BOUNDARY CONDITIONS COMPATIBLE WITH THE GENERALIZED SYMMETRIES

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By

T. Burak Gürel

April 1995

THESIS
QC
19.2
.G87
1995

BOUNDARY CONDITIONS COMPATIBLE WITH THE GENERALIZED SYMMETRIES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
T. Burak Gürel
April 1995

T. Burak GÜREL
tarafından bağışlanmıştır

QC
19.2
.G87
1995
1031662

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Metin Gürses(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Varga Kalantarov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Uğurhan Muğan

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Mehmet Baray
Director of Institute of Engineering and Sciences

ABSTRACT

BOUNDARY CONDITIONS COMPATIBLE WITH THE GENERALIZED SYMMETRIES

T. Burak Gürel
M.S. in Mathematics
Supervisor: Prof. Dr. Metin Gürses
April 1995

In this work evolution type integrable equations and systems are considered. An efficient method is given to construct their boundary conditions and hence boundary value problems which are compatible with the generalized symmetries. This method is applied to some well-known nonlinear partial differential equations.

Keywords : Integrability, symmetry, generalized symmetry, recursion operator, boundary conditions compatible with symmetries.

ÖZET

GENEL SİMETRİLERLE UYUMLU SINIR DEĞER ŞARTLARI

T. Burak Gürel
Matematik Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Metin Gürses
Nisan 1995

Bu çalışmada entegre edilebilir lineer olmayan diferansiyel denklemler ve sistemler ele alındı. Bu denklemlerin genel simetrileriyle uyumlu sınır değer şartları ve sınır değer problemlerinin oluşturulması için verimli ve algoritmik bir yöntem geliştirildi. Bu yöntem bazı, çok iyi bilinen, entegre edilebilir diferansiyel denklemlere uygulandı.

Anahtar Kelimeler : Entegre edilebilirlik, simetri, genel simetri, rekörşın operatörü, simetrilerle uyumlu sınır şartları.

ACKNOWLEDGMENT

I am greatful to Prof. Dr. Metin Gürses who introduced me the marvellous world of integrable systems and, expertly and patiently guided my research up to this point.

I would like to thank Assoc. Prof. Dr. Ismagil Habibullin for his encouregement and guidance during my first steps.

I would also like to thank my family for their unfailing support and influence in my life.

I owe a lot to Esra for her patience and encouregement during my dazed and confused times.

Finally, I would like to thank all my friends for all they have done for me.

TABLE OF CONTENTS

Chapter 1

Introduction

In this work, [1], we developed a method to construct the boundary value problems of the form

$$u_t = f(u, u_1, u_2, ..., u_n) \tag{1.1}$$

$$\mathbf{p}(u, u_1, u_2, ..., u_{n-1}) = 0 \;\text{ at }\; x = x_0 \tag{1.2}$$

completely compatible with the integrablility property of equation (1.1). Here $u = u(x,t)$, $u_t = \partial u/\partial t$, $u_i = \partial^i u/\partial x^i$ and f is a scalar or vector field. It is well-known [2], [14] that for some classes of completely integrable nonlinear evolution equations (1.1), there exist boundary conditions of the form (1.2) compatible with the inverse scattering transform or any other attribute of integrability. The following remarks can be extended to the vector field case of f, see chapter 4.

Let the equation

$$u_\tau = g(u, u_1, ..., u_m) \ , \tag{1.3}$$

where $u_\tau = \partial u/\partial \tau$, for a fixed value of m, be a symmetry of the equation (1.1). Now introducing new dynamical variables $\mathbf{v} = (u, u_1, u_2, ..., u_{n-1})$ we can pass to a system of n-equations. All higher x-derivatives of u can be determined from the original equation (1.1) in terms of the new dynamical variables and their t-derivatives. So the symmetry (1.3) takes the form

$$\mathbf{v}_\tau = \mathbf{G}(\mathbf{v}, \mathbf{v}_t, \mathbf{v}_{tt}, ..., \mathbf{v}_{tt...t}) \ . \tag{1.4}$$

We call the boundary value problem defined by equations (1.1) and (1.2) as compatible with the symmetry (1.3) if the constraint $\mathbf{p}(\mathbf{v}) = 0$ (or the constraints $p^\alpha = 0$, $\alpha = 1,2,...,N$, N is the number of the constraints), is consistent with the τ-evolution

$$\frac{\partial \mathbf{p}}{\partial \tau} = 0 \,, (\mathrm{mod}\ \ \mathbf{p} = 0) \tag{1.5}$$

Equation (1.5), by virtue of the equations in (1.4) must be automatically satisfied. In fact (1.5) means that the constraint $\mathbf{p} = 0$ defines an invariant surface in the manifold with local coordinates $\mathbf{v}$, or a compatible boundary condition with (1.4).

We call the boundary condition (1.2) as compatible with the equation (1.1) if it is compatible with at least one of its higher symmetries. Here comes our main observation saying that if a boundary condition is compatible with one higher symmetry then it is compatible with an infinite number of symmetries, not necessarily with all of them. For example for the Burgers equation the boundary condition is compatible with the even order time-independent symmetries. Also note that, all known boundary conditions of the form (1.2) consistent with the inverse scattering method are compatible with an infinite series of generalized (higher) symmetries.

In this work, we deal with the boundary conditions of the form given in (1.2). We propose a method to construct such boundary conditions compatible with the time-independent generalized local symmetries of integrable nonlinear partial differential equations. We give several examples, containing the Burgers, Korteweg de Vries equations and some systems like nonlinear Schrödinger, Boussinesq. An effective investigation of boundary conditions involving an explicit t-dependence is essentially more complicated. Such a problem has been studied in [15].

Chapter 2

Symmetries and Integrability

Here in this chapter we give some definitions, related to our interest, on symmetries and integrability. For interested readers we refer [9], [6].

2.1 Lie and Generalized Symmetries

We will deal with the evolutionary type partial differential equations in $1+1$ i.e. independent variables are x, t. Consider a system of evolution equations

$$u_t^\alpha = \varphi_\alpha(x, t, u_{n_1}^1, ..., u_{n_q}^q)\,,\ \alpha = 1, 2, ..., q \tag{2.1}$$

where $u_{n_\nu}^\nu$ corresponds to the x-derivatives of u^ν up to the order n_ν containing u^ν itself; $\forall \nu$ such that $1 \leq \nu \leq q$. The above system (2.1) is exactly determined since we will deal with them, but it can be, of course, over or under determined, [9]. We can represent (2.1) in a more compact and short form as the following

$$u_t^\alpha = \varphi_\alpha[u_{n_\nu}^\nu]\ . \tag{2.2}$$

We will assume the functions φ_α to be enough differentiable and defined on a smooth manifold M, $\forall \alpha$. Now we can give the definition of an important concept; the Frèchet derivative.

