HOMOGENEOUS, ANISOTROPIC SOLUTIONS OF TOPOLOGICALLY MASSIVE GRAVITY INCLUDING A COSMOLOGICAL CONSTANT

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Kâmuran Saygılı

July 22, 1993

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Y. Nutku

Prof. Dr. Yavuz Nutku(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

HOMOGENEOUS,ANISOTROPIC SOLUTIONS OF TOPOLOGICALLY MASSIVE GRAVITY INCLUDING A COSMOLOGICAL CONSTANT

Kâmuran Saygılı
M.S. in Mathematics
Supervisor: Prof. Dr. Yavuz Nutku
July 22, 1993

Exact solutions to the field equations of Topologically massive gravity with a cosmological constant are presented. These are homogeneous,anisotropic Bianchi Type VIII and Type IX manifolds and generalize the finite action vacuum solutions of topologically massive gravity.We find that only those solutions in which two of the constant scale factors are the same admit a cosmological constant.We also find that,depending on the signature and the sign of the cosmological constant,these solutions point to the existence of a critical value for the topological mass which is determined by the cosmological constant.

ÖZET

KOSMOLOJİK SABİT İÇEREN TOPOLOJİK KÜTLELİ GRAVİTASYONDA HOMOJEN,ANİSOTROPİK ÇÖZÜMLER

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22 Temmuz 1993

2+1 boyutlu uzay-zamanda geçerli Topolojik kütleli gravitasyon teorisinin homojen,anisotropik çözümleri kozmolojik sabit içerecek şekilde genelleştirildi. Problemde kozmolojik genişleme faktörleri sabit alındı.Bu çözümler Bianchi-VIII ve Bianchi-IX tipi olup aynı zamanda sonlu eylemlidirler.Sonuç olarak sadece,iki,sabit kosmolojik genişleme faktörünün eşit olduğu durumlarda çözüm lerin kozmolojik sabit kabul ettiği gösterildi.Ayrıca bu çözümler topolojik kütle için kozmolojik sabitle belirlenen kritik bir değerin varlığınada işaret etmektedir.

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I would like to thank to Prof.Dr.Y.Nutku,my thesis advisor, who introduced me in to the subject of lower dimensional gravity teories.

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Contents

Chapter 1

Introduction

In this thesis,exact solutions to the field equations of Topologically Massive Gravity including a cosmological constant will be presented.

Lower dimensional gravity theories are attractive for their interesting properties [1].In low dimensions, although the fundamental properties of the higher dimensional field theories still remain,their mathemathical complexity disappears. Therefore lower dimensional gravity theories are studied as toy models,to gain insight in to higher dimensional theories.They are also studied in their own right,for understanding lower dimensional systems.Besides that these theories are interesting mathematically.

In three spacetime dimensions gravity theories have different structure.Their structure relies on properties unique to three dimensional Riemannian manifolds.Therefore they exhibit attractive mathematical properties.Also parity,spin and statistical behaviour of those theories are different than the real four dimensional theory,General Relativity. For a general discussion of the three dimensional field theories see references [2],[3].

In 2+1 spacetime dimensions Einstein gravity is somewhat different than the 3+1 one.It exhibits some unusual features,which can be deduced from the properties of the Einstein field equations and the curvature tensor [6].In 2+1 dimensions the curvature and the Einstein tensor are equivalent.Therefore the curvature is determined locally by the matter distribution and the cosmological constant. Spacetime is flat,de Sitter,or anti-de Sitter depending on the value of the cosmological constant.As a result there are no gravitational waves outside the matter.However point particles may have non-trivial global effects on the spacetime surrounding them.Correspondence with the Newtonian gravity also breaks down [4] [5].

