PRICE DEPENDENT PROCUREMENT DECISIONS IN ONE-PERIOD INVENTORY PROBLEM

A THESIS

SUBMITTED TO THE DEPARTMENT OF INDUSTRIAL ENGINEERING

AND THE INSTITUTE OF ENGINEERING AND SCIENCES

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Hakan Polatoğlu

June, 1989

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

PRICE DEPENDENT PROCUREMENT DECISIONS IN ONE-PERIOD INVENTORY PROBLEM

Hakan Polatoğlu
M.S. in Industrial Engineering
Supervisor: Prof. İzzet Şahin
June, 1989

In this work the classical newsboy model is extended by introducing a price dependent demand pattern. It is intended to obtain optimal procurement and pricing decisions for maximizing the expected profit. It is shown that, both decisions can be made simultaneously if we are able to identify the effects of price on the demand process.

ÖZET

TEK DÖNEMLİ ENVANTER PROBLEMİNDE FİYATA BAGLI ENİYİ TEDARİK MİKTARI

Hakan Polatoğlu
Endüstri Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. İzzet Şahin
Haziran, 1989

Tek dönemli klasik envanter probleminde beklenen eniyi kârı veren tedarik miktarı bulunurken malın satış fiyatı sabit olarak alınmaktadır. Ancak, ideal olmayan pazar şartlarında istem ile fiyat arasındaki ilişki göz önüne alındığında stok miktarının fiyattan etkilenebileceği düşünülmektedir. Bu çalışmada, en büyük kâr değerini veren stok ve fiyat kararları yukarıdaki fikirden hareket ederek eşzamanlı olarak bulunmuştur. Çözülen sayısal örnekler ve kuramsal sonuçlar göstermektedir ki, stok miktarına ek olarak verilecek fiyat kararı probleme yeni bir yönetsel boyut getirebilmektedir.

To my Mother,

ACKNOWLEDGEMENT

The author welcomes this opportunity to express his gratitude to Professor İzzet Şahin for his supervision of this thesis and his continual interest. He is also indebted to the members of his thesis committee: Associate Professor Nesim Erkip, Assistant Professor Cemal Dinçer, and Assistant Professor Levent Onur for their advice and support.

A special note of thanks is due to Associate Professor Ömer Benli, on behalf of the Bilkent University Industrial Engineering Department for providing an excellent research environment for the entire study.

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Chapter 1

INTRODUCTION AND LITERATURE REVIEW

The literature on inventory systems abounds with studies that address the question of how much inventory to hold and how. In almost all of these studies the product selling price is taken to be an exogeneous variable. There is a need to incorporate the pricing issue into the decision problem, however, to achieve an increased level of economic soundness in modelling.

The price-demand relation is one of the fundamental concepts of the *theory of the firm* in the neoclassical economic theory. Traditionally, this relation was studied under economic equilibrium where the demand becomes deterministic once the price is given. Later (see [2,3,5,6,7,9]), uncertainty was introduced into the problem where demand to be realized at any price level was taken to be random.

A number of researchers have been interested in the pricing concept as it relates to inventory theory (e.g., [3,10,12,13,14,15,16,17,18,19,20,21,22,23,24,25]). Few among those, however, were interested in the price-demand interaction. Most of them studied the effects of various procurement or selling price adjustment scenarios on the EOQ formulation. Since the amount of inventory to be held is strongly dependent on demand distribution, which in turn is affected by the product price, there is a need to determine the best procurement quantity and pricing decisions simultaneously.

To discuss the price-demand relationship, the structure of the market should be identified first. In perfect competition the market price is determined by various equilibria present in the economic system; the atomistic firm is a price taker. For such a firm it remains to make output decisions in view of expected rises and falls in the market price and demand. Hence, given the optimal output decision, the price will be determined by market equilibrium which is assumed to be reached instantaneously. On the other hand, in imperfect competition the firm can set the product price. However, the demand to be realized at a given price level is uncertain. Naturally, the firms should undertake a forecasting effort in order to have an idea about the characteristic price-demand relationship of the market.

In his pioneering work, Mills [3] studied short-term output and pricing decisions of a firm producing a single product and facing imperfect competition. For maximizing the expected profit objective, he showed that, in the case of demand uncertainty the optimal strategy differs from the deterministic case. Baron [6] elaborated on the same subject releasing more structural results. Sandmo [7] and Baron [5] studied the same problem by introducing the idea of risk. In their models the utility (cf. Von Neumann and Morgenstern [1]) of the expected profit is maximized. They showed that pricing and output decisions are affected by the risk attitude of the firm. Leland [9] contributed to the subject by generalizing the problem for three cases, namely, the quantity-setting firm, the price-setting firm, and the price- and quantity-setting firm. For the risk-neutral case the results of the above studies have implications on the pricing issues related to the one-period inventory problem.

In a recent paper, Lau & Lau [22] attacked the pricing issues related to the one-period inventory problem. They showed how the pricing decision can be studied as a new problem in addition to the procurement decision. For maximizing the expected profit, they indicated that there exists no analytical solution for the optimal price value for a general demand distribution. Instead, they proposed a numerical solution procedure. Their results will be referred to in the sequel.

Abad [25] studied the joint price and lot-size determination problem for a multi-period inventory system facing deterministic demand. In his work, demand is expressed as a function of price. He solves the tradeoff between low price high demand and high price low demand under certainty. We shall study this deterministic situation for one-period problem in detail in section 3.4.

Gerchak and Parlar [20] approached to the one-period problem from a different view point. They considered a random potential market size. The firm captures a certain percentage of this allowable potential due to its sales effort. Therefore, under a constant market price there exists a tradeoff between high sales effort high demand and low sales effort low demand. Although, in this problem they have no pricing decision the main idea is to alter the demand distribution by a decision variable, which is similar to our basic understanding.

We close this section by noting that, in this study we assume that the decision maker has a neutral risk attitude. Hence, his objective is to maximize his expected wealth (profit). The idea of risk is, therefore, beyond the scope of this work. Interested reader can refer to [11] for more information.

Chapter 2

BASIC PROBLEM AND ASSUMPTIONS

The physical model of the one-period problem is depicted in Figure-2.1. The economic model is built on this physical system with the following considerations and assumptions :

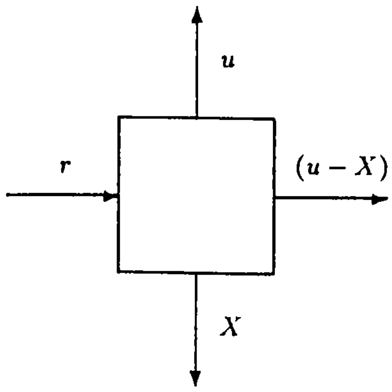

Figure 2.1: Physical Model of the One-Period Problem where

r	initial inventory level ($r \geq 0$),
u	beginning inventory level,
X	random demand with p.d.f. of $f(\cdot)$,
$(u - X)$	leftovers if $u > X$, or shortages if $u < X$.

(A1): The order quantity is $(u - r)$ but, u is taken to be the decision variable where $u \geq r$, and there is no capacity limit on u.

(A2): The costs associated with the problem are expressed in the following functional forms:

$$\text{Procurement Cost} = \begin{cases} C(u-r) & \text{if } u-r>0 \\ 0 & \text{if } u-r\le 0 \end{cases}$$

$$\text{Shortage Cost} = \begin{cases} S(X-u) & \text{if } X-u>0 \\ 0 & \text{if } X-u\le 0 \end{cases}$$

$$\text{Holding Cost} = \begin{cases} H(u-X) & \text{if } u-X>0 \\ 0 & \text{if } u-X\le 0 \end{cases}$$

where $C(\cdot), S(\cdot)$, and $H(\cdot)$ are positive valued functions. Note that we can incorporate the salvage value into the function $H(\cdot)$.

(A3): We assume that the vendor is in imperfect competition. That is, he can set any price between P_ℓ and P_u ($P_\ell > 0$), where P_ℓ is the lower bound and P_u is the upper bound on the product price that are induced by economic conditions or set exogeneously. We also assume that demand can be realized at any price between P_ℓ and P_u.

(A4): The vendor is certain on forecast results; that is, he believes the forecasters 100 %.

(A5): In general, the quantity demanded (X) and the price level (p) are dependent through the following implicit function

$$\mathcal{F}(X, p, \varepsilon) = 0 \tag{2.1}$$

where ε is a random variable. [9] . Therefore, we can solve for X

$$X = \mathcal{X}(p, \varepsilon)$$

in terms of p and ε. Furthermore, we can assume that the random term is additively separable in the form :

$$X = \bar{X}(p) + \varepsilon(p) \tag{2.2}$$

where $\bar{X}(p)$ is the expected demand at the price level of p. Equation (2.2) makes sense because, $\varepsilon(p)$ becomes the forecast error term which is added to the mean regression curve $\bar{X}(p)$ to yield the random demand. In his work, Mills [3] further assumes that the forecast error term is independent of p. That is, the forecast is equally predictive at any price level. In this study, we shall consider a more general relation given by (2.2). Lau & Lau [22] also consider a similar relation to (2.2).

(A6): Expected value of the quantity demanded at the price level of p is expressed as $\bar{X}(p)$ which is a continuous and differentiable function of p. We assume that $\bar{X}(p)$ is a monotone decreasing function; i.e.,

$$\frac{\partial \bar{X}(p)}{\partial p} < 0 \ ,$$

meaning that, on the average the customers demand less at higher prices.

(A7): The forecast error $\varepsilon(p)$ is assumed to be a continuous random variable with a p.d.f. of $g(x;p)$, $\varepsilon_1(p) \leq x \leq \varepsilon_2(p)$, and zero mean for all $p \in [P_\ell, P_u]$:

$$E[\varepsilon(p);p] = \int_{\varepsilon_1(p)}^{\varepsilon_2(p)} x \cdot g(x;p) \cdot dx = 0 \ ,$$

where $\varepsilon_1(p)$ and $\varepsilon_2(p)$, being continuous functions of p , are the bounds on $\varepsilon(p)$, if they exist, and p is a parameter of $g(\cdot;p)$. Also, we assume that $g(x;p)$ is a unimodal p.d.f. and is differentiable in p and x. Consequently, the c.d.f. of the forecast error,

$$G(x;p) = \int_{\varepsilon_1(p)}^{x} g(t;p) \cdot dt \ ,$$

becomes continuous and differentiable in both x and p , and monotone increasing in x.