Definition 2.1 *The Frèchet derivative of the function*

$$v_\alpha[u_t^\alpha, u_{n_\nu}^\nu] = u_t^\alpha - \varphi_\alpha[u_{n_\nu}^\nu] = 0 \tag{2.3}$$

in the direction of the vector $\mathbf{v} = (v^1, v^2, ..., v^q)$, *denoted by* $\psi_{\alpha *}(\mathbf{v})$ *is*

$$\psi_{\alpha *}(\mathbf{v}) = \frac{\partial}{\partial \epsilon} \psi_\alpha[(u^\alpha + \epsilon\, v^\alpha)_t, (u^\nu + \epsilon\, v^\nu)_{n_\nu}]\bigg|_{\epsilon = 0} \tag{2.4}$$

where ϵ *is a real parameter.*

Example Consider the Korteweg de Vries equation

$$u_t = u_{xxx} + 6uu_x \ . \tag{2.5}$$

Here $q = 1$, $n_1 = 3$ and

$$\psi[u_t, u_3] = u_t - u_{xxx} - 6uu_x = 0 \ . \tag{2.6}$$

First of all we shall perform the transformation $u \rightarrow u + \epsilon\, v$ according to the definition. Now write (2.6) for the new variable $u + \epsilon\, v$

$$\begin{aligned} \psi[(u + \epsilon\, v)_t, (u + \epsilon\, v)_3] \ &= \ (u + \epsilon\, v)_t - (u + \epsilon\, v)_{xxx} - \\ & \qquad 6(u + \epsilon\, v)(u + \epsilon\, v)_x = 0 \ . \end{aligned} \tag{2.7}$$

Differentiating (2.7) with respect to ϵ and putting $\epsilon = 0$ we get

$$\psi_*(v) = v_t - v_{xxx} - 6vu_x - 6uv_x = 0 \tag{2.8}$$

which is the Frèchet derivative of (2.5) in direction of v. It should be observed that (2.8) is a linear equation for v.

This is true generally also, i.e. (2.4) is an evolution type linear partial differential equation for each v^α, $\alpha = 1, 2, ..., q$.

Another important concept is the symmetry of partial differential equations which is generally defined by, [6] :

Definition 2.2 *A vector valued function* $\sigma = (\sigma^1, \sigma^2, ..., \sigma^q)$ *is a symmetry of (2.1) if and only if it satisfies the differential equations*

$$\frac{\partial}{\partial t} \sigma^\alpha = \varphi_{\alpha *}(\sigma) \tag{2.9}$$

modulo the original equation (2.1), for all $\alpha = 1, 2, ..., q$.

Example Again consider the Korteweg de Vries (KdV) equation (2.5).

$$u_\tau = 1 + 6tu_x \tag{2.10}$$

is a symmetry of KdV since it satisfies (2.8). This symmetry comes from the Galilean invariance of the KdV. In fact KdV has three more invariances i.e. x-translation, t-translation, and scaling. These four symmetries of KdV have orders less than or equal to the order of the original equation (2.5).

Let us consider another symmetry of KdV

$$u_\tau = u_{xxxxx} + 10uu_{xxx} + 20u_x u_{xx} + 30u^2 u_x \tag{2.11}$$

which has order greater than the order of KdV. This symmetry does not come from any invariance of the equation.

Definition 2.3 *The symmetries coming from the invariances of the differential equation are called the Lie or classical symmetries. The symmetries which do not come from any invariance of the equation are called the generalized or higher symmetries.*

The Lie symmetries have x-derivative order less than or equal to the x-derivative order of the original equation. Using this definition we can say that (2.10) is a Lie symmetry of KdV, and (2.11) is a generalized symmetry of KdV.

2.2 Integrability in $1 + 1$

There are many definitions of integrability and an integrable equation, but the most practical and useful one is the following, [6] :

Definition 2.4 *An equation is said to be integrable if it possesses infinitely many time-independent higher symmetries.*

Taking into account that there exists an algorithmic way of finding symmetries, it follows that the above definition can lead to a simple test for integrability, provided the following conjecture is valid, [6].

Conjecture 2.5 *If a system of equations possesses at least one time-independent higher symmetry, then it possesses infinitely many.*

The above conjecture holds for all known integrable equations. One easy way of determining whether a given equation possesses infinitely many symmetries, is the existence of a recursion operator.

Definition 2.6 *A recursion operator is an integro-differential operator which sends symmetries to symmetries, i.e. if R is a recursion operator and u_{τ_n} is a symmetry then $R u_{\tau_n} = u_{\tau_{n+1}}$ is the next symmetry.*

It can be obviously seen that if an equation admits a Lie symmetry and a recursion operator then we can generate infinitely many higher symmetries. Besides these some of the Lie symmetries can be generated from the other Lie symmetries by applying the recursion operator.

Example For the KdV equation the recursion operator is

$$R = D^2 + 4u + 2u_x D^{-1} \tag{2.12}$$

where D is total x-derivative. The Lie symmetries of KdV corresponding to the x-translation invariance and the t-translation invariance are respectively u_x and u_t. Here the interesting observation is $R u_x = u_t$, i.e. u_t is generated from u_x by using the KdV itself. The first time-independent higher symmetry (2.11) of KdV is, in fact, equal to $R^2 u_x$. Continuing this process or by showing generally that $R^n u_x$ satisfies (2.8) for any n, we can prove that KdV admits infinitely many higher symmetries, hence it is an integrable equation.

Chapter 3

Integrable Boundary Value Problems

It is now a well-known fact that the nonlinear partial differential equations integrable by the inverse scattering technique have infinitely many symmetries. Hence we expect that the boundary conditions consistent with the inverse scattering method must be compatible with these symmetries of the partial differential equations. Here in this chapter our aim is to propose a method to construct such boundary conditions for integrable equations.

3.1 Boundary Conditions Compatible with Symmetries

In the sequel we suppose that equation (1.1) admits a recursion operator of the form, [4], [7]

$$R = \sum_{i=0}^{i_1} \alpha_i D^i + \sum_{i=0}^{k_1} \alpha_{-1_i} D^{-1}\, \alpha_{-2_i} \ , \ i_1 \geq 0 \ , \ k_1 \geq 0 \tag{3.1}$$

where $\alpha_i, \alpha_{-1_i}, \alpha_{-2_i}$ are functions of the dynamical variables, D is the total derivative with respect to x. Passing to the new dynamical variables $\mathbf{v}$, $\mathbf{v}_t$, $\mathbf{v}_{tt}, \ldots$ we can obtain, by using (3.1), the recursion operator of the system of equations (1.4) takes the form:

$$\mathbf{R} = \sum_{i=0}^{M} \mathbf{a}_i (\partial_t)^i + \sum_{i=0}^{K} \mathbf{a}_{-\mathbf{1}_i} (\partial_t)^{-1}\, \mathbf{a}_{-\mathbf{2}_i} \ , \ M \geq 0 \ , \ K \geq 0 \tag{3.2}$$

where the coefficient matrices $\mathbf{a}_i$, $\mathbf{a}_{-1_i}$ and $\mathbf{a}_{-2_i}$ depend on $\mathbf{v}$ and on a finite number of its t-derivatives, ∂_t is the operator of total derivative with respect to t. If (1.1) is a scalar equation, R is a scalar operator, then $\mathbf{R}$ is an $n \times n$ matrix valued operator. To find boundary conditions compatible with a higher symmetry $\mathbf{v}_\tau = \mathbf{R}^{n_0}\, \mathbf{v}_t$ of the equation (1.4), we need the coefficient matrix $\mathbf{b_N}$ in the expression

$$\mathbf{R}^{n_0} = \mathbf{b_N}(\partial_t)^N + \mathbf{b_{N-1}}(\partial_t)^{N-1} + \ldots \tag{3.3}$$

to be a scalar multiple of the identity matrix. If $\mathbf{b_N}$ is not so, the impossibility of finding consistent boundary conditions is shown in the section 3.2 for a special case. The rest of this work depends highly on the following theorem:

