ROBUST SAMPLED DATA CONTROL

A THESIS
SUBMITTED TO THE DEPARTMENT OF ELECTRICAL
ENGINEERING
AND THE INSTITUTE OF GRADUATE STUDIES
OF BILKENT UNIVERSITY
IN PARTIAL FULFILMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
OGAN OCALI
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. M. Erol Sezer(Principal Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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Contents

ABSTRACT

ROBUST SAMPLED DATA CONTROL

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M.S. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. M. Erol Sezer
June, 1990

Robust control of uncertain plants is a major area of interest in control theory. In this thesis, robust stabilization of plants under a class of structural perturbations using sampled-data controllers is considered. It is shown that a controllable system under bounded perturbations that satisfy matching conditions can be stabilized using high-gain sampled-data control, provided that the sampling period is sufficiently small. This result is then applied to robust stabilization of interconnected systems using decentralized sampled-data control, where both single-rate and multi-rate sampling schemes are considered.

Keywords: Robust Stability, Sampled-Data Control, Additive Perturbations, Interconnected Systems, Multirate Sampling, Matching Condition.

ÖZET

ÖRNEKLENMİŞ VERİ İLE GÜRBÜZ KONTROL

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Haziran, 1990

Belirsiz sistemlerin gürbüz kontrolü kontrol teorisinin geniş bir ilgi alanıdır. Bu tezde yapısal belirsizliği olan sistemlerin örneklenmiş durum geribeslemesi ile gürbüz kararlılaştırılması incelenmiştir. Sistem belirsizliği sonlu olduğu ve uyum koşulunu (matching condition) sağladığı zaman, yüksek kazançlı örneklenmiş veri geribeslemesi ile gürbüz kararlılığın sağlanabileceği gösterilmiştir. Bu sonuç, bazı tür bileşik sistemlerin ayrışık denetim probleminede genelleştirilerek, tekli ya da çoklu örnekleme hızlarında gürbüz kararlılığı sağlayan, ayrışık geribesleme yapısı elde edilmiştir.

Anahtar Sözcükler: Gürbüz Kararlılık; Örneklenmiş Veri ile Kontrol; Ayrışık Kontrol.

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Chapter 1

INTRODUCTION

In all control applications, because of some practical and theoretical reasons the plant that is to be controlled is not completely certain. For some cases, those uncertainties can be modelled as additive perturbations to a completely known *nominal* system. Due to both physical reasons and our method of modelling we may have *a priori* information about the structure and/or the bounds of these interconnections.

It is known that a large class of uncertain systems can be robust stabilized by high gain-continuous time feedback. Systems under perturbations which satisfy the so called matching conditions are included in this class [8], where the effect of perturbations can be beaten by designing the controllers to make the nominal system highly stable.

The aim of this thesis is to investigate the same problem for sampled-data control systems. Assuming that the perturbations satisfy the matching conditions, we try to answer the following questions: Can robust stability be achieved by sampled-data control? In the case of decentralized sampled-data control of interconnected systems, does multirate sampling change the nature of the problem?

The problem with sampled-data control is that the controller operates open-loop between the sampling instants. In other words, the controller is unaware of the errors that are generated by perturbations between the sampling instants, and has to wait until the next sampling instant to accommodate for those errors.

In all practical sampled-data control applications it is desirable to have large sampling periods. When the controller is a digital computer, it must be given enough time between the sampling instants for the necessary data processing. More frequent sampling requires faster and more expensive hardware. It may even be a practical impossibility to design hardware for a very fast sampled-data controller. On the other hand, as the plant to be controlled becomes more uncertain, faster correction action is needed, which necessitates more frequent sampling. If it is is possible to stabilize a plant with continuous-time feedback, one expects to find a sampling period for which a sampled-data controller exists, which achieves robust stability. An extensively general problem could be stated as follows: Given a plant uncertainty set $\mathcal{S}_u$, and a set of allowable sampled-data controllers $\mathcal{C}_a$, what is the largest possible sampling period T such that one can find a controller $C \in \mathcal{C}$ which exponentially stabilizes all systems $S \in \mathcal{S}_u$?

As we noted before, when the plant uncertainties satisfy the matching conditions, then robust stability can be achieved using high-gain continuous-time feedback. This is made possible by increasing the gain margin of the nominal system sufficiently so that it can tolerate the destabilizing effect of the perturbations. However, when a shift-invariant sampled-data controller is used, the system will have a finite gain margin that depends on the sampling period, so that one can not employ arbitrarily high gains in the feedback loop. Although it is possible to achieve arbitrarily large gain margins by using periodically varying feedback gains [4] , destabilizing effect of perturbations are also amplified preventing robust stabilization. A practical solution has been given in [5] where it was shown that robustness bounds can be improved using generalized hold functions. However, no class of perturbations for which robust stabilization can be achieved was identified. Obviously, the main difficulty is due to the fact that sampling process changes the structure of the perturbations completely.

In this thesis we provide a complete solution to the problem for the case of additive perturbations that satisfy the matching conditions. The organization of the thesis is as follows. In Chapter 2, we consider robust stabilization of a single-input system under perturbations. Using a generalized sampled-data-hold function in the feedback loop, which simulates continuous-time high-gain state feedback control in the absence of perturbations, we show that robust stability can be achieved for all sampling periods smaller than a critical value. This critical value of the sampling period is shown to depend only on the system parameters and the bound of the perturbations.

In Chapter 3, decentralized sampled-data control of interconnected systems are considered, where the interconnections are treated as perturbations on the decoupled subsystems. The results of Chapter 2 are shown to apply also to interconnected systems despite the additional decentralization constraint on the control structure. A technical difficulty due to subsystems having nonuniform dimensions is resolved by employing an artificial expansion procedure [10].

In Chapter 4, the same problem in Chapter 3 is considered with a further constraint of multirate sampling. It is shown that the structure of the decentralized controller allows for robust stabilization even though each local controller is using sampled information on subsystem states taken at different rates.

Finally, further research topics are mentioned in the conclusions.

Chapter 2

ROBUST SAMPLED-DATA CONTROL OF SINGLE-INPUT SYSTEMS

As a preparation for stabilization of interconnected systems using sampled-data control, we first investigate in this chapter, the robust stabilization problem for a single system under perturbations.

2.1 System and Controller Structure

We consider a single-input system $\mathcal{S}$ described as:

$$\mathcal{S} : \dot{x}(t) = (A + bh^{\mathrm{T}})x(t) + bu(t) \tag{2.1}$$

where $x(t) \in \mathcal{R}^n$ and $u(t) \in \mathcal{R}$ are respectively the state and the input of S, and $A \in \mathcal{R}^{n \times n}$ and $b \in \mathcal{R}^n$ are constant matrices representing the parameters of a nominal system described as

$$\mathcal{S}_N : \dot{x}(t) = Ax(t) + bu(t) \tag{2.2}$$

The additional term bh^{T} in (2.1) represents an unknown perturbation to $\mathcal{S}_N$, which satisfies the so called matching condition. For the time being we take h to be constant, that is, $h \in \mathcal{R}^n$.

We assume that the nominal system $\mathcal{S}_N$ is controllable and the pair (A, b) is in controllable canonical form

$$A = \begin{bmatrix} 0 & 1 & & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & 1 \\ a_1 & a_2 & \dots & a_n \end{bmatrix} \quad , \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.3}$$

We partition the perturbation term accordingly as

$$h^{\mathrm{T}} = \left[\begin{array}{cccc} h_1 & h_2 & \cdot\ \cdot & h_n \end{array} \right] . \tag{2.4}$$

The only information about h is that its elements are bounded.

To the system $\mathcal{S}$ of (2.1) we apply a sampled-data state control described as

$$u(t) = k^{\mathrm{T}}(t)x(mT) \; , \quad mT \leq t < (m+1)T \tag{2.5}$$

where $T \in \mathcal{R}$ is the sampling period, and $k(\cdot)$ is a periodically varying feedback gain with period T, that is,

$$k(t+T) = k(t) \;\; , \;\; t \geq 0 \; . \tag{2.6}$$

Thus the controller consists of an impulse sampler, a zero-order hold, and a periodic gain.

Our purpose is to choose the feedback gain $k(\cdot)$ so as to make the resulting closed-loop sampled-data system stable under any unknown, but bounded perturbation term h. Since the control law in (2.5) is essentially an open-loop control between sampling instants, stability of the closed-loop system will depend on both the sampling period and the bound of the perturbations. However, for a given perturbation bound, we would like to choose the gain $k(\cdot)$ to keep the closed-loop system stable for the largest possible sampling period.

It is well known [9] that the system $\mathcal{S}$ of (2.1) can be stabilized by a suitable high-gain constant state feedback of the form

$$u(t) = k^{\mathrm{T}}x(t) \tag{2.7}$$

where $k \in \mathcal{R}^n$. It is therefore, reasonable to choose the periodic gain $k(\cdot)$ of the sampled-data control of (2.5) so that it produces the same effect as the constant state feedback of (2.7) when the perturbations are absent. For this purpose, we note that the solution of the closed-loop system nominal system

$$\hat{\mathcal{S}}_N : \quad \dot{x}(t) = (A + bk^{\mathrm{T}})x(t) \tag{2.8}$$

under the control (2.7) is given by

$$x(t) = \Phi_k(t)x(0) \; , \quad t \geq 0 \tag{2.9}$$

where

$$\Phi_k(t) = e^{(A+bk^{\mathrm{T}})t} \; . \tag{2.10}$$

Thus in the absence of perturbations, the feedback control of (2.7) is equivalent to the open-loop control

$$u(t) = k^{\mathrm{T}}\Phi_k(t)x(0) \; , \quad t \geq 0 \; . \tag{2.11}$$

We now choose the periodic feedback gain $k(\cdot)$ of the sampled data control law in (2.5) as

$$k^{\mathrm{T}}(t) = k^{\mathrm{T}}\Phi_k(t) \quad , \quad 0 \leq t < T \; . \tag{2.12}$$

With the feedback gain chosen as in (2.12), the sampled-data closed-loop nominal system is described by

$$\hat{\mathcal{S}}_N : \quad \dot{x}(t) = Ax(t) + bk^{\mathrm{T}}\Phi_k(t - mT)x(mT) \; , \quad mT \leq t < (m+1)T \tag{2.13}$$

the solution of which can easily be shown to satisfy

$$x(t) = \Phi_k(t - mT)x(mT) \quad , \quad mT \leq t < (m+1)T \; . \tag{2.14}$$

This verifies that the nominal system does not differentiate the sampled-data control of (2.5) with $k(\cdot)$ as in (2.12) from the continuous-time control of (2.7).

