# MODELING THE SUPPLIER UNCERTAINTY WITH PHASE-TYPE DISTRIBUTIONS IN INVENTORY PROBLEMS

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
Ahmet Barış Balcıoğlu

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Ülkü Gürler (Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Halim Doğrusöz

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## MODELING THE SUPPLIER UNCERTAINTY WITH PHASE-TYPE DISTRIBUTIONS IN INVENTORY PROBLEMS

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M.S. in Industrial Engineering
Supervisor: Assoc. Prof. Ülkü Gürler
September, 1996

This study considers a stochastic inventory model where the supply availability is subject to random fluctuations. The periods in which the supplier is available (ON) or unavailable (OFF) are modeled as a semi-Markov process. During ON periods the $(q, r)$ policy is applied. During OFF periods, the amount enough to bring the inventory position to $q + r$ is ordered as soon as the supplier becomes available again. The regenerative cycles are identified by observing the inventory position and using the renewal reward theorem the average cost per time objective function is derived. In our study, a K-stage Phase-Type distribution for ON periods and a general distribution for OFF periods are assumed. In our study, the problem is theoretically solved for K-stage Phase-Type distributions; additionally numerical computations are made for 2-stage Phase-Type distributions. For large $q$ values the structure of the objective function is investigated.

*Key words*: Inventory Models, Phase-Type Distribution, Semi-Markov Processes, Supplier Availability

# ÖZET

## ENVANTER PROBLEMLERİNDE SUNUCUNUN BELİRSİZLİĞİNİN EVRE-TÜRÜ DAĞLIMLARLA MODELLENMESİ

Ahmet Barış Balcıoğlu
Endüstri Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Doç. Ülkü Gürler
Eylül, 1996

Bu çalışmada çeşitli nedenlerden ötürü arzın rassal dalgalanmalar gösterdiği bir envanter modeli anlatılmaktadır. Sunucunun hizmet verdiği (AÇIK) ve veremediği (KAPALI) süreler bir yarı-Markov süreç olarak modellenmiştir. AÇIK durumlarda $(q, r)$ politikası uygulanmaktadır. KAPALI durumda ise sunucu tekrar çalışabilir duruma gelince, envanter pozisyonunun $q + r$'ye çıkması için yetecek miktarda ısmarlama yapılır. Yeniden tekrarlanabilir çevrimler, envanter pozisyonu gözlemlenerek belirlenir ve yenileme ödül kuramı kullanılarak birim zaman ortalama maliyet işlevi türetilir. Çalışmamızda, AÇIK dönemler için K- aşamalı Evre- Türü, KAPALI dönemler içinse genel bir dağılım varsayılmaktadır. Bu çalışmada, problem K-aşamalı Evre-Türü dağılım için kuramsal olarak çözülmüş, ayrıca 2-aşamalı Evre-Türü dağılımlar kullanılarak sayısal çözümlemelere gidilmiştir. Büyük $q$ değerleri için amaç işlevinin yapısı da incelenmiştir.

*Anahtar sözcükler.* Envanter Modelleri, Evre-Türü dağılımlar, Yarı-Markov Süreçleri, Arzın Karşılanabilirliği.

To my family
and to my closest friend on earth; Aslıhan Özlem Polat

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# Chapter 1

# Introduction and Literature Review

Inventory problems are as old as human history, but introduction of analytical tools to study these problems has started since the beginning of this century. The importance of studying inventory problems arises from the fact that, we can not avoid carrying inventories due to several reasons, the main one being that it is either physically or economically impossible to obtain and distribute goods whenever demand occurs. If inventories are not kept then the customers should wait until their orders are supplied which will result in low customer satisfaction. Other than this, to cope with the effects of inflational or seasonal fluctuations of demand and prices, manufacturers are forced to hold inventories. Several other reasons may be listed similarly.

The basic questions that inventory managers are faced with are:

- How often should the inventory status be checked (i.e. what should be the review interval)?
- When to replenish the inventory?
- How much to order for replenishment?

These issues are handled by introducing mathematical models for inventory processes. A good mathematical model should capture the main features of the real problem, while avoiding analytical and numerical complexities. Inventory systems differ in size and complexity, in the types and nature of the items they carry, in the nature of information available to decision makers, in the costs related with operating systems. Most of the inventory models aim to minimize an objective function with respect to costs, although there may be other objectives such as profit maximization etc. Basically four types of costs relevant to an inventory problem:

**(i) Replenishment Costs**

This is the cost incurred each time a replenishment action is taken. It can be considered in two parts: (i) the fixed amount, often called setup or ordering cost, which must be paid to the source independent of order size, (ii) a component that depends on the size of the replenishment.

**(ii) Inventory Carrying Costs**

Holding stocks include several costs such as: (i) the opportunity cost which is the cost of capital tied up in inventory rather than having it invested elsewhere, (ii) warehouse operation costs, (iii) insurance, (iv) taxes, (v) potential spoilage or obselecence of goods. Usually these costs are accepted to be proportional to the average inventory level, where, in fact some components may be related to inventory level in a more complicated manner.

**(iii) Stockout Costs**

When stocks in hand are insufficient to meet customer demand, costs are incurred as costs of back ordering and/or lost profit on sales other than loosing the good will of the customer due to poor service.

**(iv) System Control Costs**

It includes the costs of acquiring the data necessary for the adopted decision rules, the computational costs and costs of implementation. However in this sequel this cost type is ignored.

Most of the research that has appeared in the literature implicitly assumes that the goods are available from the supplier at any time an order is placed. Even in the models which include a (possibly random) lead-time, the assumption is that the supplier will start production of the order and will deliver the amount that has been required as soon as the lead-time ends.

This assumption may be approved only if the supplier is 'always' available to meet the demand requested. However in practice, supply of the product may be disrupted due to several reasons as discussed below. Therefore, in this study we consider a model where the supplier could also go through ON and OFF times with random durations.

Following examples given by Gürler and Parlar [9] may illustrate the ON/OFF structure of the suppliers: If the supplier has its own inventory process, then we can say that the supplier is ON if ordered quantity $q$ is available in its inventory, and OFF otherwise. Or, as in a frequently encountered example, supplier is considered as a production process which is under statistical process control. The process may start production of items out of specification limits beyond an acceptable proportion and inevitably the process should be stopped before reaching the desired capability. In this case the OFF times of the supplier will be the counterpart of the termination of production while system is being inspected. Similar to this case, machine breakdowns or some maintenance policies may also yield in disruptions in production process and a need for studying supplier unavailability may arise. Rare events such as strikes, embargoes or forced shutdowns of the plants are other possible reasons for disruptions.

When such examples are considered, i.e., in cases when outside supplier may not meet the supply at random times for random durations, the implicit assumption of continuous supply availability would not be valid and new models should be constructed to handle the disruptions of supply.

## 1.1 Literature Review

There is a vast literature on modeling inventory problems. It is therefore not attempted here to give an extensive survey of such studies. The interested reader could refer to Lee and Nahmias [11], Porteus [23], Peterson and Silver [22], Silver [27] and the references therein. Instead, we present below the main studies where supplier unavailability is considered.

Silver [27] is recognized to be the first author who discussed the need of studying supplier unavailability while constructing inventory models. In his review paper, which is also important as it points out the 'serious gaps existing between the theory and practice in many organizations', Silver says that while considering the nature of the supply process, most of the literature ignores that 'only a random portion (including 0, perhaps caused by a strike or poor quality conditions) of the ordered material is received'. This is why he suggests finding simple decision rules that must be valid under these circumstances. While explaining the motivation for holding inventories, Nahmias [16] lists three important uncertainties that play a major role as (i) uncertainty of external demand, (ii) uncertainty of lead time and (iii) uncertainty of the supply. To make the third one to be understood more clearly, Nahmias gives the OPEC oil embargo of the late 70's as an example when the electric utilities and the airlines had to cope with curtailing operations due to fuel shortages. Other important uncertainties are uncertainty of yield and uncertainty of capacity. We suggest interested reader to read the review article of Yano and Lee [28] on random production and procurement yields.

In order to represent disruptive events such that in our case it is the supplier availability, alternating renewal process models are used. Meyer *et al* [14] used this approach while analyzing a single stage production-storage system of fixed capacity, with a constant known demand which is subject to stochastic failure and repair processes. In this paper, after examining the simple deterministic case corresponding to constant inter-failure and repair times, the case with random inter-failure and repair times are considered. Although a general solution of formulated recurrence equations have not been obtained, the exponential case is solved.

An article of Parlar and Berkin [20] which is more related with the present study analyzes the supplier uncertainty problem for the classical deterministic (EOQ) model, with a single supplier whose ON and OFF periods follow exponential distribution. In the model presented, it is assumed that the entire ordered amount will be available during the ON periods of the supplier. But there is a positive probability that at any given time the supplier may be unavailable (OFF) for a random duration. Applying concepts of renewal theory, an objective function (average cost/time) is constructed to find the optimal order quantities when orders are placed during the ON periods of the supplier. Two special cases with (i) "memoryless" ON and OFF periods and (ii) "memoryless" ON and deterministic OFF periods are discussed with sensitivity analysis on the cost and non-cost parameters.

A critique to this previous paper comes from Berk and Arreola-Risa [3]. They point out that Parlar and Berkin [20] make an implicit assumption that a stock out occurs during every OFF period while there is a finite probability that at the end of a cycle there may be positive stock especially when the OFF periods are much shorter than the ON periods. They also state that when the total cost per cycle is derived as if the shortage cost is incurred per unit time will not be valid when sales are lost. Keeping these in mind they develop the 'correct' model for the special case of memoryless ON and OFF periods and investigate its characteristics and additionally they study the sensitivity of the optimal order quantity to different values of the model parameters.

Karaesmen *et al* [7] extend the model of Parlar and Berkin [20] assuming that supply availability periods and disruption durations of supplier are random variables which need not to be independent. They provide two different approaches to compute the expected cost per unit time while formulating the general model. They evaluate the special cases when (i) the supply availability periods and disruption periods are deterministic, (ii) the supply availability periods and disruption periods are memoryless having a certain dependence structure, (iii) the supply availability periods are memoryless and disruption periods depending on supply availability follow a two-point distribution. They find out that the effect of correlation is case dependent for case (iii) and almost "invisible" for case (ii). They observe that, as the length of the expected length of disruption durations increases and the number of orders in a supply cycle is one, the problem can be approximated by a single period problem which is easier to solve.

In a recent paper, Parlar and Perry [21] extend the model of Parlar and Berkin [20] and develop average cost objective function models for single, two and multiple suppliers. In the case of two suppliers, in order to derive explicit expressions for the transient probabilities of a four-state continuous-time Markov chain representing the status of the system, spectral theory is used. The probabilities found in this way are used in the computation of the exact form of the average cost expression. For the multiple case, it is assumed that all the suppliers are similar in availability characteristics and in a simple model, they show that as the number of the suppliers increases, the model reduces to the classical EOQ model.

Gupta [6] analyzes a continuous-review, order quantity/reorder point inventory system with an unreliable supplier whose ON/OFF periods are distributed exponentially. It is assumed that the unit demands are generated according to a Poisson process and whenever shortages occur, they are lost. Moinzadeh and Aggarwal [15] study an unreliable bottleneck production/inventory system subject to random disruptions assuming that the demand and production rates are constant. They propose an $(s, S)$ production policy and find expressions for the operating characteristics of the system. They develop a procedure to find the optimal values of policy parameters minimizing the expected total cost. In addition they propose a heuristic procedure to find near optimal production policies.

Güllü *et al* [8] analyze a periodic inventory model assuming that demand is deterministic and dynamic where the ordered quantity can be either delivered or cancelled if the supplier can not meet the order on time. Therefore in a given period the supplier can be either available or unavailable with given probabilities which are nonstationary and independent from one period to another. Their contribution with this study are (i) demonstrating the optimality of an order-up-to policy, (ii) characterizing explicitly the optimal order-up-to levels, and (iii) providing a newsboy-like formula to compute the optimal order-up-to levels over the planning horizon.

In another study, Parlar [19] considers a continuous-review stochastic inventory problem subject to suppler unavailability. It is assumed that the demand and the lead-times are random variables. He assumes that the ON period of the supplier has a k-stage Erlang distribution and the OFF period is general. The supplier availability is modeled as a semi-Markov process. When the supplier is ON, the $(q, r)$ policy is used conveniently. But whenever the supplier is OFF, the policy changes and an amount necessary to hit a target value $r + q$ is ordered as soon as the supplier becomes available again and this results in order quantity to be a random variable. Parlar constructs the objective function (average cost/time) by first identifying the regenerative cycles of the inventory position process. Employing "method of stages" causes the problem to have a larger state space for the ON/OFF stochastic process. However, the process is analyzed using Markovian techniques. The special case when the order quantity $q$ is large is also considered.

Gürler and Parlar [9] enlarge the previous problem to the case of a duopoly of two suppliers who may be ON and OFF independent of each other for random durations. Each supplier's availability is modeled as a semi-Markov (alternating renewal) process. The durations of ON periods for the two suppliers are assumed to be distributed as Erlang random variables while the OFF periods of each supplier are general. The benefit of this approach comes from the fact that any non-exponential random variable with coefficient of variation less than one can be approximated by an Erlang random variable if the choice of stage parameter of Erlang can be made in a proper way and as a result the ON/OFF stochastic process becomes general under these assumptions. The regenerative cycles of the inventory level process are identified and applying renewal reward theorem the long-run average cost objective function is obtained. Finite time transition functions for the semi-Markov process are computed numerically using a direct method of solving a system of integral equations representing these functions. Then two particular case (i) a problem in which the ON periods of both suppliers follow a 2-stage Erlang distribution and OFF period of each supplier is exponentially distributed, and (ii) the problem where the optimal order quantity $q$ may be 'large' are described. In the latter case, it is shown that the objective function assumes a very simple form to be used to analyze the optimality conditions. The paper ends with discussion of alternative inventory policy for modeling the random supplier availability problem.

The remainder of the thesis can be outlined as follows. In Chapter 2, the main properties of Phase-type distributions are reviewed. Their closure properties are stated and some special Phase-type distributions are examined. Then the equivalence relations between some classes of these distributions are presented. The second chapter ends with the methods of approximating any general distribution with a Phase-type distribution. In Chapter 3, the mathematical model of a continuous-review stochastic inventory problem with deterministic demand and random lead-times where the single supplier is subject to disruptions is constructed and the objective cost function is derived. Chapter 4 includes the analytical solution of a special problem such that the ON periods of the supplier is distributed with 2-stage Coxian distribution. Then the model proposed in previous chapter is re-evaluated for large $q$ values. The numerical results of special problems are displayed and discussed in Chapter 5. Chapter 6 gives the conclusion and possible future research areas with the topic presented here.

# Chapter 2

# Phase-Type Distributions

In stochastic modeling, the assumption of exponential interarrival times with Poisson arrivals is frequently used mostly for mathematical convenience due to the lack-of memory property of the exponential distribution. For complex models, exponential assumption is used to obtain tractable steady-state results which avoid the cost of time-consuming simulations. However, for relatively simple models, it is still desirable to obtain exact results under general distributional assumptions.

Analytic approaches to models with general distributions rapidly become complicated and intractable. An alternative approach is to consider probability distributions and processes, which are computationally tractable while being sufficiently versatile to reflect the essential qualitative features of the model. The family of Phase-type distributions is an example of such alternatives.

The advantage of using Phase-type distributions is that their structures give rise to a Markovian state description. Their potential for algorithmic analysis is usually carried out using matrix algebra. The phase (or stage) concept was first introduced by Erlang [5]. An Erlang distribution consists of a series of $m$ exponential distributions with common rate $\mu$. Therefore the random variable associated with Erlang distribution is the sum of $m$ independent exponential random variables with rate $\mu$.

A distribution even more general is the Coxian distribution. A Coxian distribution with $m$ *stages*, also termed as *phases*, is represented in Figure 2.1. The Coxian distribution is more general than the Erlang distribution since it allows non-identical rates and branching probabilities. This distribution may be better understood by the following physical interpretation. Suppose that the overall processing time of a task is decomposed into a set of $m$ subtasks. The processing time of subtask $j$ is exponentially distributed with rate $\mu_j$. Upon completion of subtask $j$, either subtask $j+1$ is performed, with probability $a_j$, or the overall task is completed, with probability $b_j = 1 - a_j$. $b_m = 1$ explains that at most $m$ subtasks are performed. In the most general form of the Coxian distribution, it is also possible to have a zero processing time with a non-zero probability. This is achieved by adding a branching probability $(a_0, b_0)$ before stage 1.

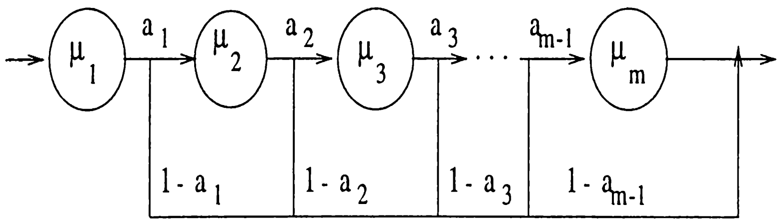


Figure 2.1: Coxian distribution with $m$ phases

Cox [4] showed that any distribution having a rational Laplace-Stieltjes Transform (LST) can be represented by a set of exponential phases. The LST of any distribution function can be approximated arbitrarily closely by a rational function (Newman and Reddy, [18]). Therefore, in principle, Coxian distributions may represent any distribution either exactly or approximately. The most general form of a distribution that are mixtures of exponential distributions is the family of phase-type distributions. A phase-type distribution with $m$ stages (or phases) is represented in Figure 2.2. The following physical interpretation can be considered: Suppose that an overall task is decomposed into a set of $m$ exponential subtasks. (The processing time of subtask $j$ is exponentially distributed with rate $\mu_j$.) The first subtask to be processed is $j$th one with probability $c_{0,j}$. Upon completion of subtask $j$, either subtask $k$ is performed, with probability $c_{j,k}$, or the overall task is completed, with probability $c_{j,0}$. The branching and transition probabilities satisfy,

$$c_{0,1} + ... + c_{0,m} = 1 \quad \text{and} \quad c_{j,1} + ... + c_{j,m} + c_{j,0} = 1$$

Again the possibility of having a zero processing time with non-zero probability may be added. Note that a Coxian distribution is a special case of phase-type distribution.

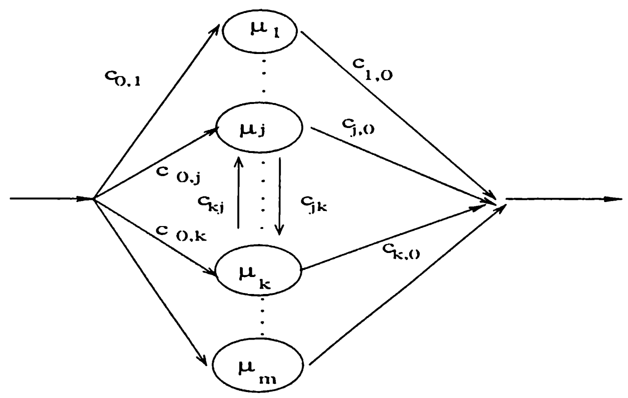


Figure 2.2: Phase-type distribution with $m$ phases.

## 2.1 Definitions and Closure Properties

A phase-type distribution can be considered as the distribution of the time until absorption in an absorbing Markov chain with the states $\{1,..., m+1\}$ with $m+1$ being the single absorbing state. Note that since the feasibility and complexity of numerical solutions of Markov processes are very much dependent on the size of the state space, the number of stages of phase-type distributions should be kept as small as possible for modeling purposes. Let Q be the infinitesimal generator of this Markov chain,

$$Q = \begin{vmatrix} T & T^o \\ \underline{O} & 0 \end{vmatrix} \tag{1}$$

where T is an $m$ x $m$ matrix with $T_{ij} \geq 0$ for i$\neq$j and $T_{ii} < 0$ for i=1,...,$m$. In this representation, m is said to be the order of the phase-type distribution. Then, T$\underline{e}$ + T$^{o}$ = 0, where $\underline{e}$ is a column vector of ones and the initial probability vector of Q is given by ($\underline{\alpha}$, $\alpha_{m+1}$), with $\underline{\alpha} = [\alpha_1, ..., \alpha_m]$, satisfying $\underline{\alpha e} + \alpha_{m+1}$= 1. Then T can be considered as the matrix of the rates of transition among the phases and T$^{o}$ is the vector of rates of transition from the transient states [1,...,$m$] to the absorbing state $m$+1.

**Definition:** Let T be a square matrix. The matrix exp(Tx) is given by the following Taylor series expansion:

$$\exp(\mathrm{Tx}) = \sum_{k=0}^{\infty} T^k \frac{x^k}{k!} = I + Tx + ... + T^k \frac{x^k}{k!} + ...$$

for all x $\in$ R.

**Lemma 2.2.2:(Neuts, [17], p.45)** The distribution function of the time until absorption in the state m+1, corresponding to the initial vector($\underline{\alpha}$, $\alpha_{m+1}$) is given by,

$$\mathrm{F(x)} = 1 - \underline{\alpha}\exp(\mathrm{Tx})\underline{e}$$

for x$\geq$ 0.

**Lemma 2.2.1:(Neuts, [17], p.45)** The states 1,...,$m$ are transient if and only if the matrix T is nonsingular.

**Definition:(Neuts, [17],p.45)** A probability distribution F(.) on [0,$\infty$) is a distribution of phase-type (PH-distribution) if and only if it is the distribution of the time until absorption in a finite Markov process of the type defined in (1). The pair ($\underline{\alpha}$,T) is called a representation of F(.).

The phase-type distribution presented in Figure 2.2 can be represented in matrix notation and any PH-type distribution given in matrix notation can be represented as shown in Figure 2.2. First we are going to find the matrix representation of the PH-type distribution presented in Figure 2.2: It is obvious that $\underline{\alpha} = [c_{0,1}, ..., c_{0,m}]$. In the T matrix, $T_{ii} = -\mu_i$ for i =1,...,$m$ while $T_{ij} = c_{ij}\mu_i$ for i,j = 1,...,$m$ and i$\neq$ j. Then the following matrix of transitions among phases is obtained:

$$T = \begin{vmatrix} -\mu_1 & c_{12}\mu_1 & c_{13}\mu_1 & . & c_{1m}\mu_1 \\ c_{21}\mu_2 & -\mu_2 & c_{23}\mu_2 & . & c_{2m}\mu_2 \\ & & & & \\ c_{m1}\mu_m & c_{m2}\mu_m & c_{m3}\mu_m & . & -\mu_m \end{vmatrix}$$

With the same idea, $T^o$, the vector of rates of transition from the transient states [1,...,$m$] to the absorbing state is obtained as follows: $T^o = [c_{10}\mu_1, c_{20}\mu_2, ..., c_{m0}\mu_m]$. Now assume that we have the T matrix of order m, the $T^o$ vector, and the initial probability vector $\underline{\alpha}$. What we aim is to find the transition probabilities $c_{ij}$ shown in Figure 2.2. We can directly equate $[c_{01}, c_{02}, ..., c_{0m}] = \underline{\alpha}$. The transition probabilities among the transient phases, $c_{ij} = -\frac{T_{ij}}{T_{ii}}$ for i,j =1,...,$m$ and i$\neq$ j. The transition probabilities from any transient state to the absorbing state, $c_{i0} = -\frac{T_i^o}{T_{ii}}$. This brief discussion shows the equivalence of the graphical and matrix representation of a phase-type distribution. We now present some well-known properties of PH-type distributions:

**Some properties:**

**a.** The distribution F(.) has a jump of hight $\alpha_{m+1}$ at x = 0, and its density function $F'(x)$ on (0,$\infty$) is given by $F'(x) = \underline{\alpha}\exp(Tx)T^o$

**b.** The Laplace-Stieltjes transform f(s) of F(.) is given by $f(s) = \alpha_{m+1} + \underline{\alpha}(sI-T)^{-1}T^o$, for Re s$\geq$0

**c.** The noncentral moments $\mu_i'$ of F(.) are all finite and given by

$\mu_i' = (-1)^i i!(\underline{\alpha}T^{-i}\underline{e})$, for i$\geq$ 0.

