	8.36	0	2	1	2.35
170	360	0	0	0	19.79	745	22.033	9.874	3	19.79	0	2	1	2.23
180	360	0	0	0	12.15	777	23.912	10.990	3	12.15	0	2	1	2.18
190	360	0	0	0	9.36	744	24.616	12.324	3	9.36	0	2	1	2.00
200	360	0	0	0	18.06	818	28.043	13.515	3	18.06	0	2	1	2.08
210	360	0	0	0	16.17	953	34.483	14.972	3	16.17	0	2	1	2.30
220	360	0	0	0	19.09	842	31.970	16.583	3	19.09	0	2	1	1.93
230	360	0	0	0	16.14	870	34.761	18.109	3	16.14	0	2	1	1.92
240	360	0	0	0	7.81	863	35.932	29.810	5	7.81	0	3	2	1.21
250	360	0	0	0	9.48	741	31.630	31.934	5	9.48	0	3	2	0.99
260	360	0	0	0	1.44	863	38.730	34.148	5	1.44	0	3	2	1.13
270	360	0	0	0	16.14	737	34.055	36.562	5	16.14	0	3	2	0.93

Table 7: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
280	360	0	0	0	14.68	736	35.542	38.408	5	14.68	0	3	2	0.93
290	360	0	0	0	10.48	721	36.709	41.542	5	10.48	0	3	2	0.88
300	360	0	0	0	7.16	706	36.390	43.816	5	7.16	0	3	2	0.83
310	360	0	0	0	0.13	697	38.199	42.451	5	0.13	0	2	2	0.90
320	360	0	0	0	3.85	680	38.713	43.203	5	3.85	0	2	2	0.90
330	360	0	0	0	19.32	560	33.165	44.269	5	19.32	0	2	2	0.75
340	360	0	0	0	7.25	481	29.733	45.297	5	7.25	0	2	2	0.66
350	360	0	0	0	13.19	406	26.305	45.605	4	13.19	0	2	1	0.58
10	370	0	0	0	5.77	32	0.155	0.131	3	5.77	0	2	1	1.19
20	370	0	0	0	19.16	62	0.394	0.277	3	19.16	0	2	1	1.42
30	370	0	0	0	16.93	100	0.785	0.472	3	16.93	0	2	1	1.66
40	370	0	0	0	14.99	177	1.648	0.737	3	14.99	0	2	1	2.24
50	370	0	0	0	13.08	221	2.496	1.073	3	13.08	0	2	1	2.33
60	370	0	0	0	0.83	283	3.580	1.454	3	0.83	0	2	1	2.46
70	370	0	0	0	5.82	367	5.224	1.916	3	5.82	0	2	1	2.73
80	370	0	0	0	14.15	373	5.786	2.474	3	14.15	0	2	1	2.34
90	370	0	0	0	12.08	440	7.623	3.170	3	12.08	0	2	1	2.40
100	370	0	0	0	4.02	520	9.742	3.671	3	4.02	0	2	1	2.65
110	370	0	0	0	1.39	573	11.538	4.388	3	1.39	0	2	1	2.63
120	370	0	0	0	18.06	607	13.319	5.117	3	18.06	0	2	1	2.60
130	370	0	0	0	5.63	667	15.625	5.947	3	5.63	0	2	1	2.63
140	370	0	0	0	8.60	705	17.691	6.868	3	8.60	0	2	1	2.58
150	370	0	0	0	15.90	678	18.181	7.801	3	15.90	0	2	1	2.33
160	370	0	0	0	11.24	767	21.424	8.924	3	11.24	0	2	1	2.40
170	370	0	0	0	2.09	848	25.393	10.048	3	2.09	0	2	1	2.53
180	370	0	0	0	7.53	797	25.321	11.229	3	7.53	0	2	1	2.25
190	370	0	0	0	3.30	759	25.350	12.510	3	3.30	0	2	1	2.03
200	370	0	0	0	8.86	840	29.429	14.165	3	8.86	0	2	1	2.08
210	370	0	0	0	11.47	872	32.110	15.325	3	11.47	0	2	1	2.10
220	370	0	0	0	14.48	909	35.130	16.800	3	14.48	0	2	1	2.09
230	370	0	0	0	10.10	835	33.544	18.555	3	10.10	0	2	1	1.81
240	370	0	0	0	7.33	815	33.982	30.840	5	7.33	0	3	2	1.10
250	370	0	0	0	1.01	887	38.978	33.247	5	1.01	0	3	2	1.17
260	370	0	0	0	7.04	816	37.813	35.232	5	7.04	0	3	2	1.07
270	370	0	0	0	12.57	956	45.615	37.600	5	12.57	0	3	2	1.21
280	370	0	0	0	17.85	793	39.273	39.788	5	17.85	0	3	2	0.99
290	370	0	0	0	16.91	828	42.612	42.594	5	16.91	0	3	2	1.00
300	370	0	0	0	14.97	774	41.914	45.228	5	14.97	0	3	2	0.93
310	370	0	0	0	6.37	774	43.471	48.174	5	6.37	0	3	2	0.90
320	370	0	0	0	8.07	720	42.622	46.097	5	8.07	0	2	2	0.92
330	370	0	0	0	6.56	685	41.858	47.036	5	6.56	0	2	2	0.89
340	370	0	0	0	7.32	691	44.061	47.954	5	7.32	0	2	2	0.92
350	370	0	0	0	6.61	509	33.990	48.984	5	6.61	0	2	2	0.69
360	370	0	0	0	1.50	484	33.544	50.928	4	1.50	0	2	1	0.66
10	380	0	0	0	4.68	28	0.139	0.131	3	4.68	0	2	1	1.06
20	380	0	0	0	2.21	76	0.473	0.276	3	2.21	0	2	1	1.71
30	380	0	0	0	6.80	114	0.904	0.485	3	6.80	0	2	1	1.86
40	380	0	0	0	17.71	167	1.574	0.750	3	17.71	0	2	1	2.10
50	380	0	0	0	11.91	200	2.229	1.099	3	11.91	0	2	1	2.03
60	380	0	0	0	5.04	302	3.954	1.505	3	5.04	0	2	1	2.63
70	380	0	0	0	0.14	356	5.146	1.994	3	0.14	0	2	1	2.58
80	380	0	0	0	6.16	413	6.563	2.494	3	6.16	0	2	1	2.63
90	380	0	0	0	10.91	419	7.455	3.096	3	10.91	0	2	1	2.41
100	380	0	0	0	7.09	495	9.571	3.746	3	7.09	0	2	1	2.56
110	380	0	0	0	7.85	569	11.785	4.481	3	7.85	0	2	1	2.63
120	380	0	0	0	19.68	638	14.636	5.377	3	19.68	0	2	1	2.72
130	380	0	0	0	8.70	725	17.513	6.102	3	8.70	0	2	1	2.87
140	380	0	0	0	13.69	692	17.748	7.009	3	13.69	0	2	1	2.53
150	380	0	0	0	19.64	727	20.014	8.042	3	19.64	0	2	1	2.49
160	380	0	0	0	14.28	674	19.470	9.154	3	14.28	0	2	1	2.13
170	380	0	0	0	14.00	765	23.462	10.223	3	14.00	0	2	1	2.30
180	380	0	0	0	2.39	775	25.256	11.442	3	2.39	0	2	1	2.21
190	380	0	0	0	0.79	831	28.240	12.757	3	0.79	0	2	1	2.21
200	380	0	0	0	11.16	900	32.881	14.379	3	11.16	0	2	1	2.29
210	380	0	0	0	3.14	941	35.479	15.643	3	3.14	0	2	1	2.27
220	380	0	0	0	11.98	939	37.392	17.354	3	11.98	0	2	1	2.15
230	380	0	0	0	19.25	943	39.334	18.966	3	19.25	0	2	1	2.07
240	380	0	0	0	19.53	846	36.748	20.773	3	19.53	0	2	1	1.77
250	380	0	0	0	0.67	911	41.830	36.949	17	0.67	0	3	8	1.13
260	380	0	0	0	11.83	868	41.468	36.450	5	11.83	0	3	2	1.14
270	380	0	0	0	19.15	929	45.918	38.905	5	19.15	0	3	2	1.18
280	380	0	0	0	1.24	801	41.788	41.184	5	1.24	0	3	2	1.01
290	380	0	0	0	17.65	833	45.136	44.270	5	17.65	0	3	2	1.02
300	380	0	0	0	18.69	839	46.784	46.686	5	18.69	0	3	2	1.00
310	380	0	0	0	5.05	811	47.079	49.800	5	5.05	0	3	2	0.95
320	380	0	0	0	10.27	745	45.693	53.146	5	10.27	0	3	2	0.86
330	380	0	0	0	10.48	716	45.609	50.052	5	10.48	0	2	2	0.91
340	380	0	0	0	3.32	692	46.115	51.031	5	3.32	0	2	2	0.90
350	380	0	0	0	15.54	606	42.386	53.422	5	15.54	0	2	2	0.79
360	380	0	0	0	10.61	571	41.624	54.271	5	10.61	0	2	2	0.77
370	380	0	0	0	6.32	491	36.634	53.940	4	6.32	0	2	1	0.68
10	390	0	0	0	13.09	24	0.133	0.135	3	13.09	0	2	1	0.99
20	390	0	0	0	1.11	79	0.526	0.291	3	1.11	0	2	1	1.81
30	390	0	0	0	4.16	116	0.950	0.504	3	4.16	0	2	1	1.89
40	390	0	0	0	4.75	176	1.716	0.773	3	4.75	0	2	1	2.22
50	390	0	0	0	4.62	215	2.596	1.136	3	4.62	0	2	1	2.29
60	390	0	0	0	8.19	265	3.537	1.566	3	8.19	0	2	1	2.26
70	390	0	0	0	3.80	313	4.900	2.020	3	3.80	0	2	1	2.43
80	390	0	0	0	0.97	433	6.973	2.582	3	0.97	0	2	1	2.70

Table 8: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
90	390	0	0	0	7.54	448	8.172	3.170	3	7.54	0	2	1	2.58
100	390	0	0	0	3.03	505	9.922	3.852	3	3.03	0	2	1	2.58
110	390	0	0	0	14.13	534	11.352	4.622	3	14.13	0	2	1	2.46
120	390	0	0	0	14.83	645	14.804	5.417	3	14.83	0	2	1	2.73
130	390	0	0	0	14.09	632	15.765	6.233	3	14.09	0	2	1	2.53
140	390	0	0	0	10.54	655	17.357	7.198	3	10.54	0	2	1	2.41
150	390	0	0	0	15.28	706	19.911	8.190	3	15.28	0	2	1	2.43
160	390	0	0	0	9.57	739	22.483	9.347	3	9.57	0	2	1	2.41
170	390	0	0	0	11.43	784	25.029	10.532	3	11.43	0	2	1	2.38
180	390	0	0	0	17.25	733	25.120	11.743	3	17.25	0	2	1	2.14
190	390	0	0	0	18.04	823	29.916	13.483	3	18.04	0	2	1	2.22
200	390	0	0	0	9.78	878	32.868	14.902	3	9.78	0	2	1	2.21
210	390	0	0	0	1.58	893	34.852	17.172	9	1.58	0	2	4	2.03
220	390	0	0	0	12.62	888	36.347	17.666	3	12.62	0	2	1	2.06
230	390	0	0	0	3.06	946	39.918	19.342	3	3.06	0	2	1	2.06
240	390	0	0	0	18.07	902	39.533	21.095	3	18.07	0	2	1	1.87
250	390	0	0	0	7.21	857	39.551	23.168	3	7.21	0	2	1	1.71
260	390	0	0	0	15.31	865	42.522	37.425	5	15.31	0	3	2	1.14
270	390	0	0	0	10.52	873	45.190	40.467	5	10.52	0	3	2	1.12
280	390	0	0	0	0.35	981	51.858	45.435	13	0.35	0	3	6	1.14
290	390	0	0	0	7.07	877	48.427	45.764	5	7.07	0	3	2	1.06
300	390	0	0	0	2.84	848	48.552	48.205	5	2.84	0	3	2	1.01
310	390	0	0	0	4.06	910	54.264	51.589	5	4.06	0	3	2	1.05
320	390	0	0	0	15.75	806	50.617	54.448	5	15.75	0	3	2	0.93
330	390	0	0	0	11.06	774	51.251	58.219	5	11.06	0	3	2	0.88
340	390	0	0	0	17.25	687	47.878	54.694	5	17.25	0	2	2	0.88
350	390	0	0	0	6.64	649	47.207	56.049	5	6.64	0	2	2	0.84
360	390	0	0	0	1.96	675	50.602	57.361	5	1.96	0	2	2	0.88
370	390	0	0	0	0.70	618	48.215	63.568	15	0.70	0	2	7	0.76
380	390	0	0	0	8.40	480	38.429	59.508	4	8.40	0	2	1	0.65
10	400	0	0	0	1.34	28	0.146	0.140	3	1.34	0	2	1	1.04
20	400	0	0	0	5.36	68	0.478	0.293	3	5.36	0	2	1	1.63
30	400	0	0	0	15.08	105	0.942	0.523	3	15.08	0	2	1	1.80
40	400	0	0	0	6.82	176	1.760	0.792	3	6.82	0	2	1	2.22
50	400	0	0	0	6.77	250	2.943	1.164	3	6.77	0	2	1	2.53
60	400	0	0	0	1.87	292	3.972	1.578	3	1.87	0	2	1	2.52
70	400	0	0	0	9.55	334	5.125	2.097	3	9.55	0	2	1	2.44
80	400	0	0	0	17.06	396	6.729	2.629	3	17.06	0	2	1	2.56
90	400	0	0	0	2.05	487	9.456	3.286	3	2.05	0	2	1	2.88
100	400	0	0	0	14.19	549	10.922	3.930	3	14.19	0	2	1	2.78
110	400	0	0	0	4.73	492	10.752	4.683	3	4.73	0	2	1	2.30
120	400	0	0	0	0.25	602	14.494	5.490	3	0.25	0	2	1	2.64
130	400	0	0	0	18.18	692	17.569	6.486	3	18.18	0	2	1	2.71
140	400	0	0	0	9.15	721	19.592	7.365	3	9.15	0	2	1	2.66
150	400	0	0	0	1.77	737	21.493	8.457	3	1.77	0	2	1	2.54
160	400	0	0	0	14.16	774	23.931	9.649	3	14.16	0	2	1	2.48
170	400	0	0	0	0.43	817	27.139	10.791	3	0.43	0	2	1	2.51
180	400	0	0	0	16.80	789	27.229	12.222	3	16.80	0	2	1	2.23
190	400	0	0	0	0.40	845	31.423	13.463	3	0.40	0	2	1	2.33
200	400	0	0	0	0.81	876	33.693	15.186	3	0.81	0	2	1	2.22
210	400	0	0	0	18.50	936	37.290	16.496	3	18.50	0	2	1	2.26
220	400	0	0	0	2.35	1036	43.629	18.090	3	2.35	0	2	1	2.41
230	400	0	0	0	16.28	941	41.818	19.830	3	16.28	0	2	1	2.11
240	400	0	0	0	16.46	961	44.523	21.857	3	16.46	0	2	1	2.04
250	400	0	0	0	5.23	950	46.164	23.703	3	5.23	0	2	1	1.95
260	400	0	0	0	1.10	982	49.539	28.311	13	1.10	0	2	6	1.75
270	400	0	0	0	10.35	949	49.762	41.818	5	10.35	0	3	2	1.19
280	400	0	0	0	16.95	903	50.439	43.953	5	16.95	0	3	2	1.15
290	400	0	0	0	15.30	1013	58.471	47.363	5	15.30	0	3	2	1.23
300	400	0	0	0	11.60	898	53.133	50.072	5	11.60	0	3	2	1.06
310	400	0	0	0	3.32	983	60.988	57.355	15	3.32	0	3	7	1.06
320	400	0	0	0	18.81	823	53.287	56.148	5	18.81	0	3	2	0.95
330	400	0	0	0	6.33	815	55.590	59.205	5	6.33	0	3	2	0.94
340	400	0	0	0	17.89	808	57.533	62.730	5	17.89	0	3	2	0.92
350	400	0	0	0	11.26	775	57.883	59.088	5	11.26	0	2	2	0.98
360	400	0	0	0	4.27	698	54.127	59.756	5	4.27	0	2	2	0.91
370	400	0	0	0	3.35	625	50.289	62.738	5	3.35	0	2	2	0.80
380	400	0	0	0	16.33	550	45.654	63.721	5	16.33	0	2	2	0.72
390	400	0	0	0	11.61	539	46.430	63.926	4	11.61	0	2	1	0.73

Table 9: Comparison between LINFSOL and B-P on nondegenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
10	20	5	0	0	2.72	18	0.005	0.078	14	2.72	0	3	1	0.07
10	30	10	0	0	2.02	23	0.009	0.045	6	2.02	0	3	1	0.20
20	30	5	0	0	15.53	37	0.019	0.132	17	15.53	0	3	1	0.15
10	40	15	0	0	14.37	29	0.014	0.101	15	14.37	0	4	1	0.13
20	40	10	0	0	8.83	39	0.025	0.073	6	8.83	0	3	1	0.35
30	40	5	0	0	5.92	46	0.041	0.194	16	5.92	0	3	1	0.21
10	50	20	0	0	6.90	26	0.016	0.065	7	6.90	0	4	1	0.25
20	50	15	0	0	18.37	40	0.034	0.088	6	18.37	0	3	1	0.39
30	50	10	0	0	15.06	55	0.059	0.155	8	15.06	0	3	1	0.38
40	50	5	0	0	10.15	57	0.075	0.528	33	10.15	0	5	1	0.14
10	60	25	0	0	11.47	30	0.021	0.070	7	11.47	0	3	1	0.30
20	60	20	0	0	14.64	58	0.054	0.136	9	14.64	0	4	1	0.39
30	60	15	0	0	19.79	66	0.081	0.283	18	19.79	0	4	1	0.28
40	60	10	0	0	13.25	78	0.118	0.272	10	13.25	0	3	1	0.43
50	60	5	0	0	4.08	81	0.149	0.567	22	4.08	0	4	1	0.26
10	70	30	0	0	11.12	30	0.025	0.070	6	11.12	0	3	1	0.36
20	70	25	0	0	18.91	49	0.056	0.123	7	18.91	0	3	1	0.45
30	70	20	0	0	3.35	103	0.144	0.233	10	3.35	0	3	1	0.62
40	70	15	0	0	9.03	96	0.165	0.418	18	9.03	0	3	1	0.40
50	70	10	0	0	6.36	77	0.168	0.395	8	6.36	0	3	1	0.42
60	70	5	0	0	8.65	90	0.225	0.906	24	8.65	0	4	1	0.25
10	80	35	0	0	2.39	35	0.033	0.076	6	2.39	0	3	1	0.43
20	80	30	0	0	1.97	62	0.080	0.134	6	1.97	0	3	1	0.60
30	80	25	0	0	10.29	113	0.182	0.283	11	10.29	0	4	1	0.64
40	80	20	0	0	14.81	108	0.218	0.347	8	14.81	0	3	1	0.63
50	80	15	0	0	4.22	104	0.246	1.130	44	4.22	0	5	1	0.22
60	80	10	0	0	5.82	108	0.293	0.960	24	5.82	0	3	1	0.30
70	80	5	0	0	16.65	106	0.330	1.371	25	16.65	0	5	1	0.24
10	90	40	0	0	17.58	32	0.034	0.093	7	17.58	0	4	1	0.37
20	90	35	0	0	2.45	60	0.086	0.177	9	2.45	0	3	1	0.49
30	90	30	0	0	16.33	87	0.159	0.288	10	16.33	0	3	1	0.55
40	90	25	0	0	10.16	105	0.231	0.349	8	10.16	0	2	1	0.66
50	90	20	0	0	0.19	113	0.299	0.704	15	0.19	0	4	1	0.42
60	90	15	0	0	6.92	135	0.406	0.879	13	6.92	0	4	1	0.46
70	90	10	0	0	18.96	143	0.484	1.219	17	18.96	0	4	1	0.40
80	90	5	0	0	11.65	125	0.483	2.729	43	11.65	0	7	1	0.18
10	100	45	0	0	18.25	31	0.037	0.103	7	18.25	0	4	1	0.36
20	100	40	0	0	1.79	58	0.092	0.179	8	1.79	0	3	1	0.51
30	100	35	0	0	10.78	102	0.206	0.298	9	10.78	0	3	1	0.69
40	100	30	0	0	16.75	98	0.251	0.468	11	16.75	0	3	1	0.54
50	100	25	0	0	3.36	149	0.423	0.646	11	3.36	0	3	1	0.66
60	100	20	0	0	17.08	135	0.446	0.736	8	17.08	0	2	1	0.61
70	100	15	0	0	1.01	164	0.616	1.280	15	1.01	0	4	1	0.48
80	100	10	0	0	3.56	144	0.605	1.818	17	3.56	0	5	1	0.33
90	100	5	0	0	13.95	128	0.616	2.099	20	13.95	0	4	1	0.29
10	110	50	0	0	17.83	39	0.050	0.112	7	17.83	0	4	1	0.45
20	110	45	0	0	16.60	64	0.111	0.195	8	16.60	0	3	1	0.57
30	110	40	0	0	19.92	105	0.230	0.328	9	19.92	0	3	1	0.70
40	110	35	0	0	16.39	122	0.326	0.578	15	16.39	0	3	1	0.56
50	110	30	0	0	14.97	126	0.411	0.693	9	14.97	0	3	1	0.59
60	110	25	0	0	6.29	140	0.522	0.769	7	6.29	0	2	1	0.68
70	110	20	0	0	16.60	188	0.760	1.062	10	16.60	0	2	1	0.72
80	110	15	0	0	6.89	170	0.778	7.250	144	6.89	0	14	1	0.11
90	110	10	0	0	15.94	168	0.862	1.452	10	15.94	0	2	1	0.59
100	110	5	0	0	14.35	144	0.806	3.172	25	14.35	0	5	1	0.25
10	120	55	0	0	15.23	28	0.039	0.116	7	15.23	0	4	1	0.34
20	120	50	0	0	11.71	64	0.120	0.196	7	11.71	0	3	1	0.61
30	120	45	0	0	5.73	80	0.194	0.380	10	5.73	0	4	1	0.51
40	120	40	0	0	15.84	146	0.420	0.541	10	15.84	0	3	1	0.78
50	120	35	0	0	15.60	147	0.504	0.737	9	15.60	0	3	1	0.68
60	120	30	0	0	4.44	189	0.729	0.912	9	4.44	0	2	1	0.80
70	120	25	0	0	13.12	183	0.816	1.405	14	13.12	0	3	1	0.58
80	120	20	0	0	2.64	185	0.913	1.295	7	2.64	0	2	1	0.70
90	120	15	0	0	6.91	213	1.156	1.376	5	6.91	0	2	1	0.84
100	120	10	0	0	0.72	158	0.963	1.722	7	0.72	0	2	1	0.56
110	120	5	0	0	14.13	160	1.114	3.800	21	14.13	0	5	1	0.29
10	130	60	0	0	2.51	33	0.051	0.134	8	2.51	0	4	1	0.38
20	130	55	0	0	2.03	78	0.161	0.213	7	2.03	0	3	1	0.76
30	130	50	0	0	8.21	103	0.272	0.414	10	8.21	0	3	1	0.66
40	130	45	0	0	4.35	131	0.414	0.555	8	4.35	0	3	1	0.75
50	130	40	0	0	0.57	170	0.620	0.798	9	0.57	0	3	1	0.78
60	130	35	0	0	3.12	210	0.884	1.000	9	3.12	0	2	1	0.88
70	130	30	0	0	18.02	212	1.025	1.363	10	18.02	0	2	1	0.75
80	130	25	0	0	10.78	264	1.383	1.463	9	10.78	0	2	1	0.95
90	130	20	0	0	13.49	227	1.329	2.036	9	13.49	0	3	1	0.65
100	130	15	0	0	18.66	205	1.327	2.932	15	18.66	0	4	1	0.45
110	130	10	0	0	6.67	199	1.399	6.655	61	6.67	0	6	1	0.21
120	130	5	0	0	12.75	170	1.344	9.406	75	12.75	0	8	1	0.14
10	140	65	0	0	18.47	39	0.065	0.144	8	18.47	0	4	1	0.45
20	140	60	0	0	8.23	69	0.158	0.251	8	8.23	0	3	1	0.63
30	140	55	0	0	7.35	100	0.281	0.496	11	7.35	0	4	1	0.57
40	140	50	0	0	9.06	131	0.444	0.602	9	9.06	0	3	1	0.74
50	140	45	0	0	19.65	181	0.708	.0.826	8	19.65	0	3	1	0.86
60	140	40	0	0	1.16	219	0.992	1.044	9	1.16	0	2	1	0.95
70	140	35	0	0	17.13	207	1.058	1.366	10	17.13	0	2	1	0.77
80	140	30	0	0	13.45	194	1.116	1.748	11	13.45	0	2	1	0.64
90	140	25	0	0	19.83	255	1.589	1.854	8	19.83	0	2	1	0.86
100	140	20	0	0	2.47	238	1.646	2.277	10	2.47	0	2	1	0.72

Table 10: Comparison between LINFSOL and B-P on primal degenerate problems

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
110	140	15	0	0	10.22	252	1.880	2.261	6	10.22	0	2	1	0.83
120	140	10	0	0	5.05	193	1.605	2.621	6	5.05	0	2	1	0.61
130	140	5	0	0	7.10	172	1.580	4.412	19	7.10	0	3	1	0.36
10	150	70	0	0	13.09	32	0.059	0.159	8	13.09	0	4	1	0.37
20	150	65	0	0	18.82	64	0.158	0.269	8	18.82	0	3	1	0.59
30	150	60	0	0	19.67	107	0.324	0.538	12	19.67	0	4	1	0.60
40	150	55	0	0	8.81	150	0.539	0.638	9	8.81	0	3	1	0.85
50	150	50	0	0	4.02	205	0.859	0.901	9	4.02	0	3	1	0.95
60	150	45	0	0	18.55	186	0.897	1.319	10	18.55	0	3	1	0.68
70	150	40	0	0	10.66	243	1.327	1.618	10	10.66	0	3	1	0.82
80	150	35	0	0	17.67	271	1.669	1.729	9	17.67	0	2	1	0.97
90	150	30	0	0	8.69	231	1.561	2.161	9	8.69	0	2	1	0.72
100	150	25	0	0	7.44	238	1.771	2.335	8	7.44	0	2	1	0.76
110	150	20	0	0	3.20	273	2.166	2.815	10	3.20	0	2	1	0.77
120	150	15	0	0	17.05	253	2.253	5.213	18	17.05	0	5	1	0.43
130	150	10	0	0	4.35	261	2.447	4.635	8	4.35	0	4	1	0.53
140	150	5	0	0	15.65	216	2.169	7.169	25	15.65	0	5	1	0.30
10	160	75	0	0	5.12	31	0.059	0.176	9	5.12	0	5	1	0.34
20	160	70	0	0	16.90	67	0.167	0.273	8	16.90	0	3	1	0.61
30	160	65	0	0	10.13	98	0.310	0.542	11	10.13	0	4	1	0.57
40	160	60	0	0	17.77	128	0.494	0.678	9	17.77	0	3	1	0.73
50	160	55	0	0	19.79	152	0.688	0.976	10	19.79	0	3	1	0.71
60	160	50	0	0	19.15	208	1.072	1.396	10	19.15	0	3	1	0.77
70	160	45	0	0	7.04	229	1.340	1.674	9	7.04	0	3	1	0.80
80	160	40	0	0	1.41	250	1.669	1.863	9	1.41	0	2	1	0.90
90	160	35	0	0	4.55	233	1.672	2.268	10	4.55	0	2	1	0.74
100	160	30	0	0	17.37	299	2.394	2.796	10	17.37	0	2	1	0.86
110	160	25	0	0	17.61	305	2.626	12.620	105	17.61	0	12	1	0.21
120	160	20	0	0	10.03	289	2.704	3.375	9	10.03	0	2	1	0.80
130	160	15	0	0	4.50	311	3.062	5.618	17	4.50	0	4	1	0.54
140	160	10	0	0	8.47	279	3.016	18.096	105	8.47	0	10	1	0.17
150	160	5	0	0	12.52	180	2.099	9.199	19	12.52	0	6	1	0.23
10	170	80	0	0	16.54	29	0.060	0.199	10	16.54	0	5	1	0.30
20	170	75	0	0	2.21	82	0.225	0.297	8	2.21	0	3	1	0.76
30	170	70	0	0	2.42	131	0.439	0.611	12	2.42	0	4	1	0.72
40	170	65	0	0	8.38	147	0.599	0.713	9	8.38	0	3	1	0.84
50	170	60	0	0	16.28	218	1.064	1.096	11	16.28	0	3	1	0.97
60	170	55	0	0	1.67	221	1.206	1.415	10	1.67	0	3	1	0.85
70	170	50	0	0	4.09	232	1.429	1.825	10	4.09	0	3	1	0.78
80	170	45	0	0	14.12	274	1.897	2.040	10	14.12	0	2	1	0.93
90	170	40	0	0	15.13	274	2.089	2.485	10	15.13	0	2	1	0.84
100	170	35	0	0	6.19	286	2.354	2.754	9	6.19	0	2	1	0.85
110	170	30	0	0	6.60	329	3.034	3.970	12	6.60	0	3	1	0.76
120	170	25	0	0	18.33	301	2.909	5.085	20	18.33	0	3	1	0.57
130	170	20	0	0	15.96	318	3.343	7.317	25	15.96	0	5	1	0.46
140	170	15	0	0	16.95	283	3.280	6.485	13	16.95	0	4	1	0.51
150	170	10	0	0	3.88	251	3.046	8.486	26	3.88	0	4	1	0.36
160	170	5	0	0	3.45	220	2.910	9.977	22	3.45	0	5	1	0.29
10	180	85	0	0	6.11	30	0.063	0.196	9	6.11	0	5	1	0.32
20	180	80	0	0	0.13	68	0.194	0.348	9	0.13	0	4	1	0.56
30	180	75	0	0	6.12	102	0.370	0.590	10	6.12	0	4	1	0.63
40	180	70	0	0	1.57	157	0.677	0.819	11	1.57	0	3	1	0.83
50	180	65	0	0	12.72	222	1.155	1.147	11	12.72	0	3	1	1.01
60	180	60	0	0	17.46	236	1.349	1.510	11	17.46	0	3	1	0.89
70	180	55	0	0	19.92	259	1.704	1.984	12	19.92	0	3	1	0.86
80	180	50	0	0	8.13	271	1.993	2.121	9	8.13	0	2	1	0.94
90	180	45	0	0	7.90	299	2.420	2.678	11	7.90	0	2	1	0.90
100	180	40	0	0	15.28	336	2.952	2.922	9	15.28	0	2	1	1.01
110	180	35	0	0	10.33	300	2.873	3.387	9	10.33	0	2	1	0.85
120	180	30	0	0	16.77	330	3.482	4.164	11	16.77	0	2	1	0.84
130	180	25	0	0	1.45	371	4.138	4.604	10	1.45	0	2	1	0.90
140	180	20	0	0	10.36	356	4.179	15.329	85	10.36	0	7	1	0.27
150	180	15	0	0	18.78	310	3.945	6.476	11	18.78	0	3	1	0.61
160	180	10	0	0	16.78	246	3.411	5.535	6	16.78	0	2	1	0.62
170	180	5	0	0	16.36	246	3.544	10.227	21	16.36	0	4	1	0.35
10	190	90	0	0	10.71	30	0.069	0.207	9	10.71	0	5	1	0.33
20	190	85	0	0	6.28	69	0.219	0.373	9	6.28	0	4	1	0.59
30	190	80	0	0	13.07	105	0.396	0.630	10	13.07	0	4	1	0.63
40	190	75	0	0	0.37	161	0.725	0.859	11	0.37	0	3	1	0.84
50	190	70	0	0	2.17	203	1.077	1.200	11	2.17	0	3	1	0.90
60	190	65	0	0	10.53	231	1.400	1.577	10	10.53	0	3	1	0.89
70	190	60	0	0	8.61	279	1.968	2.012	10	8.61	0	3	1	0.98
80	190	55	0	0	18.16	293	2.294	2.289	10	18.16	0	2	1	1.00
90	190	50	0	0	3.33	342	2.915	2.748	11	3.33	0	2	1	1.06
100	190	45	0	0	18.88	344	3.186	3.306	11	18.88	0	2	1	0.96
110	190	40	0	0	6.55	355	3.599	4.077	12	6.55	0	2	1	0.88
120	190	35	0	0	18.84	334	3.873	4.330	10	18.84	0	2	1	0.89
130	190	30	0	0	16.39	365	4.280	4.610	6	16.39	0	2	1	0.93
140	190	25	0	0	13.88	359	4.493	5.463	10	13.88	0	2	1	0.82
150	190	20	0	0	5.50	354	4.715	8.536	17	5.50	0	4	1	0.55
160	190	15	0	0	0.20	333	4.802	7.004	14	0.20	0	2	1	0.69
170	190	10	0	0	4.52	284	4.408	6.742	7	4.52	0	2	1	0.65
180	190	5	0	0	8.48	237	3.876	26.639	87	8.48	0	8	1	0.15
10	200	95	0	0	12.39	34	0.080	0.191	8	12.39	0	4	1	0.42
20	200	90	0	0	19.64	62	0.199	0.381	9	19.64	0	4	1	0.52
30	200	85	0	0	12.21	86	0.346	0.649	10	12.21	0	4	1	0.53
40	200	80	0	0	13.58	126	0.607	0.858	9	13.58	0	3	1	0.71
50	200	75	0	0	7.98	246	1.367	1.262	11	7.98	0	3	1	1.08
60	200	70	0	0	18.86	243	1.548	1.645	10	18.86	0	3	1	0.94

