ANALYTIC AND ASYMPTOTIC PROPERTIES OF NON - SYMMETRIC LINNIK'S PROBABILITY DENSITIES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE INSTITUTE OF ENGINEERING AND SCIENCES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
M. Burak Erdoğan
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Iossif V. Ostrovskii(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

ANALYTIC AND ASYMPTOTIC PROPERTIES OF NON-SYMMETRIC LINNIK'S PROBABILITY DENSITIES

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M.S. in Mathematics
Supervisor: Prof. Iossif V. Ostrovskii
August 1995

We prove that the function

$$\varphi_\alpha^\theta(t) = \frac{1}{1 + e^{-i\theta \operatorname{sgn} t}|t|^\alpha}\,,\ \alpha \in (0,2)\,,\ \theta \in \mathbb{R},$$

is a characteristic function of a probability distribution if and only if $(\alpha,\theta) \in PD = \{(\alpha,\theta) : \alpha \in (0,2), |\theta| \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}) \pmod{2\pi}\}$. This distribution is absolutely continuous, its density is denoted by $p_\alpha^\theta(x)$. For $\theta = 0 \pmod{2\pi}$, it is symmetric and was introduced by Linnik (1953). Under another restrictions on θ it was introduced by Laha (1960), Pillai (1990), Pakes (1992).

In the work, it is proved that $p_\alpha^\theta(\pm x)$ is completely monotonic on $(0,\infty)$ and is unimodal on $\mathbb{R}$ for any $(\alpha,\theta) \in PD$. Monotonicity properties of $p_\alpha^\theta(x)$ with respect to θ are studied. Expansions of $p_\alpha^\theta(x)$ both into asymptotic series as $x \to \pm\infty$ and into conditionally convergent series in terms of $\log|x|$, $|x|^{k\alpha}$, $|x|^k$ $(k = 0,1,2,...)$ are obtained. The last series are absolutely convergent for almost all but not for all values of $(\alpha,\theta) \in PD$. The corresponding subsets of PD are described in terms of Liouville numbers.

Keywords : Cauchy type integral, Characteristic function, Completely monotonicity, Liouville numbers, Plemelj-Sokhotskii formula, Unimodality

ÖZET

SİMETRİK OLMAYAN LİNNİK OLASILIK YOĞUNLUKLARININ ANALİTİK VE ASİMTOTİK ÖZELLİKLERİ

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$$\varphi_\alpha^\theta(t) = \frac{1}{1 + e^{-i\theta \operatorname{sgn} t}|t|^\alpha}, \; \alpha \in (0,2), \; \theta \in \mathbb{R},$$

fonksiyonu bir olasılık dağılımının karakteristik fonksiyonudur, ancak ve ancak $(\alpha,\theta) \in PD = \{(\alpha,\theta) : \alpha \in (0,2), |\theta| \leq \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}) \pmod{2\pi}\}$. Bu dağılım mutlak süreklidir ve yoğunluğu $p_\alpha^\theta(x)$ ile gosterilir. $\theta = 0 \pmod{2\pi}$ için simetriktir ve Linnik (1953) tarafından ortaya atılmıştır. Ayrıca, Laha (1960), Pillai (1990) ve Pakes (1992) tarafından θ üzerine başka sınırlamalar getirilerek incelenmiştir.

Bu çalışmada, her $(\alpha,\theta) \in PD$ için, $p_\alpha^\theta(\pm x)$'in $(0,\infty)$ üzerinde tam monoton ve $\mathbb{R}$ üzerinde unimodel olduğu ispatlanmiştır. $p_\alpha^\theta(x)$'in θ'ya göre monotonluk özellikleri incelenmiştir. $p_\alpha^\theta(x)$'in x sonsuza giderken asimtotik seri açılımıyla, $\log|x|$, $|x|^{k\alpha}$, $|x|^k$ $(k = 0,1,2,...)$ terimleri cinsinden koşulsal yakınsak seriye açılımı elde edilmiştir. Bunlardan ikincisi $(\alpha,\theta) \in PD$ nin hemen hemen bütün değerleri (fakat tümü değil) için mutlak yakınsaktir. PD'nin karşılık gelen altkümeleri Liouville sayıları cinsinden ifade edilir.

Anahtar Kelimeler : Cauchy tipi integral, Karakteristik fonksiyon, Tam monotonluk, Liouville sayıları, Plemelj-Sokhotskii formülü, Unimodellik

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Chapter 1

Introduction

In 1953, Ju. V. Linnik [1] proved that the function

$$\varphi_\alpha(t) = \frac{1}{1+|t|^\alpha}\,,\ \alpha \in (0,2), \tag{1.1}$$

is a characteristic function of a symmetric probability density $p_\alpha(x)$. Since then, the family of symmetric Linnik's densities $\{p_\alpha(x) : \alpha \in (0,2)\}$ had several probabilistic applications (see, e.g. [2]-[7]). In 1994, S. Kotz, I. V. Ostrovskii and A. Hayfavi [8] carried out a detailed investigation of analytic and asymptotic properties of $p_\alpha(x)$.

In 1992, A. G. Pakes [9] showed that, in some characterization problems of Mathematical Statistics, the probability densities with characteristic functions

$$\varphi_\alpha^{\theta,\nu}(t) = \left(\frac{1}{1+e^{i\theta \operatorname{sgn} t}|t|^\alpha}\right)^\nu\,,\ \alpha \in (0,2)\,,\ |\theta| \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})\,,\ \nu > 0, \tag{1.2}$$

play an important role. These densities can be viewed as generalizations of symmetric Linnik's densities. For $\nu = 1$, $|\theta| = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, these densities had appeared in the papers by R. G. Laha [10] and R. N. Pillai [11]. Therefore, the problem of study of analytic and asymptotic properties of the densities with characteristic function (1.2) seems to be of interest. In this paper, we restrict ourselves to the study of the case $\nu = 1$.

We show that the function

$$\varphi_\alpha^\theta(t) = \frac{1}{1+e^{-i\theta \operatorname{sgn} t}|t|^\alpha},\ \alpha \in (0,2)\,,\ \theta \in \mathbb{R}, \tag{1.3}$$

is a characteristic function of a probability distribution iff $|\theta| \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}) \pmod{2\pi}$. The distribution is absolutely continuous. We denote its density by $p_\alpha^\theta(x)$. Surely, for $\theta = 0 \pmod{2\pi}$, $p_\alpha^\theta(x)$ coincides with symmetric Linnik's density $p_\alpha(x)$. For $\theta \ne 0 \pmod{2\pi}$, we call $p_\alpha^\theta(x)$ non-symmetric Linnik's density. Applying some ideas of [8], we study analytic and asymptotic properties of $p_\alpha^\theta(x)$ and obtain generalizations of results of [8]. As in the symmetric case, convergence of series expansions of $p_\alpha^\theta(x)$ depends on the arithmetical nature of the parameter α, but several new phenomena appear connected with the non-symmetry parameter θ.

Chapter 2

Statement of Results

When considering the function $\varphi_\alpha^\theta(t)$ defined by (1.3), we shall assume without loss of generality that the parameter θ satisfies the additional condition $\theta \in (-\pi, \pi)$. This follows from 2π-periodicity of $\varphi_\alpha^\theta(t)$ with respect to θ and its discontinuity in t for $\theta = \pi$.

The fact that the function $\varphi_\alpha(t)$ $(= \varphi_\alpha^0(t))$ defined by (1.1) is a characteristic function of a probability density was deduced by Linnik [1] from the following theorem.

Theorem A (Linnik) *The following formula is valid.*

$$\varphi_\alpha(t) = \int_{-\infty}^{\infty} e^{itx} p_\alpha(x)\, dx \ , \ t \in \mathbb{R} \, ,$$

where the function $p_\alpha(x) \in \mathrm{L}^1(\mathbb{R})$ is representable in the form

$$p_\alpha(x) = \frac{\sin(\frac{\pi\alpha}{2})}{\pi} \int_0^\infty \frac{e^{-y|x|} y^\alpha dy}{|1 + y^\alpha e^{\frac{i\pi\alpha}{2}}|^2} \ , x \in \mathbb{R}.$$

To study the function $\varphi_\alpha^\theta(t)$ for all values of $\theta \in (-\pi, \pi)$, we generalize Theorem A in the following way.

Theorem 2.1 *The following formula is valid.*

$$\varphi_\alpha^\theta(t) = \int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x)\, dx \, , \, t \in \mathbb{R} \, ,$$

where the function $p_\alpha^\theta(x) \in L^1(\mathbb{R})$ is representable in the form
(i) for $0 \le \theta < \pi - \frac{\pi\alpha}{2}$:

$$p_\alpha^\theta(x) = \frac{\sin(\frac{\pi\alpha}{2} + \theta \operatorname{sgn} x)}{\pi} \int_0^\infty \frac{e^{-y|x|} y^\alpha dy}{|1 + e^{i\theta \operatorname{sgn} x} y^\alpha e^{\frac{i\pi\alpha}{2}}|^2} \ , x \in \mathbb{R}, \tag{2.1}$$

(ii) for $\pi - \frac{\pi\alpha}{2} < \theta < \pi$;

$$p_\alpha^\theta(x) = \frac{\sin(\frac{\pi\alpha}{2} + \theta \operatorname{sgn} x)}{\pi} \int_0^\infty \frac{e^{-y|x|} y^\alpha dy}{|1 + e^{i\theta \operatorname{sgn} x} y^\alpha e^{\frac{i\pi\alpha}{2}}|^2} + G(x)\,, x \in \mathbb{R}; \qquad (2.2)$$

where

$$G(x) = \frac{1 + \operatorname{sgn} x}{\alpha} \operatorname{Im} \exp\left(i\frac{\pi - \theta}{\alpha} + ixe^{i\frac{\pi-\theta}{\alpha}}\right), x \in \mathbb{R}, \qquad (2.3)$$

(iii) for $\theta = \pi - \frac{\pi\alpha}{2}$;

$$p_\alpha^\theta(x) = \begin{cases} -\dfrac{\sin(\pi\alpha)}{\pi} \displaystyle\int_0^\infty \frac{e^{yx} y^\alpha dy}{|1 - e^{i\pi\alpha} y^\alpha|^2} & , x < 0, \\ e^{-x}/\alpha & , x > 0, \end{cases} \qquad (2.4)$$

(iv) for $-\pi < \theta < 0$ we have

$$p_\alpha^\theta(x) = p_\alpha^{-\theta}(-x)\,, x \in \mathbb{R}. \qquad (2.5)$$

In virtue of Theorem 2.1, the function $\varphi_\alpha^\theta(t)$ is a characteristic function of a probability distribution iff the corresponding function $p_\alpha^\theta(x)$ is non-negative for all $x \in \mathbb{R}$. The following theorem determines the values of θ.

Theorem 2.2 *The function $\varphi_\alpha^\theta(t)$ defined by (1.3) is a characteristic function of a probability distribution iff θ satisfies the following condition:*

$$|\theta| \leq \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2}). \qquad (2.6)$$

If this condition is satisfied, then the corresponding distribution is absolutely continuous and its density $p_\alpha^\theta(x)$ is given by one of the formulas (2.1)-(2.5).

The sufficiency of the condition (2.6) was proved by A. G. Pakes [9] (by R. G. Laha [10] and R. N. Pillai [11] in the case of equality sign in (2.6)) by a quite different method based on the properties of stable distributions. Our proof is immediate.

The set of all pairs (α, θ) for which $\varphi_\alpha^\theta(t)$ is a characteristic function of a probability distribution is visualized on fig.2.1 (i) (p.5) as a closed diamond-shaped region without the points $(0,0)$ and $(2,0)$. Note that the point $(2,0)$ can be interpreted as the well-known Laplace distribution with the characteristic function $\varphi_2(t) = (1 + t^2)^{-1}$ and the density $p_2(x) = e^{-|x|}/2$. We shall

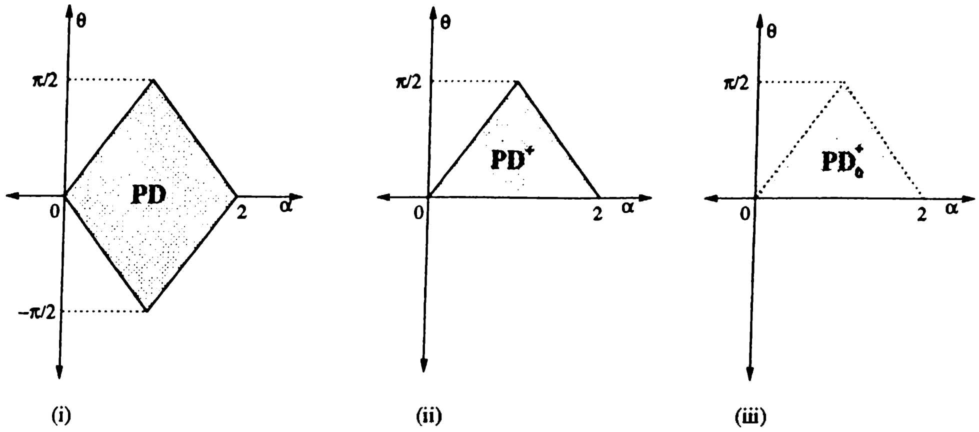

Figure 2.1:

denote this set by PD and call it *the parametrical domain.* Denote by PD^+ the part of PD consisting of pairs (α, θ) such that $\theta \geq 0$ (see fig.2.1 (ii), p.5). Without loss of generality, we can restrict our study of $p_\alpha^\theta(x)$ to pairs $(\alpha, \theta) \in PD^+$ since one can obtain $p_\alpha^\theta(x)$ for $(\alpha, \theta) \in PD \backslash PD^+$ from (2.5).

Recall that a function $f(x)$ defined on an interval $I \subset \mathbb{R}$ is called *completely monotonic* (resp. *absolutely monotonic*) if it is infinitely differentiable on I and, moreover, $(-1)^k f^{(k)}(x) \geq 0$ (resp. $f^{(k)}(x) \geq 0$) for any $x \in I$ and any k =0,1,...

The following theorem related to analytic properties of $p_\alpha^\theta(x)$ was proved in the symmetric case $\theta = 0$ in [8].

