ESSAYS ON
ENDOGENOUS TIME PREFERENCE
AND
STRATEGIC INTERACTION

A Ph. D. Dissertation

by
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September 2013

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To Orhun and Damla

ESSAYS ON ENDOGENOUS TIME PREFERENCE AND STRATEGIC INTERACTION

The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

AGAH REHA TURAN

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DOCTOR OF PHILOSOPHY

in

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September 2013

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ABSTRACT

ESSAYS ON ENDOGENOUS TIME PREFERENCE and STRATEGIC INTERACTION

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This thesis includes three self contained essays on the existence and qualitative properties of equilibrium dynamics under endogenous time preference. In the first essay, we reconsider the optimal growth model proposed by Stern (2006). We prove the almost everywhere differentiability of the value function and uniqueness of the optimal path, which were left as open questions and show how a small perturbation to the price of future oriented capital qualitatively changes the equilibrium dynamics. Almost none of the studies on endogenous time preference consider the strategic interaction among the agents. In the second essay, by considering a strategic growth model with endogenous time preference, we provide the sufficient conditions of supermodularity for dynamic games with open-loop strategies and show that the stationary state

Nash equilibria tend to be symmetric. We numerically show that the initially rich can pull the poor out of poverty trap even when sustaining a higher level of steady state capital stock for itself. Lastly, in the third essay, we consider the socially determined time preference which depends on the level of fish stock and characterize the basic fishery model under this setup. We provide existence of collusive and open-loop Nash equilibria and compare the efficiency and qualitative properties of them.

Keywords: Endogenous Time Preference, Supermodular Games, Lattice Programming, Dynamic Resource Games

ÖZET

ENDOJEN ZAMAN TERCİHİ ve STRATEJİK ETKİLEŞİM ÜZERİNE MAKALELER

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Bu tez endojen zaman tercihi modellerinde denge dinamiklerinin varlığı ve özelliklerinin çalışıldığı üç ayrı makale içermektedir. İlk makalede, Stern (2006) tarafından ortaya konan optimal büyüme modeli yeniden ele alınarak, bu makalede cevaplanmak üzere bırakılan değer fonksiyonun hemen her yerde türevinin alınabilirliği ve optimal yolun hemen her yerde tek olması hususları ispat edilmiştir. Ayrıca, geleceğe odaklı sermayenin fiyatındaki küçük değişikliklerin denge dinamikleri üzerindeki niteliksel etkisi gösterilmiştir.

Zaman tercihinin endojen olduğu çalışmaların hemen hemen hiçbirinde aktörler arasındaki stratejik etkileşim göz önüne alınmamaktadır. İkinci makalede, endojen zaman tercihli stratejik büyüme modeli altında, aktörlerin birbirlerinin stratejilerine kendi stratejilerini artırarak cevap verdikleri Süpermoduler Oyunların açık döngülü bilgi yapısı altında yeter şartları ortaya konmuş, bu oyun yapısı altında başlangıç şartlarındaki farklılıkların uzun dönem içerisinde ortadan kaybolduğu ispatlanmıştır. Ayrıca, sayısal olarak, tek başına ele alındıklarında yoksulluk kapanına takılacak aktörlerin, başlangıç koşulları açısından daha zengin bir aktörle stratejik etkileşime girdiklerinde bu kapandan kurtulabildikleri gösterilmiştir. Son olarak, üçüncü makalede mülkiyet haklarının tanımlı olmadığı bir ortamda, aktörlerin zaman tercihlerinin ekonomideki kaynaklarla belirlendiği durumda işbirliği dengesinin ve açık döngülü bilgi yapısı altında işbirliğinin olmadığı Nash dengesinin varlığı gösterilmiş, bu Nash dengesinin etkinliği ve işbirliği dengesinden niteliksel farklılıkları incelenmiştir.

Anahtar Kelimeler: Endojen Zaman Tercihi, Süpermoduler Oyunlar, Latis Programlama, Dinamik Kaynak Oyunları

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CHAPTER 1

INTRODUCTION

Poverty trap is a self–perpetuating condition where poverty is its own cause. Many different feedback mechanisms from demography to the lack of financial development, from the non-convexities in technology caused by externalities and fixed costs to social norms are highlighted as the sources of the vicious cycle that an economy is trapped in. (For an recent survey, see Azariadis and Stachurski, 2005.)

The main message of the poverty trap literature is that a long term performance of an economy may depend on initial conditions, by suggesting that long run performance of an economy could be much better if its initial condition were better. (Matsuyama, 2008) Having said that initial conditions are not the only factor behind cross country income differences, in this literature it is demonstrated that the initially underendowed economies may lag permanently behind the otherwise identical economies. The dependence on initial conditions are shown by the emergence of threshold dynamics according to which the economies with low initial capital stock or income converge to a steady state with low per capita income, while economies with high initial capital stock or income converge to a steady state with high per capita income.