Table 11: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
70	200	65	0	0	1.11	307	2.216	2.130	10	1.11	0	3	1	1.04
80	200	60	0	0	0.54	332	2.653	2.291	9	0.54	0	2	1	1.16
90	200	55	0	0	2.16	288	2.619	2.839	10	2.16	0	2	1	0.92
100	200	50	0	0	0.34	401	3.926	3.388	10	0.34	0	2	1	1.16
110	200	45	0	0	4.02	443	4.693	3.873	9	4.02	0	2	1	1.21
120	200	40	0	0	19.16	373	4.365	4.187	8	19.16	0	2	1	1.04
130	200	35	0	0	1.50	429	5.256	5.439	12	1.50	0	2	1	0.97
140	200	30	0	0	19.00	382	5.094	5.679	10	19.00	0	2	1	0.90
150	200	25	0	0	11.72	352	4.927	7.344	10	11.72	0	3	1	0.67
160	200	20	0	0	6.25	337	5.191	6.886	9	6.25	0	2	1	0.75
170	200	15	0	0	10.15	330	5.290	57.698	217	10.15	0	21	1	0.09
180	200	10	0	0	14.89	320	5.392	14.519	21	14.89	0	5	1	0.37
190	200	5	0	0	0.64	294	5.230	14.493	26	0.64	0	4	1	0.36
10	210	100	0	0	13.01	32	0.080	0.231	10	13.01	0	5	1	0.35
20	210	95	0	0	5.28	78	0.256	0.414	10	5.28	0	4	1	0.62
30	210	90	0	0	5.28	115	0.489	0.699	11	5.28	0	4	1	0.70
40	210	85	0	0	6.06	173	0.854	0.924	10	6.06	0	3	1	0.92
50	210	80	0	0	3.17	194	1.136	1.396	12	3.17	0	3	1	0.81
60	210	75	0	0	12.22	242	1.629	1.737	10	12.22	0	3	1	0.94
70	210	70	0	0	19.63	263	2.017	2.209	10	19.63	0	3	1	0.91
80	210	65	0	0	16.07	357	3.048	2.829	11	16.07	0	3	1	1.08
90	210	60	0	0	3.79	363	3.387	3.486	11	3.79	0	3	1	0.97
100	210	55	0	0	15.26	336	3.452	4.305	12	15.26	0	3	1	0.80
110	210	50	0	0	3.19	363	4.078	4.272	11	3.19	0	2	1	0.95
120	210	45	0	0	16.49	417	5.071	4.945	11	16.49	0	2	1	1.03
130	210	40	0	0	9.72	421	5.424	5.155	9	9.72	0	2	1	1.05
140	210	35	0	0	2.48	395	5.436	6.093	11	2.48	0	2	1	0.89
150	210	30	0	0	14.06	409	6.172	6.829	11	14.06	0	2	1	0.90
160	210	25	0	0	12.26	372	5.950	7.100	8	12.26	0	2	1	0.84
170	210	20	0	0	15.91	338	5.589	10.382	17	15.91	0	3	1	0.54
180	210	15	0	0	9.75	355	6.197	26.962	73	9.75	0	8	1	0.23
190	210	10	0	0	12.35	294	5.539	63.227	191	12.35	0	17	1	0.09
200	210	5	0	0	11.55	307	6.056	15.440	17	11.55	0	4	1	0.39
10	220	105	0	0	8.37	41	0.105	0.244	10	8.37	0	5	1	0.43
20	220	100	0	0	13.04	61	0.212	0.435	10	13.04	0	4	1	0.49
30	220	95	0	0	14.99	119	0.516	0.734	11	14.99	0	4	1	0.70
40	220	90	0	0	14.13	143	0.745	1.065	11	14.13	0	3	1	0.70
50	220	85	0	0	4.00	215	1.315	1.421	11	4.00	0	3	1	0.93
60	220	80	0	0	10.30	244	1.768	1.937	12	10.30	0	3	1	0.91
70	220	75	0	0	2.86	296	2.334	2.369	11	2.86	0	3	1	0.99
80	220	70	0	0	16.63	362	3.227	2.977	11	16.63	0	3	1	1.08
90	220	65	0	0	3.30	313	3.049	3.616	11	3.30	0	3	1	0.84
100	220	60	0	0	2.37	416	4.425	3.798	11	2.37	0	2	1	1.17
110	220	55	0	0	0.40	431	5.049	4.386	10	0.40	0	2	1	1.15
120	220	50	0	0	18.27	368	4.592	4.949	10	18.27	0	2	1	0.93
130	220	45	0	0	16.04	480	6.514	5.666	10	16.04	0	2	1	1.15
140	220	40	0	0	16.16	450	6.422	6.525	11	16.16	0	2	1	0.98
150	220	35	0	0	10.46	447	6.899	6.771	8	10.46	0	2	1	1.02
160	220	30	0	0	9.20	449	7.408	7.823	10	9.20	0	2	1	0.95
170	220	25	0	0	13.81	392	6.734	8.813	11	13.83	0	2	1	0.76
180	220	20	0	0	4.61	418	7.662	9.283	10	4.61	0	2	1	0.83
190	220	15	0	0	13.05	360	7.077	12.227	10	13.05	0	3	1	0.58
200	220	10	0	0	6.81	395	8.113	10.261	8	6.81	0	2	1	0.79
210	220	5	0	0	12.14	302	6.600	18.061	20	12.14	0	4	2	0.37
10	230	110	0	0	3.60	32	0.087	0.251	10	3.60	0	5	1	0.35
20	230	105	0	0	18.09	69	0.250	0.451	10	18.09	0	4	1	0.55
30	230	100	0	0	18.40	121	0.559	0.770	11	18.40	0	4	1	0.73
40	230	95	0	0	12.96	150	0.826	1.234	11	12.96	0	3	1	0.67
50	230	90	0	0	18.00	222	1.413	1.529	12	18.00	0	3	1	0.92
60	230	85	0	0	15.60	214	1.587	2.009	12	15.60	0	3	1	0.79
70	230	80	0	0	12.75	303	2.520	2.535	12	12.75	0	3	1	0.99
80	230	75	0	0	19.34	320	3.023	3.106	10	19.34	0	3	1	0.97
90	230	70	0	0	16.71	345	3.549	3.773	11	16.71	0	3	1	0.94
100	230	65	0	0	9.10	392	4.423	3.921	10	9.10	0	2	1	1.13
110	230	60	0	0	5.95	445	5.435	4.635	11	5.95	0	2	1	1.17
120	230	55	0	0	13.66	453	6.061	5.682	14	13.66	0	2	1	1.07
130	230	50	0	0	13.47	471	6.684	6.310	11	13.47	0	2	1	1.06
140	230	45	0	0	6.71	471	7.162	6.688	10	6.71	0	2	1	1.07
150	230	40	0	0	0.70	491	7.987	8.031	12	0.70	0	2	1	0.99
160	230	35	0	0	10.34	423	7.312	8.101	9	10.34	0	2	1	0.90
170	230	30	0	0	19.17	426	7.818	9.098	10	19.17	0	2	1	0.86
180	230	25	0	0	0.70	423	8.182	9.361	8	0.70	0	2	1	0.87
190	230	20	0	0	19.42	381	7.976	10.715	10	19.42	0	2	1	0.74
200	230	15	0	0	9.14	345	7.451	14.214	13	9.14	0	3	1	0.52
210	230	10	0	0	11.05	333	7.760	17.520	25	11.05	0	3	1	0.44
220	230	5	0	0	12.85	277	6.798	26.242	37	12.85	0	5	1	0.26
10	240	115	0	0	8.88	34	0.100	0.290	11	8.88	0	6	1	0.34
20	240	110	0	0	14.57	79	0.306	0.479	10	14.57	0	4	1	0.64
30	240	105	0	0	0.01	141	0.665	0.814	11	0.01	0	4	1	0.82
40	240	100	0	0	6.03	176	1.001	1.705	11	6.03	0	3	1	0.59
50	240	95	0	0	3.93	245	1.689	1.582	12	3.93	0	3	1	1.07
60	240	90	0	0	18.51	210	1.622	2.100	11	18.51	0	3	1	0.77
70	240	85	0	0	1.32	306	2.689	2.601	11	1.32	0	3	1	1.03
80	240	80	0	0	13.35	323	3.317	3.177	10	13.35	0	3	1	1.04
90	240	75	0	0	15.05	391	4.164	3.970	11	15.05	0	3	1	1.05
100	240	70	0	0	17.70	456	5.349	4.133	10	17.70	0	2	1	1.29
110	240	65	0	0	9.91	496	6.445	4.835	10	9.91	0	2	1	1.33
120	240	60	0	0	10.05	409	5.638	5.609	11	10.05	0	2	1	1.01
130	240	55	0	0	14.70	523	7.699	6.512	12	14.70	0	2	1	1.18

Table 12: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
140	240	50	0	0	14.70	438	6.870	6.864	10	14.70	0	2	1	1.00
150	240	45	0	0	13.04	504	8.434	7.946	11	13.04	0	2	1	1.06
160	240	40	0	0	8.56	477	8.541	8.113	8	8.56	0	2	1	1.05
170	240	35	0	0	2.56	486	9.283	9.466	9	2.56	0	2	1	0.98
180	240	30	0	0	8.20	456	9.141	9.606	7	8.20	0	2	1	0.95
190	240	25	0	0	10.67	446	9.403	11.052	9	10.67	0	2	1	0.85
200	240	20	0	0	16.49	425	9.675	12.040	9	16.49	0	2	1	0.80
210	240	15	0	0	16.81	358	8.585	19.218	16	16.81	0	4	1	0.45
220	240	10	0	0	19.29	361	9.140	13.149	7	19.29	0	2	1	0.70
230	240	5	0	0	4.33	309	8.207	47.482	57	4.33	0	9	1	0.17
10	250	120	0	0	19.86	37	0.109	0.299	11	19.86	0	6	1	0.36
20	250	115	0	0	3.87	65	0.262	0.512	11	3.87	0	4	1	0.51
30	250	110	0	0	17.93	131	0.650	0.843	11	17.93	0	4	1	0.77
40	250	105	0	0	2.67	161	0.966	1.378	10	2.67	0	3	1	0.70
50	250	100	0	0	13.25	216	1.526	1.596	11	13.25	0	3	1	0.96
60	250	95	0	0	11.24	261	2.092	2.185	11	11.24	0	3	1	0.96
70	250	90	0	0	15.50	296	2.706	2.783	11	15.50	0	3	1	0.97
80	250	85	0	0	7.31	356	3.603	3.248	9	7.31	0	3	1	1.11
90	250	80	0	0	18.31	385	4.327	4.057	11	18.31	0	3	1	1.07
100	250	75	0	0	18.50	361	4.529	4.190	10	18.50	0	2	1	1.08
110	250	70	0	0	13.60	443	5.826	5.009	10	13.60	0	2	1	1.16
120	250	65	0	0	1.82	482	6.883	5.774	11	1.82	0	2	1	1.19
130	250	60	0	0	4.19	552	8.575	6.716	11	4.19	0	2	1	1.28
140	250	55	0	0	0.89	521	8.553	7.168	10	0.89	0	2	1	1.19
150	250	50	0	0	11.49	501	8.940	8.133	10	11.49	0	2	1	1.10
160	250	45	0	0	10.02	501	9.427	9.236	10	10.02	0	2	1	1.02
170	250	40	0	0	2.64	604	12.049	10.142	10	2.64	0	2	1	1.19
180	250	35	0	0	9.19	457	9.651	10.824	10	9.19	0	2	1	0.89
190	250	30	0	0	1.99	473	10.751	12.065	10	1.99	0	2	1	0.89
200	250	25	0	0	4.92	493	11.491	12.203	8	4.92	0	2	1	0.94
210	250	20	0	0	1.68	450	11.055	13.917	10	1.68	0	2	1	0.79
220	250	15	0	0	13.03	415	10.869	26.781	25	13.03	0	5	1	0.41
230	250	10	0	0	11.41	409	11.155	48.280	84	11.41	0	7	1	0.23
240	250	5	0	0	15.82	320	9.462	29.792	20	15.82	0	5	1	0.32
10	260	125	0	0	17.85	29	0.098	0.331	11	17.85	0	6	1	0.30
20	260	120	0	0	15.31	89	0.362	0.532	11	15.31	0	4	1	0.68
30	260	115	0	0	17.07	122	0.653	0.918	12	17.07	0	4	1	0.71
40	260	110	0	0	3.00	194	1.188	1.847	12	3.00	0	4	1	0.64
50	260	105	0	0	15.78	217	1.567	1.871	12	15.78	0	4	1	0.84
60	260	100	0	0	4.99	247	2.044	2.351	13	4.99	0	3	1	0.87
70	260	95	0	0	3.96	313	2.946	2.884	12	3.96	0	3	1	1.02
80	260	90	0	0	13.57	361	3.771	3.522	11	13.57	0	3	1	1.07
90	260	85	0	0	14.50	360	4.213	4.262	12	14.50	0	3	1	0.99
100	260	80	0	0	3.51	425	5.319	5.154	12	3.51	0	3	1	1.03
110	260	75	0	0	8.35	456	6.423	5.200	11	8.35	0	2	1	1.24
120	260	70	0	0	0.38	522	7.779	6.350	12	0.38	0	2	1	1.23
130	260	65	0	0	2.56	525	8.475	6.977	11	2.56	0	2	1	1.21
140	260	60	0	0	0.40	475	8.076	7.996	11	0.40	0	2	1	1.01
150	260	55	0	0	1.84	573	10.650	8.923	11	1.84	0	2	1	1.19
160	260	50	0	0	8.58	586	11.477	9.307	9	8.58	0	2	1	1.23
170	260	45	0	0	1.35	498	10.441	10.528	10	1.35	0	2	1	0.99
180	260	40	0	0	4.05	507	11.115	11.095	9	4.05	0	2	1	1.00
190	260	35	0	0	0.83	551	12.620	12.031	8	0.83	0	2	1	1.05
200	260	30	0	0	2.02	562	13.551	13.638	10	2.02	0	2	1	0.99
210	260	25	0	0	5.08	484	12.251	18.144	12	5.08	0	3	1	0.68
220	260	20	0	0	19.42	472	12.699	20.059	16	19.42	0	3	1	0.63
230	260	15	0	0	18.05	515	14.570	15.355	6	18.05	0	2	1	0.95
240	260	10	0	0	4.38	373	10.882	36.681	57	4.38	0	4	1	0.30
250	260	5	0	0	1.45	325	10.282	29.314	20	1.45	0	4	1	0.35
10	270	130	0	0	4.40	33	0.109	0.340	11	4.40	0	6	1	0.32
20	270	125	0	0	17.89	95	0.408	0.557	11	17.89	0	4	1	0.73
30	270	120	0	0	17.90	129	0.686	0.900	11	17.90	0	4	1	0.76
40	270	115	0	0	7.76	148	0.954	1.539	12	7.76	0	4	1	0.62
50	270	110	0	0	11.47	227	1.713	1.937	12	11.47	0	4	1	0.88
60	270	105	0	0	4.49	292	2.616	2.511	13	4.49	0	3	1	1.04
70	270	100	0	0	16.00	316	3.145	3.122	12	16.00	0	3	1	1.01
80	270	95	0	0	1.74	338	3.799	3.691	12	1.74	0	3	1	1.03
90	270	90	0	0	8.84	393	4.722	4.438	12	8.84	0	3	1	1.06
100	270	85	0	0	13.29	435	5.759	5.380	11	13.29	0	3	1	1.07
110	270	80	0	0	1.75	497	7.128	6.442	12	1.75	0	3	1	1.11
120	270	75	0	0	15.94	505	7.863	6.469	12	15.94	0	2	1	1.22
130	270	70	0	0	3.20	500	8.448	7.455	12	3.20	0	2	1	1.13
140	270	65	0	0	18.37	538	9.632	8.004	11	18.37	0	2	1	1.20
150	270	60	0	0	13.73	568	10.701	9.238	12	13.73	0	2	1	1.16
160	270	55	0	0	2.80	619	12.467	9.678	10	2.80	0	2	1	1.29
170	270	50	0	0	0.12	592	12.583	10.897	10	0.12	0	2	1	1.15
180	270	45	0	0	8.40	568	12.924	11.289	8	8.40	0	2	1	1.14
190	270	40	0	0	0.30	586	14.042	13.367	10	0.30	0	2	1	1.05
200	270	35	0	0	18.67	525	13.087	13.535	8	18.67	0	2	1	0.97
210	270	30	0	0	13.34	584	15.345	15.477	10	13.34	0	2	1	0.99
220	270	25	0	0	7.26	586	16.500	45.049	63	7.26	0	8	1	0.37
230	270	20	0	0	8.57	513	14.660	18.209	12	8.57	0	2	2	0.81
240	270	15	0	0	9.39	482	14.468	24.060	16	9.39	0	3	1	0.60
250	270	10	0	0	19.63	351	11.155	18.467	6	19.63	0	2	1	0.60
260	270	5	0	0	11.23	360	11.880	56.874	40	11.23	0	8	1	0.21
10	280	135	0	0	9.12	33	0.113	0.358	12	9.12	0	6	1	0.32
20	280	130	0	0	0.96	87	0.384	0.644	13	0.96	0	5	1	0.60
30	280	125	0	0	5.36	111	0.647	1.085	14	5.36	0	5	1	0.60
40	280	120	0	0	16.05	180	1.206	1.423	11	16.05	0	4	1	0.85

Table 13: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
50	280	115	0	0	8.09	206	1.681	2.013	12	8.09	0	4	1	0.84
60	280	110	0	0	17.01	246	2.248	2.505	13	17.01	0	3	1	0.90
70	280	105	0	0	2.16	281	3.017	3.207	12	2.16	0	3	1	0.94
80	280	100	0	0	8.35	399	4.516	3.747	11	8.35	0	3	1	1.21
90	280	95	0	0	2.70	391	4.937	4.555	12	2.70	0	3	1	1.08
100	280	90	0	0	19.74	473	6.423	5.575	12	19.74	0	3	1	1.15
110	280	85	0	0	0.44	428	6.348	6.499	12	0.44	0	3	1	0.98
120	280	80	0	0	7.71	489	7.821	6.496	11	7.71	0	2	1	1.20
130	280	75	0	0	9.63	536	9.152	7.511	12	9.63	0	2	1	1.22
140	280	70	0	0	1.24	528	9.648	8.559	12	1.24	0	2	1	1.13
150	280	65	0	0	18.02	597	11.652	9.195	11	18.02	0	2	1	1.27
160	280	60	0	0	4.68	661	13.706	10.496	12	4.68	0	2	1	1.31
170	280	55	0	0	12.21	601	13.313	11.835	12	12.21	0	2	1	1.12
180	280	50	0	0	13.78	594	13.819	12.583	11	13.78	0	2	1	1.10
190	280	45	0	0	6.89	602	14.698	13.769	10	6.89	0	2	1	1.07
200	280	40	0	0	4.80	627	16.409	15.363	12	4.80	0	2	1	1.07
210	280	35	0	0	6.73	610	16.653	15.988	10	6.73	0	2	1	1.04
220	280	30	0	0	13.96	580	16.729	17.730	11	13.96	0	2	1	0.94
230	280	25	0	0	3.74	563	17.066	22.639	12	3.74	0	3	1	0.75
240	280	20	0	0	9.00	564	17.893	19.195	8	9.00	0	2	1	0.93
250	280	15	0	0	12.10	417	13.943	25.763	12	12.10	0	3	1	0.54
260	280	10	0	0	13.34	461	15.911	31.850	12	13.34	0	4	1	0.50
270	280	5	0	0	6.94	360	12.946	49.426	47	6.94	0	5	6	0.26
10	290	140	0	0	14.29	35	0.120	0.355	12	14.29	0	6	1	0.34
20	290	135	0	0	6.47	74	0.357	0.657	12	6.47	0	5	1	0.54
30	290	130	0	0	14.50	118	0.681	1.131	14	14.50	0	5	1	0.60
40	290	125	0	0	10.93	170	1.178	1.438	11	10.93	0	4	1	0.82
50	290	120	0	0	2.38	185	1.508	2.085	12	2.38	0	4	1	0.72
60	290	115	0	0	5.45	248	2.393	2.555	12	5.45	0	3	1	0.94
70	290	110	0	0	15.62	337	3.547	3.262	12	15.62	0	3	1	1.09
80	290	105	0	0	6.72	399	4.608	3.868	10	6.72	0	3	1	1.19
90	290	100	0	0	19.69	361	4.702	4.722	11	19.69	0	3	1	1.00
100	290	95	0	0	8.02	403	5.721	5.701	12	8.02	0	3	1	1.00
110	290	90	0	0	16.19	536	8.247	6.832	13	16.19	0	3	1	1.21
120	290	85	0	0	7.22	494	8.259	7.856	13	7.22	0	3	1	1.05
130	290	80	0	0	13.62	549	9.775	7.792	12	13.62	0	2	1	1.25
140	290	75	0	0	6.27	570	10.908	8.877	12	6.27	0	2	1	1.23
150	290	70	0	0	7.22	574	11.586	9.995	12	7.22	0	2	1	1.16
160	290	65	0	0	14.62	582	12.541	10.650	10	14.62	0	2	1	1.18
170	290	60	0	0	16.57	679	15.658	12.197	11	16.57	0	2	1	1.28
180	290	55	0	0	8.64	638	15.379	12.874	10	8.64	0	2	1	1.19
190	290	50	0	0	18.06	622	15.783	14.157	10	18.06	0	2	1	1.11
200	290	45	0	0	8.63	628	16.856	15.899	10	8.63	0	2	1	1.06
210	290	40	0	0	5.85	597	16.928	15.922	8	5.85	0	2	1	1.06
220	290	35	0	0	3.51	642	19.073	17.803	9	3.51	0	2	1	1.07
230	290	30	0	0	18.35	599	18.359	19.288	9	18.35	0	2	1	0.95
240	290	25	0	0	8.02	563	18.318	21.500	12	8.02	0	2	1	0.85
250	290	20	0	0	17.41	574	19.639	25.771	9	17.41	0	3	1	0.76
260	290	15	0	0	7.73	469	16.594	34.180	16	7.73	0	4	1	0.49
270	290	10	0	0	18.53	471	17.739	36.296	15	18.53	0	4	1	0.49
280	290	5	0	0	18.73	380	14.581	76.967	64	18.73	0	8	1	0.19
10	300	145	0	0	10.16	27	0.100	0.364	12	10.16	0	6	1	0.27
20	300	140	0	0	14.39	73	0.347	0.672	12	14.39	0	5	1	0.52
30	300	135	0	0	0.32	123	0.734	1.136	13	0.32	0	5	1	0.65
40	300	130	0	0	16.46	134	0.958	1.539	13	16.46	0	4	1	0.62
50	300	125	0	0	14.19	234	1.965	2.121	12	14.19	0	4	1	0.93
60	300	120	0	0	1.79	301	2.894	2.634	12	1.79	0	3	1	1.10
70	300	115	0	0	4.51	354	3.835	3.535	14	4.51	0	3	1	1.09
80	300	110	0	0	18.70	362	4.394	4.120	12	18.70	0	3	1	1.07
90	300	105	0	0	10.66	379	5.102	4.825	11	10.66	0	3	1	1.06
100	300	100	0	0	10.65	483	7.021	5.843	12	10.65	0	3	1	1.20
110	300	95	0	0	18.62	512	8.085	6.807	12	18.62	0	3	1	1.19
120	300	90	0	0	15.66	572	9.671	7.964	12	15.66	0	3	1	1.21
130	300	85	0	0	8.42	627	11.521	7.978	11	8.42	0	2	1	1.44
140	300	80	0	0	8.76	632	12.646	9.060	11	8.76	0	2	1	1.40
150	300	75	0	0	17.17	637	13.159	10.153	11	17.17	0	2	1	1.30
160	300	70	0	0	15.66	684	15.182	11.024	11	15.66	0	2	1	1.38
170	300	65	0	0	0.85	684	16.208	12.113	10	0.85	0	2	1	1.34
180	300	60	0	0	3.50	702	17.566	13.263	10	3.50	0	2	1	1.32
190	300	55	0	0	11.22	629	16.504	14.229	9	11.22	0	2	1	1.16
200	300	50	0	0	15.49	659	18.249	15.623	9	15.49	0	2	1	1.17
210	300	45	0	0	1.30	793	23.197	17.378	10	1.30	0	2	1	1.33
220	300	40	0	0	13.91	657	19.898	19.015	11	13.91	0	2	1	1.05
230	300	35	0	0	15.41	672	21.538	19.648	9	15.41	0	2	1	1.10
240	300	30	0	0	19.73	552	18.482	23.012	13	19.73	0	2	1	0.80
250	300	25	0	0	11.20	611	21.463	49.384	46	11.20	0	6	1	0.43
260	300	20	0	0	4.08	541	19.749	23.955	9	4.08	0	2	1	0.82
270	300	15	0	0	17.39	502	19.021	38.351	18	17.39	0	4	1	0.50
280	300	10	0	0	13.57	405	16.345	31.971	7	13.57	0	3	1	0.51
290	300	5	0	0	8.65	377	15.553	36.087	16	8.65	0	3	1	0.43
10	310	150	0	0	4.16	35	0.130	0.377	12	4.16	0	6	1	0.35
20	310	145	0	0	12.95	81	0.390	0.695	12	12.95	0	5	1	0.56
30	310	140	0	0	2.79	146	0.894	1.185	13	2.79	0	5	1	0.75
40	310	135	0	0	1.43	160	1.186	1.613	13	1.43	0	4	1	0.74
50	310	130	0	0	3.18	210	1.848	2.248	13	3.18	0	4	1	0.82
60	310	125	0	0	15.41	254	2.517	2.663	11	15.41	0	3	1	0.95
70	310	120	0	0	19.50	289	3.292	3.583	13	19.50	0	3	1	0.92
80	310	115	0	0	9.14	377	4.669	4.253	12	9.14	0	3	1	1.10
90	310	110	0	0	17.24	430	5.873	5.143	12	17.24	0	3	1	1.14