Theorem 2.3 *(i) For any pair $(\alpha, \theta) \in PD^+$, the function $p_\alpha^\theta(x)$ is completely monotonic on $(0, \infty)$ and is absolutely monotonic on $(-\infty, 0)$.*
(ii) For $1 < \alpha < 2$, $0 \leq \theta \leq \pi - \frac{\pi\alpha}{2}$, $p_\alpha^\theta(x)$ is a continuous function on $\mathbb{R}$ and

$$p_\alpha^\theta(0) := \lim_{x \to 0^+} p_\alpha^\theta(x) = \lim_{x \to 0^-} p_\alpha^\theta(x) = \frac{1}{\alpha} \frac{\cos \frac{\theta}{\alpha}}{\sin \frac{\pi}{\alpha}}.$$

For $0 < \alpha \leq 1$, $0 \leq \theta < \frac{\pi\alpha}{2}$, we have

$$\lim_{x \to 0^+} p_\alpha^\theta(x) = \lim_{x \to 0^-} p_\alpha^\theta(x) = +\infty.$$

For $0 < \alpha \leq 1$, $\theta = \frac{\pi\alpha}{2}$, we have

$$\lim_{x \to 0^+} p_\alpha^\theta(x) = \infty; \; p_\alpha^\theta(x) = 0, \; for \; x < 0.$$

(iii) For $1 < \alpha < 2$, $0 \le \theta \le \pi - \frac{\pi\alpha}{2}$ *and* $0 < \alpha \le 1$, $0 \le \theta < \frac{\pi\alpha}{2}$, *we have*

$$\lim_{x\to 0^+} (-1)^k \left(p_\alpha^\theta(x)\right)^{(k)} = \infty\,,\ \lim_{x\to 0^-} \left(p_\alpha^\theta(x)\right)^{(k)} = \infty\,,\ k = 1,2,3,\ldots$$

The first of these equalities remains be true for $0 < \alpha \le 1$, $\theta = \pi\alpha/2$.

Recall that an absolutely continuous distribution is called *unimodal with mode 0* if its density is non-decreasing on $(-\infty, 0)$ and is non-increasing on $(0,\infty)$. The following theorem is an immediate corollary of Theorem 2.3.

Theorem 2.4 *For any pair* $(\alpha,\theta) \in PD$, *the distribution with the characteristic function* (1.3) *is unimodal with mode 0.*

Note that, in the case $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, this theorem was proved by R. G. Laha [10] in 1961.

The following theorem measures the non-symmetry of $p_\alpha^\theta(x)$. Surely, this non-symmetry increases with $|\theta|$. We shall denote by PD_0^+ the part of PD^+ which is obtained by removing the pairs (α,θ) with $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$ (see fig.2.1 (iii), p.5).

Theorem 2.5 *(i) For any pair* $(\alpha,\theta) \in PD^+$, *we have*

$$\int_0^\infty p_\alpha^\theta(\pm x)dx = \frac{1}{2} \pm \frac{\theta}{\pi\alpha}.$$

(ii) For any pair $(\alpha,\theta) \in PD^+$, *we have*

$$p_\alpha^\theta(x)\sin(\frac{\pi\alpha}{2} - \theta) \ge p_\alpha^\theta(-x)\sin(\frac{\pi\alpha}{2} + \theta)\ ,\ x > 0.$$

(iii) For any pair $(\alpha,\theta) \in PD^+$ *such that* $\alpha \in (0,1)$, *we have*

$$p_\alpha^\theta(x) \ge p_\alpha^\theta(-x)\,,\ x > 0.$$

For any pair $(\alpha,\theta) \in PD^+$ *such that* $\alpha \in (1,2)$, $\theta > 0$, *this assertion is false.*

(iv) As a function of θ, $0 \le \theta < \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2})$, $p_\alpha^\theta(x)$ *increases and* $p_\alpha^\theta(-x)$ *decreases for any fixed* $\alpha \in (0,1)$ *and* $x > 0$.

For any pair $(\alpha,\theta) \in PD^+$ *such that* $\alpha \in (1,2)$, $\theta > 0$, *this assertion is false.*

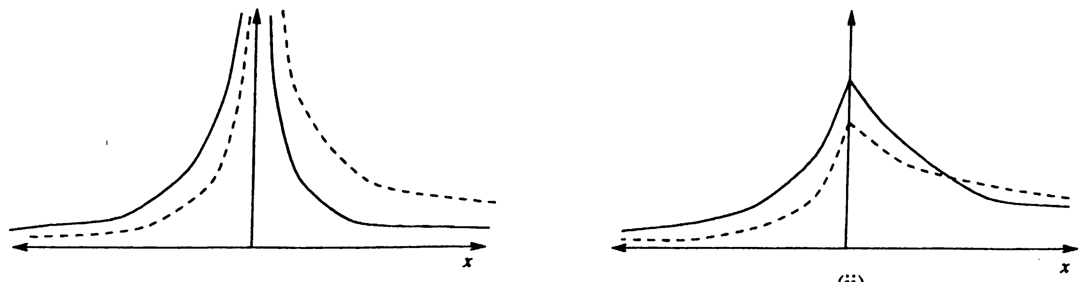

Figure 2.2:

On fig.2.2, there are pictured graphs of $p_\alpha^{\theta_1}(x)$ and $p_\alpha^{\theta_2}(x)$: (i) for $0 < \alpha < 1$, $0 < \theta_1 < \theta_2 < \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2})$, (ii) for $1 < \alpha < 3/2$, $\frac{\pi\alpha}{2} - \frac{\pi}{2} < \theta_1 < \theta_2 < \pi - \frac{\pi\alpha}{2}$. The graphs of $p_\alpha^{\theta_1}(x)$ are pictured by continuous lines, the graphs of $p_\alpha^{\theta_2}(x)$ are pictured by dotted lines.

The following two theorems characterize the asymptotic behaviour of $p_\alpha^\theta(x)$ at ∞. For $\theta = 0$, they were proved in [8]

Theorem 2.6 *For any pair $(\alpha, \theta) \in PD_0^+$ the following asymptotic (divergent) series describes the asymptotic behaviour of $p_\alpha^\theta(x)$ at ∞.*

$$p_\alpha^\theta(x) \sim \frac{1}{\pi}\sum_{k=1}^{\infty} \Gamma(1+\alpha k)(-1)^{k+1} \sin(\frac{\pi\alpha k}{2} + k\theta \mathrm{sgn} x)\,|x|^{-1-\alpha k},\ |x| \to \infty \quad (2.7)$$

This theorem is an immediate consequence of the following more informative theorem:

Theorem 2.7 *For any pair $(\alpha, \theta) \in PD_0^+$ and $N = 1, 2, 3, ..,$ the following formula is valid:*

$$p_\alpha^\theta(x) = \frac{1}{\pi}\sum_{k=1}^{N} \Gamma(1+\alpha k)(-1)^{k+1} \sin(\frac{\pi\alpha k}{2} + k\theta \mathrm{sgn} x)\,|x|^{-1-\alpha k} + R_{N,\alpha}(x), \quad (2.8)$$

where

$$|R_{N,\alpha}(x)| \le \frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|\sin(\frac{\pi\alpha}{2} + \theta \mathrm{sgn} x)|}|x|^{-1-\alpha(N+1)}. \quad (2.9)$$

Corollary 1. *For any pair $(\alpha, \theta) \in PD_0^+$, the following representation is valid:*

$$p_\alpha^\theta(x) = \frac{1}{\pi}\Gamma(1+\alpha)\sin(\frac{\pi\alpha}{2} + \theta \mathrm{sgn} x)\,|x|^{-1-\alpha} + O(|x|^{-1-2\alpha}),\ |x| \to \infty.$$

Corollary 2. *For any pair $(\alpha,\theta)\in PD^+$, the following equality is valid:*

$$\lim_{x\to\infty}\frac{p_\alpha^\theta(x)}{p_\alpha^\theta(-x)}=\frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\sin(\frac{\pi\alpha}{2}-\theta)}\ ,\ x>0,$$

(the right hand side is equal to $+\infty$ if $\theta=\frac{\pi\alpha}{2}$).

Corollary 3. *For any pairs (α,θ_1), $(\alpha,\theta_2)\in PD_0^+$, the following equality is valid:*

$$\lim_{x\to\infty}\frac{p_\alpha^{\theta_1}(x)}{p_\alpha^{\theta_2}(x)}=\frac{\sin(\frac{\pi\alpha}{2}+\theta_1)}{\sin(\frac{\pi\alpha}{2}+\theta_2)}\ ,\ x>0.$$

Corollary 4. *For N= $1,2,3,..$, the following formulas are valid:*
(i) For $\alpha\in(0,1]$, $\theta=\frac{\pi\alpha}{2}$;

$$p_\alpha^\theta(x)=\begin{cases}0 & ,x<0,\\ \frac{1}{\pi}\sum_{k=1}^N\Gamma(1+\alpha k)(-1)^{k+1}\sin(\pi\alpha k)\,|x|^{-1-\alpha k}+ & \\ +R_{N,\alpha}(x) & ,x>0,\end{cases}$$

(ii) For $\alpha\in(1,2)$, $\theta=\pi-\frac{\pi\alpha}{2}$;

$$p_\alpha^\theta(x)=\begin{cases}\frac{1}{\pi}\sum_{k=1}^N\Gamma(1+\alpha k)(-1)\sin(\pi\alpha k)\,|x|^{-1-\alpha k}+ & \\ +R_{N,\alpha}(x) & ,x<0,\\ e^{-x}/\alpha & ,x>0,\end{cases}$$

where

$$|R_{N,\alpha}(x)|\le\frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|\sin(\pi\alpha)|}|x|^{-1-\alpha(N+1)}.$$

The analytic structure of $p_\alpha^\theta(x)$ depends on the arithmetic nature of the parameter α. Firstly we will deal with the case $\alpha=1/n$, where n is an integer.

Theorems 2.8-2.11 were proved for $\theta=0$ in [8].

Theorem 2.8 *For any n= $1,2,3,..$, and $0\le\theta<\frac{\pi}{2n}$ the following formula is valid:*

$$p_{1/n}^\theta(x)\ =\ \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin N}^{\infty}\Gamma(1-\frac{k}{n})(-1)^{k+1}\sin(\frac{\pi k}{2n}+k\theta\mathrm{sgn}x)|x|^{\frac{k}{n}-1}$$

$$+\frac{1}{\pi}\sum_{j=1}^{\infty}(-1)^{(n+1)j}\frac{\Gamma'(j)}{\Gamma^2(j)}\sin(\frac{\pi j}{2}+\theta nj\mathrm{sgn}x)|x|^{j-1} \tag{2.10}$$
$$+\frac{n(-1)^n}{\pi}(\log|x|)e^{(x(-1)^n\sin(\theta n))}\cos(x\cos(\theta n)-\theta n(-1)^n)$$
$$-(-1)^n\frac{2\theta n+\pi\mathrm{sgn}x}{2\pi}e^{(x(-1)^n\sin(\theta n))}\sin(x\cos(\theta n)+\theta n).$$

Corollary 1. *For any $0\le\theta\le\frac{\pi}{2}$ the following representation is valid:*

$$\begin{aligned} p_1^\theta(x) &= \frac{1}{\pi}\sum_{j=1}^{\infty}\frac{\Gamma\prime(j)}{\Gamma^2(j)}\sin(\frac{\pi j}{2}+\theta j\mathrm{sgn}x)|x|^{j-1} \\ &-\frac{1}{\pi}(\log|x|)e^{-x\sin\theta}\cos(x\cos\theta+\theta) \\ &-\frac{1}{\pi}(\frac{\pi\mathrm{sgn}x+2\theta}{2})e^{-x\sin\theta}\sin(x\cos\theta+\theta). \end{aligned}$$

Following theorem deals with the general case of a rational α:

Theorem 2.9 *Let $\alpha\in(0,2)$ be a rational number. Set $\alpha=m/n$ where m and n are relatively prime integers both greater than 1. The following representation is valid for $\alpha=m/n\in(0,2)$, $0\le\theta<\min(\frac{\pi\alpha}{2},\pi-\frac{\pi\alpha}{2})$:*

$$\begin{aligned} p_\alpha^\theta(x) &= \sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}\frac{(-1)^{k+1}\sin(\frac{\pi k\alpha}{2}+k\theta\mathrm{sgn}x)}{\Gamma(k\alpha)\sin(\pi k\alpha)}|x|^{k\alpha-1} \\ &+\frac{1}{\pi}\log\frac{1}{|x|}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\sin(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x)|x|^{mt-1} \\ &-(\frac{\theta\mathrm{sgn}x}{\pi\alpha}+\frac{1}{2})\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\cos(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x)|x|^{mt-1} \\ &+\frac{1}{\alpha}\sum_{j=1,\frac{j}{m}\notin\mathbb{N}}^{\infty}\frac{(-1)^{j-1}}{\Gamma(j)}\frac{\sin(\frac{\pi j}{2}+\frac{\theta}{\alpha}j\mathrm{sgn}x)}{\sin\frac{\pi j}{\alpha}}|x|^{j-1} \\ &+\frac{1}{\pi}\sum_{t=1}^{\infty}(-1)^{(m+n)t}\frac{\Gamma\prime(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x)|x|^{mt-1}. \end{aligned} \tag{2.11}$$

All the series in (2.11) can be represented by entire functions. Following theorem is an immediate corollary of Theorem 2.9.

Theorem 2.10 *Under the conditions of Theorem 2.9, the following representation holds for $x>0$*

$$p_\alpha^\theta(\pm x)=\frac{1}{|x|}A_\pm(|x|^\alpha)+\frac{1}{\pi}\log\frac{1}{|x|}B_\pm(|x|^m)+C_\pm(|x|)$$

where $A_\pm(z)$, $B_\pm(z)$, $C_\pm(z)$ are entire functions of finite order.

Note that the term with $\log|x|$ in (2.11) vanishes identically if $\theta = \pi l/n$, for some integer l and, moreover, m is even, n is odd.

The following theorem deals with the general case of irrational α:

Theorem 2.11 *If the number $\alpha \in (0,2)$ is not a rational number, then the following representation is valid for $0 \le \theta \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$.*

$$p_\alpha^\theta(x) = \frac{1}{|x|}\lim_{s\to\infty}\left\{\sum_{k=1}^{s}\frac{(-1)^{k+1}}{\Gamma(k\alpha)}\frac{\sin(\frac{\pi k\alpha}{2} + k\theta\operatorname{sgn}x)}{\sin(\pi k\alpha)}|x|^{k\alpha}\right.$$
$$\left.+\frac{1}{\alpha}\sum_{1\le k<\alpha(s+\frac{1}{2})}\frac{(-1)^{k+1}}{\Gamma(k)}\frac{\sin(\frac{\pi k}{2} + \frac{k\theta}{\alpha}\operatorname{sgn}x)}{\sin(\frac{\pi k}{\alpha})}|x|^{k}\right\} \qquad (2.12)$$

The limit is uniform with respect to x on any compact subset of $\mathbb{R}$.