Theorem 3.1 *Suppose* $\mathbf{p}(\mathbf{v}) = 0$ *is a constraint of rank* $n-1$ *for the equation (1.4), and* $\mathbf{R}$ *be the recursion operator (3.2). Assume that* $\mathbf{p}(\mathbf{v}) = 0$ *is compatible with a higher symmetry* $\mathbf{v}_\tau = \mathbf{R}^{n_0}\, \mathbf{v}_t$. *Then it is compatible with* $\mathbf{v}_\tau = H(\mathbf{R}^{n_0})\, \mathbf{v}_t$ *where* H *is a scalar polynomial with constant coefficients.*

Proof Introduce new variables $w^i = p^i$, $\forall i \leq n-1$ and $w^n = v^{j_0}$, for a fixed j_0 such that $1 \leq j_0 \leq n$, where p^i, $i \leq n-1$, are coordinates of the vector $\mathbf{p}$. So we have $\mathbf{w}_\tau = \tilde{\mathbf{R}}^{n_0}\, \mathbf{w}_t$ in the new variables and $\tilde{\mathbf{R}} = A\mathbf{R}A^{-1}$ where A is the Jacobian matrix of the transformation $\mathbf{v} \to \mathbf{w}$. Notice that under this change of variables the constraint $\mathbf{p}(\mathbf{v}) = 0$ turns into $w^i = 0$ for $i = 1, 2, \ldots, n-1$. Imposing this constraint reduces the equation $\mathbf{w}_\tau = \tilde{\mathbf{R}}^{n_0}\, \mathbf{w}_t$ to the form:

$$\begin{pmatrix} 0 \\ \ldots \\ 0 \\ w^n_\tau \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{R}}^{n_0}_{1,1} & \ldots & \tilde{\mathbf{R}}^{n_0}_{1,n-1} & \tilde{\mathbf{R}}^{n_0}_{1,n} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{\mathbf{R}}^{n_0}_{n-1,1} & \ldots & \tilde{\mathbf{R}}^{n_0}_{n-1,n-1} & \tilde{\mathbf{R}}^{n_0}_{n-1,n} \\ \tilde{\mathbf{R}}^{n_0}_{n,1} & \ldots & \tilde{\mathbf{R}}^{n_0}_{n,n-1} & \tilde{\mathbf{R}}^{n_0}_{n,n} \end{pmatrix} \begin{pmatrix} 0 \\ \ldots \\ 0 \\ w^n_t \end{pmatrix}$$

Here the condition $\tilde{\mathbf{R}}^{n_0}_{j,n} = 0\ \forall j,\ j = 1, 2, \ldots, n-1$ must hold. Really, letting $\tilde{\mathbf{R}}^{n_0}_{j,n} \neq 0$ for some $j \leq n-1$, the equation $\tilde{\mathbf{R}}^{n_0}_{j,n}\, w^n_t = 0$ gives a connection between the variables $w^n, w^n_t, \ldots$ which are supposed to be independent. So the nonlocal operator $\tilde{\mathbf{R}}^{no}$ is of the following form :

$$\begin{pmatrix} \tilde{\mathbf{R}}^{no}_{1,1} & \ldots & \tilde{\mathbf{R}}^{no}_{1,n-1} & 0 \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{\mathbf{R}}^{no}_{n-1,1} & \ldots & \tilde{\mathbf{R}}^{no}_{n-1,n-1} & 0 \\ \tilde{\mathbf{R}}^{no}_{n,1} & \ldots & \tilde{\mathbf{R}}^{no}_{n,n-1} & \tilde{\mathbf{R}}^{no}_{n,n} \end{pmatrix}$$

Its any power is also in the same form; it can be shown by using the equations

$$\sum_{j=1}^{n-1} \tilde{\mathbf{R}}^{no}_{i,j} 0 = 0 \quad \forall i = 1, 2, ..., n-1$$

since $\tilde{\mathbf{R}}^{no}$ contains integral operators also. We conclude that the boundary condition $\mathbf{p}(\mathbf{v}) = 0$ is compatible with the equation $\mathbf{w}_\tau = H(\tilde{\mathbf{R}}^{no})\,\mathbf{w}_t$, where $H = H(z)$ is any scalar polynomial of z with constant coefficients. But this is equivalent to the compatibility of the constraint with the equation $\mathbf{v}_\tau = H(\mathbf{R}^{no})\,\mathbf{v}_t$. □

3.2 Boundary Value Problem for the Burgers Equation and Its Uniqueness

For illustration let us give the Burgers equation in a detailed way. It is

$$u_t = u_{xx} + 2uu_x \tag{3.4}$$

which possesses the recursion operator

$$R = D + u + u_x\, D^{-1}\ , \tag{3.5}$$

[8]. The simplest symmetry of this equation is $u_\tau = u_x$. In terms of the new dynamical variables u, u_x this symmetry takes the form

$$\begin{aligned} u_\tau &= u_1 \\ u_{1\tau} &= u_t - 2uu_1 \end{aligned} \tag{3.6}$$

where $u_n = \partial^n u/\partial x^n$. This equation does not admit any invariant surface of the form $p(u, u_1) = 0$. Differentiating this constraint with respect to τ we get

$$\frac{\partial p}{\partial u} u_1 + \frac{\partial p}{\partial u_1}(u_t - 2uu_1) = 0 \ . \tag{3.7}$$

Since u_1 and u_t are independent, we can equate their coefficients to zero,

$$\Longrightarrow \ \frac{\partial p}{\partial u} = \frac{\partial p}{\partial u_1} = 0 \ . \tag{3.8}$$

This implies $p \equiv$ *constant* which is a trivial solution. As a conclusion we don't have any invariant surface in (u, u_1)-plane. The third order symmetry $u_\tau = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2 u_x$ of (3.4), rewritten in terms of the new variables (u, u_1) gives the following system of two equations

$$\begin{aligned} u_\tau &= u_{1t} + uu_t + (u^2 + u_1)u_1 \\ u_{1\tau} &= u_{tt} - uu_{1t} + (u^2 + u_1)u_t - 2uu_1(u^2 + u_1) \end{aligned} \tag{3.9}$$

which also does not possess any invariant surface of the form $p(u, u_1) = 0$. It may easily be proved that the same argument holds for every symmetry of odd order of (3.4), i.e. $u_\tau = u_{2m+1} + h(u_{2m}, ..., u)$, because the correspondent system of equations has different orders in the highest t-derivatives

$$u_\tau = \partial_t^m u_1 + \dots \ , \quad u_{1\tau} = \partial_t^{m+1} u + \dots \ . \tag{3.10}$$

But unlike the symmetries of odd order, for the symmetries of even order the correspondent system of equations has the same orders in the highest t-derivatives. This leads us to show that the symmetries of even order admit an invariant surface $p(u, u_1) = 0$, depending upon two arbitrary parameters.