Suppose that upon absorption into the state $m+1$, we instantaneously perform independent multinomial trials with probabilities $\alpha_1, ..., \alpha_m, \alpha_{m+1}$, until one of the alternatives 1,...,$m$ occurs. Restarting the process Q in the corresponding state, we consider the time of next absorption and repeat the same procedure there. By continuing this procedure indefinitely a new Markov process is constructed such that $(m+1)^{st}$ state becomes an instantaneous state. This new Markov process with states 1,...,m has an infinitesimal generator,

$$Q^* = T + \tilde{T}^o A^o$$

where $\tilde{T}^o$ is an $m$ x $m$ matrix with identical columns $T^o$ and $A^o = (1 - \alpha_{m+1})^{-1}\text{diag}(\alpha_1, ..., \alpha_m)$. Without loss of generality, we assume that $\alpha_{m+1} = 0$. The following definition is a characterization PH-type distributions in terms of this modified process:

**Definition:(Neuts, [17], p.48)** The representation ($\underline{\alpha}$,T) is called irreducible if and only if the matrix $Q^*$ is irreducible. (From now on, we restrict our attention to irreducible representation.)

### 2.1.1 Discrete Phase-Type Distributions

Discrete PH-distributions are defined by considering an $(m+1)$-state Markov chain P of the form,

$$P = \begin{vmatrix} T & T^o \\ \underline{0} & 1 \end{vmatrix}$$

where I - T is nonsingular. The probability distribution $\{p_k\}$ of PH-type is given by:

$$p_0 = \alpha_{m+1} \quad p_k = \underline{\alpha} T^{k-1} T^o \ \text{ for } k \geq 1.$$

The corresponding probability generating is the following:

$$P(z) = \alpha_{m+1} + z\underline{\alpha}(I - zT)^{-1}T^o$$

and the factorial moments are given by:

$$P^k(1) = k!\underline{\alpha}T^{k-1}(I-T)^{-k}\underline{e}$$

### 2.1.2 Closure Properties

A number of operations on PH-distributions lead again to distributions of PH-type. In each case, a representation for the new distribution is obtained. Before stating the theorems, a notational convention will be presented. If $T^0$ is an $m$-vector and $\underline{\beta}$ is an $n$-vector, the $m$ x $n$ matrix $T^o\underline{\beta}$ with elements $T_i^o\beta_j$, 1 $\leq i \leq m$, 1$\leq j \leq n$, is denoted by $T^oB^o$. The following theorem states that the convolution of two continuous (or discrete) phase-type distributions is also a phase-type distribution.

**Theorem 2.2.2:(Neuts, [17], p.51)** If F(.) and G(.) are both continuous (or both discrete) PH-distributions with representations ($\underline{\alpha}$,T) and ($\underline{\beta}$,S) of orders m and n respectively, then their convolution F*G(.) is a PH-distribution with representation ($\underline{\gamma}$,L) given by (in the continuous case):

$$\underline{\gamma} = [\underline{\alpha}, \alpha_{m+1}\underline{\beta}]$$

$$L = \begin{vmatrix} T & T^oB^o \\ \underline{O} & S \end{vmatrix}$$

**Theorem2.2.4:(Neuts, [17], p.53)** A finite mixture of PH-distributions is a PH-distribution. If $(p_1, ..., p_k)$ is the mixing density and $F_j(.)$ has the representation $[\underline{\alpha}(j), T(j)]$, 1$\leq j \leq k$, then the mixture has the representation $\underline{\alpha} = [p_1\underline{\alpha}(1), p_2\underline{\alpha}(2), ..., p_k\underline{\alpha}(k)]$, and

$$T = \begin{vmatrix} T(1) & 0 & & 0 \\ 0 & T(2) & \cdots & 0 \\ & & & \\ . & . & & . \\ 0 & 0 & \cdots & T(k) \end{vmatrix}$$

Infinite mixtures of PH-distributions are generally not of phase-type. The following theorem gives an important and useful exception, for which the concept of the *Kronecker product* of matrices should be introduced.

**Definition:(Neuts, [17], p.53)** If L and M are rectangular matrices of dimension $k_1$ x $k_2$ and $k_1'$ x $k_2'$, their Kronecker product $L \otimes M$ is the matrix of dimensions $k_1 k_1'$ x $k_2 k_2'$, written in block-partitioned form as

$$\begin{vmatrix} L_{11}M & L_{12}M & & L_{1k_2}M \\ \cdot & & & \\ L_{k_1 1}M & L_{k_1 2}M & \dots & L_{k_1 k_2}M \end{vmatrix}$$

Product property: If L, M, U and V are rectangular matrices such that the ordinary matrix products LU and MV are defined, then $(L \otimes M)(U \otimes V) = LU \otimes MV$

**Theorem2.2.5:(Neuts, [17], p.53)** Let $\{s_v\}$ be a discrete PH-density with representation $(\underline{\beta}, S)$ of order n, and F(.) a continuous PH-distribution with representation $(\underline{\alpha}, T)$ of order $m$, then the mixture $\sum_{v=0}^{\infty} s_v . F^v(.)$, of the successive convolutions of F(.), is of phase type with representations $(\underline{\gamma}, L)$ of order mn, given by

$$\underline{\gamma} = \underline{\alpha} \otimes \underline{\beta}(I - \alpha_{m+1} S)^{-1}$$
$$L = T \otimes I + (1 - \alpha_{m+1})\, T^o A^o \otimes (I - \alpha_{m+1} + S)^{-1} S$$
$$\gamma_{mn+1} = \beta_{n+1} + \alpha_{m+1} \underline{\beta}(I - \alpha_{m+1} S)^{-1} S^o$$
$$L^o = T^o \otimes (I - \alpha_{m+1} S)^{-1} S^o$$

The following theorem gives the corresponding result of the theorem2.2.5 when F(.) is a discrete PH-distribution.

**Theorem2.2.6:(Neuts, [17], p.56)** Let $\{s_v\}$ and $\{p_k\}$ be discrete PH-densities with representations of $(\underline{\beta}, S)$ and $(\underline{\alpha}, T)$ of orders $n$ and $m$ respectively. $\sum_{v=0}^{\infty} s_v.\{p_k\}^{(v)}$ is of phase type with representation $(\underline{\gamma}, L)$ of order $mn$, given by,

$$\underline{\gamma} = \underline{\alpha} \otimes \underline{\beta}(I - \alpha_{m+1}S)^{-1}$$
$$L = T \otimes I + (1 - \alpha_{m+1})T^{o}A^{o} \otimes (I - \alpha_{m+1} + S)^{-1}S$$

If X and Y are independent random variables with PH-distributions F(.) and G(.), then the distributions $F_1$(.) =F(.)G(.) and $F_2$(.)= 1 - [1 - F(.)][1 - G(.)], corresponding to max(X,Y) and min (X,Y), are also of phase type. The following theorem provides their phase-type representations:

**Theorem2.2.9:(Neuts, [17], p.60)**: Let F(.) and G(.) have representations $(\underline{\alpha}, T)$ and $(\underline{\beta}, S)$ of orders $m$ and $n$ respectively, then $F_1$(.) has the representation $(\underline{\gamma}, L)$ of order $mn + m + n$, given by $\underline{\gamma} = [\underline{\alpha} \otimes \underline{\beta}, \beta_{n+1}\underline{\alpha}, \alpha_{m+1}\underline{\beta}]$

$$L = \begin{vmatrix} T \otimes I + I \otimes S & I \otimes S^{o} & T^{o} \otimes I \\ 0 & T & 0 \\ 0 & 0 & S \end{vmatrix}$$

and $F_2$(.) has the representation $[\underline{\alpha} \otimes \underline{\beta}, T \otimes I + I \otimes S]$

## 2.2 Special PH-Type Distributions

### 2.2.1 Mixtures of Generalized Erlang (Coxian) Distributions (MGE)

Graphical representation of a MGE distribution is shown in Figure 2.3.

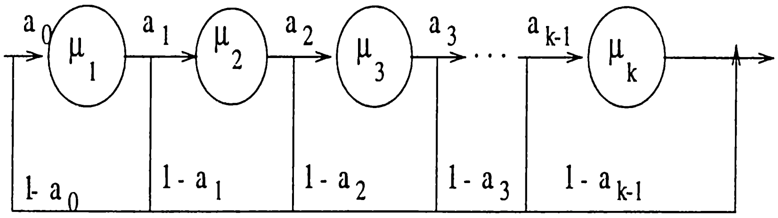


Figure 2.3: Graphical representation of MGE distribution

Holding time in each phase is exponentially distributed with a rate $\mu_i$ in phase $i$. Here, $a_i$ is the conditional probability that the process visits phase $i+1$ given that phase $i$ is completed. This probability $a_o$ is usually taken to be 1. MGE distribution has the following $(\underline{\alpha}, T)$ representation:

$$T = \begin{vmatrix} -\mu_1 & \mu_1 a_1 & & & \\ & -\mu_2 & \mu_2 a_2 & & \\ & & -\mu_3 & \mu_3 a_3 & \\ & & & & \\ & & & & \mu_{k-1} a_{k-1} \\ & & & & -\mu_k \end{vmatrix}, \quad \underline{\alpha} = (1, 0, ..., 0)$$

$T^o = [\mu_1(1-a_1), \mu_2(1-a_2), ..., \mu_k]^T$. For k=2, when $\mu_1 \neq \mu_2$,

$$f_X(x) = c_1 \mu_1 e^{-\mu_1 x} + c_2 \mu_2 e^{-\mu_2 x}, \quad x \geq 0$$

where $c_1 = [\mu_1(1-a_1) - \mu_2]/[\mu_1 - \mu_2], and c_2 = 1 - c_1$ (See Appendix A for calculations.)

A well-known special case of MGE is the Erlang-distribution with the following graphical representation, for which the density corresponds to that of a Gamma density with parameters $k$ and $\mu$.

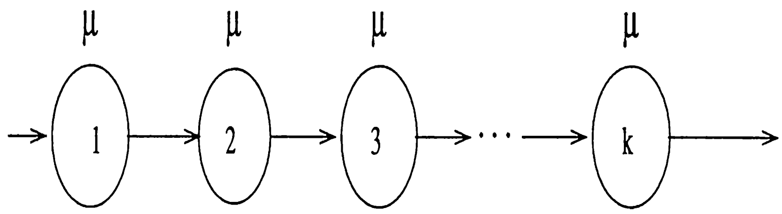


Figure 2.4: Graphical representation of the Erlang distribution

### 2.2.2 Hyperexponential Distribution

A hyperexponential random variable is a proper mixture of exponential random variables with graphical representation shown in Figure 2.5.

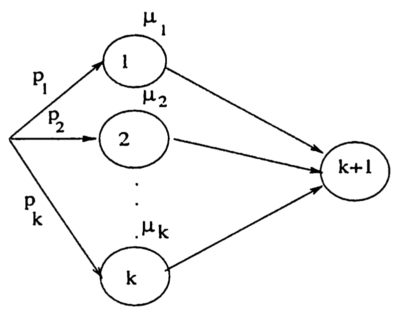


Figure 2.5: Graphical representation of the hyperexponential distribution

The $i^{\text{th}}$ exponential random variable with rate $\mu_i$ is selected with probability $p_i$, $1 \leq i \leq k$. For $x \geq 0$, its density function is given by the following function, where the details can be found in the Appendix A:

$$f_X(x) = \sum_{i=1}^{k} p_i.\mu_i e^{-\mu_i x}$$

Notice that the MGE distribution shown in Figure 2.3 can represent the hyperexponential distribution by taking $a_i = 0$ for all $i$ and $\underline{\alpha} = [p_1, ..., p_k]$.

## 2.3 Equivalence Relations Between Some Classes of PH-Type Distributions

Definition: Two distributions are said to be equivalent if the LST of their density functions are identical.

### 2.3.1 Exponential and Arbitrary Phase-type Distributions

Proposition:(Altıok, [1]) A k-phase phase-type distribution is equivalent to an exponential (obviously a single-phase type distribution) distribution with mean $\gamma^{-1}$ provided that the transition rate from every phase to phase k+1 (absorbing phase) is $\gamma$. No restriction is imposed on the structure of the phase-type distribution.

Corollary1:(Altıok, [1]) A k-phase MGE distribution is equivalent to an exponential distribution with a mean $\gamma^{-1}$ provided that the rate into state $k+1$ from any state is $\gamma$.

Corollary2:(Altıok, [1]) In a trivial case, a hyperexponential distribution is equivalent to an exponential distribution with a mean $\gamma^{-1}$, if all the phases have the same mean $\gamma^{-1}$.

### 2.3.2 Hyperexponential and MGE Distributions

An MGE equivalent will be found of a given k-phase hyperexponential random variable using the transform techniques. We assume that both hyperexponential and the MGE distributions have the same number (k) of phases. First we are going to find an MGE equivalent to a given hyperexponential distribution. For the MGE distribution, $\mu_i$ is the rate of the $i^{th}$ exponential phase and $a_i$ gives the conditional probability that the process visits phase $i+1$ given that phase $i$ is completed, i=1,...,k. For the hyperexponential distribution, $i^{\text{th}}$ exponential random variable having a rate $\lambda_i$ is chosen with probability $p_i$, $1 \leq i \leq k$. In order to achieve a better insight, before stating the conditions when the equivalent MGE can be found for the given hyperexponential random variable, a mathematical procedure will be shown.

Let the LST of hyperexponential density function be,

$$H^*(s) = \frac{N_h(s)}{D_h(s)}$$

where

$$N_h(s) = \sum_{i=1}^{k} \lambda_i p_i \prod_{j=1, j \neq i}^{k} (s + \lambda_j)$$

and

$$D_h(s) = \prod_{i=1}^{k} (s + \lambda_i)$$

Let the LST of the MGE density function be,

$$C^*(s) = \frac{N_c(s)}{D_c(s)}$$

where

$$N_c(s) = \sum_{i=1}^{k} (1 - a_i) \mu_i \prod_{l=1}^{i-1} a_l \mu_l \sum_{j=i+1}^{k} (s + \mu_j)$$

and

$$\prod_{l=1}^{0} a_l \mu_l = 1, \prod_{j=k+1}^{k} (s + \mu_j) = 1, a_k = 0$$

and

$$D_c(s) = \prod_{i=1}^{k} (s + \mu_i)$$

Our assumption which forces both distributions to have the same number of phases enable us to equate the polynomials. As stated previously, in order that two distributions are equivalent, their LST's must be equal. One way to achieve this is to equate denominators and numerators by matching the coefficients of the corresponding terms. The fact that there is a one-to-one correspondence in $D_h(s)$ and $D_c(s)$ in terms involving $s^n, n = 0, ..., k$ necessitates $\mu_i = \lambda_i$. So the denominators of the two LST's become the same. Now the $a_i$s in the MGE distribution need to be identified. This can be done by equating the coefficients of the corresponding terms in the numerators of the two LSTs. Let,

$$N_h(s) = \sum_{i=0}^{k-1} c_i.s^i$$

and

$$N_c(s) = \sum_{i=0}^{k-1} c_i'.s^i$$

Then, $a_i$s will be found by solving the set of $k-1$ nonlinear equations;

$$c_1 = c_1'$$

$$c_{k-1} = c_{k-1}' \tag{2}$$

For a given k-phase hyperexponential distribution with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k$, there always exists a unique equivalent MGE distributions with $\mu_i = \lambda_i$, for which $a_i$, for $i < k$ will be found solving the $k-1$ nonlinear equations. Now suppose a k-phase MGE distribution with $\mu = (\mu_1, ..., \mu_k)$ and $a = (a_1, ..., a_{k-1})$ is given and a hyperexponential equivalent is sought (with $\lambda_i = \mu_i$ for all $i$). The $p_i$'s will be found from (2) coupled with the equation $\sum_{i=1}^{k} p_i = 1$. This can happen only if the $c_v^2$ (squared coefficient of variation) of the MGE distribution is greater than or equal to 1 because $c_v^2$ of hyperexponential is always greater than 1.0. This equivalent hyperexponential is unique.

## 2.4 Moment Approximations

In this section, the issue of fitting MGE distributions using the method of moments will be summarized. Since the LST of any distribution function can be approximated arbitrarily closely by a rational function, in principle, phase-type distributions may be used to approximate any general distribution. (For convenience, $c$ will denote $c_v^2$ from now on.) It is known that under certain regularity conditions, two distributions coincide if and only if all of their moments coincide. Therefore, in a phase-type approximation, it is desirable to equate as many moments of the phase-type distribution as possible with those of a given general distribution. However, including large number of moments makes the process of characterizing the approximation of phase-type distribution difficult. Therefore usually the first three moments are used for approximation purposes. But it must be noted that the use of the third moment may not always result in an improvement over the use of two moments.

### 2.4.1 Three-moment approximations ($c > 1$)

Altıok [2] suggested a three-moment approximate representation of general distributions. For practical purposes, distributions are distinguished by dividing the range of the squared coefficient of variation into two: $c < 1$ and $c \geq 1$. According to the existing empirical results, it does not seem necessary to include the third moment if $c < 1$. Therefore, the main concern will be the general distributions with $c > 1$, and a set of expressions for their two-phase approximation MGE representations will be developed. The LST of the probability density function of a two stage MGE distribution (with $a_0 = 1$ and a is used in place of $a_1$) is given by;

$$L^*(s) = \frac{s\mu_1(1-a) + \mu_1\mu_2}{s^2 + s(\mu_1 + \mu_2) + \mu_1\mu_2}$$

The first three moments can be found by taking successive derivatives of $L^*(s)$, where the set of moments of the original distribution that will be approximated is denoted by $(m_1, m_2, m_3)$ and the three unknown parameters of the two-stage MGE distribution that will be identified are $(\mu_1, \mu_2, a)$.

$$m_1 = \frac{1}{\mu_1} + \frac{a}{\mu_2} \tag{3}$$

which implies

$$a = \frac{\mu_2(m_1\mu_1 - 1)}{\mu_1} \tag{4}$$

$$m_2 = \frac{2(1-a)}{\mu_1^2} - \frac{[2a\mu_1\mu_2 - 2a(\mu_1+\mu_2)^2]}{\mu_1^2\mu_2^2} \tag{5}$$

$$m_3 = \frac{6(1-a)}{\mu_1^3} - \frac{[12a\mu_1\mu_2(\mu_1+\mu_2) - 6a(\mu_1+\mu_2)^3]}{\mu_1^3\mu_2^3} \tag{6}$$

Substituting the unknowns $(\mu_1, \mu_2, a)$ into known moments $(m_1, m_2, m_2)$ from the original distribution, we obtain

$$2m_1(\overbrace{\mu_1+\mu_2}^{X}) - m_2\overbrace{\mu_1\mu_2}^{Y} = 2 \text{ and } 6m_1(\mu_1+\mu_2)^2 - 6m_1(\mu_1\mu_2) - 6(\mu_1+\mu_2) - m_3\mu_1^2\mu_2^2 = 0$$

Rewriting the equations in terms of X and Y results in

$$Y = \frac{(6m_1 - 3m_2/m_1)}{[(6m_2^2/4m_1) - m_3]} \tag{7}$$

$$X = \frac{1}{m_1} + \frac{m_2 Y}{2m_1} \tag{8}$$

Which implies

$$\mu_1 = (X + \sqrt{X^2 - 4Y})/2 \tag{9}$$

and

$$\mu_2 = X - \mu_1 \tag{10}$$

The positive root is taken as $\mu_1$ so that $\mu_1 > \mu_2$ will hold for $c > 1$. For the resulting two-stage MGE distribution to be legitimate, $\mu_1$ and $\mu_2$ must be positive and real, and $a$ must be between 0 and 1. For $\mu_1$ and $\mu_2$ to be positive and real, $Y$ and $X$ should satisfy $Y > 0$ and $X^2 \geq 4Y$. If Y is positive, X is always positive, since $m_1, m_2, m_3$ are positive numbers and the following condition must hold:

$$3m_2^2 < 2m_1m_3 \tag{11}$$

The squared coefficient of variation $c$ is $c = (m_2/m_1^2) - 1$, which implies

$$m_3/m_1^3 > \frac{3}{2}(c+1)^2 \tag{12}$$

Now let us analyze the second necessary condition, namely $X^2 \geq 4Y$. From (8), we have:

$$X^2 = \frac{1}{m_1^2} + \frac{m_2 Y}{m_1^2} + \frac{m_2^2 Y^2}{4m_1^2} \tag{13}$$

In order that this expression is greater or equal to 4Y the following inequality should hold,

$$g(Y) = \frac{1}{m_1^2} + \frac{m_1(c+1)Y^2}{4} - (3-c)Y \geq 0, \tag{14}$$

where g(Y) is convex in Y and its minimum is attained at

$$Y^* = \frac{(6-2c)}{m_1^2(c+1)^2} \tag{15}$$

By inserting (15) into (14) we derive

$$\frac{[(c+1)^2 - (3-c)^2]}{m_1^2(c+1)^2} \geq 0 \tag{16}$$

For (16) to hold we arrive at the condition that $c > 1$. So, we have shown that $\mu_1$ and $\mu_2$ are real and $\mu_1 > \mu_2 > 0$.

Now we are going to analyze possible values of $a$. Using (3) and (5), we can write

$$\mu_1 = (\frac{2}{m_1})(\frac{1}{2-a-aD}) \tag{17}$$

$$\mu_2 = (\frac{2}{m_1})(\frac{1}{1+D}) \tag{18}$$

where, $D = \sqrt{1 + (2/a)(c-1)}$.

For $\mu_1 > \mu_2$ the following inequality is obtained

$$a > \frac{1-D}{1+D} \tag{19}$$

In order that (19) should hold, $a > 0$ must be satisfied which is also a necessary and sufficient condition for $\mu_1 > \mu_2$. From (9) and (10) we know that $\mu_1 > \mu_2$. Hence $a > 0$ in (17) and $\mu_1$ is always positive if

$$a < \frac{2}{1+D} \tag{20}$$

Since $a > 0$, $D$ is always greater than 1.0 and therefore $a$ <1. Clearly, (20) is a tighter bound than 1.0. As $D$ is always greater than 1, (19) gives a negative lower bound, therefore practically the lower bound for a is zero. Hence, we reach the conclusion that the resulting unique two-stage phase-type distribution is legitimate.

The discussion presented above states that (12) is necessary and sufficient to approximate a distribution with known first three moments and with $c >$ 1.0. If (12) is not satisfied either a three-moment approximation is used at the expense of adding more phases in the MGE distribution or one can choose the nearest acceptable third moment to the original moment or a two-moment approximation can be resorted.