We next investigate the behaviour of the perturbed system under sampled-data control. Using (2.5) and (2.12) in (2.1), the description of the closed-loop sampled data control system is obtained as

$$\hat{\mathcal{S}} : \quad \dot{x}(t) = (A + bh^{\mathrm{T}})x(t) + bk^{\mathrm{T}}\Phi_k(t - mT)x(mT) \; , \quad mT \leq t < (m+1)T \tag{2.15}$$

Lemma 2.1 *The solution of $\hat{\mathcal{S}}$ is given by*

$$x(t) = \Phi(t - mT)x(mT) \quad , \quad mT \le t < (m+1)T \tag{2.16}$$

where

$$\Phi(t) = \Phi_k(t) + \int_0^t \Phi_h(t-s)bh^{\mathrm{T}}\Phi_k(s)\mathrm{d}s \tag{2.17}$$

with $\Phi_k(\cdot)$ as in (2.10), and Φ_h defined as

$$\Phi_h(t) = e^{(A+bh^{\mathrm{T}})t} \quad . \tag{2.18}$$

Proof: Let

$$e(t) = x(t) - \Phi_k(t-mT)x(mT) \quad , \quad mT \le t < (m+1)T \quad . \tag{2.19}$$

Then from (2.10) and (2.15) we obtain

$$\begin{aligned} \dot{e}(t) &= (A + bh^{\mathrm{T}})x(t) + bk^{\mathrm{T}}\Phi_k(t-mT)x(mT) \\ &\quad -(A+bk^{\mathrm{T}})\Phi_k(t-mT)x(mT) \\ &= (A+bh^{\mathrm{T}})e(t) + bh^{\mathrm{T}}\Phi_k(t-mT)x(mT) \end{aligned} \tag{2.20}$$

with $e(mT) = 0$. Thus $e(t)$ is given by

$$e(t) = \int_{mT}^t \Phi_h(t-s)bh^{\mathrm{T}}\Phi_k(s-mT)x(mT)\mathrm{d}s \tag{2.21}$$

and the proof follows by substituting (2.21) into (2.19).

From Lemma 2.1, it follows that the behaviour of the closed-loop sampled data system $\hat{\mathcal{S}}$ at the sampling instants can be described by a discrete-time system

$$\hat{\mathcal{D}} : x[(m+1)T] = \Phi(T)x(mT) \tag{2.22}$$

where from (2.16) and (2.17)

$$\Phi(T) = \Phi_k(T) + \int_0^T \Phi_h(T-s)bh^{\mathrm{T}}\Phi_k(s)\mathrm{d}s \quad . \tag{2.23}$$

Since $\Phi(t)$ is continuous, and therefore bounded on the interval $(0,T)$, it follows that $\hat{\mathcal{S}}$ in (2.15) is continuous-time stable if and only if $\hat{\mathcal{D}}$ in (2.22) is discrete-time stable. The rest of this chapter is devoted to the stability analysis of $\hat{\mathcal{D}}$ through a detailed investigation of the matrix $\Phi(T)$ in (2.23).

2.2 Structure of $\Phi(T)$

Given a bounded perturbation matrix h, our purpose is to investigate the possibility of choosing the sampling period T and the gain matrix k such that $\Phi(T)$ of (2.23) is stable in discrete sense. From our experience with continuous-time systems, we know that for any given T it is possible to choose k to make the norm of $\Phi_k(T)$ arbitrarily small. Hence, when $h = 0$, $\Phi(T)$ can be made stable with arbitrary degree of stability (in discrete sense) by using high feedback gains. However, high gains in k results in impulsive terms in $\Phi_k(\cdot)$, which give rise to nonvanishing terms in the integral in (2.23). As a result, when $h \neq 0$, the norm of $\Phi(T)$ can not be made arbitrarily small by choosing a large k.

The purpose of this section is to derive an alternative expression for $\Phi(T)$, which provides more insight to its behaviour. For this purpose we define the following matrices

$$U = \begin{bmatrix} 0 & 0 & & & 0 \\ 1 & 0 & \cdot & & 0 \\ 0 & 1 & & & \cdot \\ \cdot & & & & \cdot \\ 0 & & \cdot & 1 & 0 \end{bmatrix} \quad , \quad U^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & \cdot & & 0 \\ \cdot & 0 & 1 & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & & 1 \\ 0 & & \cdot & \cdot & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} h_n & 0 & \cdot & \cdot & 0 \\ h_{n-1} & 0 & \cdot & \cdot & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ h_1 & 0 & \cdot & \cdot & 0 \end{bmatrix} \quad , \quad \bar{G} = \begin{bmatrix} h_n & 0 & & 0 \\ h_{n-1} & h_n & & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ h_1 & h_2 & \cdot & h_n \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & & & 0 \\ \cdot & \cdot & & & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & & & 0 \\ c_1 & c_2 & \cdot & \cdot & c_n \end{bmatrix} \quad , \quad \bar{C} = \begin{bmatrix} c_n & 0 & & 0 \\ c_{n-1} & c_n & & 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ c_1 & c_2 & \cdot & c_n \end{bmatrix} \tag{2.24}$$

where

$$c_i = a_i + h_i \quad , \quad i = 1, 2, \ldots, n \tag{2.25}$$

and state the following.

Lemma 2.2 *The matrices in (2.24) satisfy the following identities*

a) $\bar{G}U = U\bar{G}$, $\bar{C}U = U\bar{C}$

b) $\bar{G}\bar{C} = \bar{C}\bar{G}$

c) $UA = UU^{\mathrm{T}}$, $Ub = 0$, $UC = 0$

d) $\bar{G}(I - UU^{\mathrm{T}}) = G$, $(I - U^{\mathrm{T}}U)\bar{G} = bh^{\mathrm{T}}, (I - U^{\mathrm{T}}U)\bar{C} = C$

e) $A + bh^{\mathrm{T}} = C + U^{\mathrm{T}}$

Proof: All these identities follow directly from the definitions and from

$$\bar{G} = \sum_{i=0}^{n-1} h_{n-i}U^i \quad , \quad \bar{C} = \sum_{i=0}^{n-1} c_{n-i}U^i \tag{2.26}$$

which may be considered as defining equations for $\bar{G}$ and $\bar{C}$. The details are omitted.

We finally let

$$\hat{G} = \bar{G}U = U\bar{G} \quad , \quad \hat{C} = \bar{C}U = U\bar{C} \tag{2.27}$$

and prove the following.

Lemma 2.3 $\Phi(t)$ *in (2.17) can be expressed as*

$$\Phi(t) = \Phi_k(t) - \Phi_h(t)\hat{\Sigma} + \hat{\Sigma}\Phi_k(t) + \int_0^t \Phi_h(t-s)\Sigma\Phi_k(s)\mathrm{d}s \tag{2.28}$$

where

$$\hat{\Sigma} = \sum_{i=0}^{n-1} \hat{C}^i\hat{G} \quad , \quad \Sigma = \sum_{i=0}^{n-1} \hat{C}^i G \ . \tag{2.29}$$

Proof: From (2.17) it can easily be shown that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\Phi(t) &= (A + bh^{\mathrm{T}})\Phi(t) + bk^{\mathrm{T}}\Phi_k(t) \\ \Phi(0) &= I_n \end{aligned} \tag{2.30}$$

Let us denote the right-hand side of (2.28) by $\Psi(t)$. Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Psi(t) &= (A+bk^{\mathrm{T}})\Phi_k(t) - (A+bh^{\mathrm{T}})\Phi_h(t)\hat{\Sigma} \\
&\quad +\hat{\Sigma}(A+bk^{\mathrm{T}})\Phi_k(t) + (A+bh^{\mathrm{T}})\int_0^t \Phi_h(t-s)\Sigma\Phi_k(s)\mathrm{d}s \\
&\quad +\Sigma\Phi_k(t) \\
&= (A+bh^{\mathrm{T}})\Psi(t) + bk^{\mathrm{T}}\Phi_k(t) + \bar{\Sigma}\Phi_k(t) \qquad (2.31)
\end{aligned}$$

where

$$\bar{\Sigma} = \Sigma - (A+bh^{\mathrm{T}})\hat{\Sigma} + \hat{\Sigma}(A+bk^{\mathrm{T}}) - bh^{\mathrm{T}} \qquad (2.32)$$

We now claim that $\bar{\Sigma} = 0$. If the claim is true, then from (2.31)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}}\Psi(t) &= (A+bh^{\mathrm{T}})\Psi(t) + bk^{\mathrm{T}}\Phi_k(t) \\
\Psi(0) &= I_n \qquad (2.33)
\end{aligned}$$

and the proof follows by comparing (2.30) and (2.33). To prove the claim, let us rewrite $\hat{\Sigma}$, using the explicit expressions in (2.29) as

$$\begin{aligned}
\bar{\Sigma} &= G - (A+bh^{\mathrm{T}})\hat{G} + \hat{G}(A+bk^{\mathrm{T}}) - bh^{\mathrm{T}} \\
&\quad + \sum_{i=1}^{n-1}[\hat{C}^i G - (A+bh^{\mathrm{T}})\hat{C}^i\hat{G} + \hat{C}^i\hat{G}(A+bk^{\mathrm{T}})] \\
&= \bar{\Sigma}_0 + \sum_{i=1}^{n-1}\bar{\Sigma}_i \ . \qquad (2.34)
\end{aligned}$$