Proposition 3.2 *If the boundary condition $p(u, u_1) = 0$ is compatible with an even order higher symmetry of the Burgers equation, then it is of the form $c(u_1 + u^2) + c_1 u + c_2 = 0$ and is compatible with every symmetry of the form $u_\tau = P(\mathbf{R}^2)u_t$ where P denotes any polynomial with constant coefficients.*

Proof The Frèchet derivative of (3.4) gives us the symmetry equation of the Burgers equation

$$\partial_t \sigma = (D^2 + 2uD + 2w)\,\sigma \tag{3.11}$$

where $w = u_1$. Since the operators are acting on the symmetries in (3.11), we may take

$$D^{-1} = \partial_t^{-1}(D + 2u) \tag{3.12}$$

in the recursion operator (3.5). After this substitution the recursion formula $u_{\tau_{i+1}} = R\, u_{\tau_i}$ becomes

$$u_{\tau_{i+1}} = (u + 2w\partial_t^{-1} u)\, u_{\tau_i} + (1 + w\partial_t^{-1})\, w_{\tau_i} \ . \tag{3.13}$$

Differentiating this equation with respect to x and replacing $w_x = u_2 = u_t - 2uw$ we can easily get

$$w_{\tau_{i+1}} = [\partial_t + 2(u_t - 2uw)\partial_t^{-1} u]\, u_{\tau_i} + [-u + (u_t - 2uw)\partial_t^{-1}]\, w_{\tau_i} \tag{3.14}$$

for $i = 1, 2, \ldots$. Hence our new recursion operator $\mathbf{R}$ is in the 2×2 matrix form as we proposed before

$$\mathbf{R} = \begin{pmatrix} u + 2\,w\,\partial_t^{-1} u & 1 + w\,\partial_t^{-1} \\ \partial_t + 2\,(u_t - 2\,u\,w)\,\partial_t^{-1} u & -u + (u_t - 2\,u\,w)\,\partial_t^{-1} \end{pmatrix} . \tag{3.15}$$

To apply the algorithm we shall square the recursion operator $\mathbf{R}$ given in the equation (3.15) so that the coefficient matrix of highest order ∂_t becomes the identity matrix. In fact we have

$$\mathbf{R}^2 = \mathbf{I}\,\partial_t + \begin{pmatrix} u^2 + w + 2u_t\,\partial_t^{-1} u & u_t\,\partial_t^{-1} \\ 2u_t + 2w_t\,\partial_t^{-1} u & u^2 + w + w_t\,\partial_t^{-1} \end{pmatrix}$$

where $\mathbf{I}$ is the 2×2 identity matrix. So the constraint $p(u,\, w)$ describes an invariant surface for the following system

$$\begin{pmatrix} u \\ w \end{pmatrix}_\tau = \mathbf{R}^2 \begin{pmatrix} u \\ w \end{pmatrix}_t \tag{3.16}$$

which is exactly the coupled Burgers type integrable system, [9]

$$\begin{aligned} u_\tau &= u_{tt} + 2(w + u^2)\, u_t \\ w_\tau &= w_{tt} + 2u_t^2 + 2(w + u^2)\, w_t \ . \end{aligned} \tag{3.17}$$

It is straightforward to show that the above system (3.17) is compatible with the constraint $p(u,w)=0$ only if $p=w+u^2+c_1u+c_2$ or $u=constant$. □

Now we will show the uniqueness of the boundary condition $p=w+u^2+c_1u+c_2$ by proving and using a new property of the Burgers hierarchy. We have the following proposition :

Proposition 3.3 *The function $u(x,t,\tau_n)$, $\forall n\geq 1$ satisfies infinitely many Burgers like equations*

$$u_{\tau_i\tau_i}-u_{\tau_{2i+2}}=-2\,u_{\tau_i}\,D^{-1}\,u_{\tau_i} \tag{3.18}$$

for all $i=-1,0,1,2,\ldots$. The Burgers equation, itself, corresponds to $i=-1$, ($\tau_{-1}=x$ and $\tau_0=t$). All u_{τ_i} for $i>-1$ correspond to higher symmetries.

It is very straightforward to determine the even numbered symmetries of the Burgers equation from (3.18) recursively. Here the interesting observation is the equation (3.18). According to it u satisfies the Burgers like equations with respect to the variables (τ_i,τ_{2i+2}) for all $i=-1,0,1,2,\ldots$.

Proof The proof of this proposition depends crucially on the definition of the higher symmetries of the Burgers equation. We define them through the relation

$$u_{\tau_n}=R^{n+1}\,u_x \tag{3.19}$$

where R is the recursion operator given in the equation (3.5) and $n\geq -1$. Equation (3.19) is equivalent to the equation

$$u_{\tau_n}=R\,u_{\tau_{n-1}}\ . \tag{3.20}$$

To prove (3.18) we will use induction.
For $m=-1$ (3.18) turns out to be the Burgers equation obviously. So assume it holds for $m=k$ and show holds also for $m=k+1$ i.e. show that the equation

$$u_{\tau_{k+1}\tau_{k+1}}-u_{\tau_{2k+4}}=-2u_{\tau_{k+1}}D^{-1}\,u_{\tau_{k+1}} \tag{3.21}$$

holds under the assumption. To perform this first of all we need to calculate $u_{\tau_{k+1}\tau_{k+1}}$ and $u_{\tau_{2k+4}}$ in terms of u_{τ_k} and $u_{\tau_{2k+2}}$ respectively. By using (3.20) we get

$$u_{\tau_{k+1}\tau_{k+1}} = \frac{\partial (R\, u_{\tau_k})}{\partial \tau_{k+1}}$$

where R is the recursion operator. Then writing R explicitely and taking the partial derivative

$$u_{\tau_{k+1}\tau_{k+1}} = (u_{\tau_{k+1}} + u_{x,\tau_{k+1}} D^{-1})\, u_{\tau_k} + R\, \frac{\partial\, u_{\tau_{k+1}}}{\partial \tau_k}$$

can easily be derived. Applying the same argument up to transforming all $u_{\tau_{k+1}}$'s to u_{τ_k}'s we conclude that

$$u_{\tau_{k+1}\tau_{k+1}} = R^2\, u_{\tau_k \tau_k} + D\,[(R\, u_{\tau_k})(D^{-1}\, u_{\tau_k})] + R\,[D\,(u_{\tau_k} D^{-1}\, u_{\tau_k})]\,. \quad (3.22)$$

Again by virtue of (3.20)

$$u_{\tau_{2k+4}} = R^2\, u_{\tau_{2k+2}} \quad (3.23)$$

is satisfied trivially. Now we can subtract $u_{\tau_{2k+4}}$ from $u_{\tau_{k+1}\tau_{k+1}}$ to prove the claim. But when we make this subtraction with the help of (3.22) and (3.23) and then use the assumption of induction, saying that the relation holds for $m = k$, it is straightforward to reach the desired result. □

Now by letting the most general boundary conditions $p = f(u, u_x) = 0$ at $x = x_0$ and taking τ_i and τ_{2i+2} derivatives for $i \geq 0$ of f and using the equation (3.18) we obtain

$$f_{u_x}^2\, f_{uu} + f_u^2\, f_{u_x u_x} - 2\, f_{u_x}^3 - 2\, f_u\, f_{u_x}\, f_{u u_x} = 0\ . \quad (3.24)$$

Making a change of variables $u = x_1$ and $u_x + u^2 + c_1 u + c_2 = x_2$ then equation (3.24) becomes

$$f_{x_2}^2\, f_{x_1 x_1} + f_{x_1}^2\, f_{x_2 x_2} - 2\, f_{x_1}\, f_{x_2}\, f_{x_1 x_2} = 0\ . \quad (3.25)$$

Assume $f_{x_2} \neq 0$ and let $g = f_{x_1}/f_{x_2}$, so we get that

$$g_{x_1} = g\, g_{x_2}\ . \quad (3.26)$$

This simple equation can be solved but we will not follow this way, in fact the general solution of (3.26) is $g = \rho(x_1 g + x_2)$. Since $f(u, u_x) = 0$ this , in principle, implies that either

a) $u_x = \phi(u)$

which implies $f = u_x - \phi(u)$ at $x = x_0$. Or

b) $u = \psi(u_x)$

which implies $f = u - \psi(u_x)$ at $x = x_0$.