### 2.4.2 Two moment approximation

The case with $c > 1$

If first two moments are available, the two-phase MGE distribution can always be found (Marie, [13]). Given that the mean is $m_1$ and the squared coefficient of variation c is known, the parameters are found by the following formulae;

$$\begin{aligned} \mu_1 &= \frac{2}{m_1} \\ \mu_2 &= \frac{c}{m_1} \\ a &= 0.5c \end{aligned}$$

For a two-moment approximation, a hyperexponential distribution with two-phases and a weighted mixing distribution for the branching probabilities in terms of phase rates $\mu_1$ and $\mu_2$ is also possible to be found as shown in Figure 2.6. In this case, the equations

$$\begin{aligned} \mu_1 + \mu_1 &= \frac{2}{m_1} \\ \mu_1 \mu_2 &= \frac{2}{m_2} \end{aligned}$$

have always a solution for $c \geq 1$

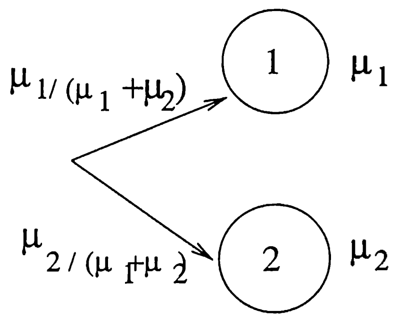


Figure 2.6: Weighted Hyperexponential distribution

The case with $c < 1$

For the case $c < 1$ Erlang distribution is proposed and used as an approximation. (See eg. Sauer and Chandy, [26], Gürler and Parlar, [9] ). The number of stages, k, should satisfy $1/k \leq c \leq 1/k - 1$. Once k is determined, a Generalized Erlang distribution with $a$ and $\mu$ can be found by,

$$\begin{aligned} 1 - a &= \frac{2kc + k - 2 - \sqrt{k^2 + 4 - 4c}}{2(c+1)(k-1)} \\ \mu &= \frac{[1 + (k-1)a]}{m_1} \end{aligned}$$

For 0.5$\leq c < 1$, $\mu_1 = \frac{c}{m_1}$, $\mu_2 = \frac{2}{m_1}$, and $a = 2(1-c)$, (Marie, [13]).

# Chapter 3

# The Model and Notations

In this chapter, we consider a continuous-review stochastic inventory problem with constant demand and random lead times where the supply is subject to random disruptions. It is assumed that the disruptions of the supplier follow an ON/OFF sequence. When the supplier is ON, the $(q, r)$ policy of Hadley and Whitin [10] is used, i.e., when IP hits the reorder point $r$, $q$ units are ordered and the target value $R = q + r$ is reached. Here IP is the amount on hand plus on order minus back orders. When the supplier becomes unavailable (OFF), the policy changes so that one orders enough to bring IP to the target level $R$ as soon as the supplier becomes available again. As a consequence, the order quantity becomes a random variable in the model.

The supplier ON/OFF status is modeled as a semi-Markov process and the regenerative cycles are defined in the following way: Every time the IP reaches $R = r + q$ right after the completion of an OFF period, the regenerative cycle starts. We split the regenerative cycles to a random number of sub-cycles which start when the IP is raised to $R$ during the ON period of the supplier. Let $\check{\mathrm{N}}(\mathrm{q})$ be the number of such sub-cycles which are identical except the last one.

This model is similar to that of Parlar [19] where he assumes that ON periods follow Erlang distribution and OFF periods are general. As an extension of his work, we assume in this study that the ON period is distributed

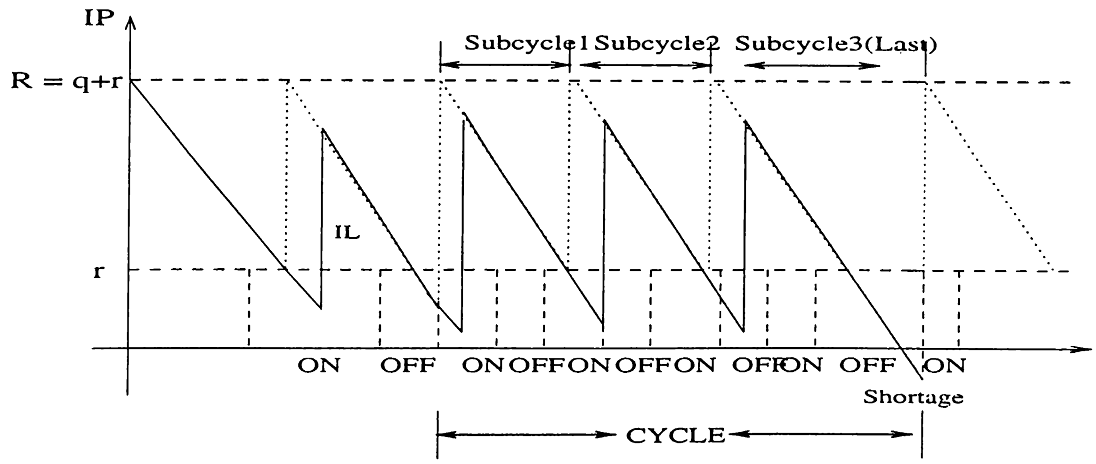


Figure 3.1: Regenerative Cycles of the Inventory Position

with k-stage phase-type distribution (Figure 3.2). This extension is motivated by the approximating properties of Phase-type distributions as reviewed in Chapter 2. The situation can be interpreted as follows: When the period is ON and inventory drops to $r$, it can be at any stage $j$, $j = 1, ..., k$. Time to stay in stage $j$ is exponentially distributed with rate $\mu_j$. Here $c_{0,j}$ is the probability that $j^{th}$ stage will be the initial stage of an ON period after an OFF period. Upon departure from stage j, either the ON period continues with stage i with probability $c_{j,i}$ or the ON period finishes and an OFF period starts with probability $c_{j,0}$. The branching probabilities satisfy,

$$c_{0,1} + ... + c_{0,k} = 1 \quad \text{and} \quad c_{j,1} + ... + c_{j,k} + c_{j,0} = 1$$

The OFF period which is assumed to follow a general distribution is denoted by O.

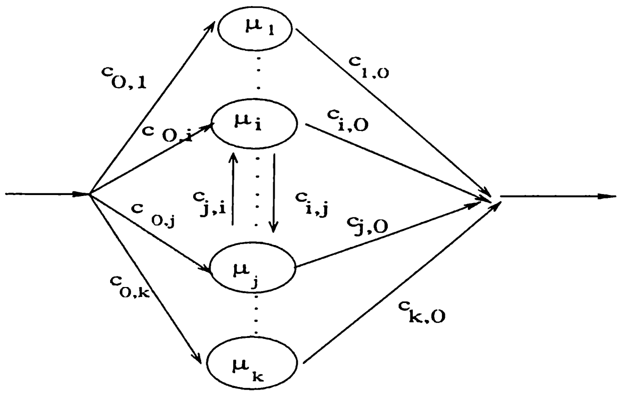


Figure 3.2: Phase-type distribution with k phases.

## 3.1 The Semi-Markov Processes and The Objective Function

Let $\{\zeta(t),\ t \geq 0\}$ be the semi-Markov process representing ON/OFF status of the supplier such that $\zeta(t) = 0$ corresponds to the OFF period and $\zeta(t) = j$ indicates that the supplier is at the $j^{\text{th}}$ stage of the ON period. Note that the duration of stay in any state $j$, $j = 1, ..., k$ is exponential. We define $P_{ij}(t) = P\{\zeta(t) = j \mid \zeta(0) = i\}$, i,j = 1,2,...,k,0 as the transition functions of the SMP.

Our aim now is to find the $(q, r)$ values which minimize the long run average cost function. The representation in terms of a semi-Markov process allows us to use the renewal reward theorem (Ross, [25]) which states that the long run average cost function is the ratio of the expected cycle cost to expected cycle length. We first consider below the cycle length.

### 3.1.1 Cycle Length

The Theorem of Parlar [19] below is useful in our case to find the expected value of the cycle length therefore we present it together with its proof for the sake of completeness. Let $\tilde{T}_i$ be the time required to complete the cycle if the process is at state i, i =1,...,k, and $T_i$ denote the expectation of $\tilde{T}_i$ and $\tilde{T}_0$ be the 'waiting time' for the supplier to return to the ON state. For any vector

V, let $V^T$ denote its transpose. We denote the constant demand rate with D.

**Theorem 1.** **(Parlar, [19])** The $T_i$, i = 1,...,k, values are obtained from the solution of the linear system,

$$(I - P)T = b \tag{1}$$

where I is the identity matrix, and

$$\begin{aligned} P &= [P_{ij}(q/D)], \ i,j = 1,\dots,k, \\ T^T &= [T_1,\dots,T_k], \\ b^T &= [q/D + T_0P_{10}(q/D),\dots,q/D + T_0P_{k0}(q/D)]. \end{aligned}$$

**Proof.** Conditioning on the state found when the inventory reaches $r$ after q/D time units and using the renewal argument, we obtain, for i =1,...,k

$$\tilde{T}_i = \begin{cases} q/D + \tilde{T}_i & \text{if } \zeta(q/D) = i \\ q/D + \tilde{T}_j & \text{if } \zeta(q/D) = j, j = 1,\dots,k, j \neq i \\ q/D + \tilde{T}_0 & \text{if } \zeta(q/D) = 0 \end{cases} \tag{2}$$

Taking expectations in 2, we have

$$T_i = \sum_{j=1}^{k}[q/D + T_j]P_{ij}(q/D) + [q/D + T_0]P_{i0}(q/D), i = 1,\dots,k \tag{3}$$

Collecting terms containing the unknowns on the left-hand side gives for i=1,...,k,

$$T_i(1 - P_{ii}(q/D)) - \sum_{j=1,j\neq i}^{k} T_jP_{ij}(q/D) = q/D + T_0P_{i0}(q/D), \tag{4}$$

Writing 4 using matrix notation, we obtain (I - P)T = b. □

**Remark3.1.** The results can easily be modified for the case when the demand is random. In particular equation 4 would be replaced by,

$$T_i(1 - E[P_{ii}(q/D)]) - \sum_{j=1,j\neq i}^{k} T_jE[P_{ij}(q/D)] = E[q/D] + T_0E[P_{i0}(q/D)]$$

and the rest of the modification would be obvious. However to keep the presentation simple we take the demand as constant.

**Remark3.2.** Note that in Parlar's model, every time a regenerative cycle starts, the process is in the first stage of the ON period. But in our case the first stage the cycle starts is random. We therefore need the following proposition.

**Proposition1.** Expected cycle length T(q) is,

$$T(q) = [c_{0,1}, \ldots., c_{0,k}](I - P)^{-1}b$$

**Proof.** When Figure 3.2 is examined it is seen that a cycle may start with any stage i, i =1,...,k with probability $c_{0,i}$. Therefore the cycle length T(q) is found by summing the products of the initial state probabilities with the corresponding expected cycle lengths. □

### 3.1.2 Cycle Cost

A complete cycle consists of a random number of sub-cycles of length q/D and a final one of a longer length due to the 'waiting' time until the supplier becomes ON again.

Let K be the ordering cost per order, h be the holding cost per unit per unit time and b be the backorder cost per unit which are same for all sub-cycles. Other than the ordering cost, the expected inventory (or average inventory) carrying cost should be calculated. By definition, the net inventory is the on hand inventory minus the backorders. Then the expected on hand inventory is equal to the expected net inventory plus the expected number of backorders. The cost incurred in the shorter sub-cycles is the cost of a cycle in the standard Hadley/Whitin $(q, r)$ model, (i.e., a 'standard cycle'). While computing the inventory holding cost, we will use the assumption made by Hadley and Whitin that the expected number of backorders is negligible.

Let $\tilde{c}(q,r)$ be the cost of an arbitrary sub-cycle before the last one with $c(q,r)=E[\tilde{c}(q,r)]$. Besides the ordering cost, the other components of this cost are as follows:

i)Holding Cost:

Let L be the (possibly random) lead-time with a p.d.f of $g_L(l)$ and Z be the demand during L. By definition, safety stock $S_Z$ is the expected value of the inventory level (IL, which is inventory on hand minus the backorders) just before an order arrives, i.e, $S_Z = E[IL(Z, r)]$
where IL(Z,r) = r - Z = r - LD. Then

$$S_Z = E[IL(Z, r)] = \int_0^\infty (r - z)k_z(z)dz = r - \phi$$

where

$$k_z(z) = \frac{1}{D} g_L(z/D) \tag{5}$$

and $\phi = \int_0^\infty z k_z dz = E[Z]$ is the expected demand during L. The expected net inventory immediately after the delivery of the ordered q units is $q + S_Z$. Then at the start of these cycles, the net inventory will be $q + S_Z$ and at the end of the cycle it will be just $S_Z$. These will be also the expected values of the on hand inventory as we assume that the expected number of back orders can be neglected. We derive that the average inventory during a cycle is

$$\frac{1}{2}q + S_Z$$

In order to find the total expected inventory during a cycle, we should multiply the average inventory with the cycle length q/D, which gives

$$\frac{1}{2}q^2/D + S_Z q/D.$$

Then,

$$E[\text{holding cost}] = h[\frac{1}{2}q^2/D + (r - \phi)q/D] \tag{6}$$

ii) Backorder Cost:

Actual number of back orders $\eta_L(Z,r)$ during a standard cycle depends on the reorder point r and the demand Z during the lead time L. That is, $\eta_L(z,r) = I(z \geq r)(z-r)$ where $I(z \geq r)$ is the indicator function. Then, letting $\bar{\eta}_L(r) = \int_r^\infty (z-r)k_z(z)dz$ be the expected number of back orders during a standard cycle, we have

$$E[\text{back order cost}] = b\bar{\eta}_L(r) \quad (7)$$

Combining all the individual expected costs, i.e., ordering cost K, (6) and (7) will give us the expected cost per standard cycle as,

$$c(q,r) = K + h[\frac{1}{2}q^2/D + (r-\phi)q/D] + b\bar{\eta}_L(r)$$

We now consider the cost of the last sub-cycle. Suppose $\tilde{C}(q,r)$ is the cost of the last sub-cycle in the model and let $C(q,r) = E[\tilde{C}(q,r)]$. Besides the ordering cost, the other components of this cost are as follows:

iii) Holding cost:

Let $\psi = \tilde{T}_0 + L$ be the length of the total delay, i.e., the lead time and the 'waiting' time for the supplier to return to the ON state after an order attempt is made. Also let W be the demand during $\tilde{T}_0$. Then $U = W + Z$ be the combined demand during $\psi$. Then the safety stock $S_U$ is the expected value of the inventory level (IL) just before the order arrives when the supplier returns to the ON state, i.e., $S_U = E[IL(U,r)] = r - U$.
If $g_\psi(\tau)$ is the p.d.f of the total delay random variable $\psi$, then

$$S_U = \int_0^\infty (r-u)k_u(u)du = r - \xi,$$

where

$$k_u(u) = \frac{1}{D}g_\psi(u/D) \quad (8)$$

is the marginal density of the demand during the total delay $\psi$ and $\xi = \int_0^\infty uk_u(u)du = E[U]$ is the expected demand during the total delay. It must be noted that in the last sub-cycle the order quantity is a random variable since the actual amount ordered depends on the remaining time the supplier stays in the OFF state. Then let us define $\tilde{Q}(q)$ as the random order quantity so that $\tilde{Q}(q) = q + D\tilde{T}_0$, since an additional $D\tilde{T}_0$ units will be demanded while waiting for the supplier to return to the ON state. Let

$$Q(q) = E[\tilde{Q}(q)] = q + DT_0$$

with $T_0 = E[\tilde{T}_0]$. If $\tilde{T}_l(q) = q/D + \tilde{T}_0$ is the length of the last sub-cycle, then

$$T_l(q) = E[\tilde{T}_l(q)] = q/D + T_0$$

Therefore,

$$E[\text{holding cost}] = h[\frac{1}{2}Q(q)T_l(q) + (r - \xi)T_l(q)] \tag{9}$$

iv) Backorder cost:

Number of back orders $\eta_\psi(U,r)$ during the last sub-cycle depends on the reorder point r, and the demand U during the total delay $\psi$,i.e, $\eta_\psi(u,r) = I(u \geq r)(u - r)$ where $I(u \geq r)$ is the indicator function. Then, letting $\bar{\eta}_\psi(r) = \int_r^\infty (u - r)k_u(u)du$ is the expected number of back orders during the last sub-cycle

$$E[\text{back order cost}] = b\bar{\eta}_\psi(r). \tag{10}$$

Combining all the individual expected costs, i.e., ordering cost K, (9) and (10) will give us the expected cost for the last sub-cycle as:

$$C(q,r) = K + h[\frac{1}{2}Q(q)T_l(q) + (r - \xi)T_l(q)] + b\bar{\eta}_\psi(r)$$

If we define $\tilde{N}(q)$ as the total number of sub-cycles in a cycle then the random cycle cost $\tilde{\mathcal{C}}(q,r)$ will be,

$$\tilde{\mathcal{C}}(q,r) = \sum_{i=1}^{\tilde{N}(q)-1} \tilde{c}_i(q,r) + \tilde{C}(q,r) \tag{11}$$

where $\tilde{c}_i(q,r)$, i =1,..., $\tilde{N}(q)$ - 1, is the cycle cost of a sub-cycle before the last one, $\tilde{C}(q,r)$ is the cost of the last sub-cycle.

### 3.1.3 Computation of $E[\tilde{N}(q)]$

Let $\tilde{N}_i$ be the number of sub-cycles required to complete the cycle if the process is at state i and $N_i = E[\tilde{N}_i]$ be its expected value for i = 1,...,k. We again refer to the following result of Parlar. The proof is skipped since it is similar to that of Theorem 1.

**Theorem 2.** **(Parlar, [19])** The values of $N_i$, i=1,...,k are obtained from the solution of the system,

$$(I - P)N = e$$

where $N^T = [N_1,...,N_k]$ and $e^T = [1,...,1]$.

Due to the same reason indicated in Remark3.2. we need the following result.

**Proposition2.**Expected number of sub-cycles in a cycle $N(q)=E[\tilde{N}(q)]$ is,

$$N(q) = [c_{0,1}, ...., c_{0,k}](I - P)^{-1}e$$

where $c_{0,i}$ gives the probability of starting with state i to an ON period whose distribution is Phase-type.

**Proof.** When Figure 3.2 is examined it is seen that a cycle may start with any stage i, i =1,...,k with probability $c_{0,i}$. Therefore the number of sub-cycles in a cycle, N(q) is found summing the products of the initial state probabilities with the corresponding expected number of cycles.□

Taking expectations of $\tilde{\mathcal{C}}(q,r)$ in 11 and noting that $\tilde{N}(q)$ - 1 is a stopping time for $\tilde{c}_i(q,r)$, we can use Wald's equation (Ross, [25]) and write

$$E[\tilde{\mathcal{C}}(q,r)] = \mathcal{C}(q,r) = E[\tilde{N}(q) - 1]E[\tilde{c}_i(q,r)] + E[\tilde{C}(q,r)]$$

$$= n(q)c(q,r) + C(q,r)$$

as the expected cycle cost where $n(\mathrm{q}) = \mathrm{N(q)}$ -1.
We can now construct the objective function $\mathcal{K}(\mathrm{q},\mathrm{r})$ as follows:

$$\mathcal{K}(\mathrm{q},\mathrm{r}) = \frac{\mathrm{E}[\tilde{\mathcal{C}}(\mathrm{q},\mathrm{r})]}{\mathrm{E}[\tilde{\mathrm{T}}(\mathrm{q})]} = \frac{n(\mathrm{q})\mathrm{c}(\mathrm{q},\mathrm{r}) + \mathrm{C}(\mathrm{q},\mathrm{r})}{\mathrm{T}(\mathrm{q})} \tag{12}$$

## 3.2 Integral Equations of The Transition Probabilities

In order to find the expected cycle length, T(q) and expected number of subcycles in a cycle, N(q), we need to invert the (I - P) matrix where $\mathrm{P} = [\mathrm{P_{ij}(q/D)}]$, i,j = 1,...,k. Therefore we need to identify $\mathrm{P_{ij}(t)}$ of the SMP $\{\zeta(\mathrm{t}), \mathrm{t} \geq 0\}$ where $\mathrm{P_{ij}(t)} = \mathrm{P}\{\zeta(\mathrm{t}) = \mathrm{j} \mid \zeta(0) = \mathrm{i}\}$.
Let $\mathrm{F_i} = 1 - \mathrm{e}^{-\mu_i t}$, $\mathrm{t} \geq 0$ be the cumulative distribution function of the time of stay in state i having rate $\mu_i$ with the density $\mathrm{dF_i(t)/dt} = \mathrm{f_i(t)} = \mu_i \mathrm{e}^{-\mu_i t}$ and let $\bar{\mathrm{F}}_i = 1 - \mathrm{F_i}$, i = 1,...,k.

**Theorem3.** The transition functions $\mathrm{P_{ij}(t)}$ $t \geq 0$ of the semi-Markov process representing the supplier availability are the solutions of the following integral equations:

$$\mathrm{P_{ii}(t)} = \bar{\mathrm{F}}_i(\mathrm{t}) + \sum_{m=1,m\neq i}^{k} \mathrm{c_{i,m}} \int_0^t \mathrm{f_i(x)P_{mi}(t-x)} + \mathrm{c_{i,0}} \int_0^t \mathrm{f_i(x)P_{0i}(t-x)} \tag{13}$$

$$\mathrm{P_{ij}(t)} = \sum_{m=1,m\neq i}^{k} \mathrm{c_{i,m}} \int_0^t \mathrm{f_i(x)P_{mj}(t-x)} + \mathrm{c_{i,0}} \int_0^t \mathrm{f_i(x)P_{0j}(t-x)} \tag{14}$$

$$\mathrm{P_{0j}(t)} = \sum_{m=1}^{k} \mathrm{c_{0,m}} \int_0^t \mathrm{dG(x)P_{mj}(t-x)} \tag{15}$$

**Proof.** For the equation (13), we condition on the state visited at time x of the first transition and add the probability that no transition occurs by time t, i.e., $1-F_i(t)$. For equation (14), note that the probability of a transition out of state i in the time interval $(x, x + dx]$ is $dF_i(x)$. The conditional probability of ending up in state j after $t - x$ time units starting at m is $P_{mj}(t-x)$. The probability of passing from state i to m, $c_{i,m}$ must be multiplied with this integral (see Figure 3.2). The result is obtained by summing over the possible values of m. For (15) a similar idea is used. □

### 3.2.1 Transient Solutions of $P_{ij}(t)$ for special Phase-type distributions

a) For 2-stage Phase-type distribution

There are three stages where 1,2 represent the ON states and 0 represents the OFF state.