Table 14: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
100	310	105	0	0	18.43	450	6.763	6.119	13	18.43	0	3	1	1.11
110	310	100	0	0	6.29	503	8.134	7.048	12	6.29	0	3	1	1.15
120	310	95	0	0	19.12	483	8.500	8.448	14	19.12	0	3	1	1.01
130	310	90	0	0	7.99	617	11.732	8.537	13	7.99	0	2	1	1.37
140	310	85	0	0	16.79	543	11.145	9.291	11	16.79	0	2	1	1.20
150	310	80	0	0	17.91	635	13.715	10.608	12	17.91	0	2	1	1.29
160	310	75	0	0	13.82	626	14.364	11.462	11	13.82	0	2	1	1.25
170	310	70	0	0	19.82	707	17.221	12.551	10	19.82	0	2	1	1.37
180	310	65	0	0	3.05	724	18.566	14.072	11	3.05	0	2	1	1.32
190	310	60	0	0	18.82	744	20.319	15.373	11	18.82	0	2	1	1.32
200	310	55	0	0	4.36	745	21.164	16.220	10	4.36	0	2	1	1.30
210	310	50	0	0	13.81	671	20.521	17.970	10	13.81	0	2	1	1.14
220	310	45	0	0	15.13	689	22.206	19.324	9	15.13	0	2	1	1.15
230	310	40	0	0	8.55	745	25.447	20.420	9	8.55	0	2	1	1.25
240	310	35	0	0	6.58	755	26.278	22.813	10	6.58	0	2	1	1.15
250	310	30	0	0	11.94	711	25.936	24.417	10	11.94	0	2	1	1.06
260	310	25	0	0	13.41	623	23.567	25.382	9	13.41	0	2	1	0.93
270	310	20	0	0	18.82	563	22.245	53.922	45	18.82	0	5	1	0.41
280	310	15	0	0	8.33	536	21.743	89.438	71	8.33	0	9	1	0.24
290	310	10	0	0	1.48	499	21.087	28.362	6	1.48	0	2	1	0.74
300	310	5	0	0	5.92	425	19.077	68.695	55	5.92	0	5	1	0.28
10	320	155	0	0	9.75	28	0.107	0.438	13	9.75	0	7	1	0.24
20	320	150	0	0	7.95	64	0.323	0.719	12	7.95	0	5	1	0.45
30	320	145	0	0	1.82	143	0.903	1.193	12	1.82	0	5	1	0.76
40	320	140	0	0	2.26	181	1.369	1.616	12	2.26	0	4	1	0.85
50	320	135	0	0	14.28	224	1.987	2.317	13	14.28	0	4	1	0.86
60	320	130	0	0	14.14	282	2.936	2.903	13	14.14	0	3	1	1.01
70	320	125	0	0	11.29	290	3.446	3.699	13	11.29	0	3	1	0.93
80	320	120	0	0	9.56	356	4.616	4.381	11	9.56	0	3	1	1.05
90	320	115	0	0	3.58	446	6.304	5.383	13	3.58	0	3	1	1.17
100	320	110	0	0	13.97	456	7.227	6.261	12	13.97	0	3	1	1.15
110	320	105	0	0	18.43	483	8.131	7.312	12	18.43	0	3	1	1.11
120	320	100	0	0	7.40	600	10.963	8.725	14	7.40	0	3	1	1.26
130	320	95	0	0	10.29	645	12.569	9.778	12	10.29	0	3	1	1.29
140	320	90	0	0	15.76	618	12.997	9.420	10	15.76	0	2	1	1.38
150	320	85	0	0	8.59	741	16.794	11.035	12	8.59	0	2	1	1.52
160	320	80	0	0	14.34	704	16.698	12.007	11	14.34	0	2	1	1.39
170	320	75	0	0	12.99	693	17.371	12.914	10	12.99	0	2	1	1.35
180	320	70	0	0	13.22	736	19.817	14.664	11	13.22	0	2	1	1.35
190	320	65	0	0	5.95	769	21.515	15.943	11	5.95	0	2	1	1.35
200	320	60	0	0	13.96	726	21.302	17.701	11	13.96	0	2	1	1.20
210	320	55	0	0	15.21	828	25.818	18.589	10	15.21	0	2	1	1.39
220	320	50	0	0	4.84	700	23.005	20.246	10	4.84	0	2	1	1.14
230	320	45	0	0	15.32	715	24.751	21.628	10	15.32	0	2	1	1.14
240	320	40	0	0	4.78	631	22.881	23.359	10	4.78	0	2	1	0.98
250	320	35	0	0	7.10	688	25.721	24.448	9	7.10	0	2	1	1.05
260	320	30	0	0	18.49	677	26.770	26.498	9	18.49	0	2	1	1.01
270	320	25	0	0	15.40	613	25.163	27.136	8	15.40	0	2	1	0.93
280	320	20	0	0	5.64	562	23.865	29.437	9	5.64	0	2	1	0.81
290	320	15	0	0	7.38	515	22.886	106.735	94	7.38	0	9	1	0.21
300	320	10	0	0	15.37	505	23.476	245.251	204	15.37	0	22	1	0.10
310	320	5	0	0	1.94	441	21.064	127.644	72	1.94	0	11	1	0.17
10	330	160	0	0	15.48	30	0.118	0.439	13	15.48	0	7	1	0.27
20	330	155	0	0	15.54	89	0.472	0.785	14	15.54	0	5	1	0.60
30	330	150	0	0	6.88	140	0.926	1.276	14	6.88	0	5	1	0.73
40	330	145	0	0	2.42	204	1.580	1.681	12	2.42	0	4	1	0.94
50	330	140	0	0	14.74	228	2.164	2.363	12	14.74	0	4	1	0.92
60	330	135	0	0	4.63	253	2.684	2.946	12	4.63	0	3	1	0.91
70	330	130	0	0	17.58	312	3.718	4.340	14	17.58	0	4	1	0.86
80	330	125	0	0	6.41	394	5.250	4.755	13	6.41	0	3	1	1.10
90	330	120	0	0	17.44	450	6.587	5.561	11	17.44	0	3	1	1.18
100	330	115	0	0	9.68	457	7.402	6.607	13	9.68	0	3	1	1.12
110	330	110	0	0	18.31	503	8.848	7.750	13	18.31	0	3	1	1.14
120	330	105	0	0	15.38	599	11.318	8.938	14	15.38	0	3	1	1.27
130	330	100	0	0	19.77	632	12.863	8.861	12	19.77	0	2	1	1.45
140	330	95	0	0	15.03	626	13.735	10.307	13	15.03	0	2	1	1.33
150	330	90	0	0	17.23	645	14.857	11.531	12	17.23	0	2	1	1.29
160	330	85	0	0	10.05	670	16.325	12.586	12	10.05	0	2	1	1.30
170	330	80	0	0	9.87	765	20.090	13.751	11	9.87	0	2	1	1.46
180	330	75	0	0	16.95	681	18.811	15.527	12	16.95	0	2	1	1.21
190	330	70	0	0	5.06	729	21.183	16.173	10	5.06	0	2	1	1.31
200	330	65	0	0	8.45	728	22.208	18.087	10	8.45	0	2	1	1.23
210	330	60	0	0	13.40	751	24.188	18.690	9	13.40	0	2	1	1.29
220	330	55	0	0	0.36	786	26.768	21.837	11	0.36	0	2	1	1.23
230	330	50	0	0	17.06	729	26.039	22.408	10	17.06	0	2	1	1.16
240	330	45	0	0	2.55	801	29.812	23.935	9	2.55	0	2	1	1.25
250	330	40	0	0	17.83	746	29.009	24.887	8	17.83	0	2	1	1.17
260	330	35	0	0	6.69	673	27.087	27.011	9	6.69	0	2	1	1.00
270	330	30	0	0	11.77	667	28.184	29.424	10	11.77	0	2	1	0.96
280	330	25	0	0	15.56	653	28.732	31.976	11	15.56	0	2	1	0.90
290	330	20	0	0	15.59	552	25.572	32.615	9	15.59	0	2	1	0.78
300	330	15	0	0	2.86	596	28.237	50.385	15	2.86	0	4	1	0.56
310	330	10	0	0	15.16	519	26.091	34.351	7	15.16	0	2	1	0.76
320	330	5	0	0	2.60	420	21.730	57.675	19	2.60	0	4	1	0.38
10	340	165	0	0	10.92	43	0.174	0.448	13	10.92	0	7	1	0.39
20	340	160	0	0	16.77	92	0.489	0.803	14	16.77	0	5	1	0.61
30	340	155	0	0	10.22	114	0.771	1.336	14	10.22	0	5	1	0.58
40	340	150	0	0	12.74	192	1.591	1.822	14	12.74	0	4	1	0.87
50	340	145	0	0	3.59	231	2.217	2.566	14	3.59	0	4	1	0.86

Table 15: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
60	340	140	0	0	17.40	275	3.002	3.472	15	17.40	0	4	1	0.86
70	340	135	0	0	9.81	376	4.620	4.480	15	9.81	0	4	1	1.03
80	340	130	0	0	17.57	362	4.956	4.835	13	17.57	0	3	1	1.03
90	340	125	0	0	7.68	477	7.159	5.846	13	7.68	0	3	1	1.22
100	340	120	0	0	7.13	413	6.873	7.052	15	7.13	0	3	1	0.97
110	340	115	0	0	16.53	525	9.433	7.953	13	16.53	0	3	1	1.19
120	340	110	0	0	3.87	574	11.193	9.125	14	3.87	0	3	1	1.23
130	340	105	0	0	19.64	620	13.024	9.136	12	19.64	0	2	1	1.43
140	340	100	0	0	4.85	612	13.786	10.483	12	4.85	0	2	1	1.32
150	340	95	0	0	19.07	690	16.381	11.675	12	19.07	0	2	1	1.40
160	340	90	0	0	2.89	669	16.942	12.646	11	2.89	0	2	1	1.34
170	340	85	0	0	11.72	731	19.604	14.413	11	11.72	0	2	1	1.36
180	340	80	0	0	2.94	765	21.620	15.944	12	2.94	0	2	1	1.36
190	340	75	0	0	10.82	785	23.579	17.605	12	10.82	0	2	1	1.34
200	340	70	0	0	11.62	701	22.208	18.861	11	11.62	0	2	1	1.18
210	340	65	0	0	16.52	835	27.604	20.273	10	16.52	0	2	1	1.36
220	340	60	0	0	12.09	810	28.284	22.815	12	12.09	0	2	1	1.24
230	340	55	0	0	3.19	830	30.883	24.151	12	3.19	0	2	1	1.28
240	340	50	0	0	7.89	844	32.757	24.881	10	7.89	0	2	1	1.32
250	340	45	0	0	15.42	738	30.009	26.843	10	15.42	0	2	1	1.12
260	340	40	0	0	2.54	710	30.779	27.221	8	2.54	0	2	1	1.13
270	340	35	0	0	14.06	732	32.544	30.571	9	14.06	0	2	1	1.06
280	340	30	0	0	2.51	779	35.972	33.017	10	2.51	0	2	1	1.09
290	340	25	0	0	3.68	586	27.983	48.620	11	3.68	0	4	1	0.58
300	340	20	0	0	4.42	613	30.270	54.177	20	4.42	0	4	2	0.56
310	340	15	0	0	6.24	577	29.522	54.627	13	6.24	0	4	1	0.54
320	340	10	0	0	10.66	504	26.440	37.445	7	10.66	0	2	1	0.71
330	340	5	0	0	14.83	473	26.720	61.946	16	14.83	0	4	1	0.43
10	350	170	0	0	2.18	37	0.174	0.479	13	2.18	0	7	1	0.36
20	350	165	0	0	19.76	89	0.488	0.840	14	19.76	0	5	1	0.58
30	350	160	0	0	19.37	121	0.853	1.371	13	19.37	0	5	1	0.62
40	350	155	0	0	3.13	193	1.632	1.838	13	3.13	0	4	1	0.89
50	350	150	0	0	10.12	238	2.333	2.690	14	10.12	0	4	1	0.87
60	350	145	0	0	4.57	268	3.007	3.542	14	4.57	0	4	1	0.85
70	350	140	0	0	16.77	330	4.225	4.634	14	16.77	0	4	1	0.91
80	350	135	0	0	13.01	362	5.402	4.974	13	13.01	0	3	1	1.09
90	350	130	0	0	5.87	448	7.051	6.229	14	5.87	0	3	1	1.13
100	350	125	0	0	2.15	449	7.741	6.794	12	2.15	0	3	1	1.14
110	350	120	0	0	9.47	549	10.340	8.521	15	9.47	0	3	1	1.21
120	350	115	0	0	11.34	666	13.332	9.575	14	11.34	0	3	1	1.39
130	350	110	0	0	19.16	625	13.545	10.832	13	19.16	0	3	1	1.25
140	350	105	0	0	19.52	666	15.282	10.870	13	19.52	0	2	1	1.41
150	350	100	0	0	10.03	666	16.271	12.640	14	10.03	0	2	1	1.29
160	350	95	0	0	16.86	736	19.110	13.486	12	16.86	0	2	1	1.42
170	350	90	0	0	8.19	699	19.370	14.433	11	8.19	0	2	1	1.34
180	350	85	0	0	17.80	872	25.368	16.207	11	17.80	0	2	1	1.57
190	350	80	0	0	14.07	780	24.141	17.934	12	14.07	0	2	1	1.35
200	350	75	0	0	13.56	791	26.623	20.002	12	13.56	0	2	1	1.33
210	350	70	0	0	6.76	759	25.985	20.902	11	6.76	0	2	1	1.24
220	350	65	0	0	3.95	895	32.326	22.605	10	3.95	0	2	1	1.43
230	350	60	0	0	12.14	838	31.285	24.920	11	12.14	0	2	1	1.26
240	350	55	0	0	8.56	753	30.856	27.171	12	8.56	0	2	1	1.14
250	350	50	0	0	7.11	849	36.866	28.155	10	7.11	0	2	1	1.31
260	350	45	0	0	19.77	775	34.060	29.708	9	19.77	0	2	1	1.15
270	350	40	0	0	2.28	844	38.878	32.419	10	2.28	0	2	1	1.20
280	350	35	0	0	16.33	760	37.154	33.356	9	16.33	0	2	1	1.11
290	350	30	0	0	15.46	813	40.933	36.590	10	15.46	0	2	1	1.12
300	350	25	0	0	4.72	678	35.210	35.865	7	4.72	0	2	1	0.98
310	350	20	0	0	14.05	724	39.269	38.832	8	14.05	0	2	1	1.01
320	350	15	0	0	10.37	537	30.135	210.226	143	10.37	0	15	1	0.14
330	350	10	0	0	1.54	551	32.218	41.696	8	1.54	0	2	1	0.77
340	350	5	0	0	17.51	425	26.027	79.147	40	17.51	0	4	1	0.33
10	360	175	0	0	12.85	43	0.183	0.494	14	12.85	0	7	1	0.37
20	360	170	0	0	18.90	84	0.479	0.899	15	18.90	0	5	1	0.53
30	360	165	0	0	3.20	118	0.854	1.443	15	3.20	0	5	1	0.59
40	360	160	0	0	6.60	195	1.690	1.916	14	6.60	0	4	1	0.88
50	360	155	0	0	1.44	264	2.638	2.742	14	1.44	0	4	1	0.96
60	360	150	0	0	9.41	290	3.324	3.671	14	9.41	0	4	1	0.91
70	360	145	0	0	17.31	373	4.868	4.698	14	17.31	0	4	1	1.04
80	360	140	0	0	12.96	374	5.536	5.036	12	12.96	0	3	1	1.10
90	360	135	0	0	16.65	398	6.510	6.141	12	16.65	0	3	1	1.06
100	360	130	0	0	11.34	544	9.518	7.268	13	11.34	0	3	1	1.31
110	360	125	0	0	1.44	553	10.546	8.739	15	1.44	0	3	1	1.21
120	360	120	0	0	15.10	561	11.674	9.548	12	15.10	0	3	1	1.22
130	360	115	0	0	7.97	648	14.545	11.313	13	7.97	0	3	1	1.29
140	360	110	0	0	18.24	670	15.643	10.718	11	18.24	0	2	1	1.46
150	360	105	0	0	7.69	641	16.291	14.695	14	7.69	0	3	1	1.11
160	360	100	0	0	8.36	710	18.870	13.833	12	8.36	0	2	1	1.36
170	360	95	0	0	19.79	801	22.756	15.351	12	19.79	0	2	1	1.48
180	360	90	0	0	12.15	839	25.205	16.778	12	12.15	0	2	1	1.50
190	360	85	0	0	9.36	876	27.989	18.266	11	9.36	0	2	1	1.53
200	360	80	0	0	18.06	837	27.973	20.233	11	18.06	0	2	1	1.38
210	360	75	0	0	16.17	830	28.893	21.833	11	16.17	0	2	1	1.32
220	360	70	0	0	19.09	893	33.243	24.619	13	19.09	0	2	1	1.35
230	360	65	0	0	16.14	845	33.103	25.530	12	16.14	0	2	1	1.30
240	360	60	0	0	7.81	930	38.143	27.267	11	7.81	0	2	1	1.40
250	360	55	0	0	9.48	926	39.836	29.355	12	9.48	0	2	1	1.36
260	360	50	0	0	1.44	886	39.176	30.589	10	1.44	0	2	1	1.28
270	360	45	0	0	16.14	846	39.534	32.734	9	16.14	0	2	1	1.21

Table 16: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

					LINFSOL					B-P				
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
280	360	40	0	0	14.68	815	39.042	32.935	8	14.68	0	2	1	1.19
290	360	35	0	0	10.48	765	38.244	36.323	9	10.48	0	2	1	1.05
300	360	30	0	0	7.16	777	41.245	38.316	9	7.16	0	2	1	1.08
310	360	25	0	0	0.13	770	41.679	38.142	6	0.13	0	2	1	1.09
320	360	20	0	0	3.85	698	39.792	56.108	19	3.85	0	3	1	0.71
330	360	15	0	0	19.32	568	33.538	55.721	15	19.32	0	3	1	0.60
340	360	10	0	0	7.25	512	31.697	47.794	15	7.25	0	2	6	0.66
350	360	5	0	0	13.19	454	29.597	73.753	17	13.19	0	4	1	0.40
10	370	180	0	0	5.77	34	0.152	0.506	14	5.77	0	7	1	0.30
20	370	175	0	0	19.16	91	0.520	0.876	14	19.16	0	5	1	0.59
30	370	170	0	0	16.93	119	0.880	1.808	20	16.93	0	6	2	0.49
40	370	165	0	0	14.99	157	1.381	1.947	13	14.99	0	4	1	0.71
50	370	160	0	0	13.08	220	2.279	2.782	14	13.08	0	4	1	0.82
60	370	155	0	0	0.83	300	3.679	3.705	13	0.83	0	4	1	0.99
70	370	150	0	0	5.82	328	4.374	4.812	14	5.82	0	4	1	0.91
80	370	145	0	0	14.15	440	6.528	5.438	14	14.15	0	3	1	1.20
90	370	140	0	0	12.08	433	7.165	6.469	14	12.08	0	3	1	1.11
100	370	135	0	0	4.02	491	8.834	7.365	12	4.02	0	3	1	1.20
110	370	130	0	0	1.39	552	10.736	8.911	15	1.39	0	3	1	1.20
120	370	125	0	0	18.06	617	13.093	10.211	15	18.06	0	3	1	1.28
130	370	120	0	0	5.63	622	14.103	11.406	13	5.63	0	3	1	1.24
140	370	115	0	0	8.60	672	16.256	13.113	13	8.60	0	3	1	1.24
150	370	110	0	0	15.90	763	19.908	12.925	13	15.90	0	2	1	1.54
160	370	105	0	0	11.24	799	22.036	14.074	12	11.24	0	2	1	1.57
170	370	100	0	0	2.09	802	23.723	16.198	14	2.09	0	2	1	1.46
180	370	95	0	0	7.53	824	25.642	17.703	12	7.53	0	2	1	1.45
190	370	90	0	0	3.30	827	27.250	18.710	11	3.30	0	2	1	1.46
200	370	85	0	0	8.86	878	29.877	21.615	14	8.86	0	2	1	1.38
210	370	80	0	0	11.47	947	34.309	22.345	11	11.47	0	2	1	1.54
220	370	75	0	0	14.48	909	34.641	24.602	12	14.48	0	2	1	1.41
230	370	70	0	0	10.10	930	37.165	25.564	10	10.10	0	2	1	1.45
240	370	65	0	0	7.33	891	37.469	27.480	10	7.33	0	2	1	1.36
250	370	60	0	0	1.01	931	41.015	29.634	10	1.01	0	2	1	1.38
260	370	55	0	0	7.04	917	42.094	31.264	10	7.04	0	2	1	1.35
270	370	50	0	0	12.57	1029	49.069	33.266	9	12.57	0	2	1	1.48
280	370	45	0	0	17.85	875	43.611	35.953	11	17.85	0	2	1	1.21
290	370	40	0	0	16.91	819	42.629	37.240	9	16.91	0	2	1	1.14
300	370	35	0	0	14.97	812	44.134	39.185	9	14.97	0	2	1	1.13
310	370	30	0	0	6.37	800	45.015	43.502	12	6.37	0	2	1	1.03
320	370	25	0	0	8.07	755	44.154	54.809	12	8.07	0	3	1	0.81
330	370	20	0	0	6.56	626	39.096	45.404	8	6.56	0	2	1	0.86
340	370	15	0	0	7.32	633	40.766	99.319	25	7.32	0	6	1	0.41
350	370	10	0	0	6.61	502	33.801	49.326	9	6.61	0	2	2	0.69
360	370	5	0	0	1.50	518	35.956	103.477	29	1.50	0	5	6	0.35
10	380	185	0	0	4.68	33	0.154	0.513	14	4.68	0	7	1	0.30
20	380	180	0	0	2.21	80	0.485	0.907	14	2.21	0	5	1	0.53
30	380	175	0	0	6.80	119	0.945	1.544	15	6.80	0	5	1	0.61
40	380	170	0	0	17.71	176	1.594	2.132	15	17.71	0	4	1	0.75
50	380	165	0	0	11.91	230	2.479	2.799	13	11.91	0	4	1	0.89
60	380	160	0	0	5.04	276	3.396	3.826	13	5.04	0	4	1	0.89
70	380	155	0	0	0.14	408	5.792	4.830	13	0.14	0	4	1	1.20
80	380	150	0	0	6.16	433	6.630	5.522	14	6.16	0	3	1	1.20
90	380	145	0	0	10.91	416	7.022	6.689	14	10.91	0	3	1	1.05
100	380	140	0	0	7.09	509	9.463	7.947	15	7.09	0	3	1	1.19
110	380	135	0	0	7.85	611	12.384	9.129	14	7.85	0	3	1	1.36
120	380	130	0	0	19.68	666	14.646	10.496	14	19.68	0	3	1	1.40
130	380	125	0	0	8.70	588	13.957	11.655	13	8.70	0	3	1	1.20
140	380	120	0	0	13.69	650	16.156	13.432	13	13.69	0	3	1	1.20
150	380	115	0	0	19.64	779	20.629	15.475	14	19.64	0	3	1	1.33
160	380	110	0	0	14.28	817	23.176	15.146	14	14.28	0	2	1	1.53
170	380	105	0	0	14.00	878	26.264	16.312	13	14.00	0	2	1	1.61
180	380	100	0	0	2.39	830	27.036	17.968	12	2.39	0	2	1	1.50
190	380	95	0	0	0.79	893	30.131	19.872	13	0.79	0	2	1	1.52
200	380	90	0	0	11.16	960	34.311	21.821	13	11.16	0	2	1	1.57
210	380	85	0	0	3.14	888	32.530	23.536	13	3.14	0	2	1	1.38
220	380	80	0	0	11.98	976	37.703	25.266	12	11.98	0	2	1	1.49
230	380	75	0	0	19.25	932	38.819	27.226	11	19.25	0	2	1	1.43
240	380	70	0	0	19.53	948	41.159	28.654	11	19.53	0	2	1	1.44
250	380	65	0	0	0.67	982	45.562	32.123	12	0.67	0	2	1	1.42
260	380	60	0	0	11.83	915	43.082	32.937	10	11.83	0	2	1	1.31
270	380	55	0	0	19.15	1011	49.825	36.058	11	19.15	0	2	1	1.38
280	380	50	0	0	1.24	894	46.769	36.976	10	1.24	0	2	1	1.26
290	380	45	0	0	17.65	931	50.266	40.190	11	17.65	0	2	1	1.25
300	380	40	0	0	18.69	849	49.653	40.904	8	18.69	0	2	1	1.21
310	380	35	0	0	5.05	891	52.206	44.953	11	5.05	0	2	1	1.16
320	380	30	0	0	10.27	746	45.429	45.641	9	10.27	0	2	1	1.00
330	380	25	0	0	10.48	684	43.141	49.166	11	10.48	0	2	1	0.88
340	380	20	0	0	3.32	780	52.415	133.559	48	3.32	0	8	1	0.39
350	380	15	0	0	15.54	693	48.457	91.292	15	15.54	0	5	1	0.53
360	380	10	0	0	10.61	514	36.941	53.641	7	10.61	0	2	1	0.69
370	380	5	0	0	6.32	478	35.990	103.105	19	6.32	0	5	1	0.35
10	390	190	0	0	13.09	34	0.158	0.508	13	13.09	0	7	1	0.31
20	390	185	0	0	1.11	79	0.502	0.955	14	1.11	0	5	1	0.53
30	390	180	0	0	4.16	132	1.027	1.600	15	4.16	0	5	1	0.64
40	390	175	0	0	4.75	204	1.914	2.180	16	4.75	0	4	1	0.88
50	390	170	0	0	4.62	266	2.957	3.018	15	4.62	0	4	1	0.98
60	390	165	0	0	8.19	307	3.877	3.898	13	8.19	0	4	1	0.99
70	390	160	0	0	3.80	340	4.955	5.191	15	3.80	0	4	1	0.95
80	390	155	0	0	0.97	419	6.619	5.574	13	0.97	0	3	1	1.19

Table 17: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
90	390	150	0	0	7.54	454	8.098	7.223	17	7.54	0	3	1	1.12
100	390	145	0	0	3.03	549	10.453	8.125	14	3.03	0	3	1	1.29
110	390	140	0	0	14.13	650	13.516	9.711	16	14.13	0	3	1	1.39
120	390	135	0	0	14.83	597	13.379	10.988	15	14.83	0	3	1	1.22
130	390	130	0	0	14.09	711	17.150	12.250	14	14.09	0	3	1	1.40
140	390	125	0	0	10.54	698	18.324	13.848	13	10.54	0	3	1	1.32
150	390	120	0	0	15.28	799	22.592	15.701	13	15.28	0	3	1	1.44
160	390	115	0	0	9.57	803	23.852	15.385	13	9.57	0	2	1	1.55
170	390	110	0	0	11.43	792	24.873	17.281	14	11.43	0	2	1	1.44
180	390	105	0	0	17.25	890	29.785	18.529	12	17.25	0	2	1	1.61
190	390	100	0	0	18.04	929	32.074	21.090	14	18.04	0	2	1	1.52
200	390	95	0	0	9.78	904	33.510	21.609	11	9.79	0	2	1	1.55
210	390	90	0	0	1.58	1009	39.166	24.039	12	1.58	0	2	1	1.63
220	390	85	0	0	12.62	867	36.074	26.598	12	12.62	0	2	1	1.36
230	390	80	0	0	3.06	996	42.500	27.996	11	3.06	0	2	1	1.52
240	390	75	0	0	18.07	872	39.072	30.461	12	18.07	0	2	1	1.28
250	390	70	0	0	7.21	939	44.357	31.496	10	7.21	0	2	1	1.41
260	390	65	0	0	15.31	902	44.586	34.207	10	15.31	0	2	1	1.30
270	390	60	0	0	10.52	980	50.807	36.986	11	10.52	0	2	1	1.37
280	390	55	0	0	0.35	1069	57.334	39.604	12	0.35	0	2	2	1.45
290	390	50	0	0	7.07	868	48.234	40.783	9	7.07	0	2	1	1.18
300	390	45	0	0	2.84	859	49.918	42.996	10	2.84	0	2	1	1.16
310	390	40	0	0	4.06	868	53.455	47.150	12	4.06	0	2	1	1.13
320	390	35	0	0	15.75	858	54.363	49.590	11	15.75	0	2	1	1.10
330	390	30	0	0	11.06	725	47.957	51.633	11	11.06	0	2	1	0.93
340	390	25	0	0	17.25	825	57.016	50.441	7	17.25	0	2	1	1.13
350	390	20	0	0	6.64	720	51.742	54.635	9	6.64	0	2	1	0.95
360	390	15	0	0	1.96	673	49.770	69.779	8	1.96	0	3	1	0.71
370	390	10	0	0	0.70	672	52.809	60.275	12	0.70	0	2	3	0.88
380	390	5	0	0	8.40	503	40.479	66.143	16	8.40	0	2	1	0.61
10	400	195	0	0	1.34	32	0.154	0.584	15	1.34	0	8	1	0.26
20	400	190	0	0	5.36	93	0.586	1.033	15	5.36	0	6	1	0.57
30	400	185	0	0	15.08	122	0.984	1.617	14	15.08	0	5	1	0.61
40	400	180	0	0	6.82	176	1.750	2.156	14	6.82	0	4	1	0.81
50	400	175	0	0	6.77	262	3.004	3.127	15	6.77	0	4	1	0.96
60	400	170	0	0	1.87	290	3.805	4.096	15	1.87	0	4	1	0.93
70	400	165	0	0	9.55	390	5.670	5.254	15	9.55	0	4	1	1.08
80	400	160	0	0	17.06	432	7.035	6.020	15	17.06	0	3	1	1.17
90	400	155	0	0	2.05	478	8.623	7.223	15	2.05	0	3	1	1.19
100	400	150	0	0	14.19	463	9.113	8.352	14	14.19	0	3	1	1.09
110	400	145	0	0	4.73	579	12.459	9.955	15	4.73	0	3	1	1.25
120	400	140	0	0	0.25	680	15.600	10.949	15	0.25	0	3	1	1.42
130	400	135	0	0	18.18	731	18.184	12.621	14	18.18	0	3	1	1.44
140	400	130	0	0	9.15	822	21.961	14.256	15	9.15	0	3	1	1.54
150	400	125	0	0	1.77	813	23.135	16.357	15	1.77	0	3	1	1.41
160	400	120	0	0	14.16	739	22.400	15.639	13	14.16	0	2	1	1.43
170	400	115	0	0	0.43	798	25.547	17.385	13	0.43	0	2	1	1.47
180	400	110	0	0	16.80	948	32.064	19.506	14	16.80	0	2	1	1.64
190	400	105	0	0	0.40	881	31.756	21.033	12	0.40	0	2	1	1.51
200	400	100	0	0	0.81	856	32.135	22.764	12	0.81	0	2	1	1.41
210	400	95	0	0	18.50	936	36.863	25.061	12	18.50	0	2	1	1.47
220	400	90	0	0	2.35	971	40.320	26.863	12	2.35	0	2	1	1.50
230	400	85	0	0	16.28	1000	43.332	29.613	13	16.28	0	2	1	1.46
240	400	80	0	0	16.46	944	43.518	31.182	11	16.46	0	2	1	1.40
250	400	75	0	0	5.23	1005	48.576	33.607	12	5.23	0	2	1	1.45
260	400	70	0	0	1.10	1052	51.756	36.018	12	1.10	0	2	1	1.44
270	400	65	0	0	10.35	998	51.588	38.971	13	10.35	0	2	1	1.32
280	400	60	0	0	16.95	1020	55.322	40.387	12	16.95	0	2	1	1.37
290	400	55	0	0	15.30	926	52.471	40.969	9	15.31	0	2	1	1.28
300	400	50	0	0	11.60	917	53.938	43.541	9	11.60	0	2	1	1.24
310	400	45	0	0	3.32	908	55.762	46.252	10	3.32	0	2	1	1.21
320	400	40	0	0	18.81	940	60.117	49.443	10	18.81	0	2	1	1.22
330	400	35	0	0	6.33	891	59.953	50.254	9	6.33	0	2	1	1.19
340	400	30	0	0	17.89	873	60.900	50.568	7	17.89	0	2	1	1.20
350	400	25	0	0	11.26	751	54.820	70.158	11	11.27	0	3	1	0.78
360	400	20	0	0	4.27	691	52.152	59.092	9	4.27	0	2	1	0.88
370	400	15	0	0	3.35	621	49.039	58.988	5	3.35	0	2	1	0.83
380	400	10	0	0	16.33	631	51.766	60.975	7	16.33	0	2	1	0.85
390	400	5	0	0	11.61	497	42.502	103.902	20	11.61	0	4	1	0.41

Table 18: Comparison between LINFSOL and B-P on primal degenerate problems (continued).