If we insert corresponding f's in (3.24), respectively we get

a) $\phi'' + 2 = 0$

implying that $u_x + u^2 + c_1 u + c_2 = 0$ at $x = x_0$

b) $\psi'' + 2(\psi')^3 = 0$

implying that $u = constant$ for $\psi' = 0$ and a special case of **a** for $\psi' \neq 0$. Hence we found all possible boundary conditions compatible with symmetries.

Remark On the invariant surface $p(u, w) = 0$ the system (3.17) turns into the Burgers like equation $u_\tau = u_{tt} - 2(c_1 u + c_2)u_t$ which is also integrable, [5].

Chapter 4

Applications to Other Partial Differential Equations

After giving the method and an illustrative example with details, now we shall apply this technique to some other nonlinear partial differential equations to obtain the compatible boundary conditions.

4.1 The Nonlinear Schrödinger Equation

Our first equation is the following system

$$\begin{aligned} u_t &= u_2 + 2u^2 v \\ -v_t &= v_2 + 2uv^2 \end{aligned} \tag{4.1}$$

which is, under the substitution $v \to u^*$ and $t \to it$, equivalent to the well-known nonlinear Schrödinger equation, where $*$ is the complex conjugation.

Since we are dealing with a system of two equations, the initial recursion operator will be the following 2×2 operator matrix

$$R = \begin{pmatrix} D + 2uD^{-1}v & 2uD^{-1}u \\ -2vD^{-1}v & -D - 2vD^{-1}u \end{pmatrix} \tag{4.2}$$

with respect to the column vector $(u_\tau, v_\tau)^T$, where T is the transposition operation. Our dynamical variables, of course according to the algorithm, are

u, u_1, v, v_1. Here u_1, v_1 denote u_x, v_x respectively. Trivially higher derivatives of u and v can be represented in terms these dynamical variables and their t-derivatives, with the aid of the system (4.1). Now after transforming R into t dependent form with respect to the above dynamical variables, we get the following 4×4 matrix

$$\mathbf{R} = \begin{pmatrix} -2u\partial_t^{-1}v_1 & 1 + 2u\partial_t^{-1}v & 2u\partial_t^{-1}u_1 & -2u\partial_t^{-1}u \\ \xi - 2u_1\partial_t^{-1}v_1 & 2u_1\partial_t^{-1}v & 2u_1\partial_t^{-1}u_1 & -2u_1\partial_t^{-1}u \\ 2v\partial_t^{-1}v_1 & -2v\partial_t^{-1}v & -2v\partial_t^{-1}u_1 & -1 + 2v\partial_t^{-1}u \\ 2v_1\partial_t^{-1}v_1 & -2v_1\partial_t^{-1}v & \eta - 2v_1\partial_t^{-1}u_1 & 2v_1\partial_t^{-1}u \end{pmatrix}$$

where $\xi = \partial_t - 2uv$ and $\eta = \partial_t + 2uv$. To obtain the coefficient matrix of the highest power of ∂_t as a scalar multiple of the identity matrix, we shall square $\mathbf{R}$, then apply to the column matrix $(u, u_1, v, v_1)^T$. Then we get a system of four equations

$$\begin{aligned} u_\tau &= u_{tt} - 2u^2 v_t - 4u\, v_1\, u_1 + 2\, v\, u_1^2 - 2\, u^3\, v^2 , \\ u_{1\tau} &= u_{1,tt} - 2u^2 v_{1t} - 2\, u_1^2\, v_1 - 6\, u^2\, v^2\, u_1 - \\ &\quad 4\, u\, v_1\, u_t + 4\, v\, u_1\, u_t + 4\, v\, u^3\, v_1 , \\ v_\tau &= -v_{tt} - 2\, v^2\, u_t + 4\, v\, u_1\, v_1 - 2\, u\, v_1^2 + 2\, v^3\, u^2 , \\ v_{1\tau} &= -v_{1,tt} - 2\, v^2\, u_{1,t} + 2\, v_1^2\, u_1 + 6\, v_1\, v^2\, u^2 - \\ &\quad 4\, v\, u_1\, v_t + 4\, u\, v_1\, v_t - 4\, v^3\, u\, u_1 \end{aligned} \tag{4.3}$$

which is supposed to admit a boundary condition of the form $u_1 = p^1(u, v)$, $v_1 = p^2(u, v)$ at $x = x_0$. Using the fourth order symmetry (4.3) we can directly determine the compatible constraints $p^1(u, v)$, $p^2(u, v)$ solving some differential equations.

So for nonlinear Schrödinger equation we get the consistent boundary conditions to be

$$u_1 = p^1 = cu \text{ and } v_1 = p^2 = cv \text{ at } x = x_0 . \tag{4.4}$$

Since the system (4.3) is of the form

$$(u, u_1, v, v_1)^T_\tau = \mathbf{R}^2 (u, u_1, v, v_1)^T_t \tag{4.5}$$

we have the following corollary to the theorem 3.1:

Corollary 4.1 *The boundary conditions (4.4) are compatible with the symmetries of the form* $(u, u_1, v, v_1)^T_\tau = H(\mathbf{R}^2)(u, u_1, v, v_1)^T_t$ *where* H *is any scalar polynomial with constant coefficients.*

The analytical properties of this boundary value problem are studied previously ([2], [10], [11]) by means of the inverse scattering technique.

Remark On the invariant surface $u_1 = cu$, $v_1 = cv$ the system (4.3) is reduced to a system of two equations

$$
\begin{aligned}
u_\tau &= u_{tt} - 2u^2 v_t - 2c^2 u^2 v - 2u^3 v^2 \,, \\
v_\tau &= v_{tt} - 2v^2 u_t + 2c^2 v^2 u + 2v^3 u^2 \,.
\end{aligned}
\tag{4.6}
$$

The integrability of (4.6) is shown in [4]. Under a suitable change of variable (4.6) becomes the famous derivative nonlinear Schrödinger equation.

4.2 The Harry-Dym Equation

Among the integrable nonlinear partial differential equations, the Harry-Dym equation

$$u_t + u^3 u_3 = 0 \tag{4.7}$$

is of special interest. It is not quasilinear and so its analytical properties are not typical. It has the recursion operator

$$R = u^3 D^3 \, u D^{-1} \, \frac{1}{u^2} \,, \tag{4.8}$$

given in [13]. For the Harry-Dym equation, the new dynamical variables are u, u_1, u_2 but unfortunately passing to this set of variables is not regular since $u_3 = -\frac{u_t}{u^3}$ has a singular surface given by $u = 0$ which should be examined seperately. The reflection symmetry $x \to -x$, $u \to -u$, $t \to t$ exists in the Harry-Dym equation itself and its all higher symmetries so the trivial boundary condition $u(0, t) = 0$ is consistent with the integrability.