(A8): From (A5) and (A7) the demand distribution can be obtained as

$$f(x;p) = \begin{cases} g(x - \bar{X}(p);p) & , \quad \varepsilon_1(p) + \bar{X}(p) \leq x \leq \varepsilon_2(p) + \bar{X}(p), \\ 0 & , \quad \text{otherwise,} \end{cases}$$

and the c.d.f. of demand becomes

$$F(x;p) = \int_{\varepsilon_1(p)+\bar{X}(p)}^{x} f(t;p) \cdot dt = \int_{\varepsilon_1(p)}^{x-\bar{X}(p)} g(y;p) \cdot dy = G(x - \bar{X}(p);p).$$

Note that

$$E[X;p] = \int_{\varepsilon_1(p)+\bar{X}(p)}^{\varepsilon_2(p)+\bar{X}(p)} t \cdot f(t;p) \cdot dt = \int_{\varepsilon_1(p)}^{\varepsilon_2(p)} [y + \bar{X}(p)] \cdot g(y;p) \cdot dy = \bar{X}(p).$$

It became a common practice in the field to use a normal (or any other Pearson type) distribution to represent the demand process. This practice however, is only an approximation because demand can never be negative or infinite in reality. The approximation is based on the negligable probability of occurrences of very large or negative values. In this study we will prefer to work with a non-negative and finite demand process. Therefore, we require that the limits of the demand distribution are positive and finite at all feasible price levels. That is:

$$0 \leq \varepsilon_1(p) + \bar{X}(p) < \varepsilon_2(p) + \bar{X}(p) < \infty \qquad , \forall p \in [P_\ell, P_u].$$

Chapter 3

MATHEMATICAL MODEL

3.1 General Expected Profit Function

Under the assumptions, profit $\Pi(p,u)$ of the vendor becomes

$$\Pi(p,u) = \begin{cases} p \cdot X - C(u-r) - H(u-X) & \text{, if} \quad \varepsilon_1(p) + \bar{X}(p) \le X \le u, \\ p \cdot u - C(u-r) - S(X-u) & \text{, if} \quad u \le X \le \varepsilon_2(p) + \bar{X}(p). \end{cases}$$

Using (A8) the expected profit can be obtained from

$$\begin{aligned} E[\Pi(p,u)] = & \int_{\varepsilon_1(p)+\bar{X}(p)}^{u} [p \cdot x - C(u-r) - H(u-x)] \cdot f(x;p) \cdot dx \\ & + \int_{u}^{\varepsilon_2(p)+\bar{X}(p)} [p \cdot u - C(u-r) - S(x-u)] \cdot f(x;p) \cdot dx. \end{aligned} \tag{3.1}$$

3.2 Expected Profit Function With Constant Unit Costs

To facilitate the mathematical analysis, general expressions in (A2) will be simplified as follows:

$$\left. \begin{aligned} C(x) &= c \cdot x \\ S(x) &= s \cdot x \\ H(x) &= h \cdot x \end{aligned} \right\} \tag{3.2}$$

where c, s, and h are constant unit costs given in \$/unit. Note that no fixed cost terms are allowed in (3.2). The case of additional fixed procurement cost will be considered later. Using (3.2) in (3.1) we write

$$\begin{aligned} E[\Pi(p,u)] &= p\cdot\bar{X}(p) - c\cdot(u-r) - h\cdot\int_{\epsilon_1(p)+\bar{X}(p)}^{u}(u-x)\cdot f(x;p)\cdot dx \\ &\quad -(p+s)\cdot\int_{u}^{\epsilon_2(p)+\bar{X}(p)}(x-u)\cdot f(x;p)\cdot dx. \end{aligned} \tag{3.3}$$

Using (A8) and substituting $y = x - \bar{X}(p)$ in (3.3) we obtain

$$\begin{aligned} E[\Pi(p,u)] &= p\cdot\bar{X}(p) - c\cdot(u-r) - h\cdot\int_{\epsilon_1(p)}^{u-\bar{X}(p)}[u-\bar{X}(p)-y]\cdot g(y;p)\cdot dy \\ &\quad -(p+s)\cdot\int_{u-\bar{X}(p)}^{\epsilon_2(p)}[y-u+\bar{X}(p)]\cdot g(y;p)\cdot dy. \end{aligned} \tag{3.4}$$

Rewriting (3.4) we get

$$E[\Pi(p,u)] = c\cdot r + (p+s-c)\cdot u - s\cdot\bar{X}(p) - (p+s+h)\cdot\Theta(u,p), \tag{3.5}$$

where

$$\Theta(u,p) = \int_{\epsilon_1(p)}^{u-\bar{X}(p)}[u-\bar{X}(p)-y]\cdot g(y;p)\cdot dy = \int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy. \tag{3.6}$$

For details of the derivation of equation (3.5) see Appendix-A. The function $\Theta(u,p)$ is studied in Appendix-B.

3.3 Mathematical Programming Formulation and General Solution Procedure

The objective of the vendor is to maximize $E[\Pi(p,u)]$ by making the best procurement and price decisions.

The problem can be formulated as a nonlinear optimization problem as

$$\begin{aligned} E[\Pi(p^*,u^*)] = &\ \max_{p,u}\ \{E[\Pi(p,u)] : (p,u)\in\mathcal{Y}\} \\ &\ s.t.\quad \mathcal{Y} = \{(p,u) : r\le u<\infty\ ,\ P_l\le p\le P_u\}. \end{aligned} \tag{3.7}$$

Since $E[\Pi(p,u)]$ is continuous in u and p, it has a global maximum over $\mathcal{Y}$. By assumption (A6), $\bar{X}(p)$ is differentiable in p and from Appendix-B it follows that $\Theta(u,p)$ is differentiable in both u and p. Therefore, $E[\Pi(p,u)]$ becomes differentiable over $\mathcal{Y}$ so that the first and second order optimality conditions can be studied by taking partial derivatives. However, since the expressions in the hessian becomes impractical for a clear analytical study, we shall prefer an alternative approach.

Considering (3.5), we write

$$\frac{\partial E[\Pi(p,u)]}{\partial u} = (p+s-c)-(p+s+h)\cdot G(u-X(p);p), \tag{3.8}$$

$$\frac{\partial^2 E[\Pi(p,u)]}{\partial u^2} = -(p+s+h)\cdot g(u-X(p);p) < 0. \tag{3.9}$$

which immediately proves that for any p , $E[\Pi(p,u)]$ is concave in u. Therefore, the optimal procurement decision u^* can be obtained from the following as a function of p

$$\left.\begin{array}{ll} (i) & G(u^*-X(p);p)=\dfrac{p+s-c}{p+s+h}=1-\dfrac{h+c}{p+s+h} \\[2ex] (ii) & \text{if } u^*<r \text{ then set } u^*=r. \end{array}\right\} \tag{3.10}$$

It is essential to note that, we obtain a unique solution for (3.10) for any price value between zero and infinity, if it exists.

Again from (3.5) we obtain

$$\frac{\partial E[\Pi(p,u)]}{\partial p} = u-s\cdot\frac{\partial \bar{X}(p)}{\partial p}-(p+s+h)\cdot\frac{\partial\Theta(u,p)}{\partial p}-\Theta(u,p), \tag{3.11}$$

$$\frac{\partial^2 E[\Pi(p,u)]}{\partial p^2} = -s\cdot\frac{\partial^2 \bar{X}(p)}{\partial p^2}-(p+s+h)\cdot\frac{\partial^2\Theta(u,p)}{\partial p^2}-2\cdot\frac{\partial\Theta(u,p)}{\partial p}. \tag{3.12}$$

It is evident from (3.11) and (3.12) that for the general case it may not be trivial to solve for p^* which maximizes $E[\Pi(u,p)]$ for a given u. For a special problem, however, we can solve for u^* in (3.10) as a function of p and substitute it in (3.5). Then, through a functional analysis over p we can evaluate the global maximizer p^*. We shall illustrate the use of this method in the example problems.

3.4 Certainty Profit

If there is no uncertainty, then we have $\varepsilon = 0$ with probability one and the demand becomes deterministic at any price level. In this case we write the certainty profit as

$$\Pi_c(p,u)=\begin{cases} p\cdot u-c\cdot(u-r)-s\cdot[\bar{X}(p)-u] & \text{, if } u\le\bar{X}(p), \\ p\cdot\bar{X}(p)-c\cdot(u-r)-h\cdot[u-\bar{X}(p)] & \text{, if } u\ge\bar{X}(p). \end{cases}$$

Therefore, we obtain the optimal decisions from

$$\Pi_c(p_c^*,u_c^*)=\underset{p,u}{Max}\ \{\Pi_c(p,u):\ r\le u<\infty\ ,\ P_l\le p\le P_u\}. \tag{3.13}$$

Given p, suppose $u \leq \bar{X}(p)$. Then we have

$$\Pi_c(p,u) = (p+s-c)\cdot u - s\cdot\bar{X}(p) + c\cdot r. \tag{3.14}$$

Since, (3.14) is a linear increasing function of u, we get $u_c^* = \bar{X}(p)$. Therefore, we write

$$\Pi_c(p,u_c^*) = (p-c)\cdot\bar{X}(p) + c\cdot r. \tag{3.15}$$

Now, suppose $u \geq \bar{X}(p)$. Then we have

$$\Pi_c(p,u) = (p+h)\cdot\bar{X}(p) - u\cdot(c+h) + c\cdot r \tag{3.16}$$

which is a linear decreasing function of u. Therefore, u_c^* becomes the smallest possible u which is given by:

$$u_c^* = \begin{cases} \bar{X}(p) & \text{, if } \bar{X}(p) \geq r, \\ r & \text{, if } \bar{X}(p) \leq r. \end{cases} \tag{3.17}$$

Consequently, we write

$$\Pi_c(p,u_c^*) = \begin{cases} (p-c)\cdot\bar{X}(p) + c\cdot r & \text{, if } \bar{X}(p) \geq r, \\ (p+h)\cdot\bar{X}(p) - h\cdot r & \text{, if } \bar{X}(p) \leq r. \end{cases} \tag{3.18}$$

Let p_1 be defined by

$$\bar{X}(p_1) = r,$$

such that $p_1 \in [P_\ell, P_u]$. If $r < \bar{X}(P_u)$ then set $p_1 = P_u$, and if $r > \bar{X}(P_\ell)$ then set $p_1 = P_\ell$.

If $p \geq p_1$ then, $\bar{X}(p) \leq \bar{X}(p_1) = r$. We can not have $u \leq \bar{X}(p) \leq r$. Therefore, we only consider the case of $u \geq \bar{X}(p)$. From (3.17) and (3.18) we get

$$\Pi_c(p,u_c^*) = (p+h)\cdot\bar{X}(p) - h\cdot r, \tag{3.19}$$

and $u_c^* = r$.