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
10	20	0	5	1	2.72	21	0.007	0.030	4	2.72	0	2	1	0.22
10	30	0	5	1	2.02	22	0.010	0.023	3	2.02	0	2	1	0.43
20	30	0	10	1	15.53	42	0.023	0.051	5	15.53	0	2	2	0.45
10	40	0	5	1	14.37	21	0.013	0.026	3	14.37	0	2	1	0.49
20	40	0	10	1	8.83	47	0.033	0.039	3	8.83	0	2	1	0.84
30	40	0	15	1	5.92	48	0.044	0.092	5	5.92	0	2	2	0.48
10	50	0	5	1	6.90	23	0.016	0.028	3	6.90	0	2	1	0.58
20	50	0	10	1	18.37	60	0.050	0.046	3	18.37	0	2	1	1.08
30	50	0	15	1	15.06	82	0.087	0.074	3	15.06	0	2	1	1.17
40	50	0	20	1	10.15	71	0.095	0.157	5	10.15	0	2	2	0.60
10	60	0	5	1	11.47	28	0.022	0.033	3	11.47	0	2	1	0.68
20	60	0	10	1	14.64	55	0.057	0.053	3	14.64	0	2	1	1.08
30	60	0	15	1	19.79	93	0.115	0.121	5	19.79	0	2	2	0.95
40	60	0	20	1	13.25	97	0.153	0.137	3	13.25	0	2	1	1.12
50	60	0	25	1	4.08	94	0.174	0.265	5	4.08	0	2	2	0.66
10	70	0	5	1	11.12	26	0.025	0.035	3	11.12	0	2	1	0.70
20	70	0	10	1	18.91	56	0.067	0.060	3	18.91	0	2	1	1.11
30	70	0	15	1	3.35	72	0.108	0.099	3	3.35	0	2	1	1.09
40	70	0	20	1	9.03	95	0.169	0.149	3	9.03	0	2	1	1.14
50	70	0	25	1	6.36	125	0.264	0.216	3	6.36	0	2	1	1.22
60	70	0	30	1	8.65	118	0.287	0.395	5	8.65	0	2	2	0.73
10	80	0	5	1	2.39	27	0.029	0.037	3	2.39	0	2	1	0.78
20	80	0	10	1	1.97	58	0.078	0.067	3	1.97	0	2	1	1.17
30	80	0	15	1	10.29	96	0.160	0.112	3	10.29	0	2	1	1.43
40	80	0	20	1	14.81	118	0.236	0.169	3	14.81	0	2	1	1.39
50	80	0	25	1	4.22	119	0.284	0.317	5	4.22	0	2	2	0.90
60	80	0	30	1	5.82	141	0.384	0.329	3	5.82	0	2	1	1.17
70	80	0	35	1	16.65	133	0.415	0.569	5	16.65	0	2	2	0.73
10	90	0	5	1	17.58	24	0.030	0.041	3	17.58	0	2	1	0.75
20	90	0	10	1	2.45	72	0.110	0.075	3	2.45	0	2	1	1.46
30	90	0	15	1	16.33	123	0.241	0.126	3	16.33	0	2	1	1.92
40	90	0	20	1	10.16	126	0.302	0.195	3	10.16	0	2	1	1.54
50	90	0	25	1	0.19	137	0.374	0.354	5	0.19	0	2	2	1.06
60	90	0	30	1	6.92	188	0.578	0.480	5	6.92	0	2	2	1.20
70	90	0	35	1	18.96	150	0.536	0.488	3	18.96	0	2	1	1.10
80	90	0	40	1	11.65	168	0.690	0.801	5	11.65	0	2	2	0.86
10	100	0	5	1	18.25	27	0.038	0.045	3	18.25	0	2	1	0.83
20	100	0	10	1	1.79	75	0.124	0.082	3	1.79	0	2	1	1.52
30	100	0	15	1	10.78	85	0.186	0.140	3	10.78	0	2	1	1.33
40	100	0	20	1	16.75	145	0.363	0.210	3	16.75	0	2	1	1.73
50	100	0	25	1	3.36	172	0.506	0.301	3	3.36	0	2	1	1.68
60	100	0	30	1	17.08	172	0.594	0.523	5	17.08	0	2	2	1.14
70	100	0	35	1	1.01	217	0.819	0.530	3	1.01	0	2	1	1.55
80	100	0	40	1	3.56	189	0.829	0.661	3	3.56	0	2	1	1.25
90	100	0	45	1	13.95	165	0.791	1.078	5	13.95	0	2	2	0.73
10	110	0	5	1	17.83	24	0.035	0.048	3	17.83	0	2	1	0.74
20	110	0	10	1	16.60	74	0.137	0.088	3	16.60	0	2	1	1.55
30	110	0	15	1	19.92	83	0.196	0.151	3	19.92	0	2	1	1.30
40	110	0	20	1	16.39	162	0.449	0.229	3	16.39	0	2	1	1.96
50	110	0	25	1	14.97	144	0.469	0.329	3	14.97	0	2	1	1.43
60	110	0	30	1	6.29	188	0.714	0.445	3	6.29	0	2	1	1.60
70	110	0	35	1	16.60	154	0.672	0.596	3	16.60	0	2	1	1.13
80	110	0	40	1	6.89	240	1.130	0.761	3	6.89	0	2	1	1.48
90	110	0	45	1	15.94	224	1.156	0.894	3	15.94	0	2	1	1.29
100	110	0	50	1	14.35	166	0.947	1.411	5	14.35	0	2	2	0.67
10	120	0	5	1	15.23	26	0.041	0.051	3	15.23	0	2	1	0.80
20	120	0	10	1	11.71	64	0.128	0.094	3	11.71	0	2	1	1.37
30	120	0	15	1	5.73	117	0.290	0.161	3	5.73	0	2	1	1.80
40	120	0	20	1	15.84	151	0.457	0.247	3	15.84	0	2	1	1.85
50	120	0	25	1	15.60	191	0.663	0.356	3	15.60	0	2	1	1.86
60	120	0	30	1	4.44	186	0.742	0.481	3	4.44	0	2	1	1.54
70	120	0	35	1	13.12	208	0.930	0.787	5	13.12	0	2	2	1.18
80	120	0	40	1	2.64	284	1.417	1.000	5	2.64	0	2	2	1.42
90	120	0	45	1	6.91	225	1.267	0.995	3	6.91	0	2	1	1.27
100	120	0	50	1	0.72	225	1.382	1.185	3	0.72	0	2	1	1.17
110	120	0	55	1	14.13	195	1.327	1.806	5	14.13	0	2	2	0.73
10	130	0	5	1	2.51	29	0.050	0.054	3	2.51	0	2	1	0.92
20	130	0	10	1	2.03	82	0.178	0.103	3	2.03	0	2	1	1.72
30	130	0	15	1	8.21	119	0.318	0.178	3	8.21	0	2	1	1.79
40	130	0	20	1	4.35	159	0.509	0.268	3	4.35	0	2	1	1.90
50	130	0	25	1	0.57	190	0.711	0.387	3	0.57	0	2	1	1.84
60	130	0	30	1	3.12	217	0.928	0.518	3	3.12	0	2	1	1.79
70	130	0	35	1	18.02	222	1.071	0.842	5	18.02	0	2	2	1.27
80	130	0	40	1	10.78	224	1.222	0.858	3	10.78	0	2	1	1.42
90	130	0	45	1	13.49	266	1.588	1.329	5	13.49	0	2	2	1.19
100	130	0	50	1	18.66	305	1.970	1.594	5	18.66	0	2	2	1.24
110	130	0	55	1	6.67	244	1.733	1.482	3	6.67	0	2	1	1.17
120	130	0	60	1	12.75	199	1.557	2.289	5	12.75	0	2	2	0.68
10	140	0	5	1	18.47	25	0.048	0.057	3	18.47	0	2	1	0.85
20	140	0	10	1	8.23	74	0.172	0.108	3	8.23	0	2	1	1.59
30	140	0	15	1	7.35	105	0.313	0.186	3	7.35	0	2	1	1.69
40	140	0	20	1	9.06	171	0.590	0.292	3	9.06	0	2	1	2.02
50	140	0	25	1	19.65	178	0.730	0.415	3	19.65	0	2	1	1.76
60	140	0	30	1	1.16	285	1.321	0.558	3	1.16	0	2	1	2.37
70	140	0	35	1	17.13	191	1.020	0.723	3	17.13	0	2	1	1.41
80	140	0	40	1	13.45	319	1.881	0.927	3	13.45	0	2	1	2.03
90	140	0	45	1	19.83	297	1.980	1.122	3	19.83	0	2	1	1.76
100	140	0	50	1	2.47	301	2.265	1.400	3	2.47	0	2	1	1.62

Table 19: Comparison between LINFSOL and B-P on dual degenerate problems

				LINFSOL				B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
110	140	0	55	1	10.22	282	2.415	2.065	5	10.22	0	2	2	1.17
120	140	0	60	1	5.05	237	2.151	1.916	3	5.05	0	2	1	1.12
130	140	0	65	1	7.10	211	2.339	2.823	5	7.10	0	2	2	0.83
10	150	0	5	1	13.09	29	0.055	0.061	3	13.09	0	2	1	0.91
20	150	0	10	1	18.82	70	0.174	0.117	3	18.82	0	2	1	1.48
30	150	0	15	1	19.67	118	0.367	0.197	3	19.67	0	2	1	1.86
40	150	0	20	1	8.81	170	0.633	0.302	3	8.81	0	2	1	2.10
50	150	0	25	1	4.02	212	0.922	0.441	3	4.02	0	2	1	2.09
60	150	0	30	1	18.55	207	1.062	0.616	3	18.55	0	2	1	1.72
70	150	0	35	1	10.66	249	1.399	0.773	3	10.66	0	2	1	1.81
80	150	0	40	1	17.67	300	1.875	0.978	3	17.67	0	2	1	1.92
90	150	0	45	1	8.69	299	2.157	1.199	3	8.69	0	2	1	1.80
100	150	0	50	1	7.44	317	2.463	1.837	5	7.44	0	2	2	1.34
110	150	0	55	1	3.20	330	2.919	1.704	3	3.20	0	2	1	1.71
120	150	0	60	1	17.05	284	2.695	2.537	5	17.05	0	2	2	1.06
130	150	0	65	1	4.35	303	3.063	2.302	3	4.35	0	2	1	1.33
140	150	0	70	1	15.65	242	2.554	3.479	5	15.65	0	2	2	0.73
10	160	0	5	1	5.12	31	0.062	0.065	3	5.12	0	2	1	0.95
20	160	0	10	1	16.90	77	0.206	0.123	3	16.90	0	2	1	1.68
30	160	0	15	1	10.13	103	0.345	0.209	3	10.13	0	2	1	1.65
40	160	0	20	1	17.77	144	0.576	0.321	3	17.77	0	2	1	1.79
50	160	0	25	1	19.79	217	0.993	0.471	3	19.79	0	2	1	2.11
60	160	0	30	1	19.15	261	1.386	0.631	3	19.15	0	2	1	2.20
70	160	0	35	1	7.04	250	1.707	0.880	3	7.04	0	2	1	1.94
80	160	0	40	1	1.41	324	2.188	1.052	3	1.41	0	2	1	2.08
90	160	0	45	1	4.55	306	2.293	1.296	3	4.55	0	2	1	1.77
100	160	0	50	1	17.37	328	2.704	1.616	3	17.37	0	2	1	1.67
110	160	0	55	1	17.61	364	3.323	1.818	3	17.61	0	2	1	1.83
120	160	0	60	1	10.03	376	3.828	2.679	5	10.03	0	2	2	1.43
130	160	0	65	1	4.50	289	3.272	3.129	5	4.50	0	2	2	1.05
140	160	0	70	1	8.47	301	3.354	2.844	3	8.47	0	2	1	1.18
150	160	0	75	1	12.52	255	3.082	4.225	5	12.52	0	2	2	0.73
10	170	0	5	1	16.54	30	0.065	0.066	3	16.54	0	2	1	1.00
20	170	0	10	1	2.21	75	0.209	0.129	3	2.21	0	2	1	1.62
30	170	0	15	1	2.42	118	0.417	0.220	3	2.42	0	2	1	1.90
40	170	0	20	1	8.38	171	0.707	0.340	3	8.38	0	2	1	2.08
50	170	0	25	1	16.28	206	1.045	0.503	3	16.28	0	2	1	2.08
60	170	0	30	1	1.67	242	1.347	0.674	3	1.67	0	2	1	2.00
70	170	0	35	1	4.09	310	1.961	0.887	3	4.09	0	2	1	2.21
80	170	0	40	1	14.12	350	2.496	1.101	3	14.12	0	2	1	2.27
90	170	0	45	1	15.13	409	3.289	1.375	3	15.13	0	2	1	2.39
100	170	0	50	1	6.19	420	3.747	1.994	5	6.19	0	2	2	1.88
110	170	0	55	1	6.60	352	3.474	1.937	3	6.60	0	2	1	1.79
120	170	0	60	1	18.33	438	4.818	2.804	5	18.33	0	2	2	1.72
130	170	0	65	1	15.96	343	4.107	3.296	5	15.96	0	2	2	1.25
140	170	0	70	1	16.95	373	4.434	3.776	5	16.95	0	2	2	1.17
150	170	0	75	1	3.88	289	3.682	3.502	3	3.88	0	2	1	1.05
160	170	0	80	1	3.45	262	3.596	4.973	5	3.45	0	2	2	0.72
10	180	0	5	1	6.11	31	0.069	0.069	3	6.11	0	2	1	1.01
20	180	0	10	1	0.13	71	0.213	0.135	3	0.13	0	2	1	1.57
30	180	0	15	1	6.12	109	0.406	0.236	3	6.12	0	2	1	1.72
40	180	0	20	1	1.57	214	0.943	0.366	3	1.57	0	2	1	2.58
50	180	0	25	1	12.72	230	1.177	0.524	3	12.72	0	2	1	2.25
60	180	0	30	1	17.46	278	1.623	0.706	3	17.46	0	2	1	2.30
70	180	0	35	1	19.92	267	1.832	0.921	3	19.92	0	2	1	1.99
80	180	0	40	1	8.13	320	2.512	1.177	3	8.13	0	2	1	2.13
90	180	0	45	1	7.90	349	2.926	1.435	3	7.90	0	2	1	2.04
100	180	0	50	1	15.28	409	3.801	1.724	3	15.28	0	2	1	2.21
110	180	0	55	1	10.33	380	3.749	2.507	5	10.33	0	2	2	1.50
120	180	0	60	1	16.77	443	4.932	2.958	5	16.77	0	2	2	1.67
130	180	0	65	1	1.45	404	5.158	3.507	5	1.45	0	2	2	1.47
140	180	0	70	1	10.36	409	5.085	3.165	3	10.36	0	2	1	1.61
150	180	0	75	1	18.78	387	5.091	4.584	5	18.78	0	2	2	1.11
160	180	0	80	1	16.78	331	4.701	4.042	3	16.78	0	2	1	1.16
170	180	0	85	1	16.36	254	3.812	5.942	5	16.36	0	2	2	0.64
10	190	0	5	1	10.71	23	0.060	0.072	3	10.71	0	2	1	0.83
20	190	0	10	1	6.28	85	0.268	0.142	3	6.28	0	2	1	1.89
30	190	0	15	1	13.07	111	0.437	0.244	3	13.07	0	2	1	1.79
40	190	0	20	1	0.37	153	0.714	0.378	3	0.37	0	2	1	1.89
50	190	0	25	1	2.17	217	1.183	0.548	3	2.17	0	2	1	2.16
60	190	0	30	1	10.53	275	1.762	0.753	3	10.53	0	2	1	2.34
70	190	0	35	1	8.61	271	1.910	0.974	3	8.61	0	2	1	1.96
80	190	0	40	1	18.16	370	3.020	1.235	3	18.16	0	2	1	2.44
90	190	0	45	1	3.33	412	3.695	1.557	3	3.33	0	2	1	2.37
100	190	0	50	1	18.88	435	4.294	1.842	3	18.88	0	2	1	2.33
110	190	0	55	1	6.55	440	4.805	2.150	3	6.55	0	2	1	2.23
120	190	0	60	1	18.84	413	4.756	2.527	3	18.84	0	2	1	1.88
130	190	0	65	1	16.39	436	5.420	2.916	3	16.39	0	2	1	1.86
140	190	0	70	1	13.88	477	6.211	3.360	3	13.88	0	2	1	1.85
150	190	0	75	1	5.50	491	6.701	3.778	3	5.50	0	2	1	1.77
160	190	0	80	1	0.20	416	6.124	6.353	11	0.20	0	2	5	0.96
170	190	0	85	1	4.52	344	5.353	4.805	3	4.52	0	2	1	1.11
180	190	0	90	1	8.48	343	5.698	6.859	5	8.48	0	2	2	0.83
10	200	0	5	1	12.39	32	0.081	0.077	3	12.39	0	2	1	1.05
20	200	0	10	1	19.64	88	0.289	0.149	3	19.64	0	2	1	1.93
30	200	0	15	1	12.21	110	0.451	0.259	3	12.21	0	2	1	1.74
40	200	0	20	1	13.58	180	0.877	0.404	3	13.58	0	2	1	2.17
50	200	0	25	1	7.98	252	1.507	0.585	3	7.98	0	2	1	2.58
60	200	0	30	1	18.86	310	2.047	0.779	3	18.86	0	2	1	2.63

Table 20: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
70	200	0	35	1	1.11	305	2.344	1.045	3	1.11	0	2	1	2.24
80	200	0	40	1	0.54	391	3.265	1.314	3	0.54	0	2	1	2.49
90	200	0	45	1	2.16	381	3.462	1.609	3	2.16	0	2	1	2.15
100	200	0	50	1	0.34	486	5.025	1.905	3	0.34	0	2	1	2.64
110	200	0	55	1	4.02	456	5.237	2.257	3	4.02	0	2	1	2.32
120	200	0	60	1	19.16	475	6.599	2.632	3	19.16	0	2	1	2.51
130	200	0	65	1	1.50	424	5.817	3.777	5	1.50	0	2	2	1.54
140	200	0	70	1	19.00	521	7.064	4.309	5	19.00	0	2	2	1.64
150	200	0	75	1	11.72	534	7.701	4.908	5	11.72	0	2	2	1.57
160	200	0	80	1	6.25	468	7.136	4.553	3	6.25	0	2	1	1.57
170	200	0	85	1	10.15	417	6.698	6.422	5	10.15	0	2	2	1.04
180	200	0	90	1	14.89	427	7.495	5.658	3	14.89	0	2	1	1.32
190	200	0	95	1	0.64	323	5.826	7.971	5	0.64	0	2	2	0.73
10	210	0	5	1	13.01	29	0.077	0.079	3	13.01	0	2	1	0.97
20	210	0	10	1	5.28	75	0.259	0.159	3	5.28	0	2	1	1.63
30	210	0	15	1	5.28	132	0.561	0.272	3	5.28	0	2	1	2.06
40	210	0	20	1	6.06	181	0.928	0.416	3	6.06	0	2	1	2.23
50	210	0	25	1	3.17	225	1.345	0.607	3	3.17	0	2	1	2.21
60	210	0	30	1	12.22	273	1.876	0.819	3	12.22	0	2	1	2.29
70	210	0	35	1	19.63	338	2.679	1.086	3	19.63	0	2	1	2.47
80	210	0	40	1	16.07	337	2.947	1.348	3	16.07	0	2	1	2.19
90	210	0	45	1	3.79	408	3.925	1.667	3	3.79	0	2	1	2.35
100	210	0	50	1	15.26	467	4.941	2.008	3	15.26	0	2	1	2.46
110	210	0	55	1	3.19	435	5.015	2.394	3	3.19	0	2	1	2.10
120	210	0	60	1	16.49	476	5.944	2.801	3	16.49	0	2	1	2.12
130	210	0	65	1	9.72	528	7.099	3.203	3	9.72	0	2	1	2.22
140	210	0	70	1	2.48	522	7.509	4.501	5	2.48	0	2	2	1.67
150	210	0	75	1	14.06	535	8.186	4.207	3	14.06	0	2	1	1.95
160	210	0	80	1	12.26	536	8.643	5.851	5	12.26	0	2	2	1.48
170	210	0	85	1	15.91	437	7.485	5.287	3	15.91	0	2	1	1.42
180	210	0	90	1	9.75	460	8.144	7.365	5	9.75	0	2	2	1.11
190	210	0	95	1	12.35	345	6.622	6.568	3	12.35	0	2	1	1.01
200	210	0	100	1	11.55	331	6.738	9.275	5	11.55	0	2	2	0.73
10	220	0	5	1	8.37	25	0.074	0.083	3	8.37	0	2	1	0.89
20	220	0	10	1	13.04	75	0.283	0.169	3	13.04	0	2	1	1.68
30	220	0	15	1	14.99	120	0.552	0.287	3	14.99	0	2	1	1.92
40	220	0	20	1	14.13	176	0.954	0.441	3	14.13	0	2	1	2.16
50	220	0	25	1	4.00	221	1.383	0.645	3	4.00	0	2	1	2.15
60	220	0	30	1	10.30	287	2.079	0.855	3	10.30	0	2	1	2.43
70	220	0	35	1	2.86	343	2.778	1.120	3	2.86	0	2	1	2.48
80	220	0	40	1	16.63	419	3.763	1.418	3	16.63	0	2	1	2.65
90	220	0	45	1	3.30	430	4.330	1.765	3	3.30	0	2	1	2.45
100	220	0	50	1	2.37	532	5.900	2.095	3	2.37	0	2	1	2.82
110	220	0	55	1	0.40	432	5.640	2.525	3	0.40	0	2	1	2.23
120	220	0	60	1	18.27	474	6.134	2.936	3	18.27	0	2	1	2.09
130	220	0	65	1	16.04	570	8.033	3.648	3	16.04	0	2	1	2.20
140	220	0	70	1	16.16	530	7.933	3.834	3	16.16	0	2	1	2.07
150	220	0	75	1	10.46	540	8.579	4.422	3	10.46	0	2	1	1.94
160	220	0	80	1	9.20	576	9.616	6.086	5	9.20	0	2	2	1.58
170	220	0	85	1	13.81	561	9.926	5.514	3	13.81	0	2	1	1.80
180	220	0	90	1	4.61	525	9.878	7.715	5	4.61	0	2	2	1.28
190	220	0	95	1	13.05	481	9.285	8.651	5	13.05	0	2	2	1.07
200	220	0	100	1	6.81	378	7.909	7.601	3	6.81	0	2	1	1.04
210	220	0	105	1	12.14	406	8.852	10.680	5	12.14	0	2	2	0.83
10	230	0	5	1	3.60	27	0.085	0.086	3	3.60	0	2	1	0.98
20	230	0	10	1	18.09	76	0.309	0.174	3	18.09	0	2	1	1.77
30	230	0	15	1	18.40	123	0.590	0.301	3	18.40	0	2	1	1.96
40	230	0	20	1	12.96	180	1.015	0.460	3	12.96	0	2	1	2.20
50	230	0	25	1	18.00	221	1.460	0.673	3	18.00	0	2	1	2.17
60	230	0	30	1	15.60	305	2.342	0.896	3	15.60	0	2	1	2.62
70	230	0	35	1	12.75	336	2.885	1.184	3	12.75	0	2	1	2.44
80	230	0	40	1	19.34	438	4.220	1.509	3	19.34	0	2	1	2.80
90	230	0	45	1	16.71	464	4.891	1.877	3	16.71	0	2	1	2.61
100	230	0	50	1	9.10	470	5.564	2.206	3	9.10	0	2	1	2.52
110	230	0	55	1	5.95	417	5.310	2.610	3	5.95	0	2	1	2.03
120	230	0	60	1	13.66	526	7.188	3.061	3	13.66	0	2	1	2.35
130	230	0	65	1	13.47	629	9.193	3.567	3	13.47	0	2	1	2.58
140	230	0	70	1	6.71	557	8.647	4.014	3	6.71	0	2	1	2.15
150	230	0	75	1	0.70	630	10.406	5.632	5	0.70	0	2	2	1.85
160	230	0	80	1	10.34	601	10.488	6.351	5	10.34	0	2	2	1.65
170	230	0	85	1	19.17	626	11.580	5.856	3	19.17	0	2	1	1.98
180	230	0	90	1	0.70	616	11.797	6.359	3	0.70	0	2	1	1.86
190	230	0	95	1	19.42	575	11.917	8.920	5	19.42	0	2	2	1.34
200	230	0	100	1	9.14	567	12.321	10.007	5	9.14	0	2	2	1.23
210	230	0	105	1	11.05	499	11.425	8.687	3	11.05	0	2	1	1.32
220	230	0	110	1	12.85	391	9.628	12.310	5	12.85	0	2	2	0.78
10	240	0	5	1	8.88	26	0.085	0.087	3	8.88	0	2	1	0.98
20	240	0	10	1	14.57	75	0.303	0.180	3	14.57	0	2	1	1.68
30	240	0	15	1	0.01	128	0.629	0.310	3	0.01	0	2	1	2.03
40	240	0	20	1	6.03	209	1.222	0.472	3	6.03	0	2	1	2.59
50	240	0	25	1	3.93	267	1.840	0.709	3	3.93	0	2	1	2.60
60	240	0	30	1	18.51	271	2.109	0.932	3	18.51	0	2	1	2.26
70	240	0	35	1	1.32	337	2.965	1.240	3	1.32	0	2	1	2.39
80	240	0	40	1	13.35	335	3.322	1.543	3	13.35	0	2	1	2.15
90	240	0	45	1	15.05	441	4.787	1.899	3	15.05	0	2	1	2.52
100	240	0	50	1	17.70	574	6.969	2.291	3	17.70	0	2	1	3.04
110	240	0	55	1	9.91	536	7.025	2.712	3	9.91	0	2	1	2.59
120	240	0	60	1	10.05	556	8.034	3.216	3	10.05	0	2	1	2.50
130	240	0	65	1	14.70	576	8.954	3.701	3	14.70	0	2	1	2.42