**Result1.**The transition functions $P_{ij}(t)$ $t \geq 0$ of the semi-Markov process representing the supplier availability when the ON period is distributed with 2-stage Phase-type distribution, are the solutions of the following integral equations:

$$
\begin{aligned}
P_{11}(t) &= 1 - F_1(t) + c_{12}\int_0^t dF_1(x)P_{21}(t-x) + c_{10}\int_0^t dF_1(x)P_{01}(t-x) \\
P_{22}(t) &= 1 - F_2(t) + c_{21}\int_0^t dF_2(x)P_{12}(t-x) + c_{20}\int_0^t dF_2(x)P_{02}(t-x) \\
P_{12}(t) &= c_{12}\int_0^t dF_1(x)P_{22}(t-x) + c_{10}\int_0^t dF_1(x)P_{02}(t-x) \\
P_{21}(t) &= c_{21}\int_0^t dF_2(x)P_{11}(t-x) + c_{20}\int_0^t dF_2(x)P_{01}(t-x) \\
P_{01}(t) &= c_{01}\int_0^t dG(x)P_{11}(t-x) + c_{0,2}\int_0^t dG(x)P_{21}(t-x) \\
P_{02}(t) &= c_{01}\int_0^t dG(x)P_{12}(t-x) + c_{0,2}\int_0^t dG(x)P_{22}(t-x)
\end{aligned}
$$

b) For k-stage Erlang distribution

Graphical representation of Erlang distribution is given in Figure 3.3. $F_i = A(t) = 1-e^{-\mu t}$, $t \geq 0$ is the cumulative distribution function of the time of stay in state i with rate $\mu$ and the density $dF_i(t)/dt = \mu e^{-\mu t}$, $i = 1,...,k$.
$c_{0,1} = 1$, $c_{0,j} = 0$ for $j = 2,...,k$.
$c_{i,i+1} = 1$, while $c_{i,j} = 0$ for $j \neq i+1$
$c_{k,0} = 1$, while $c_{j,0} = 0$ for $j = 1,...,k-1$.

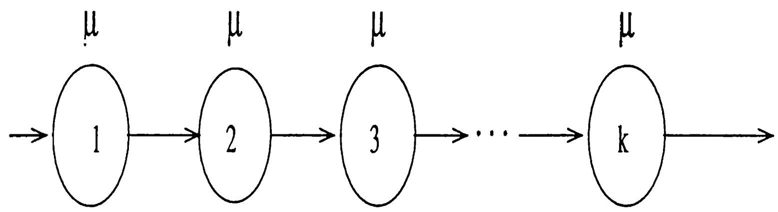


Figure 3.3: Graphical representation of the Erlang distribution

**Result2.** The transition functions $P_{ij}(t)$ $t \geq 0$ of the semi-Markov process representing the supplier availability when the ON period is distributed with k-stage Erlang are the solutions of the following integral equations:

$$\begin{aligned}
P_{ii}(t) &= 1 - A(t) + \int_0^t dA(x)P_{i+1,i}(t-x) \\
P_{ij}(t) &= \int_0^t dA(x)P_{i+1,j}(t-x) \\
P_{0j}(t) &= \int_0^t dG(x)P_{1j}(t-x)
\end{aligned}$$

c) For k-stage Coxian distribution

Graphical representation of k-stage Coxian distribution is shown in Figure 3.4. As a special case of phase-type distribution, the Coxian distribution has the following properties:
$c_{01} = 1$, while $c_{0j} = 0$ for $j = 2,...,k$
$c_{j,j+1} = a_j$, while $c_{j,i} = 0$ for $i \neq j+1$, $j=1,...,k - 1$
$c_{j0} = 1 - a_j$ $j \neq k$, $c_{k0} = 1$
**Result3.** The transition functions $P_{ij}(t)$ $t \geq 0$ of the semi-Markov process representing the supplier availability when the ON period is distributed with

k-stage Coxian distribution are the solutions of the following integral equations:

$$\begin{aligned}
P_{ii}(t) &= 1 - F_i(t) + a_i \int_0^t dF_i(x) P_{i+1,i}(t-x) + (1-a_i) \int_0^t dF_i(x) P_{0,i}(t-x), \\
P_{ij}(t) &= a_i \int_0^t dF_i(x) P_{i+1,j}(t-x) + (1-a_i) \int_0^t dF_i(x) P_{0,j}(t-x) \;\; j \neq i \\
P_{0j}(t) &= \int_0^t dG(x) P_{1j}(t-x)
\end{aligned}$$

i , j = 1,...,k.

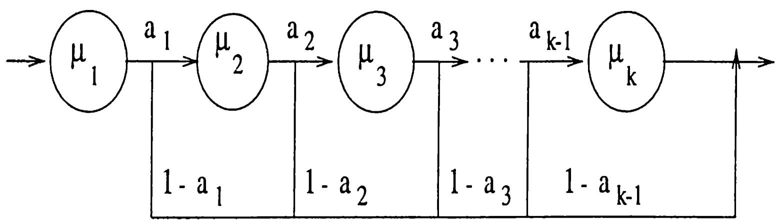


Figure 3.4: Graphical representation of the Coxian distribution

In chapter 4, we are going to describe a detailed numerical example where we compute the transition functions and optimize the inventory model.

# Chapter 4

# Analytical Results

In this chapter, we are going to describe in detail the analysis of a special problem in which the ON periods are distributed with 2-stage Coxian distribution.

## 4.1 Analysis of 2-stage Coxian distribution

There are 3 stages where 1,2 show the ON stages and 0 represents the OFF period. Let $F_i(t) = 1 - e^{-\mu_i t}$, $t \geq 0$ be the cumulative distribution function of the time of stay in state i having rate $\mu_i$ with the density $dF_i(t)/dt = f_i(t) = \mu_i e^{-\mu_i t}$, $i = 1,\ldots,k$. The OFF period has a general distribution with the cumulative distribution function $G(t)$ and density function $dG(t)/dt = g(t)$.

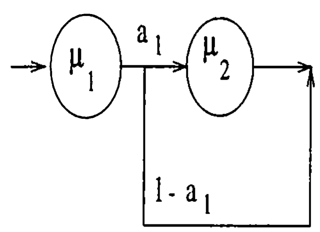


Figure 4.1: 2-Stage Coxian Distribution

The transition functions $P_{ij}(t)$ $t \geq 0$ of the semi-Markov process representing the supplier availability when the ON period is distributed with 2-stage Coxian distribution are the solutions of the following integral equations:

$$\begin{aligned}
P_{11}(t) &= 1 - F_1(t) + a_1 \int_0^t dF_1(x)P_{21}(t-x) + (1-a_1)\int_0^t dF_1(x)P_{0,2}(t-x) \\
P_{22}(t) &= 1 - F_2(t) + \int_0^t dF_2(x)P_{0,2}(t-x) \\
P_{12}(t) &= a_1 \int_0^t dF_1(x)P_{22}(t-x) + (1-a_1)\int_0^t dF_1(x)P_{0,2}(t-x) \\
P_{21}(t) &= \int_0^t dF_2(x)P_{0,1}(t-x) \\
P_{01}(t) &= \int_0^t dG(x)P_{11}(t-x) \\
P_{02}(t) &= \int_0^t dG(x)P_{12}(t-x)
\end{aligned}$$

By a simple change of variable, we can represent these integrals in an equivalent form. For instance, we can write

$$P_{02}(t) = \int_0^t dG(x)P_{12}(t-x) = \int_0^t g(t-u)P_{12}(u)du$$

Writing the other integrals similarly we have these six integral equations written in a matrix format as

$$P(t) = \int_0^t H(t-u)P(u)du + v(t) \tag{1}$$

where

$$P(t) = [P_{01}, P_{02}, P_{11}, P_{12}, P_{21}, P_{22}]^T$$

and

$$H(t) = \begin{vmatrix} 0 & 0 & g(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & g(t) & 0 & 0 \\ 0 & (1-a_1)f_1(t) & 0 & 0 & a_1 f_1(t) & 0 \\ 0 & (1-a_1)f_1(t) & 0 & 0 & 0 & a_1 f_1(t) \\ f_2(t) & 0 & 0 & 0 & 0 & 0 \\ 0 & f_2(t) & 0 & 0 & 0 & 0 \end{vmatrix} \tag{2}$$

finally,

$$v(t) = [0, 0, \bar{F}_1(t), 0, 0, \bar{F}_2(t)]^T$$

The integral equations in (1) are classified as 'Volterra type of second kind' (Linz, [12]). There are several numerical solution methods that can be used to compute these integrals one of which is direct numerical integration. Direct method of solving the integral equations is based on approximating an integral using one of many classical formulae such as, trapezoidal rule, Simpson's rule and Bode's rule (Press *et al.* [24]). For a scalar integral equation such as,

$$P(t) = \bar{F}(t) + \int_0^t f(t,u)P(u)du \tag{3}$$

$P(0) = \bar{F}(0) = 1$ for given $\bar{F}(t)$ and $f(t,u) \equiv f(t - u)$ with the unknown function $P(t)$, $t \geq 0$, the integral is approximated using the trapezoidal rule as,

$$\begin{aligned} \int_0^t f(t,u)P(u)du &\simeq \Delta t[\frac{1}{2}f(t,u_1)P(u_1) + f(t,u_2)P(u_2) \\ &\quad +\ldots + f(t,u_{n-1})P(u_{n-1}) + \frac{1}{2}f(t,u_n)P(u_n)] \end{aligned}$$

where the interval of integration [0,t] is divided into n equal subintervals of length $\Delta t = t/n$; $u_j \leq t$, $j \geq 1$; $u_1 = 0$ and $u_n = t$. [The integration is over u, $0 \leq u \leq t$; therefore for $u > t$, $f(t,u) = 0$.]

The integral equation in (3) can now be approximated by the sum

$$\begin{aligned} P(t) &= \bar{F}(t) + \Delta t[\frac{1}{2}f(t,u_1)P(u_1) + f(t,u_2)P(u_2) \\ &\quad +\ldots + f(t,u_{n-1})P(u_{n-1}) + \frac{1}{2}f(t,u_n)P(u_n)] \end{aligned}$$

If we return to the equation (3) and consider n sample values of $P(t)$, such that $P(u_i) = P_i$, i =1,...,n, equation (3) becomes a a system of n linear equations in n unknowns $P_i$, (Gürler and Parlar, [9]) as,

$P_1 = \bar{F}_1$

$P_2 = \bar{F}_2 + \Delta t[\frac{1}{2}f_{21}P_1 + \frac{1}{2}f_{22}P_2]$

$P_i = \bar{F}_i + \Delta t[\frac{1}{2}f_{i1}P_1 + f_{i2}P_2 + \ldots + f_{i,i-1}P_{i-1} + \frac{1}{2}f_{ii}P_i] \quad i = 2, \ldots, n$

where $\bar{F}_i = \bar{F}(u_i)$, $f_{ij} = f(t_i, u_j)$, $j \leq i$, $u_j \leq t_i$. With this approach, the numerical solution of an integral equation can be reduced to solving a system of n linear equations in n unknowns. Then we can express this linear system as $\tilde{P} = \tilde{H}\check{P} + \tilde{v}$, with the solution, $\tilde{P} = (I - \tilde{H})^{-1}\tilde{v}$, where $\tilde{P} = [P_1, P_2, ..., P_n]^T$ and $\check{v} = [\bar{F}_1, \bar{F}_2, ..., \bar{F}_n]^T$ are nx1 column vectors and

$$\tilde{H} = \Delta t \begin{vmatrix} 0 & 0 & & 0 & 0 \\ \frac{1}{2}f_{21} & \frac{1}{2}f_{22} & & 0 & 0 \\ & & & & \\ . & . & & & . \\ \frac{1}{2}f_{n1} & f_{n2} & \cdots & f_{n,n-1} & \frac{1}{2}f_{nn} \end{vmatrix}$$

is an nxn matrix.

In our case, as we have a system of integral equations in N (N = 6, for 2-stage Coxian distribution) unknowns, the problem becomes a little bit more complicated but still the trapezoidal rule can be applied after dividing the [0,t] interval into n subintervals of equal length. Since there are N unknown functions each of which being sampled at n points, the resulting system now has nN unknowns. So we obtain,

$$\hat{P} = \hat{H}\hat{P} + \hat{v}$$

where

$$\hat{P} = [P_{01}(u_1), ..., P_{01}(u_n)|P_{02}(u_1), ..., P_{02}(u_n)|...|P_{22}(u_1), ..., P_{22}(u_n)]$$

$$\hat{v} = [0, ..., 0|0, ..., 0|\bar{F}_1(u_1), ..., \bar{F}_1(u_n)|...|\bar{F}_2(u_1), ..., \bar{F}_2(u_n)]$$

are nN dimensional column vectors and $\hat{H}$ is a suitably constructed sparse nNxnN matrix whose non-zero sub-matrices are positioned in a manner similar to the non-zero entries in (2). Then $\hat{H}$ will be a matrix of matrices as follows:

$$\begin{vmatrix} H_{6x6} & H_{6x6} & G_{6x6} & H_{6x6} & H_{6x6} & H_{6x6} \\ H_{6x6} & H_{6x6} & H_{6x6} & G_{6x6} & H_{6x6} & H_{6x6} \\ H_{6x6} & (1-a_1)F_{1,6x6} & H_{6x6} & H_{6x6} & a_1F_{1,6x6} & H_{6x6} \\ H_{6x6} & (1-a_1)F_{1,6x6} & H_{6x6} & H_{6x6} & H_{6x6} & a_1F_{1,6x6} \\ F_{2,6x6} & H_{6x6} & H_{6x6} & H_{6x6} & H_{6x6} & H_{6x6} \\ H_{6x6} & F_{2,6x6} & H_{6x6} & H_{6x6} & H_{6x6} & H_{6x6} \end{vmatrix}$$

where $H_{6x6}$ is a 6 x 6 null matrix, and

$$G_{6x6} = \Delta t \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}g_{21} & \frac{1}{2}g_{22} & 0 & 0 & 0 & 0 \\ . & & & & & . \\ \frac{1}{2}g_{n1} & g_{n2} & & & & \frac{1}{2}g_{nn} \end{vmatrix}$$

and for i = 1,2

$$F_{i,6x6} = \Delta t \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}f_{i,21} & \frac{1}{2}f_{i,22} & 0 & 0 & 0 & 0 \\ . & . & & & & . \\ \frac{1}{2}f_{i,n1} & f_{i,n2} & & & & \frac{1}{2}f_{i,nn} \end{vmatrix}$$

where $f_{i,jk} = f_i(t_j, u_k)$ , i=1,2, j,k=1,...,n.

Solving $\hat{P}$ would give us the numerically estimated solution of the transition functions $P_{ij}$ however, since both $n$ and T(q) are not available in closed form, it is not possible to analyze K(q,r) in (13) in Chapter 3.

Gürler and Parlar [9] developed a computer program to overcome a similar problem with two suppliers where the ON periods are distributed with K-stage Erlang. Therefore we adopted the program for our problem and reached out the optimal values for this problem. Also we solved additional problems with several 2-Stage Phase-Type Distributions. The analysis of the results will be carried out in the next chapter.

## 4.2 Analysis of The Model for Large $q$

When the model is analyzed for asymptotically 'large' values of $q$ (compared with D), the transient probabilities can be replaced by the limiting values $P_j \equiv \lim_{t\to\infty} P\{\zeta(t) = j|\zeta(0) = i\}$ in order to simplify the problem. If optimal $q$ is not likely to be large, this approach would provide a solution which is easy to compute but probably poor in approximation.

In order to have large $q$ values, order cost K should be large and/or holding cost h should be small compared to the backorder cost b. When these conditions hold, the time-dependent (transient) conditional probabilities can be replaced by their constant limiting values. In order to find the limiting probabilities, we are going to use the proposition made by Ross.

Let $T_{ii}$ denote the time between successive transitions into state i and let $\mu_{ii} = E[T_{ii}]$ and $\mu_j$ be the mean of the $j^{th}$ stage which is exponentially distributed.

**Proposition 4.8.1. (Ross [25], p. 131)**
If the semi-Markov process is irreducible and $T_{ii}$ has a non-lattice distribution with finite mean, then

$$P_j \equiv \lim_{t\to\infty} P\{\zeta(t) = j | \zeta(0) = i\}$$

exists and is independent of the initial state. Furthermore,

$$P_j = \frac{\mu_j}{\mu_{jj}}$$

Now, we are going to compute these limiting values for Phase-type, Coxian and Erlang Distributions. Note that a random variable Y will denote the durations of absorbing states of all these distributions which is assumed to be a general distributions with E[Y] as its expected value.

### 4.2.1 The Limiting Probabilities of Phase-Type Distributions

**Lemma1.** For k-stage Phase-type distributions for j = 1,...,k,

$$\mu_{jj} = \mu_j + \sum_{i=0, i\neq j}^{k} c_{ji} T_{ij}$$

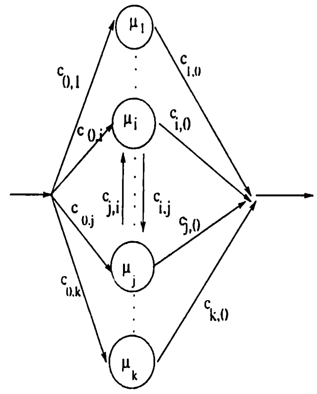


Figure 4.2: k-Stage Phase-Type Distribution

where $T_{ij}$ values are obtained from the solution of the systems

$$T_j = (I - A_j)^{-1} B_j$$

with $T_j = [T_{0j}, T_{1j}, \ldots, T_{ij}, \ldots, T_{kj}]$ $i \neq j$ $i=0,\ldots,k$

$$A_j = \begin{vmatrix} a_{0j} & a_{1j} & a_{2j} & & a_{kj} \\ c_{10} & 0 & c_{12} & & c_{1k} \\ \ldots & & c_{il} & & \ldots \\ c_{k0} & c_{k1} & \ldots & c_{k,k-1} & 0 \end{vmatrix}$$

$B_j^T = [b_{0j}, \mu_1, \ldots, \mu_i, \ldots]$ $i \neq j$ $i=1,\ldots,k$

$$b_{0j} = E[Y] + \sum_{i=1, i\neq j}^{k} c_{0i}\mu_i$$

$$a_{ij} = \sum_{i=0, i\neq j}^{k} \sum_{l=1, l\neq i, l\neq j}^{k} c_{0l}c_{li}$$

**Proof.** If we examine Figure 4.2 we can write,

$$\mu_{11} \quad = \quad \mu_1 + c_{10}T_{01} + c_{12}T_{21} + \ldots + c_{1k}T_{k1}$$

$$\mu_{jj} = \mu_j + c_{j0}T_{0j} + c_{j1}T_{1j} + \ldots + c_{j,j-1}T_{j-1,j} + c_{j,j+1}T_{j+1,j} + \ldots + c_{j,k}T_{k,j} \qquad (4)$$

which is

$$\mu_{jj} = \mu_j + \sum_{i=0,i\neq j}^{k} c_{ji} T_{ij}$$

for j = 1,...,k.

$$\begin{aligned}
T_{0j} &= c_{0j}E[Y] + c_{01}\{E[Y] + \mu_1 + c_{10}T_{0j} + \ldots + c_{1,j-1}T_{j-1,j} + c_{1,j+1}T_{j+1,j} \\
&+ \ldots + c_{1k}T_{kj}\} + \ldots \\
&+ c_{0,j-1}\{E[Y] + \mu_{j-1} + c_{j-1,0}T_{0j} + \ldots + c_{j-1,k}T_{k,j}\} \\
&+ c_{0,j+1}\{E[Y] + \mu_{j+1} + c_{j+1,0}T_{0j} + \ldots + c_{j+1,k}T_{k,j}\}\ldots \\
&+ c_{0,k}\{E[Y] + \mu_k + c_{k,0}T_{0j} + \ldots + c_{k,k-1}T_{k-1,j}\}
\end{aligned}$$

$$\Rightarrow T_{0j} = \underbrace{E[Y] + \sum_{i=1,i\neq j}^{k} c_{0i}\mu_i}_{b_{0j}} + \underbrace{\sum_{i=0,i\neq j}^{k} \sum_{l=1,l\neq i,l\neq j}^{k} c_{0l}c_{li}\, T_{ij}}_{a_{ij}} \tag{5}$$

$$\begin{aligned}
T_{ij} &= c_{ij}\mu_i + c_{i0}(\mu_i + T_{0j}) + c_{i1}(\mu_i + T_{1j}) + \ldots + c_{i,j-1}(\mu_i + T_{j-1,j}) \\
&+ c_{i,j+1}(\mu_i + T_{j+1,j}) + \ldots + c_{i,k}(\mu_i + T_{k,j})
\end{aligned}$$

$$\Rightarrow T_{ij} = \mu_i + \sum_{l=0,l\neq j}^{k} c_{i,l} T_{l,j} \tag{6}$$

i,j $\neq$ 0, i,j =1,...,k □

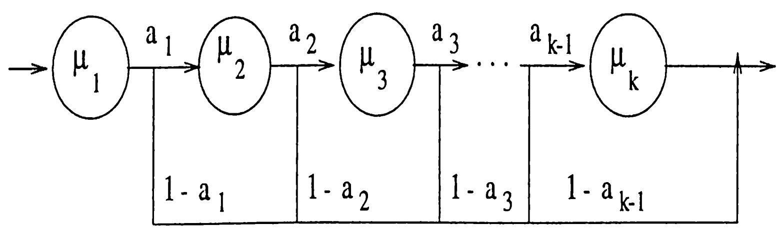


Figure 4.3: k-Stage Coxian Distribution

**Corollary1.** For k-stage Coxian distribution,

$$\mu_{jj} = \mu_j + a_j\mu_{j+1} + \ldots + a_ja_{j+1}\ldots a_{k-1}\mu_k + T_{0j} \tag{7}$$

$$T_{0j} = \frac{E[Y] + \mu_1 + a_1\mu_2 + \ldots + a_1a_2\ldots a_{j-2}\mu_{j-1}}{a_1a_2\ldots a_{j-1}} \tag{8}$$

$$T_{0j} = \frac{E[Y] + \sum_{i=1}^{j-1} \mu_i \prod_{n=1}^{i-1} a_n}{\prod_{i=1}^{j-1} a_i} \quad j > 1 \tag{9}$$

**Proof.** For k-stage Coxian Distribution, if we examine Figure 4.3 $c_{0,1} = 1$, $c_{0,i} = 0$, i=2,...,k and $c_{i,i+1} = a_i$, $c_{i,0} = 1 - a_i$ , $c_{i,j} = 0$ $i \neq 0, i+1$. Then (4) turns into

$$\begin{aligned}
\mu_{jj} &= \mu_j + \underbrace{c_{j,0}}_{1-a_j} T_{0j} + \underbrace{c_{j,j+1}}_{a_j} T_{j+1,j} \\
T_{j+1,j} &= \mu_{j+1} + \underbrace{c_{j+1,0}}_{1-a_{j+1}} T_{0j} + \underbrace{c_{j+1,j+2}}_{a_{j+1}} T_{j+2,j}
\end{aligned}$$

$$T_{k,j} = \mu_k + \underbrace{(1-a_k)}_{0} T_{0j} + \underbrace{a_k}_{1} T_{0,j}$$

Now let us apply (5) to find $T_{0j}$

$$\begin{aligned}
\Rightarrow T_{0j} &= \underbrace{c_{0,1}}_{1}\{E[Y] + \mu_1 + \underbrace{c_{1,0}}_{1-a_1} T_{0j} + \underbrace{c_{1,2}}_{a_1} T_{2j}\} \\
\text{Applying (6) } T_{2j} &= \mu_2 + (1-a_2)T_{0j} + a_2 T_{3j}
\end{aligned}$$

$$\begin{aligned}
\Rightarrow T_{0j} &= E[Y] + \mu_1 + a_1\mu_2 + ... + a_1..a_{j-2}\mu_{j-1} \\
&\quad + \{1 \underbrace{-a_1 + a_1}_{0} - a_1 a_2 ... - a_1 ... a_{j-1}\} T_{0j}
\end{aligned}$$

From which the results follow. □

**Corollary2. (Parlar [19])** For k-stage Erlang Distribution,

$$\begin{aligned}
\mu_{jj} &= E[Y] + \sum_{i=1}^{k} \mu_i \\
P_j &= \frac{\mu_j}{\mu_{jj}}
\end{aligned}$$

Then if $q$ is large, we can replace the transition probabilities $P_{ij}(q/D)$ by the limiting probabilities which are independent of the decision variable $q$.