A general tendency in these studies is assuming that an agent's period utility is discounted with a constant rate. In this thesis, we depart from this assumption and study the implications of endogenous time preference on threshold dynamics. Following the empirically supported assumption (see Lawrence, 1991, and Samwick, 1998) in recent theoretical studies we let the rate of time preference decrease with the level of wealth, i.e., the rich are more patient than the poor. We assume first that an economy admits a representative household (chapter 2) and study the implications of the endogenous time preference on equilibrium dynamics. Then we analyze how the results under representative agent framework would change, if we consider the conflicting interests of the agents (chapter 3 and 4).

Imperfect ability of people to imagine the future can be ameliorated by spending resources. These resources range from time and efforts that increase the anticipation of future to goods that support or enforce considering future benefits. This idea has been formally introduced by Becker and Mulligan (1997) in a finite horizon model where the discount function depends on allocated resources called "future oriented capital". Stern (2006) uses this idea under the classic optimal growth model and provides numerical examples in which multiple steady states and a conditionally sustained growth path may occur.

In chapter two, we extend the analysis provided by Stern (2006). Stern's effort to adapt the classical optimal growth framework to include endogenous discounting provides us a more flexible framework regarding the discounting of time, while maintaining time consistency. For that reason it is important to provide a comprehensive analysis of this model.

In a standard optimal growth model with geometric discounting and the usual concavity assumptions on preferences and technology, the optimal policy correspondence is single valued and the properties of the optimal path are easily found by using the first order conditions together with envelope theorem by differentiating the value function. However, in endogenous discounting models, the objective function includes multiplication of a discount function which generally destroys the concavity of value function. Under non-convex technology, Amir, Mirman, and Perkins (1991) were able to deal with this situation by employing lattice programming techniques. By finding a partial order that will turn budget sets into lattice spaces, Stern (2006) utilizes this technique to show monotonicity and convergence result of any optimal path.

We prove the convergence of the optimal path of future oriented capital and consumption rather than just the optimal path of capital and provide conditions under which the system does not converge to zero. Moreover, we prove the almost everywhere differentiability of the value function, and the almost everywhere uniqueness of the optimal path, which were left as open questions in Stern (2006).

The price of the future oriented capital is assumed to be constant in Becker and Mulligan (1997) and Stern (2006) transferred it into the optimal growth model by defining a parameter, that merely act as a price that converts future oriented capital stock to the units of consumption and capital goods. Since the resources to increase appreciation of the future cannot be used in production, any factors related to either the cost of producing future oriented capital or efficiency in it affect this parameter. However, the implications of this parameter on the dynamic behavior of the system are left unexplored.

We show how a small perturbation to the price of future oriented capital qualitatively changes the equilibrium dynamics. We demonstrate the occurrence of a saddle-node bifurcation with respect to the price of future oriented capital stock. By using the same functional forms and the parameter set as Stern (2006) did while giving an example to multiple steady states and divergence, we show that only by changing the value of cost of future oriented capital, one can also obtain global convergence.

There are cases such that more than one player can and does manipulate the system for his own benefit, their interests don't always coincide and no single player has an exclusive control over the turn of events (Clemhout & Wan, 1979). In such cases, one decision maker assumption as a first approximation to reality may not be applicable i.e. the analyses assuming that the agents are isolated may not be robust to the considerations of strategic interactions among agents in the economy.

In Chapter three, in line with the Erol et. al. (2011) we let the discount factor be increasing in the stock of wealth and analyze to what extent the strategic complementarity inherent in agents' strategies can alter the non-convergence results being found under a single agent optimal growth model.

We adopt the non-cooperative open loop Nash equilibrium concept, in which players choose their strategies as simple time functions and they are able to commit themselves to time paths as equilibrium strategies. In this setup, agents choose their strategies simultaneously and face with a single criterion optimization problem constrained by the strategies of the rival taken as given. We focus on the qualitative properties of the open-loop Nash equilibria and the dynamic implications of the strategic interaction.

Due to the potential lack of both concavity and the differentiability of the value functions associated with each agent's problem, topological arguments cannot be used while proving the existence of Nash equilibria and characterizing their properties. Instead, we employ the theory of monotone comparative statics and the supermodular games based on order and monotonicity properties on lattices (see Topkis, 1998).

In this chapter, we first provide the sufficient conditions of supermodularity for dynamic games with open-loop strategies based on two fundamental elements: the ability to order elements in the strategy space of the agents and the strategic complementarity which implies upward sloping best responses. The supermodular game structure in our model enables us provide the existence and the monotonicity results on the greatest and the least equilibria. We sharpen these results by showing the differentiability of the value function and the uniqueness of the best response correspondences almost everywhere.