The following theorem deals with the "extremely" non-symmetric case. In the case $0 < \alpha < 1$, it was proved by R. N. Pillai [11].

Theorem 2.12 *The following representations are valid:*
(i) for $0 < \alpha < 1$, $\theta = \frac{\pi\alpha}{2}$;

$$\begin{aligned} p_\alpha^\theta(x) &= 0 \qquad\qquad , x < 0,\\ p_\alpha^\theta(x) &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{\Gamma(k\alpha)}|x|^{k\alpha-1}, x > 0,\end{aligned}$$

(ii) for $1 \le \alpha < 2$, $\theta = \pi - \frac{\pi\alpha}{2}$;

$$\begin{aligned} p_\alpha^\theta(x) &= \frac{e^{-x}}{\alpha} \qquad\qquad ; x > 0,\\ p_\alpha^\theta(x) &= \frac{e^{-x}}{\alpha} - \sum_{k=1}^{\infty}\frac{|x|^{k\alpha-1}}{\Gamma(k\alpha)}, x < 0.\end{aligned}$$

The representations above can also be written in the following form
(i) for $0 < \alpha < 1$, $\theta = \frac{\pi\alpha}{2}$;

$$p_\alpha^\theta(x) = -\frac{1+\operatorname{sgn}x}{2}(E_\alpha(-x^\alpha))'$$

(ii) for $1 \le \alpha < 2$, $\theta = \pi - \frac{\pi\alpha}{2}$;

$$p_\alpha^\theta(x) = \frac{e^{-x}}{\alpha} + \frac{1-\operatorname{sgn}x}{2}(E_\alpha(|x|^\alpha))'$$

where the function $E_\alpha(z)$ is the well-known Mittag-Leffler's function defined as

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}.$$

It is natural to ask whether the limits of each of the two sums in the right hand side of (2.12) exist. We prove that it is the case for almost all $(\alpha, \theta) \in PD$ in the sense of the planar Lebesgue measure. To describe the corresponding set we need Liouville numbers. Recall that an irrational number l is called a Liouville number if, for any $r = 2, 3, 4, \ldots$, there exists a pair of integers p, $q \geq 2$, such that

$$0 < |l - \frac{p}{q}| < \frac{1}{q^r}.$$

We denote the set of all Liouville numbers by L. By the famous Liouville theorem (see, e.g. [12], p.7), all numbers in L are transcendental. Moreover ([12], p.8), the set L has the Lebesgue measure zero.

Theorem 2.13 *If $(\alpha, \theta) \in \{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$, then the following representation is valid*

$$\begin{aligned} p_\alpha^\theta(x) &= \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k\alpha)} \frac{\sin(\frac{\pi k \alpha}{2} + k\theta \mathrm{sgn} x)}{\sin(\pi k \alpha)} |x|^{k\alpha - 1} \\ &\quad + \frac{1}{\alpha} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\Gamma(k)} \frac{\sin(\frac{\pi k}{2} + \frac{k\theta}{\alpha} \mathrm{sgn} x)}{\sin(\frac{\pi k}{\alpha})} |x|^{k-1} \end{aligned} \qquad (2.13)$$

where both of the series converge absolutely and uniformly on any compact set.

The following theorem is an immediate corollary of Theorem 2.13.

Theorem 2.14 *If $(\alpha, \theta) \in \{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$, then the following representation holds for $x > 0$*

$$p_\alpha^\theta(\pm x) = \frac{1}{|x|} G_\pm(|x|^\alpha) + \frac{1}{\alpha} H_\pm(|x|)$$

where $G_\pm(z)$, $H_\pm(z)$ are entire functions of finite order.

Since the set $L \bigcup \mathbb{Q}$ has zero linear Lebesgue measure, the set $\{(\alpha, \theta) \in PD : \alpha \notin L \bigcup \mathbb{Q}\}$ is of full planar measure in PD. Thus, (2.13) is valid almost everywhere in PD. But it turns out that the set where both of the series in the right hand side of (2.13) diverge is non-empty and, moreover, it is large in some sense.

Theorem 2.15 *Both of the series in* (2.13) *diverge on a dense subset of* PD *of the continuum power.*

This theorem is a generalization of a theorem of I.V. Ostrovskii [13] related to the case $\theta = 0$ (when the role of PD is played by the interval $(0, 2)$).

Chapter 3

Description of the Parametrical Domain and Some Analytic Properties of Non-Symmetric Linnik's Probability Densities

Proof of Theorem 2.1. Case (i): $0 \le \theta < \pi - \frac{\pi\alpha}{2}$;
Firstly we will prove that $p_\alpha^\theta(x) \in L^1(\mathbb{R})$. It is evident from (2.1) that $p_\alpha^\theta(x) \ge 0$ and we have

$$\begin{aligned}\int_{-\infty}^{\infty} p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_{-\infty}^{0} dx \int_0^\infty \frac{e^{yx}y^\alpha dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &\quad + \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty dx \int_0^\infty \frac{e^{-yx}y^\alpha dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}\end{aligned}$$

Since the integrands in the in the right hand side are measurable and non-negative, by Fubini's theorem we have

$$\begin{aligned}\int_{-\infty}^{\infty} p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}\int_{-\infty}^{0} e^{yx}dx \\ &\quad + \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}\int_0^\infty e^{-yx}dx \\ &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{y^{\alpha-1} dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &\quad + \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^{\alpha-1} dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} < \infty.\end{aligned}$$

Now we will prove that $\int_{-\infty}^{\infty} e^{itx}p_\alpha^\theta(x)dx = \varphi_\alpha^\theta(t)$. From (2.1) we derive

$$\begin{aligned}\int_{-\infty}^{\infty} e^{itx}p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_{-\infty}^{0} e^{itx}dx \int_0^\infty \frac{e^{yx}y^\alpha dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &\quad + \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty e^{itx}dx \int_0^\infty \frac{e^{-yx}y^\alpha dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}.\end{aligned}$$

We have proved that the integrands in the right hand side are in $L^1(\mathbb{R}^2)$. Using Fubini's theorem again, we obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}\int_{-\infty}^0 e^{itx}e^{yx}dx \\
&\quad +\frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}\int_0^\infty e^{itx}e^{-yx}dx \\
&= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{-i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2)(y+it)} \\
&\quad +\frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{i\theta}y^\alpha e^{i\frac{\pi\alpha}{2}}|^2)(y-it)} \\
&= \frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{ydy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)}\right\} \\
&\quad +\frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{ydy}{(1+e^{-i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)}\right\} \\
&\quad -\frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)}\right\} \\
&\quad +\frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{ydy}{(1+e^{-i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)}\right\} \\
&=: \frac{1}{\pi}[\mathrm{Im}A+\mathrm{Im}B-it\mathrm{Im}C+it\mathrm{Im}D]. \qquad (3.1)
\end{aligned}$$

In the complex y-plane, we consider the region

$$G_R=\{y=\xi+i\eta : |y|<R, \eta>0\}\,,\ R>|t| \qquad (3.2)$$

and define the branch of multivalued function y^α as

$$y^\alpha=|y|^\alpha e^{i\alpha \arg y}\,,\ 0\le \arg y\le \pi. \qquad (3.3)$$

The integrands of A and C are analytic in the closure of G_R except the simple pole at $y=i|t|$. From the residue theory we have

$$\oint_{\partial G_R}\frac{y\ dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)}=2\pi i\mathrm{Res}_{i|t|}=\frac{\pi i}{1+e^{i\theta}|t|^\alpha}.$$

Setting $C_R=\{y=\xi+i\eta\ :\ |y|=R\,,\,\eta\ge 0\}$, we have

$$\begin{aligned}
\frac{\pi i}{1+e^{i\theta}|t|^\alpha} &= \int_0^R \frac{\xi\ d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\
&\quad +\int_{C_R}\frac{y\ dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)} \\
&\quad -\int_0^R \frac{\xi\ d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)}.
\end{aligned}$$

Letting $R \to \infty$, the integral along C_R obviously tends to 0, so we have

$$\begin{aligned}
&\int_0^\infty \frac{\xi\, d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\
&- \int_0^\infty \frac{\xi\, d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \quad = \quad \frac{\pi i}{1+e^{i\theta}|t|^\alpha}.
\end{aligned}$$

Using the notations A and B, we can rewrite this equality in the following form

$$A - \overline{B} = \frac{\pi i}{1+e^{i\theta}|t|^\alpha}$$

whence

$$\mathrm{Im}A + \mathrm{Im}B = \pi \mathrm{Re}\frac{1}{1+e^{i\theta}|t|^\alpha}. \tag{3.4}$$

For evaluating $-\mathrm{Im}C + \mathrm{Im}D$, we have in the similar way:

$$\oint_{\partial G_R} \frac{dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)} = 2\pi i \mathrm{Res}_{i|t|} = \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|};$$

$$\begin{aligned}
&\int_0^R \frac{\xi\, d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\
&+ \int_{C_R} \frac{dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)} \\
&+ \int_0^R \frac{d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \quad = \quad \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|};
\end{aligned}$$

$$\begin{aligned}
&\int_0^\infty \frac{d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\
&+ \int_0^\infty \frac{d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \quad = \quad \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|};
\end{aligned}$$

$$C + \overline{D} = \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|},$$

$$-\mathrm{Im}C + \mathrm{Im}D = -\frac{\pi}{|t|}\mathrm{Im}\frac{1}{1+e^{i\theta}|t|^\alpha}. \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1), we have

$$\begin{aligned}
\int_{-\infty}^\infty e^{itx} p_\alpha^\theta(x)\, dx &= \mathrm{Re}\frac{1}{1+e^{i\theta}|t|^\alpha} + i\,\mathrm{sgn}t\,\mathrm{Im}\frac{1}{1+e^{i\theta}|t|^\alpha} \\
&= \frac{1}{1+e^{-i\theta\,\mathrm{sgn}t}|t|^\alpha}.
\end{aligned}$$

Case (ii): $\pi - \frac{\pi\alpha}{2} < \theta < \pi$;

As in the case (i), firstly we will prove that $p_\alpha^\theta(x) \in L^1(\mathbb{R})$. From (2.2) we have

$$\begin{aligned}\int_{-\infty}^{\infty} |p_\alpha^\theta(x)|dx &= \frac{|\sin(\frac{\pi\alpha}{2} - \theta)|}{\pi} \int_{-\infty}^{0} dx \int_0^\infty \frac{e^{yx} y^\alpha dy}{|1 + e^{-i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &+ \int_{-\infty}^{\infty} |G(x)|\, dx \qquad (3.6) \\ &+ \frac{|\sin(\frac{\pi\alpha}{2} + \theta)|}{\pi} \int_0^\infty dx \int_0^\infty \frac{e^{-yx} y^\alpha dy}{|1 + e^{i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}.\end{aligned}$$

It is evident from (2.3) that $G(x) \in L^1(\mathbb{R})$. Since the integrands of first and third integrals in the right hand side of (3.6) are measurable and nonnegative, from Fubini's theorem we have

$$\begin{aligned}\int_{-\infty}^{\infty} |p_\alpha^\theta(x)|dx &= \frac{|\sin(\frac{\pi\alpha}{2} - \theta)|}{\pi} \int_0^\infty \frac{y^{\alpha-1} dy}{|1 + e^{-i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &+ \int_{-\infty}^{\infty} |G(x)|\, dx \\ &+ \frac{|\sin(\frac{\pi\alpha}{2} + \theta)|}{\pi} \int_0^\infty \frac{y^{\alpha-1} dy}{|1 + e^{i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} < \infty.\end{aligned}$$

Now we will prove that $\int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x)dx = \varphi_\alpha^\theta(t)$. From (2.2) we derive

$$\begin{aligned}\int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2} - \theta)}{\pi} \int_{-\infty}^{0} e^{itx} dx \int_0^\infty \frac{e^{yx} y^\alpha dy}{|1 + e^{-i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2} \\ &+ \int_{-\infty}^{\infty} e^{itx} G(x)\, dx \\ &+ \frac{\sin(\frac{\pi\alpha}{2} + \theta)}{\pi} \int_0^\infty e^{itx} dx \int_0^\infty \frac{e^{-yx} y^\alpha dy}{|1 + e^{i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}}|^2}.\end{aligned}$$

We have proved that the integrands in the right hand side are in $L^1(\mathbb{R}^2)$, thus, from Fubini's theorem we have as in the case (i)

$$\begin{aligned}\int_{-\infty}^{\infty} e^{itx} p_\alpha^\theta(x)dx &= \frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{y dy}{(1 + e^{-i\theta} y^\alpha e^{i\frac{\pi\alpha}{2}})(y^2 + t^2)}\right\} \\ &+ \frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{y dy}{(1 + e^{-i\theta} y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2 + t^2)}\right\} \\ &- \frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{(1 + e^{i\theta} y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2 + t^2)}\right\} \\ &+ \frac{it}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{(1 + e^{-i\theta} y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2 + t^2)}\right\} \\ &+ \int_{-\infty}^{\infty} e^{itx} G(x)\, dx \\ &=: \frac{1}{\pi}[\mathrm{Im}A + \mathrm{Im}B - it\mathrm{Im}C + it\mathrm{Im}D] + \tilde{G}(t). \quad (3.7)\end{aligned}$$

As in the case (i), in the complex y-plane we consider the region (3.2) and define the branch of the multivalued function y^α by (3.3).