Table 21: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
140	240	0	70	1	14.70	644	10.597	4.267	3	14.70	0	2	1	2.48
150	240	0	75	1	13.04	621	10.731	4.872	3	13.04	0	2	1	2.20
160	240	0	80	1	8.56	751	13.699	5.506	3	8.56	0	2	1	2.49
170	240	0	85	1	2.56	689	13.432	6.235	3	2.56	0	2	1	2.15
180	240	0	90	1	8.20	595	12.400	6.721	3	8.20	0	2	1	1.85
190	240	0	95	1	10.67	596	12.813	7.621	3	10.67	0	2	1	1.68
200	240	0	100	1	16.49	555	12.654	10.377	5	16.49	0	2	2	1.22
210	240	0	105	1	16.81	512	12.429	9.130	3	16.81	0	2	1	1.36
220	240	0	110	1	19.29	464	11.635	10.091	3	19.29	0	2	1	1.15
230	240	0	115	1	4.33	384	10.161	13.980	5	4.33	0	2	2	0.73
10	250	0	5	1	19.86	32	0.102	0.090	3	19.86	0	2	1	1.13
20	250	0	10	1	3.87	82	0.333	0.184	3	3.87	0	2	1	1.81
30	250	0	15	1	17.93	115	0.596	0.323	3	17.93	0	2	1	1.85
40	250	0	20	1	2.67	177	1.076	0.492	3	2.67	0	2	1	2.19
50	250	0	25	1	13.25	236	1.664	0.724	3	13.25	0	2	1	2.30
60	250	0	30	1	11.24	302	2.481	0.981	3	11.24	0	2	1	2.53
70	250	0	35	1	15.50	356	3.253	1.287	3	15.50	0	2	1	2.53
80	250	0	40	1	7.31	443	4.540	1.613	3	7.31	0	2	1	2.82
90	250	0	45	1	18.31	434	5.073	1.977	3	18.31	0	2	1	2.57
100	250	0	50	1	18.50	508	6.371	2.461	3	18.50	0	2	1	2.59
110	250	0	55	1	13.60	505	6.961	2.855	3	13.60	0	2	1	2.44
120	250	0	60	1	1.82	525	7.918	3.306	3	1.82	0	2	1	2.40
130	250	0	65	1	4.19	562	8.949	3.886	3	4.19	0	2	1	2.30
140	250	0	70	1	0.89	629	10.513	4.410	3	0.89	0	2	1	2.38
150	250	0	75	1	11.49	705	12.695	4.941	3	11.49	0	2	1	2.57
160	250	0	80	1	10.02	623	11.775	5.671	3	10.02	0	2	1	2.08
170	250	0	85	1	2.64	632	12.667	6.330	3	2.64	0	2	1	2.00
180	250	0	90	1	9.19	614	12.991	8.643	5	9.19	0	2	2	1.50
190	250	0	95	1	1.99	622	13.871	9.583	5	1.99	0	2	2	1.45
200	250	0	100	1	4.92	612	14.470	10.644	5	4.92	0	2	2	1.36
210	250	0	105	1	1.68	614	15.500	11.666	5	1.68	0	2	2	1.33
220	250	0	110	1	13.03	525	13.628	10.340	3	13.03	0	2	1	1.32
230	250	0	115	1	11.41	539	15.182	11.341	3	11.41	0	2	1	1.34
240	250	0	120	1	15.82	420	12.080	15.756	5	15.82	0	2	2	0.77
10	260	0	5	1	17.85	26	0.085	0.094	3	17.85	0	2	1	0.91
20	260	0	10	1	15.31	91	0.386	0.191	3	15.31	0	2	1	2.02
30	260	0	15	1	17.07	132	0.696	0.330	3	17.07	0	2	1	2.11
40	260	0	20	1	3.00	179	1.123	0.512	3	3.00	0	2	1	2.19
50	260	0	25	1	15.78	238	1.776	0.753	3	15.78	0	2	1	2.36
60	260	0	30	1	4.99	261	2.205	1.010	3	4.99	0	2	1	2.18
70	260	0	35	1	3.96	334	3.189	1.348	3	3.96	0	2	1	2.37
80	260	0	40	1	13.57	417	4.475	1.689	3	13.57	0	2	1	2.65
90	260	0	45	1	14.50	496	6.003	2.071	3	14.50	0	2	1	2.90
100	260	0	50	1	3.51	496	6.433	2.484	3	3.51	0	2	1	2.59
110	260	0	55	1	8.35	528	7.430	2.951	3	8.35	0	2	1	2.52
120	260	0	60	1	0.38	638	10.261	3.430	3	0.38	0	2	1	2.99
130	260	0	65	1	2.56	610	9.948	3.984	3	2.56	0	2	1	2.50
140	260	0	70	1	0.40	675	11.926	4.595	3	0.40	0	2	1	2.60
150	260	0	75	1	1.84	647	12.090	5.137	3	1.84	0	2	1	2.35
160	260	0	80	1	8.58	682	13.753	5.921	3	8.58	0	2	1	2.32
170	260	0	85	1	1.35	720	15.259	6.554	3	1.35	0	2	1	2.33
180	260	0	90	1	4.05	666	14.518	8.925	5	4.05	0	2	2	1.63
190	260	0	95	1	0.83	741	17.153	8.068	3	0.83	0	2	1	2.13
200	260	0	100	1	2.02	732	17.872	11.081	5	2.02	0	2	2	1.61
210	260	0	105	1	5.08	649	16.694	12.543	5	5.08	0	2	2	1.33
220	260	0	110	1	19.42	683	18.657	13.402	5	19.42	0	2	2	1.39
230	260	0	115	1	18.05	571	16.195	11.977	3	18.05	0	2	1	1.35
240	260	0	120	1	4.38	572	16.869	12.694	3	4.38	0	2	1	1.33
250	260	0	125	1	1.45	399	12.348	23.447	17	1.45	0	2	8	0.53
10	270	0	5	1	4.40	28	0.099	0.097	3	4.40	0	2	1	1.02
20	270	0	10	1	17.89	70	0.312	0.202	3	17.89	0	2	1	1.55
30	270	0	15	1	17.90	119	0.669	0.342	3	17.90	0	2	1	1.96
40	270	0	20	1	7.76	192	1.283	0.540	3	7.76	0	2	1	2.38
50	270	0	25	1	11.47	215	1.721	0.784	3	11.47	0	2	1	2.19
60	270	0	30	1	4.49	303	2.645	1.063	3	4.49	0	2	1	2.49
70	270	0	35	1	16.00	332	3.342	1.364	3	16.00	0	2	1	2.45
80	270	0	40	1	1.74	445	4.937	1.768	3	1.74	0	2	1	2.79
90	270	0	45	1	8.84	475	5.911	2.140	3	8.84	0	2	1	2.76
100	270	0	50	1	13.29	582	7.916	2.572	3	13.29	0	2	1	3.08
110	270	0	55	1	1.75	662	9.669	3.041	3	1.75	0	2	1	3.18
120	270	0	60	1	15.94	609	9.999	3.561	3	15.94	0	2	1	2.81
130	270	0	65	1	3.20	723	12.461	4.164	3	3.20	0	2	1	2.99
140	270	0	70	1	18.37	681	12.467	4.819	3	18.37	0	2	1	2.59
150	270	0	75	1	13.73	703	13.679	5.369	3	13.73	0	2	1	2.55
160	270	0	80	1	2.80	689	14.215	6.163	3	2.80	0	2	1	2.31
170	270	0	85	1	0.12	792	17.224	8.338	5	0.12	0	2	2	2.07
180	270	0	90	1	8.40	714	16.205	7.481	3	8.40	0	2	1	2.17
190	270	0	95	1	0.30	732	17.245	8.431	3	0.30	0	2	1	2.05
200	270	0	100	1	18.67	661	16.512	11.413	5	18.67	0	2	2	1.45
210	270	0	105	1	13.34	616	16.227	10.136	3	13.34	0	2	1	1.60
220	270	0	110	1	7.26	650	18.214	13.868	5	7.26	0	2	2	1.31
230	270	0	115	1	8.57	686	19.951	15.056	5	8.57	0	2	2	1.33
240	270	0	120	1	9.39	627	19.155	13.335	3	9.39	0	2	1	1.44
250	270	0	125	1	19.63	471	15.071	14.156	3	19.63	0	2	1	1.06
260	270	0	130	1	11.23	404	13.933	19.671	5	11.23	0	2	2	0.71
10	280	0	5	1	9.12	30	0.108	0.098	3	9.12	0	2	1	1.10
20	280	0	10	1	0.96	69	0.320	0.204	3	0.96	0	2	1	1.57
30	280	0	15	1	5.36	140	0.816	0.355	3	5.36	0	2	1	2.30
40	280	0	20	1	16.05	182	1.238	0.550	3	16.05	0	2	1	2.25

Table 22: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
50	280	0	25	1	8.09	244	1.937	0.807	3	8.09	0	2	1	2.40
60	280	0	30	1	17.01	297	2.711	1.094	3	17.01	0	2	1	2.48
70	280	0	35	1	2.16	343	3.594	1.426	3	2.16	0	2	1	2.52
80	280	0	40	1	8.35	413	4.771	1.816	3	8.35	0	2	1	2.63
90	280	0	45	1	2.70	497	6.466	2.222	3	2.70	0	2	1	2.91
100	280	0	50	1	19.74	530	7.337	2.660	3	19.74	0	2	1	2.76
110	280	0	55	1	0.44	607	9.320	3.214	3	0.44	0	2	1	2.90
120	280	0	60	1	7.71	765	12.974	3.681	3	7.71	0	2	1	3.52
130	280	0	65	1	9.63	698	12.477	4.272	3	9.63	0	2	1	2.92
140	280	0	70	1	1.24	703	13.160	4.907	3	1.24	0	2	1	2.68
150	280	0	75	1	18.02	765	15.229	5.580	3	18.02	0	2	1	2.73
160	280	0	80	1	4.68	773	16.400	6.322	3	4.68	0	2	1	2.59
170	280	0	85	1	12.21	782	17.516	7.039	3	12.21	0	2	1	2.49
180	280	0	90	1	13.78	736	17.201	9.494	5	13.78	0	2	2	1.81
190	280	0	95	1	6.89	748	18.804	10.680	5	6.89	0	2	2	1.76
200	280	0	100	1	4.80	831	21.789	11.746	5	4.80	0	2	2	1.86
210	280	0	105	1	6.73	711	19.427	10.555	3	6.73	0	2	1	1.84
220	280	0	110	1	13.96	760	21.932	11.616	3	13.96	0	2	1	1.89
230	280	0	115	1	3.74	701	21.328	12.586	3	3.74	0	2	1	1.69
240	280	0	120	1	9.00	734	23.413	16.997	5	9.00	0	2	2	1.38
250	280	0	125	1	12.10	609	20.080	14.807	3	12.10	0	2	1	1.36
260	280	0	130	1	13.34	516	17.673	16.004	3	13.34	0	2	1	1.10
270	280	0	135	1	6.94	414	14.795	22.006	5	6.94	0	2	2	0.67
10	290	0	5	1	14.29	27	0.102	0.103	3	14.29	0	2	1	0.99
20	290	0	10	1	6.47	73	0.347	0.212	3	6.47	0	2	1	1.63
30	290	0	15	1	14.50	125	0.734	0.370	3	14.50	0	2	1	1.98
40	290	0	20	1	10.93	181	1.267	0.569	3	10.93	0	2	1	2.23
50	290	0	25	1	2.38	233	1.908	0.836	3	2.38	0	2	1	2.28
60	290	0	30	1	5.45	318	3.001	1.135	3	5.45	0	2	1	2.64
70	290	0	35	1	15.62	352	3.745	1.482	3	15.62	0	2	1	2.53
80	290	0	40	1	6.72	403	4.843	1.863	3	6.72	0	2	1	2.60
90	290	0	45	1	19.69	501	6.608	2.330	3	19.69	0	2	1	2.84
100	290	0	50	1	8.02	563	8.217	2.746	3	8.02	0	2	1	2.99
110	290	0	55	1	16.19	585	9.067	3.276	3	16.19	0	2	1	2.77
120	290	0	60	1	7.22	618	10.552	3.901	3	7.22	0	2	1	2.70
130	290	0	65	1	13.62	669	12.617	4.457	3	13.62	0	2	1	2.83
140	290	0	70	1	6.27	651	12.722	5.224	3	6.27	0	2	1	2.44
150	290	0	75	1	7.22	759	15.875	5.712	3	7.22	0	2	1	2.78
160	290	0	80	1	14.62	764	16.601	6.567	3	14.62	0	2	1	2.53
170	290	0	85	1	16.57	694	16.044	7.270	3	16.57	0	2	1	2.21
180	290	0	90	1	8.64	816	19.899	8.063	3	8.64	0	2	1	2.47
190	290	0	95	1	18.06	812	21.120	8.941	3	18.06	0	2	1	2.36
200	290	0	100	1	8.63	879	23.844	12.150	5	8.63	0	2	2	1.96
210	290	0	105	1	5.85	800	22.805	10.885	3	5.85	0	2	1	2.10
220	290	0	110	1	3.51	787	23.612	11.961	3	3.51	0	2	1	1.97
230	290	0	115	1	18.35	883	27.680	16.152	5	18.35	0	2	2	1.71
240	290	0	120	1	8.02	736	24.226	14.166	3	8.02	0	2	1	1.71
250	290	0	125	1	17.41	715	24.419	15.493	3	17.41	0	2	1	1.58
260	290	0	130	1	7.73	663	23.636	16.610	3	7.73	0	2	1	1.42
270	290	0	135	1	18.53	498	18.609	17.808	3	18.53	0	2	1	1.04
280	290	0	140	1	18.73	453	17.464	24.554	5	18.73	0	2	2	0.71
10	300	0	5	1	10.16	26	0.103	0.105	3	10.16	0	2	1	0.98
20	300	0	10	1	14.39	73	0.367	0.219	3	14.39	0	2	1	1.68
30	300	0	15	1	0.32	131	0.802	0.381	3	0.32	0	2	1	2.10
40	300	0	20	1	16.46	203	1.482	0.593	3	16.46	0	2	1	2.50
50	300	0	25	1	14.19	262	2.307	0.858	3	14.19	0	2	1	2.69
60	300	0	30	1	1.79	286	2.804	1.162	3	1.79	0	2	1	2.41
70	300	0	35	1	4.51	345	3.903	1.520	3	4.51	0	2	1	2.57
80	300	0	40	1	18.70	460	5.731	1.938	3	18.70	0	2	1	2.96
90	300	0	45	1	10.66	464	6.403	2.404	3	10.66	0	2	1	2.66
100	300	0	50	1	10.65	522	7.854	2.853	3	10.65	0	2	1	2.75
110	300	0	55	1	18.62	581	9.388	3.458	3	18.62	0	2	1	2.71
120	300	0	60	1	15.66	578	10.103	3.971	3	15.66	0	2	1	2.54
130	300	0	65	1	8.42	682	13.375	4.701	3	8.42	0	2	1	2.84
140	300	0	70	1	8.76	769	15.426	5.318	3	8.76	0	2	1	2.90
150	300	0	75	1	17.17	698	15.117	5.949	3	17.17	0	2	1	2.54
160	300	0	80	1	15.66	810	18.523	6.833	3	15.66	0	2	1	2.71
170	300	0	85	1	0.85	865	20.626	7.541	3	0.85	0	2	1	2.74
180	300	0	90	1	3.50	843	21.342	8.495	3	3.50	0	2	1	2.51
190	300	0	95	1	11.22	894	23.836	9.212	3	11.22	0	2	1	2.59
200	300	0	100	1	15.49	817	23.167	12.549	5	15.49	0	2	2	1.85
210	300	0	105	1	1.30	789	23.241	15.183	15	1.30	0	2	7	1.53
220	300	0	110	1	13.91	733	22.741	15.066	5	13.91	0	2	2	1.51
230	300	0	115	1	15.41	749	24.147	16.570	5	15.41	0	2	2	1.46
240	300	0	120	1	19.73	844	28.255	17.870	5	19.73	0	2	2	1.58
250	300	0	125	1	11.20	794	27.878	15.858	3	11.20	0	2	1	1.76
260	300	0	130	1	4.08	748	27.238	17.089	3	4.08	0	2	1	1.59
270	300	0	135	1	17.39	612	23.147	18.415	3	17.39	0	2	1	1.26
280	300	0	140	1	13.57	591	23.223	19.777	3	13.57	0	2	1	1.17
290	300	0	145	1	8.65	495	20.297	27.308	5	8.65	0	2	2	0.74
10	310	0	5	1	4.16	33	0.138	0.110	3	4.16	0	2	1	1.26
20	310	0	10	1	12.95	92	0.474	0.232	3	12.95	0	2	1	2.04
30	310	0	15	1	2.79	106	0.685	0.393	3	2.79	0	2	1	1.74
40	310	0	20	1	1.43	213	1.614	0.606	3	1.43	0	2	1	2.66
50	310	0	25	1	3.18	252	2.228	0.886	3	3.18	0	2	1	2.52
60	310	0	30	1	15.41	323	3.270	1.200	3	15.41	0	2	1	2.72
70	310	0	35	1	19.50	387	4.471	1.568	3	19.50	0	2	1	2.85
80	310	0	40	1	9.14	513	6.597	1.978	3	9.14	0	2	1	3.34
90	310	0	45	1	17.24	479	6.807	2.467	3	17.24	0	2	1	2.76

Table 23: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

				LINFSOL				B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
100	310	0	50	1	18.43	554	8.596	2.957	3	18.43	0	2	1	2.91
110	310	0	55	1	6.29	685	11.717	3.462	3	6.29	0	2	1	3.38
120	310	0	60	1	19.12	725	13.290	4.064	3	19.12	0	2	1	3.27
130	310	0	65	1	7.99	731	14.424	4.721	3	7.99	0	2	1	3.06
140	310	0	70	1	16.79	753	15.792	5.469	3	16.79	0	2	1	2.89
150	310	0	75	1	17.91	826	18.621	6.149	3	17.91	0	2	1	3.03
160	310	0	80	1	13.82	797	18.625	6.912	3	13.82	0	2	1	2.69
170	310	0	85	1	19.82	893	21.980	7.825	3	19.82	0	2	1	2.81
180	310	0	90	1	3.05	800	20.938	8.601	3	3.05	0	2	1	2.43
190	310	0	95	1	18.82	923	25.313	9.788	3	18.82	0	2	1	2.59
200	310	0	100	1	4.36	875	25.369	10.628	3	4.36	0	2	1	2.39
210	310	0	105	1	13.81	898	27.228	14.352	5	13.81	0	2	2	1.90
220	310	0	110	1	15.13	963	30.985	15.605	5	15.13	0	2	2	1.99
230	310	0	115	1	8.55	886	29.594	13.903	3	8.55	0	2	1	2.13
240	310	0	120	1	6.58	878	30.399	14.944	3	6.58	0	2	1	2.03
250	310	0	125	1	11.94	916	32.986	16.173	3	11.94	0	2	1	2.04
260	310	0	130	1	13.41	747	28.087	17.630	3	13.41	0	2	1	1.59
270	310	0	135	1	18.82	627	24.624	19.182	3	18.82	0	2	1	1.28
280	310	0	140	1	8.33	721	29.392	20.425	3	8.33	0	2	1	1.44
290	310	0	145	1	1.48	659	28.193	22.051	3	1.48	0	2	1	1.28
300	310	0	150	1	5.92	517	22.849	31.353	7	5.92	0	2	3	0.73
10	320	0	5	1	9.75	24	0.108	0.112	3	9.75	0	2	1	0.97
20	320	0	10	1	7.95	74	0.407	0.233	3	7.95	0	2	1	1.75
30	320	0	15	1	1.82	121	0.823	0.403	3	1.82	0	2	1	2.04
40	320	0	20	1	2.26	187	1.499	0.626	3	2.26	0	2	1	2.39
50	320	0	25	1	14.28	249	2.371	0.934	3	14.28	0	2	1	2.54
60	320	0	30	1	14.14	343	3.673	1.245	3	14.14	0	2	1	2.95
70	320	0	35	1	11.29	386	4.612	1.623	3	11.29	0	2	1	2.84
80	320	0	40	1	9.56	463	6.134	2.034	3	9.56	0	2	1	3.02
90	320	0	45	1	3.58	542	7.966	2.539	3	3.58	0	2	1	3.14
100	320	0	50	1	13.97	615	9.962	3.031	3	13.97	0	2	1	3.29
110	320	0	55	1	18.43	608	10.917	3.583	3	18.43	0	2	1	3.05
120	320	0	60	1	7.40	650	12.321	4.212	3	7.40	0	2	1	2.93
130	320	0	65	1	10.29	685	14.105	4.916	3	10.29	0	2	1	2.87
140	320	0	70	1	15.76	706	15.245	5.587	3	15.76	0	2	1	2.73
150	320	0	75	1	8.59	844	19.507	6.318	3	8.59	0	2	1	3.09
160	320	0	80	1	14.34	757	18.489	7.218	3	14.34	0	2	1	2.56
170	320	0	85	1	12.99	878	22.558	8.105	3	12.99	0	2	1	2.78
180	320	0	90	1	13.22	958	26.186	8.874	3	13.22	0	2	1	2.95
190	320	0	95	1	5.95	926	26.563	9.859	3	5.95	0	2	1	2.69
200	320	0	100	1	13.96	930	27.922	11.036	3	13.96	0	2	1	2.53
210	320	0	105	1	15.21	945	29.695	11.936	3	15.21	0	2	1	2.49
220	320	0	110	1	4.84	1006	34.047	13.206	3	4.84	0	2	1	2.58
230	320	0	115	1	15.32	947	32.982	14.700	3	15.32	0	2	1	2.24
240	320	0	120	1	4.78	977	35.009	19.143	5	4.78	0	2	2	1.83
250	320	0	125	1	7.10	982	37.484	16.764	3	7.10	0	2	1	2.24
260	320	0	130	1	18.49	845	33.552	18.318	3	18.49	0	2	1	1.83
270	320	0	135	1	15.40	888	36.274	24.594	5	15.40	0	2	2	1.47
280	320	0	140	1	5.64	810	34.877	21.131	3	5.64	0	2	1	1.65
290	320	0	145	1	7.38	728	32.274	22.928	3	7.38	0	2	1	1.41
300	320	0	150	1	15.37	627	28.601	25.533	5	15.37	0	2	2	1.12
310	320	0	155	1	1.94	507	24.194	32.417	5	1.94	0	2	2	0.75
10	330	0	5	1	15.48	28	0.128	0.117	3	15.48	0	2	1	1.10
20	330	0	10	1	15.54	84	0.479	0.241	3	15.54	0	2	1	1.99
30	330	0	15	1	6.88	125	0.910	0.420	3	6.88	0	2	1	2.16
40	330	0	20	1	2.42	225	1.868	0.658	3	2.42	0	2	1	2.84
50	330	0	25	1	14.74	276	2.650	0.949	3	14.74	0	2	1	2.79
60	330	0	30	1	4.63	376	4.166	1.282	3	4.63	0	2	1	3.25
70	330	0	35	1	17.58	386	4.855	1.711	3	17.58	0	2	1	2.84
80	330	0	40	1	6.41	494	6.833	2.155	3	6.41	0	2	1	3.17
90	330	0	45	1	17.44	555	8.530	2.595	3	17.44	0	2	1	3.29
100	330	0	50	1	9.68	580	9.665	3.205	3	9.68	0	2	1	3.02
110	330	0	55	1	18.31	625	11.332	3.724	3	18.31	0	2	1	3.04
120	330	0	60	1	15.38	588	12.088	4.390	3	15.38	0	2	1	2.75
130	330	0	65	1	19.77	893	19.004	5.024	3	19.77	0	2	1	3.78
140	330	0	70	1	15.03	784	17.800	5.771	3	15.03	0	2	1	3.08
150	330	0	75	1	17.23	728	17.345	6.626	3	17.23	0	2	1	2.62
160	330	0	80	1	10.05	897	22.503	7.463	3	10.05	0	2	1	3.02
170	330	0	85	1	9.87	913	24.314	8.351	3	9.87	0	2	1	2.91
180	330	0	90	1	16.95	909	25.568	9.204	3	16.95	0	2	1	2.78
190	330	0	95	1	5.06	1037	30.979	10.250	3	5.06	0	2	1	3.02
200	330	0	100	1	8.45	930	29.069	11.378	3	8.45	0	2	1	2.55
210	330	0	105	1	13.40	1084	35.277	12.383	3	13.40	0	2	1	2.85
220	330	0	110	1	0.36	1030	35.244	16.492	5	0.36	0	2	2	2.14
230	330	0	115	1	17.06	1029	37.199	14.717	3	17.06	0	2	1	2.53
240	330	0	120	1	2.55	1027	38.702	19.673	5	2.55	0	2	2	1.97
250	330	0	125	1	17.83	966	38.087	17.388	3	17.83	0	2	1	2.19
260	330	0	130	1	6.69	953	39.438	23.261	5	6.69	0	2	2	1.70
270	330	0	135	1	11.77	940	40.555	20.399	3	11.77	0	2	1	1.99
280	330	0	140	1	15.56	1006	44.474	27.292	5	15.56	0	2	2	1.63
290	330	0	145	1	15.59	747	34.157	23.439	3	15.59	0	2	1	1.46
300	330	0	150	1	2.86	874	41.240	33.979	15	2.86	0	2	7	1.21
310	330	0	155	1	15.16	688	33.899	26.353	3	15.16	0	2	1	1.29
320	330	0	160	1	2.60	485	25.376	36.085	5	2.60	0	2	2	0.70
10	340	0	5	1	10.92	34	0.164	0.119	3	10.92	0	2	1	1.38
20	340	0	10	1	16.77	68	0.411	0.246	3	16.77	0	2	1	1.67
30	340	0	15	1	10.22	148	1.064	0.428	3	10.22	0	2	1	2.49
40	340	0	20	1	12.74	200	1.711	0.663	3	12.74	0	2	1	2.58
50	340	0	25	1	3.59	241	2.398	0.973	3	3.59	0	2	1	2.47

Table 24: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
60	340	0	30	1	17.40	339	3.927	1.318	3	17.40	0	2	1	2.98
70	340	0	35	1	9.81	391	4.999	1.782	3	9.81	0	2	1	2.80
80	340	0	40	1	17.57	511	7.414	2.177	3	17.57	0	2	1	3.41
90	340	0	45	1	7.68	535	8.584	2.696	3	7.68	0	2	1	3.18
100	340	0	50	1	7.13	642	11.375	3.204	3	7.13	0	2	1	3.55
110	340	0	55	1	16.53	593	11.332	3.861	3	16.53	0	2	1	2.93
120	340	0	60	1	3.87	729	15.050	4.463	3	3.87	0	2	1	3.37
130	340	0	65	1	19.64	675	14.702	5.365	3	19.64	0	2	1	2.74
140	340	0	70	1	4.85	838	19.316	5.902	3	4.85	0	2	1	3.27
150	340	0	75	1	19.07	858	21.440	6.790	3	19.07	0	2	1	3.16
160	340	0	80	1	2.89	902	23.521	7.636	3	2.89	0	2	1	3.08
170	340	0	85	1	11.72	929	25.666	8.495	3	11.72	0	2	1	3.02
180	340	0	90	1	2.94	1007	29.351	9.504	3	2.94	0	2	1	3.09
190	340	0	95	1	10.82	973	29.521	10.471	3	10.82	0	2	1	2.82
200	340	0	100	1	11.62	989	31.874	11.702	3	11.62	0	2	1	2.72
210	340	0	105	1	16.52	1045	34.831	12.638	3	16.52	0	2	1	2.76
220	340	0	110	1	12.09	1117	39.153	13.931	3	12.09	0	2	1	2.81
230	340	0	115	1	3.19	1239	45.885	15.275	3	3.19	0	2	1	3.00
240	340	0	120	1	7.89	1095	42.649	20.340	5	7.89	0	2	2	2.10
250	340	0	125	1	15.42	1056	43.137	17.839	3	15.42	0	2	1	2.42
260	340	0	130	1	2.54	989	41.329	19.404	3	2.54	0	2	1	2.13
270	340	0	135	1	14.06	963	42.681	25.874	5	14.06	0	2	2	1.65
280	340	0	140	1	2.51	1008	46.002	28.319	5	2.51	0	2	2	1.62
290	340	0	145	1	3.68	886	42.280	30.412	5	3.68	0	2	2	1.39
300	340	0	150	1	4.42	743	37.059	25.533	3	4.42	0	2	1	1.45
310	340	0	155	1	6.24	782	40.464	27.572	3	6.24	0	2	1	1.47
320	340	0	160	1	10.66	628	33.508	29.270	3	10.66	0	2	1	1.14
330	340	0	165	1	14.83	507	28.593	39.693	5	14.83	0	2	2	0.72
10	350	0	5	1	2.18	34	0.172	0.125	3	2.18	0	2	1	1.38
20	350	0	10	1	19.76	81	0.534	0.272	3	19.76	0	2	1	1.96
30	350	0	15	1	19.37	147	1.099	0.463	3	19.37	0	2	1	2.37
40	350	0	20	1	3.13	190	1.726	0.718	3	3.13	0	2	1	2.40
50	350	0	25	1	10.12	246	2.653	1.053	3	10.12	0	2	1	2.52
60	350	0	30	1	4.57	307	3.711	1.360	3	4.57	0	2	1	2.73
70	350	0	35	1	16.77	412	5.469	1.848	3	16.77	0	2	1	2.96
80	350	0	40	1	13.01	413	6.114	2.240	3	13.01	0	2	1	2.73
90	350	0	45	1	5.87	504	8.291	2.777	3	5.87	0	2	1	2.99
100	350	0	50	1	2.15	626	11.196	3.386	3	2.15	0	2	1	3.31
110	350	0	55	1	9.47	652	12.731	3.925	3	9.47	0	2	1	3.24
120	350	0	60	1	11.34	691	14.369	4.644	3	11.34	0	2	1	3.09
130	350	0	65	1	19.16	811	18.125	5.322	3	19.16	0	2	1	3.41
140	350	0	70	1	19.52	834	19.896	6.075	3	19.52	0	2	1	3.28
150	350	0	75	1	10.03	769	19.415	7.019	3	10.03	0	2	1	2.77
160	350	0	80	1	16.86	892	23.925	7.750	3	16.86	0	2	1	3.09
170	350	0	85	1	8.19	1070	30.086	8.821	3	8.19	0	2	1	3.41
180	350	0	90	1	17.80	1014	30.201	9.877	3	17.80	0	2	1	3.06
190	350	0	95	1	14.07	1038	32.361	10.839	3	14.07	0	2	1	2.99
200	350	0	100	1	13.56	987	32.932	12.106	3	13.56	0	2	1	2.72
210	350	0	105	1	6.76	1189	41.316	13.278	3	6.76	0	2	1	3.11
220	350	0	110	1	3.95	1139	41.585	15.477	3	3.95	0	2	1	2.69
230	350	0	115	1	12.14	1090	41.873	15.575	3	12.14	0	2	1	2.69
240	350	0	120	1	8.56	1035	41.508	17.058	3	8.56	0	2	1	2.43
250	350	0	125	1	7.11	1235	52.412	18.638	3	7.11	0	2	1	2.81
260	350	0	130	1	19.77	1032	44.896	24.462	5	19.77	0	2	2	1.84
270	350	0	135	1	2.28	1056	48.559	26.229	5	2.28	0	2	2	1.85
280	350	0	140	1	16.33	1020	48.511	23.246	3	16.33	0	2	1	2.09
290	350	0	145	1	15.46	913	45.220	31.142	5	15.46	0	2	2	1.45
300	350	0	150	1	4.72	919	46.925	34.896	7	4.72	0	2	3	1.34
310	350	0	155	1	14.05	834	43.919	27.872	3	14.05	0	2	1	1.58
320	350	0	160	1	10.37	775	43.062	29.819	3	10.37	0	2	1	1.44
330	350	0	165	1	1.54	738	42.843	43.374	15	1.54	0	2	7	0.99
340	350	0	170	1	17.51	524	32.255	43.616	5	17.51	0	2	2	0.74
10	360	0	5	1	12.85	28	0.146	0.125	3	12.85	0	2	1	1.16
20	360	0	10	1	18.90	78	0.491	0.264	3	18.90	0	2	1	1.86
30	360	0	15	1	3.20	131	1.086	0.458	3	3.20	0	2	1	2.37
40	360	0	20	1	6.60	195	1.784	0.706	3	6.60	0	2	1	2.53
50	360	0	25	1	1.44	265	2.836	1.062	3	1.44	0	2	1	2.67
60	360	0	30	1	9.41	371	4.473	1.400	3	9.41	0	2	1	3.19
70	360	0	35	1	17.31	399	5.641	1.862	3	17.31	0	2	1	3.03
80	360	0	40	1	12.96	464	7.088	2.311	3	12.96	0	2	1	3.07
90	360	0	45	1	16.65	586	9.845	2.837	3	16.65	0	2	1	3.47
100	360	0	50	1	11.34	637	11.742	3.501	3	11.34	0	2	1	3.35
110	360	0	55	1	1.44	697	13.812	4.041	3	1.44	0	2	1	3.42
120	360	0	60	1	15.10	733	15.805	4.747	3	15.10	0	2	1	3.33
130	360	0	65	1	7.97	824	18.960	5.436	3	7.97	0	2	1	3.49
140	360	0	70	1	18.24	873	21.114	6.313	3	18.24	0	2	1	3.34
150	360	0	75	1	7.69	874	22.621	7.101	3	7.69	0	2	1	3.19
160	360	0	80	1	8.36	1016	27.965	8.035	3	8.36	0	2	1	3.48
170	360	0	85	1	19.79	1017	30.058	9.031	3	19.79	0	2	1	3.33
180	360	0	90	1	12.15	1125	34.854	10.172	3	12.15	0	2	1	3.43
190	360	0	95	1	9.36	1070	34.954	11.297	3	9.36	0	2	1	3.09
200	360	0	100	1	18.06	1062	36.497	12.595	3	18.06	0	2	1	2.90
210	360	0	105	1	16.17	1044	38.021	13.814	3	16.17	0	2	1	2.75
220	360	0	110	1	19.09	1097	40.783	45.250	3	19.09	0	2	1	0.90
230	360	0	115	1	16.14	1110	44.318	16.489	3	16.14	0	2	1	2.69
240	360	0	120	1	7.81	1161	49.261	17.614	3	7.81	0	2	1	2.80
250	360	0	125	1	9.48	1110	48.795	23.063	5	9.48	0	2	2	2.12
260	360	0	130	1	1.44	1148	53.454	25.436	5	1.44	0	2	2	2.10
270	360	0	135	1	16.14	1033	49.854	22.475	3	16.14	0	2	1	2.22