Using (2.27) and the identities in Lemma 2.2, we have

$$\begin{aligned}
\bar{\Sigma}_0 &= G - (C+U^{\mathrm{T}})U\bar{G} + \bar{G}U(A+bk^{\mathrm{T}}) - bh^{\mathrm{T}} \\
&= G + \bar{G}UU^{\mathrm{T}} - (U^{\mathrm{T}}U\bar{G} + bh^{\mathrm{T}}) - CU\bar{G} \\
&= \bar{G} - \bar{G} - CU\bar{G} \\
&= -C\hat{G} \qquad (2.35)
\end{aligned}$$

and for $i = 1, 2, \ldots, n-1$,

$$\bar{\Sigma}_i = \hat{C}^i G - (C+U^{\mathrm{T}})\hat{C}^i U\bar{G} + \hat{C}^i\bar{G}U(A+bk^{\mathrm{T}})$$

$$
\begin{aligned}
&= \hat{C}^i(G + \bar{G}UU^{\mathrm{T}}) - (C + U^{\mathrm{T}})\hat{C}^i U\bar{G} \\
&= (\hat{C}^i - U^{\mathrm{T}}\hat{C}^i U)\bar{G} - C\hat{C}^i U\bar{G} \\
&= (I - U^{\mathrm{T}}U)\bar{C}\hat{C}^{i-1}U\bar{G} - C\hat{C}^i U\bar{G} \\
&= C(\hat{C}^{i-1} - \hat{C}^i)\hat{G} \qquad (2.36)
\end{aligned}
$$

Hence, from (2.34)-(2.36)

$$
\begin{aligned}
\bar{\Sigma} &= -C\hat{G} + C\sum_{i=1}^{n-1}(\hat{C}^{i-1} - \hat{C}^i)\hat{G} \\
&= -C\hat{C}^{n-1}\hat{G} \\
&= -C\bar{C}^{n-1}U^n\bar{G} \\
&= 0\ . \qquad (2.37)
\end{aligned}
$$

This proves the claim, and completes the proof.

Using the expression in Lemma 2.3, we can write

$$
\Phi(T) = -\Phi_h(T)\hat{\Sigma}R(T) \qquad (2.38)
$$

where

$$
R(T) = (I_n + \hat{\Sigma})\Phi_k(T) + \int_0^T \Phi_h(T-s)\Sigma\Phi_k(s)\mathrm{d}s \qquad (2.39)
$$

which is to be used in the next section for the stability analysis of $\Phi(T)$.

2.3 Stability Analysis

A comparison of the expressions (2.23) and (2.38) shows that the term $-\Phi_h(T)\hat{\Sigma}$ in (2.38) is separated from the integral term in (2.23), with the remaining terms being lumped in $R(T)$. In this section, we show that stability of $\Phi(T)$ under high feedback gains depends crucially on the term $-\Phi_h(T)\hat{\Sigma}$, which is itself independent of k. This allows us to derive stabilizability conditions, which concern only the sampling period T and the perturbation matrix h .

We start with a preliminary result.

Lemma 2.4 *Let the feedback gain $k = k(\gamma)$ be chosen such that the eigenvalues of $A+bk^{\mathrm{T}}$ are placed at $\gamma\mu_i$, where μ_i are arbitrarily fixed, distinct negative real numbers, and γ is a positive parameter. Then for any $B_h > 0$, any $T > 0$ and any $\epsilon > 0$, there exists a $\gamma_0 > 0$ such that*

$$\|R(T)\| < \epsilon \tag{2.40}$$

for all $\gamma > \gamma_0$ and all h with $\|h\| \leq B_h$, where $\|\cdot\|$ denotes the spectral norm of the indicated matrix.

Proof: With k chosen as in the statement of the Lemma, the modal matrix of $A + bk^{\mathrm{T}}$ is ΓM, where

$$\Gamma = \mathrm{diag}\{1, \gamma, \cdot\cdot, \gamma^{n-1}\} \tag{2.41}$$

and

$$M = \begin{bmatrix} 1 & 1 & & 1 \\ \mu_1 & \mu_2 & & \mu_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \mu_1^{n-1} & \mu_2^{n-1} & & \mu_n^{n-1} \end{bmatrix} \tag{2.42}$$

Hence, $A + bk^{\mathrm{T}} = \Gamma M D M^{-1}\Gamma^{-1}$, and therefore

$$\Phi_k(T) = \Gamma M e^{DT} M^{-1}\Gamma^{-1} \tag{2.43}$$

where $D = \mathrm{diag}\{\gamma\mu_1, \gamma\mu_2, \cdot\cdot, \gamma\mu_n\}$. Since $\|M\|$ and $\|M^{-1}\|$ are bounded, and since $\|\Gamma\| \leq \gamma^{n-1}$, $\|\Gamma^{-1}\| \leq 1$ and $\|e^{DT}\| \leq e^{\gamma\mu T}$ for all $\gamma > 1$, where $\mu = \max\{\mu_i\}$, it follows from (2.43) that for every $T > 0$,

$$\lim_{\gamma\to\infty} \|\Phi_k(T)\| = 0 \tag{2.44}$$

independent of h.

On the other hand, with

$$I = \int_0^T \Phi_h(T-s)\Sigma\Phi_k(s)\mathrm{d}s \quad , \tag{2.45}$$

we have

$$\begin{aligned} \|I\| &= \| \int_0^T \Phi_h(T-s) \sum_{i=0}^{n-1} \hat{C}^i G \Gamma M e^{Ds} M^{-1} \Gamma^{-1} \mathrm{d}s \| \\ &\leq \int_0^T \|\Phi_h(T-s) \Sigma M e^{Ds} M^{-1} \Gamma^{-1}\| \mathrm{d}s \end{aligned} \tag{2.46}$$

where the identity $G\Gamma = G$ is used in passing from the second line to the third. Since for a fixed $T > 0$, $\|\Phi_h(T-s)\|$ is bounded for all $s \in [0,T]$ and for all $\|h\| \leq B_h$, it follows from (2.46) that

$$\|I\| \leq B_I \int_0^T e^{\gamma \mu_s} \mathrm{d}s \tag{2.47}$$

for some $B_I < \infty$. Therefore, for every $T > 0$, and any $\|h\| \leq B_h$,

$$\lim_{\gamma \to \infty} \| \int_0^T \Phi_h(T-s) \Sigma \Phi_k(s) \mathrm{d}s \| = 0 \tag{2.48}$$

The proof then follows from (2.39), (2.44) and (2.48).

Next we investigate the behaviour of the term $\Phi_h(T)\hat{\Sigma}$ in (2.38).

Lemma 2.5 *For a given $B_h > 0$, there exists a $T_0 > 0$ such that $\Phi_h(T)\hat{\Sigma}$ is stable in discrete sense (that is, it has all eigenvalues in the unit circle) for all $0 \leq T < T_0$ and all $\|h\| \leq B_h$.*

Proof: Since both $\hat{C}$ and $\hat{G}$ are triangular matrices with zero diagonals, by (2.29) $\hat{\Sigma}$ has the same structure, and therefore has all its eigenvalues at the origin. For every fixed h, since $\Phi_h(0) = I$, and the eigenvalues of $\Phi_h(T)\hat{\Sigma}$ depend continuously on T, there exists $T_h > 0$ such that maximum of modulus of eigenvalues of $\Phi_h(T)\hat{\Sigma}$ is unity. On the other hand, for every fixed T , the eigenvalues of $\Phi_h(T)\hat{\Sigma}$ are continuous functions of h. Thus T_h depends continuously on h. Letting $T_0 = \inf T_h$, where the infimum is on the bounded set $\|h\| \leq B_h$, the proof follows.

We are now ready to prove our main result on stabilizability of the closed-loop sampled-data system $\hat{\mathcal{S}}$ of (2.15).

Theorem 2.1 *For every $B_h > 0$, there exists a $T_0 > 0$ such that for any $0 < T \leq T_0$ there exists a sampled-data state-feedback controller of the form (2.5) which stabilizes $\hat{S}$ for all $\|h\| < B_h$.*

Proof: Choose T_0 as in Lemma 2.5 and fix $0 < T \leq T_0$. Then $\Phi_h(T)\hat{\Sigma}$ is stable for all $\|h\| < B_h$, so that for any fixed h there exists a positive definite matrix P_h such that

$$\hat{\Sigma}^{\mathrm{T}}\Phi_h^{\mathrm{T}}(T)P_h\Phi_h(T)\hat{\Sigma} - P_h = -I_n \ . \tag{2.49}$$

Choose $\epsilon_h > 0$ such that

$$\|P_h\|\epsilon_h^2 + 2\|P_h\|\|\Phi_h(T)\hat{\Sigma}\|\epsilon_h < 1 \tag{2.50}$$

and let $\epsilon = \inf \epsilon_h$. Since both $\|P_h\|$ and $\|\Phi_h(T)\hat{\Sigma}\|$ are contin[illegible] h, $\epsilon > 0$, and from (2.50)

$$\|P_h\|\epsilon^2 + 2\|P_h\|\|\Phi_h(T)\hat{\Sigma}\|\epsilon < 1 \tag{2.51}$$

for all $\|h\| \leq B_h$.

We now choose γ_0 as in Lemma 2.4, let $k = k(\gamma_0)$, and construct the sampled-data controller as in (2.12). Then, for any $T > 0$ and for any fixed h,

$$\begin{aligned} \Phi^{\mathrm{T}}(T)P_h\Phi(T) - P_h &= -I_n - R^T(T)P_hR(T) - 2R^T(T)P_h\Phi_h(T)\hat{\Sigma} \\ &< 0 \end{aligned} \tag{2.52}$$

where the last inequality follows from (2.40), (2.51), and (2.52). This shows that once $T < T_0$ is fixed and γ_0 and k are chosen accordingly, $\Phi(T)$ is stable for all $\|h\| \leq B_h$. The proof then follows from the fact that stability of the discrete-time system $\hat{\mathcal{D}}$ in (2.22) implies stability of the sampled-data system $\hat{\mathcal{S}}$ in (2.15).

Theorem 2.1 also provides a constructive procedure to determine the sampling period and to compute a suitable state feedback control law, which depends only on the sampling period and the bound of the perturbations, but not on the perturbations themselves. Now, the question arises: " Does a sampled-data controller designed for a fixed sampling period $T < T_0$ work also for smaller sampling periods?" Although the proofs of Lemmas 2.4 , 2.5 and Theorem 2.1 do not allow for a positive answer, it is strongly believed that it does. In the following section, this question is investigated through an example.