**Theorem 5. (Parlar, [19])** For large q, the k x k fundamental matrix of P is

$$(\mathrm{I}-\mathrm{P})^{-1} = \frac{1}{\mathrm{P}_0}\begin{vmatrix} P_0+P_1 & P_2 & & P_k \\ P_1 & P_0+P_2 & \dots & P_k \\ . & . & & \\ P_1 & P_2 & \dots & P_0+P_k \end{vmatrix}$$

**Proof.** For large q, the (I − P) matrix turns into,

$$(\mathrm{I}-\mathrm{P}) = \begin{vmatrix} 1-P_1 & -P_2 & & -P_k \\ -P_1 & 1-P_2 & \dots & -P_k \\ . & . & & \\ -P_1 & -P_2 & \dots & 1-P_k \end{vmatrix}$$

Multiplying (I − P) by $(\mathrm{I}-\mathrm{P})^{-1}$ gives the identity matrix I. □

**Proposition1.**$\mathrm{N}(\mathrm{q}) = 1/\mathrm{P}_0$

**Proof.** $\mathrm{N}(\mathrm{q}) = [\mathrm{c}_{0,1},\dots,\mathrm{c}_{0,\mathrm{k}}](\mathrm{I}-\mathrm{P})^{-1}\mathrm{e}$ where

$\mathrm{e} = [1,\dots,1]^{\mathrm{T}}$, and

multiplying each row of $(\mathrm{I}-\mathrm{P})^{-1}$ by e gives $1/\mathrm{P}_0$ then we have

$\mathrm{N}(\mathrm{q}) = \underbrace{\sum_{i=1}^{k} c_{0,i}}_{1}(1/\mathrm{P}_0) = 1/\mathrm{P}_0$ □

**Proposition2.**$\mathrm{T}(\mathrm{q}) = \mathrm{q}/(\mathrm{D}\mathrm{P}_0) + \mathrm{T}_0$

**Proof.**$\mathrm{T}(\mathrm{q}) = [\mathrm{c}_{0,1},\dots,\mathrm{c}_{0,\mathrm{k}}](\mathrm{I}-\mathrm{P})^{-1}\mathrm{b}$ where

$\mathrm{b} = [\mathrm{q}/\mathrm{D} + \mathrm{T}_0\mathrm{P}_0,\dots,\mathrm{q}/\mathrm{D} + \mathrm{T}_0\mathrm{P}_0]^{\mathrm{T}}$ and

multiplying each row of $(\mathrm{I}-\mathrm{P})^{-1}$ by b gives $\mathrm{q}/(\mathrm{D}\mathrm{P}_0) + \mathrm{T}_0$ then we have

$\mathrm{T}(\mathrm{q}) = \underbrace{\sum_{i=1}^{k} c_{0,i}}_{1}\{\mathrm{q}/(\mathrm{D}\mathrm{P}_0) + \mathrm{T}_0\} = \mathrm{q}/(\mathrm{D}\mathrm{P}_0) + \mathrm{T}_0$ □

Then the objective cost function that we derived in Chapter 3 in 12 will be

$$\begin{aligned} \mathcal{K}(\mathrm{q},\mathrm{r}) &= \frac{n(\mathrm{q})\mathrm{c}(\mathrm{q},\mathrm{r}) + \mathrm{C}(\mathrm{q},\mathrm{r})}{\mathrm{T}(\mathrm{q})} \\ &= \frac{\frac{\mathrm{P}_0-1}{\mathrm{P}_0}\mathrm{c}(\mathrm{q},\mathrm{r}) + \mathrm{C}(\mathrm{q},\mathrm{r})}{\mathrm{q}/(\mathrm{D}\mathrm{P}_0) + \mathrm{T}_0} \end{aligned}$$

The structure of resultant objective cost function is identical to the structure of the cost function presented by Parlar [19]. Therefore his convexity analysis for the cost function will be the same for our case also under mild restrictions with his following theorem.

**Theorem 6. Parlar [19]** For large $q$, the objective function $\mathcal{K}(\mathrm{q}, \mathrm{r})$ is convex over the region $\Omega = \{\ (r, q) | r_o \leq r < \infty, 0 < q < \infty\}$ provided that $\mathrm{k}_{\mathrm{z}}'(\mathrm{r}) \leq 0$ for $\mathrm{r} \geq \mathrm{r_o}$ and $\mathrm{k}_{\mathrm{U}}'(\mathrm{r}) \leq 0$ for $\mathrm{r} \geq \mathrm{r_o}$.

# Chapter 5

# Numerical Results

In this chapter, we are going to display and discuss the numerical results of some special problems. To solve these problems we have used a program (see Appendix B) written in Microsoft QuickBasic v4.5 which was run with a clock speed 50 MHz. The aims of this implementation can be listed as follows:

- Sensitivity analysis with respect to parameters of the distributions in terms of cost and quantity.
- Consideration and comparison of cases of Phase-type distributions in terms of their effects on optimal cost and ordering quantity values.
- Verification of the analytical results.

For these special problems we assume that the demand is deterministic at a rate $D \equiv 1$ and all other assumptions of the EOQ model apply ([22], p.174). We consider 8 different cost structures with respect to K, h and b. For each case of Phase-type distribution used, we present the mean and variance of the distribution of the ON period. This may enable us to better interpret the numerical results. These numerical results are presented in Tables 5.1 to 5.14 and Figures 5.1 to 5.22.

The computer program that we used is a revised form of that Gürler and Parlar [9] developed for the case of two independent suppliers whose ON periods are distributed as Erlang. The interested reader may find more details about the algorithm being used in their paper. The summary of the algorithm to compute the optimal $(q, r)$ values is as follows:

Step 1. Start with a high positive level, such as $10^6$, as the initial value of $C^*$.

Step 2. Start with a feasible $(q, r)$ point in the 2-dimensional region that is known to contain the optimal $(q^*, r^*)$.

Step 3. Using $q$ value evaluate the transition functions using the method solving the system of integral equations for $P_{ij}(q)$.

Step 4. Generate the P matrix mentioned in Theorem 1 and 2 of Chapter 3 and compute the expected cycle length, T(q) as given by Proposition 1 in Chapter 3 and expected cycle cost by equation 12 in Chapter 3.

Step 5. Evaluate the expected $C(q, r)$. If this new value of $C$ is better than the previous one, keep the corresponding $(q, r)$ (If the improvement is negligible, stop.) Go to Step 3. Otherwise, go to Step 7.

Step 7. Generate a new feasible $q$ value in the following way: If $q^*$ is known to be in the interval $[q_l, q_h]$ and if $q_{old}$ refers to the previous point, generate the new point $q_{new}$ with $q_{new} = q_{old} + (q_h - q_l)(2\theta - 1)^v$ where $\theta$ is a random number between 0 and 1 and $v$ is an odd integer. With similar computation find a new feasible $r$ value. After the new $(q, r)$ value is generated in this way go to Step 3.

## 5.1 2-Stage Phase-Type Distribution

### 5.1.1 Moments of ON and OFF periods

Let X be the random variable denoting the duration of the ON status of the supplier which follows 2-Stage Phase-type distribution (or Coxian) and Y be the random variable denoting that of OFF periods. For 14 problems designed, $Y \sim \text{Exp}(\mu)$ where $\mu$=0.75. Therefore E[Y]=$1/\mu$=1.333 and

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Var[Y]=$1/(\mu^2)$=1.778.

The noncentral moments E[$X^i$] of X are given by
E[$X^i$] = $(-1)^i i!(\underline{\alpha} T^{-i} \underline{e})$, for i$\geq$ 0 (Neuts [17], p.46). Then,

$$E[X^1] = (-1).([c_{01}, c_{01}] \begin{vmatrix} -\lambda_1 & c_{12}\lambda_1 \\ c_{21}\lambda_2 & -\lambda_2 \end{vmatrix}^{-1} \begin{vmatrix} 1 \\ 1 \end{vmatrix})$$

$$E[X^2] = (2).([c_{01}, c_{01}] \begin{vmatrix} -\lambda_1 & c_{12}\lambda_1 \\ c_{21}\lambda_2 & -\lambda_2 \end{vmatrix}^{-2} \begin{vmatrix} 1 \\ 1 \end{vmatrix})$$

So that we can find Var[X]=E[$X^2$]$-$(E[$X^1$]$)^2$ where $\lambda_i$, i=1,2 denotes the mean of the i$^{th}$ exponential stage of ON periods. Using this approach the mean and variance of the ON periods are computed and are given under the tables displaying the optimal results of each problem.

### 5.1.2 General Case

In this section we are going to display three problems for each of which optimal $q$, $r$ and cost values for different cost values of K, h and b are computed. The problems differ from each other with their initial branching probabilities ($c_{01}$ and $c_{02}$ couples). Other than this $c_{12}$=0.5, $c_{10}$=0.5, $c_{21}$=0.5, $c_{20}$=0.5, where $c_{ij}$ represents the transition probability from state i to state j and $\lambda_1$=0.6, $\lambda_2$=0.5 for ON states, $\mu$=0.75 for the OFF state in the SMP representing the availability of the supplier will be the same for 3 cases. With this organization we will be able to observe the possible effects of initial branching probabilities on optimal $q$ and optimal cost values while we are investigating how these optimal values change when K, h and b cost triplets change.

When the results of Table 5.1 are evaluated, it is seen that optimal cost values are always greater that the corresponding EOQ costs for each case. Only when K=400, h=300, b=500 and K=400, h=300, b=1000 we see that optimal $q$ values are less than the optimal EOQ $q$ values. However, in EOQ models reorder point is equal to 0. So we can conclude that for this stochastic inventory problem, holding a safety stock may result in optimal $q$ values less

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.00037 | 0.00012 | 243.96499 | 2.000 | 200.00 |
| | | 1000 | 2.53359 | 0.00136 | 291.86826 | | |
| | 300 | 500 | 1.37249 | 0.01085 | 372.11108 | 1.154 | 346.41 |
| | | 1000 | 1.37090 | 0.00990 | 440.54296 | | |
| 400 | 100 | 500 | 2.89125 | 0.00296 | 324.89390 | 2.828 | 282.8 |
| | | 1000 | 3.31322 | 0.00008 | 361.29997 | | |
| | 300 | 500 | 1.35694 | 0.02672 | 499.50160 | 1.632 | 489.89 |
| | | 1000 | 1.38103 | 0.00705 | 564.80055 | | |

Table 5.1: Sensitivity Analysis When $c_{01}$=0.5, $c_{02}$=0.5 E[ON]=3.6667, Var[ON]=13.5183

than optimal EOQ $q$ values.

When we investigate the results displayed in Table 5.2, we are faced with almost the same results of Table 5.2.

Similar explanation made for the results of Table 5.1 can be repeated for the results of third case presented by Table 5.3.

Inspecting Figures 5.1 and 5.2 with Tables mentioned up to now brings in the following observations:

- For all three cases the trend that the optimal costs follow seem to be the same (Figure 5.1). The optimal cost curve of third case is below than that of the first two cases. This may indicate that the initial branching probabilities may have an impact on optimal costs.

- While keeping any two components of cost triplets K, h and b, an increase in the other component yields in an increase in optimal inventory holding cost.

- If we take the EOQ cost curve as a border we see that for the $2^{nd}$, $4^{th}$, $6^{th}$ and $8^{th}$ cost triplets, the deviation of results from this border is more than the deviation of the remaining cost triplets. This may be evaluated

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 1.97263 | 0.01351 | 243.89439 | 2.000 | 200.00 |
| | | 1000 | 2.52716 | 0.00056 | 290.82935 | | |
| | 300 | 500 | 1.37486 | 0.00077 | 369.06055 | 1.154 | 346.41 |
| | | 1000 | 1.38167 | 0.01731 | 440.68112 | | |
| 400 | 100 | 500 | 2.90107 | 0.00007 | 323.54681 | 2.828 | 282.8 |
| | | 1000 | 3.27863 | 0.00323 | 360.16058 | | |
| | 300 | 500 | 1.41123 | 0.00042 | 494.22071 | 1.632 | 489.89 |
| | | 1000 | 1.40820 | 0.00222 | 562.26816 | | |

Table 5.2: Sensitivity Analysis When $c_{01}$=0.4, $c_{02}$=0.6 E[ON]=3.6889, Var[ON]=13.5722

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.74536 | 0.02214 | 223.35857 | 2.000 | 200.00 |
| | | 1000 | 2.76588 | 0.00063 | 257.96447 | | |
| | 300 | 500 | 1.42143 | 0.00087 | 376.65771 | 1.154 | 346.41 |
| | | 1000 | 1.31054 | 0.01374 | 431.61638 | | |
| 400 | 100 | 500 | 2.95191 | 0.00267 | 286.34499 | 2.828 | 282.8 |
| | | 1000 | 3.29906 | 0.00176 | 318.29493 | | |
| | 300 | 500 | 1.32240 | 0.00728 | 500.60396 | 1.632 | 489.89 |
| | | 1000 | 1.35140 | 0.00496 | 562.62438 | | |

Table 5.3: Sensitivity Analysis When $c_{01}$=0.3, $c_{02}$=0.7 E[ON]=3.7111, Var[ON]=13.6252

as the effect of back ordering cost on optimal inventory costs.

- When Figure 5.2 is inspected it is found out that the optimal ordering quantity $q$ reaches its peak value for the $6^{th}$ cost triplet. This is the case when K=400, b=1000 (their peak values), h=100 (its lowest value), so it follows our expectation.

- Again Figure 5.2 shows that for $3^{rd}$, $4^{th}$, $7^{th}$ and $8^{th}$ cost triplets, the optimal ordering quantities are at their minimum. This is because the holding cost is at its peak value.

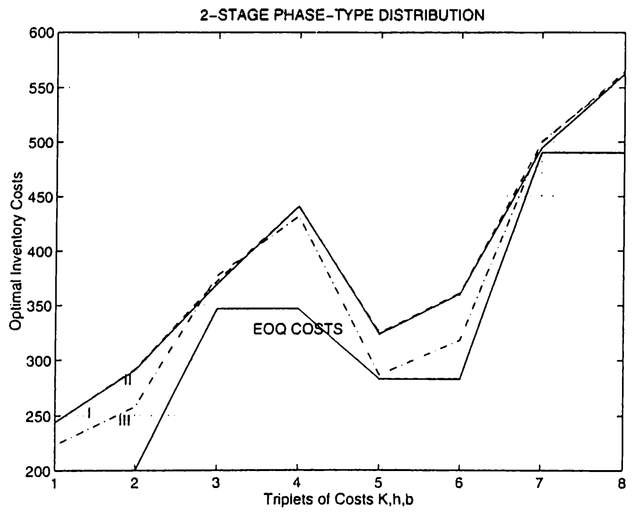


Figure 5.1: Comparison of Optimal Costs (1)

### 5.1.3 Special Case

In this section optimal $q$, $r$ and cost values for different cost values of K, h and b of 5 problems are going to be displayed such that $\mu$=0.75 for the OFF state in the SMP representing the availability of the supplier will be the same for all while $\lambda_1$ and/or $\lambda_2$ for ON states will change. The structure of the SMP for these cases is designed in such a way that the OFF period will start after the first two ON periods are sequentially passed. With this organization we will be able to see the possible effects of mean rates of ON periods on optimal cost and $q$ values. Except a few cases, optimal cost values are more than the respective

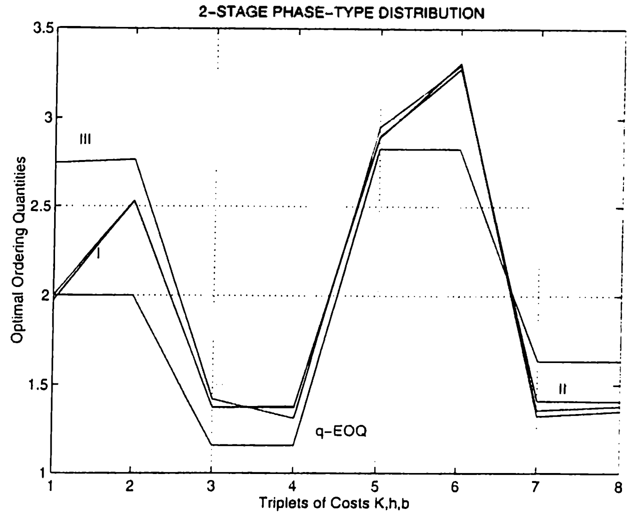


Figure 5.2: Comparison of Optimal $qs$ (1)

| K | h | b | q* | r* | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.74651 | 0.00025 | 264.47348 | 2.000 | 200.00 |
| | | 1000 | 3.02116 | 0.56784 | 340.57748 | | |
| | 300 | 500 | 1.44276 | 0.00124 | 388.49019 | 1.154 | 346.41 |
| | | 1000 | 1.56492 | 0.05780 | 539.46168 | | |
| 400 | 100 | 500 | 3.39686 | 0.00426 | 317.36452 | 2.828 | 282.8 |
| | | 1000 | 3.97894 | 0.26119 | 387.76730 | | |
| | 300 | 500 | 1.44374 | 0.00480 | 480.66034 | 1.632 | 489.89 |
| | | 1000 | 1.98704 | 0.00102 | 619.07847 | | |

Table 5.4: Sensitivity Analysis When $\lambda_1$=0.4, $\lambda_2$=0.5 E[ON]=4.5000, Var[ON]=10.25

| K | h | b | q* | r* | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.71925 | 0.01299 | 264.43303 | 2.000 | 200.00 |
| | | 1000 | 2.99937 | 0.58275 | 340.45789 | | |
| | 300 | 500 | 1.45900 | 0.00223 | 388.45973 | 1.154 | 346.41 |
| | | 1000 | 1.52000 | 0.06827 | 538.42052 | | |
| 400 | 100 | 500 | 3.36186 | 0.04843 | 318.30650 | 2.828 | 282.8 |
| | | 1000 | 3.92418 | 0.28314 | 387.72095 | | |
| | 300 | 500 | 1.50316 | 0.00515 | 480.22213 | 1.632 | 489.89 |
| | | 1000 | 1.97780 | 0.00061 | 618.31015 | | |

Table 5.5: Sensitivity Analysis When $\lambda_1$=0.5, $\lambda_2$=0.4 E[ON]=4.5000, Var[ON]=10.25

optimal EOQ costs. For the exceptions, as the difference is insignificant, we concluded that there might have been some numerical mistakes which can be ignored while discussing the results.

When all the Tables 5.4 to 5.8 and Figures 5.3 and 5.4 are evaluated, following observations can be made:

- When Figure 5.3 is inspected, it is seen that, roughly all the cost curves of 5 cases are similar. We can conclude that different $\lambda_1$ and $\lambda_2$ values do not change the optimal cost values.
- If we take the EOQ cost curve as a border we see that for the $2^{nd}$, $4^{th}$, $6^{th}$ and $8^{th}$ cost triplets, the deviation of results from this border is more than the deviation of the remaining cost triplets. This may be evaluated as the effect of back ordering cost on optimal inventory costs.
- When Figure 5.4 is inspected it is found out that the optimal ordering quantity $q$ reaches its peak value for the $6^{th}$ cost triplet. This is the case when K=400, b=1000 (their peak values), h=100 (its lowest value), so it follows our expectation.

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.74131 | 0.00003 | 250.88858 | 2.000 | 200.00 |
| | | 1000 | 3.08870 | 0.54133 | 322.95861 | | |
| | 300 | 500 | 1.85816 | 0.00172 | 390.98234 | 1.154 | 346.41 |
| | | 1000 | 1.82902 | 0.00039 | 514.28566 | | |
| 400 | 100 | 500 | 3.39182 | 0.00136 | 300.71605 | 2.828 | 282.8 |
| | | 1000 | 3.96922 | 0.26854 | 367.25358 | | |
| | 300 | 500 | 1.83786 | 0.00749 | 465.30612 | 1.632 | 489.89 |
| | | 1000 | 2.00109 | 0.00443 | 588.30607 | | |

Table 5.6: Sensitivity Analysis When $\lambda_1$=0.5, $\lambda_2$=0.5 E[ON]=4.0000, Var[ON]=8

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.83966 | 0.00343 | 238.66516 | 2.000 | 200.00 |
| | | 1000 | 3.121181 | 0.54005 | 306.99798 | | |
| | 300 | 500 | 0.86901 | 0.00612 | 429.57120 | 1.154 | 346.41 |
| | | 1000 | 2.83990 | 0.00719 | 529.70598 | | |
| 400 | 100 | 500 | 3.37374 | 0.00084 | 285.77613 | 2.828 | 282.8 |
| | | 1000 | 3.97717 | 0.26780 | 348.79973 | | |
| | 300 | 500 | 2.85591 | 0.00658 | 497.21678 | 1.632 | 489.89 |
| | | 1000 | 2.85098 | 0.00232 | 580.71838 | | |

Table 5.7: Sensitivity Analysis When $\lambda_1$=0.6, $\lambda_2$=0.5 E[ON]=3.6667, Var[ON]=6.7775

| K | h | b | q* | r* | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.83794 | 0.01640 | 238.86471 | 2.000 | 200.00 |
| | | 1000 | 3.13698 | 0.53696 | 307.07522 | | |
| | 300 | 500 | 0.84344 | 0.00443 | 430.34898 | 1.154 | 346.41 |
| | | 1000 | 2.82941 | 0.02016 | 530.44496 | | |
| 400 | 100 | 500 | 3.42736 | 0.00006 | 285.76872 | 2.828 | 282.8 |
| | | 1000 | 3.99358 | 0.26593 | 348.82037 | | |
| | 300 | 500 | 2.87332 | 0.00469 | 498.31354 | 1.632 | 489.89 |
| | | 1000 | 2.02288 | 0.03401 | 582.57175 | | |

Table 5.8: Sensitivity Analysis When $\lambda_1$=0.5, $\lambda_2$=0.6 E[ON]=3.6667, Var[ON]=6.7775

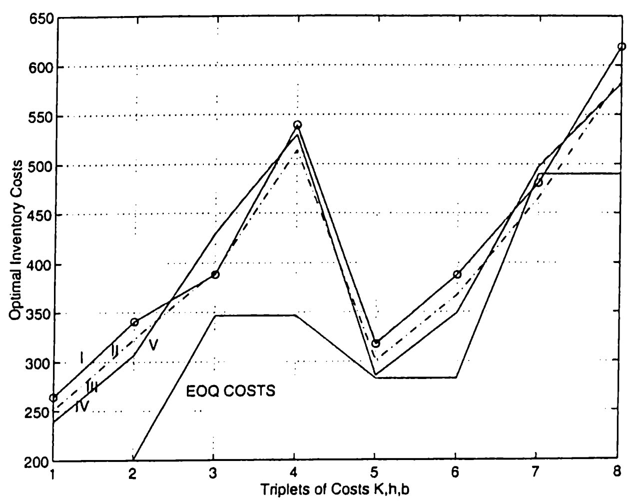


Figure 5.3: Comparison of Optimal Costs (2)

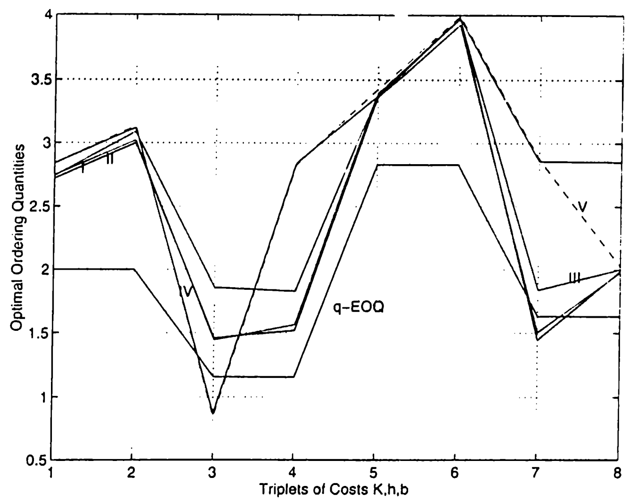


Figure 5.4: Comparison of Optimal $q$s (2)

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{mEOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.60640 | 0.00858 | 294.86797 | 2.000 | 200.00 |
| | | 1000 | 2.70397 | 0.69546 | 380.43490 | | |
| | 300 | 500 | 1.09924 | 0.02402 | 410.51730 | 1.154 | 346.41 |
| | | 1000 | 1.09889 | 0.29979 | 583.64075 | | |
| 400 | 100 | 500 | 3.33382 | 0.00010 | 355.70911 | 2.828 | 282.8 |
| | | 1000 | 3.82131 | 0.30919 | 435.91765 | | |
| | 300 | 500 | 1.25640 | 0.00289 | 522.48926 | 1.632 | 489.89 |
| | | 1000 | 1.79761 | 0.00814 | 682.75575 | | |

Table 5.9: Sensitivity analysis when $a_1$ = 0.0 E[ON]=1.6667, Var[ON]=2.7777

## 5.2 2-Stage Coxian Distribution

In this section for all cases $\lambda_1$=0.6 and $\lambda_2$=0.5 for ON states, $\mu$=0.75 for the OFF state in the SMP representing the availability of the supplier but only $a_1$ which denotes the probability of transition to the next ON state right after the first ON state will change. With this approach, we will be able to observe the possible effects of changes in $a_1$'s on optimal costs.