In 2+1 dimensions,as an alternative to Einstein gravity S.Deser,R.Jackiw and S.Templeton [2] [3] proposed Topologically massive gravity in 1982.The field equations of Topologically Massive gravity consist of the usual Einstein tensor coupled to conformal tensor.In three dimensions the Weyl tensor vanishes due to equivalence of the curvature and the Ricci tensor. [13].Therefore one uses the conformal tensor unique to three dimension [15].It is refered to as Bach tensor,York tensor,or the Cotton tensor.As Weyl tensor,it is invariant under local conformal transformations of the metric and vanishes if and only if the metric is conformally flat.It is also traceless,symmetric and covariantly constant. The action for Topologically Massive Gravity consists of the Einstein-Hilbert term coupled to a term which is proportional to Chern-Simons secondary characteristic classes [1].Einstein gravity coupled to the Chern-Simons gravity becomes a dynamical theory.Furthermore in this theory gravitational waves become massive [2].

In this thesis exact solutions to the field equations of Topologically massive gravity including a cosmological constant will be presented.These are Bianchi Type-VIII and Type-IX homogeneous, anisotropic manifolds and generalize the finite action solutions of topologically massive gravity.

The plan of this thesis will be as follows. In section 2, we shall introduce topologically massive gravity. In sections 3 and 4 solutions of the field equations of topologically massive gravity including a cosmological constant will be presented.Finally at the last section,5, we shall summarize and discuss the solutions obtained.

Chapter 2

Topologically Massive Gravity

In $2+1$ dimensional space-time Einstein's equation can be written as,[1]

$$G^i{}_k + \Lambda g^i{}_k = \kappa T^i{}_k \qquad (2.1)$$

where $G^i{}_k$ is the Einstein tensor

$$G^i{}_k = R^i{}_k - \frac{1}{2}\delta^i{}_k R \qquad (2.2)$$

$R^i{}_k$ is the Ricci tensor, R is the curvature scalar. Λ is the cosmological constant and κ is the gravitational constant with the dimension of inverse mass ($c = 1$). T^i_k is the energy momentum tensor of the matter field.

In three dimension Riemann tensor Ricci tensor and the Einstein tensor have both six components.That can be seen from the symmetries of those tensors [13] Riemann tensor can be completely written in terms of the Einstein tensor or the Ricci tensor [4] [5].Equivalently Riemann tensor can be written as the double dual of the Einstein tensor [6].

$$R^{ik}{}_{lm} = \epsilon^{ikr}\epsilon_{lmp}G^p{}_r \qquad (2.3)$$

As a result the curvature tensor is determined locally by the matter distribution and the cosmological constant. Therefore source free regions of spacetime are regions of constant curvature.As a result there are no gravitational waves in three dimensional Einstein gravity.Space is locally flat,de Sitter or anti-de Sitter depending on the value of the cosmological constant [1].In vacuum the curvature tensor is given by

$$R_{iklm} = \Lambda(g_{il}g_{km} - g_{im}g_{kl}) \tag{2.4}$$

with the trace

$$R = 6\Lambda \tag{2.5}$$

For a general discussion of the three dimensional Einstein gravity see the references [4] and [5]

As an alternative to Einstein gravity in three dimensions S.Deser,R.Jackiw and S.Templeton [2],[3] proposed topologically massive gravity,in 1982. It is mathematically attractive as it posseses properties unique to three dimensional manifolds.

In Topologically massive gravity the Einstein tensor is coupled to conformal tensor for three dimensional manifolds. It is the analog of the conformal tensor of Weyl which is not defined in three dimensions.DJT field equations for the vacuum are

$$G^i{}_k + \frac{1}{\mu}C^i{}_k = 0 \tag{2.6}$$

where $G^i{}_k$ is the Einstein tensor as defined above. $C^i{}_k$ is the conformal tensor in three dimensions and μ is the DJT coupling constant with the dimension of inverse length.

The conformal tensor $C^i{}_k$ is called the Cotton tensor,York tensor or the Bach tensor in the literature and it is the three dimensional analog of the Weyl tensor which is defined in higher dimensional spaces. It is defined as

$$C^i_{\ k} = \frac{1}{\sqrt{-g}}\epsilon^{imn}\left(R_{km} - \frac{1}{4}g_{km}R\right)_{;n} \tag{2.7}$$

It is of third order derivative with respect to the metric.It is traceless,symmetric and covariantly constant by virtue of the Bianchi identities satisfied by $G^i_{\ k}$

$$\epsilon_{kmn}\sqrt{-g}C^{mn} = G^i_{\ k;i} = 0 \tag{2.8}$$

Also it is invariant under local conformal transformations of the metric and vanishes if and only if the metric is conformally flat.For details of this subject see references [15], [7],[2].