2.4 An Example

Consider a system $\mathcal{S}$ as in (2.1) with $n = 2$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad h = [h_1, h_2] \tag{2.53}$$

with $||h|| \leq 1$.

We fix $\mu_1 = -1$, $\mu_2 = -2$ arbitrarily, and investigate the stability of $\Phi(T)$ for various T and σ values.

Let us define the normalized logarithmic degree of stability of $\Phi(T)$ as

$$\rho = \frac{1}{T} \ln \left(\sup_{||h|| \leq 1} \max_{i=1,2} \{ | \lambda_i[\Phi(T)] | \} \right) \tag{2.54}$$

where $\lambda(\cdot)$ denotes the eigenvalues of the indicated matrix. Clearly, $\Phi(T)$ is stable if and only if $\rho < 0$.

Fig. 2.1 shows a plot of ρ vs T, in the limiting case $\gamma \to \infty$. From the plot, maximum allowable sampling period is read approximately to be 0.69. This value is very close to the value $T_0 = 0.692$, which is the maximum T for $\Phi_h(T)\hat{\Sigma}$ in Lemma 3.5 to be stable.

Fig. 2.2 shows a plot of ρ vs γ for $T = T_0 = 0.692$ from which we observe that γ values as small as $1.1 \sim 1.2$ are sufficient to achieve robust stability. It is interesting to note that a value of $\gamma = 4 \sim 5$ performs better (results in better stability degrees) than a very high value.

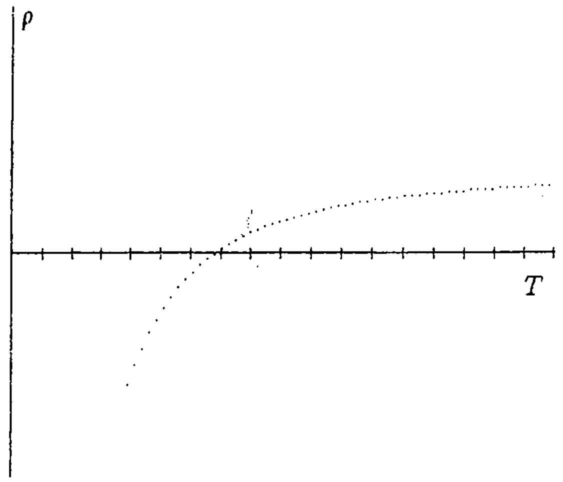


Figure 2.1. ρ vs. T for $\gamma \to \infty$

To check whether a controller corresponding to a fixed γ_0 and T_0 also works for smaller T, we plot in Fig. 2.3 ρ vs T corresponding to $\gamma = 4$. We observe that $\Phi(T)$ is stable for all $0 < T \leq 1.01$, providing a positive answer to our question.

Finally we plot in Fig. 2.4, the stability regions in the space of the disturbance parameters h_1 and h_2 for different γ and T values. We note that the largest circular region in the parameter space for $\gamma = \gamma_0 = 1.1$ and $T = T_0 = 0.69$ corresponds to $||h|| \leq 1||$, which is consistent with the above analysis.

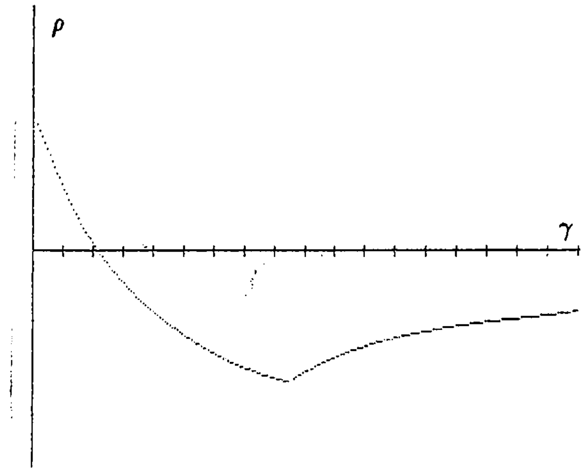


Figure 2.2. ρ vs. γ for $T = 0.692$

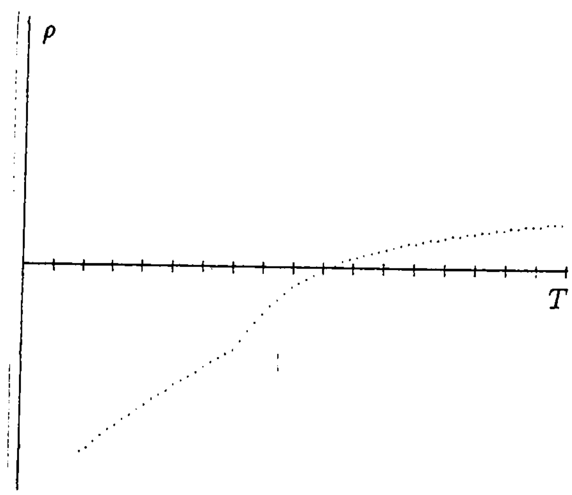
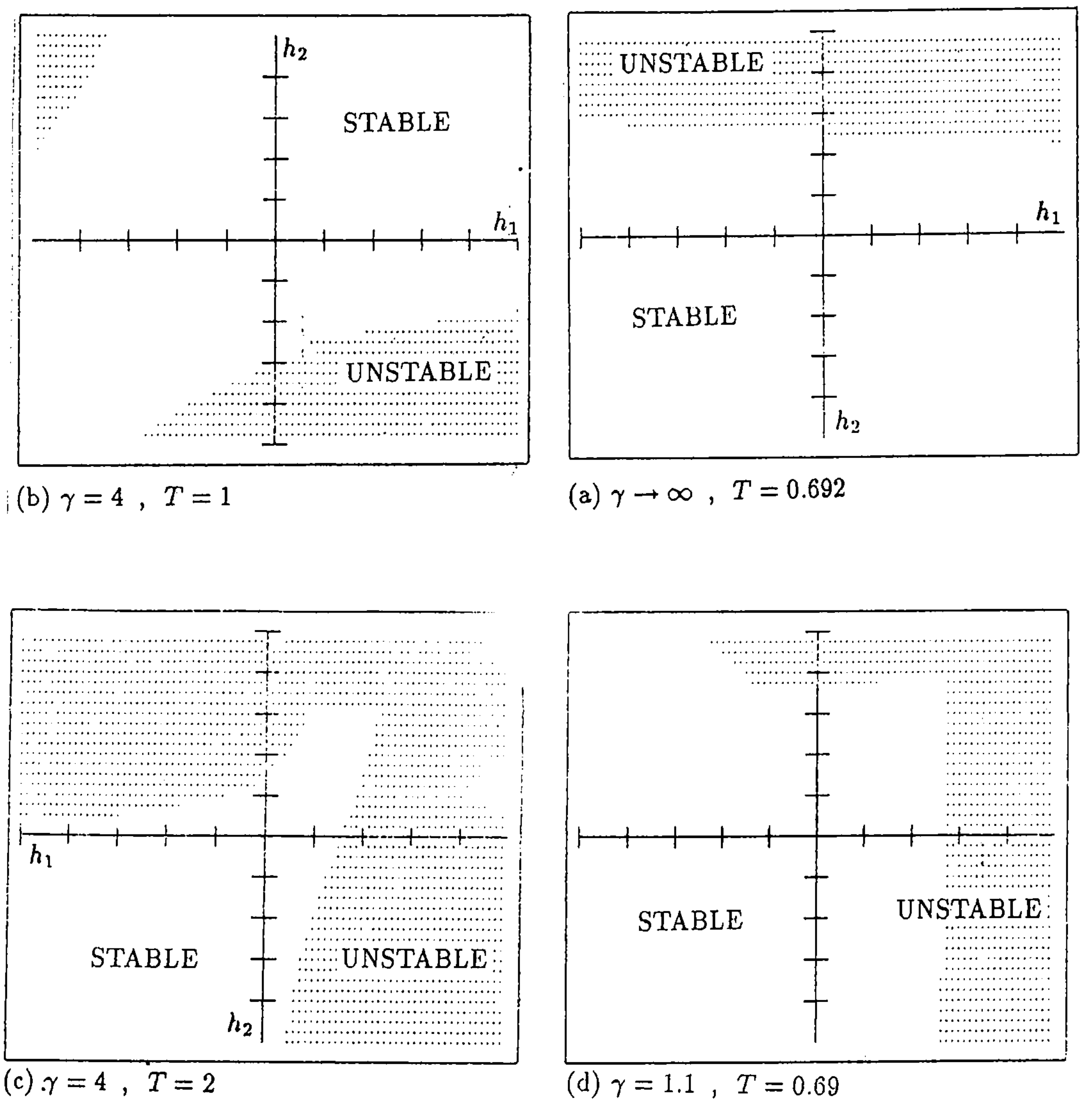


Figure 2.3. ρ vs. T for $\gamma = 4$



(b) $\gamma = 4$, $T = 1$ (a) $\gamma \to \infty$, $T = 0.692$

(c) $\gamma = 4$, $T = 2$ (d) $\gamma = 1.1$, $T = 0.69$

Fig 2.4. Stability regions in the parameter space

Chapter 3

ROBUST DECENTRALIZED SAMPLED DATA CONTROL OF INTERCONNECTED SYSTEMS

In this Chapter, we investigate using the results obtained in Chapter 2, robust stabilization of interconnected systems by decentralized single-rate sampled-data state feedback.