The integrands of A and C are analytic in the closure of G_R except the simple poles at $y = i|t|$ and at $y = ie^{i\psi}$ where $\psi = \frac{\pi-\theta}{\alpha}$. By the residue theory, we have

$$\begin{aligned}\oint_{\partial G_R} \frac{y\, dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)} &= 2\pi i \mathrm{Res}_{i|t|} + 2\pi i \mathrm{Res}_{ie^{i\psi}} \\ &= \frac{\pi i}{1+e^{i\theta}|t|^\alpha} + \frac{2\pi i}{\alpha}\frac{e^{2i\psi}}{t^2-e^{2i\psi}}.\end{aligned}$$

Letting $R \to \infty$, we have, as in the case (i),

$$\begin{aligned}&\int_0^\infty \frac{\xi\, d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\ -&\int_0^\infty \frac{\xi\, d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)} = \frac{\pi i}{1+e^{i\theta}|t|^\alpha} + \frac{2\pi i}{\alpha}\frac{e^{2i\psi}}{t^2-e^{2i\psi}}.\end{aligned}$$

Using the notations A and B, we can rewrite this equality in the following form

$$A - \overline{B} = \frac{\pi i}{1+e^{i\theta}|t|^\alpha} + \frac{2\pi i}{\alpha}\frac{e^{2i\psi}}{t^2-e^{2i\psi}}$$

whence

$$\mathrm{Im}A + \mathrm{Im}B = \pi \mathrm{Re}\frac{1}{1+e^{i\theta}|t|^\alpha} + \frac{2\pi}{\alpha}\mathrm{Re}\frac{e^{2i\psi}}{t^2-e^{2i\psi}}. \tag{3.8}$$

For evaluating $-\mathrm{Im}C + \mathrm{Im}D$ we have in the similar way:

$$\begin{aligned}\oint_{\partial G_R} \frac{dy}{(1+e^{i\theta}y^\alpha e^{-i\frac{\pi\alpha}{2}})(y^2+t^2)} &= 2\pi i \mathrm{Res}_{i|t|} + 2\pi i \mathrm{Res}_{ie^{i\psi}} \\ &= \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|} + \frac{2\pi}{\alpha}\frac{e^{i\psi}}{t^2-e^{2i\psi}},\end{aligned}$$

and, letting $R \to \infty$,

$$\begin{aligned}&\int_0^\infty \frac{d\xi}{(1+e^{i\theta}\xi^\alpha e^{-i\frac{\pi\alpha}{2}})(\xi^2+t^2)} \\ +&\int_0^\infty \frac{d\xi}{(1+e^{i\theta}\xi^\alpha e^{i\frac{\pi\alpha}{2}})(\xi^2+t^2)} = \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|} + \frac{2\pi}{\alpha}\frac{e^{i\psi}}{t^2-e^{2i\psi}},\end{aligned}$$

whence

$$C + \overline{D} = \frac{\pi}{(1+e^{i\theta}|t|^\alpha)|t|} + \frac{2\pi}{\alpha}\frac{e^{i\psi}}{t^2-e^{2i\psi}};$$

$$\mathrm{Im}C - \mathrm{Im}D = \frac{\pi}{|t|}\mathrm{Im}\frac{1}{1+e^{i\theta}|t|^\alpha} + \frac{2\pi}{\alpha}\mathrm{Im}\frac{e^{i\psi}}{t^2-e^{2i\psi}}. \tag{3.9}$$

Using (3.7), (2.3) and remembering that $\psi = \frac{\pi-\theta}{\alpha}$, we obtain

$$\begin{aligned}
\tilde{G}(t) &= \frac{2}{\alpha}\int_0^\infty e^{ixt}\mathrm{Im}\exp\{i\psi + ixe^{i\psi}\}\,dx \\
&= \frac{e^{i\psi}}{\alpha i}\int_0^\infty \exp\{ixt + ixe^{i\psi}\}\,dx - \frac{e^{-i\psi}}{\alpha i}\int_0^\infty \exp\{ixt - ixe^{-i\psi}\}\,dx \\
&= \frac{2}{\alpha}\,\frac{-1 + it\sin\psi}{-1 + 2it\sin\psi + t^2}. \qquad (3.10)
\end{aligned}$$

Substituting (3.8), (3.9), (3.10) into (3.7), we obtain

$$\int_{-\infty}^\infty e^{itx}\,p_\alpha^\theta(x)\,dx = \varphi_\alpha^\theta(t).$$

Case (iii): $\theta = \pi - \frac{\pi\alpha}{2}$

The proof of belonging of $p_\alpha^\theta(x)$ defined by (2.4) to $L^1(\mathbb{R})$ is nearly same. We will prove that $\int_{-\infty}^\infty e^{itx}p_\alpha^\theta(x)dx = \varphi_\alpha^\theta(t)$. From (2.4) we have

$$\int_{-\infty}^\infty e^{itx}p_\alpha^\theta(x)dx = -\frac{\sin(\pi\alpha)}{\pi}\int_{-\infty}^0 e^{itx}\int_0^\infty \frac{e^{yx}y^\alpha dy}{|1-e^{i\pi\alpha}y^\alpha|^2} + \int_0^\infty e^{itx}\frac{e^{-x}}{\alpha}dx.$$

The integrand of first integral in the right hand side is in $L^1(\mathbb{R}^2)$, thus, as in case (i), we have:

$$\begin{aligned}
\int_{-\infty}^\infty e^{itx}p_\alpha^\theta(x)dx &= \frac{1}{\alpha(1-it)} - \frac{\sin(\pi\alpha)}{\pi}\int_0^\infty \frac{y^\alpha\,dy}{|1-e^{i\pi\alpha}y^\alpha|^2(y+it)} \\
&= \frac{1}{\alpha(1-it)} + \frac{1}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{y\,dy}{(1-e^{-i\pi\alpha}y^\alpha)(y^2+t^2)}\right\} \\
&\quad -\frac{(it)}{\pi}\mathrm{Im}\left\{\int_0^\infty \frac{dy}{(1-e^{-i\pi\alpha}y^\alpha)(y^2+t^2)}\right\} \\
&=: \frac{1}{\alpha(1-it)} + \frac{1}{\pi}\mathrm{Im}A - \frac{it}{\pi}\mathrm{Im}B. \qquad (3.11)
\end{aligned}$$

Having defined the region G_R by (3.2) and the branch of y^α by (3.3), we have

$$\begin{aligned}
v.p.\oint_{\partial G_R}\frac{y\,dy}{(1-e^{-i\pi\alpha}y^\alpha)(y^2+t^2)} &= 2\pi i\mathrm{Res}_{i|t|} + \pi i\mathrm{Res}_{-1} \\
&= \frac{\pi i}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - \frac{\pi i}{\alpha(1+t^2)},
\end{aligned}$$

whence, letting $R\to\infty$, we obtain

$$A - v.p.\int_0^\infty \frac{\xi\,d\xi}{(1-\xi^\alpha)(\xi^2+t^2)} = \frac{\pi i}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - \frac{\pi i}{\alpha(1+t^2)}. \quad (3.12)$$

Similarly for evaluating integral B, we obtain

$$\begin{aligned} v.p. \oint_{\partial G_R} \frac{dy}{(1-e^{-i\pi\alpha}y^\alpha)(y^2+t^2)} &= 2\pi i \mathrm{Res}_{i|t|} + \pi i \mathrm{Res}_{-1} \\ &= \frac{\pi}{(1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha)|t|} + \frac{\pi i}{\alpha(1+t^2)}; \end{aligned}$$

$$B + v.p. \int_0^\infty \frac{d\xi}{(1-\xi^\alpha)(\xi^2+t^2)} = \frac{\pi}{(1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha)|t|} + \frac{\pi i}{\alpha(1+t^2)}. \tag{3.13}$$

From (3.12), (3.13) we obtain

$$\begin{aligned} \frac{1}{\pi}\mathrm{Im}A - \frac{it}{\pi}\mathrm{Im}B &= \mathrm{Re}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - i\,\mathrm{sgn}t\ \mathrm{Im}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} \\ &\quad -\frac{1}{\alpha(1+t^2)} - \frac{it}{\alpha(1+t^2)}. \end{aligned} \tag{3.14}$$

Substituting (3.14) into (3.11) we have

$$\begin{aligned} \int_{-\infty}^\infty e^{itx} p_\alpha^\theta(x)dx &= \frac{1}{\alpha(1-it)} + \mathrm{Re}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} - i\,\mathrm{sgn}t\mathrm{Im}\frac{1}{1-e^{-\frac{i\pi\alpha}{2}}|t|^\alpha} \\ &\quad -\frac{1}{\alpha(1+t^2)} - \frac{it}{\alpha(1+t^2)} \\ &= \frac{1}{1-e^{\frac{i\pi\alpha}{2}\mathrm{sgn}t}|t|^\alpha}. \end{aligned}$$

Case (iv): $-\pi < \theta < 0$;
From (1.3) it is evident that $\varphi_\alpha^\theta(t) = \varphi_\alpha^{-\theta}(-t)$. By (i)-(iii) the following formula is valid

$$\begin{aligned} \varphi_\alpha^{-\theta}(-t) &= \int_{-\infty}^\infty e^{-itx} p_\alpha^{-\theta}(x)\,dx \\ &= \int_{-\infty}^\infty e^{itx} p_\alpha^{-\theta}(-x)\,dx. \end{aligned}$$

Hence

$$\varphi_\alpha^\theta(t) = \int_{-\infty}^\infty e^{itx} p_\alpha^{-\theta}(-x)\,dx,\ t\in\mathbb{R},\ -\pi<\theta<0.$$

Thus, the representation is valid

$$\varphi_\alpha^\theta(t) = \int_{-\infty}^\infty e^{itx} p_\alpha^\theta(x)\,dx,\quad -\pi<\theta<0$$

where $p_\alpha^\theta(x) = p_\alpha^{-\theta}(-x)$. $\square$

Proof of Theorem 2.2. In Theorem 2.1 we have proved that $\varphi_\alpha^\theta(t)$ is a Fourier transform of some function $p_\alpha^\theta(x) \in L^1(\mathbb{R})$ for $(\alpha,\theta) \in (0,2)\times(-\pi,\pi)$. Hence $\varphi_\alpha^\theta(t)$ is a characteristic function for the values of (α,θ) for which $p_\alpha^\theta(x) \geq 0$ almost everywhere. It is not a characteristic function for the values of (α,θ) for which $p_\alpha^\theta(x) < 0$ on a set of positive Lebesgue measure.

Therefore it suffices to determine the values of (α,θ) for which $p_\alpha^\theta(x) \geq 0$ almost everywhere . It is evident from (2.1), (2.2), (2.4), (2.5) that it is the case iff $(\alpha,\theta) \in PD$. Since $\int_{-\infty}^{\infty} p_\alpha^\theta(x)dx = \varphi_\alpha^\theta(0) = 1$, the function $p_\alpha^\theta(x)$ is a probability density iff $(\alpha,\theta) \in PD$. It means that $\varphi_\alpha^\theta(t)$ is a characteristic function iff $(\alpha,\theta) \in PD$. $\square$

Proof of Theorem 2.3. (i) It is obvious, that for any $|x| > 0$, $(\alpha,\theta) \in PD_0^+$ and for any $k = 1,2,3...$ the integral in the formula (2.1) is k-times differentiable and we have

$$(-1)^k(p_\alpha^\theta(x))^{(k)} = \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{e^{-yx}y^{\alpha+k}dy}{|1+e^{i\frac{\pi\alpha}{2}+i\theta}y^\alpha|^2} > 0 \ , \ \ x>0, \quad (3.15)$$

$$(p_\alpha^\theta(x))^{(k)} = \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{e^{yx}y^{\alpha+k}dy}{|1+e^{i\frac{\pi\alpha}{2}-i\theta}y^\alpha|^2} > 0 \ , \ \ x<0. \quad (3.16)$$

Hence, $p_\alpha^\theta(x)$ is completely monotonic on $(0,\infty)$ and absolutely monotonic on $(-\infty,0)$ for $(\alpha,\theta) \in PD_0^+$. The proof is similar for $\theta = \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$.
(ii) By the monotonic convergence theorem, we have from (2.1)

$$\begin{aligned}\lim_{x\to 0^+} p_\alpha^\theta(x) &= \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{i\frac{\pi\alpha}{2}+i\theta}y^\alpha|^2}\\ \lim_{x\to 0^-} p_\alpha^\theta(x) &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{y^\alpha dy}{|1+e^{i\frac{\pi\alpha}{2}-i\theta}y^\alpha|^2}\end{aligned}$$

Evidently, the integrals in the right hand side are divergent for $0 < \alpha \leq 1$ and convergent for $1 < \alpha < 2$ and in the latter case we have

$$\lim_{x\to 0^+} p_\alpha^\theta(x) = \lim_{x\to 0^-} p_\alpha^\theta(x) = \frac{\cos\theta/\alpha}{\alpha\sin\pi/\alpha}$$

For the (α,θ) located on the boundary of the PD, proof is obvious.
(iii) For $0 \leq \theta < \frac{\pi\alpha}{2}$, the proof is obvious by applying monotonic convergence theorem to (3.15), (3.16) For $\theta = \frac{\pi\alpha}{2}$, it follows from (2.1) immediately. $\square$

Proof of Theorem 2.5. (i) For $(\alpha,\theta) \in PD_0^+$, we have from (2.1)

$$\int_0^\infty p_\alpha^\theta(x)dx = \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty dx\int_0^\infty \frac{e^{-yx}y^\alpha\, dy}{|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2}.$$

Since the integrand in the right hand side belongs to $L^1(\mathbb{R}^2)$, from Fubini's theorem we have

$$\begin{aligned}
\int_0^\infty p_\alpha^\theta(x)dx &= \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^{\alpha-1}\,dy}{|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2} \\
&= \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{y^{\alpha-1}\,dy}{1+2\cos(\frac{\pi\alpha}{2}+\theta)y^\alpha+y^{2\alpha}} \\
&= \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi\alpha}\int_0^\infty \frac{du}{1+2\cos(\frac{\pi\alpha}{2}+\theta)u+u^2} \\
&= \frac{1}{2}+\frac{\theta}{\pi\alpha}.
\end{aligned}$$

For $\theta=\min(\frac{\pi\alpha}{2},\pi-\frac{\pi\alpha}{2})$, proof is evident from (2.1) and (2.4).

(ii) For the pairs $(\alpha,\theta)\in PD_0^+$, from (2.1) we have for $x>0$

$$\begin{aligned}
\frac{p_\alpha^\theta(x)}{\sin(\frac{\pi\alpha}{2}+\theta)}-\frac{p_\alpha^\theta(-x)}{\sin(\frac{\pi\alpha}{2}-\theta)} &= \frac{1}{\pi}\int_0^\infty \frac{e^{-yx}y^\alpha\,dy}{|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2}-\frac{1}{\pi}\int_0^\infty \frac{e^{-yx}y^\alpha\,dy}{|1+e^{-i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2} \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-yx}2y^{2\alpha}(\cos(\frac{\pi\alpha}{2}-\theta)-\cos(\frac{\pi\alpha}{2}+\theta))\,dy}{|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2|1+e^{-i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2} \\
&= \frac{1}{\pi}\int_0^\infty \frac{e^{-yx}4y^{2\alpha}\sin\frac{\pi\alpha}{2}\sin\theta\,dy}{|1+e^{i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2|1+e^{-i\theta+i\frac{\pi\alpha}{2}}y^\alpha|^2} \\
&\geq 0.
\end{aligned}$$

For $\theta=\min(\frac{\pi\alpha}{2},\pi-\frac{\pi\alpha}{2})$, proof is evident from (2.1) and (2.4).