The DJT field equations with a cosmological constant can be derived from an action which contains the Einstein-Hilbert lagrangian coupled to a Chern-Simons term with the coupling constant μ.This term corresponds to the Chern-Simons secondary characteristic class. [2]

$$\frac{1}{\kappa^2}\int(R - 2\Lambda)\sqrt{-g}d^3x + \frac{1}{\kappa^2\mu}I_{cs} \tag{2.9}$$

where I_{cs} is the Chern-Simons lagrangian

$$-\frac{1}{4}\int\epsilon^{\mu\nu\alpha}(R_{\mu\nu ik}\omega_{\alpha}^{\ ik} + \frac{2}{3}\omega_{\mu k}^{\ \ l}\omega_{\nu l}^{\ \ i}\omega_{\alpha i}^{\ \ k})d^3x \tag{2.10}$$

Here we have used the Ricci connection definition of the curvature $R_{\mu\nu ik}$ and $\omega_{\mu k}^{\ \ l}$ is the torsion-free spin connection [16].

Upon variation with respect to the metric, the usual Einstein-Hilbert term gives the Einstein tensor and the Chern-Simons term gives the conformal tensor in three dimensions. [2].Therefore the vacuum DJT field equations express a balance between the Einstein tensor and the conformal tensor. We shall not rederive the DJT field equations here since this is not related to our problem directly.One can find the details of this derivation in reference [2]. The DJT field equations for the vacuum are

$$G^i_{\ k} + \Lambda\delta^i_{\ k} + \frac{1}{\mu}C^i_{\ k} = 0 \tag{2.11}$$

with the trace

$$R = 6\Lambda \tag{2.12}$$

where Λ is the cosmological constant.

In topologically massive gravity, due to Chern-Simons coupling,gravitational waves become massive with the mass parameter μ [2].Also the gauge invariance may lead to quantisation of the mass parameter μ,depending on the topological properties of the gauge group.

$$(\nabla_i \nabla^i + \mu^2)R_{kl} = -g_{kl}R^{mn}R_{mn} + 3R^m_k R_{lm} \tag{2.13}$$

Various exact solutions of the vacuum DJT field equations are known.Hall ,Morgan and Perjes [9] have constructed Newman-Penrose formalism and discussed the algebraic structure of DJT fields.They have presented new algebraically special exact solutions.One of these solutions,reminiscent of the Brink man metric ,describes the propagation of a gravitational wave. In particular,there are finite-action solutions [10] [11] [8] [12]. Special cases of Bianchi Type VIII and Type IX solutions have been studied by Vuorio [10] Percacci et.al. [11] and Nutku and Baekler [8],in the case of vanishing cosmological constant.Therefore It will be of interest to see if they can be generalized to include a cosmological constant. A supersymmeyric generalization of topologically massive gravity including a cosmological constant has been considered by S.Deser [17].We shall consider the field equations for the bosonic sector here.

We shall present exact solutions to the field equations of Topologically massive gravity including a cosmological constant.These are Bianchi Type VIII and Type IX homogeneous anisotropic manifolds.

Three dimensional lie algebras were classified by L.Bianchi [14] around the turn of the century.The solutions,that we present,are Bianchi Type-VIII and Type-IX. A Lie algebra can be given by its basis left-invariant 1-forms. We shall denote them with

$$\sigma^i \tag{2.14}$$

We shall consider an orthonormal frame with the metric

$$ds^2 = \eta_{ik}\omega^i \otimes \omega^k \tag{2.15}$$

where the co-frame,

$$\omega^i = \lambda_i \sigma^i \tag{2.16}$$

is proportional to the left-invariant 1-forms σ^i of either Bianchi Type-VIII or Type-IX, depending on the choice of Lorentz or Euclidean signature for η_{ik} respectively. In equation (2.16) we set the scale factors λ_i as constant and there is no summation convention over the label i.