3.1 Interconnected Systems and Decentralized Control

We consider a large-scale system consisting of N interconnected subsystems described as

$$\mathcal{S}_i \ : \ \dot{x}_i = A_i x_i + b_i u_i + \sum_{j \in \mathcal{N}} b_i h_{ij}^{\mathrm{T}} x_j \ , \ i \in \mathcal{N} \tag{3.1}$$

where $x_i(t) \in \mathcal{R}^{n_i}$ and $u_i(t) \in \mathcal{R}$ are the state and the input of the i^{th} subsystem $\mathcal{S}_i$, and $\mathcal{N} = \{1, 2, \ldots, N\}$. The constant matrices $A_i \in \mathcal{R}^{n_i \times n_i}$ and $b_i \in \mathcal{R}^{n_i}$ define the decoupled subsystems

$$\mathcal{S}_i^{\mathrm{D}} \ : \ \dot{x}_i = A_i x_i + b_i u_i \ , \ i \in \mathcal{N} \tag{3.2}$$

and $h_{ij}^{\mathrm{T}} = [h_1^{ij}\ h_2^{ij} \ldots h_{n_i}^{ij}]$ are constant bounded but unknown interconnection parameters.

We assume that the decoupled subsystems $\mathcal{S}_i^{\mathrm{D}}$ are controllable and A_i, b_i are in controllable canonical form as in (3.2)

Letting

$$
\begin{aligned}
x &= \left[x_1^{\mathrm{T}}\ x_2^{\mathrm{T}} \ldots x_N^{\mathrm{T}}\right]^{\mathrm{T}} \\
u &= [u_1\ u_2 \ldots u_N]^{\mathrm{T}} \\
A &= \mathrm{diag}\{A_1,\ A_2, \ldots, A_N\} \\
B &= \mathrm{diag}\{b_1,\ b_2, \ldots, b_N\} \\
H &= \left[h_{ij}^{\mathrm{T}}\right]_{N,N}
\end{aligned} \tag{3.3}
$$

the interconnected system in (3.1) can be described compactly as

$$
\mathcal{S}\ :\ \ \dot{x} = (A + BH)x + Bu \tag{3.4}
$$

As can clearly be seen from the description in (3.4), the perturbation term BH due to the interconnection satisfies the matching conditions. Then generalizing the result of Chapter 2 to multi-input systems, we can assert that the system $\mathcal{S}$ can be stabilized by a sampled-data state feedback control. In the following we will show that this can be achieved by a much more restricted control, namely, by decentralized control.

Imitating the structure of the control law considered in Chapter 2, we choose the decentralized sampled-data state feedback control as

$$
u_i(t) = k_i^{\mathrm{T}}(t)x_i(mT)\ \ ,\ \ mT \le t < (m+1)T\ \ ,\ \ i \in \mathcal{N} \tag{3.5}
$$

where $k_i(t)$ are T-periodic time-varying gains, that is,

$$
k_i(t+T) = k_i(t)\ \ ,\ \ t \ge 0\ \ ,\ \ i \in \mathcal{N}. \tag{3.6}
$$

Having the result of Chapter 2, a natural choice for $k_i(t)$ would be

$$
k_i(t) = k_i^{\mathrm{T}}\Phi_{ik_i}(t)\ \ ,\ \ 0 \le t < T\ \ ,\ \ i \in \mathcal{N} \tag{3.7}
$$

where

$$\Phi_{ik_i}(t) = e^{(A_i + b_i k_i^{\mathrm{T}})t} \quad , \quad i \in \mathcal{N} \tag{3.8}$$

and the matrices k_i , $i \in \mathcal{N}$ are to be determined.

Letting

$$\begin{aligned} K &= \operatorname{diag}\{k_1^{\mathrm{T}},\ k_2^{\mathrm{T}}, \ldots, k_N^{\mathrm{T}}\} \\ K(t) &= \operatorname{diag}\{k_1^{\mathrm{T}}(t),\ k_2^{\mathrm{T}}(t), \ldots, k_N^{\mathrm{T}}(t)\} \\ \Phi_K(t) &= \operatorname{diag}\{\Phi_{1k_1}(t),\ \Phi_{2k_2}(t), \ldots, \Phi_{Nk_N}(t)\} \end{aligned} \tag{3.9}$$

we observe from (3.3) , (3.7) and (3.8) that

$$\Phi_K(t) = e^{(A+BK)t} \tag{3.10}$$

and

$$K(t) = K\Phi_K(t) \quad , \quad 0 \leq t < T \tag{3.11}$$

Thus rewriting (3.5) using (3.11) in a compact form as

$$u(t) = K\Phi_K(t - mT)x(mT) \quad , \quad mT \leq t < (m+1)T \tag{3.12}$$

interconnected closed-loop sampled-data system becomes

$$\hat{\mathcal{S}} \ : \ \dot{x}(t) = (A + BH)x(t) + BK\Phi_K(t - mT)x(mT) \quad , \quad mT \leq t < (m+1)T. \tag{3.13}$$

Noting that (3.13) is just a multi-input version of (2.15), it follows from Lemma 2.1 that the solution of $\hat{\mathcal{S}}$ is given by

$$x(t) = \Phi(t - mT)x(mT) \quad , \quad mT \leq t < (m+1)T \tag{3.14}$$

where

$$\Phi(t) = \Phi_K(t) + \int_0^t \Phi_H(t-s)BH\Phi_K(s)\mathrm{d}s \tag{3.15}$$

with $\Phi_K(\cdot)$ as in (3.10) and

$$\Phi_H(t) = e^{(A+BH)t} \ . \tag{3.16}$$

Following the same argument as in Section 2.1, we observe that the closed-loop interconnected sampled-data system $\hat{\mathcal{S}}$ of (3.13) is stable if the accompanying discrete time system $\hat{\mathcal{D}}$ of (2.22) is stable, where $\Phi(t)$ is now obtained from (3.15) as

$$\Phi(T) = \Phi_K(T) + \int_0^T \Phi_H(T-s)BH\Phi_K(s)\mathrm{d}s \tag{3.17}$$

Although $\Phi(T)$ of (3.17) is almost the same as that of (2.23), extension of Lemma 2.3 to multivariable case is far from being trivial. The main difficulty in obtaining an alternate expression for $\Phi(T)$ as in (2.28) lies in the definition of the matrices Σ and $\hat{\Sigma}$ in (2.29). In the next section we propose an expansion procedure to overcome this difficulty.

3.2 Expansion of The System

Let n be an arbitrary integer satisfying $n > n_i$, $i \in \mathcal{N}$. Consider the following subsystems

$$\tilde{\mathcal{S}}_i \ : \ \dot{\tilde{x}}_i = \tilde{A}_i\tilde{x}_i + \tilde{b}_i\tilde{u}_i + \sum_{j\in\mathcal{N}} \tilde{b}_i\tilde{h}_{ij}^{\mathrm{T}}\tilde{x}_j \ , \ i \in \mathcal{N} \tag{3.18}$$

associated with the subsystems $\mathcal{S}_i$ of (3.1), where $\tilde{x}_i \in \mathcal{R}^n$ and

$$\tilde{A}_i = \begin{bmatrix} \bar{A}_i & E_i \\ 0 & A_i \end{bmatrix} \ , \ \tilde{b}_i = \begin{bmatrix} 0 \\ b_i \end{bmatrix} \ , \ \tilde{h}_{ij}^{\mathrm{T}} = [0 \ h_{ij}^{\mathrm{T}}] \tag{3.19}$$

with $\bar{A}_i \in \mathcal{R}^{(n-n_i)\times(n-n_i)}$ and $E_i \in \mathcal{R}^{(n-n_i)\times n_i}$ being arbitrary matrices, and $\tilde{b}_i, \tilde{h}_{ij} \in \mathcal{R}^n$ are obtained by padding b_i and h_{ij} with zeros. Using the compact notation of (3.4), the overall interconnected system consisting of the subsystems of $\tilde{\mathcal{S}}_i$ of (3.18) can be described as

$$\tilde{\mathcal{S}} \ : \ \dot{\tilde{x}} = (\tilde{A} + \tilde{B}\tilde{H})\tilde{x} + \tilde{B}\tilde{u} \tag{3.20}$$

where the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{H}$ and the vectors $\tilde{x}$ and $\tilde{u}$ are defined as in (3.3).

Associated with the control law in (3.12), we apply the sampled-data control

$$\tilde{u}(t) = \tilde{K}\tilde{\Phi}_K(t-mT)\tilde{x}(mT) \ , \ mT \le t < (m+1)T \tag{3.21}$$

to $\tilde{\mathcal{S}}$ to obtain

$$\hat{\tilde{\mathcal{S}}} \ : \ \dot{\tilde{x}}(t) = (\tilde{A} + \tilde{B}\tilde{H})\tilde{x}(t) + \tilde{B}\tilde{K}\tilde{\Phi}_K(t - mT)\tilde{x}(mT) \ , \ mT \leq t < (m+1)T \tag{3.22}$$

where $\tilde{K}$ and $\tilde{\Phi}_K(t)$ are defined as in (3.9) and (3.10) with

$$\tilde{k}_i^{\mathrm{T}} = [0 \ k_i^{\mathrm{T}}]. \tag{3.23}$$

We recall [10] that the system $\hat{\tilde{\mathcal{S}}}$ of (3.21) is called an expansion of the system $\tilde{\mathcal{S}}$ of (3.13) if there exists a matrix V with full row rank such that the relation

$$x(t) = V\tilde{x}(t) \ , \ t \geq 0 \tag{3.24}$$

holds between their solutions whenever $x(0) = V\tilde{x}(0)$. We now prove the following.

Lemma 3.1 *The system $\hat{\tilde{\mathcal{S}}}$ of (3.21) is an expansion of the system $\tilde{\mathcal{S}}$ of (3.13)*

Proof: Let the matrices V_i [illegible] be defined as

$$V_i = [0 \ I_{n_i}] \ , \quad i \in \mathcal{N} \tag{3.25}$$

and let

$$V = \mathrm{diag}\{V_1, V_2, \ldots, V_N\} \tag{3.26}$$

Then obviously V is of full row rank. By construction of $\tilde{A}$, $\tilde{B}$, $\tilde{K}$ and $\tilde{H}$ we have

$$V\tilde{A} = AV \ , \ V\tilde{B} = B \ , \ \tilde{K} = KV \ , \ \tilde{H} = HV \tag{3.27}$$

so that

$$\begin{aligned} V(\tilde{A} + \tilde{B}\tilde{K}) &= (A + BK)V \\ V(\tilde{A} + \tilde{B}\tilde{H}) &= (A + BH)V \end{aligned} \tag{3.28}$$

Hence

$$\begin{aligned} V\tilde{\Phi}_K(t) &= \Phi_K(t)V \\ V\tilde{\Phi}_H(t) &= \Phi_H(t)V \end{aligned} \tag{3.29}$$

which, together with (3.15), imply that

$$V\tilde{\tilde{\Phi}}(t) = \Phi(t)V \tag{3.30}$$

where $\tilde{\tilde{\Phi}}_H(t)$ is defined as in (3.17). The proof then follows from (3.14).