Table 25: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
280	360	0	140	1	14.68	1184	59.293	24.011	3	14.68	0	2	1	2.47
290	360	0	145	1	10.48	1097	57.137	25.980	3	10.48	0	2	1	2.20
300	360	0	150	1	7.16	984	53.168	34.162	5	7.16	0	2	2	1.56
310	360	0	155	1	0.13	1012	56.499	46.416	17	0.13	0	2	8	1.22
320	360	0	160	1	3.85	842	49.255	31.007	3	3.85	0	2	1	1.59
330	360	0	165	1	19.32	764	46.332	33.202	3	19.32	0	2	1	1.40
340	360	0	170	1	7.25	681	42.768	35.111	3	7.25	0	2	1	1.22
350	360	0	175	1	13.19	545	35.775	47.037	5	13.19	0	2	2	0.76
10	370	0	5	1	5.77	32	0.162	0.126	3	5.77	0	2	1	1.29
20	370	0	10	1	19.16	96	0.608	0.267	3	19.16	0	2	1	2.28
30	370	0	15	1	16.93	125	1.008	0.465	3	16.93	0	2	1	2.17
40	370	0	20	1	14.99	195	1.855	0.736	3	14.99	0	2	1	2.52
50	370	0	25	1	13.08	275	2.995	1.073	3	13.08	0	2	1	2.79
60	370	0	30	1	0.83	359	4.479	1.434	3	0.83	0	2	1	3.12
70	370	0	35	1	5.82	400	5.650	1.870	3	5.82	0	2	1	3.02
80	370	0	40	1	14.15	475	7.383	2.371	3	14.15	0	2	1	3.11
90	370	0	45	1	12.08	600	10.274	2.908	3	12.08	0	2	1	3.53
100	370	0	50	1	4.02	624	12.050	3.519	3	4.02	0	2	1	3.42
110	370	0	55	1	1.39	643	13.742	4.174	3	1.39	0	2	1	3.29
120	370	0	60	1	18.06	654	14.503	4.888	3	18.06	0	2	1	2.97
130	370	0	65	1	5.63	813	19.221	5.696	3	5.63	0	2	1	3.37
140	370	0	70	1	8.60	791	19.726	6.461	3	8.60	0	2	1	3.05
150	370	0	75	1	15.90	952	25.210	7.313	3	15.90	0	2	1	3.45
160	370	0	80	1	11.24	972	27.583	8.240	3	11.24	0	2	1	3.35
170	370	0	85	1	2.09	1102	33.205	9.315	3	2.09	0	2	1	3.56
180	370	0	90	1	7.53	1029	32.604	10.541	3	7.53	0	2	1	3.09
190	370	0	95	1	3.30	997	32.948	11.384	3	3.30	0	2	1	2.89
200	370	0	100	1	8.86	999	35.032	12.693	3	8.86	0	2	1	2.76
210	370	0	105	1	11.47	1246	45.417	14.111	3	11.47	0	2	1	3.22
220	370	0	110	1	14.48	1126	43.536	42.055	3	14.48	0	2	1	1.04
230	370	0	115	1	10.10	1235	49.884	16.495	3	10.10	0	2	1	3.02
240	370	0	120	1	7.33	1106	46.878	21.758	5	7.33	0	2	2	2.15
250	370	0	125	1	1.01	1233	54.730	19.600	3	1.01	0	2	1	2.79
260	370	0	130	1	7.04	1059	48.363	25.813	5	7.04	0	2	2	1.87
270	370	0	135	1	12.57	1114	53.928	22.851	3	12.57	0	2	1	2.36
280	370	0	140	1	17.85	1114	56.854	29.796	5	17.85	0	2	2	1.91
290	370	0	145	1	16.91	1161	60.931	26.271	3	16.91	0	2	1	2.32
300	370	0	150	1	14.97	1062	57.787	28.243	3	14.97	0	2	1	2.05
310	370	0	155	1	6.37	1046	58.643	36.427	5	6.37	0	2	2	1.61
320	370	0	160	1	8.07	958	56.086	39.183	5	8.07	0	2	2	1.43
330	370	0	165	1	6.56	941	57.896	33.713	3	6.56	0	2	1	1.72
340	370	0	170	1	7.32	828	52.739	35.536	3	7.32	0	2	1	1.48
350	370	0	175	1	6.61	825	54.774	37.666	3	6.61	0	2	1	1.45
360	370	0	180	1	1.50	586	40.559	52.648	5	1.50	0	2	2	0.77
10	380	0	5	1	4.68	32	0.167	0.130	3	4.68	0	2	1	1.29
20	380	0	10	1	2.21	88	0.573	0.281	3	2.21	0	2	1	2.04
30	380	0	15	1	6.80	130	1.073	0.479	3	6.80	0	2	1	2.24
40	380	0	20	1	17.71	228	2.188	0.742	3	17.71	0	2	1	2.95
50	380	0	25	1	11.91	270	3.053	1.089	3	11.91	0	2	1	2.80
60	380	0	30	1	5.04	305	3.868	1.461	3	5.04	0	2	1	2.65
70	380	0	35	1	0.14	368	5.295	1.908	3	0.14	0	2	1	2.77
80	380	0	40	1	6.16	460	7.437	2.409	3	6.16	0	2	1	3.09
90	380	0	45	1	10.91	603	10.645	3.005	3	10.91	0	2	1	3.54
100	380	0	50	1	7.09	629	12.230	3.621	3	7.09	0	2	1	3.38
110	380	0	55	1	7.85	623	13.087	4.258	3	7.85	0	2	1	3.07
120	380	0	60	1	19.68	717	16.116	5.035	3	19.68	0	2	1	3.20
130	380	0	65	1	8.70	800	19.954	5.828	3	8.70	0	2	1	3.42
140	380	0	70	1	13.69	828	21.497	6.643	3	13.69	0	2	1	3.24
150	380	0	75	1	19.64	896	24.797	7.576	3	19.64	0	2	1	3.27
160	380	0	80	1	14.28	956	28.467	8.510	3	14.28	0	2	1	3.35
170	380	0	85	1	14.00	992	30.626	9.565	3	14.00	0	2	1	3.20
180	380	0	90	1	2.39	1085	35.207	10.656	3	2.39	0	2	1	3.30
190	380	0	95	1	0.79	975	33.246	11.702	3	0.79	0	2	1	2.84
200	380	0	100	1	11.16	1097	39.196	12.934	3	11.16	0	2	1	3.03
210	380	0	105	1	3.14	1213	45.559	14.271	3	3.14	0	2	1	3.19
220	380	0	110	1	11.98	1216	47.807	37.981	3	11.98	0	2	1	1.26
230	380	0	115	1	19.25	1297	53.605	17.132	3	19.25	0	2	1	3.13
240	380	0	120	1	19.53	1143	50.323	18.499	3	19.53	0	2	1	2.72
250	380	0	125	1	0.67	1155	53.122	24.136	5	0.67	0	2	2	2.20
260	380	0	130	1	11.83	1297	62.134	21.709	3	11.83	0	2	1	2.86
270	380	0	135	1	19.15	1211	60.869	23.309	3	19.15	0	2	1	2.61
280	380	0	140	1	1.24	1276	66.331	30.478	5	1.24	0	2	2	2.18
290	380	0	145	1	17.65	1095	59.518	26.987	3	17.65	0	2	1	2.21
300	380	0	150	1	18.69	1201	67.356	35.519	5	18.69	0	2	2	1.90
310	380	0	155	1	5.05	1100	64.150	37.801	5	5.05	0	2	2	1.70
320	380	0	160	1	10.27	1054	64.938	32.790	3	10.27	0	2	1	1.98
330	380	0	165	1	10.48	1159	73.505	34.937	3	10.48	0	2	1	2.10
340	380	0	170	1	3.32	991	65.602	43.314	9	3.32	0	2	4	1.51
350	380	0	175	1	15.54	868	60.052	38.822	3	15.54	0	2	1	1.55
360	380	0	180	1	10.61	698	50.527	41.763	3	10.61	0	2	1	1.21
370	380	0	185	1	6.32	623	46.792	69.017	15	6.32	0	2	7	0.68
10	390	0	5	1	13.09	34	0.186	0.132	3	13.09	0	2	1	1.41
20	390	0	10	1	1.11	95	0.648	0.282	3	1.11	0	2	1	2.30
30	390	0	15	1	4.16	140	1.171	0.494	3	4.16	0	2	1	2.37
40	390	0	20	1	4.75	208	2.034	0.764	3	4.75	0	2	1	2.66
50	390	0	25	1	4.62	272	3.114	1.121	3	4.62	0	2	1	2.78
60	390	0	30	1	8.19	366	4.861	1.531	3	8.19	0	2	1	3.17
70	390	0	35	1	3.80	374	5.496	1.964	3	3.80	0	2	1	2.80
80	390	0	40	1	0.97	457	7.487	2.482	3	0.97	0	2	1	3.02

Table 26: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P					
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
90	390	0	45	1	7.54	574	10.444	3.045	3	7.54	0	2	1	3.43
100	390	0	50	1	3.03	643	12.859	3.705	3	3.03	0	2	1	3.47
110	390	0	55	1	14.13	701	15.506	4.405	3	14.13	0	2	1	3.52
120	390	0	60	1	14.83	708	16.884	5.150	3	14.83	0	2	1	3.28
130	390	0	65	1	14.09	843	21.740	5.952	3	14.09	0	2	1	3.65
140	390	0	70	1	10.54	846	22.756	6.804	3	10.54	0	2	1	3.34
150	390	0	75	1	15.28	896	25.271	7.698	3	15.28	0	2	1	3.28
160	390	0	80	1	9.57	1011	30.342	8.771	3	9.57	0	2	1	3.46
170	390	0	85	1	11.43	1083	34.833	9.860	3	11.43	0	2	1	3.53
180	390	0	90	1	17.25	1176	39.348	10.909	3	17.25	0	2	1	3.61
190	390	0	95	1	18.04	1137	39.855	12.050	3	18.04	0	2	1	3.31
200	390	0	100	1	9.78	1140	41.985	13.433	3	9.78	0	2	1	3.13
210	390	0	105	1	1.58	1091	41.855	14.724	3	1.58	0	2	1	2.84
220	390	0	110	1	12.62	1218	49.621	32.147	3	12.62	0	2	1	1.54
230	390	0	115	1	3.06	1369	58.921	17.882	3	3.06	0	2	1	3.30
240	390	0	120	1	18.07	1255	57.818	19.520	3	18.07	0	2	1	2.96
250	390	0	125	1	7.21	1244	59.625	20.768	3	7.21	0	2	1	2.87
260	390	0	130	1	15.31	1257	61.858	22.403	3	15.31	0	2	1	2.76
270	390	0	135	1	10.52	1289	65.216	29.137	5	10.52	0	2	2	2.24
280	390	0	140	1	0.35	1310	68.373	31.103	5	0.35	0	2	2	2.20
290	390	0	145	1	7.07	1289	71.843	28.126	3	7.07	0	2	1	2.55
300	390	0	150	1	2.84	1268	72.842	31.803	5	2.84	0	2	2	2.29
310	390	0	155	1	4.06	1152	69.262	38.736	5	4.06	0	2	2	1.79
320	390	0	160	1	15.75	1167	73.570	41.215	5	15.75	0	2	2	1.79
330	390	0	165	1	11.06	1221	79.592	35.738	3	11.06	0	2	1	2.23
340	390	0	170	1	17.25	1032	70.557	37.818	3	17.25	0	2	1	1.87
350	390	0	175	1	6.64	997	72.355	40.529	3	6.64	0	2	1	1.79
360	390	0	180	1	1.96	1028	76.816	42.910	3	1.96	0	2	1	1.79
370	390	0	185	1	0.70	732	57.613	55.748	11	0.70	0	2	5	1.03
380	390	0	190	1	8.40	677	55.404	61.645	5	8.40	0	2	2	0.90
10	400	0	5	1	1.34	30	0.168	0.135	3	1.34	0	2	1	1.25
20	400	0	10	1	5.36	90	0.627	0.287	3	5.36	0	2	1	2.18
30	400	0	15	1	15.08	132	1.135	0.504	3	15.08	0	2	1	2.25
40	400	0	20	1	6.82	228	2.393	0.801	3	6.82	0	2	1	2.99
50	400	0	25	1	6.77	277	3.262	1.152	3	6.77	0	2	1	2.83
60	400	0	30	1	1.87	389	5.219	1.543	3	1.87	0	2	1	3.38
70	400	0	35	1	9.55	334	5.125	2.021	3	9.55	0	2	1	2.54
80	400	0	40	1	17.06	520	8.752	2.617	3	17.06	0	2	1	3.34
90	400	0	45	1	2.05	599	11.326	3.152	3	2.05	0	2	1	3.59
100	400	0	50	1	14.19	631	12.902	3.811	3	14.19	0	2	1	3.39
110	400	0	55	1	4.73	780	17.422	4.508	3	4.73	0	2	1	3.86
120	400	0	60	1	0.25	723	17.510	5.246	3	0.25	0	2	1	3.34
130	400	0	65	1	18.18	841	21.719	6.127	3	18.18	0	2	1	3.54
140	400	0	70	1	9.15	759	20.965	7.045	3	9.15	0	2	1	2.98
150	400	0	75	1	1.77	941	27.744	8.074	3	1.77	0	2	1	3.44
160	400	0	80	1	14.16	897	27.686	9.016	3	14.16	0	2	1	3.07
170	400	0	85	1	0.43	1187	38.830	10.021	3	0.43	0	2	1	3.87
180	400	0	90	1	16.80	1087	37.756	11.164	3	16.80	0	2	1	3.38
190	400	0	95	1	0.40	1047	38.342	12.523	3	0.40	0	2	1	3.06
200	400	0	100	1	0.81	1197	45.752	13.855	3	0.81	0	2	1	3.30
210	400	0	105	1	18.50	1223	49.594	15.078	3	18.50	0	2	1	3.29
220	400	0	110	1	2.35	1254	52.297	25.484	3	2.35	0	2	1	2.05
230	400	0	115	1	16.28	1313	58.484	18.008	3	16.28	0	2	1	3.25
240	400	0	120	1	16.46	1202	56.863	19.635	3	16.46	0	2	1	2.90
250	400	0	125	1	5.23	1311	63.852	21.244	3	5.23	0	2	1	3.01
260	400	0	130	1	1.10	1186	60.734	23.961	5	1.10	0	2	2	2.53
270	400	0	135	1	10.35	1401	74.232	30.117	5	10.35	0	2	2	2.46
280	400	0	140	1	16.95	1324	72.953	26.311	3	16.95	0	2	1	2.77
290	400	0	145	1	15.30	1426	82.436	29.027	3	15.30	0	2	1	2.84
300	400	0	150	1	11.60	1263	75.744	38.809	7	11.60	0	2	3	1.95
310	400	0	155	1	3.32	1431	89.367	39.051	5	3.32	0	2	2	2.29
320	400	0	160	1	18.81	1321	84.955	34.299	3	18.81	0	2	1	2.48
330	400	0	165	1	6.33	1272	86.420	36.381	3	6.33	0	2	1	2.38
340	400	0	170	1	17.89	985	69.945	38.639	3	17.89	0	2	1	1.81
350	400	0	175	1	11.26	937	69.271	41.510	3	11.26	0	2	1	1.67
360	400	0	180	1	4.27	946	73.045	43.744	3	4.27	0	2	1	1.67
370	400	0	185	1	3.35	927	74.258	47.072	3	3.35	0	2	1	1.58
380	400	0	190	1	16.33	734	61.555	49.322	3	16.33	0	2	1	1.25
390	400	0	195	1	11.61	649	55.654	66.057	5	11.61	0	2	2	0.84

Table 27: Comparison between LINFSOL and B-P on dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
10	20	5	5	1	2.72	19	0.006	0.092	16	2.72	0	4	1	0.06
10	30	10	5	0	2.02	16	0.007	0.105	17	2.02	0	4	1	0.07
20	30	5	10	1	15.53	39	0.020	0.075	8	15.53	0	3	1	0.27
10	40	15	5	0	14.37	22	0.011	0.123	18	14.37	0	4	1	0.09
20	40	10	10	1	8.83	36	0.024	0.142	15	8.83	0	4	1	0.17
30	40	5	15	1	5.92	53	0.046	0.094	4	5.92	0	3	1	0.49
10	50	20	5	0	6.90	28	0.017	0.067	7	6.90	0	4	1	0.25
20	50	15	10	1	18.37	46	0.037	0.230	25	18.37	0	5	1	0.16
30	50	10	15	1	15.06	76	0.077	0.243	15	15.06	0	5	1	0.32
40	50	5	20	1	10.15	74	0.094	0.268	10	10.15	0	4	1	0.35
10	60	25	5	0	11.47	30	0.021	0.120	14	11.47	0	5	1	0.18
20	60	20	10	1	14.64	55	0.053	0.303	29	14.64	0	9	1	0.17
30	60	15	15	1	19.79	63	0.077	0.334	22	19.79	0	5	1	0.23
40	60	10	20	1	13.25	89	0.133	0.263	7	13.25	0	4	1	0.51
50	60	5	25	1	4.08	91	0.163	0.634	21	4.08	0	5	1	0.26
10	70	30	5	0	11.12	26	0.022	0.096	9	11.12	0	4	1	0.23
20	70	25	10	0	18.91	59	0.065	0.186	14	18.91	0	4	1	0.35
30	70	20	15	1	3.35	87	0.122	0.395	25	3.35	0	5	1	0.31
40	70	15	20	1	9.03	111	0.192	0.912	47	9.03	0	8	1	0.21
50	70	10	25	1	6.36	120	0.245	0.753	20	6.36	0	6	1	0.33
60	70	5	30	1	8.65	113	0.271	0.849	16	8.65	0	5	1	0.32
10	80	35	5	0	2.39	32	0.030	0.096	8	2.39	0	4	1	0.32
20	80	30	10	0	1.97	66	0.082	0.166	9	1.97	0	4	1	0.49
30	80	25	15	0	10.29	80	0.130	0.757	54	10.29	0	8	1	0.17
40	80	20	20	1	14.81	124	0.240	0.544	20	14.81	0	5	1	0.44
50	80	15	25	1	4.22	116	0.279	0.690	16	4.22	0	5	1	0.40
60	80	10	30	1	5.82	121	0.325	1.059	20	5.82	0	5	1	0.31
70	80	5	35	1	16.65	112	0.344	1.522	22	16.65	0	6	1	0.23
10	90	40	5	0	17.58	28	0.030	0.103	8	17.58	0	4	1	0.29
20	90	35	10	0	2.45	69	0.096	0.243	15	2.45	0	5	1	0.40
30	90	30	15	0	16.33	80	0.147	0.840	57	16.33	0	9	1	0.18
40	90	25	20	1	10.16	128	0.283	0.573	19	10.16	0	5	1	0.49
50	90	20	25	1	0.19	145	0.374	0.783	16	0.19	0	5	1	0.48
60	90	15	30	1	6.92	142	0.430	1.666	37	6.92	0	7	1	0.26
70	90	10	35	1	18.96	137	0.485	1.519	18	18.96	0	6	1	0.32
80	90	5	40	1	11.65	158	0.623	1.499	11	11.65	0	4	1	0.42
10	100	45	5	0	18.25	34	0.045	0.106	7	18.25	0	4	1	0.42
20	100	40	10	0	1.79	63	0.106	0.213	10	1.79	0	4	1	0.50
30	100	35	15	0	10.78	76	0.185	1.035	63	10.78	0	12	1	0.18
40	100	30	20	1	16.75	143	0.387	1.197	52	16.75	0	9	1	0.32
50	100	25	25	1	3.36	148	0.481	0.851	19	3.36	0	4	1	0.57
60	100	20	30	1	17.08	175	0.627	1.076	12	17.08	0	5	1	0.58
70	100	15	35	1	1.01	215	0.849	1.802	27	1.01	0	5	2	0.47
80	100	10	40	1	3.56	181	0.802	2.105	22	3.56	0	5	1	0.38
90	100	5	45	1	13.95	161	0.813	2.547	22	13.95	0	5	1	0.32
10	110	50	5	0	17.83	28	0.039	0.110	7	17.83	0	4	1	0.36
20	110	45	10	0	16.60	48	0.098	0.217	9	16.60	0	4	1	0.45
30	110	40	15	0	19.92	102	0.243	0.710	37	19.92	0	8	1	0.34
40	110	35	20	0	16.39	116	0.339	1.772	83	16.39	0	12	1	0.19
50	110	30	25	1	14.97	204	0.690	0.767	11	14.97	0	4	1	0.90
60	110	25	30	1	6.29	195	0.753	0.931	11	6.29	0	3	1	0.81
70	110	20	35	1	16.60	199	0.850	2.161	29	16.60	0	7	1	0.39
80	110	15	40	1	6.89	224	1.082	3.871	47	6.89	0	10	1	0.28
90	110	10	45	1	15.94	200	1.096	3.031	26	15.94	0	5	1	0.36
100	110	5	50	1	14.35	195	1.134	2.703	11	14.35	0	5	1	0.42
10	120	55	5	0	15.23	35	0.053	0.128	8	15.23	0	4	1	0.41
20	120	50	10	0	11.71	67	0.133	0.282	12	11.71	0	5	1	0.47
30	120	45	15	0	5.73	89	0.244	0.487	16	5.73	0	6	1	0.50
40	120	40	20	0	15.84	123	0.384	1.096	42	15.84	0	7	1	0.35
50	120	35	25	1	15.60	171	0.614	1.005	20	15.60	0	5	1	0.61
60	120	30	30	1	4.44	194	0.803	1.321	18	4.44	0	5	2	0.61
70	120	25	35	1	13.12	210	0.968	2.013	23	13.12	0	6	1	0.48
80	120	20	40	1	2.64	256	1.311	1.976	12	2.64	0	4	1	0.66
90	120	15	45	1	6.91	232	1.336	3.028	18	6.91	0	6	1	0.44
100	120	10	50	1	0.72	227	1.402	2.535	11	0.72	0	4	1	0.55
110	120	5	55	1	14.13	179	1.237	3.971	17	14.13	0	5	1	0.31
10	130	60	5	0	2.51	31	0.051	0.140	8	2.51	0	4	1	0.36
20	130	55	10	0	2.03	62	0.133	0.278	11	2.03	0	4	1	0.48
30	130	50	15	0	8.21	96	0.277	0.452	12	8.21	0	4	1	0.61
40	130	45	20	0	4.35	125	0.412	1.558	62	4.35	0	11	1	0.26
50	130	40	25	1	0.57	193	0.735	1.964	64	0.57	0	8	16	0.37
60	130	35	30	1	3.12	251	1.145	1.467	20	3.12	0	5	1	0.78
70	130	30	35	1	18.02	234	1.154	1.877	14	18.02	0	6	1	0.61
80	130	25	40	1	10.78	268	1.491	2.222	14	10.78	0	5	1	0.67
90	130	20	45	1	13.49	260	1.622	3.994	32	13.49	0	7	1	0.41
100	130	15	50	1	18.66	275	1.838	3.180	11	18.66	0	5	1	0.58
110	130	10	55	1	6.67	263	1.979	4.714	18	6.67	0	6	1	0.42
120	130	5	60	1	12.75	217	1.702	7.184	29	12.75	0	8	1	0.24
10	140	65	5	0	18.47	34	0.057	0.142	8	18.47	0	4	1	0.40
20	140	60	10	0	8.23	67	0.156	0.290	10	8.23	0	4	1	0.54
30	140	55	15	0	7.35	104	0.325	0.547	14	7.35	0	5	1	0.59
40	140	50	20	0	9.06	136	0.474	0.972	27	9.06	0	6	1	0.49
50	140	45	25	1	19.65	174	0.728	3.110	98	19.65	0	15	1	0.23
60	140	40	30	1	1.16	258	1.208	1.210	11	1.16	0	4	1	1.00
70	140	35	35	1	17.13	248	1.301	2.470	27	17.13	0	7	1	0.53
80	140	30	40	1	13.45	250	1.495	3.222	29	13.45	0	7	1	0.46
90	140	25	45	1	19.83	299	1.985	4.297	39	19.83	0	6	1	0.46
100	140	20	50	1	2.47	286	2.027	4.458	19	2.47	0	7	1	0.45

Table 28: Comparison between LINFSOL and B-P on primal-dual degenerate problems

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
110	140	15	55	1	10.22	269	2.080	5.511	18	10.22	0	7	1	0.38
120	140	10	60	1	5.05	273	2.271	5.881	18	5.05	0	6	1	0.39
130	140	5	65	1	7.10	251	2.307	7.620	24	7.10	0	7	1	0.30
10	150	70	5	0	13.09	38	0.069	0.151	8	13.09	0	4	1	0.46
20	150	65	10	0	18.82	65	0.160	0.312	10	18.82	0	4	1	0.51
30	150	60	15	0	19.67	97	0.321	0.535	13	19.67	0	4	1	0.60
40	150	55	20	0	8.81	154	0.587	0.798	15	8.81	0	4	1	0.74
50	150	50	25	1	4.02	192	0.861	2.407	60	4.02	0	14	1	0.36
60	150	45	30	1	18.55	217	1.139	1.939	27	18.55	0	6	1	0.59
70	150	40	35	1	10.66	267	1.504	2.385	25	10.66	0	6	1	0.63
80	150	35	40	1	17.67	254	1.606	3.256	30	17.67	0	6	1	0.49
90	150	30	45	1	8.69	316	2.265	3.756	24	8.69	0	6	1	0.60
100	150	25	50	1	7.44	350	2.622	4.148	14	7.44	0	6	1	0.63
110	150	20	55	1	3.20	339	2.833	6.594	30	3.20	0	8	1	0.43
120	150	15	60	1	17.05	314	2.811	7.519	22	17.05	0	8	1	0.37
130	150	10	65	1	4.35	308	2.944	8.495	18	4.35	0	8	1	0.35
140	150	5	70	1	15.65	253	2.672	12.398	32	15.65	0	10	1	0.22
10	160	75	5	0	5.12	30	0.065	0.189	9	5.12	0	5	1	0.35
20	160	70	10	0	16.90	65	0.174	0.334	10	16.90	0	4	1	0.52
30	160	65	15	0	10.13	120	0.392	0.566	13	10.13	0	4	1	0.69
40	160	60	20	0	17.77	152	0.605	0.818	13	17.77	0	4	1	0.74
50	160	55	25	0	19.79	166	0.769	1.128	12	19.79	0	4	1	0.68
60	160	50	30	1	19.15	248	1.349	5.691	135	19.15	0	19	1	0.24
70	160	45	35	1	7.04	302	1.844	1.999	14	7.04	0	5	1	0.92
80	160	40	40	1	1.41	309	2.119	4.319	62	1.41	0	6	22	0.49
90	160	35	45	1	4.55	315	2.409	3.889	21	4.55	0	6	1	0.62
100	160	30	50	1	17.37	327	2.594	5.724	39	17.37	0	6	1	0.45
110	160	25	55	1	17.61	318	2.784	5.595	25	17.61	0	5	1	0.50
120	160	20	60	1	10.03	381	3.560	8.618	32	10.03	0	8	1	0.41
130	160	15	65	1	4.50	369	3.721	6.706	16	4.50	0	5	1	0.55
140	160	10	70	1	8.47	326	3.511	9.663	20	8.47	0	7	1	0.36
150	160	5	75	1	12.52	230	2.717	11.812	20	12.52	0	8	1	0.23
10	170	80	5	0	16.54	26	0.056	0.190	9	16.54	0	5	1	0.29
20	170	75	10	0	2.21	59	0.165	0.348	10	2.21	0	4	1	0.48
30	170	70	15	0	2.42	128	0.464	0.604	12	2.42	0	4	1	0.77
40	170	65	20	0	8.38	144	0.610	0.810	11	8.38	0	3	1	0.75
50	170	60	25	0	16.28	169	0.858	1.149	11	16.28	0	3	1	0.75
60	170	55	30	0	1.67	205	1.186	2.848	55	1.67	0	7	1	0.42
70	170	50	35	1	4.09	291	1.859	2.163	18	4.09	0	4	1	0.86
80	170	45	40	1	14.12	355	2.487	2.484	13	14.12	0	4	1	1.00
90	170	40	45	1	15.13	306	2.392	4.333	29	15.13	0	6	1	0.55
100	170	35	50	1	6.19	379	3.198	5.124	26	6.19	0	6	2	0.62
110	170	30	55	1	6.60	380	3.461	6.355	23	6.60	0	7	1	0.54
120	170	25	60	1	18.33	453	4.455	8.989	43	18.33	0	7	1	0.50
130	170	20	65	1	15.96	390	4.162	11.447	37	15.96	0	9	1	0.36
140	170	15	70	1	16.95	347	3.998	11.392	29	16.95	0	7	1	0.35
150	170	10	75	1	3.88	345	4.204	13.385	23	3.88	0	9	1	0.31
160	170	5	80	1	3.45	273	3.601	11.990	14	3.45	0	7	1	0.30
10	180	85	5	0	6.11	35	0.075	0.198	9	6.11	0	5	1	0.38
20	180	80	10	0	0.13	79	0.237	0.355	9	0.13	0	4	1	0.67
30	180	75	15	0	6.12	107	0.400	0.703	14	6.12	0	5	1	0.57
40	180	70	20	0	1.57	154	0.686	0.963	14	1.57	0	4	1	0.71
50	180	65	25	0	12.72	197	1.040	1.227	13	12.72	0	3	1	0.85
60	180	60	30	0	17.46	193	1.161	2.454	36	17.46	0	6	1	0.47
70	180	55	35	1	19.92	246	1.680	4.425	68	19.92	0	8	1	0.38
80	180	50	40	1	8.13	326	2.396	3.036	20	8.13	0	5	1	0.79
90	180	45	45	1	7.90	313	2.595	3.649	16	7.90	0	5	1	0.71
100	180	40	50	1	15.28	405	3.619	4.074	17	15.28	0	3	1	0.89
110	180	35	55	1	10.33	405	3.944	5.634	17	10.33	0	6	1	0.70
120	180	30	60	1	16.77	406	4.318	7.552	23	16.77	0	6	1	0.57
130	180	25	65	1	1.45	449	5.046	9.455	18	1.45	0	8	1	0.53
140	180	20	70	1	10.36	439	5.319	12.335	25	10.36	0	9	1	0.43
150	180	15	75	1	18.78	383	4.954	16.733	35	18.78	0	10	1	0.30
160	180	10	80	1	16.78	340	4.624	14.439	24	16.78	0	7	1	0.32
170	180	5	85	1	16.36	261	3.895	13.484	16	16.36	0	6	1	0.29
10	190	90	5	0	10.71	33	0.077	0.218	10	10.71	0	5	1	0.35
20	190	85	10	0	6.28	74	0.226	0.400	11	6.28	0	4	1	0.57
30	190	80	15	0	13.07	105	0.436	0.727	13	13.07	0	4	1	0.60
40	190	75	20	0	0.37	179	0.863	1.021	14	0.37	0	4	1	0.85
50	190	70	25	0	2.17	171	0.992	1.422	14	2.17	0	4	1	0.70
60	190	65	30	0	10.53	211	1.322	2.208	24	10.53	0	5	1	0.60
70	190	60	35	1	8.61	266	1.992	6.443	108	8.61	0	14	7	0.31
80	190	55	40	1	18.16	334	2.616	4.374	40	18.16	0	7	3	0.60
90	190	50	45	1	3.33	385	3.309	5.283	31	3.33	0	8	4	0.63
100	190	45	50	1	18.88	425	3.997	5.822	26	18.88	0	7	1	0.69
110	190	40	55	1	6.55	390	3.988	7.367	22	6.55	0	9	1	0.54
120	190	35	60	1	18.84	428	4.705	7.535	21	18.84	0	6	1	0.62
130	190	30	65	1	16.39	412	4.974	10.226	20	16.39	0	8	1	0.49
140	190	25	70	1	13.88	373	4.789	13.580	31	13.88	0	9	1	0.35
150	190	20	75	1	5.50	465	6.248	11.314	18	5.50	0	6	1	0.55
160	190	15	80	1	0.20	375	5.410	15.122	37	0.20	0	6	9	0.36
170	190	10	85	1	4.52	339	5.335	17.706	20	4.52	0	8	1	0.30
180	190	5	90	1	8.48	319	5.110	21.473	24	8.48	0	9	1	0.24
10	200	95	5	0	12.39	32	0.078	0.227	10	12.39	0	5	1	0.34
20	200	90	10	0	19.64	74	0.254	0.417	11	19.64	0	4	1	0.61
30	200	85	15	0	12.21	116	0.494	0.753	12	12.21	0	4	1	0.66
40	200	80	20	0	13.58	162	0.821	0.922	11	13.58	0	3	1	0.89
50	200	75	25	0	7.98	201	1.141	1.468	13	7.98	0	4	1	0.78
60	200	70	30	0	18.86	206	1.378	1.813	14	18.86	0	3	1	0.76