Maurer-Cartan's equations of structure are

$$d\sigma^i = \frac{1}{2} C_j{}^i{}_k \sigma^i \wedge \sigma^k \tag{2.17}$$

where $C_j{}^i{}_k$ are the structure constants of Bianchi Type-VIII or Type-IX.

Connection 1-forms $\omega^i{}_k$ curvature 2-forms $\Theta^i{}_j$ and the Riemann curvature tensor R^i_{jkl} are calculated from the Cartan's equations of structure

$$
\begin{aligned}
&d\omega^i + \omega^i{}_k \wedge \omega^k = 0 \\
&\Theta^i{}_j = d\omega^i{}_j + \omega^i{}_k \wedge \omega^k_j = \tfrac{1}{2} R^i{}_{jkl} \omega^k \wedge \omega^l
\end{aligned}
\tag{2.18}
$$

Chapter 3

Bianchi Type-IX Solution

The coframe,which is proportional to the left-invariant 1-forms of the rotation group parametrized in terms of the euler angles,is

$$\begin{aligned} \omega^1 &= \lambda_1(-\sin\psi d\theta + \cos\psi \sin\theta d\phi) \\ \omega^2 &= \lambda_2(\cos\psi d\theta + \sin\psi \sin\theta d\phi) \\ \omega^3 &= \lambda_3(d\psi + \cos\theta d\phi) \end{aligned} \tag{3.1}$$

where the structure constants in equation (2.17) are totally antisymmetric

$$C_{ijk} = \epsilon_{ijk} \tag{3.2}$$

i.e. Bianchi Type IX.

The metric,(2.15),with euclidean signature,is

$$\begin{aligned} ds^2 &= (\lambda_1^2 \sin^2\psi + \lambda_2^2 \cos^2\psi) d\theta^{\,2} \\ &+2(-\lambda_1^2 + \lambda_2^2)\sin\psi\cos\psi\sin\theta d\theta d\phi \\ &+[(\lambda_1^2\cos^2\psi + \lambda_2^2\sin^2\psi)\sin^2\theta + \lambda_3^2\cos^2\theta] d\phi^{\,2} \\ &+2\lambda_3^2\cos\theta d\phi d\psi + \lambda_3^2 d\psi^{\,2} \end{aligned} \tag{3.3}$$

With the co-frame,(3.1),the field equations (2.11) become

$$\frac{1}{4\lambda_1^3\lambda_2^3\lambda_3^3}\{\,2\lambda_1^4(2\lambda_1^2-\lambda_2^2-\lambda_3^2)-2(\lambda_2^2+\lambda_3^2)(\lambda_2^2-\lambda_3^2)^2$$
$$+\mu\lambda_1\lambda_2\lambda_3\,[\,2\lambda_1^2(2\lambda_1^2-\lambda_2^2-\lambda_3^2)-(\lambda_2^2-\lambda_3^2)^2-\lambda_1^4 \tag{3.4}$$
$$+4\lambda_1^2\lambda_2^2\lambda_3^2\Lambda\,]\,\}=0$$

$$\frac{1}{4\lambda_1^3\lambda_2^3\lambda_3^3}\{\,2\lambda_2^4(2\lambda_2^2-\lambda_1^2-\lambda_3^2)-2(\lambda_1^2+\lambda_3^2)(\lambda_1^2-\lambda_3^2)^2$$
$$+\mu\lambda_1\lambda_2\lambda_3\,[\,2\lambda_2^2(2\lambda_2^2-\lambda_1^2-\lambda_3^2)-(\lambda_1^2-\lambda_3^2)^2-\lambda_2^4 \tag{3.5}$$
$$+4\lambda_1^2\lambda_2^2\lambda_3^2\Lambda\,]\,\}=0$$