Note that by the construction of V in (3.25) and (3.26) we have

$$V \; [illegible] \; I \tag{3.31}$$

and

$$\Phi(t) \; [illegible] \tag{3.32}$$

The importance of Lemma 3.1 is that using the expansion of the system we can find an alternate expression for $\Phi(t)$ as in Section 2.2. Our gain in dealing with $\tilde{\hat{\mathcal{S}}}$ rather than $\hat{\mathcal{S}}$ is that the subsystems of $\tilde{\hat{\mathcal{S}}}$ are of the same dimension n which makes a generalization of Lemma 2.3 possible.

Due to the relation (3.24) stability of $\tilde{\hat{\mathcal{S}}}$ implies the stability of $\hat{\mathcal{S}}$, but the converse is not generally true. We may have $\tilde{\hat{\mathcal{S}}}$ unstable but $\hat{\mathcal{S}}$ stable. Since we are interested in the stability of $\hat{\mathcal{S}}$ only, we use the expansion just for obtaining an alternate expression for $\Phi(t)$.

3.3 Structure of $\Phi(t)$

The aim of this section is to provide the decentralized version of the alternate expression for $\Phi(t)$ as in Lemma 3.2, which will enable us to investigate the stability properties of $\hat{\mathcal{S}}$. For this purpose we first specify the matrices $\bar{A}_i$ and E_i of in (3.19) as

$$\bar{A}_i = \begin{bmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}_{(n-n_i)\times(n-n_i)}, \quad E_i = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \end{bmatrix}_{(n-n_i)\times n_i} \tag{3.33}$$

For this specific expansion of the system, we observe that $(\ddot{A}_i, \tilde{b}_i)$ is in canonical controllable form, similar to (A_i, b_i).

We define the following matrices for the expanded system in parallel to (2.24).

$$
\begin{aligned}
U_i &= \begin{bmatrix} 0 & 0 & & 0 \\ 1 & 0 & \dots & 0 \\ & \ddots & & \\ 0 & \dots & 1 & 0 \end{bmatrix} \quad , \quad U_i^{\mathrm{T}} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & & \ddots & \\ & & & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \\
G_{ij} &= \begin{bmatrix} h_n^{ij} & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ h_1^{ij} & 0 & \dots & 0 \end{bmatrix} \quad , \quad \bar{G}_{ij} = \begin{bmatrix} h_n^{ij} & 0 & & 0 \\ \vdots & & & \\ h_1^{ij} & h_2^{ij} & & h_n^{ij} \end{bmatrix} \quad , \qquad (3.34) \\
C_{ij} &= \begin{bmatrix} 0 & 0 & & 0 \\ \vdots & & & \vdots \\ 0 & 0 & & 0 \\ \tilde{c}_1^{ij} & \tilde{c}_2^{ij} & \dots & \tilde{c}_n^{ij} \end{bmatrix} \quad , \quad \bar{C}_{ij} = \begin{bmatrix} \tilde{c}_n^{ij} & 0 & & 0 \\ \vdots & & & \\ \tilde{c}_1^{ij} & \tilde{c}_2^{ij} & \dots & \tilde{c}_n^{ij} \end{bmatrix} \\
C &= (C_{ij})_{N,N} \quad , \quad , G = (G_{ij})_{N,N} \quad , \quad U = diag\{U_i\}
\end{aligned}
$$

with

$$
\tilde{c}_l^{ij} = \begin{cases} \ddot{a}_l^{ij} + \tilde{h}_l^{ij} & i = j \\ \tilde{h}_l^{ij} & \text{else} \end{cases} \qquad (3.35)
$$

We then state the following identities which are the natural decentralized extension of Lemma 2.2.

Lemma 3.2 *The matrices in (3.34) satisfy the following identities.*

a) $\bar{G}U = U\bar{G}$, $\bar{C}U = U\bar{C}$

b) $\bar{G}\bar{C} = \bar{C}\bar{G}$

c) $U\tilde{B} = 0$, $UC = 0$, $U\tilde{A} = UU^{\mathrm{T}}$

d) $\bar{G}(I - UU^{\mathrm{T}}) = G$, $(I -$ [illegible] $U^{\mathrm{T}}U)\bar{C} = C$

e) $\tilde{A} + \tilde{B}\tilde{H} = C + U^{\mathrm{T}}$

Proof: All these identities follow from the definitions and from

$$\bar{G}_{ij} = \sum_{l=0}^{n-1} \tilde{h}^{ij}_{n-l} U_i^l \quad , \quad \bar{C}_{ij} = \sum_{l=0}^{n-1} \tilde{c}^{ij}_{n-l} U_i^l \tag{3.36}$$

We omit the details here.

Finally, letting

$$\hat{G} = \bar{G}U = U\bar{G} \quad , \quad \hat{C} = \bar{C}U = U\bar{C} \tag{3.37}$$

we prove the following fact for the expanded system $\hat{\tilde{\mathcal{S}}}$.

Lemma 3.3 *$\tilde{\Phi}(t)$ in (3.30) can be expressed as*

$$\tilde{\Phi}(t) = \tilde{\Phi}_K(t) - \tilde{\Phi}_H(t)\hat{\Sigma} + \hat{\Sigma}\tilde{\Phi}_K(t) + \int_0^t \tilde{\Phi}_H(t-s)\Sigma\tilde{\Phi}_K(s)\mathrm{d}s \tag{3.38}$$

where

$$\hat{\Sigma} = \sum_{i=0}^{n-1} \hat{C}^i\hat{G} \quad , \quad \Sigma = \sum_{i=0}^{n-1} \hat{C}^i G \tag{3.39}$$

Proof The proof follows exactly the same lines as the proof of Lemma 2.3, and is therefore omitted

Using the alternate expression in (3.37) for $\tilde{\Phi}(t)$ and (3.30), we can rewrite $\Phi(T)$ as

$$\begin{aligned} \Phi(T) &= V\tilde{\Phi}(T)V^{\mathrm{T}} \qquad (3.40) \\ &= V\left[\tilde{\Phi}_K(T) - \tilde{\Phi}_H(T)\hat{\Sigma} + \hat{\Sigma}\tilde{\Phi}_K(T) + \int_0^T \tilde{\Phi}_H(T-s)\Sigma\tilde{\Phi}_K(s)\mathrm{d}s\right]V^{\mathrm{T}} \ . \end{aligned}$$

Similar to (2.38) we decompose (3.40) as

$$\Phi(T) = -V\tilde{\Phi}_H(T)\hat{\Sigma}V^{\mathrm{T}} + R(T) \ , \tag{3.41}$$

where

$$R(T) = V(I_{nN} + \hat{\Sigma})\tilde{\Phi}_K(t)V^{\mathrm{T}} + V\left[\int_0^T \tilde{\Phi}_H(T-s)\Sigma\tilde{\Phi}_K(s)\mathrm{d}s\right]V^{\mathrm{T}} , \tag{3.42}$$

which we will use for analyzing the stability of $\hat{\mathcal{S}}$ in the next section.

3.4 Stability Analysis

In this section, similar to the case of single input systems, we will show that stability of $\Phi(T)$ depends crucially on $V\tilde{\Phi}_H(T)\hat{\Sigma}V^T$, which is itself independent of the local feedback gains. For doing this we will show that the norm of $R(T)$ in (3.41) can be made arbitrarily small by choosing high-gain local feedback controllers.

Imitating Lemma 2.4 we state the following.

Lemma 3.4 *Let the local feedback gains $k_i = k_i(\gamma)$ be chosen such that the eigenvalues of $A_i + b_i k_i^T$ are placed at $\gamma\mu_i^j$, where μ_i^j are arbitrarily fixed, distinct negative real numbers, and γ is a positive gain parameter. Then, for any $B_h > 0$, any $T > 0$, and any $\epsilon > 0$, there exists a $\gamma_0 > 0$ such that*

$$\|R(T)\| < \epsilon \tag{3.43}$$

for all $\gamma \geq \gamma_0$, where $\|\cdot\|$ denotes the spectral norm.

Proof: From (3.23) and (3.30) we have

$$\tilde{\Phi}_{ik_i}(t) = \begin{bmatrix} e^{\bar{A}_i t} & \int_0^t e^{\bar{A}_i(t-s)} E_i \Phi_{ik_i}(s) ds \\ 0 & \Phi_{ik_i}(t) \end{bmatrix} , \tag{3.44}$$

where for any fixed $T > 0$

$$\lim_{\gamma\to\infty} \|\Phi_{ik_i}(T)\| = 0 \ , \ i \in \mathcal{N} \tag{3.45}$$

as has already been shown in the proof of Lemma 4. On the other hand, letting

$$I_i(T) = \int_0^T e^{\bar{A}_i(T-s)} E_i \Phi_{ik_i}(s) ds \tag{3.46}$$

substituting

$$\Phi_{ik_i}(s) = \Gamma_i M_i e^{D_i s} \Gamma_i^{-1} M_i^{-1} \tag{3.47}$$

where M_i , Γ_i and D_i are similar to M , Γ and D defined in (2.41) - (2.43), and using the identity $E_i\Gamma_i = E_i$, we have

$$\|I_i(T)\| \leq \|E_iM_i\|\|M_i^{-1}\Gamma_i^{-1}\|\| \int_0^T \|e^{\tilde{A}_i(T-s)}\|e^{D_is}\mathrm{d}s \ . \tag{3.48}$$