(iii) The inequality $p_\alpha^\theta(x)\geq p_\alpha^\theta(-x)$ for $x>0$, $(\alpha,\theta)\in PD^+$, $\alpha\in(0,1)$ is an immediate corollary of (ii). Using Corollary 2 of Theorem 2.7 (see p.8), we conclude that $p_\alpha^\theta(x)<p_\alpha^\theta(-x)$ for x being large enough if $(\alpha,\theta)\in PD^+$, $\alpha\in(1,2)$, $\theta>0$.

(iv) From (2.1) we have

$$\begin{aligned}
p_\alpha^\theta(x) &= \frac{\sin(\frac{\pi\alpha}{2}+\theta)}{\pi}\int_0^\infty \frac{e^{-yx}y^\alpha\,dy}{1+2\cos(\frac{\pi\alpha}{2}+\theta)y^\alpha+y^{2\alpha}},\ x>0, \\
p_\alpha^\theta(x) &= \frac{\sin(\frac{\pi\alpha}{2}-\theta)}{\pi}\int_0^\infty \frac{e^{yx}y^\alpha\,dy}{1+2\cos(\frac{\pi\alpha}{2}-\theta)y^\alpha+y^{2\alpha}},\ x<0. \quad (3.17)
\end{aligned}$$

For $0\leq\theta<\min(\frac{\pi\alpha}{2},\frac{\pi}{2}-\frac{\pi\alpha}{2})$, both $\frac{\pi\alpha}{2}+\theta$ and $\frac{\pi\alpha}{2}-\theta$ are in between 0 and $\frac{\pi}{2}$. Thus as θ increases $\sin(\frac{\pi\alpha}{2}+\theta)$ increases and $\cos(\frac{\pi\alpha}{2}+\theta)$ decreases, hence $p_\alpha^\theta(x)$ increases for fixed $x>0$. Similarly $p_\alpha^\theta(x)$ decreases for fixed $x<0$.

For $\alpha\in(1,2)$, $p_\alpha^\theta(x)$ is a continuous function of x on $\mathbb{R}$ by Theorem 2.3 (ii). Moreover, for fixed $\alpha\in(1,2)$, $p_\alpha^\theta(0)$ decreases as θ increases. Hence, $p_\alpha^\theta(x)$ can not increase in θ for $x>0$ being small enough. □

Note that (3.17) yields that $p_\alpha^\theta(x)$ remains to be decreasing in $\theta \in (0, \min(\frac{\pi\alpha}{2}, \frac{\pi}{2} - \frac{\pi\alpha}{2}))$ for any fixed $\alpha \in (1, 3/2)$ and $x < 0$. Corollary 3 of Theorem 2.7 (see p.8) shows that, for $\alpha \in (1,2)$, $p_\alpha^\theta(x)$ is increasing in θ for fixed $x > 0$ being large enough. This justifies the picture on fig.2.2 (ii) (p.7).

Chapter 4

Representation by a Cauchy Type Integral

Consider the Cauchy type integral

$$f_\alpha(z) = \frac{1}{\pi}\int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}\, dv}{v - z}, \quad 0 < \alpha < 2. \tag{4.1}$$

The function is analytic in the region $C = \{z : 0 < \arg z < 2\pi\}$. Since the function $e^{-v^{1/\alpha}}\, v^{1/\alpha}$ is analytic on the open positive ray $\mathbb{R}^+$, it satisfies Lipschitz condition on this ray. Therefore by the well-known properties of Cauchy type integrals (see, e.g. [14], p.25), $f_\alpha(z)$ has boundary values $f_\alpha(x+i0)$ and $f_\alpha(x-i0)$ for any $x > 0$. Below, it will be convenient to write $f_\alpha(x)$ instead of $f_\alpha(x+i0)$ for $x > 0$.

The following lemma is a generalization of Lemma 4.1 of [8], the latter can be obtained from ours by setting $\theta = 0$.

Lemma 4.1 *For any pair $(\alpha,\theta) \in PD^+$, the following representation is valid:*

$$|x|^{1/\alpha}\, p_\alpha^\theta(\mathrm{sgn}x\, |x|^{1/\alpha}) = \frac{1}{\alpha}\, \mathrm{Im} f_\alpha(|x|\, e^{i(\pi - \frac{\pi\alpha}{2} - \theta \mathrm{sgn} x)}). \tag{4.2}$$

Proof. Except $\alpha \in (1,2)\,,\, \theta = \pi - \frac{\pi\alpha}{2}\,,\, x > 0$; for all $(\alpha,\theta) \in PD^+$ we have $|x|\, e^{i(\pi - \frac{\pi\alpha}{2} + \theta \mathrm{sgn} x)} \in C$. Firstly we will make the proof for these values of parameters α, θ and x.

Putting $\mathrm{sgn}x|x|^{1/\alpha}$ instead of x in (2.1) and multiplying by $|x|^{1/\alpha}$, we have

$$|x|^{1/\alpha}\, p_\alpha^\theta(\mathrm{sgn}x\, |x|^{1/\alpha}) = \frac{\sin(\frac{\pi\alpha}{2} + \theta \mathrm{sgn}x)}{\pi}\int_0^\infty \frac{e^{-y|x|^{1/\alpha}}\, y^\alpha\, |x|^{1/\alpha}\, dy}{|1 + e^{i\theta \mathrm{sgn} x}\, y^\alpha\, e^{i\frac{\pi\alpha}{2}}|^2}.$$

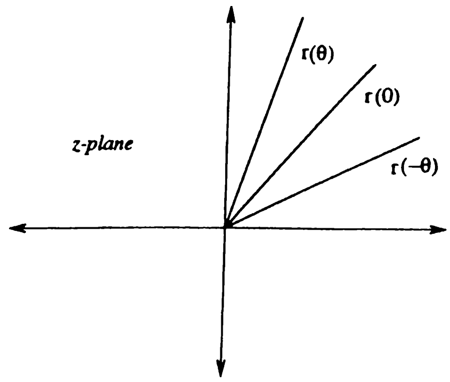

Figure 4.1:

Changing the variable $y = (v/|x|)^{1/\alpha}$, we obtain

$$\begin{aligned} |x|^{1/\alpha} p_\alpha^\theta(\mathrm{sgn}x\,|x|^{1/\alpha}) &= \frac{\sin(\frac{\pi\alpha}{2} + \theta\mathrm{sgn}x)}{\pi\alpha} \int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}\, |x|\, dv}{||x| + e^{i\theta\mathrm{sgn}x}\, v\, e^{i\frac{\pi\alpha}{2}}|^2} \\ &= \frac{1}{\alpha}\,\mathrm{Im}\frac{1}{\pi}\int_0^\infty \frac{e^{-v^{1/\alpha}}\, v^{1/\alpha}\, dv}{v + |x|\, e^{-i\theta\mathrm{sgn}x}\, e^{-i\frac{\pi\alpha}{2}}} \\ &= \frac{1}{\alpha}\,\mathrm{Im} f_\alpha(|x|\, e^{i(\pi - \frac{\pi\alpha}{2} - \theta\mathrm{sgn}x)}) \end{aligned} \tag{4.3}$$

For the exceptional values of (α, θ) and $x > 0$ we have from (2.4)

$$x^{1/\alpha} p_\alpha^\theta(-x^{1/\alpha}) = x^{1/\alpha}\, \frac{e^{-x^{1/\alpha}}}{\alpha}. \tag{4.4}$$

By the Plemelj-Sokhotski theorem ([14], p.25), the following equality holds

$$f_\alpha(x + i0) - f_\alpha(x - i0) = 2ie^{-x^{1/\alpha}}\, x^{1/\alpha}. \tag{4.5}$$

Evidently, for any $x, y \in \mathbb{R}$, $y \neq 0$, $f_\alpha(x + iy)$ and $f_\alpha(x - iy)$ are complex conjugate. Hence, $f_\alpha(x)\,(:= f_\alpha(x + i0))$ and $f_\alpha(x - i0)$ are also, and (4.5) can be rewritten in the form

$$\mathrm{Im}\, f_\alpha(x) = e^{-x^{1/\alpha}}\, x^{1/\alpha}.$$

Comparing with (4.4), we obtain

$$x^{1/\alpha} p_\alpha^\theta(-x^{1/\alpha}) = \frac{1}{\alpha}\,\mathrm{Im}\, f_\alpha(x).$$

which coincides with (4.2) in this case. □

In fig.4.1 (p.24), $r(-\theta)$, $r(0)$, $r(\theta)$ denote the rays $\{z : \arg z = \pi - \frac{\pi\alpha}{2} - \theta\}$, $\{z : \arg z = \pi - \frac{\pi\alpha}{2}\}$, $\{z : \arg z = \pi - \frac{\pi\alpha}{2} + \theta\}$ respectively. In Lemma

4.1 we showed that $p_\alpha^\theta(x)$ depends on the values of the Cauchy type integral (4.1) on the rays $r(-\theta)$, $r(\theta)$ for $x > 0$, $x < 0$, respectively. When $\theta = 0$, both $r(-\theta)$, $r(\theta)$ coincide with $r(0)$. This is the symmetric case which was investigated in [8]. In our case, for $\theta \neq 0$, the rays do not coincide. It is evident that, for the $(\alpha, \theta) \in PD_0^+$, the rays are situated in the open upper half plane and, for the pairs $(\alpha, \theta) \in PD^+ \backslash PD_0^+$, either $r(-\theta)$ coincides with the positive ray or $r(\theta)$ coincides with the negative ray. For $\alpha = 1$, $\theta = \pi/2$ both of them lie on the real axis.

Chapter 5

Asymptotic Behaviour at Infinity

We are now ready to prove Theorem 2.7.

Proof of Theorem 2.7. From (4.2) we have

$$|x|^{1/\alpha} p^{\theta}_{\alpha}(\mathrm{sgn}x\,|x|^{1/\alpha}) = \frac{1}{\alpha}\,\mathrm{Im} f_{\alpha}(|x|\,e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)}). \tag{5.1}$$

As it was shown in [8], the function $f_\alpha(z)$ can be represented in the form

$$f_\alpha(z) = -\frac{\alpha}{\pi}\sum_{k=1}^{N}\frac{\Gamma(1+\alpha k)}{z^k} + f_{\alpha,N}(z), \tag{5.2}$$

where

$$|f_{\alpha,N}(z)| \le \frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|z|^{N+1}|\sin(\phi)|}\,,\ \phi = \arg z$$

for N=1,2,3,... Substituting (5.2) into (5.1) we obtain

$$\begin{aligned}|x|^{1/\alpha} p^{\theta}_{\alpha}(\mathrm{sgn}x\,|x|^{1/\alpha}) \;&=\; \frac{1}{\pi}\sum_{k=1}^{N}\frac{\Gamma(1+\alpha k)}{|x|^k}(-1)^{k+1}\sin(\frac{\pi\alpha k}{2}+k\theta\mathrm{sgn}x)\\ &\quad+\frac{1}{\alpha}\mathrm{Im} f_{\alpha,N}(|x|e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)})\end{aligned}$$

where

$$|\mathrm{Im} f_{\alpha,N}(|x|e^{i(\pi-\frac{\pi\alpha}{2}-\theta\mathrm{sgn}x)})| \le \frac{\alpha\Gamma(1+\alpha(N+1))}{\pi|x|^{N+1}|\sin(\frac{\pi\alpha}{2}+\theta\mathrm{sgn}x)|},$$

putting $|x|$ instead of $|x|^{1/\alpha}$ we obtain (2.8). □

Chapter 6

Analytic Structure of $p_\alpha^\theta(x)$

Proof of the theorems concerning the analytic structure of $p_\alpha^\theta(x)$ for the rational values of α are based on the following facts about the analytic structure of the Cauchy type integral (4.2):

Theorem *In $C = \{z : 0 < \arg z < 2\pi\}$ the following representation is valid:*

$$f_{1/n}(z) = \frac{1}{n\pi}\sum_{k=0}^{n-1} z^k \Gamma(1-\frac{k}{n}) + z^n A_n(z) + z^n B_n(z). \tag{6.1}$$

Here

$$A_n(z) = \frac{1}{\pi} e^{-z^n}[\log\frac{1}{z} + \pi i], \tag{6.2}$$

(the branch of the logarithm is defined by the condition $0 < \arg z < 2\pi$); $B_n(z)$ is an entire function representable by the power series

$$B_n(z) = \sum_{k=0}^{\infty} \beta_k^{(n)} z^k$$

where

$$\beta_k^{(n)} = \begin{cases} \Gamma(-kn)/\pi n & , k/n \notin \mathbb{N} \\ \dfrac{(-1)^j}{\pi n}\dfrac{\Gamma'(1+j)}{\Gamma^2(1+j)} & , k/n = j\ , j = 0, 1, 2, \ldots \end{cases} \tag{6.3}$$

Theorem *Assume, $\alpha \in (0.2)$ is represented in the form $\alpha = m/n$, where m. n are relatively prime integers. The following formula is valid in $C = \{z : 0 < \arg z < 2\pi\}$;*

$$f_{m/n}(z) = \frac{m}{n\pi}\sum_{k=0}^{q} \Gamma(1-\frac{km}{n}) z^k$$

$$
\begin{aligned}
&+[\log\frac{1}{|z|}+i(\pi-\arg z)]\sum_{s=1}^{\infty}\xi_{ms-r-1}^{(n)}z^{s+q}\\
&-\pi\sum_{\substack{k=0\\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty}\xi_k^{(n)}\frac{e^{-\frac{\pi i}{m}(k+r+1)}}{\sin(\frac{\pi}{m}(k+r+1))}z^{\frac{k+r+1}{m}+q}\\
&+m\sum_{s=1}^{\infty}\beta_{ms-r-1}^{(n)}z^{s+q} \qquad (6.4)
\end{aligned}
$$

where q is the greatest integer strictly less then $\frac{n}{m}$, $r=n-qm-1$ and

$$
\xi_k^{(n)}=\begin{cases}0 & ,\ k/n\notin\mathbb{N}\\ \dfrac{(-1)^j}{\pi j!} & ,\ k/n=j\,,\ j=0,1,2,\ldots\end{cases} \qquad (6.5)
$$

$\beta_k^{(n)}$ was defined by (6.3).

The first of above theorems is a combination of Lemma 6.1 and Lemma 7.1 of [8]. The second one is a combination of Lemma 10.1 and Lemma 11.1 of [8].