$$\frac{1}{4\lambda_1^3\lambda_2^3\lambda_3^3}\{\,2\lambda_3^4(2\lambda_3^2-\lambda_2^2-\lambda_1^2)-2(\lambda_1^2+\lambda_2^2)(\lambda_1^2-\lambda_2^2)^2$$
$$+\mu\lambda_1\lambda_2\lambda_3\,[\,2\lambda_3^2(2\lambda_3^2-\lambda_2^2-\lambda_1^2)-(\lambda_1^2-\lambda_2^2)^2-\lambda_3^4 \tag{3.6}$$
$$+4\lambda_1^2\lambda_2^2\lambda_3^2\Lambda\,]\,\}=0$$

which are the diagonal components of the field equations (2.11) The Ricci scalar (2.12) is given by

$$\begin{aligned} R &= \frac{1}{2\lambda_1^2\lambda_2^2\lambda_3^2}(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)\\ &= 6\Lambda\,. \end{aligned} \tag{3.7}$$

Solving equation (3.4) for Λ and using this in equation (3.5) one finds Λ and μ interms of λ_1 , λ_2 and λ_3.Using these Λ and μ,equation (3.6) reduces to

$$\begin{aligned} &(\lambda_1^2-\lambda_2^2)(\lambda_1^2-\lambda_3^2)(\lambda_2^2-\lambda_3^2)\\ &(\lambda_1+\lambda_2+\lambda_3)(\lambda_1+\lambda_2-\lambda_3)(\lambda_1-\lambda_2+\lambda_3)(-\lambda_1+\lambda_2+\lambda_3)=0 \end{aligned} \tag{3.8}$$

If $\Lambda=0$,equation (3.7),the Ricci scalar gives

$$\begin{array}{ll} \lambda_1+\lambda_2+\lambda_3=0 & \lambda_1+\lambda_2-\lambda_3=0 \\ \lambda_1-\lambda_2+\lambda_3=0 & -\lambda_1+\lambda_2+\lambda_3=0 \end{array} \tag{3.9}$$

This case was studied by Y.Nutku,P.Baekler [8],I.Vuorio [10] and R.Percacci, et.al [11]

For $\Lambda \neq 0$ case,consistency of the field equations,(3.8) and the Ricci scalar equation (3.7) require that any two of the directions should be equivalent.For $\lambda_2 = \lambda_3$ the metric (3.3) becomes

$$\begin{aligned} ds^2 = & (\lambda_1^2 \sin^2\psi + \lambda_2^2\cos^2\psi)d\theta^2 \\ & +2(-\lambda_1^2+\lambda_2^2)\sin\psi\cos\psi\sin\theta d\theta d\phi \\ & +[(\lambda_1^2\cos^2\psi+\lambda_2^2\sin^2\psi)\sin^2\theta+\lambda_2^2\cos^2\theta]d\phi^2 \\ & +2\lambda_2^2\cos\theta d\phi d\psi + \lambda_2^2 d\psi^2 \end{aligned} \tag{3.10}$$

The field equations (3.4),(3.5),(3.6) reduce to

$$\frac{1}{4\lambda_2^6}\{\, 4\lambda_1(\lambda_1^2-\lambda_2^2)+\mu\lambda_2^2[4(\lambda_1^2-\lambda_2^2)-\lambda_1^2+4\lambda_2^4\Lambda]\,\}=0 \tag{3.11}$$

$$\frac{1}{4\lambda_2^6}\{\, -2\lambda_1(\lambda_1^2-\lambda_2^2)+\mu\lambda_2^2[-\lambda_1^2+4\lambda_2^4\Lambda]\,\}=0 \tag{3.12}$$

and the Ricci scalar (3.7) reduces to

$$R=\frac{4\lambda_2^2-\lambda_1^2}{2\lambda_2^4}=6\Lambda\,. \tag{3.13}$$

Solving the above equations (3.11),(3.12) and (3.13) one finds

$$\lambda_1=\frac{-6\mu}{\mu^2+27\Lambda}\,,\qquad \lambda_2=\lambda_3=\frac{3}{\sqrt{\mu^2+27\Lambda}} \tag{3.14}$$

or equivalently

$$\mu = \frac{-3\lambda_1}{2\lambda_2^2}\,, \qquad \Lambda = \frac{4\lambda_2^2 - \lambda_1^2}{12\lambda_2^4} \tag{3.15}$$

Clearly any permutation of the labels $1, 2, 3$ above will also yield a solution.