The supermodular games are characterized with a specific property: as one player selects higher strategies, the other players do as well. Hence interactions dominated by complementarities provide agents with an incentive to follow the behavior of the others. The key feature of our analysis is that under strategic complementarities, the initial differences tend to vanish and the stationary state Nash equilibria tend to be symmetric under open-loop strategies. We show that the initially rich can pull the poor out of poverty trap even when sustaining a higher level of steady state capital stock for itself.

In chapter four, we consider the hypothesis that while the time preference is one of the major factors on allocation of the resources, these resources can also affect the society's time preference.

There is a new but growing literature considering the dependence of time preference to the aggregate variables while studying extinction and exploitation of renewable resources. However, none of these studies investigate the implications of endogenous discounting under strategic interaction.

The fishery model has been used as a metaphor for any kind of renewable resource on which the property rights are not well defined. (see Long, 2010, for a comprehensive survey) In these models, the set of feasible strategies available to the players are interdependent and in addition, the agents' choices in the current period affect the payoffs and their choice sets in the future. We let the socially determined time preference depend on the level of resource stock and characterize the basic fishery model.

By using a discrete time formulation, we study the existence and the efficiency of the open loop Nash equilibrium (OLNE). We show first that unlike constant discounting, we cannot rely on symmetric social planner problem while showing the existence and the qualitative properties of Nash equilibrium. Instead, we use a topological fixed point theorem to show existence of OLNE.

We prove that OLNE may result in overexploitation or under exploitation of the resources relative to efficient solution depending on the return is bounded or unbounded from below.

The OLNE differs from the collusive equilibria in terms of not only efficiency but also equilibrium dynamics. We show that open loop information structure can remove indeterminacy that we may face under collusive equilibrium and be a source of multiplicity despite the uniqueness we may face under collusive equilibrium.

CHAPTER 2

SADDLE-NODE BIFURCATION IN OPTIMAL GROWTH MODELS A LA BECKER-MULLIGAN

Based on the following observations, Becker & Mulligan (1997) formally introduced a finite horizon model where the discount function depends on allocated resources which they called "future oriented capital".

- People are not all equally patient.
- Many of the differences among people are explainable: Patience seems to be associated with income, development, and education.
- Heavy discounting of the future is viewed by many people to be inappropriate, undesirable, or even "irrational".
- People are often aware of their weaknesses and may spend resources to overcome them. These resources could be time and efforts that increase the anticipation of future or goods that support or enforce considering future benefits.

Stern (2006) use this idea under the classic optimal growth model by providing two interpretations:

- (A dynasty) The discount factor is the degree of intergenerational altruism: The future oriented capital stock represents actions that the parent could take in order to strengthen the relationship with his child.

- (Single individual with an infinite lifetime) The discount factor represents the degree to which the individual appreciates future utility when making current decisions: Education, religion, self-discipline as well as time spent on imagining future utilities can effect discount factor.

The price of the future oriented capital is assumed to be constant in Becker and Mulligan (1997). Stern (2006) transferred it into the optimal growth model by defining a parameter, that merely act as a price that converts future oriented capital stock to the units of consumption and capital goods. Since the resources to increase appreciation of the future cannot be used in production, any factors related to either the cost of producing future oriented capital or efficiency in it affect this parameter. However, the implications of this parameter on the dynamic behavior of the system are left unexplored.

We show how a small perturbation to the price of future oriented capital qualitatively changes the equilibrium dynamics. In particular, we demonstrate the occurrence of a saddle-node bifurcation with respect to the price of future oriented capital stock. We use the same functional forms and the parameter set as Stern (2006) did while giving an example for the multiple steady states and divergence and show that, with the same functional forms, only by changing the value of cost of future oriented capital, one can also obtain global convergence.

Stern's effort to adapt the classical optimal growth framework to include endogenous discounting provide us a more flexible framework in regards to the discounting of time, while maintaining time consistency. For that reason it is important to provide a comprehensive analysis of this model. We contribute to this effort by proving the almost everywhere differentiability of the value function, and the almost everywhere uniqueness of the optimal path, which were left as open questions in Stern (2006). The monotonicity and convergence results of any optimal path of capital are available in Stern (2006). We prove the convergence of the optimal path of future oriented capital and consumption rather than just the optimal path of capital and provide conditions under which the system does not converge to zero.

This chapter is organized as follows. The next section introduces the model and provides the dynamic properties of optimal paths. Section 3 discusses the relation between the relative cost of future oriented capital and the long run equilibrium.