If $p \leq p_1$ then, $\bar{X}(p) \geq \bar{X}(p_1) = r$. We can have either $u \geq \bar{X}(p)$ or $u \leq \bar{X}(p)$. However, in either case, from (3.15) and (3.18) we get

$$\Pi_c(p,u_c^*) = (p-c)\cdot\bar{X}(p) + c\cdot r, \tag{3.20}$$

and $u_c^* = \bar{X}(p)$. Therefore, from (3.19) and (3.20) we obtain

$$\Pi_c(p,u_c^*) = \begin{cases} \left.\begin{matrix} (p+h)\cdot\bar{X}(p) - h\cdot r, \\ u_c^* = r. \end{matrix}\right\} & \text{, for } p \geq p_1, \\ \left.\begin{matrix} (p-c)\cdot\bar{X}(p) + c\cdot r, \\ u_c^* = \bar{X}(p). \end{matrix}\right\} & \text{, for } p \leq p_1. \end{cases} \tag{3.21}$$

Problem (3.13) can be rewritten in the following form

$$\Pi_c(p_c^*,u_c^*) = \underset{p}{Max} \; \{\Pi_c(p,u_c^*) : \; P_\ell \leq p \leq P_u\} \tag{3.22}$$

where $\Pi_c(p, u_c^*)$ is given by (3.21). Since $\bar{X}(p)$ is continuous by assumption (A7), $\Pi_c(p, u_c^*)$ is also continuous; hence, it has a global maximum over $[P_l, P_u]$, say at p_c^*.

Note that for $r = 0$ we have $p \leq p_1$, and from (3.21) we get $u_c^* = \bar{X}(p)$, and

$$\Pi_c(p, u_c^*) = (p - c)\cdot\bar{X}(p). \tag{3.23}$$

The idea of comparing the uncertainty and certainty results was introduced by Mills [3]. This enables us to see the change in the expected profit and decision variables due to the uncertainty introduced into the problem. We shall provide this comparison for the example problems.

3.5 Additional Set-Up Cost

In this case we allow for an additional (constant) term $\mathcal{K}$ in the procurement cost function

$$C(x) = \begin{cases} \mathcal{K} + c\cdot x & , \text{if} \quad x > 0, \\ 0 & , \text{if} \quad x = 0. \end{cases}$$

Therefore, the profit function becomes

$$\Pi_f(p, u) = \begin{cases} p\cdot x - \mathcal{K} - c\cdot(u - r) - h\cdot(u - x) & , \quad \varepsilon_1(p) + \bar{X}(p) \leq x < u, \\ p\cdot u - \mathcal{K} - c\cdot(u - r) - s\cdot(x - u) & , \quad u \leq x \leq \varepsilon_2(p) + \bar{X}(p), \end{cases}$$

and we can write the expected profit as

$$E[\Pi_f(p, u)] = \begin{cases} -\mathcal{K} - c\cdot(u - r) - L(u, p) + p\cdot\bar{X}(p) & , \text{if} \quad u > r, \\ -L(r, p) + p\cdot\bar{X}(p) & , \text{if} \quad u = r, \end{cases} \tag{3.24}$$

where $L(u, p)$ is the expected loss function given by :

$$\begin{aligned} L(u, p) &= h\cdot\int_{\varepsilon_1(p)+\bar{X}(p)}^{u} (u - x)\cdot f(x; p)\cdot dx \\ &\qquad + (p + s)\cdot\int_{u}^{\varepsilon_2(p)+\bar{X}(p)} (x - u)\cdot f(x; p)\cdot dx, \end{aligned} \tag{3.25}$$

$$= (p + s)\cdot[\bar{X}(p) - u] + (p + s + h)\cdot\Theta(u, p). \tag{3.26}$$

See Appendix-C for the derivation of (3.26).

Furthermore, defining a function $M(u, p)$ as :

$$M(u, p) = L(u, p) + c\cdot(u - r) - p\cdot\bar{X}(p), \tag{3.27}$$

we can write

$$-E[\Pi_f(p,u)] = \begin{cases} M(u,p)+\mathcal{K} & \text{, if} \quad u > r, \\ M(r,p) & \text{, if} \quad u = r. \end{cases} \tag{3.28}$$

Note that

$$M(u,p) = -E[\Pi(p,u)].$$

In this regard, the optimization problem becomes

$$\begin{aligned} -E[\Pi_f(p^*,u^*)] = \; & \underset{p,u}{Min} \quad \{-E[\Pi_f(p,u)] : (p,u) \in \mathcal{Y}\} \\ & s.t. \quad \mathcal{Y} = \{(p,u) : r \leq u < \infty \;\;,\;\; P_\ell \leq p \leq P_u\}. \end{aligned} \tag{3.29}$$

Since $E[\Pi_f(p,u)]$ is discontinuous at $u = r$, we need a different analysis than we have developed for (3.7).

For the classical newsboy problem with a constant set-up cost we define the optimal policy by two parameters namely, the reorder point (s) and the order-up-to level ($\mathcal{S}$). This is known as the (s,$\mathcal{S}$) policy. For our model the presence of an (s,$\mathcal{S}$) type policy is an important fact because, it might have useful implications for the multi-period extension of the theory. In this regard, we shall reveal the conditions under which an (s,$\mathcal{S}$) type policy would yield the optimal decisions.

Let $\overset{o}{u}$ and $\overset{o}{p}_u$ be given by the relaxed problem

$$M(\overset{o}{u},\overset{o}{p}_u) = \underset{u,p}{Min} \quad \{M(u,p) : 0 \leq u < \infty \;\;,\;\; P_\ell \leq p \leq P_u\}. \tag{3.30}$$

Since $M(u,p)$ is continuous in p and u, there exists a solution $(\overset{o}{u},\overset{o}{p}_u)$ for the problem (3.30). This problem can be rewritten as:

$$M(\overset{o}{u},\overset{o}{p}_u) = \underset{u}{Min} \quad \{M(u,p_u) : \;\; 0 \leq u < \infty\}, \tag{3.31}$$

where p_u is the best price value at u and it can be determined from

$$M(u,p_u) = \underset{p}{Min} \quad \{M(u,p) : \;\; P_\ell \leq p \leq P_u\}. \tag{3.32}$$

We claim that, we can device an (s,$\mathcal{S}$) type policy which will operate on $M(u,p_u)$,and we can utilize that to obtain the optimal decisions. This requires $M(u,p_u)$ to be $\mathcal{K}$-convex in u. We shall prove our claim but, before getting into details of the proof we need to study $M(u,p_u)$ further.

Using the information given in Appendix-B we can rewrite (3.27) as

$$M(u,p) = \begin{cases} s\cdot\bar{X}(p) - u\cdot(p+s-c) - c\cdot r & , \ \varepsilon_1(p) + \bar{X}(p) \geq u > 0, \\ s\cdot\bar{X}(p) - u\cdot(p+s-c) - c\cdot r + (p+s+h)\cdot\Theta(u,p) & , \text{otherwise}, \\ -(p+h)\cdot\bar{X}(p) + u\cdot(c+h) - c\cdot r & , \ \varepsilon_2(p) + \bar{X}(p) \leq u. \end{cases} \tag{3.33}$$

In this representation, $M(u,p)$ becomes a monotone decreasing function of p for $u \leq \varepsilon_1(p) + \bar{X}(p)$. Therefore, for $0 < u \leq \varepsilon_1(P_u) + \bar{X}(P_u)$ we have $p_u = P_u$ and

$$M(u,p_u) = s\cdot\bar{X}(P_u) - u\cdot(P_u + s - c) - c\cdot r \tag{3.34}$$

which is a linear (decreasing) function of u with a slope of $-(P_u+s-c)$. We call this function as the left tail of $M(u,p_u)$.

If, on the other hand, $u \geq \varepsilon_2(p) + \bar{X}(p)$ then we have

$$M(u,p) = -(p+h)\cdot\bar{X}(p) + u\cdot(c+h) - c\cdot r. \tag{3.35}$$

For any u, the price which minimizes (3.35), say p_u, can either be a boundary value, or an interior value. If former then, $p_u = P_\ell$, or $p_u = P_u$: $M(u,p_u)$ becomes a linear function of u. If latter then, p_u is a relative minimum point which should satisfy

$$\frac{\partial M(u,p_u)}{\partial p_u} = 0 = -\bar{X}(p_u) - (p_u + h)\cdot\frac{\partial \bar{X}(p_u)}{\partial p_u}. \tag{3.36}$$

Note that the condition (3.36) is independent of u, hence, for all $u \geq \varepsilon_2(p) + \bar{X}(p)$ we have a unique p_u, if it exists.

Therefore, for $u \geq \varepsilon_2(P_\ell) + \bar{X}(P_\ell)$ we have a unique p_u. In this range, $M(u,p_u)$ becomes a linear (increasing) function of u with a slope of $(c+h)$. We call this linear function as the right tail of $M(u,p_u)$.

We can therefore, interpret the right and left tails of $M(u,p_u)$ as its "asymptotes" in a sense that above the maximum and below the minimum beginning inventory levels the function is linear (convex) in u.

We also note that, the first and third form of $M(u,p_u)$ in equation (3.33) are equal to equations (3.14) and (3.16), respectively, with a sign change. This proves us that the right and left tails of the function $M(u,p_u)$ become negative of the certainty profit function in the appropriate u ranges.

When we solve problem (3.32) we get p_u either as a boundary value, that is, it becomes either P_ℓ or P_u, or as an interior value. If there exists no interior point solution to (3.32) then the problem (3.31) becomes trivial. Therefore, we assume that there exists an interior point solution for some u.

We already know that, if p_u is a boundary value then $M(u,p_u)$ is convex in u. On the other hand, for interior point solutions we need to know the behaviour of $M(u,p_u)$ in u.

If $p_u \in (P_\ell, P_u)$ then, it has to be a relative minimum point. Therefore, the best price at a given u has to satisfy

$$\frac{\partial M(u,p_u)}{\partial p_u} = 0. \tag{3.37}$$

Writing the chain rule for differentiating $M(u,p_u)$ w.r.t. u as

$$\frac{dM(u,p_u)}{du} = \frac{\partial M(u,p_u)}{\partial u} + \frac{\partial M(u,p_u)}{\partial p_u} \cdot \frac{dp_u}{du}$$

and using (3.37) we obtain

$$\frac{dM(u,p_u)}{du} = -(p_u + s - c) + (p_u + s + h) \cdot G(u - \bar{X}(p_u); p_u). \tag{3.38}$$

We note that the first order condition on $M(u,p_u)$ yields

$$G(u - \bar{X}(p_u); p_u) = \frac{p_u + s - c}{p_u + s + h}, \tag{3.39}$$

which implies that p_u is the best price at u, and u is the best procurement at p_u. Therefore, (3.39) gives the global minimum point $(\overset{o}{u}, \overset{o}{p}_u)$ for the problem (3.31). That is, every first order point of $M(u,p_u)$ is a global minimum point of (3.31).