Chapter 4

Bianchi Type-VIII Solution

For the lorentz signature,the coframe,which is proportional to the left-invariant 1-forms of Bianchi Type-VIII ,is

$$\begin{aligned}
\omega^0 &= \lambda_0(d\psi + \sinh\theta d\phi) \\
\omega^1 &= \lambda_1(-\sin\psi d\theta + \cos\psi \cosh\theta d\phi) \\
\omega^2 &= \lambda_2(\cos\psi d\theta + \sin\psi \cosh\theta d\phi)
\end{aligned} \tag{4.1}$$

where the structure constants of equation (2.17) are

$$C^0_{\ 12} = -1 \quad C^1_{\ 20} = 1 \quad C^2_{\ 01} = 1 \tag{4.2}$$

The metric, (2.15),with lorentz signature,is

$$\begin{aligned}
ds^2 = (\lambda_1^2 \sin^2\psi + \lambda_2^2 \cos^2\psi)d\theta^{\,2} \\
+2(-\lambda_1^2 + \lambda_2^2)\sin\psi\cos\psi\cosh\theta d\theta d\phi \\
+[(\lambda_1^2\cos^2\psi + \lambda_2^2\sin^2\psi)\cosh^2\theta - \lambda_0^2\sinh^2\theta]d\phi^{\,2} \\
-2\lambda_0^2\sinh\theta d\phi d\psi - \lambda_0^2 d\psi^{\,2}
\end{aligned} \tag{4.3}$$

With this co-frame,(4.1), the field equations (2.11) become

$$\frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3}\{-2\lambda_0^4(2\lambda_0^2-\lambda_1^2-\lambda_2^2)+2(\lambda_1^2+\lambda_2^2)(\lambda_1^2-\lambda_2^2)^2$$
$$+\mu\lambda_0\lambda_1\lambda_2[2\lambda_0^2(2\lambda_0^2-\lambda_1^2-\lambda_2^2)-(\lambda_1^2-\lambda_2^2)^2-\lambda_0^4 \tag{4.4}$$
$$-4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\}=0$$

$$\frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3}\{2\lambda_1^4(2\lambda_1^2-\lambda_0^2-\lambda_2^2)-2(\lambda_0^2+\lambda_2^2)(\lambda_0^2-\lambda_2^2)^2$$
$$+\mu\lambda_0\lambda_1\lambda_2[-2\lambda_1^2(2\lambda_1^2-\lambda_0^2-\lambda_2^2)+(\lambda_0^2-\lambda_2^2)^2+\lambda_1^4 \tag{4.5}$$
$$+4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\}=0$$

$$\frac{1}{4\lambda_0^3\lambda_1^3\lambda_2^3}\{2\lambda_2^4(2\lambda_2^2-\lambda_1^2-\lambda_0^2)-2(\lambda_0^2+\lambda_1^2)(\lambda_0^2-\lambda_1^2)^2$$
$$+\mu\lambda_0\lambda_1\lambda_2[-2\lambda_2^2(2\lambda_2^2-\lambda_1^2-\lambda_0^2)+(\lambda_0^2-\lambda_1^2)^2+\lambda_2^4 \tag{4.6}$$
$$+4\lambda_0^2\lambda_1^2\lambda_2^2\Lambda]\}=0$$

which are the diagonal components of the equation (2.11).The Ricci scalar (2.12) is given by

$$\begin{aligned} R &= \frac{-1}{2\lambda_0^2\lambda_1^2\lambda_2^2}(\lambda_0+\lambda_1+\lambda_2)(\lambda_0+\lambda_1-\lambda_2)(\lambda_0-\lambda_1+\lambda_2)(-\lambda_0+\lambda_1+\lambda_2) \\ &= 6\Lambda\,. \end{aligned} \tag{4.7}$$