Table 29: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
70	200	65	35	1	1.11	261	2.007	5.004	68	1.11	0	12	1	0.40
80	200	60	40	1	0.54	340	2.785	4.530	35	0.54	0	9	2	0.61
90	200	55	45	1	2.16	412	3.774	4.591	23	2.16	0	6	1	0.82
100	200	50	50	1	0.34	444	4.417	5.607	21	0.34	0	7	1	0.79
110	200	45	55	1	4.02	454	4.977	5.932	18	4.02	0	5	1	0.84
120	200	40	60	1	19.16	481	5.578	7.866	22	19.16	0	6	1	0.71
130	200	35	65	1	1.50	491	6.054	7.854	15	1.50	0	5	1	0.77
140	200	30	70	1	19.00	507	6.788	11.718	25	19.00	0	7	1	0.58
150	200	25	75	1	11.72	461	6.587	15.867	33	11.72	0	8	1	0.42
160	200	20	80	1	6.25	447	6.664	17.536	23	6.25	0	9	1	0.38
170	200	15	85	1	10.15	473	7.709	18.606	30	10.15	0	7	1	0.41
180	200	10	90	1	14.89	413	7.059	24.202	25	14.89	0	10	1	0.29
190	200	5	95	1	0.64	325	6.001	23.113	20	0.64	0	8	1	0.26
10	210	100	5	0	13.01	29	0.075	0.237	10	13.01	0	5	1	0.32
20	210	95	10	0	5.28	54	0.186	0.432	11	5.28	0	4	1	0.43
30	210	90	15	0	5.28	114	0.502	0.679	10	5.28	0	4	1	0.74
40	210	85	20	0	6.06	152	0.817	1.092	13	6.06	0	4	1	0.75
50	210	80	25	0	3.17	241	1.517	1.576	13	3.17	0	4	1	0.96
60	210	75	30	0	12.22	229	1.618	2.001	14	12.22	0	3	1	0.81
70	210	70	35	1	19.63	297	2.339	9.667	168	19.63	0	19	1	0.24
80	210	65	40	1	16.07	319	2.805	7.620	88	16.07	0	13	1	0.37
90	210	60	45	1	3.79	407	3.931	4.733	27	3.79	0	5	2	0.83
100	210	55	50	1	15.26	437	4.555	5.186	18	15.26	0	5	1	0.88
110	210	50	55	1	3.19	484	5.536	6.954	25	3.19	0	6	2	0.80
120	210	45	60	1	16.49	483	5.836	7.259	19	16.49	0	5	1	0.80
130	210	40	65	1	9.72	449	6.051	11.661	36	9.72	0	7	1	0.52
140	210	35	70	1	2.48	516	7.151	14.440	41	2.48	0	8	2	0.50
150	210	30	75	1	14.06	504	7.485	12.399	20	14.06	0	6	1	0.60
160	210	25	80	1	12.26	539	8.450	20.542	42	12.26	0	9	1	0.41
170	210	20	85	1	15.91	457	7.802	20.693	24	15.91	0	9	1	0.38
180	210	15	90	1	9.75	478	8.401	19.871	24	9.75	0	7	1	0.42
190	210	10	95	1	12.35	385	7.481	24.783	22	12.35	0	8	2	0.30
200	210	5	100	1	11.55	330	6.664	26.911	23	11.55	0	8	1	0.25
10	220	105	5	0	8.37	33	0.103	0.252	10	8.37	0	5	1	0.41
20	220	100	10	0	13.04	82	0.305	0.441	10	13.04	0	4	1	0.69
30	220	95	15	0	14.99	116	0.527	0.786	13	14.99	0	4	1	0.67
40	220	90	20	0	14.13	167	0.921	1.116	12	14.13	0	4	1	0.83
50	220	85	25	0	4.00	219	1.398	1.596	12	4.00	0	4	1	0.88
60	220	80	30	0	10.30	210	1.532	2.085	15	10.30	0	3	1	0.73
70	220	75	35	0	2.86	315	2.608	7.000	102	2.86	0	15	1	0.37
80	220	70	40	1	16.63	381	3.484	3.869	27	16.63	0	4	1	0.90
90	220	65	45	1	3.30	399	4.016	5.229	28	3.30	0	6	1	0.77
100	220	60	50	1	2.37	470	5.141	6.335	27	2.37	0	6	2	0.81
110	220	55	55	1	0.40	465	5.532	7.970	31	0.40	0	7	5	0.69
120	220	50	60	1	18.27	560	7.133	8.446	22	18.27	0	6	1	0.84
130	220	45	65	1	16.04	529	7.197	9.654	21	16.04	0	6	1	0.75
140	220	40	70	1	16.16	570	8.384	11.979	24	16.16	0	6	1	0.70
150	220	35	75	1	10.46	546	8.607	17.499	41	10.46	0	8	1	0.49
160	220	30	80	1	9.20	533	8.952	15.859	21	9.20	0	7	1	0.56
170	220	25	85	1	13.81	548	9.811	22.729	25	13.81	0	10	1	0.43
180	220	20	90	1	4.61	596	11.015	27.323	30	4.61	0	10	1	0.40
190	220	15	95	1	13.05	516	10.033	28.755	21	13.05	0	10	1	0.35
200	220	10	100	1	6.81	429	8.802	36.150	30	6.81	0	11	1	0.24
210	220	5	105	1	12.14	379	8.155	37.107	27	12.14	0	10	1	0.22
10	230	110	5	0	3.60	29	0.086	0.282	11	3.60	0	6	1	0.30
20	230	105	10	0	18.09	56	0.230	0.504	11	18.09	0	4	1	0.46
30	230	100	15	0	18.40	134	0.638	0.801	12	18.40	0	4	1	0.80
40	230	95	20	0	12.96	178	1.039	1.187	12	12.96	0	4	1	0.87
50	230	90	25	0	18.00	187	1.321	1.734	14	18.00	0	4	1	0.76
60	230	85	30	0	15.60	233	1.808	2.174	14	15.60	0	3	1	0.83
70	230	80	35	0	12.75	262	2.273	8.692	139	12.75	0	16	1	0.26
80	230	75	40	0	19.34	312	3.092	9.099	115	19.34	0	13	1	0.34
90	230	70	45	1	16.71	376	3.939	4.961	24	16.71	0	5	1	0.79
100	230	65	50	1	9.10	448	5.170	6.663	30	9.10	0	6	1	0.78
110	230	60	55	1	5.95	576	7.179	7.386	23	5.95	0	6	1	0.97
120	230	55	60	1	13.66	478	6.893	10.635	38	13.66	0	7	1	0.65
130	230	50	65	1	13.47	558	8.023	10.636	20	13.47	0	7	1	0.75
140	230	45	70	1	6.71	561	8.605	11.549	22	6.71	0	5	1	0.75
150	230	40	75	1	0.70	592	9.762	17.838	35	0.70	0	9	4	0.55
160	230	35	80	1	10.34	550	9.412	18.963	29	10.34	0	8	1	0.50
170	230	30	85	1	19.17	545	10.054	23.433	37	19.17	0	8	1	0.43
180	230	25	90	1	0.70	626	12.241	20.133	21	0.70	0	6	1	0.61
190	230	20	95	1	19.42	563	11.400	39.390	54	19.42	0	12	1	0.29
200	230	15	100	1	9.14	427	9.065	32.374	25	9.14	0	9	1	0.28
210	230	10	105	1	11.05	464	10.496	29.907	24	11.05	0	7	1	0.35
220	230	5	110	1	12.85	401	9.708	35.188	23	12.85	0	8	1	0.28
10	240	115	5	0	8.88	34	0.105	0.299	11	8.88	0	6	1	0.35
20	240	110	10	0	14.57	71	0.302	0.507	11	14.57	0	4	1	0.60
30	240	105	15	0	0.01	142	0.735	0.853	12	0.01	0	4	1	0.86
40	240	100	20	0	6.03	175	1.127	1.216	12	6.03	0	4	1	0.93
50	240	95	25	0	3.93	207	1.516	1.808	14	3.93	0	4	1	0.84
60	240	90	30	0	18.51	232	2.326	2.448	15	18.51	0	4	1	0.95
70	240	85	35	0	1.32	343	3.243	3.508	25	1.32	0	5	1	0.92
80	240	80	40	0	13.35	336	3.698	4.346	23	13.35	0	5	1	0.85
90	240	75	45	1	15.05	363	4.304	4.891	18	15.05	0	5	1	0.88
100	240	70	50	1	17.70	415	5.199	6.327	21	17.70	0	6	1	0.82
110	240	65	55	1	9.91	494	6.688	9.276	42	9.91	0	7	1	0.72
120	240	60	60	1	10.05	570	8.212	8.311	20	10.05	0	5	1	0.99
130	240	55	65	1	14.70	576	9.001	12.861	33	14.70	0	8	1	0.70

Table 30: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
140	240	50	70	1	14.70	613	10.358	14.407	27	14.70	0	8	1	0.72
150	240	45	75	1	13.04	658	11.684	14.337	22	13.04	0	6	1	0.81
160	240	40	80	1	8.56	698	12.949	18.100	27	8.56	0	7	1	0.72
170	240	35	85	1	2.56	556	10.923	25.130	40	2.56	0	9	1	0.43
180	240	30	90	1	8.20	713	15.140	30.884	25	8.20	0	12	1	0.49
190	240	25	95	1	10.67	594	13.130	24.066	25	10.67	0	6	1	0.55
200	240	20	100	1	16.49	525	12.168	33.422	37	16.49	0	8	1	0.36
210	240	15	105	1	16.81	534	13.348	34.830	27	16.81	0	8	1	0.38
220	240	10	110	1	19.29	428	11.282	40.971	30	19.29	0	9	1	0.28
230	240	5	115	1	4.33	441	13.400	39.398	20	4.33	0	8	1	0.34
10	250	120	5	0	19.86	32	0.109	0.302	11	19.86	0	6	1	0.36
20	250	115	10	0	3.87	70	0.314	0.525	11	3.87	0	4	1	0.60
30	250	110	15	0	17.93	105	0.589	0.970	14	17.93	0	5	1	0.61
40	250	105	20	0	2.67	156	1.034	1.286	13	2.67	0	4	1	0.80
50	250	100	25	0	13.25	211	1.653	1.832	13	13.25	0	4	1	0.90
60	250	95	30	0	11.24	281	2.395	2.498	15	11.24	0	4	1	0.96
70	250	90	35	0	15.50	305	3.033	3.751	24	15.50	0	5	1	0.81
80	250	85	40	0	7.31	352	3.724	8.193	83	7.31	0	12	1	0.45
90	250	80	45	1	18.31	401	4.754	6.270	36	18.31	0	6	1	0.76
100	250	75	50	1	18.50	513	6.845	7.284	34	18.50	0	6	1	0.94
110	250	70	55	1	13.60	509	7.044	10.473	44	13.60	0	9	1	0.67
120	250	65	60	1	1.82	602	8.832	9.937	24	1.82	0	7	1	0.89
130	250	60	65	1	4.19	499	7.967	11.854	24	4.19	0	7	1	0.67
140	250	55	70	1	0.89	667	11.641	13.067	22	0.89	0	7	1	0.89
150	250	50	75	1	11.49	652	11.844	15.816	30	11.49	0	6	1	0.75
160	250	45	80	1	10.02	558	10.670	22.875	35	10.02	0	10	1	0.47
170	250	40	85	1	2.64	680	14.031	21.554	30	2.64	0	7	1	0.65
180	250	35	90	1	9.19	691	15.044	31.042	35	9.19	0	11	1	0.48
190	250	30	95	1	1.99	663	15.141	31.053	32	1.99	0	9	1	0.49
200	250	25	100	1	4.92	611	15.098	31.898	27	4.92	0	8	1	0.47
210	250	20	105	1	1.68	591	14.771	35.876	36	1.68	0	7	2	0.41
220	250	15	110	1	13.03	604	15.756	41.793	25	13.03	0	9	1	0.38
230	250	10	115	1	11.41	534	14.629	48.708	25	11.41	0	10	1	0.30
240	250	5	120	1	15.82	410	11.562	56.550	27	15.82	0	11	1	0.20
10	260	125	5	0	17.85	37	0.122	0.316	11	17.85	0	6	1	0.39
20	260	120	10	0	15.31	74	0.340	0.617	13	15.31	0	5	1	0.55
30	260	115	15	0	17.07	127	0.730	1.014	14	17.07	0	5	1	0.72
40	260	110	20	0	3.00	153	1.052	1.341	13	3.00	0	4	1	0.79
50	260	105	25	0	15.78	190	1.633	1.929	14	15.78	0	4	1	0.85
60	260	100	30	0	4.99	261	2.344	2.577	15	4.99	0	4	1	0.91
70	260	95	35	0	3.96	312	3.028	3.368	17	3.96	0	4	1	0.90
80	260	90	40	0	13.57	361	4.017	9.117	97	13.57	0	12	1	0.44
90	260	85	45	1	14.50	423	5.186	10.232	93	14.50	0	10	1	0.51
100	260	80	50	1	3.51	495	6.637	15.745	136	3.51	0	13	21	0.42
110	260	75	55	1	8.35	530	7.679	8.179	25	8.35	0	6	1	0.94
120	260	70	60	1	0.38	601	9.384	12.752	64	0.38	0	6	20	0.74
130	260	65	65	1	2.56	616	10.211	11.312	26	2.56	0	6	1	0.90
140	260	60	70	1	0.40	642	11.810	15.939	46	0.40	0	7	10	0.74
150	260	55	75	1	1.84	648	12.247	16.019	28	1.84	0	6	1	0.76
160	260	50	80	1	8.58	644	12.846	18.716	27	8.58	0	7	1	0.69
170	260	45	85	1	1.35	664	13.980	17.773	21	1.35	0	5	1	0.79
180	260	40	90	1	4.05	735	16.249	22.120	26	4.05	0	6	1	0.73
190	260	35	95	1	0.83	767	18.032	28.802	30	0.83	0	7	2	0.63
200	260	30	100	1	2.02	622	15.399	36.517	32	2.02	0	9	1	0.42
210	260	25	105	1	5.08	664	17.335	25.758	23	5.08	0	4	1	0.67
220	260	20	110	1	19.42	674	18.553	49.687	29	19.42	0	11	1	0.37
230	260	15	115	1	18.05	617	17.672	53.778	38	18.05	0	10	2	0.33
240	260	10	120	1	4.38	512	14.901	54.514	26	4.38	0	10	1	0.27
250	260	5	125	1	1.45	404	12.416	62.395	35	1.45	0	10	1	0.20
10	270	130	5	0	4.40	34	0.118	0.340	12	4.40	0	6	1	0.35
20	270	125	10	0	17.89	61	0.288	0.625	12	17.89	0	5	1	0.46
30	270	120	15	0	17.90	111	0.654	1.021	13	17.90	0	5	1	0.64
40	270	115	20	0	7.76	176	1.241	1.418	14	7.76	0	4	1	0.88
50	270	110	25	0	11.47	248	1.992	1.989	14	11.47	0	4	1	1.00
60	270	105	30	0	4.49	260	2.413	2.698	15	4.49	0	4	1	0.89
70	270	100	35	0	16.00	314	3.273	3.533	17	16.00	0	4	1	0.93
80	270	95	40	0	1.74	343	4.031	4.396	17	1.74	0	4	1	0.92
90	270	90	45	1	8.84	409	5.306	8.900	63	8.84	0	10	1	0.60
100	270	85	50	1	13.29	508	7.067	6.471	19	13.29	0	5	1	1.09
110	270	80	55	1	1.75	548	8.246	9.041	26	1.75	0	7	1	0.91
120	270	75	60	1	15.94	571	9.045	10.722	30	15.94	0	7	1	0.84
130	270	70	65	1	3.20	622	11.018	12.261	29	3.20	0	6	1	0.90
140	270	65	70	1	18.37	689	13.152	14.368	33	18.37	0	6	1	0.92
150	270	60	75	1	13.73	626	12.244	15.486	25	13.73	0	6	1	0.79
160	270	55	80	1	2.80	688	14.381	18.662	24	2.80	0	7	1	0.77
170	270	50	85	1	0.12	727	15.749	23.216	34	0.12	0	7	3	0.68
180	270	45	90	1	8.40	792	18.161	26.405	27	8.40	0	8	1	0.69
190	270	40	95	1	0.30	779	18.894	25.751	40	0.30	0	5	9	0.73
200	270	35	100	1	18.67	744	18.830	36.716	34	18.67	0	9	1	0.51
210	270	30	105	1	13.34	692	19.033	38.063	30	13.34	0	8	1	0.50
220	270	25	110	1	7.26	645	18.245	39.773	30	7.26	0	7	1	0.46
230	270	20	115	1	8.57	642	18.718	46.792	31	8.57	0	8	1	0.40
240	270	15	120	1	9.39	643	19.339	71.376	41	9.39	0	13	1	0.27
250	270	10	125	1	19.63	606	19.106	62.665	26	19.63	0	10	1	0.30
260	270	5	130	1	11.23	365	12.161	66.116	25	11.23	0	10	1	0.18
10	280	135	5	0	9.12	31	0.110	0.359	12	9.12	0	6	1	0.31
20	280	130	10	0	0.96	68	0.345	0.662	12	0.96	0	5	1	0.52
30	280	125	15	0	5.36	140	0.848	1.105	15	5.36	0	5	1	0.77
40	280	120	20	0	16.05	183	1.332	1.480	14	16.05	0	4	1	0.90

Table 31: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL					B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
50	280	115	25	0	8.09	246	2.052	2.061	13	8.09	0	4	1	1.00
60	280	110	30	0	17.01	224	2.151	2.810	16	17.01	0	4	1	0.77
70	280	105	35	0	2.16	348	3.646	3.515	15	2.16	0	4	1	1.04
80	280	100	40	0	8.35	353	4.218	4.212	16	8.35	0	3	1	1.00
90	280	95	45	1	2.70	423	5.697	11.132	87	2.70	0	13	1	0.51
100	280	90	50	1	19.74	479	6.811	6.392	18	19.74	0	4	1	1.07
110	280	85	55	1	0.44	563	8.796	13.664	82	0.44	0	8	21	0.64
120	280	80	60	1	7.71	596	9.893	11.534	33	7.71	0	8	4	0.86
130	280	75	65	1	9.63	613	10.863	10.385	22	9.63	0	4	1	1.05
140	280	70	70	1	1.24	752	14.635	16.856	29	1.24	0	9	2	0.87
150	280	65	75	1	18.02	722	14.644	20.593	44	18.02	0	8	1	0.71
160	280	60	80	1	4.68	747	16.091	22.604	30	4.68	0	9	1	0.71
170	280	55	85	1	12.21	787	17.772	21.750	30	12.21	0	6	1	0.82
180	280	50	90	1	13.78	874	20.537	25.817	27	13.78	0	7	1	0.80
190	280	45	95	1	6.89	866	21.661	30.165	33	6.89	0	7	1	0.72
200	280	40	100	1	4.80	767	20.003	34.307	29	4.80	0	8	1	0.58
210	280	35	105	1	6.73	804	21.702	36.289	28	6.73	0	7	1	0.60
220	280	30	110	1	13.96	782	22.704	40.530	30	13.96	0	7	1	0.56
230	280	25	115	1	3.74	739	22.299	49.864	27	3.74	0	9	1	0.45
240	280	20	120	1	9.00	741	23.063	54.599	27	9.00	0	9	1	0.42
250	280	15	125	1	12.10	630	20.583	67.252	34	12.10	0	10	1	0.31
260	280	10	130	1	13.34	485	16.719	88.469	43	13.34	0	13	1	0.19
270	280	5	135	1	6.94	442	15.899	85.992	29	6.94	0	12	1	0.18
10	290	140	5	0	14.29	30	0.110	0.371	12	14.29	0	6	1	0.30
20	290	135	10	0	6.47	62	0.315	0.672	12	6.47	0	5	1	0.47
30	290	130	15	0	14.50	136	0.838	1.135	14	14.50	0	5	1	0.74
40	290	125	20	0	10.93	200	1.477	1.549	14	10.93	0	4	1	0.95
50	290	120	25	0	2.38	272	2.458	2.186	14	2.38	0	4	1	1.12
60	290	115	30	0	5.45	261	2.609	2.926	16	5.45	0	4	1	0.89
70	290	110	35	0	15.62	326	3.617	3.644	15	15.62	0	4	1	0.99
80	290	105	40	0	6.72	375	4.669	4.846	19	6.72	0	4	1	0.96
90	290	100	45	0	19.69	363	4.960	5.895	19	19.69	0	4	1	0.84
100	290	95	50	1	8.02	427	6.526	23.171	201	8.02	0	20	16	0.28
110	290	90	55	1	16.19	576	9.285	8.798	21	16.19	0	6	1	1.06
120	290	85	60	1	7.22	614	10.743	11.035	24	7.22	0	7	1	0.97
130	290	80	65	1	13.62	575	10.565	13.485	31	13.62	0	7	1	0.78
140	290	75	70	1	6.27	691	13.738	18.443	40	6.27	0	9	1	0.74
150	290	70	75	1	7.22	792	16.833	18.849	29	7.22	0	8	1	0.89
160	290	65	80	1	14.62	747	16.672	21.936	25	14.62	0	9	1	0.76
170	290	60	85	1	16.57	756	17.630	24.065	33	16.57	0	7	1	0.73
180	290	55	90	1	8.64	873	21.615	25.725	27	8.64	0	7	1	0.84
190	290	50	95	1	18.06	782	20.233	31.878	32	18.06	0	8	1	0.63
200	290	45	100	1	8.63	852	23.428	38.638	34	8.63	0	9	1	0.61
210	290	40	105	1	5.85	823	23.827	41.370	30	5.85	0	8	1	0.58
220	290	35	110	1	3.51	789	23.549	44.138	31	3.51	0	8	1	0.53
230	290	30	115	1	18.35	828	26.468	49.686	26	18.35	0	9	1	0.53
240	290	25	120	1	8.02	693	22.688	65.611	32	8.02	0	11	1	0.35
250	290	20	125	1	17.41	639	21.520	60.414	34	17.41	0	8	1	0.36
260	290	15	130	1	7.73	622	21.923	93.897	37	7.73	0	14	1	0.23
270	290	10	135	1	18.53	536	19.553	88.237	28	18.53	0	12	1	0.22
280	290	5	140	1	18.73	455	17.180	88.310	27	18.73	0	11	1	0.19
10	300	145	5	0	10.16	27	0.104	0.372	12	10.16	0	6	1	0.28
20	300	140	10	0	14.39	68	0.339	0.736	15	14.39	0	5	1	0.46
30	300	135	15	0	0.32	130	0.831	1.156	14	0.32	0	5	1	0.72
40	300	130	20	0	16.46	157	1.205	1.600	14	16.46	0	4	1	0.75
50	300	125	25	0	14.19	241	2.348	2.276	14	14.19	0	4	1	1.03
60	300	120	30	0	1.79	256	2.590	3.015	15	1.79	0	4	1	0.86
70	300	115	35	0	4.51	329	3.804	3.830	15	4.51	0	4	1	0.99
80	300	110	40	0	18.70	384	4.924	4.481	16	18.70	0	3	1	1.10
90	300	105	45	0	10.66	379	5.423	9.206	56	10.66	0	7	1	0.59
100	300	100	50	0	10.65	465	7.218	8.000	32	10.65	0	4	1	0.90
110	300	95	55	1	18.62	577	9.686	10.370	38	18.62	0	6	1	0.93
120	300	90	60	1	15.66	535	9.421	9.676	20	15.66	0	5	1	0.97
130	300	85	65	1	8.42	713	13.861	13.031	23	8.42	0	7	1	1.06
140	300	80	70	1	8.76	656	13.396	14.535	25	8.76	0	6	1	0.92
150	300	75	75	1	17.17	689	15.124	20.802	26	17.17	0	10	1	0.73
160	300	70	80	1	15.66	800	18.347	21.938	33	15.66	0	7	1	0.84
170	300	65	85	1	0.85	903	22.004	28.095	27	0.85	0	10	1	0.78
180	300	60	90	1	3.50	860	21.898	29.690	26	3.50	0	9	1	0.74
190	300	55	95	1	11.22	947	25.375	32.071	37	11.22	0	7	1	0.79
200	300	50	100	1	15.49	852	24.193	34.325	33	15.49	0	7	1	0.70
210	300	45	105	1	1.30	910	26.553	39.902	32	1.30	0	7	2	0.67
220	300	40	110	1	13.91	897	27.818	46.944	28	13.91	0	9	1	0.59
230	300	35	115	1	15.41	772	25.555	53.763	33	15.41	0	9	1	0.48
240	300	30	120	1	19.73	765	26.144	63.402	34	19.73	0	10	1	0.41
250	300	25	125	1	11.20	734	25.728	71.457	28	11.20	0	11	1	0.36
260	300	20	130	1	4.08	781	28.259	77.261	25	4.08	0	11	1	0.37
270	300	15	135	1	17.39	619	23.484	87.833	32	17.39	0	11	1	0.27
280	300	10	140	1	13.57	586	23.438	87.145	31	13.57	0	10	1	0.27
290	300	5	145	1	8.65	496	20.605	98.340	30	8.65	0	11	1	0.21
10	310	150	5	0	4.16	34	0.131	0.421	13	4.16	0	7	1	0.31
20	310	145	10	0	12.95	95	0.481	0.735	13	12.95	0	5	1	0.65
30	310	140	15	0	2.79	122	0.820	1.202	14	2.79	0	5	1	0.68
40	310	135	20	0	1.43	168	1.345	1.678	14	1.43	0	4	1	0.80
50	310	130	25	0	3.18	208	1.936	2.335	14	3.18	0	4	1	0.83
60	310	125	30	0	15.41	246	2.671	3.239	17	15.41	0	4	1	0.82
70	310	120	35	0	19.50	360	4.311	4.003	16	19.50	0	4	1	1.08
80	310	115	40	0	9.14	354	4.552	4.956	17	9.14	0	4	1	0.92
90	310	110	45	0	17.24	434	6.192	6.207	19	17.24	0	4	1	1.00