It follows by (3.38) that,

$$\frac{dM(u,p_u)}{du} < 0 \;\Leftrightarrow\; G(u - \bar{X}(p_u); p_u) < \frac{p_u + s - c}{p_u + s + h}. \tag{3.40}$$

Suppose that we choose a u, and obtain the best price at u as p_u. Having p_u we now evaluate the best beginning inventory level u_b from

$$G(u_b - \bar{X}(p_u); p_u) = \frac{p_u + s - c}{p_u + s + h}.$$

Since $G(\cdot)$ is a monotone increasing function we write

$$\begin{aligned} G(u - \bar{X}(p_u); p_u) \quad &< \quad G(u_b - \bar{X}(p_u); p_u) \\ &\Leftrightarrow \quad u < u_b. \end{aligned}$$

Consequently, (3.40) becomes

$$\frac{dM(u,p_u)}{du} < 0 \Leftrightarrow u < u_b. \tag{3.41}$$

Proposition :

$M(u, p_u)$ is convex in u.

Proof by contradiction :

First we shall show that

$$\frac{dM(u,p_u)}{du} < 0 \quad , \forall u \in (0, \overset{o}{u}). \tag{3.42}$$

Let $u_1 \in (0, \overset{o}{u})$ and p_{u_1} be the best price at u_1. Suppose that we construct $M(u, p_{u_1})$ in u and observe that it is increasing at u_1 therefore, the best procurement at p_{u_1}, say u_{b_1}, is less than u_1. Furthermore, suppose that we move an infinitesimal amount to the right of u_1, say to u_2, and construct $M(u, p_{u_2})$ where again $u_{b_2} < u_2$. See the proposed scheme in Figure-3.1.

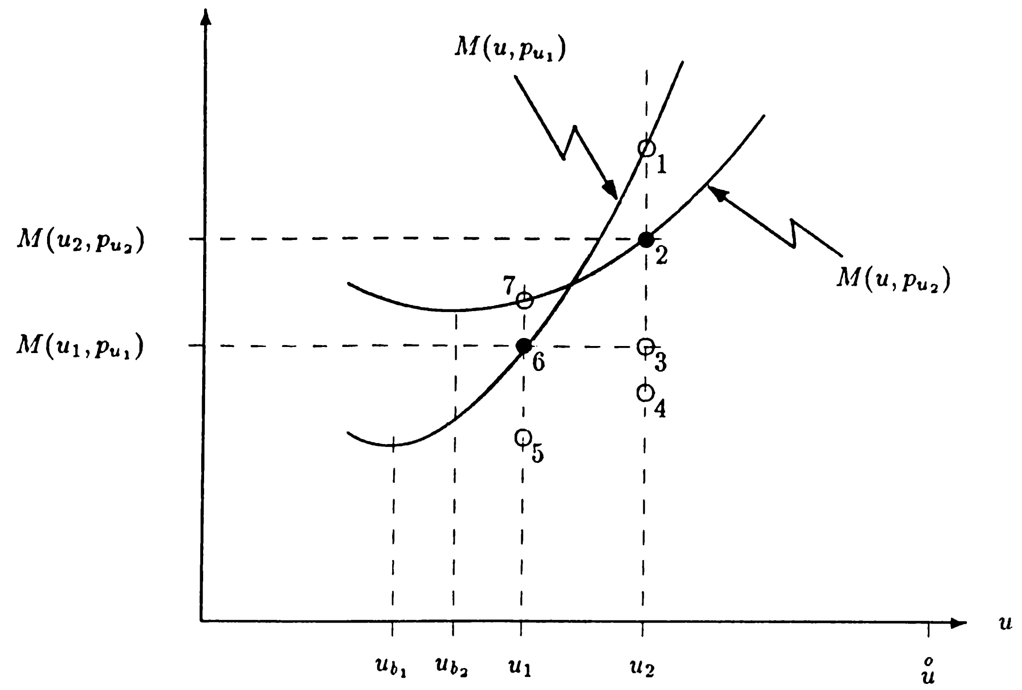

Figure 3.1: Construction Of $M(u, p_u)$.

We see in the figure that, when we make a pricing decision at u_2 we can only reduce $M(u_2, p_{u_1})$, that is, we can be at points 2, 3, or 4. Moreover, since $M(u, p_{u_2})$ is increasing at u_2, it can pass through points 5 or 7. We have stated that point 6 was the best price point at u_1 hence, $M(u, p_{u_2})$ can not pass through 5, because otherwise p_{u_2} would have been better than p_{u_1} at u_1. Since point 7 is above 6 ,$M(u, p_{u_2})$ can not pass through 3 or 4.

Therefore, $M(u, p_{u_2})$ contains the points 7 and 2. It follows that, point 2 should be above point 6 and this behaviour reproduce itself everytime we move to the right. When it happens that $M(u, p_{u_2})$ contains points 6 and 3 we say that it attains an extremum point in between u_1 and u_2. This, however can not be a relative minimum because, $M(u, p_u)$ can not rise to such a point. Thus, $M(u, p_u)$ increases until it reaches its right tail, which is described before.

Concludingly, it becomes a contradiction to have $u_b < u$ for any $u \in (0, \overset{o}{u})$. Hence, we have $u_b > u$ and from (3.41) it follows that

$$\frac{dM(u, p_u)}{du} < 0 \quad , \forall u \in (0, \overset{o}{u}).$$

The same construction of the proof applies for $u \in (\overset{o}{u}, \infty)$ where we have

$$\frac{dM(u, p_u)}{du} > 0 \tag{3.43}$$

by (3.38).

Consequently, by combining the results (3.42) and (3.43) we have $M(u, p_u)$ decreasing in $u \in (0, \overset{o}{u})$ and increasing in $u \in (\overset{o}{u}, \infty)$. Thus, $M(u, p_u)$ becomes convex in u. A typical $M(u, p_u)$ function is shown in Figure-3.2.

□

Now, suppose that we start with r units of initial inventory, and do not procure anything. Then, the optimal pricing decision will be determined from

$$M(r, \overset{o}{p}_r) = \underset{p}{Min} \quad \{M(r, p) : P_\ell \leq p \leq P_u\}. \tag{3.44}$$

Depending on the values of r, $\overset{o}{u}$, $M(\overset{o}{u}, \overset{o}{p}_u)$, and $M(r, \overset{o}{p}_r)$ the procurement decision, whether to order up to $\overset{o}{u}$ or do not order, is made according to (3.28). It is clear with (3.28) that, whenever we place an order we incur a fixed cost of $\mathcal{K}$. Therefore, given an initial stock of r to start with, the optimal policy will be determined by the following (s,S) type procedure :

$$\left.\begin{array}{lll} (i) & \text{Find } M(\overset{o}{u}, \overset{o}{p}_u) \text{ and } M(r, \overset{o}{p}_r), & \\ (ii) & \text{If } \overset{o}{u} \leq r \text{ then, do not order and set } p^* = \overset{o}{p}_r, & \\ (iii) & \text{If } \overset{o}{u} > r \text{ then, } & \text{if } M(\overset{o}{u}, \overset{o}{p}_u) + \mathcal{K} < M(r, \overset{o}{p}_r) \text{ then} \\ & & \text{order up to } \overset{o}{u} \text{ and set } p^* = \overset{o}{p}_u \text{ else,} \\ & & \text{do not order and set } p^* = \overset{o}{p}_r. \end{array}\right\} \tag{3.45}$$

This procedure will be demonstrated for an example problem in the next chapter.

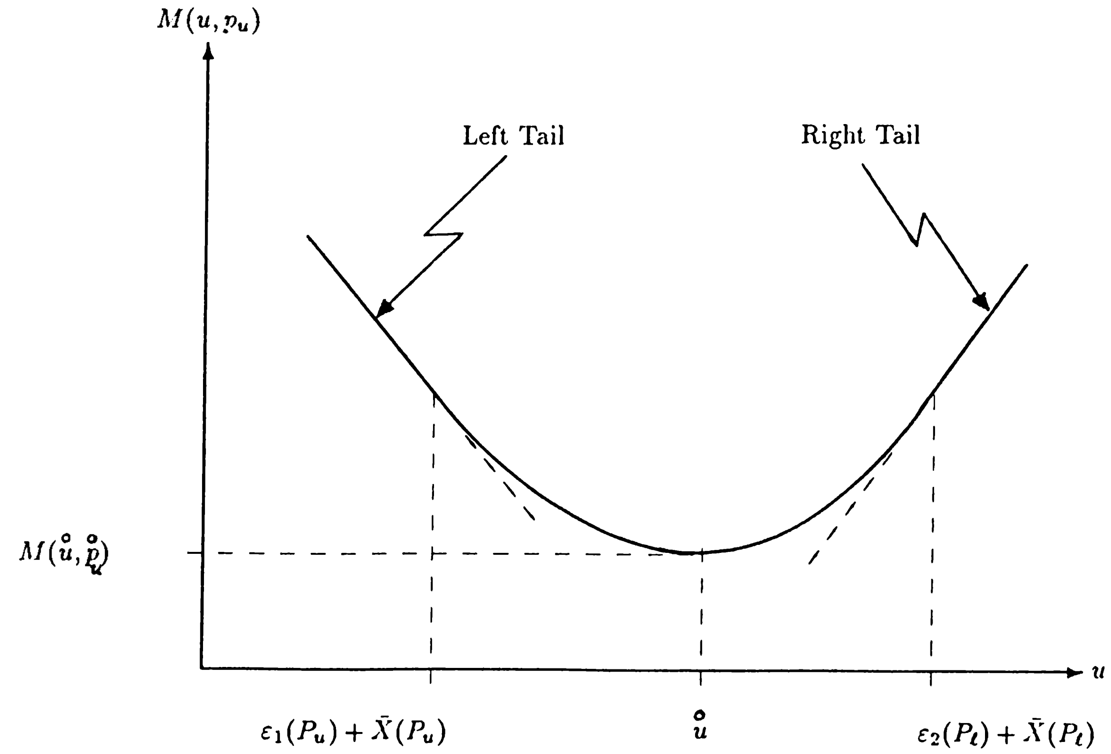

Figure 3.2: $M(u, p_u)$ Function.

Chapter 4

EXAMPLES

We shall assume that, the expected value of demand has a linear relation with price. That is,

$$\bar{X}(p) = a - b \cdot p, \tag{4.1}$$

where a and b are non-negative parameters. The parameter b can be interpreted as the price sensitivity of the expected demand.

Mills [3] studied the linear case as it is given by the equation (4.1). Lau & Lau [22], however, considered a slightly different equation

$$\bar{X}(p) = a - b \cdot (p - p_m) \tag{4.2}$$

where p_m is the mid-price given by

$$p_m = (P_l + P_u)/2.$$

In equation (4.2) we note that $\bar{X}(p)$ rotates about the point (p_m, a) for different b values, and with $p_m = 0$, equation (4.2) becomes (4.1). In this study we shall consider the case given by equation (4.2).