When Tables 5.9 to 5.14 and Figures 5.5 and 5.6 are inspected the following

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.57949 | 0.00864 | 303.26127 | 2.000 | 200.00 |
| | | 1000 | 2.57040 | 0.75951 | 391.38859 | | |
| | 300 | 500 | 1.03955 | 0.00537 | 415.92314 | 1.154 | 346.41 |
| | | 1000 | 1.03865 | 0.42244 | 592.71544 | | |
| 400 | 100 | 500 | 3.22057 | 0.00287 | 366.08145 | 2.828 | 282.8 |
| | | 1000 | 3.76562 | 0.32601 | 449.71431 | | |
| | 300 | 500 | 1.21618 | 0.00601 | 533.40165 | 1.632 | 489.89 |
| | | 1000 | 1.75041 | 0.03965 | 700.08646 | | |

Table 5.10: Sensitivity Analysis When $a_1 = 0.1$ E[ON]=1.8667, Var[ON]=3.5376

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.61102 | 0.00024 | 294.83332 | 2.000 | 200.00 |
| | | 1000 | 2.75418 | 0.67513 | 380.43160 | | |
| | 300 | 500 | 1.09789 | 0.00508 | 409.56417 | 1.154 | 346.41 |
| | | 1000 | 1.09985 | 0.38974 | 583. 09872 | | |
| 400 | 100 | 500 | 3.31321 | 0.00209 | 355.74443 | 2.828 | 282.8 |
| | | 1000 | 3.81420 | 0.31042 | 435.91661 | | |
| | 300 | 500 | 1.24020 | 0.00008 | 522.32685 | 1.632 | 489.89 |
| | | 1000 | 1.84538 | 0.00152 | 682.52086 | | |

Table 5.11: Sensitivity Analysis When $a_1 = 0.2$ E[ON]=2.0667, Var[ON]=4.2176

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.60186 | 0.00163 | 286.67082 | 2.000 | 200.00 |
| | | 1000 | 2.79141 | 0.66130 | 369.71076 | | |
| | 300 | 500 | 1.18281 | 0.00361 | 403.32932 | 1.154 | 346.41 |
| | | 1000 | 1.16869 | 0.31546 | 572.01900 | | |
| 400 | 100 | 500 | 3.34861 | 0.00191 | 345.31467 | 2.828 | 282.8 |
| | | 1000 | 3.83991 | 0.30704 | 422.81796 | | |
| | 300 | 500 | 1.29965 | 0.00245 | 511.07453 | 1.632 | 489.89 |
| | | 1000 | 1.87796 | 0.00000 | 665.421666 | | |

Table 5.12: Sensitivity Analysis When $a_1 = 0.3$ E[ON]=2.2667, Var[ON]=4.8176

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.70640 | 0.00137 | 264.07675 | 2.000 | 200.00 |
| | | 1000 | 2.97718 | 0.58887 | 340.20399 | | |
| | 300 | 500 | 1.47741 | 0.00280 | 387.7750 | 1.154 | 346.41 |
| | | 1000 | 1.47820 | 0.10879 | 536.19191 | | |
| 400 | 100 | 500 | 3.39869 | 0.00058 | 317.08702 | 2.828 | 282.8 |
| | | 1000 | 3.88426 | 0.29348 | 387.66018 | | |
| | 300 | 500 | 1.47827 | 0.00086 | 477.66348 | 1.632 | 489.89 |
| | | 1000 | 1.92845 | 0.017000 | 617.33572 | | |

Table 5.13: Sensitivity Analysis When $a_1 = 0.6$ E[ON]=2.8667, Var[ON]=6.1376

| K | h | b | $q^*$ | $r^*$ | $C^*$ | $q^*_{EOQ}$ | $C^*_{EOQ}$ |
|---|---|---|---|---|---|---|---|
| (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) |
| 200 | 100 | 500 | 2.86001 | 0.00101 | 238.68121 | 2.000 | 200.00 |
| | | 1000 | 3.08924 | 0.54965 | 307.00102 | | |
| | 300 | 500 | 0.86056 | 0.00018 | 429.05224 | 1.154 | 346.41 |
| | | 1000 | 2.84530 | 0.00178 | 529.55575 | | |
| 400 | 100 | 500 | 3.40458 | 0.00301 | 285.79782 | 2.828 | 282.8 |
| | | 1000 | 4.00676 | 0.24446 | 348.81030 | | |
| | 300 | 500 | 2.84265 | 0.00105 | 495.45303 | 1.632 | 489.89 |
| | | 1000 | 2.84744 | 0.00604 | 580.91564 | | |

Table 5.14: Sensitivity Analysis When $a_1 = 1.0$ E[ON]=3.6667, Var[ON]=6.7775

observations can be made:

- In Figure 5.5, roughly all the cost curves of 6 cases follow the same trend. When the $a_1$ values increase the cost value drops.
- When Figures 5.1, 5.3 and 5.5 are compared, we see that the cost curves have parallel tendencies. This observation can be repeated for optimal $q$ values when Figures 5.2, 5.4 and 5.6 are investigated altogether.
- When the back ordering cost gets higher, the total cost increases dramatically. We can conclude that the cost function is sensitive to back ordering cost in cases where supply is subject to disruptions.

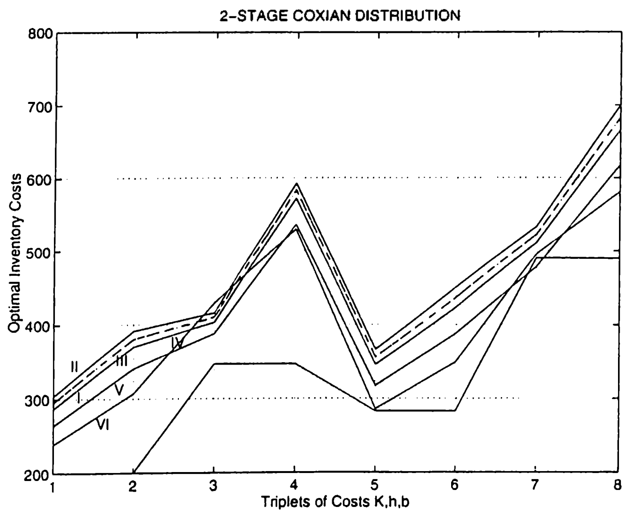


Figure 5.5: Comparison of Optimal Costs (3)

## 5.3 Comparison of Several Phase-Type Distributions

This section comprises of 16 figures such that, in each figure, the points on X-axis represent the Tables 5.1 to 5.14, while the Y-axis represents the corresponding Cost Values (on Figures 5.7 to 5.14) and $q$ values (on Figures 5.15 to 5.22) for the Triplet of Costs K,h,b given under each graph.

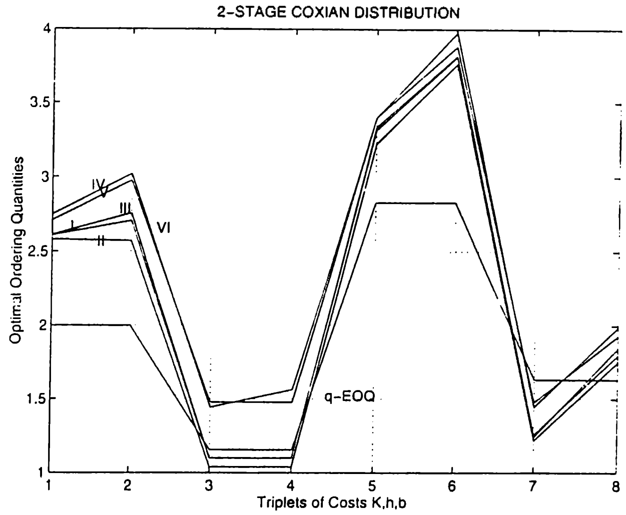


Figure 5.6: Comparison of Optimal $q$s (3)

The following results can be seen:

- Graphics in Figures 5.7 and 5.8; Figures 5.11 and 5.12 follow the same patterns. This helps us to see the effect of increases in back ordering cost on total costs.

- Graphics in Figures 5.7 and 5.11; Figures 5.8 and 5.12 follow the same pattern. This helps us to see the effect of increases in ordering cost on total costs.

- When Figures 5.7 to 5.14, we can see that generally optimal costs are highest for cases with Coxian Distribution and lowest for cases with general Phase-Type distributions.

- When Figures 5.15 to 5.22, we can see that generally optimal $q$ values are highest for cases with special Phase-Type Distribution and lowest for cases with general Phase-Type distributions.

- These last two observations lead us to the following conclusion: The Phase-Type Distributions are so versatile that with different structures, branching probabilities and rates, we can represent the same ON periods but have different values. This shows that parameters of these distributions have an impact on optimal cost and $q$ values.