Proof of Theorem 2.8. From (4.2), putting $\alpha=1/n$ we obtain

$$
|x|^n p_{1/n}^{\theta}(\mathrm{sgn}x|x|^n)=n\,\mathrm{Im}f_{1/n}(|x|\,e^{i(\pi-\frac{\pi}{2n}-\theta\mathrm{sgn}x)}). \qquad (6.6)
$$

Substituting (6.1) into (6.6) we have

$$
\begin{aligned}
|x|^n p_{1/n}^{\theta}(\mathrm{sgn}x|x|^n) &= \frac{1}{\pi}\sum_{k=0}^{n-1}(-1)^{k+1}\Gamma(1-\frac{k}{n})\sin(\frac{\pi k}{2n}+\theta k\mathrm{sgn}x)|x|^k\\
&\quad+n|x|^n(-1)^{n+1}\cos(\theta n)\mathrm{Re}\{A_n(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\mathrm{sgn}x)})\}\\
&\quad+n|x|^n(-1)^{n+1}\cos(\theta n)\mathrm{Re}\{B_n(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\mathrm{sgn}x)})\}\\
&\quad-n|x|^n(-1)^{n}\sin(\theta n\mathrm{sgn}x)\mathrm{Im}\{A_n(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\mathrm{sgn}x)})\}\\
&\quad-n|x|^n(-1)^{n}\sin(\theta n\mathrm{sgn}x)\mathrm{Im}\{B_n(|x|e^{i(\pi-\frac{\pi}{2n}-\theta\mathrm{sgn}x)})\}\\
&=: \Sigma+R_A+R_B+I_A+I_B. \qquad (6.7)
\end{aligned}
$$

Utilizing (6.2), we obtain

$$
\begin{aligned}
R_A &= \frac{n}{\pi}|x|^n(-1)^{n+1}\cos(\theta n)\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n))\\
&\quad\cdot[-\log|x|\cos(|x|^n\cos(\theta n))-(\theta\mathrm{sgn}x+\frac{\pi}{2n})(-1)^n\sin(|x|^n\cos(\theta n))];\\
I_A &= -\frac{n}{\pi}|x|^n(-1)^n\sin(\theta n\mathrm{sgn}x)\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n))\\
&\quad\cdot[-\log|x|(-1)^n\sin(|x|^n\cos(\theta n))+(\theta\mathrm{sgn}x+\frac{\pi}{2n})\cos(|x|^n\cos(\theta n))],
\end{aligned}
$$

hence

$$\begin{aligned}
R_A + I_A &= \frac{n|x|^n(-1)^n}{\pi}\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n)) \\
&\cdot[\log|x|\cos(|x|^n(-1)^n\cos(\theta n) - \theta n\mathrm{sgn}x)] \qquad (6.8)\\
&-\frac{n|x|^n(-1)^n}{\pi}\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n)) \\
&\cdot[(\theta\mathrm{sgn}x + \frac{\pi}{2n})\sin(|x|^n\cos(\theta n) + \theta n\mathrm{sgn}x)].
\end{aligned}$$

Utilizing (6.3), we obtain

$$\begin{aligned}
R_B &= \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}|x|^{k+n}(-1)^{k+n+1}\Gamma(-\frac{k}{n})\cos(\theta n)\cos(\frac{\pi k}{2n} + \theta k\mathrm{sgn}x) \\
&+\frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj+n}(-1)^{(j+1)(n+1)}\frac{\Gamma'(1+j)}{\Gamma^2(1+j)}\cos(\theta n)\cos(\frac{\pi j}{2} + \theta jn\mathrm{sgn}x); \\
I_B &= \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}|x|^{k+n}(-1)^{k+n}\Gamma(-\frac{k}{n})\sin(\theta n\mathrm{sgn}x)\sin(\frac{\pi k}{2n} + \theta k\mathrm{sgn}x) \\
&-\frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj+n}(-1)^{(j+1)(n+1)}\frac{\Gamma'(1+j)}{\Gamma^2(1+j)}\sin(\theta n\mathrm{sgn}x)\sin(\frac{\pi j}{2} + \theta jn\mathrm{sgn}x),
\end{aligned}$$

hence

$$\begin{aligned}
R_B + I_B &= \frac{1}{\pi}\sum_{k=1,\frac{k}{n}\notin\mathbb{N}}^{\infty}|x|^{k+n}(-1)^{k+n+1}\Gamma(-\frac{k}{n})\sin((k+n)(\frac{\pi}{2n} + \theta\mathrm{sgn}x)) \\
&+\frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj+n}(-1)^{(j+1)(n+1)}\frac{\Gamma'(1+j)}{\Gamma^2(1+j)}\sin((j+1)(\frac{\pi}{2} + \theta n\mathrm{sgn}x)).
\end{aligned}$$

Putting $s = k + n$ in the first sum and substituting $j + 1$ for j in the second one, we have

$$\begin{aligned}
R_B + I_B &= \frac{1}{\pi}\sum_{s=n+1,\frac{s}{n}\notin\mathbb{N}}^{\infty}|x|^{s}(-1)^{s+1}\Gamma(1-\frac{s}{n})\sin(\frac{\pi s}{2n} + \theta s\mathrm{sgn}x) \\
&+\frac{1}{\pi}\sum_{j=0}^{\infty}|x|^{nj}(-1)^{j(n+1)}\frac{\Gamma'(j)}{\Gamma^2(j)}\sin(\frac{\pi j}{2} + \theta jn\mathrm{sgn}x). \qquad (6.9)
\end{aligned}$$

Putting (6.8), (6.9) into (6.7), we get

$$\begin{aligned}
|x|^n p^{\theta}_{1/n}(\mathrm{sgn}x|x|^n) &= \frac{1}{\pi}\sum_{k=0,\frac{k}{n}\notin\mathbb{N}}^{\infty}(-1)^{k+1}\Gamma(1-\frac{k}{n})\sin(\frac{\pi k}{2n} + \theta k\mathrm{sgn}x)|x|^k \\
&+\frac{1}{\pi}\sum_{j=0}^{\infty}(-1)^{j(n+1)}\frac{\Gamma'(j)}{\Gamma^2(j)}\sin(\frac{\pi j}{2} + \theta jn\mathrm{sgn}x)|x|^{nj} \\
&+\frac{n|x|^n(-1)^n}{\pi}\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n))
\end{aligned}$$

$$\cdot[\log|x|\cos(|x|^n(-1)^n\cos(\theta n)-\theta n\mathrm{sgn}x)]$$
$$-\frac{n|x|^n(-1)^n}{\pi}\exp(|x|^n(-1)^n\mathrm{sgn}x\sin(\theta n))$$
$$\cdot[(\theta\mathrm{sgn}x+\frac{\pi}{2n})\sin(|x|^n\cos(\theta n)+\theta n\mathrm{sgn}x)].$$

Substituting $|x|$ for $|x|^n$ we obtain (2.10). □

Proof of Theorem 2.9. From (4.2) we have

$$|x|^{n/m}p^{\theta}_{m/n}(\mathrm{sgn}x\,|x|^{n/m})=\frac{n}{m}\mathrm{Im}f_{n/m}(|x|e^{i(\pi-\frac{\pi m}{2n}-\theta\mathrm{sgn}x)}). \tag{6.10}$$

Substituting (6.4) into (6.10) we obtain

$$\begin{aligned}
&|x|^{n/m}p^{\theta}_{m/n}(\mathrm{sgn}x\,|x|^{n/m})=\\
&\frac{1}{\pi}\sum_{k=0}^{q}\Gamma(1-\frac{km}{n})(-1)^{k+1}\sin(\frac{\pi mk}{2n}+\theta k\mathrm{sgn}x)|x|^k\\
&+\frac{n}{m}\log\frac{1}{|x|}\sum_{s=1}^{\infty}\xi^{(n)}_{ms-r-1}(-1)^{s+q+1}\sin((s+q)(\frac{\pi m}{2n}+\theta\mathrm{sgn}x))|x|^{s+q}\\
&+(\frac{\pi}{2}+\frac{n\theta\mathrm{sgn}x}{m})\sum_{s=1}^{\infty}\xi^{(n)}_{ms-r-1}(-1)^{s+q}\cos((s+q)(\frac{\pi m}{2n}+\theta\mathrm{sgn}x))|x|^{s+q}\\
&-\frac{\pi n}{m}\sum_{\substack{k=0\\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty}\xi^{(n)}_k\frac{\sin(\pi q-(\theta\mathrm{sgn}x+\frac{\pi m}{2n})(\frac{k+r+1}{m}+q))}{\sin(\frac{\pi}{m}(k+r+1))}|x|^{\frac{k+r+1}{m}+q}\\
&+n\sum_{s=1}^{\infty}\beta^{(n)}_{ms-r-1}(-1)^{s+q+1}\sin((s+q)(\frac{\pi m}{2n}+\theta\mathrm{sgn}x))|x|^{s+q}\\
&=:\Sigma_1+\frac{n}{m}\log\frac{1}{|x|}\Sigma_2+(\frac{\pi}{2}+\frac{n\theta\mathrm{sgn}x}{m})\Sigma_3-\frac{n\pi}{m}\Sigma_4+n\Sigma_5,
\end{aligned} \tag{6.11}$$

say. Now we shall transform Σ_2, Σ_3, Σ_4, Σ_5 by substituting $\xi^{(n)}_k$, $\beta^{(n)}_k$.

The coefficients $\xi^{(n)}_k$ differ from zero only if $\frac{k}{n}$ is an integer, hence $\xi^{(n)}_{ms-r-1}$ is nonzero iff $\frac{ms-r-1}{n}$ is an integer. Remembering the definition of r we have

$$\frac{ms-r-1}{n}=\frac{m(s+q)}{n}-1.$$

Since m,n are relatively prime, $\frac{ms-r-1}{n}$ is an integer iff $\frac{s+q}{n}$ is. Hence $\xi^{(n)}_{ms-r-1}\neq 0$ iff $s\in\{nt-q\}_{t=1}^{\infty}$. When $s=nt-q$, using (6.5), we obtain

$$\xi^{(n)}_{ms-r-1}=\xi^{(n)}_{n(mt-1)}=\frac{(-1)^{mt-1}}{\pi(mt-1)!},\ t=1,2,\dots.$$

Thus,

$$\Sigma_2=\frac{1}{\pi}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t}}{(mt-1)!}\sin(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x))|x|^{nt}. \tag{6.12}$$

Similarly,

$$\Sigma_3 = \frac{1}{\pi}\sum_{t=1}^{\infty}\frac{(-1)^{(m+n)t-1}}{(mt-1)!}\cos(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x))|x|^{nt}. \qquad (6.13)$$

Substituting $r = n - qm - 1$, we obtain

$$\Sigma_4 = -\sum_{\substack{k=0\\ k\notin\{ms-r-1\}_{s=1}^{\infty}}}^{\infty}\xi_k^{(n)}\frac{\sin((\frac{\pi m}{2n}+\theta\mathrm{sgn}x)(\frac{k+n}{m}))}{\sin(\frac{\pi(k+n)}{m})}|x|^{\frac{k+n}{m}}$$

This sum is taken over the values of k such that $k \notin \{ms-r-1\}_{s=1}^{\infty}$, and summand vanish if $\frac{k}{n}$ is of the form $k = nj$, j =0,1,2... Now the relation is $nj \notin \{ms-r-1\}_{s=1}^{\infty}$ and it is equivalent to $j \notin \{\frac{m(s+q)}{n}-1\}_{s=1}^{\infty}$. But the numbers $\frac{m(s+q)}{n}-1$ are integers iff $s = nt - q$, t =1,2,..., hence the relation $nj \notin \{ms-r-1\}_{s=1}^{\infty}$ is equivalent to $j \notin \{mt-1\}_{s=1}^{\infty}$. Using (6.5), we can rewrite Σ_4

$$\Sigma_4 = -\sum_{\substack{j=0\\ j\notin\{mt-1\}_{t=1}^{\infty}}}^{\infty}\frac{(-1)^j}{j!}\frac{\sin((j+1)(\frac{\pi}{2}+\frac{\theta n}{m}\mathrm{sgn}x))}{\sin(\frac{\pi n}{m}(j+1))}|x|^{\frac{n(j+1)}{m}}.$$

Substituting $j+1$ for j, we obtain

$$\Sigma_4 = -\sum_{j=1,\frac{j}{m}\notin\mathbb{N}}^{\infty}\frac{(-1)^{j-1}}{(j-1)!}\frac{\sin(\frac{\pi j}{2}+\frac{\theta nj}{m}\mathrm{sgn}x)}{\sin(\frac{\pi nj}{m})}|x|^{\frac{nj}{m}}. \qquad (6.14)$$

Using the same argument, we shall divide Σ_5 into two parts. The first summation is taken over the values of s for which $\frac{ms-r-1}{n}$ is an integer, i.e. $s = nt - q$, $t = 1,2,3....$ The second summation is taken over the values of s for which $\frac{ms-r-1}{n}$ is non-integer, i.e. the values of s for which $\frac{s+q}{n} \notin \mathbb{N}$. Remembering the formulas for $\beta_k^{(n)}$ we can rewrite Σ_5, in the form

$$\begin{aligned}\Sigma_5 &= \sum_{t=1}^{\infty}\frac{(-1)^{mt+nt}}{\pi n}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x)|x|^{nt}\\ &+\sum_{\substack{s=1\\ \frac{s+q}{n}\notin\mathbb{N}}}^{\infty}\frac{(-1)^{s+q+1}}{\pi n}\Gamma(1-\frac{m(s+q)}{n})\sin((s+q)(\frac{m\pi}{2n}+\theta\mathrm{sgn}x))|x|^{s+q}.\end{aligned}$$

Putting $p = s + q$ in the second sum we have,

$$\begin{aligned}\Sigma_5 &= \sum_{t=1}^{\infty}\frac{(-1)^{mt+nt}}{\pi n}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2}+\theta nt\mathrm{sgn}x)|x|^{nt}\\ &+\sum_{p=q+1,\frac{p}{n}\notin\mathbb{N}}^{\infty}\frac{(-1)^{p+1}}{\pi n}\Gamma(1-\frac{mp}{n})\sin(\frac{mp\pi}{2n}+\theta p\mathrm{sgn}x)|x|^{p}. \qquad (6.15)\end{aligned}$$