4.1 Example 1

In this example we assume that the forecast error is independent of the price and has a uniform p.d.f. given by

$$g(x;p) = g(x) = \begin{cases} \dfrac{1}{2 \cdot \lambda} & , \ -\lambda \leq x \leq \lambda, \\ \\ 0 & , \text{ otherwise,} \end{cases} \tag{4.3}$$

where $\lambda \geq 0$. Figure-4.1 displays possible price-demand realizations under conditions (4.2) and (4.3).

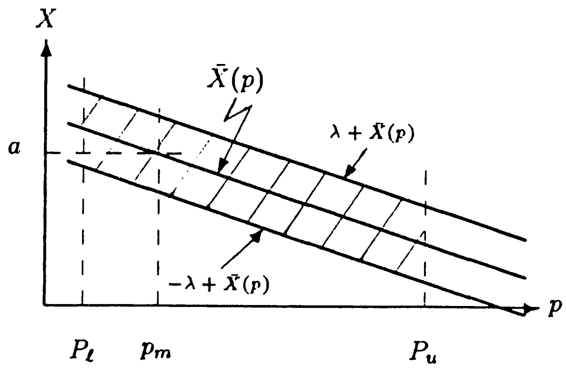

Figure 4.1: The Cross-Hatched Area Represents the Price-Demand Values That Can be Realized Under the Assumptions (4.2) and (4.3) .

Note that, the uniform distribution is not continuous at its limit points. Although, this contradicts with assumption (A7) we shall show that it is still possible to utilize the proposed method for obtaining the solution.

From (A7) it follows that

$$G(x) = \begin{cases} 0 \,, & x < -\lambda, \\ \dfrac{x+\lambda}{2\cdot\lambda} \,, & -\lambda \le x \le \lambda, \\ 1 \,, & \lambda < x, \end{cases} \tag{4.4}$$

where the function becomes continuous in x, but not differentiable at $x = -\lambda$, and $x = \lambda$.

Taking $r = 0$ in (3.10) and using (4.4) it follows that

$$u^* = \bar{X}(p) + \lambda - \frac{2\cdot\lambda\cdot(c+h)}{(p+s+h)}. \tag{4.5}$$

Furthermore, we evaluate

$$\Theta(u,p) = \begin{cases} 0 & \text{, for} \quad u < \bar{X}(p) - \lambda, \\ \frac{1}{4\lambda}\cdot[u - \bar{X}(p) + \lambda]^2 & \text{, otherwise,} \\ u - \bar{X}(p) & \text{, for} \quad u > \bar{X}(p) + \lambda, \end{cases} \tag{4.6}$$

from (3.6), which becomes a continuous function of u and p. On substituting (4.6) in equation (3.5) with (4.5) we obtain

$$E[\Pi(p,u^*)] = c\cdot r + (p-c)\cdot\bar{X}(p) - \frac{\lambda\cdot(c+h)\cdot(p+s-c)}{(p+s+h)} \tag{4.7}$$

which is a continuous and differentiable function of p. Finally, using (4.7), we get

$$\frac{\partial E[\Pi(p,u^*)]}{\partial p} = \bar{X}(p) + (p-c)\cdot\frac{\partial \bar{X}(p)}{\partial p} - \frac{\lambda\cdot(c+h)^2}{(p+s+h)^2}, \tag{4.8}$$

$$\frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} = 2\cdot\frac{\partial \bar{X}(p)}{\partial p} + (p-c)\cdot\frac{\partial^2 \bar{X}(p)}{\partial p^2} + \frac{2\cdot\lambda\cdot(c+h)^2}{(p+s+h)^3}. \tag{4.9}$$

Under the linearity assumption given by (4.2), and by utilizing (4.8) and (4.9), we employ a functional analysis to determine p^*, the maximizer of (4.7). See Appendix-D for the details.

For the certainty profit, from (3.23) we write

$$\frac{\partial \Pi_c(p)}{\partial p} = \bar{X}(p) + (p-c)\cdot\frac{\partial \bar{X}(p)}{\partial p}, \tag{4.10}$$

$$\frac{\partial^2 \Pi_c(p)}{\partial p^2} = 2\cdot\frac{\partial \bar{X}(p)}{\partial p} + (p-c)\cdot\frac{\partial^2 \bar{X}(p)}{\partial p^2}. \tag{4.11}$$

Using (4.2) in (4.11) we find that $\Pi_c(p)$ is concave in p and from (4.10) we get

$$p_c^* = \frac{a+b\cdot p_m + b\cdot c}{2\cdot b}.$$

Moreover, we obtain

$$u^* = \frac{a+b\cdot p_m - b\cdot c}{2},$$

$$\Pi_c(p_c^*) = \frac{(a+b\cdot p_m - b\cdot c)^2}{4\cdot b} + c\cdot r.$$

Table-4.1 displays the numerical results obtained for the following parameter set

$$\begin{aligned} c &= 1, h=0.5, s=1, r=0, a=102, P_\ell = 1.6, P_u = 4, p_m = 2.8, \\ \lambda &= 17.32, 34.64, 51.96, 69.28, \\ b &= 25, 35, 45, 55. \end{aligned}$$

It is seen in Table-4.1 that as λ (or standard deviation of the error term σ_ϵ) is increased, the optimal price declines slightly, the procurement quantity increases, and the expected profit decreases. It is intuitive to realize less profit as uncertainty in the problem increases, and order more in order to cope with uncertainties. Also, as b (i.e. the price sensitivity of the expected demand) is increased the optimal price and the expected profit decrease but the procurement quantity keeps increasing. This result is intuitive too, because as price sensitivity increases the effect of uncertainties would also be coupled with it.

which is a continuous and differentiable function of p. Finally, using (4.7), we get

$$\frac{\partial E[\Pi(p,u^*)]}{\partial p} = \bar{X}(p) + (p-c)\cdot\frac{\partial \bar{X}(p)}{\partial p} - \frac{\lambda\cdot(c+h)^2}{(p+s+h)^2}, \tag{4.8}$$

$$\frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} = 2\cdot\frac{\partial \bar{X}(p)}{\partial p} + (p-c)\cdot\frac{\partial^2 \bar{X}(p)}{\partial p^2} + \frac{2\cdot\lambda\cdot(c+h)^2}{(p+s+h)^3}. \tag{4.9}$$

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Using (4.2) in (4.11) we find that $\Pi_c(p)$ is concave in p and from (4.10) we get

$$p_c^* = \frac{a+b\cdot p_m+b\cdot c}{2\cdot b}.$$

Moreover, we obtain

$$u^* = \frac{a+b\cdot p_m-b\cdot c}{2},$$

$$\Pi_c(p_c^*) = \frac{(a+b\cdot p_m-b\cdot c)^2}{4\cdot b} + c\cdot r.$$

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Certainty values are listed in Table-4.1 under $\lambda = 0$ column. Comparing the first column and the others we conclude that, uncertainty brings a loss in expected profit, the percentage of which is smaller when b is small. Also, uncertainty requires to hold more stocks in order to cope with possible demand fluctuations. We note, however, that the optimal price setting is almost unaffected by uncertainty. The reason for this negligible change in p^* can be explained as follows

Using (3.23) in equation (3.5) we write

$$E[\Pi(p,u)] = \Pi_c(p,u_c^*) + (p+s-c)\cdot[u - \bar{X}(p)] - (p+s+h)\cdot\Theta(u,p) \tag{4.12}$$

then, considering (3.10) we modify (4.12) as

$$E[\Pi(p,u^*)] = \Pi_c(p,u_c^*) + (p+s+h)\cdot\{G(u^* - \bar{X}(p);p)\cdot[u^* - \bar{X}(p)] - \Theta(u^*,p)\}$$

which becomes after some manipulations:

$$E[\Pi(p,u^*)] = \Pi_c(p,u_c^*) + (p+s+h)\cdot\int_{\varepsilon_1(p)}^{u^*-\bar{X}(p)} y\cdot g(y;p)\cdot dy. \tag{4.13}$$

Note that the second term in (4.13) is the partial expectation of the forecast error which is negative because the expected value of it is zero. Consequently, certainty profit exceeds the expected profit for the uncertainty case. For a detailed treatment of partial expectations the reader may refer to Winkler et al. [8].

The negligible difference between the p^* and p_c^* values that we encounter in our example can now be explained by equation (4.13). If the contribution of the second term for the optimality conditions is small, then we would expect a slight deviation of p^* values from p_c^*. Although this is the case in our example, it might not be true for other realizations. See, for instance, the results of Example-2 in Table-4.4.

Lau & Lau [22] studied the same problem using a normal forecast error distribution with zero mean and σ_ε^2. The only difference in their model is that they assumed a unit salvage value instead of a unit holding cost. Letting $h = -0.5$ and retaining the rest of the parameter values in the above consideration, we obtain the results listed in Table-4.2. On the other hand, Table-4.3 shows their results for the same parameter set. Comparing Table-4.2 and Table-4.3 we see that two different forecast error distributions yield similar results. Also, the discussion above of Table-4.1 is true for their case.

b						
	$\lambda =$	0	17.32	34.64	51.96	69.28
	$\sigma_\varepsilon =$	0	10	20	30	40
25	$p^* =$	3.940	3.913	3.886	3.859	3.830
	$u^* =$	73.500	81.887	90.190	98.406	106.531
	$E[\Pi(p^*, u^*)] =$	216.090	197.291	178.528	159.802	141.113
35	$p^* =$	3.357	3.333	3.309	3.284	3.259
	$u^* =$	82.500	89.904	97.216	104.432	111.547
	$E[\Pi(p^*, u^*)] =$	194.464	176.527	158.630	140.775	122.962
45	$p^* =$	3.033	3.012	2.990	2.968	2.946
	$u^* =$	91.500	98.261	104.930	111.502	117.973
	$E[\Pi(p^*, u^*)] =$	186.050	168.686	151.364	134.084	116.848
55	$p^* =$	2.827	2.808	2.789	2.769	2.749
	$u^* =$	100.500	106.809	113.028	119.153	125.180
	$E[\Pi(p^*, u^*)] =$	183.641	166.686	149.772	132.900	116.070

Table 4.1: Results of Example-1.

$$c = 1, h = 0.5, s = 1, r = 0, a = 102,$$
$$P_\ell = 1.6, P_u = 4, p_m = 2.8.$$

b	$\lambda =$ $\sigma_\epsilon =$	0 0	17.32 10	34.64 20	51.96 30	69.28 40
25	$p^* =$	3.940	3.936	3.931	3.927	3.922
	$u^* =$	73.500	87.025	100.543	114.054	127.557
	$E[\Pi(p^*, u^*)] =$	216.090	208.406	200.722	193.040	185.359
35	$p^* =$	3.357	3.353	3.349	3.345	3.340
	$u^* =$	82.500	95.470	108.432	121.384	134.327
	$E[\Pi(p^*, u^*)] =$	194.464	186.927	179.392	171.858	164.324
45	$p^* =$	3.033	3.029	3.026	3.022	3.018
	$u^* =$	91.500	104.087	116.663	129.229	141.786
	$E[\Pi(p^*, u^*)] =$	186.050	178.616	171.184	163.752	156.323
55	$p^* =$	2.827	2.824	2.820	2.817	2.813
	$u^* =$	100.500	112.805	125.100	137.384	149.657
	$E[\Pi(p^*, u^*)] =$	183.641	176.283	168.926	161.571	154.218