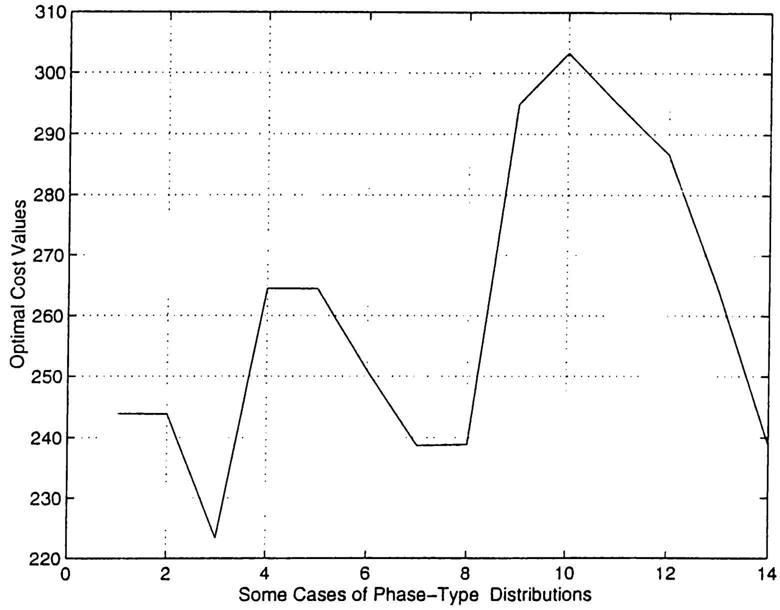


Figure 5.7: When K=200, h=100, b=500

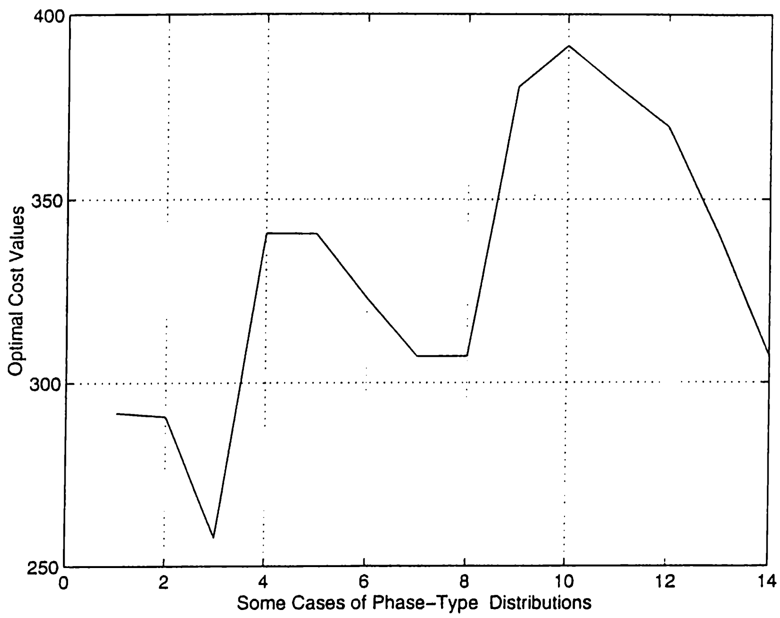


Figure 5.8: When K=200, h=100, b=1000

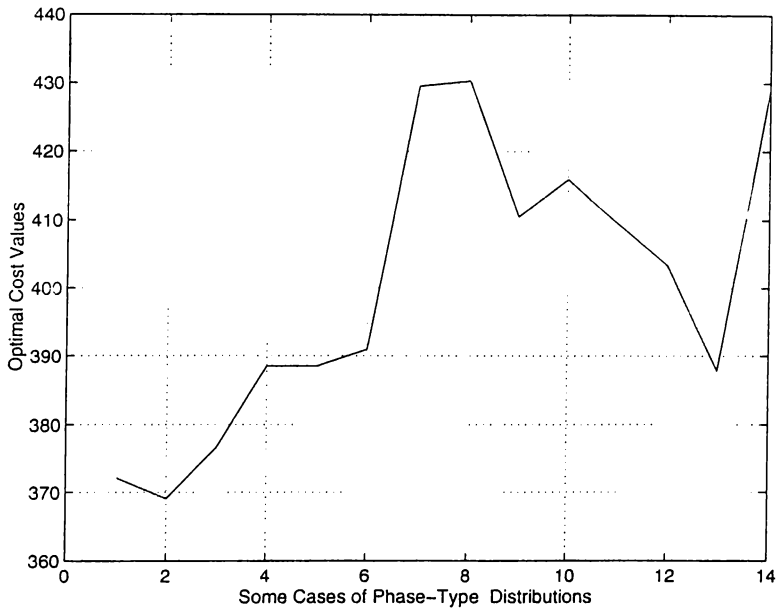


Figure 5.9: When K=200, h=300, b=500

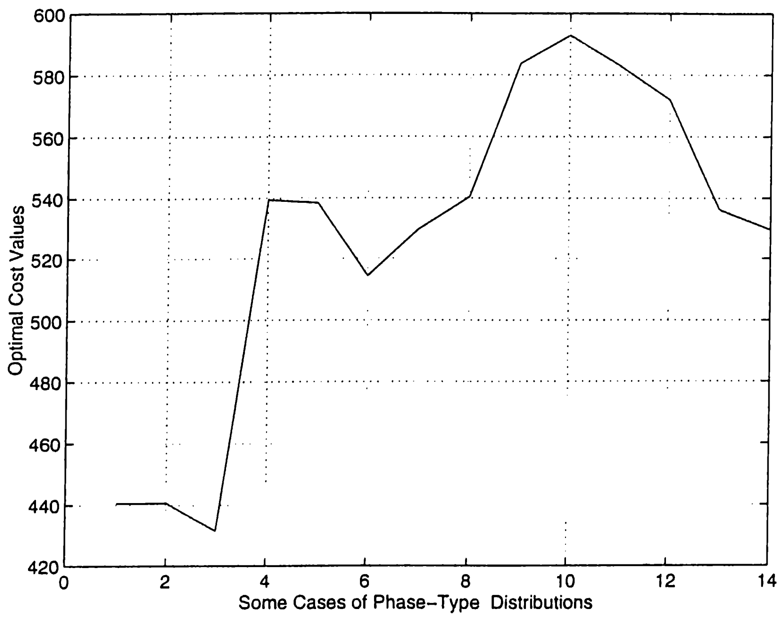


Figure 5.10: When K=200, h=300, b=1000

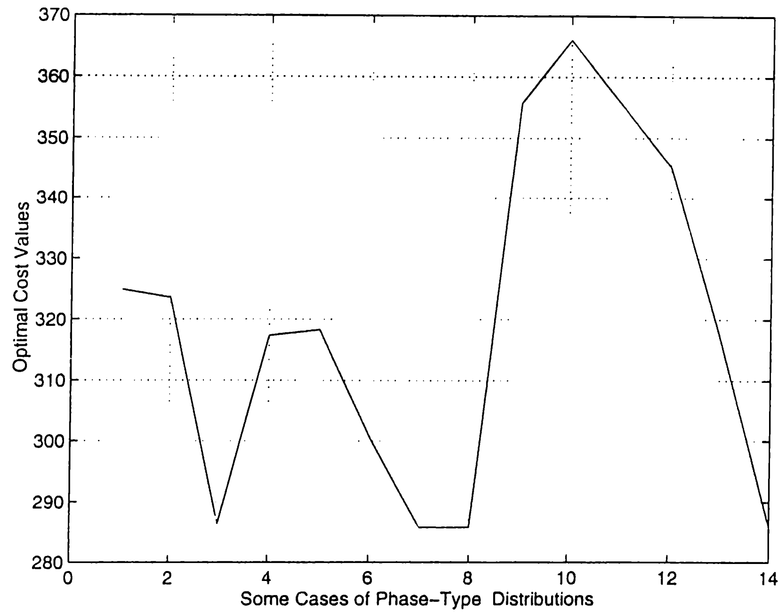


Figure 5.11: When K=400, h=100, b=500

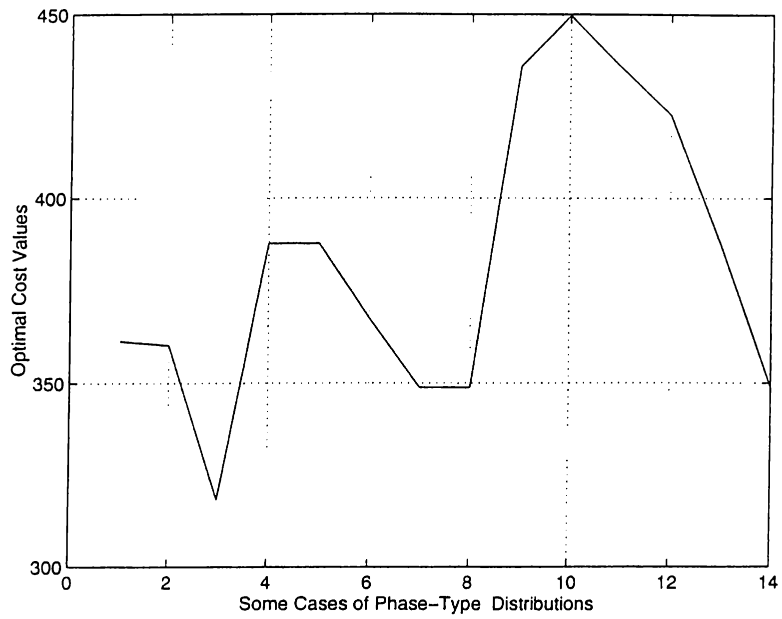


Figure 5.12: When K=400, h=100, b=1000

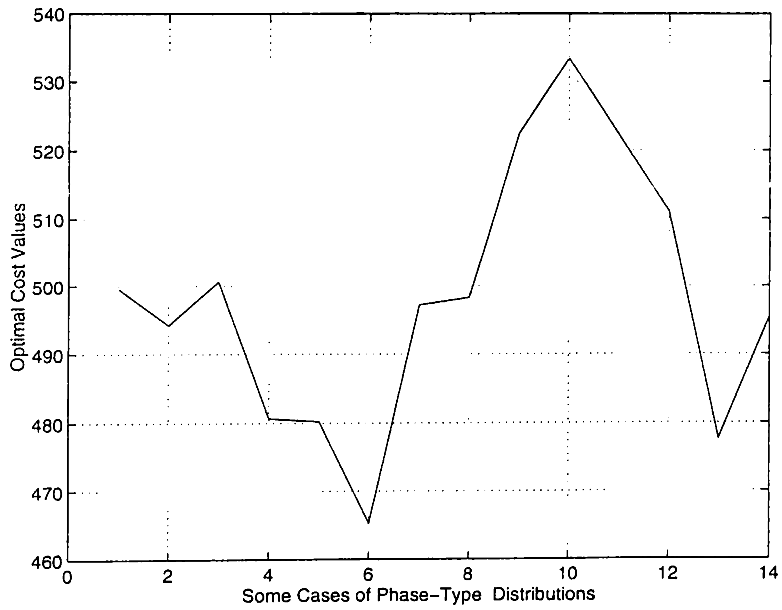


Figure 5.13: When K=400, h=300, b=500

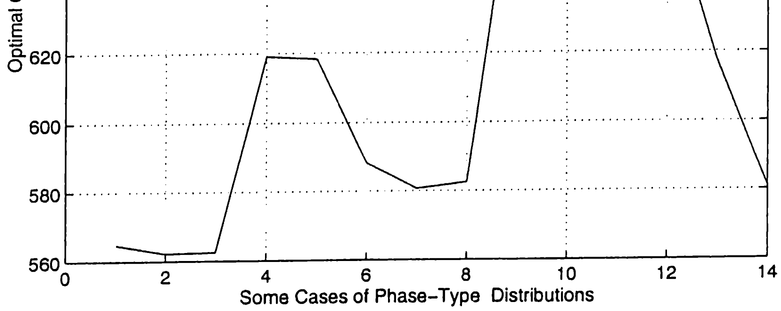


Figure 5.14: When K=400, h=300, b=1000

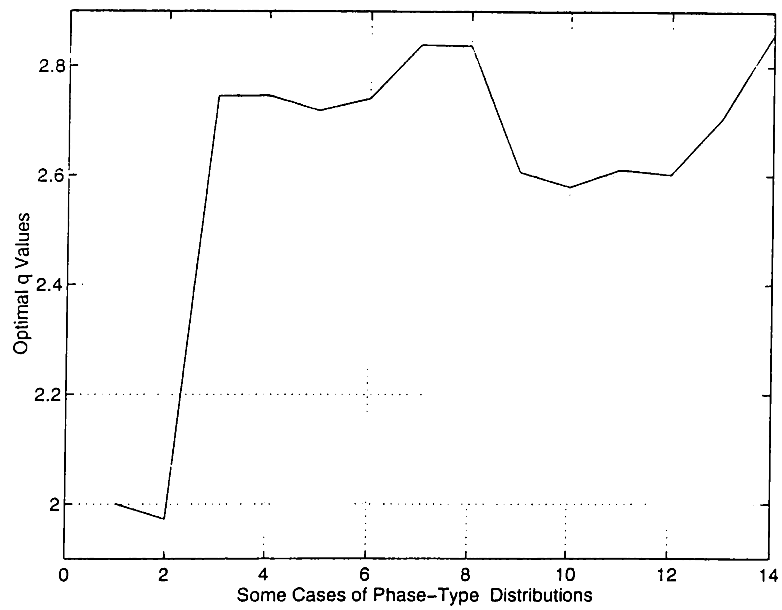


Figure 5.15: When K=200, h=100, b=500

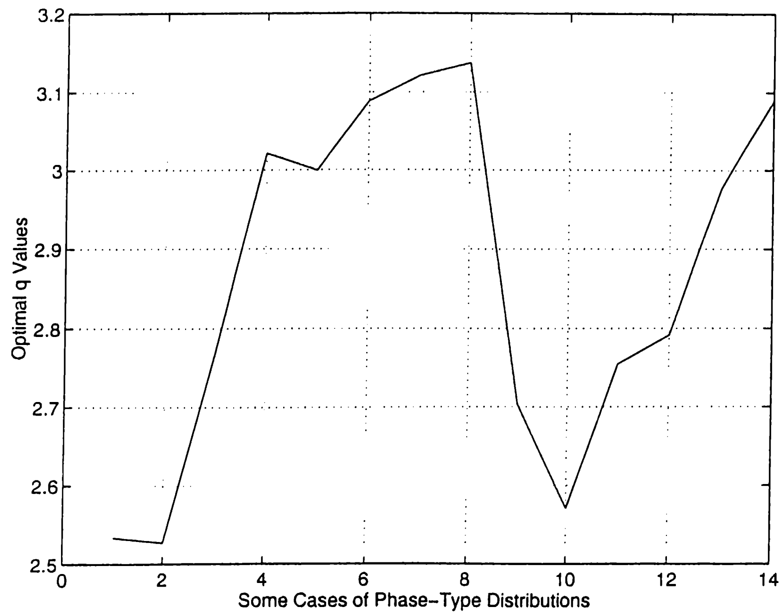


Figure 5.16: When K=200, h=100, b=1000

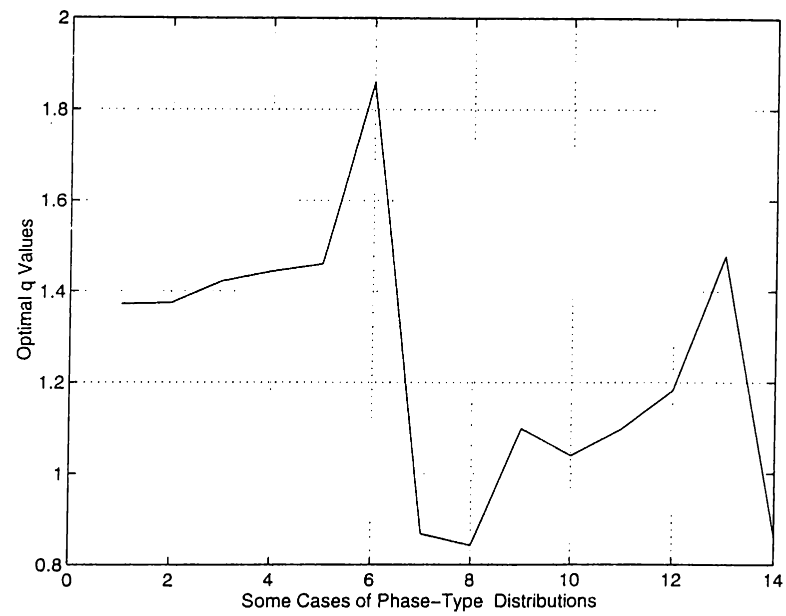


Figure 5.17: When K=200, h=300, b=500

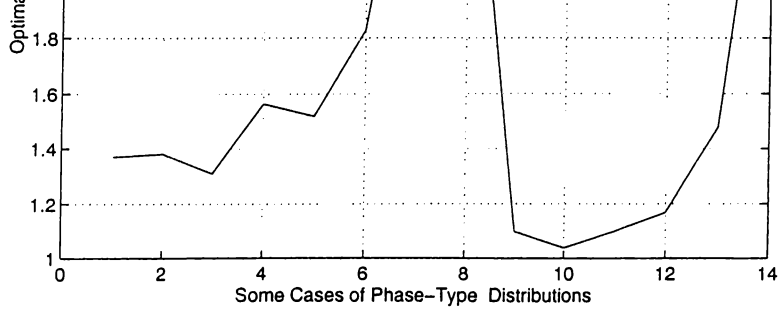


Figure 5.18: When K=200, h=300, b=1000

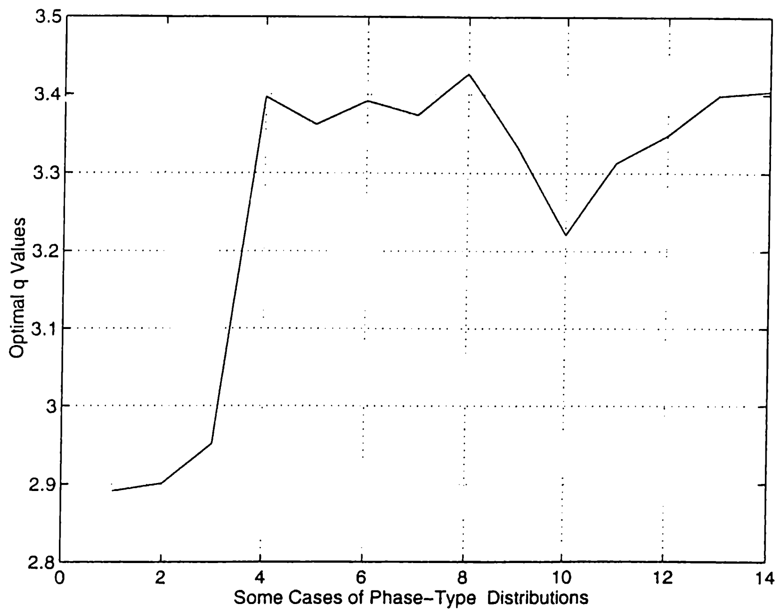


Figure 5.19: When K=400, h=100, b=500

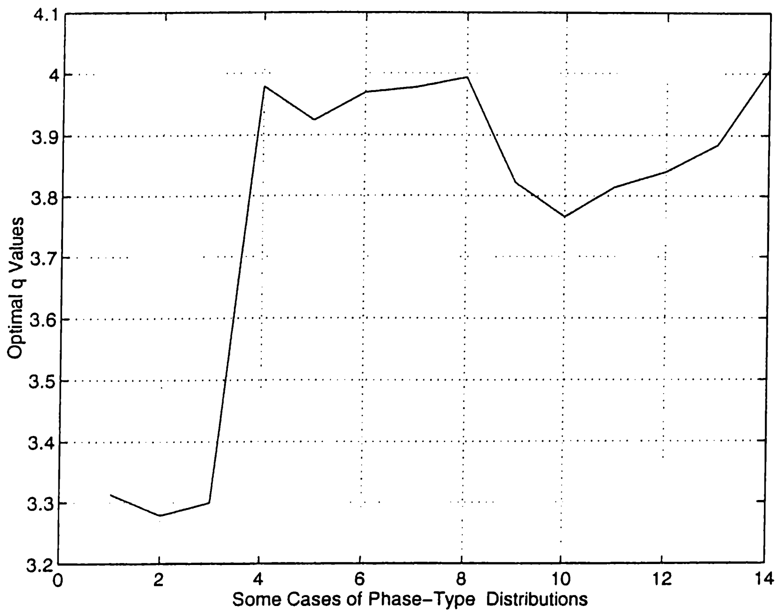


Figure 5.20: When K=400, h=100, b=1000

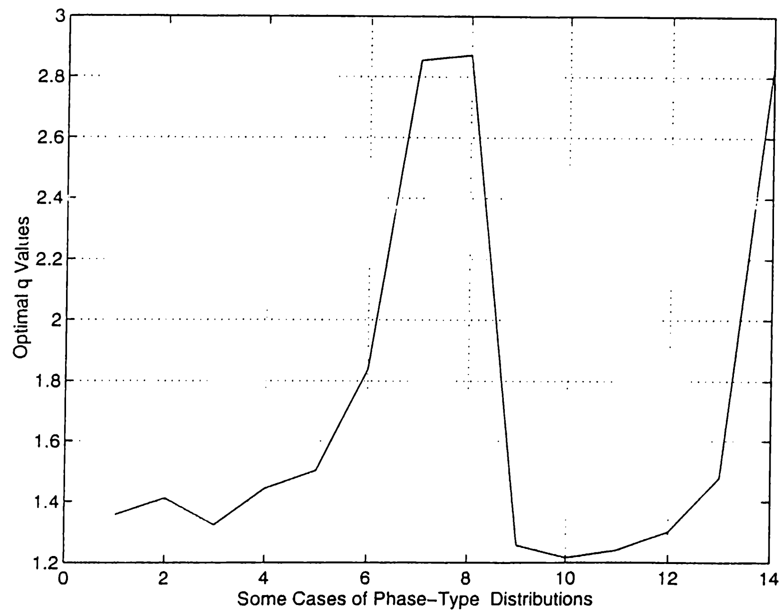


Figure 5.21: When K=400, h=300, b=500

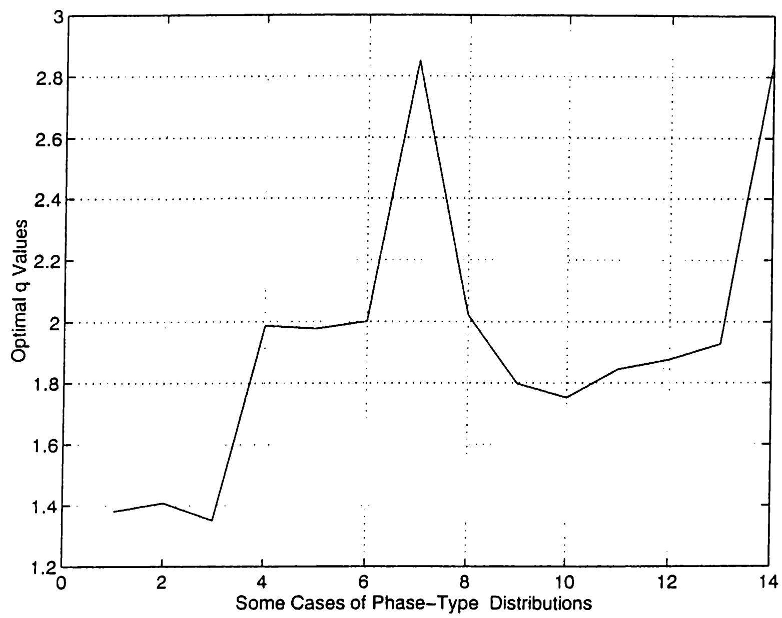


Figure 5.22: When K=400, h=300, b=1000

# Chapter 6

# Conclusion

In this research our basic motivation was to analyze a continuous-review stochastic inventory problem with deterministic demand and random lead times where the single supplier is subject to unusual circumstances such as machine breakdowns, strikes, political upheavals. The supplier availability is modeled as a semi-Markov process. It is assumed that the supplier availability (ON) periods are distributed with k-stage phase-type distributions while the OFF periods follow a general distribution. The nature of phase-type distribution gives rise to transform the non-Markovian stochastic process of supplier's availability into a Markovian one. After the regenerative cycles are identified, the expressions for the expected cycle cost and cycle length are obtained and using the renewal reward theorem we become able to construct the objective function of the long-run average cost per time.

Although our assumption on the distribution of ON periods is k-stage phase-type distribution, we also evaluate the cases for special phase-type distributions such as k-stage Coxian and k-stage Erlang distribution. We discuss several special cases where the problem is solved numerically. We also investigate the problem for large $q$ values and while constructing the objective cost function for this case, we compute the limiting values of the transition probabilities for k-stage phase-type and k-stage Coxian distributions. We find out that whichever of these distributions rule the ON periods, the structure of the cost function is the same with what Parlar [19] found for the case when ON periods follow k-stage Erlang distribution.

The reason that we use phase-type distributions is that in principle any general distribution may be approximated by a phase-type distribution. Additionally as their structures give rise to a Markovian state description they become sufficiently versatile to reflect the essential qualitative features of the model. However for our numerical problems we assume that parameters of the distributions are known in advance. As a possible future research topic, a case with general ON and OFF distributions can be considered and these distributions may be approximated with phase-type distributions. The results of the approximation can then be compared with the real ones.

When numerical results are evaluated we see that the optimal cost and $q$ values are sensitive to type and parameters of the Phase-Type Distribution.

Theoretically, the research presented here can be extended for multiple supplier cases. This enables us to see how the model is effected with increasing number of suppliers. Another extension can be made to analyze the model presented here for lost sales case which will be a trivial one as the only difference will be in the expected cost expression including an extra term. The case with random demand can be considered as a future research.

# Appendix A

# Some Computational Issues

In this section, some computational aspects related to phase type distributions will be illustrated via some standard examples.

1) MGE: k=2 $\mu_1 \neq \mu_2$

$$T = \begin{vmatrix} -\mu_1 & \mu_1 a_1 \\ & -\mu_2 \end{vmatrix}$$

where $\underline{\alpha} = (1, 0)$,

$$T^o = [\mu_1(1 - a_1, \mu_2]^T,$$

and f(x)=$\underline{\alpha}.exp(Tx)T^o$

In order to find exp(Tx); go through diagonalization,i.e.,

exp(Tx)=$P^{-1}.exp(Dx).P$

So first find the eigenvectors of matrix T.

$det \left| \lambda I - T \right| = 0$

$$det \begin{vmatrix} \lambda + \mu_1 & -\mu_1 a_1 \\ & \lambda + \mu_2 \end{vmatrix} = 0$$

$(\lambda + \mu_1)(\lambda + \mu_2) = 0$, $\lambda = -\mu_1$,

$\lambda = -\mu_2$ are eigenvalues.

In order to find the eigenvectors,

$$\begin{vmatrix} \lambda + \mu_1 & -\mu_1 a_1 \\ & \lambda + \mu_2 \end{vmatrix} \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix}$$

If $\lambda = -\mu_1 \Rightarrow -\mu_1 a_1 x_2 = 0$ and $(\mu_2 - \mu_1)x_2 = 0$

let $x_1 = t, x_2 = 0$

$$X = t \begin{vmatrix} 1 \\ 0 \end{vmatrix}$$

If $\lambda = -\mu_2$, then $(\mu_1 - \mu_2)x_1 - \mu_1 a_1 x_2 = 0$,

let $x_2 = t, x_1 = \frac{\mu_1 a_1 t}{\mu_1 - \mu_2}$

$$X = t \begin{vmatrix} \frac{\mu_1 a_1 t}{\mu_1 - \mu_2} \\ 1 \end{vmatrix}$$

$$\Rightarrow P = \begin{vmatrix} \frac{\mu_1 a_1}{\mu_1 - \mu_2} & 1 \\ 1 & 0 \end{vmatrix}, P^{-1} = \begin{vmatrix} 0 & 1 \\ 1 & -\frac{\mu_1 a_1}{\mu_1 - \mu_2} \end{vmatrix}, D = \begin{vmatrix} -\mu_2 & \\ & -\mu_1 \end{vmatrix}$$

$$\exp(\mathrm{Tx}) = \mathrm{P}\exp(\mathrm{Dx})P^{-1} = \begin{vmatrix} e^{-\mu_1 x} & -\frac{\mu_1 a_1}{\mu_1 - \mu 2} e^{-\mu_1 x} + \frac{\mu_1 a_1}{\mu_1 - \mu 2} e^{-\mu_2 x} \\ & e^{-\mu_2 x} \end{vmatrix}$$

$$f(x) = (1,0) \begin{vmatrix} e^{-\mu_1 x} & -\frac{\mu_1 a_1}{\mu_1 - \mu 2} e^{-\mu_1 x} + \frac{\mu_1 a_1}{\mu_1 - \mu 2} e^{-\mu_2 x} \\ & e^{-\mu_2 x} \end{vmatrix} \begin{vmatrix} \mu_1(1 - a_1) \\ \mu_2 \end{vmatrix}$$

$$= \mu_1 e^{-\mu_1 x} \underbrace{\frac{\mu_1(1 - a_1) - \mu_2}{\mu_1 - \mu_2}}_{c_1} + \mu_2 e^{-\mu_2 x} \underbrace{\frac{\mu_1 a_1}{\mu_1 - \mu_2}}_{c_2}$$

2) Hyperexponential distribution.

$$T = \begin{vmatrix} -\lambda_1 & & \\ & -\lambda_2 & \\ & & \\ & & -\lambda_m \end{vmatrix}$$

$$\underline{\alpha} = (p_1, ..., p_m), T^o = [\lambda_1, ..., \lambda_m]^T$$

$$\exp(\mathrm{Tx}) = \begin{vmatrix} e^{-\lambda_1 x} & & \\ & e^{-\lambda_2 x} & \\ & & \\ & & e^{-\lambda_m x} \end{vmatrix}$$

$$f_x(x) = \underline{\alpha}.exp(Tx).T^o = \begin{vmatrix} p_1 e^{-\lambda_1 x} & & \\ & p_2 e^{-\lambda_2 x} & \\ & & \\ & & p_m e^{-\lambda_m x} \end{vmatrix} \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ . \\ \lambda_m \end{vmatrix}$$