Solving equation (4.4) for Λ and using this in equation (4.5) one finds Λ and μ in terms of λ_0 , λ_1 and λ_2.Using these Λ and μ equation (4.6) reduces to

$$\begin{aligned} &(\lambda_0^2-\lambda_1^2)(\lambda_0^2-\lambda_2^2)(\lambda_1^2-\lambda_2^2) \\ &(\lambda_0+\lambda_1+\lambda_2)(\lambda_0+\lambda_1-\lambda_2)(\lambda_0-\lambda_1+\lambda_2)(-\lambda_0+\lambda_1+\lambda_2)=0 \end{aligned} \tag{4.8}$$

If $\Lambda=0$,the Ricci scalar (4.7),gives

$$\begin{aligned} \lambda_0+\lambda_1+\lambda_2=0 \qquad & \lambda_0+\lambda_1-\lambda_2=0 \\ \lambda_0-\lambda_1+\lambda_2=0 \qquad & -\lambda_0+\lambda_1+\lambda_2=0 \end{aligned} \tag{4.9}$$

This case was also studied before,see the references [8],[10] and [11].

For $\Lambda \neq 0$ case,consistency of the field equations (4.8) and the Ricci scalar equation (4.7) requires that any two of the directions should be equivalent. For $\lambda_1 = \lambda_2$ the metric (4.3) reduces to

$$\begin{aligned} ds^2 = \lambda_1^2 d\theta^{\,2} + (\lambda_1^2 \cosh^2\theta - \lambda_0^2 \sinh^2\theta) d\phi^{\,2} \\ -2\lambda_0^2 \sinh\theta d\phi d\psi - \lambda_0^2 d\psi^{\,2} \end{aligned} \tag{4.10}$$

The field equations (4.4),(4.5),(4.6) reduce to

$$\frac{1}{4\lambda_1^6}\{ -4\lambda_0(\lambda_0^2-\lambda_1^2)+\mu\lambda_1^2[4(\lambda_0^2-\lambda_1^2)-\lambda_0^2-4\lambda_1^4\Lambda] \} = 0 \tag{4.11}$$

$$\frac{1}{4\lambda_1^6}\{ -2\lambda_0(\lambda_0^2-\lambda_1^2)+\mu\lambda_1^2[\lambda_0^2+4\lambda_1^4\Lambda] \} = 0 \tag{4.12}$$

and the Ricci scalar (4.7) reduces to

$$R = -\frac{4\lambda_1^2-\lambda_0^2}{2\lambda_1^4} = 6\Lambda \tag{4.13}$$

Solving the above equations (4.11),(4.12) and (4.13) one finds

$$\lambda_0 = \frac{6\mu}{\mu^2-27\Lambda} \qquad \lambda_1 = \lambda_2 = \frac{3}{\sqrt{\mu^2-27\Lambda}} \tag{4.14}$$

or equivalently

$$\mu = \frac{3\lambda_0}{2\lambda_1^2} \qquad \Lambda = \frac{\lambda_0^2 - 4\lambda_1^2}{12\lambda_1^4} \tag{4.15}$$

Similarly, any permutation of the labels $0, 1, 2$ above will also yield a solution.

Chapter 5

Conclusion

As a result we find that only those solutions in which two of the scale factors are the same admit a cosmological constant.

For euclidean signature case,we had chosen $\lambda_2 = \lambda_3$. Therefore the coframe

$$
\begin{aligned}
\omega^1 &= \lambda_1(-\sin\psi d\theta + \cos\psi \sin\theta d\phi) \\
\omega^2 &= \lambda_2(\cos\psi d\theta + \sin\psi \sin\theta d\phi) \\
\omega^3 &= \lambda_2(d\psi + \cos\theta d\phi)
\end{aligned}
\tag{5.1}
$$

which is giving the metric

$$
\begin{aligned}
ds^2 = (\lambda_1^2 \sin^2\psi + \lambda_2^2 \cos^2\psi) d\theta^{\,2} \\
+2(-\lambda_1^2 + \lambda_2^2)\sin\psi\cos\psi\sin\theta d\theta d\phi \\
+[(\lambda_1^2\cos^2\psi + \lambda_2^2\sin^2\psi)\sin^2\theta + \lambda_2^2\cos^2\theta] d\phi^{\,2} \\
+2\lambda_2^2\cos\theta d\phi d\psi + \lambda_2^2 d\psi^{\,2}
\end{aligned}
\tag{5.2}
$$