Table 4.2: Results of Example-1.

$$c = 1, h = -0.5, s = 1, r = 0, a = 102,$$
$$P_\ell = 1.6, P_u = 4, p_m = 2.8.$$

b	$\sigma_\varepsilon =$	10	20	30	40
25	$p^* =$	3.93	3.92	3.91	3.89
	$u^* =$	85.9	98.2	110.6	122.9
	$E[\Pi(p^*, u^*)] =$	207.6	199.1	190.6	182.2
35	$p^* =$	3.35	3.34	3.33	3.32
	$u^* =$	94.1	105.7	117.1	128.7
	$E[\Pi(p^*, u^*)] =$	186.3	178.2	170.1	161.9
45	$p^* =$	3.03	3.02	3.01	3.00
	$u^* =$	102.6	113.6	124.7	135.6
	$E[\Pi(p^*, u^*)] =$	178.1	170.2	162.3	154.4
55	$p^* =$	2.82	2.81	2.806	2.80
	$u^* =$	111.2	122.0	132.6	143.2
	$E[\Pi(p^*, u^*)] =$	175.9	168.1	160.4	152.6

Table 4.3: Results of Lau & Lau.

$c = 1, h = -0.5, s = 1, r = 0, a = 102,$
$P_l = 1.6, P_u = 4, p_m = 2.8,$
$g(\cdot;p)$ is normal with zero mean and σ_ε.

4.2 Example 2

In this example, the previous assumptions are preserved. However, we assume that the forecast error has a price dependent uniform p.d.f.

$$g(x;p) = \begin{cases} \dfrac{1}{m\cdot(p-\hat{p})^2+\varepsilon_0} & , -\frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0) \leq x \leq \frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0), \\ 0, & \text{otherwise ,} \end{cases} \tag{4.14}$$

where m, ε_0 ,and $\hat{p} \geq 0$. Therefore, we require that the forecast error is minimized at $\hat{p}$ and it is increasing away from it. Note that, with this adjustment we bring a flexibility to the price dependency of the forecast error distribution. For example, by letting $\hat{p} = P_\ell$ we imply the condition that the range of forecast error values increases quadratically in p. In this regard, possible realizations of price-demand values are shown in Figure-4.2.

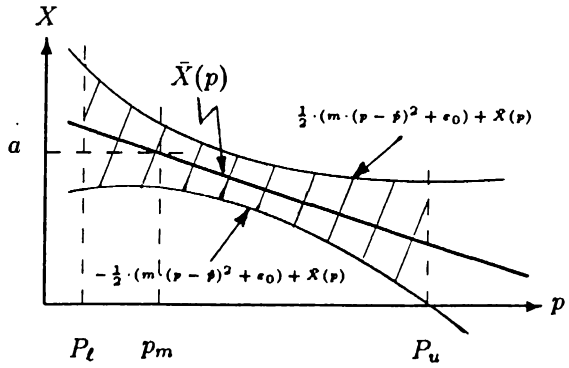

Figure 4.2: The Cross-Hatched Area Represents the Price-Demand Values That Can be Realized Under the Assumptions (4.2) and (4.14) .

Evaluating

$$G(x;p) = \frac{1}{2} + \frac{x}{m\cdot(p-\hat{p})^2+\varepsilon_0} \tag{4.15}$$

from (4.14) and taking $r = 0$ in (3.10) it follows that,

$$u^* = \bar{X}(p) + \frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0) - \frac{(m\cdot(p-\hat{p})^2+\varepsilon_0)\cdot(c+h)}{(p+s+h)}. \tag{4.16}$$

Moreover, evaluating

$$\Theta(u^*,p) = \frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0)\cdot\frac{(p+s-c)^2}{(p+s+h)^2} \tag{4.17}$$

from (3.5) and (4.16), and using it in (3.4) we obtain

$$\begin{aligned} E[\Pi(p,u^*)] &= c\cdot r + (p-c)\cdot \bar{X}(p) \\ &\quad -\frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0)\cdot\frac{(c+h)\cdot(p+s-c)}{(p+s+h)} \end{aligned} \tag{4.18}$$

Differentiating (4.18) w.r.t. p once and twice we get

$$\begin{aligned} \frac{\partial E[\Pi(p,u^*)]}{\partial p} &= \bar{X}(p)-(p-c)\cdot b - m\cdot(p-\hat{p})\cdot\frac{(c+h)\cdot(p+s-c)}{(p+s+h)} \\ &\quad -\frac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0)\cdot\frac{(c+h)^2}{(p+s+h)^2}, \end{aligned} \tag{4.19}$$

$$\begin{aligned} \frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} &= -2\cdot b - m\cdot(c+h) \\ &\quad +\frac{m\cdot(c+h)^2}{(p+s+h)^3}\cdot\left[(s+h)\cdot(2\cdot\hat{p}+s+h)+\hat{p}+\frac{\varepsilon_0}{m}\right] \end{aligned} \tag{4.20}$$

We can use (4.19) and (4.20) in a functional analysis to determine p^* which maximizes (4.18). For details see Appendix-E.

Table-4.4 and Table-4.5 display the numerical results obtained for the same parameter set used in Example-1 with the following additions

$$\begin{aligned} \hat{p} &= 1.5, \\ \varepsilon_0 &= 10, 20, 30, 40, \\ m &= 2, 4, 6, 8. \end{aligned}$$

Similar conclusions can be drawn as in the first example. An additional observation is the considerable change in p^* from its riskless value in Table-4.4. This agrees with the conclusion we made for Example-1.

b						
	$\varepsilon_0 =$	0	10	20	30	40
	$m =$	0	8	8	8	8
25	$p^* =$	3.940	3.555	3.547	3.540	3.533
	$u^* =$	73.500	92.030	94.173	96.309	98.436
	$E[\Pi(p^*, u^*)] =$	216.090	189.290	184.018	178.748	173.482
35	$p^* =$	3.357	3.143	3.136	3.130	3.123
	$u^* =$	82.500	95.587	97.536	99.477	101.409
	$E[\Pi(p^*, u^*)] =$	194.464	176.818	171.743	166.671	161.603
45	$p^* =$	3.033	2.894	2.888	2.883	2.877
	$u^* =$	91.500	101.809	103.622	105.427	107.225
	$E[\Pi(p^*, u^*)] =$	186.050	172.558	167.619	162.685	157.753
55	$p^* =$	2.827	2.728	2.723	2.718	2.712
	$u^* =$	100.500	109.154	110.868	112.573	114.272
	$E[\Pi(p^*, u^*)] =$	183.641	172.422	167.584	162.750	157.919

Table 4.4: Results of Example-2.

$$c = 1, h = 0.5, s = 1, r = 0, a = 102,$$
$$P_l = 1.6, P_u = 4, p_m = 2.8, \hat{p} = 1.5.$$

b	$\varepsilon_0 =$ $m =$	0 0	40 2	40 4	40 6	40 8
25	$p^* =$	3.940	3.801	3.703	3.614	3.533
	$u^* =$	73.500	87.964	92.012	95.465	98.436
	$E[\Pi(p^*, u^*)] =$	216.090	188.401	182.972	178.021	173.482
35	$p^* =$	3.357	3.272	3.219	3.170	3.123
	$u^* =$	82.500	94.060	96.767	99.204	101.409
	$E[\Pi(p^*, u^*)] =$	194.464	170.411	167.285	164.355	161.603
45	$p^* =$	3.033	2.973	2.939	2.907	2.877
	$u^* =$	91.500	101.516	103.564	105.462	107.225
	$E[\Pi(p^*, u^*)] =$	186.050	163.783	161.674	159.667	157.753
55	$p^* =$	2.827	2.780	2.757	2.734	2.712
	$u^* =$	100.500	109.550	111.217	112.788	114.272
	$E[\Pi(p^*, u^*)] =$	183.641	162.436	160.870	159.366	157.919

Table 4.5: Results of Example-2.

$$c = 1, h = 0.5, s = 1, r = 0, a = 102,$$
$$P_\ell = 1.6, P_u = 4, p_m = 2.8, \hat{p} = 1.5.$$

4.3 Example 3

In this case we allow for an additional set-up cost ($\mathcal{K} = 3$) and an initial stock ($r = 100$) for the problem defined in Example-1.

Since $M(u,p) = -E[\Pi(p,u)]$ and the presence of initial stock does not alter the optimal decisions for the problem (3.30), we can use Table-4.1 to obtain $(\overset{o}{u}, \overset{o}{p}_u)$ values directly. It remains to compute $\overset{o}{p}_r$ from (3.44) in order to run the procedure (3.45). The certainty values are computed by solving (3.22) where we place a set-up cost in the function (3.21).

The numerical results we get for our problem are displayed in Table-4.6. The class $\star$ indicate the optimal procurement values which turned out to be less than the initial stock. Therefore, we do not order but we set a price which maximizes the expected profit at a beginning stock of 100. For the class $\diamond$ we have optimal procurement decisions being greater than the initial stock. However, when we add $\mathcal{K} = 3$ to $M(\overset{o}{u}, \overset{o}{p}_u)$ we obtain larger values than $M(r, \overset{o}{p}_r)$. For that reason we do not order but we set the price to $\overset{o}{p}_r$. Finally, for the class $\bullet$ we order up to $\overset{o}{u}$ and set the price to $\overset{o}{p}_u$.

For $s = 0, h = 0$ and no set-up cost case Mills [3] showed that the optimal price can not be higher than the certainty price, p_c^*. This behaviour is also observed for our example problems where s and h are nonzero. See Table-4.1 through Table-4.5. However, we see in Table-4.6 that the presence of a fixed set-up cost violates this conclusion. Also, we have stated for Example-1 that the optimal price slightly declines as uncertainty in the problem is increased. But, it is seen in Table-4.6 that this is not true for the *do not order* case.