Substituting (6.12), (6.13), (6.14), (6.15) into (6.11) we obtain

$$
\begin{aligned}
&|x|^{n/m} p^{\theta}_{m/n}(\mathrm{sgn} x\, |x|^{n/m}) = \\
&\frac{1}{\pi} \sum_{k=1, \frac{k}{n} \notin \mathbb{N}}^{\infty} \Gamma(1 - \frac{km}{n})(-1)^{k+1} \sin(\frac{\pi mk}{2n} + \theta k \mathrm{sgn} x)|x|^{k} \\
&+\frac{n}{m\pi} \log \frac{1}{|x|} \sum_{t=1}^{\infty} \frac{(-1)^{(m+n)t}}{(mt-1)!} \sin(\frac{\pi mt}{2} + \theta nt \mathrm{sgn} x)|x|^{nt} \\
&+(\frac{1}{2} + \frac{n\theta \mathrm{sgn} x}{\pi m}) \sum_{t=1}^{\infty} \frac{(-1)^{(m+n)t-1}}{(mt-1)!} \cos(\frac{\pi mt}{2} + \theta nt \mathrm{sgn} x))|x|^{nt} \\
&+\frac{n}{m} \sum_{j=1, \frac{j}{m} \notin \mathbb{N}}^{\infty} \frac{(-1)^{j-1}}{(j-1)!} \frac{\sin(\frac{\pi j}{2} + \frac{\theta nj}{m} \mathrm{sgn} x)}{\sin(\frac{\pi nj}{m})} |x|^{\frac{nj}{m}} \\
&+\frac{1}{\pi} \sum_{t=1}^{\infty} (-1)^{(m+n)t} \frac{\Gamma'(mt)}{\Gamma^2(mt)} \sin(\frac{\pi mt}{2} + \theta nt \mathrm{sgn} x))|x|^{nt}.
\end{aligned}
$$

Substituting $|x|$ for $|x|^{n/m}$ and using the well-known equalities

$$
\Gamma(1-z) = \frac{\pi}{\Gamma(z)\sin(\pi z)}, \quad \Gamma(n) = (n-1)! \ , \ n \in \mathbb{N}
$$

we obtain (2.11). □

Chapter 7

Representation of $p_\alpha^\theta(x)$ by a Contour Integral

In this chapter we shall represent $p_\alpha^\theta(x)$ by a contour integral. This representation plays the key role in the transition from rational to irrational α's. For $\theta = 0$, this representation was obtained in [8].

Fix a positive $\delta < \frac{1}{2}$ and consider the integral

$$I_\delta(x;\alpha,\theta) = \frac{i}{2\alpha}\int_{L(\delta)} \frac{e^{z\log|x|}\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)\,dz}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z} \tag{7.1}$$

where $L(\delta)$ is the boundary of the region

$$G(\delta) = \{z : |z| > \frac{1}{2}\delta,\ |\arg z| < \frac{\pi}{4}\}.$$

The transition on the boundary is in the direction such that the region $G(\delta)$ remains to the left.

The contour representation mentioned above is given in the following theorem:

Theorem 7.1 *The following representation is valid for* $(\alpha,\theta) \in PD^+\backslash\{(\alpha,\theta) : \theta = \pi - \pi\alpha/2\}$, $x > 0$, *and for* $(\alpha,\theta) \in PD^+\backslash\{(\alpha,\theta) : \theta = \pi\alpha/2\}$, $x < 0$;

$$p_\alpha^\theta(x) = \frac{1}{|x|}I_\delta(x;\alpha,\theta) \tag{7.2}$$

where δ *is such that* $\alpha \in [\delta, 2-\delta]$.

This theorem is a generalization of Theorem 13.1 of [8], the latter can be obtained from ours by setting $\theta = 0$.

Firstly we will prove the following two lemmas.

Lemma 7.1 *For any fixed $0 \le \theta \le \min(\frac{\pi\alpha}{2}, \pi - \frac{\pi\alpha}{2})$, $0 < \delta < \frac{1}{2}$, $1 < M < \infty$, the integral $I_\delta(x;\alpha,\theta)$ converges absolutely and uniformly with respect to both $\alpha \in [\delta, 2-\delta]$ and $|x| < M$.*

Proof. Note that

$$|\sin\frac{\pi z}{\alpha}| \ge \sinh(\frac{\pi}{\alpha}|\mathrm{Im}z|)\,,\ |\sin \pi z| \ge \sinh(\pi|\mathrm{Im}z|),$$

moreover, on the rays $\{z : |z| \ge \frac{\delta}{2}, \arg z = \mp\frac{\pi}{4}\}$ we have $|\mathrm{Im}z| \ge \frac{\delta}{2\sqrt{2}}$. Hence $\sin\frac{\pi z}{\alpha}$, $\sin \pi z$ are bounded on $L(\delta)$ from below by a positive constant C not depending on $\alpha \in [\delta, 2-\delta]$.

Using the Stirling formula ([15],p.249)

$$\log\Gamma(z) - (z-\frac{1}{2})\log z + z - \frac{1}{2}\log 2\pi = O(|z|^{-1})\,,\ z \to \infty\,,\ \mathrm{Re}z > 0,$$

we obtain

$$\log|\Gamma(z)| = (\mathrm{Re}z)\log|z| + O(|z|)\,,\ z \to \infty.$$

Hence, there are positive constants ε and B such that

$$|\Gamma(z)| \ge Be^{\epsilon|z|\log|z|}\,,\ z \in G(\delta).$$

Noting that

$$\begin{aligned} |\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)| &\le e^{|\frac{\pi}{2}+\frac{\theta}{\alpha}\mathrm{sgn}x|.|\mathrm{Im}z|} \\ &\le e^{\pi|\mathrm{Im}z|}, \end{aligned}$$

we see that for $|x| < M$ the integrand in (7.1) can be estimated as follows

$$\left|\frac{e^{z\log|x|}\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z}\right| \le \frac{\exp(\mathrm{Re}z\log|x| + \pi|\mathrm{Im}z|)}{Be^{\epsilon|z|\log|z|}C^2}. \qquad (7.3)$$

This yields the assertion of the lemma. □

Lemma 7.2 *$p_\alpha^\theta(x)$ is a continuous function of*
(i) α on $[\frac{2\theta}{\pi}, 2-\frac{2\theta}{\pi})$ for any fixed $\theta \in (0, \frac{\pi}{2})$ and $x > 0$,
(ii) α on $(\frac{2\theta}{\pi}, 2-\frac{2\theta}{\pi}]$ for any fixed $\theta \in (0, \frac{\pi}{2})$ and $x < 0$,
(iii) α on $(0,2)$ for $\theta = 0$, and any fixed $x \neq 0$.

Proof. Comparing (2.1) and (2.4), we see that the formula (2.1) is valid for the intervals mentioned in the statement of the lemma.
(i) Take $0 < \delta < \theta$ and consider $\alpha \in [\frac{2\theta}{\pi}, 2 - \frac{2(\theta+\delta)}{\pi}]$ we have the following bound for the integrand staying in the right hand side of the formula (2.1) for fixed $x > 0$

$$\begin{aligned}\frac{e^{-yx}y^\alpha}{|1+e^{i\theta}y^\alpha e^{\frac{i\pi\alpha}{2}}|^2} &= \frac{e^{-yx}y^\alpha}{|e^{-i\theta}e^{-\frac{i\pi\alpha}{2}}+y^\alpha|^2} \\ &\leq \frac{e^{-yx}y^\alpha}{(\sin(\frac{\pi\alpha}{2}+\theta))^2} \leq \frac{e^{-yx}(1+y)^2}{(\sin\delta)^2}.\end{aligned}$$

Therefore the integral in (2.1) converges uniformly with respect to $\alpha \in [\frac{2\theta}{\pi}, 2-\frac{2(\theta+\delta)}{\pi}]$ for fixed $\theta \in (0, \pi/2)$ and fixed $x > 0$, hence is a continuous function of α.
(ii) As in the case (i), take $0 < \delta < \theta$ and consider $\alpha \in [\frac{2(\theta+\delta)}{\pi}, 2-\frac{2\theta}{\pi}]$ we have the following bound for the integrand staying in the right hand side of the formula (2.1) for fixed $x < 0$

$$\frac{e^{yx}y^\alpha}{|1+e^{-i\theta}y^\alpha e^{\frac{i\pi\alpha}{2}}|^2} \leq \frac{e^{yx}y^\alpha}{(\sin(\frac{\pi\alpha}{2}-\theta))^2} \leq \frac{e^{yx}(1+y)^2}{(\sin\delta)^2}.$$

Therefore the integral in (2.1) converges uniformly with respect to $\alpha \in [\frac{2(\theta+\delta)}{\pi}, 2-\frac{2\theta}{\pi}]$ for fixed $\theta \in (0, \pi/2)$ and fixed $x < 0$, hence is a continuous function of α.
(iii) This part was proved in [8] and the proof is similar to the cases (i), (ii).
□

Now we can prove Theorem 7.1

Proof of Theorem 7.1. Firstly we will prove the validity of the formula (7.2) for rational α's. Since the rational numbers are dense in $(0,2)$, by the continuity of $p_\alpha^\theta(x)$ and of the integral $I_\delta(x;\alpha,\theta)$ as functions of α, theorem will be proved for (α, θ, x)'s mentioned in the statement of the theorem.

Since α is rational, it has a unique representation $\alpha = m/n$ where m and n are relatively prime integers. The functions $\sin\frac{\pi z}{\alpha}$, $\sin\pi z$ vanish on the set $\{k\alpha\}_{k=-\infty}^{\infty}$ and $\{k\}_{k=-\infty}^{\infty}$ correspondingly. These sets have an intersection which is the set $\{mt\}_{t=-\infty}^{\infty}$, and they are contained in the set $\{\frac{s}{n}\}_{s=-\infty}^{\infty}$. Taking a positive $\nu < 1/2n$, both functions are bounded from below on the set $\mathbb{C}\setminus \bigcup_{s=-\infty}^{\infty}\{z : |z - s/n| < \nu\}$ by a positive constant C.

Set $X_s = s/n + 1/2n$, $s = 1,2,\ldots$ and consider the integral

$$I_\delta(x;\alpha,\theta,X_s) = \frac{i}{2\alpha}\int_{L(\delta,X_s)} \frac{e^{z\log|x|}\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)\,dz}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z} \tag{7.4}$$

where $L(\delta, X_s)$ is the boundary of the region

$$G(\delta)\bigcap\{z : \mathrm{Re}z < X_s\}.$$

By the residue theory we have

$$\begin{aligned} I_\delta(x;\alpha,\theta,X_s) &= -\frac{\pi}{\alpha}\sum_{\substack{\alpha\le k\alpha<X_s \\ k/n\notin\mathbb{N}}} (\text{Residue at } z = k\alpha) \\ &\quad -\frac{\pi}{\alpha}\sum_{\substack{1\le k<X_s \\ k/m\notin\mathbb{N}}} (\text{Residue at } z = k) \\ &\quad -\frac{\pi}{\alpha}\sum_{1\le mt<X_s} (\text{Residue at } z = mt) \\ &=: -\frac{\pi}{\alpha}(\Sigma_1 + \Sigma_2 + \Sigma_3). \end{aligned} \tag{7.5}$$

At the points $z = k\alpha$, $k = 1,2,\ldots$, $k/n \notin \mathbb{N}$, we have the residue

$$\mathrm{Res}_{k\alpha} = \frac{\alpha}{\pi}\frac{(-1)^k\sin(\frac{\pi\alpha k}{2} + k\theta\mathrm{sgn}x)|x|^{k\alpha}}{\Gamma(k\alpha)\sin\pi k\alpha},$$

hence

$$\Sigma_1 = -\frac{\alpha}{\pi}\sum_{\substack{\alpha\le k\alpha<X_s \\ k/n\notin\mathbb{N}}} \frac{(-1)^{k+1}\sin(\frac{\pi\alpha k}{2} + k\theta\mathrm{sgn}x)|x|^{k\alpha}}{\Gamma(k\alpha)\sin\pi k\alpha}. \tag{7.6}$$

At the points $z = k$, $k = 1,2,\ldots$, $k/m \notin \mathbb{N}$, we have the residue

$$\mathrm{Res}_k = \frac{1}{\pi}\frac{(-1)^k\sin(\frac{\pi k}{2} + \frac{k\theta}{\alpha}\mathrm{sgn}x)|x|^{k-1}}{\Gamma(k)\sin\frac{\pi k}{\alpha}},$$

hence

$$\Sigma_2 = -\frac{1}{\pi}\sum_{\substack{1\le k<X_s \\ k/m\notin\mathbb{N}}} \frac{(-1)^{k+1}\sin(\frac{\pi k}{2} + \frac{k\theta}{\alpha}\mathrm{sgn}x)|x|^{k-1}}{\Gamma(k)\sin\frac{\pi k}{\alpha}}. \tag{7.7}$$

To evaluate the residues at the points $z = mt$, $t = 1,2,\ldots$, put

$$f(z) := \frac{\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)|x|^z}{\Gamma(z)}.$$

Evidently $f(z)$ is analytic at $z = mt$, $t = 1,2,\ldots$, and we have

$$\begin{aligned}
\mathrm{Res}_{mt} &= \lim_{z\to mt}\left[\frac{(z-mt)^2 f(z)}{\sin\frac{\pi z}{\alpha}\sin\pi z}\right]' \\
&= \frac{\alpha}{\pi^2}(-1)^{(m+n)t} f'(mt) \\
&= \frac{\alpha}{\pi^2}(-1)^{(m+n)t}(\frac{\pi}{2} + \frac{\theta}{\alpha}\mathrm{sgn}x)\frac{\cos(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}}{\Gamma(mt)} \\
&\quad + \frac{\alpha}{\pi^2}(-1)^{(m+n)t}\log|x|\frac{\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}}{\Gamma(mt)} \\
&\quad - \frac{\alpha}{\pi^2}(-1)^{(m+n)t}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}.
\end{aligned}$$

Hence

$$\begin{aligned}
\Sigma_3 &= (\frac{\alpha}{2\pi} + \frac{\theta}{\pi^2}\mathrm{sgn}x)\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}\cos(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}}{\Gamma(mt)} \\
&\quad + \frac{\alpha}{\pi^2}\log|x|\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}}{\Gamma(mt)} \\
&\quad - \frac{\alpha}{\pi^2}\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}\Gamma'(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}. \qquad (7.8)
\end{aligned}$$