The transformed recursion operator $\mathbf{R}$ is given by the matrix

$$\mathbf{R} = \begin{pmatrix} uw + u_t\,\partial_t^{-1}\,w & -uv - u_t\,\partial_t^{-1}\,v & u^2 + u_t\,\partial_t^{-1}\,u \\ \xi + vw + v_t\,\partial_t^{-1}\,w & -v^2 - v_t\,\partial_t^{-1}\,v & uv + v_t\,\partial_t^{-1}\,u \\ w^2 + w_t\,\partial_t^{-1}\,w & \xi - vw - w_t\,\partial_t^{-1}\,v & uw + w_t\,\partial_t^{-1}\,u \end{pmatrix}$$

where $v = u_x$, $w = u_{xx}$ and $\xi = \frac{1}{u}\,\partial_t - \frac{u_t}{u^2}$. In this case to have the coefficient matrix of highest order ∂_t as a scalar multiple of the identity matrix, we should cube $\mathbf{R}$. Now we can assume a boundary condition $p(u,v,w) = 0$ compatible with the ninth order symmetry given by

$$(u,v,w)_\tau^T = \mathbf{R}^3\,(u,v,w)_t^T\ , \tag{4.9}$$

the expressions of u_τ, v_τ, w_τ are very long and so they are not written here explicitely.

We shall note that for the constraint $p = 0$ we have two choices of its rank; either one or two. If it is one; we don't have any regular invariant surface. The second choice leads to the invariant surfaces

$$u_x|_{x=x_0} = cu\ \ ,\ \ u_{xx}|_{x=x_0} = \frac{c^2 u}{2} \tag{4.10}$$

which is compatible with an infinite number of symmetries.

Remark On the invariant surface (4.10), the ninth order symmetry u_τ

$$\begin{aligned} u_\tau \;=\; & -u_{ttt} + 3u_{tt}u_t\frac{1}{u} - \frac{3}{2}u_{tt}u_1 h - \frac{3}{2}\frac{u_t^3}{u^2} + \frac{3}{2}uu_{1tt}h + \\ & \frac{3}{2}uu_{1t}h_t - \frac{15}{16}uh^2h_t - \frac{5}{16}h^3u_t - \frac{3}{2}u_1u_th_t \end{aligned} \tag{4.11}$$

where $h = 2uu_2 - u_1^2$, takes the form

$$u_\tau = -u_{ttt} + \frac{3u_tu_{tt}}{u} - \frac{3u_t^3u^2}{2} \tag{4.12}$$

which is equivalent to the Modified Korteweg de Vries equation.

We have the following corollary to the theorem 3.1 for the Harry-Dym equation:

Corollary 4.2 *The boundary condition (4.10) of (4.7) is compatible with every symmetry of the form* $(u,v,w)_\tau^T = H(\mathbf{R}^3)\,(u,v,w)_t^T$, *where* H *is any scalar polynomial with constant coefficients.*

4.3 The Korteweg de Vries and the Modified Korteweg de Vries Equations

Now we will consider the well-known Korteweg de Vries (KdV) equation. It is the following equation

$$u_t = u_3 + 6uu_1 \tag{4.13}$$

possessing the recursion operator

$$R = D^2 + 4u + 2u_1 D^{-1}\ . \tag{4.14}$$

This recursion operator R may be represented in the form

$$\mathbf{R} = \begin{pmatrix} 4u + 12v\,\partial_t^{-1} u & 0 & 1 + 2v\,\partial_t^{-1} \\ \partial_t + 12w\,\partial_t^{-1} u & -2u & 2w\,\partial_t^{-1} \\ 2w + 12(u_t - 6uv)\,\partial_t^{-1} u & \partial_t - 2v & -2u + 2(u_t - 6uv)\,\partial_t^{-1} \end{pmatrix}$$

To apply the technique we shall take the cube of $\mathbf{R}$, i.e. we are looking for the invariant surface of the system

$$(u,v,w)_\tau^T = \mathbf{R}^3\,(u,v,w)_t^T\ , \tag{4.15}$$

where $v = u_1$ and $w = u_2$ which are our new dynamical variables with u itself. It is straightforward to determine the boundary condition compatible with the symmetries is

$$u = 0,\ w = 0 \text{ at } x = x_0 \tag{4.16}$$

and of course v is any function of t at $x = x_0$.

Corollary 4.3 *The boundary condition (4.16) for KdV equation is compatible with all symmetries of the form* $(u,v,w)_\tau^T = H(\mathbf{R}^3)\,(u,v,w)_t^T$ *where* H *is any scalar polynomial with constant coefficients.*

The Modified KdV equation

$$u_t = u_3 + 6u^2 u_1 \tag{4.17}$$

can be handled very similary. Its recursion operator is the following operator

$$R = D^2 + 4u^2 + 4u_1 D^{-1} u \, . \tag{4.18}$$

We can write R in the following matrix form with same new dynamical variables of KdV

$$\mathbf{R} = \begin{pmatrix} 4u^2 + 24v\,\partial_t^{-1} u^3 & -4v\,\partial_t^{-1} v & 1 + 4v\,\partial_t^{-1} u \\ \partial_t + 24w\,\partial_t^{-1} u^3 + 4w\,\partial_t^{-1} w & -2u^2 - 4w\,\partial_t^{-1} v & 4w\,\partial_t^{-1} u \\ 4\xi\,\partial_t^{-1} w + 24\xi\,\partial_t^{-1} u^3 + 4uw & \partial_t - 4uv - 4\xi\,\partial_t^{-1} v & -2u^2 + 4\xi\,\partial_t^{-1} u \end{pmatrix}$$

where $\xi = u_t - 6u^2 v$. Also in the Modified KdV equation we will work with $\mathbf{R}^3$. Here the consistent boundary conditions turn out to be

$$u|_{x=x_0} = 0, \; u_x|_{x=x_0} = 0 \tag{4.19}$$

and $u_{xx}|_{x=x_0}$ is any function of t.

Corollary 4.4 *The consistent boundary conditions (4.19) for the Modified KdV equation are compatible with all symmetries of the form* $(u,v,w)_\tau^T = H(\mathbf{R}^3)\,(u,v,w)_t^T$ *where* H *is any scalar polynomial with constant coefficients.*

4.4 The Boussinesq Equation

The Boussinesq equation

$$u_{tt} = \frac{1}{3} u_{xxxx} + \frac{4}{3}(u^2)_{xx} \tag{4.20}$$

can be converted into an equivalent evolutionary system, [9]

$$\begin{aligned} u_t &= v_x \\ v_t &= \tfrac{1}{3} u_{xxx} + \tfrac{8}{3} u u_x \, . \end{aligned} \tag{4.21}$$