Finally, in Example-1 we have concluded that uncertainty favours holding more stocks than the certainty stock u_c^*. However, this conclusion is also violated by the presence of a fixed set-up cost. See for instance, u^* values for $b = 55$ in Table-4.6.

b	$\lambda =$ / $\sigma_\epsilon =$	0 / 0	17.32 / 10	34.64 / 20	51.96 / 30	69.28 / 40
25			⋆	⋆	⋆	◇
	$\overset{o}{p}_u=$	- - -	3.913	3.886	3.859	3.830
	$\overset{o}{p}_r=$	- - -	3.434	3.695	3.835	3.904
	$\overset{o}{u}=$	- - -	81.887	90.190	98.406	106.531
	$M(\overset{o}{u},\overset{o}{p}_u) =$	- - -	-297.291	-278.528	-259.802	-241.113
	$M(r,\overset{o}{p}_r) =$	- - -	-288.057	-276.397	-259.755	-240.452
	$p^* =$	3.190	3.434	3.695	3.835	3.904
	$u^* =$	100.000	100	100	100	100
	$E_f[\Pi(p^*,u^*)] =$	290.403	288.057	276.397	259.755	240.452
35			⋆	⋆	◇	◇
	$\overset{o}{p}_u=$	- - -	3.333	3.309	3.284	3.259
	$\overset{o}{p}_r=$	- - -	3.125	3.265	3.339	3.373
	$\overset{o}{u}=$	- - -	89.904	97.216	104.432	111.547
	$M(\overset{o}{u},\overset{o}{p}_u) =$	- - -	-276.527	-258.630	-240.775	-222.962
	$M(r,\overset{o}{p}_r) =$	- - -	-274.302	-258.492	-240.479	-221.231
	$p^* =$	2.857	3.125	3.265	3.339	3.373
	$u^* =$	100.000	100	100	100	100
	$E_f[\Pi(p^*,u^*)] =$	285.714	274.302	258.492	240.479	221.231
45			⋆	◇	◇	•
	$\overset{o}{p}_u=$	- - -	3.012	2.990	2.968	2.946
	$\overset{o}{p}_r=$	- - -	2.982	3.057	3.095	3.106
	$\overset{o}{u}=$	- - -	98.261	104.930	111.502	117.973
	$M(\overset{o}{u},\overset{o}{p}_u) =$	- - -	-268.686	-251.364	-234.084	-216.848
	$M(r,\overset{o}{p}_r) =$	- - -	-268.632	-250.999	-232.371	-213.185
	$p^* =$	2.844	2.982	3.057	3.095	2.946
	$u^* =$	100.000	100	100	100	117.973
	$E_f[\Pi(p^*,u^*)] =$	284.444	268.632	250.999	232.371	213.848
55			◇	◇	•	•
	$\overset{o}{p}_u=$	- - -	2.808	2.789	2.769	2.749
	$\overset{o}{p}_r=$	- - -	2.907	2.944	2.958	2.956
	$\overset{o}{u}=$	- - -	106.809	113.028	119.153	125.180
	$M(\overset{o}{u},\overset{o}{p}_u) =$	- - -	-266.686	-249.772	-232.900	-216.070
	$M(r,\overset{o}{p}_r) =$	- - -	-265.982	-247.557	-228.709	-209.646
	$p^* =$	2.836	2.907	2.944	2.769	2.749
	$u^* =$	100.000	100	100	119.153	125.180
	$E_f[\Pi(p^*,u^*)] =$	283.636	265.982	247.557	229.900	213.0700

Table 4.6: Results of Example-3.

$$c = 1, \mathcal{K} = 3, h = 0.5, s = 1, r = 100, a = 102,$$
$$P_\ell = 1.6, P_u = 4, p_m = 2.8.$$

Chapter 5

SUMMARY AND CONCLUSIONS

Upon setting our assumptions, we derived a mathematical expression for the expected profit in terms of the decision variables. In order to make it attractive for theoretical analyses we studied this function in detail. Then, we formulated the optimization problem to determine the optimal decisions for maximizing the expected profit. By employing functional analyses we proposed a general solution procedure for the problem. We supplied numerical evidence for the applicability of this procedure by solving three different examples.

We also discussed the certainty profit. In Example-1, we proved that in general it constitutes an upper bound for the uncertainty profit. For $r = 0$ and linear $\bar{X}(p)$ we obtained the optimal procurement and pricing decisions.

For an additional set-up cost, we showed that we can utilize an (s,S) type policy to obtain the optimal decisions. This policy is related to the classical (s,S) policy in a sense that we determine the optimal procurement by operating over the curve $M(u, p_u)$. Moreover, we determine the optimal price as a by-product.

We believe that, with this theoretical study we explain major issues related to the one-period problem. This was our intention to start with so that, we would have a basis to extend the theory. An essential feature of the study is the proof of optimality of an (s,S) type policy. This has useful implications for the multi-period case. Yet, another feature was the analytical approach to handling the price dependence.

Appendix A

First consider the following two integrals

$$
\begin{aligned}
\int_{\epsilon_1(p)}^{u-\bar{X}(p)} [u-\bar{X}(p)-y]\cdot g(y;p)\cdot dy &= [u-\bar{X}(p)-y]\cdot G(y;p)|_{\epsilon_1(p)}^{u-\bar{X}(p)} \\
&\quad + \int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy \\
&= \int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy, \qquad \text{(A.1)}
\end{aligned}
$$

and,

$$
\begin{aligned}
\int_{u-\bar{X}(p)}^{\epsilon_2(p)} [y-u+\bar{X}(p)]\cdot g(y;p)\cdot dy &= \int_{\epsilon_1(p)}^{\epsilon_2(p)} [y-u+\bar{X}(p)]\cdot g(y;p)\cdot dy \\
&\quad - \int_{\epsilon_1(p)}^{u-\bar{X}(p)} [y-u+\bar{X}(p)]\cdot g(y;p)\cdot dy \\
= \int_{\epsilon_1(p)}^{\epsilon_2(p)} y\cdot g(y;p)\cdot dy - u + \bar{X}(p) &+ \int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy \\
&= -u+\bar{X}(p)+\int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy\,. \qquad \text{(A.2)}
\end{aligned}
$$

Therefore, substituting (A.1) and (A.2) in (3.4) we obtain

$$
E[\Pi(p,u)] = -s\cdot\bar{X}(p) + (p+s-c)\cdot u + c\cdot r - (p+s+h)\cdot\int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy\,. \qquad \text{(A.3)}
$$

Appendix B

The $\Theta(u,p)$ function given by (3.6) is an important term which appears in all functional analyses. Therefore, its meaning and behaviour should be studied before going into detailed analyses. Two alternatively meaningful representations of $\Theta(u,p)$ are given below.

$$\Theta(u,p) = \int_{\epsilon_1(p)}^{u-\bar{X}(p)} G(y;p)\cdot dy, \tag{B.1}$$

$$\Theta(u,p) = \int_{\epsilon_1(p)+\bar{X}(p)}^{u} F(x;p)\cdot dx. \tag{B.2}$$

Equation (B.1) involves the forecast error distribution, and equation (B.2) uses the demand distribution induced by (A8). Note that, in (B.1) the lower and upper limits of the integral as well as the integrand may depend on price, whereas only the upper limit is a function of the procurement quantity u. A useful treatment is to see the integrals (B.1) and (B.2) over a plot. See Figure-B.1 below.

It is clear with (B.1) and (B.2) that $\Theta(u,p)$ is positive for all feasible u and p, that is :

$$\Theta(u,p) \geq 0 \ , \ \forall u,p. \tag{B.3}$$

Furthermore, the first derivative of $\Theta(u,p)$ with respect to p and u becomes

$$\frac{\partial\Theta(u,p)}{\partial u} = G(u-\bar{X}(p);p) = F(u;p), \tag{B.4}$$

$$\frac{\partial\Theta(u,p)}{\partial p} = -\frac{\partial\bar{X}(p)}{\partial p}\cdot G(u-\bar{X}(p);p) + \int_{\epsilon_1(p)}^{u-\bar{X}(p)} \frac{\partial G(y;p)}{\partial p}\cdot dy, \tag{B.5}$$

$$= \int_{\epsilon_1(p)+\bar{X}(p)}^{u} \frac{\partial F(x;p)}{\partial p}\cdot dx. \tag{B.6}$$

For a general demand distribution, it is not possible to study the behaviour of the derivative of $\Theta(u,p)$ w.r.t. p. It is evident in (B.6) that, besides the integrand, the limits of the

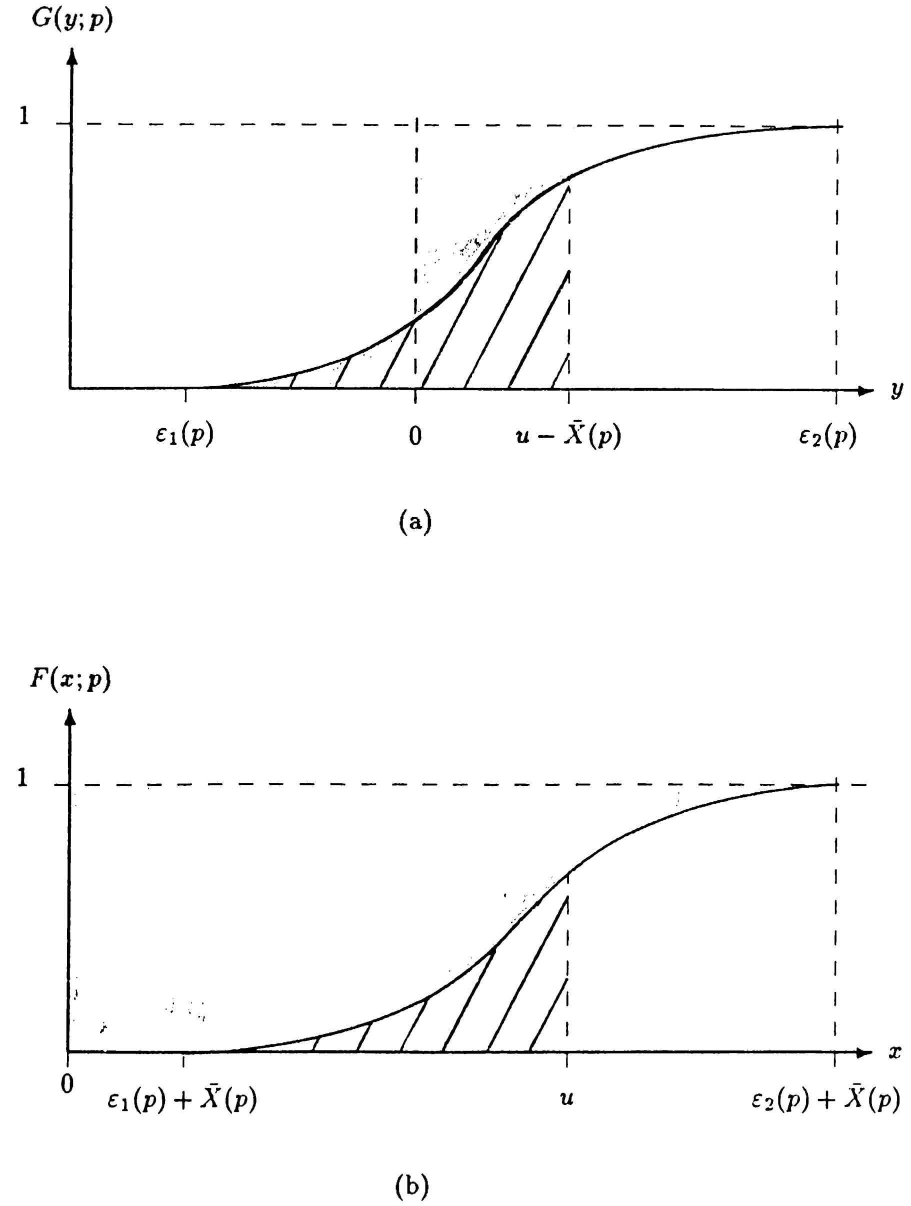

Figure B.1: (a) The shaded area is the expected value of the forecast error which is known to be zero. The cross-hatched region is $\Theta(u,p)$ given by (B.1). (b) The shaded area is the expected value of the demand which is known to be $\bar{X}(p)$. The cross-hatched region is $\Theta(u,p)$ given by (B.2).