Substituting (7.6), (7.7), (7.8) into (7.5), we obtain

$$\begin{aligned}
I_\delta(x;\alpha,\theta,X_s) &= \sum_{\substack{\alpha\le k\alpha<X_s \\ k/n\notin\mathbb{N}}}\frac{(-1)^{k+1}\sin(\frac{\pi k\alpha}{2} + k\theta\mathrm{sgn}x)}{\Gamma(k\alpha)\sin(\pi k\alpha)}|x|^{k\alpha} \\
&\quad + \frac{1}{\pi}\log\frac{1}{|x|}\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt} \\
&\quad - (\frac{\theta\mathrm{sgn}x}{\pi\alpha} + \frac{1}{2})\sum_{1\le mt<X_s}\frac{(-1)^{(m+n)t}}{\Gamma(mt)}\cos(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt} \\
&\quad + \frac{1}{\alpha}\sum_{\substack{1\le k<X_s \\ k/m\notin\mathbb{N}}}\frac{(-1)^{k+1}}{\Gamma(k)}\frac{\sin(\frac{\pi k}{2} + \frac{k\theta}{\alpha}\mathrm{sgn}x)}{\sin\frac{\pi k}{\alpha}}|x|^k \qquad (7.9) \\
&\quad + \frac{1}{\pi}\sum_{1\le mt<X_s}(-1)^{(m+n)t}\frac{\Gamma'(mt)}{\Gamma^2(mt)}\sin(\frac{\pi mt}{2} + \theta nt\mathrm{sgn}x)|x|^{mt}.
\end{aligned}$$

On the other hand, we have

$$I_\delta(x;\alpha,\theta,X_s) = \frac{i}{2\alpha}\left\{\int_{L(\delta,X_s)} + \int_{C(X_s)}\right\}\frac{e^{z\log|x|}\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\mathrm{sgn}x)\,dz}{\Gamma(z)\sin\frac{\pi z}{\alpha}\sin\pi z} \qquad (7.10)$$

where

$$\begin{aligned} L(\delta, X_s) &= L(\delta)\bigcap\{z : \mathrm{Re} z < X_s\}, \\ C(X_s) &= \{z : \mathrm{Re} z = X_s\,,\,|\arg z| \le \frac{\pi}{4}\}. \end{aligned} \tag{7.11}$$

Using the bound (7.3), we have

$$\left|\frac{e^{z\log|x|}\sin(\frac{\pi z}{2}+\frac{z\theta}{\alpha}\mathrm{sgn}x)}{\Gamma(z)\,\sin\frac{\pi z}{\alpha}\,\sin\pi z}\right| \le \frac{\exp(X_s(\log|x|+\pi))}{Be^{\epsilon X_s \log X_s}C^2},\ z\in C(X_s) \tag{7.12}$$

Hence, integral along $C(X_s)$ tends to zero as $s\to\infty$. Therefore

$$\lim_{s\to\infty} I_\delta(x;\alpha,\theta,X_s) = I_\delta(x;\alpha,\theta)$$

Taking the limit in (7.9) and from (2.11) we obtain

$$I_\delta(x;\alpha,\theta) = \frac{1}{|x|}p_\alpha^\theta(x)\ ,\ \alpha = \frac{m}{n}. \quad \square$$

Chapter 8

The Case of Irrational α

Proof of Theorem 2.11. We shall evaluate the integral $I_\delta(x;\alpha,\theta)$ by means of the Cauchy residue theorem. Substitution of the result into (7.2) of Theorem 7.1 yields (2.12). Under the conditions of Theorem 2.11, the sets $\{k\alpha\}_{k=-\infty}^{\infty}$, $\{k\}_{k=-\infty}^{\infty}$ have the empty intersection since α is irrational.

We construct a sequence $\{Q_s\}_{s=1}^{\infty}$ which plays the role of $\{X_s\}_{s=1}^{\infty}$ in the proof of Theorem 7.1 as follows. Since $\alpha \in (0,2)$, each of the intervals $(s\alpha,(s+1)\alpha)$ contains none, one or two points from the $\{k\}_{k=1}^{\infty}$. In the first case we define $Q_s = (s+\frac{1}{2})\alpha$. In the second and third cases we choose $Q_s \in (s\alpha,(s+1)\alpha)$ so that the distance from Q_s to the nearest of the three points $s\alpha$, $(s+1)\alpha$, $k \in (s\alpha,(s+1)\alpha)$ be at least $\alpha/4$.

Taking a positive $\nu < \alpha/4$, we observe that the modulus of both functions $\sin\frac{\pi z}{\alpha}$, $\sin\pi z$ are bounded from below by a positive constant C on the set

$$\mathbb{C}\setminus\bigcup_{k=-\infty}^{\infty}\left[\{z:|z-k\alpha|<\nu\}\bigcup\{z:|z-k|<\nu\}\right]$$

The vertical lines $\{z:\mathrm{Re}\,z=Q_s\}$, $s=1,2,\ldots$, are located in the interior of this set.

Consider the integral $I_\delta(x;\alpha,\theta,Q_s)$ defined by (7.4) with Q_s instead of X_s. Analogously to (7.9), (7.10), we have

$$\begin{aligned} I_\delta(x;\alpha,\theta,Q_s) &= \sum_{\alpha\le k\alpha<Q_s}\frac{(-1)^{k+1}\sin(\frac{\pi k\alpha}{2}+k\theta\,\mathrm{sgn}\,x)}{\Gamma(k\alpha)\sin\pi k\alpha}|x|^{k\alpha} \\ &\quad+\frac{1}{\alpha}\sum_{0<k<Q_s}\frac{(-1)^{k+1}\sin(\frac{\pi k}{2}+\frac{k\theta}{\alpha}\mathrm{sgn}\,x)}{\Gamma(k)\sin\frac{\pi k}{\alpha}}|x|^{k} \end{aligned} \tag{8.1}$$

$$I_\delta(x;\alpha,\theta,Q_s) = \frac{i}{2\alpha}\left\{\int_{L(\delta,Q_s)} + \int_{C(Q_s)}\right\} \frac{e^{z\log|x|}\sin(\frac{\pi z}{2} + \frac{z\theta}{\alpha}\operatorname{sgn} x)\,dz}{\Gamma(z)\sin\frac{\pi z}{\alpha}\sin\pi z} \tag{8.2}$$

where $L(\delta, Q_s)$, $C(Q_s)$ are defined by (7.11) with Q_s instead of X_s. Obviously the inequality (7.12) is valid with Q_s instead of X_s. The integral along $C(Q_s)$ tends to zero as $s \to \infty$ uniformly. By Lemma 7.1, the integral along $L(\delta, Q_s)$ approaches to the integral along $L(\delta)$ as $s \to \infty$ uniformly with respect to x on any compact subset of $\mathbb{R}$. Taking the limits as $s \to \infty$ in (8.1), (8.2), we arrive at the assertion of Theorem 2.7 except the cases $\alpha \in (1,2)\backslash\mathbb{Q}$, $\theta = \pi - \frac{\pi\alpha}{2}$, $x > 0$ and $\alpha \in (0,1)\backslash\mathbb{Q}$, $\theta = \frac{\pi\alpha}{2}$, $x < 0$. But by comparing the series expansion in (2.12) with (2.1), (2.4) for exceptional values of (α, θ, x) we see that the series expansion in (2.12) is also valid for the exceptional cases. □

Proof of Theorem 2.12. For irrational values of α, proof is evident from (2.12). For rational values of α, $\alpha \in (0,1)$, $\theta = \frac{\pi\alpha}{2}$, $x > 0$ and $\alpha \in [1,2)$, $\theta = \pi - \frac{\pi\alpha}{2}$, $x < 0$ assertion of Theorem 2.9 is valid and same with the assertion of the theorem. For the remaining values of the parameters α, θ, proof is obvious by (2.1) and (2.4). □

Proof of Theorem 2.13. For any integer $k \geq 2$, there exists an integer l_k such that

$$|\alpha - \frac{l_k}{k}| < \frac{1}{2k}. \tag{8.3}$$

Since α is not a Liouville number, there is an integer $r \geq 2$ such that for any pair of integers p, $q \geq 2$

$$|\alpha - \frac{p}{q}| \geq \frac{1}{q^r}. \tag{8.4}$$

Using (8.4), we have

$$|\alpha - \frac{l_k}{k}| \geq k^{-r}. \tag{8.5}$$

From (8.3) and (8.5), we obtain

$$k^{1-r} \leq |\alpha k - l_k| < \frac{1}{2}.$$

Using the inequality

$$\sin x \geq \frac{2}{\pi}x, \quad 0 \leq x \leq \frac{\pi}{2}, \tag{8.6}$$

we obtain

$$|\sin \pi k\alpha| = |\sin \pi(k\alpha - l_k)| \geq 2|k\alpha - l_k| \geq 2k^{1-r}.$$

Hence the first of series in (2.12) converges absolutely and uniformly on any compact subset of $\mathbb{R}$.

Similarly, as above, for any integer $k \geq 2$, there exists an integer l_k such that

$$|\frac{1}{\alpha} - \frac{l_k}{k}| < \frac{1}{2k}. \tag{8.7}$$

It follows that

$$\frac{l_k}{k} \leq \frac{1}{\alpha} + |\frac{1}{\alpha} - \frac{l_k}{k}| < \frac{1}{\alpha} + \frac{1}{2k} < \frac{2}{\alpha},$$

hence

$$l_k \leq \frac{2k}{\alpha}. \tag{8.8}$$

Since α is not a Liouville number, we have

$$|\alpha - \frac{k}{l_k}| \geq l_k^{-r}$$

Multiplying the inequality by l_k/α and using (8.7) and (8.8), we obtain

$$\frac{1}{2} > |l_k - \frac{k}{\alpha}| \geq \frac{l_k^{1-r}}{\alpha} \geq \frac{1}{\alpha}(\frac{2}{\alpha})^{1-r}k^{1-r}.$$

Hence, using (8.6), we obtain

$$|\sin \frac{\pi k}{\alpha}| = |\sin \pi(\frac{k}{\alpha} - l_k)| \geq 2|\frac{k}{\alpha} - l_k| \geq (\frac{2}{\alpha})^{2-r}k^{1-r}.$$

Hence the second of series in (2.12) converges absolutely and uniformly on any compact subset of $\mathbb{R}$. □

Proof of Theorem 2.15. We shall construct a subset D of PD^+ which (i) is dense in PD^+, (ii) has the power continuum, (iii) is such that, for $(\alpha, \theta) \in D$, both of the series in (2.13) diverges.

Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of rapidly increasing integers defined by the equations

$$\sigma_1 = 2. \quad \sigma_{n+1} = 2^{3\sigma_n}, \quad n = 1, 2, ... \tag{8.9}$$

Denote by Δ the set of all sequences $\{\delta_j\}_{j=1}^{\infty}$ with terms δ_j having values 0 or 1 only and satisfying the conditions
(i) δ_j is allowed to be equal to 1 if $j \in \{\sigma_n\}_{n=1}^{\infty}$ only;
(ii) infinitely many of δ_j's equal to 1.

Let $\Omega = \{y : y = \sum_{j=1}^{\infty} \delta_j 2^{-j}, \{\delta_j\}_{j=1}^{\infty} \in \Delta\}$. Let Λ be the set of numbers in $(0,2)$ representable by finite binary fractions. Set

$$E = \{\alpha \in (0,2) : \alpha = x + y, x \in \Lambda, y \in \Omega\}.$$

Evidently E is dense in $(0,2)$ and it is of power continuum. Set

$$D = \{(\alpha,\theta) \in PD : \alpha \in E, (\alpha + \frac{2\theta}{\pi}) \notin L \bigcup \mathbb{Q}\}.$$

It is easy to see that D is dense in PD and it is of power continuum.

It suffices to prove that for any $(\alpha,\theta) \in D$ the first of series in (2.12) diverges.

If $\alpha \in E$ then there is an integer m such that

$$\alpha = b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\infty} \delta_j 2^{-j}$$

where b, a_j take values 0 or 1, and $\{\delta_j\}_{j=1}^{\infty} \in \Delta$. Denote by $\{\eta_n\}_{n=1}^{\infty}$ the subsequence of $\{\sigma_n\}_{n=1}^{\infty}$ such that

$$\begin{aligned} \delta_j &= 1 \quad \text{for } j \in \{\eta_n\}_{n=1}^{\infty}, \\ \delta_j &= 0 \quad \text{for } j \notin \{\eta_n\}_{n=1}^{\infty}. \end{aligned}$$

Then for any $\eta_n > m$, we have

$$0 < \alpha - (b + \sum_{j=1}^{m} a_j 2^{-j} + \sum_{j=m+1}^{\eta_n} \delta_j 2^{-j}) = \sum_{j=\eta_{n+1}}^{\infty} \delta_j 2^{-j} < 2^{-\eta_{n+1}+1}$$

Multiplying this inequality by 2^{η_n}, we see that there is an integer p_n such that

$$0 < \alpha 2^{\eta_n} - p_n < 2^{\eta_n - \eta_{n+1}+1} < 2^{-\frac{1}{2}\eta_{n+1}}. \qquad (8.10)$$

for sufficiently large n.

Consider the terms of the first of series in (2.12) possessing the numbers $q = q_n = 2^{\eta_n}$. From (8.10) we obtain

$$|\sin \pi q_n \alpha| = |\sin(\pi q_n \alpha - \pi p_n)| < \pi 2^{-\frac{1}{2}\eta_{n+1}} \qquad (8.11)$$

Since $\alpha + \frac{2\theta}{\pi}$ is an irrational, non-Liouville number, as in the proof of the previous theorem, there is an integer $r \geq 2$ such that

$$|\sin(\frac{\pi q_n \alpha}{2} + q_n\theta)| = |\sin \frac{\pi 2^{\eta_n}}{2}(\alpha + \frac{2\theta}{\pi})| \geq 2(2^{\eta_n - 1})^{1-r} \tag{8.12}$$

Hence, for sufficiently large n we have

$$\left|\frac{(-1)^{q_n+1}\sin(\frac{\pi q_n \alpha}{2} + q_n\theta)|x|^{q_n\alpha}}{\Gamma(q_n\alpha)\sin \pi q_n \alpha}\right| \geq \frac{2}{\pi}2^{(\eta_n - 1)(1-r)}|x|^{q_n\alpha}2^{-q_n^2\alpha^2}2^{\frac{1}{2}\eta_{n+1}} \tag{8.13}$$

Since $\{\eta_n\}_{n=1}^{\infty}$ is a subsequence of $\{\sigma_n\}_{n=1}^{\infty}$, the following inequality holds

$$\eta_{n+1} \geq 2^{3\eta_n} = q_n^3.$$

Hence from (8.13) the series diverges. □

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