is a Bianchi Type-IX solution of the eucledianized DJT field equations. In this case the parameters are

$$
\lambda_1 = \frac{-6\mu}{\mu^2 + 27\Lambda}, \qquad \lambda_2 = \lambda_3 = \frac{3}{\sqrt{\mu^2 + 27\Lambda}} \tag{5.3}
$$

or equivalently

$$\mu = \frac{-3\lambda_1}{2\lambda_2^2}\,, \qquad \Lambda = \frac{4\lambda_2^2 - \lambda_1^2}{12\lambda_2^4} \tag{5.4}$$

The spacetime is de Sitter if $-2b < a < 2b$,anti-de Sitter if $a < -2b$ and $2b < a$ and flat if $a = \pm 2b$.

Any permutation of the scale factors λ_1,λ_2, λ_3 will also yield a solution. In the limit $\Lambda \to 0$, equation (5.3) reduce to results which were obtained by I.Vuorio [10].

$$\lambda_2 = \lambda_3 = \mp\frac{1}{2}\lambda_1 = \frac{3}{\mu}\,. \tag{5.5}$$

Similarly,in the lorentz signature case for $\lambda_1 = \lambda_2$, the coframe

$$\begin{aligned}
\omega^0 &= \lambda_0(d\psi + \sinh\theta d\phi) \\
\omega^1 &= \lambda_1(-\sin\psi d\theta + \cos\psi\cosh\theta d\phi) \\
\omega^2 &= \lambda_1(\cos\psi d\theta + \sin\psi\cosh\theta d\phi)
\end{aligned} \tag{5.6}$$

which is giving the metric

$$\begin{aligned}
ds^2 = \lambda_1^2 d\theta^{\,2} &+ (\lambda_1^2\cosh^2\theta - \lambda_0^2\sinh^2\theta)d\phi^{\,2} \\
&-2\lambda_0^2\sinh\theta d\phi d\psi - \lambda_0^2 d\psi^{\,2}
\end{aligned} \tag{5.7}$$

is a Bianchi Type-VIII solution.Here the parameters are

$$\lambda_0 = \frac{6\mu}{\mu^2 - 27\Lambda}\,, \qquad \lambda_1 = \lambda_2 = \frac{3}{\sqrt{\mu^2 - 27\Lambda}} \tag{5.8}$$

or equivalently

$$\mu = \frac{3\lambda_0}{2\lambda_1^2}\,, \qquad \Lambda = \frac{\lambda_0^2 - 4\lambda_1^2}{12\lambda_1^4} \tag{5.9}$$

In this case,the spacetime is de Sitter if $a < -2b$ and $a > 2b$, anti-de Sitter if $-2b < a < 2b$ and flat if $a = \pm 2b$.

Similarly any permutation of the labels $0, 1, 2$ above will also yield another solution.The limit $\Lambda \to 0$ in equation (5.8) was also obtained by I.Vuorio [10].

$$\lambda_1 = \lambda_2 = \pm\frac{1}{2}\lambda_0 = \frac{3}{\mu}. \tag{5.10}$$

One can get the lorentz signature field equations and the parameters μ and Λ from the euclidean ones by formal replacements

$$\begin{array}{ll} \lambda_1 \to -\lambda_0 & \Lambda \to -\Lambda \\ \lambda_2 \to \lambda_1 & \lambda_3 \to \lambda_2 \end{array} \tag{5.11}$$

In both cases,for vacuum spheroidal solutions the DJT coupling constant enters as a conformal factor into the metric and could be removed by scaling the range of the Euler angles that enter into the representation of the left-invariant 1-forms σ^i. With the introduction of the cosmological constant this is no longer the case.

From equations (5.3) and (5.8),there exists a critical value for the mass parameter μ which is determined by the cosmological constant.

$$\mu = \pm 3\sqrt{3\Lambda} \tag{5.12}$$

However in this case all the scale factors diverge to infinity.

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