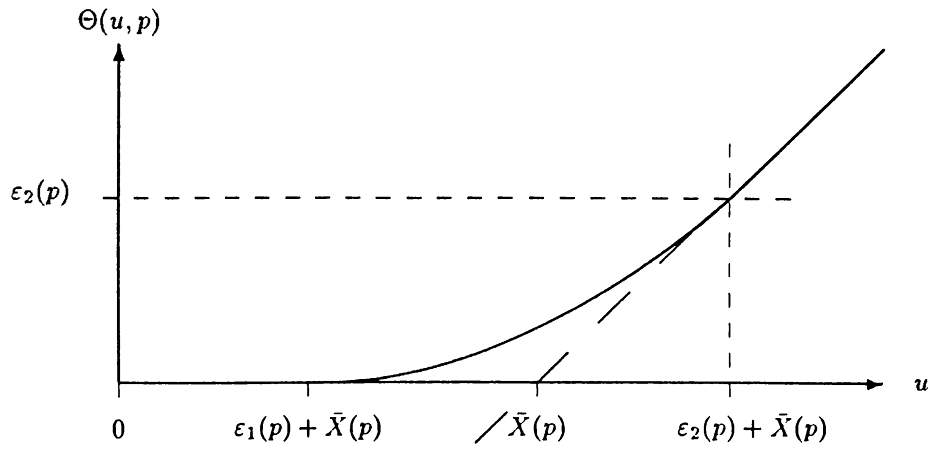

Figure B.2: A General $\Theta(u,p)$ Function in u.

integration plays a role in this behaviour. This is the mojor drawback in making a theoretical generalization for the overall problem. Nevertheless, we still can make some assertions for the $\Theta(u,p)$ function :

Utilizing the information supplied by (B.4) we obtain

$$\frac{\partial\Theta(u,p)}{\partial u}=0 \qquad \text{for} \quad u\le\varepsilon_1(p)+\bar{X}(p),$$

$$0<\frac{\partial\Theta(u,p)}{\partial u}<1 \qquad \text{for} \quad \varepsilon_1(p)+\bar{X}(p)<u<\varepsilon_2(p)+\bar{X}(p),$$

$$\frac{\partial\Theta(u,p)}{\partial u}=1 \qquad \text{for} \quad \varepsilon_2(p)+\bar{X}(p)\le u.$$

Using (B.1) and considering Figure-B.1.(a) we have

$$\int_{\varepsilon_1(p)}^{\varepsilon_2(p)} G(y;p)\cdot dy=\varepsilon_2(p)=\Theta(\varepsilon_2(p)+\bar{X}(p),p).$$

Again from (B.1) it follows that

$$\Theta(u,p)=0 \qquad \text{for} \quad u\le\varepsilon_1(p)+\bar{X}(p),$$

$$\Theta(u,p)=u-\bar{X}(p) \qquad \text{for} \quad \varepsilon_2(p)+\bar{X}(p)\le u.$$

Therefore, we can plot a general $\Theta(u,p)$ function w.r.t. u as in Figure-B.2. Considering the figure we have

$$\Theta(u,p)\ge\ u-\bar{X}(p)\quad,\forall u,p, \tag{B.7}$$

$$\Theta(u,p) \leq \quad u \quad ,\forall u,p, \tag{B.8}$$

$$\Theta(u,p) = 0 \quad ,\forall p, \quad \text{and} \quad 0 \leq u \leq \varepsilon_1(p) + \bar{X}(p), \tag{B.9}$$

$$\Theta(u,p) \leq u - \varepsilon_1(p) - \bar{X}(p) \quad ,\forall p, \quad \text{and} \quad u \geq \varepsilon_1(p) + \bar{X}(p), \tag{B.10}$$

$$\Theta(u,p) = u - \bar{X}(p) \quad ,\forall p, \quad \text{and} \quad u \geq \varepsilon_2(p) + \bar{X}(p), \tag{B.11}$$

$$\Theta(u,p) \leq \frac{[u - \varepsilon_1(p) - \bar{X}(p)]\cdot\varepsilon_2(p)}{\varepsilon_2(p) - \varepsilon_1(p)} \quad , \begin{cases} \forall p, & \text{and} \\ \varepsilon_1(p) + \bar{X}(p) \leq u \leq \varepsilon_2(p) + \bar{X}(p), \end{cases} \tag{B.12}$$

$$\Theta(u,p) \geq G(0;p)\cdot[u - \bar{X}(p)] + \int_{\varepsilon_1(p)}^{0} G(y;p)\cdot dy \quad ,\forall p,u. \tag{B.13}$$

Consequently, we can say that $\Theta(u,p)$ is a convex function in u, but it has a general behaviour in p. Given a special problem we can utilize (B.2) and (B.6) to study the price dependence of $\Theta(u,p)$. The conditions (B.7) through (B.13) are essential in attempts for generalizing the results involving $\Theta(u,p)$.

Appendix C

Utilizing (A8) and substituting $y = x - \bar{X}(p)$, we can rewrite (3.26) as

$$L(u,p) = \quad h \cdot \int_{\epsilon_1(p)}^{u-\bar{X}(p)} [u - \bar{X}(p) - y] \cdot g(y;p) \cdot dy$$

$$+(p+s) \cdot \int_{u-\bar{X}(p)}^{\epsilon_2(p)} [y - u + \bar{X}(p)] \cdot g(y;p) \cdot dy \tag{C.1}$$

which becomes

$$L(u,p) = \quad h \cdot \int_{\epsilon_1(p)}^{u-X(p)} G(y;p) \cdot dy + (p+s) \cdot [X(p) - u]$$

$$+(p+s) \cdot \int_{\epsilon_1(p)}^{u-X(p)} G(y;p) \cdot dy \tag{C.2}$$

by using (A.1) and (A.2) in (C.1). Therefore, from (3.6) we obtain

$$L(u,p) = (p+s) \cdot [\bar{X}(p) - u] + (p+s+h) \cdot \Theta(u,p). \tag{C.3}$$

Appendix D

Using (4.2) in (4.9) we obtain

$$\frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} = -2\cdot b + \frac{2\cdot\lambda\cdot(c+h)^2}{(p+s+h)^3}, \tag{D.1}$$

and it follows that

$$\frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} < 0 \iff p > p^-$$

where

$$p^- = \left[\frac{\lambda}{b}\cdot(c+h)^2\right]^{1/3} - (s+h).$$

Therefore, $E[\Pi(p,u^*)]$ (which is given by (4.7)) is found to be convex for $0 < p < p^-$ and concave for $p > p^-$.

From (4.8) we obtain $p = \overset{o}{p}$ such that,

$$\frac{\partial E[\Pi(p,u^*)]}{\partial p}\Big|_{p=\overset{o}{p}} = a - b\cdot(\overset{o}{p} - p_m) - b\cdot(\overset{o}{p} - c) - \frac{\lambda\cdot(c+h)^2}{(\overset{o}{p} + s + h)^2} = 0 \tag{D.2}$$

where $\overset{o}{p}$ is the largest root of (D.2).

Consequently, we determine the minimizer p^* of $E[\Pi(p,u^*)]$ from the following procedure:

(i) if $p^- < 0$ then set $p^- = 0$.

(ii) if $p^- < P_\ell$ then :

- if $P_\ell < \overset{o}{p} < P_u$ then set $p^* = \overset{o}{p}$,
- ∘ else if $E[\Pi(P_\ell, u^*)] > E[\Pi(P_u, u^*)]$ then set $p^* = P_\ell$ else set $p^* = P_u$.

(iii) if $P_\ell < p^- < P_u$ then :

- if $p^- < \overset{o}{p} < P_u$ then set $p_1 = \overset{o}{p}$,
- ∘ else if $E[\Pi(p^-, u^*)] > E[\Pi(P_u, u^*)]$ then set $p_1 = p^-$ else set $p_1 = P_u$.
- if $E[\Pi(P_\ell, u^*)] > E[\Pi(p_1, u^*)]$ then set $p^* = P_\ell$ else set $p^* = p_1$.

(iv) if $P_u < p^-$ then :

- if $E[\Pi(P_\ell, u^*)] > E[\Pi(P_u, u^*)]$ then set $p^* = P_\ell$ else set $p^* = P_u$.

Appendix E

This material will closely follow Appendix-D.

From (4.20) we have

$$\frac{\partial^2 E[\Pi(p,u^*)]}{\partial p^2} < 0 \iff p > p^-$$

where

$$p^- = \left[\frac{m\cdot(c+h)^2\cdot[(s+h)\cdot(2\cdot\hat{p}+s+h)+\hat{p}+\varepsilon_0/m]}{2\cdot b+m\cdot(c+h)}\right]^{1/3} - (s+h)$$

Therefore, $E[\Pi(p,u^*)]$ is found to be convex for $0 < p < p^-$,and concave for $p > p^-$. Taking this into account and by equation (4.19) we obtain $\mathring{p}$ from

$$\begin{aligned} \frac{\partial E[\Pi(p,u^*)]}{\partial p}\Big|_{p=\mathring{p}} = {} & \hat{X}(\mathring{p}) - b\cdot(\mathring{p}-c) - m\cdot(p-\hat{p})\cdot\frac{(c+h)\cdot(\mathring{p}+s-c)}{(\mathring{p}+s+h)} \\ & -\tfrac{1}{2}\cdot(m\cdot(p-\hat{p})^2+\varepsilon_0)\cdot\frac{(c+h)^2}{(\mathring{p}+s+h)^2} = 0 \end{aligned} \qquad \text{(E.1)}$$

where $\mathring{p}$ is the largest root of (E.1).

Consequently, p^* can be obtained by the same procedure described in Appendix-D. Note that $E[\Pi(p,u^*)]$ is given by (4.18) in this case.

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