Since for a fixed $T > 0$, $\|e^{\tilde{A}_i(T-s)}\|$ is bounded for all $s \in [0 \ , \ T]$, it follows from (3.50) that

$$\|I_i(T)\| \leq B_{iI}\int_0^T e^{\gamma\mu_is}\mathrm{d}s \tag{3.49}$$

for some finite B_{iI}, where $\mu_i = \max\{\mu_i^j\}$. Therefore, for every $T > 0$ and any $\|H\| \leq B_h$

$$\lim_{\gamma\to\infty} \| \int_0^T e^{\tilde{A}_i(T-s)}E_i\Phi_{ik_i}(s)\mathrm{d}s\| = 0 \ . \tag{3.50}$$

Then (3.25), (3.44) and (3.50) imply that

$$\lim_{\gamma\to\infty} \tilde{\Phi}_{ik_i}(T)V_i^{\mathrm{T}} = \lim_{\gamma\to\infty} \begin{bmatrix} \int_0^T e^{\tilde{A}_i(T-s)}E_i\Phi_{ik_i}(s)\mathrm{d}s \\ \Phi_{ik_i}(T) \end{bmatrix} = 0 \quad , \quad i \in \mathcal{N} \ . \tag{3.51}$$

Hence, for all H satisfying $\|H\| \leq B_h$ and for all $T > 0$, we have

$$\lim_{\gamma\to\infty} V(I + \hat{\Sigma})\tilde{\Phi}_K(T)V^{\mathrm{T}} = 0 \tag{3.52}$$

To bound the integral term in (3.42), we first note that

$$\tilde{\Phi}_K(s)V^{\mathrm{T}} = \mathrm{diag}\left\{\begin{bmatrix} I_i(s) \\ \Phi_{ik_i}(s) \end{bmatrix}\right\} \ , \tag{3.53}$$

where $I_i(s)$ is defined in (3.46). Using the expression for Σ given in (3.39), we have

$$\begin{aligned} \Sigma\tilde{\Phi}_K(s)V^{\mathrm{T}} &= \sum_{j=0}^{n-1} \hat{C}^jG\mathrm{diag}\left\{\begin{bmatrix} I_i(s) \\ \Phi_{ik_i}(s) \end{bmatrix}\right\} \\ &= \sum_{j=0}^{n-1} \hat{C}^jG\mathrm{diag}\left\{\begin{bmatrix} I_i(s) \\ 0 \end{bmatrix}\right\} \end{aligned} \tag{3.54}$$

where the second equality follows from the definition of $G = (G_{ij})_{N,N}$ in (3.35). Therefore

$$V\int_0^T \tilde{\Phi}_H(T-s)\Sigma\tilde{\Phi}_K(s)\mathrm{d}sV^{\mathrm{T}} = V\int_0^T \tilde{\Phi}_H(T-s)\Sigma\mathrm{diag}\left\{\left[\begin{array}{c} I_i(s) \\ 0 \end{array}\right]\right\}\mathrm{d}s \tag{3.55}$$

so that by (3.49) and boundedness of $\tilde{\Phi}_H(T-s)$ on $[0,T]$, we have

$$\lim_{\gamma\to\infty} V\int_0^T \tilde{\Phi}_H(T-s)\Sigma\tilde{\Phi}_K(s)\mathrm{d}sV^{\mathrm{T}} = 0 \tag{3.56}$$

for all $T > 0$ and all H with $\|H\| \le B_h$. This completes the proof.

Lemma 3.5 *For a given $B_h > 0$ there corresponds a $T_0 > 0$ such that for all $0 \le T < T_0$ and all $\|H\| \le B_h$, $V\tilde{\Phi}_H(T)\hat{\Sigma}V^{\mathrm{T}}$ is stable in the discrete sense.*

Proof Since every block entry of $\hat{C}$ and $\hat{G}$ are triangular matrices with zero diagonal, $\hat{\Sigma}$ has the same structure, that is, each block $\hat{\Sigma}_{ij}$ of $\hat{\Sigma} = (\hat{\Sigma}_{ij})_{N,N}$ is a triangular matrix with zero diagonal. This implies that all eigenvalues of the matrix $V\tilde{\Phi}_H(0)\hat{\Sigma}V^{\mathrm{T}} = V\hat{\Sigma}V^{\mathrm{T}} = (V_i\hat{\Sigma}_{ij}V_j^{\mathrm{T}})_{N,N}$ are at the origin.

The rest of the proof is the same as the proof of Lemma 2.5.

Now we can state a decentralized version of Theorem 2.1, which is the main result of this chapter.

Theorem 3.1 *For every $B_h \ge 0$ there exists a $T_0 > 0$ such that for any $0 < T \le T_0$ there exists a decentralized sampled-data state feedback controller of the form (3.12), which stabilizes $\hat{S}$ for all $\|H\| < B_h$.*

Proof: Having Lemmas 3.4 and 3.5 in place of Lemmas 2.4 and 2.5 respectively, proof of this theorem is exactly the same as the proof of Theorem 2.1.

Similar to the centralized case, Theorem 3.1 provides a constructive procedure to determine the sampling period and the local controllers in terms of the bound of the perturbations, but not of the perturbations themselves.

Chapter 4

MULTIRATE ROBUST SAMPLED-DATA CONTROL

In this chapter we generalize the results of Chapter 3 to the case where local controllers use values of states sampled at different rates.

4.1 Multirate Decentralized Control

Consider the interconnected system described by (3.1), or in equivalent compact notation by (3.4), which we repeat below for easy reference

$$\mathcal{S}_i \qquad \dot{x}_i(t) = A_i x_i(t) + b_i u_i(t) + \sum_{j\in\mathcal{N}} b_i h_{ij}^{\mathrm{T}} x_j \tag{4.1}$$

$$\mathcal{S} \qquad \dot{x}(t) = (A + BH)x(t) + Bu(t) \ . \tag{4.2}$$

We choose the multirate decentralized control law as

$$u_i(t) = k_i^{\mathrm{T}}(t) x_i(mT_i) \ , \quad mT_i \le t < (m+1)T_i \ , \quad i \in \mathcal{N} \tag{4.3}$$

where

$$T_i = M_i \tau \quad , \quad i \in \mathcal{N} \ , \tag{4.4}$$

is the sampling period of the i^{th} subsystem, with M_i being a positive integer. The time-varying gain $k_i(t)$ in (4.3) is periodic with period T_i, and is chosen as

$$k_i^{\mathrm{T}}(t) = k_i^{\mathrm{T}}\Phi_{ik_i}(t) \quad , \quad 0 \le t < T_i \quad , \quad i \in \mathcal{N} \tag{4.5}$$

where $\Phi_{ik_i}(t)$ is defined in (3.8).

Let

$$M = \mathrm{l.c.m}(M_i) \quad , \tag{4.6}$$

and

$$T = M\tau \quad . \tag{4.7}$$

We define τ as the basic time unit, and T as the common sampling period. Note that each T_i is a an integer multiple of τ, and T is an integer multiple of every T_i.

To describe the evolution of the state of the closed loop system consisting of $\hat{\mathcal{S}}$ in (3.17) and the feedback law in (4.3), consider an initial time instant $t_0 = mT + l\tau$, $l = 0, 1, \ldots M-1$; $m = 0, 1, 2, \ldots$. Then for each $i \in \mathcal{N}$, there exists unique integers $0 \le m_i \le (M/M_i - 1)$ and $0 \le r_i \le M_i - 1$ such that

$$l\tau = m_i T_i + r_i\tau \quad , \quad i \in \mathcal{N} \quad . \tag{4.8}$$

By (4.3), (4.5) and T_i periodicity of $k_i(t)$ in the interval $t_0 \le t < t_0 + \tau$, the input to the i^{th} subsystem is given by

$$u_i(t) = k_i^{\mathrm{T}}\Phi_{ik_i}(t - mT - m_iT_i)x_i(mT + m_iT_i) \quad . \tag{4.9}$$

Substituting (4.9) into (4.1), we obtain

$$\begin{aligned} \hat{\mathcal{S}} \ : \ \dot{x}(t) = (A + BH)x(t) \quad &+ \sum_{i \in \mathcal{N}} B_i k_i^{\mathrm{T}}\Phi_{ik_i}(t - mT - m_iT_i)x_i(mT + m_iT_i) \, , \\ & mT + l\tau \le t < mT + (l+1)\tau \quad , \end{aligned} \tag{4.10}$$

where B_i are the columns of B in (3.3). The solution of (4.10) at $t = t_0 + \tau = mT + (l+1)\tau$ is obtained as

$$\begin{aligned} x[mT + (l+1)\tau] \ &= \ \Phi_H(\tau)x(mT + l\tau) \\ &+ \ \sum_{i \in \mathcal{N}} \left[\int_0^{\tau} \Phi_H(\tau - s)B_i k_i^{\mathrm{T}}\Phi_{ik_i}(s + r_i\tau)\mathrm{d}s \right] x_i(mT + m_iT_i) \quad . \end{aligned} \tag{4.11}$$

Defining

$$J_i = \text{diag}\{0, \ldots, 0, I_{n_i}, 0, \ldots, 0\} \ , \quad i \in \mathcal{N} \tag{4.12}$$

and noting that

$$B_i k_i^{\mathrm{T}} \Phi_{ik_i}(t_1) x_i(t_2) = BK\Phi_K(t_1) J_i x(t_2) \ , \quad t_1, t_2 \geq 0 \tag{4.13}$$

where Φ_K is defined in (3.10), (4.11) can be written as

$$x[mT + (l+1)\tau] = \Phi_H(\tau) x(mT + l\tau) + \sum_{i \in \mathcal{N}} I(r_i \tau) J_i x(mT + m_i T_i) \tag{4.14}$$

where

$$I(\lambda) = \int_0^\tau \Phi_H(\tau - s) BK\Phi_K(s + \lambda) \mathrm{d}s \ , \quad \lambda \geq 0 \ . \tag{4.15}$$

To put (4.14) in a more convenient form, let us define for each $0 \leq l \leq M-1$, the index set

$$\mathcal{N}_l = \{i \in \mathcal{N} : \ l\tau = m_i T_i \ \text{for some integer } m_i\} \ . \tag{4.16}$$

In other words, $\mathcal{N}_l$ is the index set of those subsystems whose samplers are closed at time $mT + l\tau$. From (4.8), it follows that for $i \in \mathcal{N}_l$, $r_i = 0$, that is $m_i T_i = l\tau$. we now rewrite (4.14) as

$$\begin{aligned} x[mT + (l+1)\tau] &= [\Phi_H(\tau) + \sum_{i \in \mathcal{N}_l} I(0) J_i] x(mT + m_i T_i) \\ &\quad + \sum_{i \in \mathcal{N} - \mathcal{N}_l} I(r_i \tau) J_i x(mT + m_i T_i), \ l = 0, 1, \ldots, M-1. \end{aligned} \tag{4.17}$$

Using (4.17) recursively for $l = 0, 1, \ldots, M-1$, we obtain

$$\hat{\mathcal{D}} \ : \quad x[(m+1)T] = \Phi(T) x(mT) \ , \tag{4.18}$$

where

$$\begin{aligned} \Phi(T) &= \Phi_{M-1}(\tau) \cdots \Phi_1(\tau) + \Psi(\tau) \ , \\ \Phi_l(\tau) &= \Phi_H(\tau) + \sum_{i \in \mathcal{N}_l} I(0) J_i \quad , \quad l = 0, 1, \ldots, M-1 \end{aligned} \tag{4.19}$$

and $\Psi(\tau)$ is a sum of product terms each of which contains at least one $I(r\tau)$ term with $1 \leq r \leq \max(M_i) - 1$.