This is a very typical system since the orders of the x-derivatives of the equations are not equal. It has two Hamiltonian operators given in [9] which lead us to the determine the recursion operator R as the following

$$R = \begin{pmatrix} 3v + 2v_1 D^{-1} & D^2 + 2u + u_1 D^{-1} \\ \frac{1}{3}D^4 + \frac{10}{3}uD^2 + 5u_1 D + 3u_2 + \frac{16}{3}u^2 + \xi\, D^{-1} & 3v + v_1 D^{-1} \end{pmatrix}$$

where u_i and v_i denote corresponding i times x-derivatives as usual and $\xi = \frac{2}{3}u_3 + \frac{16}{3}uu_1$. Defining new dynamical variables u, z, w, v such that $z = u_1$ and $w = u_2$ we can find the transformed recursion operator $\mathbf{R}$, which is a 4×4 matrix opeartor. It is

$$\mathbf{R} = \begin{pmatrix} 3v + 8L_1 u & \partial_t & L_1 & 6K + 2u \\ \zeta + 8Su & 3v & S + \partial_t & 3z + 6L_2 \\ \eta & \mu & \kappa & \gamma \\ \delta + 8K & -\frac{1}{3}z & \frac{2}{3}u + K & 2v_t\partial_t^{-1} + 3v \end{pmatrix}$$

where

$$\begin{aligned}
\eta &= 7z_t + 8(v_t - \tfrac{8}{3}uz)\partial_t^{-1}\, u - 8\partial_t z + 5z\partial_t \ , \\
\mu &= 8u_t + 2u\partial_t - 8\partial_t u \ , \\
\kappa &= 3v + (v_t - \tfrac{8}{3}uz)\partial_t^{-1} \ , \\
\gamma &= 4w + 3\partial_t^2 + 2w_t\partial_t^{-1} \ , \\
\zeta &= 5u_t + 2u\partial_t \ , \\
\delta &= \partial_t^2 + \tfrac{1}{3}w + \tfrac{16}{3}u^2 \ , \\
L_1 &= \tfrac{1}{3}z\partial_t^{-1} \ , \\
L_2 &= \tfrac{1}{3}z_t\partial_t^{-1} \ , \\
K &= \tfrac{1}{3}u_t\partial_t^{-1} \ , \\
S &= \tfrac{1}{3}w\partial_t^{-1} \ .
\end{aligned}$$

In this case the $\mathbf{R}^3$ works, i.e. the invariant surface for system

$$(u, z, w, v)_\tau^T = \mathbf{R}^3\, (z, w, 3v_t - 8uz, u_t)^T \tag{4.22}$$

will be searched. Here the first symmetry used for the method is different from the previous ones since it is the suitable symmetry. The resulting equations of (4.22) are very long and complicated. We found the compatible boundary conditions in the form $u = c_1$, $z = c_2$, $w = c_3$, and $v = c_4$ where c_i are arbitrary constants $\forall i = 1, 2, 3, 4$, subject to satisfy some constraints. With respect to these constraints we have three distinct boundary conditions

$$\begin{array}{rl} i\ ) & u = c_1,\ z = c_2,\ w = c_3,\ v = 0 \\ ii\ ) & u = c_1,\ z = 0,\ w = 0,\ v = c_4 \\ iii\ ) & u = c_1,\ z = c_2,\ w = c_3,\ v = c_4, \text{ where} \\ & c_4^2 = \frac{1}{7}c_2^2 - \frac{16}{21}c_1^3 - \frac{18}{35}c_3c_1 \text{ when } c_4 \neq 0,\ c_2 \neq 0 \text{ or } c_4 \neq 0,\ c_3 \neq 0 \end{array} \tag{4.23}$$

Again for the Boussinesq equation we can give following corollary to the theorem 3.1:

Corollary 4.5 *The boundary conditions (4.23) are compatible with all symmetries of the form* $(u, z, w, v)_\tau^T = H(\mathbf{R}^3)\,(z, w, 3v_t - 8uz, u_t)^T$, *where* H *is any scalar polynomial with constant coefficients.*

Chapter 5

Weak Compatibility for the Burgers Equation

The concept of weak compatibility is something different from compability we discussed in the previous chapters. Here we will consider it only for the Burgers equation. In this approach we need the generalized symmetries very heavily. To this end we give the first five symmetries of the Burgers equation :

$$\begin{aligned} u_{\tau_{-1}} &= u_1 , \\ u_{\tau_0} &= u_2 + 2uu_1 , \\ u_{\tau_1} &= u_3 + 3uu_2 + 3u_1^2 + 3u^2u_1 , \\ u_{\tau_2} &= u_4 + 4uu_3 + 10u_1u_2 + 6u^2u_2 + 12uu_1^2 + 4u^3u_1 , \\ u_{\tau_3} &= u_5 + 10u^2 + 15u_1u_3 + 5uu_4 + 15u_1^3 + 50uu_1u_2 , \\ & 10u^2u_1 + 10u^3u_2 + 30u^2u_1^2 + 5u^4u_1 + 10u^2u_3 \ . \end{aligned} \tag{5.1}$$

i) Now start from a symmetry of the Burgers equation which is compatible with boundary conditions found in chapter 3 :

$$u_{\tau_2} = u_4 + 4uu_3 + 10u_1u_2 + 6u^2u_2 + 12uu_1^2 + 4u^3u_1 \ . \tag{5.2}$$

The subscript 2 of τ comes from the equation $u_{\tau_n} = R^{n+1} u_x$. It can be converted into the following form by using the Burgers equation itself

$$u_{\tau_2} = u_{tt} + 2(u_x + u^2)u_t \ . \tag{5.3}$$

This is an evolution equation in $2+1$; τ_2 is the time variable, x and t are space variables. Consider a surface $x = x_0$ in the manifold M on which the equation (5.3) is defined. Now write the equation on $x = x_0$:

$$v_{\tau_2} = v_{tt} + 2(w + v^2)v_t \tag{5.4}$$

where $v = u|_{x=x_0}$ and $w = u_x|_{x=x_0}$. The equation (5.4) is in the (t, τ_2)-space.
ii) The next step is to make a classification to determine for which w's the equation (5.4) is integrable. Classification of the above equation up to some transformation is known, [10]. If

$$w = -v^2 + c_1 v + c_2 \tag{5.5}$$

then the equation (5.4) is integrable where c_1 and c_2 are arbitrary constants. Moreover (5.5) is the only choice for integrability. The constraint (5.5) is weakly compatible with the equation (5.4). It is an interesting observation that (5.5) is the same boundary condition with the one we found in chapter 3 when we were discussing the the compatibility.

If we consider the next even order symmetry $u_{\tau_4} = R\, u_{\tau_3}$ which after using the original equation, takes the form:

$$\begin{aligned} u_{\tau_4} &= u_{ttt} + 3u_{1t}u_t + 3u_1^2 u_t + 3u_1 u_{tt} + \\ &\quad 6u_1 u^2 u_t + 3u_{tt}u^2 + 6u_t^2 u + 3u_t u^4 \ . \end{aligned} \tag{5.6}$$

We are looking for the boundary conditions on the surface $x = x_0$, hence again by letting $u_x|_{x=x_0} = w$ and $u|_{x=x_0} = v$ we bring the equation (5.6) into an equation in $1+1$. It is

$$\begin{aligned} v_{\tau_4} &= v_{ttt} + 3w_t v_t + 3w^2 v_t + 3w v_{tt} + \\ &\quad 6wv^2 v_t + 3v^2 v_{tt} + 6v v_t^2 + 3v^4 v_t \ . \end{aligned} \tag{5.7}$$