$$= \sum_{i=1}^{m} p_i \lambda_i e^{-\lambda_i x}, x \geq 0$$

# Appendix B

# Computer Program

```
' *************************** MAIN : E2E2_MM.BAS **************************

' Programmer : Mahmut Parlar & Baris Balcioglu
' Date : 96/04/17
' This program generalizes the computations for N = 3 states,
' While the ON periods are 2-STAGE PHASE-TYPE DISTRIBUTION.
' OFF periods are memoryless.
' We solve the system of integral equations to get the transient
'    probabilities. (cf. Jerri's book.)

     ' PARAMETERS AND FUNCTIONS IN THE PROBLEM:

     ' lam#()       = Parameters of the stage of the ON Periods
     ' mu#          = Parameter of the OFF Period
     ' cij          = Branching probabilities amoung the stages.
     ' NPrbs        = Number of probabilities to compute which is 6.
     '                The number of states of the SMP is 3.
     ' NPts         = Number of points in the integration interval
     '                This is kept at 10 for all cases.
     ' P()          = Transient probabilities
     ' PFit()       = Exponential fit for transient probabilities
     ' PNum()       = Numerical estimates for the transition probabilites
```

```
        '                   Obtained after solving (I - H)X = B
        ' PLim()          = Limiting probabilities
        ' JStg            = Maximum nbr. of stages in the Erlang ON r.v.
        ' Nstg            = Number of ON stages which is 2.
        ' f()             = Exponential density of Jth stage of the ON r.v.
        ' g()             = General density of the OFF r.v.
        ' GBar()          = Survival probability of of the OFF r.v.
        ' Region$         = Regions of non-zero elements in H matrix
        ' RowGroup$       = Groups of rows for survival probabilities.
        ' H()             = Intermediate matrix / We have A = I - H
        ' AMATRIX()       = The A matrix for the system solution (QPS)
        ' BVECTOR()       = The B vector for the system solution (QPS)
        ' XVECTOR()       = The solution vector (QPS)

' ======================================================================

REM $DYNAMIC

DEFINT I-N
DEFDBL A-H, O-Z

DECLARE FUNCTION aFit (K, L)
DECLARE FUNCTION bFit (K, L)
DECLARE FUNCTION bSurv (K,L,J)
DECLARE FUNCTION f (JStg, lam#(), ujPlus1, uj)
DECLARE FUNCTION FBar (JStg, lam#(), uj)
DECLARE FUNCTION From$ (I)
DECLARE FUNCTION g (mu#, ujPlus1, uj)
DECLARE FUNCTION GBar (mu#, uj)
DECLARE FUNCTION H (I, J)
DECLARE FUNCTION HSubf (JStg, I, J)
DECLARE FUNCTION HSubg (I, J)
DECLARE FUNCTION PLim (I)
DECLARE FUNCTION Region$ (I, J)
DECLARE FUNCTION RowGroup$ (K, L)

DECLARE SUB GetData ()
DECLARE SUB GetCosts ()
DECLARE SUB Initialize ()
DECLARE SUB LINEAR1 (N, A(), B(), X(), IER)
```

```
COMMON SHARED NSup, NPts, NPrbs, DeltaT, JLow, JHigh
COMMON SHARED KOrder, hHolding, bBackorder, bBackorderTime

RANDOMIZE TIMER

Start = TIMER

' Fix the number of points of integration as 10 (This won't change!)
NPts = 10

' ------------- These parameters are used across the modules ---------------

DIM SHARED lam#(2), mu#, u(NPts), TopEQ(NPts + 1), BotEQ(NPts + 1)
DIM SHARED c01, c02, c12, c21, c10, c20
DIM SHARED TopNE(NPts + 1), BotNE(NPts + 1)

CALL GetData
CALL Initialize

' NPrbs is 6
NPrbs = 6

DIM AMATRIX(NPrbs * NPts, NPrbs * NPts), IMinusP#(2, 2)
DIM BVECTOR(NPrbs * NPts), t(2), C(2)
DIM XVECTOR(NPrbs * NPts), TBar(2), CBar(2), SBar(2)

DIM SHARED PNum(2, 2, 10)
DIM PFit(2, 2)


' ------------------------------- START AMATRIX ----------------------------

FOR I = 1 TO NPts * NPrbs
FOR J = 1 TO NPts * NPrbs

        IF I = J THEN
        AMATRIX(I, J) = 1 - H(I, J)
        ELSE AMATRIX(I, J) = -H(I, J)
        END IF
```

```
NEXT J
NEXT I


' ----------------------------- START BVECTOR ------------------------------
FOR I = 1 TO NPts * NPrbs
IF (1 <= I AND I <= 10) THEN
       K = 0
       L = 1

ELSEIF (11 <= I AND I <= 20) THEN
       K = 0
       L = 2
ELSEIF (21 <= I AND I <= 30) THEN
       K = 1
       L = 1
      ' PRINT K; L; "ROWGROUP"; ROWGROUP(K,L); BVECTOR(I)
ELSEIF (31 <= I AND I <= 40) THEN
       K = 1
       L = 2
ELSEIF (41 <= I AND I <= 50) THEN
       K = 2
       L = 1
ELSEIF (51 <= I AND I <= 60) THEN
       K = 2
       L = 2
      ' PRINT K; L; "ROWGROUP"; ROWGROUP(K,L); BVECTOR(I)
END IF

BVECTOR(I) = bSurv(K,L,I)

NEXT I

' ---------------------------------------------------------------------------



' ------ Do the inversion using QuickPAK Scientific and get XVECTOR --------
```

```
CALL LINEAR1(NPrbs * NPts, AMATRIX(), BVECTOR(), XVECTOR(), IER)

' --------------------------------------------------------------------------


' ----------------------- Test the SOLUTION VECTOR -------------------------

FOR I = 1 TO NPrbs * NPts
IF (1 <= I AND I <= 10) THEN
       K = 0
       L = 1

ELSEIF (11 <= I AND I <= 20) THEN
       K = 0
       L = 2
ELSEIF (21 <= I AND I <= 30) THEN
       K = 1
       L = 1
ELSEIF (31 <= I AND I <= 40) THEN
       K = 1
       L = 2
ELSEIF (41 <= I AND I <= 50) THEN
       K = 2
       L = 1
ELSEIF (51 <= I AND I <= 60) THEN
       K = 2
       L = 2

END IF
'PRINT USING "##.##### "; K; L; I; XVECTOR(I)
NEXT I

' --------------------------------------------------------------------------


' --------- Numerical approximation for probabilities (Keep in MAIN) -------

FOR I = 1 TO NPrbs * NPts
```

```
        IF From$(I) = "01" THEN
                PNum(0, 1, I - 0) = XVECTOR(I)

        ELSEIF From$(I) = "02" THEN
                PNum(0, 2, I - 10) = XVECTOR(I)

        ELSEIF From$(I) = "11" THEN
                PNum(1, 1, I - 20) = XVECTOR(I)



        ELSEIF From$(I) = "12" THEN
                PNum(1, 2, I - 30) = XVECTOR(I)

        ELSEIF From$(I) = "21" THEN
                PNum(2, 1, I - 40) = XVECTOR(I)

        ELSEIF From$(I) = "22" THEN
                PNum(2, 2, I - 50) = XVECTOR(I)

        END IF

NEXT I

' ---------------------------------------------------------------------------
FOR K = 0 TO 2
FOR L = 1 TO 2


' --------------------- Fitting coefficients a and b ----------------------

aCoef = aFit(K, L)
bCoef = bFit(K, L)
'PRINT K; L; PLim(L); "+"; aCoef; "*"; " EXP("; bCoef; "t)"


' ---------------------------------------------------------------------------
```

```
NEXT L
NEXT K

' ||
' ||
' ============================================================================




' ========================= Start Optimization ============================
' ||
' ||


' -------------- Create the fitted curves for given (q,r) ------------------

qLo = 0: qHi = 10
rLo = 0: rHi = 10

CALL GetCosts

qEOQ = SQR(2 * KOrder / hHolding)
CostEOQ = SQR(2 * KOrder * hHolding)

PRINT "qEOQ ="; qEOQ, "CostEOQ = "; CostEOQ

qOpt = qEOQ: 'qOpt = qLo + RND * (qHi - qLo)
rOpt = 0:    'rOpt = rLo + RND * (rHi - rLo)

ECostOpt = 10 ^ 10

FOR Iteration = 1 TO 500


IF Iteration < 250 THEN
        Power = 3
        ELSE
```

```
        Power = 5
END IF

'q = 2
'r = 1

q = qOpt + (qHi - qLo) * (2 * RND - 1) ^ Power
IF q < qLo OR q > qHi THEN GOTO StartOver

r = rOpt + (rHi - rLo) * (2 * RND - 1) ^ Power
IF r < rLo OR r > rHi THEN GOTO StartOver


FOR K = 0 TO 2
FOR L = 1 TO 2

PFit(K, L) = PLim(L) + aFit(K, L) * EXP(bFit(K, L) * q)
'PRINT PFit(K,L)

NEXT L
NEXT K



' ------------------- Generate the P and I - P Matrices --------------------

PMatrix(1, 1) = PFit(1, 1)
PMatrix(1, 2) = PFit(1, 2)

PMatrix(2, 1) = PFit(2, 1)
PMatrix(2, 2) = PFit(2, 2)

'FOR K = 1 TO 3
'FOR I = 1 TO 2
'FOR J = 1 TO 2
'PRINT USING "##.#### "; I; J; PMatrix(I, J)
'NEXT J
'NEXT I
'NEXT K
```

```
' ------------------------------ t Vector ---------------------------------
' t vector refers to the B vector in the Parlar's theorem
' D (demand) is taken as unit demand.
' E[TO] = mu# ?

PFit(1, 0) = 1 - (PFit(1, 1) + PFit(1, 2))

PFit(2, 0) = 1 - (PFit(2, 1) + PFit(2, 2))

t(1) = (q + mu# * PFit(1, 0))
t(2) = (q + mu# * PFit(2, 0))


' ---------------------------- I - P Matrix ------------------------------

FOR I = 1 TO 2
FOR J = 1 TO 2

IF I = J THEN
  IMinusP#(I, J) = 1 - PMatrix(I, J)
  ELSE
  IMinusP#(I, J) = -PMatrix(I, J)
  END IF

'PRINT USING "##.#### "; I; J; IMinusP#(I, J)
NEXT J
NEXT I

' ------------------------------------------------------------------------


CALL LINEAR1(2, IMinusP#(), t(), TBar(), IER)

FOR I = 1 TO 2
'PRINT "I, TBar "; I; TBar(I)
NEXT I

' ---------------------------- T1 = E[time] -------------------------------
```

```
T1 = (c01 * TBar(1)) + (c02 * TBar(2))
'PRINT "T1 "; T1




' LINEAR1 changes the original I - P Matrix. So, recreate it!

' ---------------------------- I - P Matrix -------------------------------

FOR I = 1 TO 2
FOR J = 1 TO 2

IF I = J THEN
  IMinusP#(I, J) = 1 - PMatrix(I, J)
  ELSE
  IMinusP#(I, J) = -PMatrix(I, J)
  END IF

'PRINT USING "##.#### "; I; J; IMinusP#(I, J)
NEXT J
NEXT I

' ---------------------------------------------------------------------------


' ------------------------------- e Vector ---------------------------------

e(1) = 1
e(2) = 1
' ---------------------------------------------------------------------------


CALL LINEAR1(2, IMinusP#(), e(), SBar(), IER)

FOR I = 1 TO 2
'PRINT "I, SBar "; I; SBar(I)
```

```
NEXT I

' ---------------------------- N(q) = E[sub-cycles] ----------------------------

N1 = (c01 * SBar(1)) + (c02 * SBar(2))
'PRINT "N1 = "; N1

' ---------------------------------------------------------------------------


cqr = KOrder + hHolding * q ^ 2 / 2 + hHolding * q * r
Bothmus = mu#
gammaBarTop = EXP(-Bothmus * r) * (hHolding * EXP(Bothmus * r)
* (Bothmus * r - 1) + bBackorder * Bothmus + hHolding)
gammaBarBot = Bothmus ^ 2
gammaBar = gammaBarTop / gammaBarBot

Cq = KOrder + hHolding * q ^ 2 / 2 + hHolding * q * r + gammaBar

' ---------------------------- C1 = E[cost] -------------------------------

C1 = (N1 - 1) * cqr + Cq
'PRINT "C1 = "; C1

' ---------------------------------------------------------------------------


ECost = C1 / T1
'PRINT "q, r, ECost "; q; r; ECost

IF ECost > ECostOpt THEN GOTO StartOver

qOpt = q
rOpt = r
ECostOpt = ECost


PRINT USING "####.##### "; Iteration; qOpt; rOpt; ECostOpt
'PRINT "SumT ="; SumT; "SumC ="; SumC
```

```
StartOver:
NEXT Iteration

Finish = TIMER
PRINT "Time it took "; Finish - Start; "seconds"


' ||
' ||
' =========================== End Optimization =============================




' *************************** END OF MAIN PROGRAM *************************

REM $STATIC
'
'
' =========================== FUNCTION : aFit ==============================
'
'
' ==========================================================================
'
'
FUNCTION aFit (K, L)

IF K = L THEN
        aFit = 1 - PLim(L)
ELSE
        aFit = -PLim(L)
END IF


END FUNCTION

'
'
' =========================== FUNCTION : bFit ==============================
```

```
'
'
' ==========================================================================
'
'
FUNCTION bFit (K, L)

FOR I = 1 TO (NPts + 1)
TopEQ(I) = 0
BotEQ(I) = 0
TopNE(I) = 0
BotNE(I) = 0
NEXT I


IF K = L THEN

FOR t% = 1 TO NPts
        u(NPts) = 5
        PNum(K, L, NPts) = PLim(L) + .001
Ratio = (PNum(K, L, t%) - PLim(L)) / (1 - PLim(L))
TopEQ(t% + 1) = TopEQ(t%) + u(t%) * LOG(Ratio)
BotEQ(t% + 1) = BotEQ(t%) + u(t%) ^ 2
NEXT t%

bFit = TopEQ(NPts + 1) / BotEQ(NPts + 1)

ELSE

FOR t% = 1 TO NPts
        u(NPts) = 5
        PNum(K, L, NPts) = PLim(L) - .001
Ratio = (PLim(L) - PNum(K, L, t%)) / PLim(L)
TopNE(t% + 1) = TopNE(t%) + u(t%) * LOG(Ratio)
BotNE(t% + 1) = BotNE(t%) + u(t%) ^ 2
NEXT t%

bFit = TopNE(NPts + 1) / BotNE(NPts + 1)

END IF
```

```
END FUNCTION

'
'
' ============================ *FUNCTION : bSurv ===========================
'
' Note the way Gbar (., mu#(), ...) works. The mu# doesn't get a number!
'
'
' ==========================================================================
'
'
FUNCTION bSurv (K,L,J)

IF RowGroup$(K, L) = "3" THEN
bSurv = FBar(1, lam#(), u(J - 20))

ELSEIF RowGroup$(K, L) = "6" THEN
bSurv = FBar(2, lam#(), u(J - 50))

ELSE
bSurv = 0

END IF

END FUNCTION

'
'
' ============================== FUNCTION : f ==============================
'
'
' ==========================================================================
'
'
```

```
FUNCTION f (JStg, lam#(), ujPlus1, uj)

f = lam#(JStg) * EXP(-lam#(JStg) * (ujPlus1 - uj))

END FUNCTION

'
'
' =========================== FUNCTION : FBar ==============================
'
'
' ==========================================================================
'
'
FUNCTION FBar (JStg, lam#(), uj)

FBar = EXP(lam#(JStg) * uj)

END FUNCTION

'
'
' =========================== *FUNCTION : From$ ============================
'
' Check to see which of (I,J) is valid in ^P
'
'
' ==========================================================================
'
'
FUNCTION From$ (I)

IF 1 <= I AND I <= 10 THEN
         From$ = "01"

        ELSEIF 11 <= I AND I <= 20 THEN
         From$ = "02"

        ELSEIF 21 <= I AND I <= 30 THEN
         From$ = "11"
```

```
        ELSEIF 31 <= I AND I <= 40 THEN
         From$ = "12"

        ELSEIF 41 <= I AND I <= 50 THEN
         From$ = "21"

        ELSEIF 51 <= I AND I <= 60 THEN
         From$ = "22"

END IF


END FUNCTION

'
'
' ============================== FUNCTION : g ==============================
'
'
' ==========================================================================
'
'
FUNCTION g (mu#, ujPlus1, uj)

g = mu# * EXP(-mu# * (ujPlus1 - uj))

END FUNCTION

'
'
' ============================ FUNCTION : GBar ============================
'
'
' ==========================================================================
'
'
FUNCTION GBar (mu#, uj)
```

```
GBar = EXP(-mu# * uj)

END FUNCTION

'
'
' ========================== SUB : GetCosts ==============================
'
'
' ========================================================================
'
'
SUB GetCosts

KOrder = 200:          '[10]
hHolding = 100:         '[5]
bBackorder = 500:      '[250]

PRINT "K ="; KOrder, "h ="; hHolding, "b ="; bBackorder

'PRINT "qEOQ = "; SQR(2 * KOrder / hHolding)
'PRINT "CEOQ = "; SQR(2 * KOrder * hHolding)

END SUB

'
'
' =========================== *SUB : GetData ==============================
'
        ' PARAMETERS
        '
        ' See the MAIN PROGRAM for definitions of the paremeters
'
'
' ========================================================================
'
'
SUB GetData

CLS
```

```
lam#(1) = .6#
lam#(2) = .5#

mu# = .75

c01 = .3#
c02 = 1 - c01


c12 = .55
c10 = 1 - c12

c21 = .60
c20 = 1 - c21


PRINT "lam1 ="; lam#(1), "lam2 ="; lam#(2)
PRINT "mu ="; mu#
PRINT "c01 ="; c01, "c02 ="; c02
PRINT "c12 ="; c12, "c10 ="; c10
PRINT "c21 ="; c21, "c20 ="; c20

END SUB

' ============================= *FUNCTION : H ==============================
'
' HSubg(Ith coord, Jth coord)
' HSubf(Stage, Ith coord, Jth coord)
'
'
' ==========================================================================
'
'
FUNCTION H (I, J)

        SELECT CASE Region$(I, J)

                CASE "13"
                H = c01 * HSubg(I - 0, J - 20)
```

```
CASE "15"
H = c02 * HSubg(I - 0, J - 40)

CASE "24"
H = c01 * HSubg(I - 10, J - 30)

CASE "26"
H = c02 * HSubg(I - 10, J - 50)

CASE "31"
H = c10 * HSubf(1, I - 0, J - 10)

CASE "35"
H =  c12 * HSubf(1, I - 20, J - 40)



CASE "42"
H = c10 * HSubf(1, I - 30, J - 10)

CASE "46"
H = c12 * HSubf(1, I - 30, J - 50)



CASE "51"
H = c20 * HSubf(2, I - 40, J - 0)

CASE "53"
H = c21 * HSubf(2, I - 40, J - 20)

CASE "62"
H = c20 * HSubf(2, I - 50, J - 10)

CASE "64"
H = c21 * HSubf(2, I - 50, J - 30)

CASE ELSE
H = 0
```

```
END SELECT

END FUNCTION

'
'
' ============================ FUNCTION : HSubf ============================
'
'
' ==========================================================================
'
'
FUNCTION HSubf (JStg, I, J)

  IF (2 <= I AND I <= NPts) AND (J = 1) THEN
  HSubf = DeltaT * (1 / 2) * f(JStg, lam#(), u(I), u(J))

  ELSEIF (I > J) AND (2 <= J AND J <= NPts - 1) THEN
  HSubf = DeltaT * (1 / 1) * f(JStg, lam#(), u(I), u(J))

  ELSEIF (2 <= I AND I <= NPts) AND (J = I) THEN
  HSubf = DeltaT * (1 / 2) * f(JStg, lam#(), u(I), u(J))

  ELSE
  HSubf = 0

  END IF


END FUNCTION

'
'
' ============================ FUNCTION : HSubg ============================
'
' Generates the submatrices for H matrices before the integral equation
' solution.
'
' ==========================================================================
```

```
'
'
FUNCTION HSubg (I, J)

  IF (2 <= I AND I <= NPts) AND (J = 1) THEN
  HSubg = DeltaT * (1 / 2) * g(mu#, u(I), u(J))

  ELSEIF (I > J) AND (2 <= J AND J <= NPts - 1) THEN
  HSubg = DeltaT * (1 / 1) * g(mu#, u(I), u(J))

  ELSEIF (2 <= I AND I <= NPts) AND (J = I) THEN
  HSubg = DeltaT * (1 / 2) * g(mu#, u(I), u(J))

  ELSE
  HSubg = 0

  END IF


END FUNCTION

'
'
' =========================== SUB : Initialize ============================
'
' Computes DeltaT and the u(J) values
'
' NOTE : If tFinal is large (> 0.5) then we get inaccurate results. So, keep
' tFinal around .25
'
' =========================================================================
'
'
SUB Initialize

tInit = 0
tFinal = .05
DeltaT = (tFinal - tInit) / (NPts - 1)

FOR I = 1 TO NPts
```

```
u(I) = tInit + (I - 1) * DeltaT
NEXT I

END SUB

'
'
' ======================== SUBROUTINE : LINEAR1 ==========================
'
' QuickPack Scientific Subroutine
'
' Solution of a system of linear equations subroutine
'
        ' Solves [ A ] * { X } = { B } using LU decomposition
        '
        ' Input
        '  N   = number of equations
        '  A() = matrix of coefficients    ( N rows by N columns )
        '  B() = right hand column vector  ( N rows )
        '
        ' Output
        '  X() = solution vector ( N rows )
        '  IER = error flag
        '    0 = no error
        '    1 = singular matrix or factorization not possible
'
' ========================================================================
'
'
SUB LINEAR1 (N, A(), B(), X(), IER) STATIC

    DIM INDEX(N), SCALE(N)

    IER = 0

    FOR I = 1 TO N
        ROWMAX = 0#
        FOR J = 1 TO N
            IF (ABS(A(I, J)) > ROWMAX) THEN ROWMAX = ABS(A(I, J))
        NEXT J
```

```
        ' check for singular matrix
        IF (ROWMAX = 0#) THEN
           IER = 1
           GOTO EXITSUB
        ELSE
           SCALE(I) = 1# / ROWMAX
        END IF
        X(I) = B(I)
    NEXT I

    FOR J = 1 TO N
        IF (J > 1) THEN
           FOR I = 1 TO J - 1
               s = A(I, J)
               IF (I > 1) THEN
                  FOR K = 1 TO I - 1
                      s = s - A(I, K) * A(K, J)
                  NEXT K
                  A(I, J) = s
               END IF
           NEXT I
        END IF

        PIVOTMAX = 0#

        FOR I = J TO N
            s = A(I, J)
            IF (J > 1) THEN
               FOR K = 1 TO J - 1
                   s = s - A(I, K) * A(K, J)
               NEXT K
               A(I, J) = s
            END IF
            PIVOT = SCALE(I) * ABS(s)
            IF (PIVOT >= PIVOTMAX) THEN
               IMAX = I
               PIVOTMAX = PIVOT
            END IF
        NEXT I
```

```
    IF (J <> IMAX) THEN
       FOR K = 1 TO N
           TMP = A(IMAX, K)
           A(IMAX, K) = A(J, K)
           A(J, K) = TMP
       NEXT K
       SCALE(IMAX) = SCALE(J)
    END IF

    INDEX(J) = IMAX

    IF (J <> N) THEN
       ' check for singular matrix
       IF (A(J, J) = 0#) THEN
          IER = 1
          GOTO EXITSUB
       END IF
       TMP = 1# / A(J, J)
       FOR I = J + 1 TO N
           A(I, J) = A(I, J) * TMP
       NEXT I
    END IF
NEXT J

' check for singular matrix

IF (A(N, N) = 0#) THEN
   IER = 1
   GOTO EXITSUB
END IF

I1 = 0

FOR I = 1 TO N
    L = INDEX(I)
    s = X(L)
    X(L) = X(I)
    IF (I1 <> 0) THEN
       FOR J = I1 TO I - 1
           s = s - A(I, J) * X(J)
```

```
            NEXT J
         ELSEIF (s <> 0#) THEN
            I1 = I
         END IF
         X(I) = s
     NEXT I

     FOR I = N TO 1 STEP -1
         s = X(I)
         IF (I < N) THEN
            FOR J = I + 1 TO N
                s = s - A(I, J) * X(J)
            NEXT J
         END IF
         X(I) = s / A(I, I)
     NEXT I

EXITSUB:
     ERASE INDEX, SCALE



END SUB

'
' ==========================================================================
'
'
FUNCTION PLim (I)
IF I = 1 THEN
Plimtop = lam#(1)
Plimbot = lam#(1) + (c10 + c12c20 )
* ( (mu# + (c02 * lam#(2))) / (1 - c02c20) ) + c12 * lam#(2)
PLim = Plimtop / Plimbot
ELSE
Plimtop = lam#(2)
Plimbot = lam#(2) + (c20 + c21c10 )
 * ( (mu# + (c01 * lam#(1))) / (1 - c01c10) ) + c21 * lam#(1)
PLim = Plimtop / Plimbot
END IF
```

```
END FUNCTION

'
' ========================= *FUNCTION : Region$ =========================
'
' Determine which blocks are valid for the AMATRIX
'
' I shows row # J shows column #
' =======================================================================
'
'
FUNCTION Region$ (I, J)

        IF (1 <= I AND I <= 10) AND (21 <= J AND J <= 30) THEN
                Region$ = "13"

        ELSEIF (1 <= I AND I <= 10) AND (41 <= J AND J <= 50) THEN
                Region$ = "15"

        ELSEIF (11 <= I AND I <= 20) AND (31 <= J AND J <= 40) THEN
                Region$ = "24"

        ELSEIF (11 <= I AND I <= 20) AND (51 <= J AND J <= 60) THEN
                Region$ = "26"

        ELSEIF (21 <= I AND I <= 30) AND (1 <= J AND J <= 10) THEN
                Region$ = "31"

        ELSEIF (21 <= I AND I <= 30) AND (41 <= J AND J <= 50) THEN
                Region$ = "35"

        ELSEIF (31 <= I AND I <= 40) AND (11 <= J AND J <= 20) THEN
                Region$ = "42"

        ELSEIF (31 <= I AND I <= 40) AND (51 <= J AND J <= 60) THEN
                Region$ = "46"

        ELSEIF (41 <= I AND I <= 50) AND (1 <= J AND J <= 10) THEN
                Region$ = "51"
```

```
        ELSEIF (41 <= I AND I <= 50) AND (21 <= J AND J <= 30) THEN
                Region$ = "53"

        ELSEIF (51 <= I AND I <= 60) AND (11 <= J AND J <= 20) THEN
                Region$ = "62"

        ELSEIF (51 <= I AND I <= 60) AND (31 <= J AND J <= 40) THEN
                Region$ = "64"


        ELSE
               Region$ = "00"

        END IF



END FUNCTION

'
'
' ========================== *FUNCTION : RowGroup$ =======================
'
' Determine which rows are valid for the BVECTOR
'
'
' ========================================================================
'
'
FUNCTION RowGroup$ (K, L)

        IF K = 0 AND L = 1 THEN
                RowGroup$ = "1"
                JLow = 1
                JHigh = 10

        ELSEIF K = 0 AND L = 2 THEN
                RowGroup$ = "2"
                JLow = 11
```

```
                JHigh = 20

        ELSEIF K = 1 AND L = 1 THEN
                RowGroup$ = "3"
                JLow = 21
                JHigh = 30

        ELSEIF K = 1 AND L = 2 THEN
                RowGroup$ = "4"
                JLow = 31
                JHigh = 40

        ELSEIF K = 2 AND L = 1 THEN
                RowGroup$ = "5"
                JLow = 41
                JHigh = 50

        ELSEIF K = 2 AND L = 2 THEN
                RowGroup$ = "6"
                JLow = 51
                JHigh = 60

        END IF

'PRINT I ; JLow; JHigh
END FUNCTION
```

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# VITA

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