Table 32: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
100	310	105	50	0	18.43	455	7.167	31.957	285	18.43	0	28	1	0.22
110	310	100	55	1	6.29	576	9.883	24.867	167	6.29	0	18	17	0.40
120	310	95	60	0	19.12	491	8.900	45.292	302	19.12	0	28	1	0.20
130	310	90	65	1	7.99	702	13.769	13.648	30	7.99	0	6	1	1.01
140	310	85	70	1	16.79	680	14.252	16.124	22	16.79	0	8	1	0.88
150	310	80	75	1	17.91	817	18.307	18.571	28	17.91	0	7	1	0.99
160	310	75	80	1	13.82	819	19.444	24.628	38	13.82	0	9	1	0.79
170	310	70	85	1	19.82	868	21.632	25.119	32	19.82	0	7	1	0.86
180	310	65	90	1	3.05	880	22.945	29.451	31	3.05	0	8	1	0.78
190	310	60	95	1	18.82	878	24.517	30.141	31	18.82	0	6	1	0.81
200	310	55	100	1	4.36	1035	29.783	37.274	34	4.36	0	8	1	0.80
210	310	50	105	1	13.81	899	27.357	39.309	27	13.81	0	7	1	0.70
220	310	45	110	1	15.13	949	30.900	60.667	41	15.13	0	12	2	0.51
230	310	40	115	1	8.55	940	31.894	51.070	32	8.55	0	8	1	0.62
240	310	35	120	1	6.58	932	32.687	55.623	36	6.58	0	7	1	0.59
250	310	30	125	1	11.94	794	28.901	62.319	31	11.94	0	8	1	0.46
260	310	25	130	1	13.41	822	30.774	78.309	34	13.41	0	10	1	0.39
270	310	20	135	1	18.82	623	24.172	81.167	38	18.82	0	9	1	0.30
280	310	15	140	1	8.33	707	28.860	98.160	41	8.33	0	11	1	0.29
290	310	10	145	1	1.48	712	30.457	112.645	36	1.48	0	12	1	0.27
300	310	5	150	1	5.92	484	21.417	111.277	33	5.92	0	11	1	0.19
10	320	155	5	0	9.75	33	0.133	0.434	13	9.75	0	7	1	0.31
20	320	150	10	0	7.95	85	0.450	0.763	13	7.95	0	5	1	0.59
30	320	145	15	0	1.82	150	1.057	1.209	13	1.82	0	5	1	0.87
40	320	140	20	0	2.26	202	1.654	1.806	16	2.26	0	4	1	0.92
50	320	135	25	0	14.28	255	2.414	2.493	15	14.28	0	4	1	0.97
60	320	130	30	0	14.14	288	3.088	3.182	15	14.14	0	4	1	0.97
70	320	125	35	0	11.29	306	3.764	4.214	16	11.29	0	4	1	0.89
80	320	120	40	0	9.56	356	4.984	5.197	18	9.56	0	4	1	0.96
90	320	115	45	0	3.58	476	7.296	6.500	20	3.58	0	4	1	1.12
100	320	110	50	0	13.97	509	8.301	25.871	211	13.97	0	22	1	0.32
110	320	105	55	1	18.43	554	9.728	22.063	143	18.43	0	14	1	0.44
120	320	100	60	0	7.40	567	10.939	31.495	178	7.40	0	19	1	0.35
130	320	95	65	1	10.29	630	13.107	15.313	32	10.29	0	8	1	0.86
140	320	90	70	1	15.76	817	17.640	17.510	30	15.76	0	8	2	1.01
150	320	85	75	1	8.59	768	17.722	18.262	23	8.59	0	7	1	0.97
160	320	80	80	1	14.34	875	21.193	23.747	33	14.34	0	8	1	0.89
170	320	75	85	1	12.99	894	23.436	30.073	39	12.99	0	9	1	0.78
180	320	70	90	1	13.22	849	23.852	34.196	41	13.22	0	9	1	0.70
190	320	65	95	1	5.95	1048	31.073	31.503	34	5.95	0	6	1	0.99
200	320	60	100	1	13.96	988	31.362	38.478	33	13.96	0	8	1	0.82
210	320	55	105	1	15.21	885	28.801	46.625	28	15.21	0	9	1	0.62
220	320	50	110	1	4.84	948	31.697	41.671	32	4.84	0	6	1	0.76
230	320	45	115	1	15.32	928	32.305	49.388	31	15.32	0	7	1	0.65
240	320	40	120	1	4.78	1001	36.351	50.823	30	4.78	0	6	1	0.72
250	320	35	125	1	7.10	889	33.875	77.790	41	7.10	0	11	1	0.44
260	320	30	130	1	18.49	856	34.251	91.289	38	18.49	0	12	1	0.38
270	320	25	135	1	15.40	829	33.894	86.461	34	15.40	0	10	1	0.39
280	320	20	140	1	5.64	801	33.685	107.940	40	5.64	0	12	1	0.31
290	320	15	145	1	7.38	754	32.629	116.171	38	7.38	0	12	1	0.28
300	320	10	150	1	15.37	611	27.917	100.951	31	15.37	0	9	1	0.28
310	320	5	155	1	1.94	505	24.116	127.807	31	1.94	0	12	1	0.19
10	330	160	5	0	15.48	29	0.124	0.466	13	15.48	0	7	1	0.27
20	330	155	10	0	15.54	76	0.415	0.898	15	15.54	0	5	1	0.46
30	330	150	15	0	6.88	141	0.992	1.258	13	6.88	0	5	1	0.79
40	330	145	20	0	2.42	202	1.697	1.831	15	2.42	0	4	1	0.93
50	330	140	25	0	14.74	213	2.119	2.584	15	14.74	0	4	1	0.82
60	330	135	30	0	4.63	292	3.281	3.411	16	4.63	0	4	1	0.96
70	330	130	35	0	17.58	367	4.672	4.375	17	17.58	0	4	1	1.07
80	330	125	40	0	6.41	407	5.709	5.460	17	6.41	0	4	1	1.05
90	330	120	45	0	17.44	446	6.926	6.627	19	17.44	0	4	1	1.05
100	330	115	50	0	9.68	549	9.284	7.354	18	9.68	0	3	1	1.26
110	330	110	55	0	18.31	542	9.945	12.025	38	18.31	0	7	1	0.83
120	330	105	60	0	15.38	540	10.563	39.660	247	15.38	0	23	1	0.27
130	330	100	65	1	19.77	644	13.481	14.758	35	19.77	0	6	1	0.91
140	330	95	70	1	15.03	695	15.643	18.282	32	15.03	0	8	2	0.86
150	330	90	75	1	17.23	803	18.900	22.197	32	17.23	0	9	1	0.85
160	330	85	80	1	10.05	846	21.223	26.579	42	10.05	0	9	4	0.80
170	330	80	85	1	9.87	912	24.392	24.728	32	9.87	0	6	1	0.99
180	330	75	90	1	16.95	988	27.516	29.889	33	16.95	0	7	1	0.92
190	330	70	95	1	5.06	1055	31.212	34.243	29	5.06	0	8	1	0.91
200	330	65	100	1	8.45	911	28.636	41.019	42	8.45	0	8	1	0.70
210	330	60	105	1	13.40	956	31.388	48.351	39	13.40	0	8	2	0.65
220	330	55	110	1	0.36	1076	36.766	43.295	26	0.36	0	7	1	0.85
230	330	50	115	1	17.06	871	30.942	51.138	36	17.06	0	7	1	0.61
240	330	45	120	1	2.55	980	36.017	59.068	35	2.55	0	8	1	0.61
250	330	40	125	1	17.83	878	33.984	68.403	38	17.83	0	9	1	0.50
260	330	35	130	1	6.69	848	33.972	87.452	40	6.69	0	11	1	0.39
270	330	30	135	1	11.77	849	35.343	82.277	33	11.77	0	9	1	0.43
280	330	25	140	1	15.56	976	42.399	83.396	34	15.56	0	8	1	0.51
290	330	20	145	1	15.59	849	38.600	102.562	32	15.59	0	10	1	0.38
300	330	15	150	1	2.86	811	38.048	123.686	42	2.86	0	11	5	0.31
310	330	10	155	1	15.16	630	31.201	136.337	35	15.16	0	12	1	0.23
320	330	5	160	1	2.60	473	25.091	130.800	30	2.60	0	11	1	0.19
10	340	165	5	0	10.92	37	0.156	0.465	13	10.92	0	7	1	0.34
20	340	160	10	0	16.77	66	0.377	0.900	15	16.77	0	5	1	0.42
30	340	155	15	0	10.22	143	1.043	1.367	15	10.22	0	5	1	0.76
40	340	150	20	0	12.74	197	1.683	1.883	15	12.74	0	4	1	0.89
50	340	145	25	0	3.59	246	2.459	2.668	15	3.59	0	4	1	0.92

Table 33: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
60	340	140	30	0	17.40	256	2.993	3.493	16	17.40	0	4	1	0.86
70	340	135	35	0	9.81	317	4.100	4.450	16	9.81	0	4	1	0.92
80	340	130	40	0	17.57	399	5.884	5.434	16	17.57	0	4	1	1.08
90	340	125	45	0	7.68	446	7.117	6.705	18	7.68	0	4	1	1.06
100	340	120	50	0	7.13	504	8.771	8.281	21	7.13	0	4	1	1.06
110	340	115	55	0	16.53	582	10.849	8.819	20	16.53	0	3	1	1.23
120	340	110	60	0	3.87	596	11.935	31.521	164	3.87	0	20	1	0.38
130	340	105	65	1	19.64	730	15.970	14.104	25	19.64	0	6	1	1.13
140	340	100	70	1	4.85	836	19.038	16.221	25	4.85	0	6	1	1.17
150	340	95	75	1	19.07	780	19.115	19.577	24	19.07	0	7	1	0.98
160	340	90	80	1	2.89	895	23.227	26.145	33	2.89	0	9	1	0.89
170	340	85	85	1	11.72	912	25.172	26.672	31	11.72	0	7	1	0.94
180	340	80	90	1	2.94	954	27.844	33.873	35	2.94	0	9	1	0.82
190	340	75	95	1	10.82	948	29.015	31.523	29	10.82	0	6	1	0.92
200	340	70	100	1	11.62	998	31.957	39.991	32	11.62	0	8	1	0.80
210	340	65	105	1	16.52	1005	34.396	42.362	27	16.52	0	7	1	0.81
220	340	60	110	1	12.09	1004	35.360	46.517	32	12.09	0	7	1	0.76
230	340	55	115	1	3.19	1088	40.752	54.476	33	3.19	0	8	1	0.75
240	340	50	120	1	7.89	1086	41.874	67.867	36	7.89	0	10	1	0.62
250	340	45	125	1	15.42	1092	43.576	57.070	33	15.42	0	6	1	0.76
260	340	40	130	1	2.54	996	41.942	76.844	35	2.54	0	9	1	0.55
270	340	35	135	1	14.06	943	41.180	82.631	32	14.06	0	9	1	0.50
280	340	30	140	1	2.51	875	39.814	83.067	30	2.51	0	8	1	0.48
290	340	25	145	1	3.68	871	41.494	109.124	39	3.68	0	10	1	0.38
300	340	20	150	1	4.42	879	42.592	113.324	33	4.42	0	10	1	0.38
310	340	15	155	1	6.24	860	43.886	122.998	35	6.24	0	10	1	0.36
320	340	10	160	1	10.66	668	36.073	141.246	34	10.66	0	11	1	0.26
330	340	5	165	1	14.83	522	29.135	164.055	31	14.83	0	13	1	0.18
10	350	170	5	0	2.18	26	0.118	0.487	14	2.18	0	7	1	0.24
20	350	165	10	0	19.76	84	0.488	0.872	15	19.76	0	5	1	0.56
30	350	160	15	0	19.37	126	0.942	1.433	15	19.37	0	5	1	0.66
40	350	155	20	0	3.13	183	1.632	1.906	14	3.13	0	4	1	0.86
50	350	150	25	0	10.12	278	2.872	2.677	14	10.12	0	4	1	1.07
60	350	145	30	0	4.57	269	3.239	3.483	15	4.57	0	4	1	0.93
70	350	140	35	0	16.77	341	4.646	4.558	16	16.77	0	4	1	1.02
80	350	135	40	0	13.01	364	5.390	5.690	17	13.01	0	4	1	0.95
90	350	130	45	0	5.87	487	7.812	6.869	17	5.87	0	4	1	1.14
100	350	125	50	0	2.15	514	9.135	7.698	19	2.15	0	3	1	1.19
110	350	120	55	0	9.47	660	12.707	9.059	18	9.47	0	3	1	1.40
120	350	115	60	0	11.34	556	11.661	10.811	22	11.34	0	3	1	1.08
130	350	110	65	1	19.16	745	16.132	14.177	24	19.16	0	6	1	1.14
140	350	105	70	1	19.52	830	19.801	18.907	36	19.52	0	7	1	1.05
150	350	100	75	1	10.03	860	21.847	23.072	32	10.03	0	9	1	0.95
160	350	95	80	1	16.86	874	23.436	22.307	24	16.86	0	7	1	1.05
170	350	90	85	1	8.19	939	26.376	29.650	39	8.19	0	8	1	0.89
180	350	85	90	1	17.80	941	28.659	33.321	39	17.80	0	8	1	0.86
190	350	80	95	1	14.07	984	31.242	35.390	35	14.07	0	7	1	0.88
200	350	75	100	1	13.56	945	31.359	39.281	29	13.56	0	8	1	0.80
210	350	70	105	1	6.76	1040	36.123	50.455	42	6.76	0	8	2	0.72
220	350	65	110	1	3.95	1130	41.125	49.240	33	3.95	0	8	1	0.84
230	350	60	115	1	12.14	1053	39.896	50.533	29	12.14	0	7	1	0.79
240	350	55	120	1	8.56	1036	41.307	58.742	37	8.56	0	7	1	0.70
250	350	50	125	1	7.11	1098	46.066	69.576	29	7.11	0	9	1	0.66
260	350	45	130	1	19.77	1117	48.574	80.342	39	19.77	0	9	1	0.60
270	350	40	135	1	2.28	948	43.059	94.557	45	2.28	0	10	7	0.46
280	350	35	140	1	16.33	988	46.799	86.971	36	16.33	0	8	1	0.54
290	350	30	145	1	15.46	975	47.613	105.488	33	15.46	0	10	1	0.45
300	350	25	150	1	4.72	908	46.355	99.609	30	4.72	0	8	1	0.47
310	350	20	155	1	14.05	899	47.387	164.869	33	14.05	0	15	1	0.29
320	350	15	160	1	10.37	834	46.775	164.922	41	10.37	0	13	1	0.28
330	350	10	165	1	1.54	701	41.070	155.443	41	1.54	0	11	2	0.26
340	350	5	170	1	17.51	471	28.828	182.707	35	17.51	0	13	1	0.16
10	360	175	5	0	12.85	38	0.172	0.512	14	12.85	0	7	1	0.34
20	360	170	10	0	18.90	83	0.512	0.900	15	18.90	0	5	1	0.57
30	360	165	15	0	3.20	127	0.979	1.465	15	3.20	0	5	1	0.67
40	360	160	20	0	6.60	166	1.564	1.945	14	6.60	0	4	1	0.80
50	360	155	25	0	1.44	238	2.539	3.094	17	1.44	0	5	1	0.82
60	360	150	30	0	9.41	274	3.389	3.709	16	9.41	0	4	1	0.91
70	360	145	35	0	17.31	325	4.395	4.812	16	17.31	0	4	1	0.91
80	360	140	40	0	12.96	410	6.287	5.695	15	12.96	0	4	1	1.10
90	360	135	45	0	16.65	457	7.742	7.129	17	16.65	0	4	1	1.09
100	360	130	50	0	11.34	486	8.746	8.568	20	11.34	0	4	1	1.02
110	360	125	55	0	1.44	577	11.586	9.361	19	1.44	0	3	1	1.24
120	360	120	60	0	15.10	581	12.572	13.590	28	15.10	0	6	1	0.93
130	360	115	65	1	7.97	642	14.758	34.531	142	7.97	0	18	19	0.43
140	360	110	70	1	18.24	675	16.508	27.707	91	18.24	0	10	1	0.60
150	360	105	75	1	7.69	826	21.594	24.908	42	7.69	0	9	1	0.87
160	360	100	80	1	8.36	920	25.664	23.623	29	8.36	0	7	1	1.09
170	360	95	85	1	19.79	850	25.186	30.945	39	19.79	0	9	1	0.81
180	360	90	90	1	12.15	1053	32.549	32.842	32	12.15	0	8	1	0.99
190	360	85	95	1	9.36	1070	34.635	38.287	42	9.36	0	8	3	0.90
200	360	80	100	1	18.06	975	33.445	50.936	46	18.06	0	11	1	0.66
210	360	75	105	1	16.17	1122	40.420	45.297	31	16.17	0	7	1	0.89
220	360	70	110	1	19.09	1118	42.555	55.387	30	19.09	0	10	1	0.77
230	360	65	115	1	16.14	1159	45.502	50.902	36	16.14	0	6	1	0.89
240	360	60	120	1	7.81	1053	43.553	67.643	35	7.81	0	9	1	0.64
250	360	55	125	1	9.48	1071	45.424	73.299	36	9.48	0	9	1	0.62
260	360	50	130	1	1.44	1185	51.641	73.990	35	1.44	0	8	1	0.70
270	360	45	135	1	16.14	968	44.324	91.934	34	16.14	0	10	1	0.48

Table 34: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
280	360	40	140	1	14.68	992	48.035	112.037	48	14.68	0	11	1	0.43
290	360	35	145	1	10.48	1017	50.227	136.941	43	10.48	0	14	1	0.37
300	360	30	150	1	7.16	1016	52.883	108.057	32	7.16	0	9	1	0.49
310	360	25	155	1	0.13	1000	53.527	150.645	52	0.13	0	12	6	0.36
320	360	20	160	1	3.85	853	47.432	148.748	40	3.85	0	11	1	0.32
330	360	15	165	1	19.32	708	42.625	195.715	44	19.32	0	15	1	0.22
340	360	10	170	1	7.25	733	45.301	187.027	40	7.25	0	13	1	0.24
350	360	5	175	1	13.19	498	32.311	186.219	38	13.19	0	12	1	0.17
10	370	180	5	0	5.77	35	0.160	0.520	14	5.77	0	7	1	0.31
20	370	175	10	0	19.16	85	0.555	0.924	15	19.16	0	5	1	0.60
30	370	170	15	0	16.93	104	0.838	1.461	14	16.93	0	5	1	0.57
40	370	165	20	0	14.99	200	1.875	2.199	16	14.99	0	5	1	0.85
50	370	160	25	0	13.08	250	2.732	3.090	15	13.08	0	5	1	0.88
60	370	155	30	0	0.83	299	3.832	3.801	16	0.83	0	4	1	1.01
70	370	150	35	0	5.82	422	5.859	5.041	18	5.82	0	4	1	1.16
80	370	145	40	0	14.15	428	6.604	6.016	18	14.15	0	4	1	1.10
90	370	140	45	0	12.08	398	6.793	7.381	18	12.08	0	4	1	0.92
100	370	135	50	0	4.02	569	10.594	8.736	19	4.02	0	4	1	1.21
110	370	130	55	0	1.39	655	13.127	9.605	19	1.39	0	3	1	1.37
120	370	125	60	0	18.06	604	13.105	12.129	21	18.06	0	4	1	1.08
130	370	120	65	1	5.63	723	17.215	33.488	139	5.63	0	17	1	0.51
140	370	115	70	1	8.60	855	21.289	23.738	58	8.60	0	9	1	0.90
150	370	110	75	1	15.90	845	22.648	22.793	30	15.90	0	8	1	0.99
160	370	105	80	1	11.24	851	24.379	22.221	27	11.24	0	6	1	1.10
170	370	100	85	1	2.09	953	28.344	29.723	32	2.09	0	8	2	0.95
180	370	95	90	1	7.53	1020	32.709	31.869	27	7.53	0	8	1	1.03
190	370	90	95	1	3.30	1159	37.908	37.416	33	3.30	0	8	1	1.01
200	370	85	100	1	8.86	1115	38.590	39.777	32	8.86	0	7	1	0.97
210	370	80	105	1	11.47	1143	41.107	42.484	29	11.47	0	6	1	0.97
220	370	75	110	1	14.48	1049	39.969	61.619	30	14.48	0	12	1	0.65
230	370	70	115	1	10.10	1230	48.783	67.234	36	10.10	0	11	1	0.73
240	370	65	120	1	7.33	1186	49.289	59.814	39	7.33	0	7	1	0.82
250	370	60	125	1	1.01	1196	51.906	65.648	36	1.01	0	7	1	0.79
260	370	55	130	1	7.04	1178	53.008	75.576	36	7.04	0	8	1	0.70
270	370	50	135	1	12.57	1109	52.359	100.613	30	12.57	0	12	1	0.52
280	370	45	140	1	17.85	1224	59.596	108.256	38	17.85	0	11	1	0.55
290	370	40	145	1	16.91	1147	58.959	83.812	34	16.91	0	6	1	0.70
300	370	35	150	1	14.97	1007	53.938	129.842	43	14.97	0	11	1	0.42
310	370	30	155	1	6.37	1058	59.869	104.914	35	6.37	0	7	1	0.57
320	370	25	160	1	8.07	1062	61.941	121.311	35	8.07	0	8	1	0.51
330	370	20	165	1	6.56	906	54.479	183.215	45	6.56	0	13	1	0.30
340	370	15	170	1	7.32	868	54.170	159.906	36	7.32	0	10	1	0.34
350	370	10	175	1	6.61	796	52.102	199.867	36	6.61	0	13	1	0.26
360	370	5	180	1	1.50	672	46.057	244.186	36	1.50	0	15	1	0.19
10	380	185	5	0	4.68	36	0.170	0.512	13	4.68	0	7	1	0.33
20	380	180	10	0	2.21	87	0.570	1.016	16	2.21	0	6	1	0.56
30	380	175	15	0	6.80	152	1.201	1.646	16	6.80	0	6	1	0.73
40	380	170	20	0	17.71	217	2.146	2.273	16	17.71	0	5	1	0.94
50	380	165	25	0	11.91	256	2.896	3.174	15	11.91	0	5	1	0.91
60	380	160	30	0	5.04	310	4.152	3.869	15	5.04	0	4	1	1.07
70	380	155	35	0	0.14	341	5.217	5.242	18	0.14	0	4	1	1.00
80	380	150	40	0	6.16	443	7.158	6.279	18	6.16	0	4	1	1.14
90	380	145	45	0	10.91	458	8.139	7.438	17	10.91	0	4	1	1.09
100	380	140	50	0	7.09	556	10.658	8.924	18	7.09	0	4	1	1.19
110	380	135	55	0	7.85	592	12.074	10.229	18	7.85	0	4	1	1.18
120	380	130	60	0	19.68	602	13.453	11.545	20	19.68	0	3	1	1.17
130	380	125	65	0	8.70	743	17.896	13.707	25	8.70	0	3	1	1.31
140	380	120	70	1	13.69	797	20.367	35.246	117	13.69	0	15	1	0.58
150	380	115	75	1	19.64	834	22.996	22.812	31	19.64	0	8	1	1.01
160	380	110	80	1	14.28	864	25.371	30.369	47	14.28	0	10	2	0.84
170	380	105	85	1	14.00	907	27.807	30.205	36	14.00	0	8	1	0.92
180	380	100	90	1	2.39	994	32.047	33.764	33	2.39	0	8	1	0.95
190	380	95	95	1	0.79	1151	38.881	43.654	43	0.79	0	10	3	0.89
200	380	90	100	1	11.16	1181	41.105	44.285	31	11.16	0	9	1	0.93
210	380	85	105	1	3.14	1212	44.975	52.562	34	3.14	0	9	1	0.86
220	380	80	110	1	11.98	1127	43.074	49.025	34	11.98	0	7	1	0.88
230	380	75	115	1	19.25	1339	54.623	62.910	39	19.25	0	9	1	0.87
240	380	70	120	1	19.53	1179	50.061	73.305	36	19.53	0	10	1	0.68
250	380	65	125	1	0.67	1320	57.842	52.955	34	0.67	0	4	1	1.09
260	380	60	130	1	11.83	1308	60.270	70.820	32	11.83	0	7	1	0.85
270	380	55	135	1	19.15	1188	58.070	86.859	30	19.15	0	9	1	0.67
280	380	50	140	1	1.24	1267	63.041	108.102	34	1.24	0	11	1	0.58
290	380	45	145	1	17.65	1153	61.012	120.965	41	17.65	0	11	1	0.50
300	380	40	150	1	18.69	1168	62.383	144.385	44	18.69	0	13	1	0.43
310	380	35	155	1	5.05	1069	61.197	142.293	44	5.05	0	11	1	0.43
320	380	30	160	1	10.27	1028	59.426	140.760	40	10.27	0	10	1	0.42
330	380	25	165	1	10.48	1078	65.949	144.742	44	10.48	0	9	1	0.46
340	380	20	170	1	3.32	814	52.623	120.754	35	3.32	0	6	1	0.44
350	380	15	175	1	15.54	921	60.836	209.068	37	15.54	0	13	1	0.29
360	380	10	180	1	10.61	722	50.848	266.172	34	10.61	0	17	1	0.19
370	380	5	185	1	6.32	526	39.260	258.961	37	6.32	0	15	1	0.15
10	390	190	5	0	13.09	38	0.182	0.580	15	13.09	0	8	1	0.31
20	390	185	10	0	1.11	77	0.502	1.041	16	1.11	0	6	1	0.48
30	390	180	15	0	4.16	146	1.236	1.695	16	4.16	0	6	1	0.73
40	390	175	20	0	4.75	181	1.863	2.377	16	4.75	0	5	1	0.78
50	390	170	25	0	4.62	254	2.992	3.273	15	4.62	0	5	1	0.91
60	390	165	30	0	8.19	274	3.699	4.059	15	8.19	0	4	1	0.91
70	390	160	35	0	3.80	325	4.836	5.268	17	3.80	0	4	1	0.92
80	390	155	40	0	0.97	400	6.766	6.354	17	0.97	0	4	1	1.06

Table 35: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)

		LINFSOL						B-P						
N	M	Pdeg	Ddeg	Err	Obj	Itr	CPU	CPU	Itr	Obj	Err	Ref	Red	Rat
90	390	150	45	0	7.54	473	8.701	7.764	18	7.54	0	4	1	1.12
100	390	145	50	0	3.03	480	9.246	9.051	18	3.03	0	4	1	1.02
110	390	140	55	0	14.13	526	11.137	10.977	21	14.13	0	4	1	1.01
120	390	135	60	0	14.83	607	14.053	12.920	23	14.83	0	4	1	1.09
130	390	130	65	0	14.09	699	17.312	25.691	79	14.09	0	12	1	0.67
140	390	125	70	1	10.54	772	20.520	65.463	305	10.54	0	24	22	0.31
150	390	120	75	1	15.28	908	25.152	22.330	28	15.28	0	7	1	1.13
160	390	115	80	1	9.57	933	27.338	23.715	28	9.57	0	6	1	1.15
170	390	110	85	1	11.43	967	29.596	29.184	33	11.43	0	7	1	1.01
180	390	105	90	1	17.25	1022	34.004	35.158	28	17.25	0	9	1	0.97
190	390	100	95	1	18.04	992	34.859	35.961	30	18.04	0	7	1	0.97
200	390	95	100	1	9.78	1162	42.426	41.266	33	9.78	0	7	1	1.03
210	390	90	105	1	1.58	1199	45.527	50.422	33	1.58	0	8	1	0.90
220	390	85	110	1	12.62	1124	45.242	60.688	40	12.62	0	10	1	0.75
230	390	80	115	1	3.06	1287	53.977	55.604	32	3.06	0	7	1	0.97
240	390	75	120	1	18.07	1278	56.475	65.705	36	18.07	0	8	1	0.86
250	390	70	125	1	7.21	1284	59.484	79.871	36	7.21	0	10	1	0.74
260	390	65	130	1	15.31	1416	69.287	87.912	37	15.31	0	10	1	0.79
270	390	60	135	1	10.52	1145	57.955	89.197	33	10.52	0	9	1	0.65
280	390	55	140	1	0.35	1366	71.061	121.949	64	0.35	0	11	11	0.58
290	390	50	145	1	7.07	1327	72.711	92.889	35	7.07	0	7	1	0.78
300	390	45	150	1	2.84	1221	69.875	129.566	36	2.84	0	11	1	0.54
310	390	40	155	1	4.06	1141	68.184	147.707	36	4.06	0	12	1	0.46
320	390	35	160	1	15.75	1211	75.914	164.613	45	15.75	0	12	1	0.46
330	390	30	165	1	11.06	1115	74.463	193.320	41	11.06	0	14	1	0.39
340	390	25	170	1	17.25	1270	87.967	161.771	44	17.25	0	9	1	0.54
350	390	20	175	1	6.64	1033	73.607	191.996	38	6.64	0	11	1	0.38
360	390	15	180	1	1.96	937	69.088	268.061	41	1.96	0	16	1	0.26
370	390	10	185	1	0.70	663	51.680	216.475	44	0.70	0	11	4	0.24
380	390	5	190	1	8.40	670	56.369	265.752	36	8.40	0	14	1	0.21
10	400	195	5	0	1.34	32	0.168	0.613	15	1.34	0	8	1	0.27
20	400	190	10	0	5.36	94	0.637	1.051	15	5.36	0	6	1	0.61
30	400	185	15	0	15.08	155	1.312	1.738	16	15.08	0	6	1	0.76
40	400	180	20	0	6.82	179	1.801	2.348	15	6.82	0	5	1	0.77
50	400	175	25	0	6.77	226	2.660	3.438	17	6.77	0	5	1	0.77
60	400	170	30	0	1.87	328	4.371	4.230	17	1.87	0	4	1	1.03
70	400	165	35	0	9.55	405	6.008	5.375	16	9.55	0	4	1	1.12
80	400	160	40	0	17.06	382	6.504	6.738	19	17.06	0	4	1	0.97
90	400	155	45	0	2.05	454	8.523	8.133	19	2.05	0	4	1	1.05
100	400	150	50	0	14.19	502	10.117	9.602	21	14.19	0	4	1	1.05
110	400	145	55	0	4.73	533	11.520	11.445	22	4.73	0	4	1	1.01
120	400	140	60	0	0.25	596	14.074	13.102	24	0.25	0	4	1	1.07
130	400	135	65	0	18.18	655	16.277	15.234	25	18.18	0	4	1	1.07
140	400	130	70	1	9.15	750	19.973	38.719	129	9.15	0	16	1	0.52
150	400	125	75	1	1.77	885	24.832	22.414	27	1.77	0	7	1	1.11
160	400	120	80	1	14.16	908	27.598	24.102	28	14.16	0	6	1	1.15
170	400	115	85	1	0.43	961	30.879	42.031	103	0.43	0	8	33	0.73
180	400	110	90	1	16.80	1023	34.125	39.070	37	16.80	0	10	1	0.87
190	400	105	95	1	0.40	1044	37.465	47.113	58	0.40	0	9	10	0.80
200	400	100	100	1	0.81	1274	48.441	47.258	49	0.81	0	8	7	1.03
210	400	95	105	1	18.50	1124	44.945	48.223	37	18.50	0	7	1	0.93
220	400	90	110	1	2.35	1278	53.605	59.578	33	2.35	0	10	1	0.90
230	400	85	115	1	16.28	1277	55.656	62.523	41	16.28	0	8	2	0.89
240	400	80	120	1	16.46	1434	66.758	70.113	39	16.46	0	8	1	0.95
250	400	75	125	1	5.23	1402	68.570	80.473	44	5.23	0	9	1	0.85
260	400	70	130	1	1.10	1397	70.477	109.023	46	1.10	0	13	2	0.65
270	400	65	135	1	10.35	1217	64.871	91.828	43	10.35	0	8	1	0.71
280	400	60	140	1	16.95	1421	78.230	103.410	41	16.95	0	9	1	0.76
290	400	55	145	1	15.30	1299	74.297	99.797	32	15.30	0	8	1	0.74
300	400	50	150	1	11.60	1255	75.496	113.652	37	11.60	0	8	1	0.66
310	400	45	155	1	3.32	1279	79.012	151.242	42	3.32	0	12	1	0.52
320	400	40	160	1	18.81	1144	71.934	142.410	36	18.81	0	10	1	0.51
330	400	35	165	1	6.33	1293	86.406	178.035	44	6.33	0	12	1	0.49
340	400	30	170	1	17.89	1007	70.805	215.598	50	17.89	0	14	1	0.33
350	400	25	175	1	11.26	1083	78.750	192.465	41	11.26	0	11	1	0.41
360	400	20	180	1	4.27	1043	79.336	233.641	44	4.27	0	13	1	0.34
370	400	15	185	1	3.35	898	71.234	268.438	44	3.35	0	14	1	0.27
380	400	10	190	1	16.33	807	67.195	231.539	39	16.33	0	11	1	0.29
390	400	5	195	1	11.61	656	55.082	252.355	33	11.61	0	12	1	0.22

Table 36: Comparison between LINFSOL and B-P on primal-dual degenerate problems (continued)