We can write (5.7) in the following form in (t, τ_4)-space

$$v_{\tau_4} = v_{ttt} + 3S v_{tt} + 3S^2 v_t + 3S_t v_t \tag{5.8}$$

where $S = w + v^2$. The problem is to find S's which make (5.8) integrable. In this case we don't have a classification so we can't immediately determine the form of S. Although we do not have the classification we know an example of an integrable equation which is the third order symmetry of the Burgers equation

$$u_{\tau_1} = u_3 + 3u_1^2 + 3uu_2 + 3u^2u_1 \tag{5.9}$$

having a very similar form with (5.8). Now transform (5.9) with $u \to c_1q + c_2$ where c_1 and c_2 are arbitrary constants. Then the equation (5.9) takes the form

$$q_{\tau_1} = q_3 + 3(c_1q + c_2)q_2 + 3(c_1q + c_2)^2q_1 + 3c_1q_1^2 \tag{5.10}$$

which is of the same form with (5.8), but only it is in (x, τ_1)- space. Hence we conclude that if $S = c_1v + c_2$ then the equation (5.8) is integrable. More explicitely

$$w = -v^2 + c_1v + c_2$$

is weakly compatible with the symmetry u_{τ_4}. This can be done for all even order symmetries. We have the following proposition :

Proposition 5.1 *The boundary condition* $w = -v^2 + c_1v + c_2$ *defined at* $x = x_0$*, is weakly compatible with all even order symmetries* $u_{\tau_{2n}}$ *of the Burgers equation.*

Proof The recursion operator of the Burgers equation is

$$R = D + u + u_x D^{-1} \ .$$

Any even order symmetry $u_{\tau_{2n}}$ can be represented as

$$u_{\tau_{2n}} = R^{2n+1}\, u_x \tag{5.11}$$

which is of $(2n+2)^{nd}$ order in x-derivatives. Using the Burgers equation itself in (5.11) sufficiently many times, it turns out to be $(n+1)^{st}$ order in t-derivatives. We have the following equivalent representations to (5.11) :

$$u_{\tau_{2n}} = R^{2n}\, u_t = (R^2)^n\, u_t \ . \tag{5.12}$$

It is easily seen that R^2 plays a special role in the determination of the even order symmetries. Hence let us calculate R^2 :

$$R^2 = D^2 + 3u_x + u_t\, D^{-1} + 2u\, D + u^2 \tag{5.13}$$

where u_t is written from the original equation. From the Frèchet derivative of the Burgers equation we can see that

$$D^2 = \partial_t - 2u\, D - 2u_x \tag{5.14}$$

which is valid when acting on the symmetries. Also from section 3.2 we know that

$$D^{-1} = \partial_t^{-1}\,(D + 2u)\ . \tag{5.15}$$

Inserting (5.14) and (5.15) in the expression of R^2 we get

$$R^2 = \partial_t + u_x + u^2 + u_t\, \partial_t^{-1}\,(D + 2u)\ . \tag{5.16}$$

Now we wish to write (5.12) on $x = x_0$. Hence we shall evaluate R^2 at $x = x_0$. For this we need to know how D operates on that surface. At $x = x_0$ we have the boundary condition $p(u, u_x) = u_x + u^2 - c_1 u - c_2 = 0$. Now taking the Frèchet derivative of p we have

$$\left(\frac{\partial}{\partial x}\sigma + 2u\sigma - c_1\sigma\right)\Bigg|_{x\,=\,x_0} = 0 \tag{5.17}$$

So in the operator language

$$D + 2u - c_1 = 0 \tag{5.18}$$

at $x = x_0$, acting on the symmetry σ. Note that (5.18) is valid for any symmetry. The equation (5.18) determines how D operates on the symmetries at $x = x_0$. Subsituting this in (5.16) we get R^2 at $x = x_0$

$$R^2 = \partial_t + c_1 v + c_2 + c_1 v_t\, \partial_t^{-1} \tag{5.19}$$

where $v = u|_{x=x_0}$. Now perform a linear transformation

$$c_1 v + c_2 = \rho$$

which brings (5.19) into the following form :

$$R^2 = \partial_t + \rho + \rho_t\, \partial_t^{-1}\ . \tag{5.20}$$

We can immediately observe that (5.20) is of the same form with the original recursion operator of the Burgers equation, in (t, τ_{2n})-space. Let $R^2|_{x=x_0} = \bar{R}$ and then we have

$$\rho_{\tau_{2n}} = \bar{R}^n\, \rho_t \tag{5.21}$$

which gives the $(n+1)^{st}$ order symmetries of the new Burgers equation

$$\rho_{\tau_2} = \rho_{tt} + 2\rho\rho_t \ . \tag{5.22}$$

Since the symmetries of the Burgers equation are integrable, its even order symmetries are weakly compatible with the boundary condition given in the proposition 5.1 . □

Chapter 6

Conclusion

In this work we investigated an efficient method to construct the boundary conditions of integrable evolution equations and systems which are consistent with integrability. We first proved a theorem which mainly says that if a boundary condition is compatible with a generalized symmetry of an integrable equation then it is compatible with infinitely many generalized symmetries. Based on this theorem we gave a method to construct the compatible boundary conditions. We applied this method to some very well-known integrable equations and got the compatible boundary conditions with the symmetries. To apply the method, a new form of the recursion operators has been introduced, which leads us to determine the constraints algorithmically.

In the last chapter the notion of *weak compatibility* has been discussed only for the Burgers equation as a way to the future works. The compatibility of the boundary conditions with an infinite number of symmetries, in the weak sense, is not yet shown for all integrable equations.

REFERENCES

[1] B. Gürel, M. Gürses, I. Habibullin, *Phys.Lett.* **190A** , 231, (1994)

[2] E. K. Sklyanin, *Func. Anal. Prilozh.* **21**, 86, (1987)

[3] I. T. Habibullin, *Phys. Lett.* **178A**, 369, (1993)

[4] A. V. Mikhailov, V. V. Sokolov, A. B. Shabat, In: *What Is Integrability?*, Ed. V. E. Zakharov, Berlin, Heidelberg, New York: Springer-Verlag, (1991)

[5] A. S. Fokas, *J. Math. Phys.* **21**, 1318, (1980)

[6] A. S. Fokas, *Studies in Applied Mathematics* **77**, 253, (1987)

[7] M. Gürses, A. Karasu and A. Satır, In: *Nonlinear evolution equations and dynamical systems, NEEDS'91*, Eds. M. Boiti, L. Martinelli and F. Pempinelli, World Scientific, Singapore , (1992)

[8] M. Gürses, A. Karasu, *Variable Coefficient Third Order KdV type of Equations*, To be published in *J. Math. Phys.* (1995)

[9] P. J. Olver, *Applications of Lie groups to differential equations*, Berlin, Heidelberg, New York: Springer-Verlag, (1986)

[10] S. I. Svinolupov, *Phys. Lett.* **135A**, 32, (1989)

[11] P. N. Bibikov and V. O. Tarasov, *Teor. Mat. Fiz.* **79**, 334, (1989)

[12] I. T. Khabibullin, *Teor. Mat. Fiz.* **86**, 130, (1991)

[13] M. Leo, R. A. Leo, G. Soliani and L. Solombrino, *Lettere Al Nuovo Cimento*, **38**, 45, (1983)

[14] M. Kac and P. van Moerbeke, *Proc. Nat. Acad. Sci. USA* **72**, 2879, (1975)

[15] A. S. Fokas and A. R. Its , *Phys. Rev. Lett.* **68**, 3117, (1992)