(4.18) represents a shift-invariant discrete-time system $\hat{\mathcal{D}}$, which describes the transitions of the closed-loop sampled-data system over the common sampling period T. In the next section we investigate the stability properties of $\hat{\mathcal{D}}$.

4.2 Stability Analysis

The decomposition in (4.19) is the key to stability analysis of $\Phi(T)$. We start with an investigation of the $I(r\tau)$ terms in $\Psi(\tau)$.

Lemma 4.1 *Let $k_i = k_i(\gamma)$ be chosen as in Lemma 3.4. Then, for every fixed $\tau > 0$ and $\epsilon > 0$ there exists a γ_0 such that for every $\gamma \geq \gamma_0$, $\|k_i^{\mathrm{T}}\Phi_{ik_i}(t)\| < \epsilon$ for all $t \geq \tau$.*

Proof: From (2.24),

$$k_i^{\mathrm{T}}\Phi_{ik_i}(t) = k_i^{\mathrm{T}}\Gamma_i M_i e^{D_i t} M_i^{-1}\Gamma_i^{-1} \ . \tag{4.20}$$

Noting that

$$\|k_i^{\mathrm{T}}\Gamma_i\| \leq \mathcal{K}_i\gamma^{n_i} \ , \tag{4.21}$$

for some finite $\mathcal{K}_i$, which is independent of γ, we have

$$\|k_i^{\mathrm{T}}\Phi_{ik_i}(t)\| \leq \mathcal{K}_i\|M_i\|\|M_i^{-1}\|\gamma^{n_i}\exp(\mu_i\gamma t) \ , \tag{4.22}$$

where $\mu_i = \max\{\mu_i^j\}$ and the proof follows.

Lemma 4.2 *Let $k_i = k_i(\gamma)$ be chosen as in Lemma 3.4. Then, for all $\tau > 0$, $B_h > 0$ and $\epsilon > 0$ there exists a $\gamma_0 > 0$ such that*

$$\|I(r\tau)\| \leq \epsilon \tag{4.23}$$

for all $r \geq 1$, for all $\gamma \geq \gamma_0$ and for all H with $\|H\| \leq B_h$.

Proof: From (4.15)

$$\|I(r\tau)\| \int_0^\tau \|\Phi_H(\tau - s)\|\|B\|\|K\Phi_K(s + r\tau)\|\mathrm{d}s \ , \tag{4.24}$$

the proof follows from Lemma 4.1 , and the boundedness of $\Phi_H(\tau - s)$ for $s \in (0, \tau)$.

An immediate consequence of Lemma 4.2 is that

$$\lim_{\gamma\to\infty} \Psi(\tau) = 0 \tag{4.25}$$

or equivalently,

$$\lim_{\gamma\to\infty} \Phi(\tau) = \prod_{l=0}^{M-1} \lim_{\gamma\to\infty} \Phi_l(\tau) \tag{4.26}$$

for all $\tau > 0$ and all bounded H.

Before proceeding any further, we would like to point out that $\Phi_l(\tau)$represents an approximation to the state transition matrix on the interval $[mT+l\tau, mT+(l+1)\tau]$. This is reasonable, because under high-gain feedback the control applied to those channels whose samplers are inactive at $t_0 = mT + l\tau$ are effectively zero, and only the channels indicated by the index set $\mathcal{N}_l$ contain significant control signals. Note that in the case of identical sampling in all subsystems we have $M = 1$, $T = \tau$, $\mathcal{N}_0 = \mathcal{N}$ so that $\Psi(\tau) = 0$ and (4.19) reduces to

$$\Phi(T) = \Phi_0(\tau) = \Phi_H(\tau) + I(0) \ , \tag{4.27}$$

which is the same as (3.17) as expected.

We now turn our attention to an investigation of the individual $\Phi_l(\tau)$ matrices in (4.20). Letting

$$S_l = \sum_{i\in\mathcal{N}_l} J_i \ , \tag{4.28}$$

we write

$$\begin{aligned} \Phi_l(\tau) &= \Phi_H(\tau) + I(0)S_l \\ &= [\Phi_H(\tau) + I(0)]S_l + \Phi_H(\tau)(I - S_l) \ . \end{aligned} \tag{4.29}$$

Now using the expansion procedure in Chapter 3, we obtain

$$\Phi_H(\tau) + I(0) = -V\tilde{\Phi}_H(\tau)\hat{\Sigma}V^{\mathrm{T}} + R(\tau) \ , \tag{4.30}$$

where $V, \hat{\Sigma}, \tilde{\Phi}_H$ and $R(\tau)$ are defined in (3.26), (3.39) ,(2.18) and (3.42). Substituting (4.30) into (4.29) and rearranging the terms we get

$$\begin{aligned} \Phi_l(\tau) &= [-V\tilde{\Phi}_H(\tau)\hat{\Sigma}V^{\mathrm{T}} + R(\tau)]S_l + V\tilde{\Phi}_H(\tau)V^{\mathrm{T}}(I - S_l) \\ &= V\tilde{\Phi}_H(\tau)(\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l)V^{\mathrm{T}} + R(\tau)S_l \end{aligned} \tag{4.31}$$

where $\tilde{S}_l$ has the same structure as S_l except that its diagonal blocks have sizes $n \times n$ rather than $n_i \times n_i$. From the analysis in Chapter 2, we know that

$$\lim_{\gamma \to \infty} R(\tau) = 0 \tag{4.32}$$

for every fixed $\tau > 0$ and all H with $\|H\| \leq B_h$. Using this property, we write from (4.31)

$$\lim_{\gamma \to \infty} \Phi_l(\tau) = V \tilde{\Phi}_l(\tau) V^{\mathrm{T}} \ , \tag{4.33}$$

where

$$\tilde{\Phi}_l(\tau) = \tilde{\Phi}_H(\tau)(\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l) \ . \tag{4.34}$$

Hence from (4.26), we have

$$\lim_{\gamma \to \infty} \Phi(T) = V \left[\prod_{l=0}^{M-1} \tilde{\Phi}_H(\tau)(\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l) \right] V^{\mathrm{T}} \ , \tag{4.35}$$

which has the following stability property.

Lemma 4.3 *To every $B_h > 0$ there corresponds a $T_0 > 0$ such that the matrix in (4.35) is stable in discrete sense for all $0 \leq T < T_0$ and for all H with $\|H\| \leq B_h$.*

Proof: Consider

$$[\lim_{\gamma \to \infty} \Phi(T)]_{T=0} = V \left[\prod_{l=0}^{M-1} (\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l) \right] V^{\mathrm{T}} \tag{4.36}$$

where each term $\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l$ in the product is a block matrix with lower triangular blocks. In particular, for $l = 0$, $\tilde{S}_0 = \tilde{I}$ so that

$$\tilde{I} - \tilde{S}_l - \hat{\Sigma}\tilde{S}_l = -\hat{\Sigma} \ , \tag{4.37}$$

whose blocks have zero diagonals. This implies that the whole product in (4.36) has the same structure. The proof then follows the same lines as the proof of Lemma 3.5.

We are now ready to prove our main result on stabilizability of the system $\mathcal{S}$ in (4.1).

Theorem 4.1 *For every $B_h \geq 0$ there exists a $T_0 > 0$ such that for any $0 < T \leq T_0$ there exists a decentralized multirate sampled-data state feedback controller of the form (4.3) which stabilizes $\mathcal{S}$ for all $\|H\| \leq B_h$.*

Proof: Fix T_0 as in Lemma 4.3, so that $\lim_{\gamma \to \infty} \Phi(T)$ in (4.35) is stable for all $0 < T \leq T_0$ and for all H with $\|H\| \leq B_h$. Then choosing γ sufficiently large so that $\Psi(\tau)$ in (4.19) and $R(\tau)$ in (4.31) are small enough not to destroy the stability of $\Phi(T)$, the proof follows.

Chapter 5

CONCLUSIONS

In this thesis we considered robust sampled data stabilizability of

- single-input, linear, time-invariant systems under perturbations that satisfy matching conditions, using state feedback;
- interconnected systems composed of smaller subsystems with uncertain interconnections that satisfy the matching conditions, using decentralized state feedback; and
- interconnected systems using decentralized multirate sampling.

We have shown that robust stabilization can be achieved in all three cases for all sampling periods smaller than a critical value T_0.

The main result of Chapter 2 guarantees the existence of a stabilizing sampled-data controller for every fixed sampling period smaller than T_0. Whether a controller designed for T_0 works for all smaller sampling periods remains to be an open problem.

Another problem that is worth to investigate is to extend the results obtained in this thesis to more general perturbation structures. It is strongly believed that all perturbation structures which allow for robust stabilization using continuous-time state feedback can also be handled by sampled-data state feedback.

Finally, design of decentralized controllers based on single or multirate sampled outputs rather than states is a closely related research topic.

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