2.1 The Model

The model differs from the classic optimal growth model by the assumption on discounting. We assume that the discount rate depends on the future oriented capital stock. The amount of resources allocated to increase the appreciation of the future in period t will be denoted with the control variable s_t. The discount on the future in period t will be a real valued function β of s_t. We assume that s_t will cost the planner an amount πs_t in terms of current resources. The parameter π merely acts as a price that converts future oriented capital stock to the units of consumption and capital goods. Since the resources to increase appreciation of the future cannot be used in production, it is strictly positive. Any factors related to either the cost of producing future oriented capital or efficiency in it affect this parameter.

Formally, the model is stated as follows:

$$\max_{\{c_{t+1},s_{t+1}\}_{t=0}^{\infty}} \sum_{t=1}^{\infty} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t), \tag{1}$$

subject to

$$\forall t,\ c_{t+1} + x_{t+1} + \pi s_{t+1} \leq f(x_t), \tag{2}$$

$$\forall t,\ c_{t+1} \geq 0,\ x_{t+1} \geq 0, s_{t+1} \geq 0,$$

$$x_0 \geq 0,\ \text{given.}$$

We make the following assumptions regarding the properties of the discount, utility and the production functions.

Assumption 2.1 $u : \mathbb{R}_+ \to \mathbb{R}_+$ *is a continuously differentiable, strictly concave, strictly increasing function that satisfies* $u(0) = 0$ *and the Inada condition* $u'(0) = \infty$.

Assumption 2.2 $f : \mathbb{R}_+ \to \mathbb{R}_+$*is a continuously differentiable, strictly increasing function that satisfies* $f(0) = 0$. *Moreover there exists* x_m *such that* $f(x) < x$ *for any* $x > x_m$.

Assumption 2.3 $\beta : \mathbb{R}_+ \to \mathbb{R}_{++}$ *is a continuously differentiable, concave, strictly increasing function satisfying* $\beta(\frac{x_m}{\pi}) < 1$, $\beta(0) = 0$ *and* $\beta'(0) = +\infty$.

Assumption 2.4 $\pi \in \mathbb{R}_{++}$.

Stern (2006) assumes the complete depreciation of the future oriented capital stock. This assumption allows us to represent future oriented capital stock with a control variable instead of additional state variable and remove the additional complexity the latter would bring. This assumption is more in line with the dynastic family interpretation than with the infinitely-lived single individual interpretation where s_t can be viewed as a parental investment in the relationship with child. Strictly increasingness of the discount function assures that the more future oriented capital stock we allocate the greater current appreciation of the future we have and concavity of it promises a diminishing return to investment in s_t.

2.2 Existence of Optimal Paths, Euler Equations

For any level of total capital k,

$$\gamma(k) := \{(c, s, x) : c + \pi s + x \leq f(k),\ c \geq 0,\ s \geq 0,\ x \geq 0\}$$

For any initial level of total capital $x_0 \geq 0$, we say that $(\mathbf{c}, \mathbf{s}, \mathbf{x}) = (x_0, c_1, s_1, x_1, ...)$ is feasible from x_0, if $(c_{t+1}, s_{t+1}, x_{t+1}) \in \gamma(x_t)$, for all t. We denote the set of all feasible sequences from $x_{0,}$by $\Pi(x_0)$. For a feasible sequence $(\mathbf{c}, \mathbf{s}, \mathbf{x})$ from x_0, we denote the total discounted utility by

$$U(\mathbf{c}, \mathbf{s}, \mathbf{x}) := \sum_{t=1}^{\infty} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t).$$

We say that $(\mathbf{c}, \mathbf{s}, \mathbf{x})$ is an optimal path from x_0, if $(c, s, x) \in \Pi(x_0)$ and $U(c, s, x) \geq U(c', s', x')$ for any $(\mathbf{c}', \mathbf{s}', \mathbf{x}') \in \Pi(x_0)$. Due to the existence of a maximum level of sustainable capital stock, any feasible capital path x is bounded from above by a finite number $A(x_0)$ depending on the initial capital x_0. Therefore $\Pi(x_0)$ is compact and the continuous function U attains its maximum on $\Pi(x_0)$ at the optimal path.

Let the maximum value of U on $\Pi(x_0)$ be called $V(x_0)$, where V indeed denotes the value function. Formally

$$V(x_0) := \max\{U(\mathbf{c},\mathbf{s},\mathbf{x}) : (\mathbf{c},\mathbf{s},\mathbf{x}) \in \Pi(x_0)\}.$$

The following proposition yields some important properties of the optimal paths.

Proposition 2.1 *If* $(\mathbf{c},\mathbf{s},\mathbf{x})$ *is an optimal path from* x_0*, Then*

(i)

$$c_{t+1} + \pi s_{t+1} + x_{t+1} = f(x_t), \forall t. \tag{3}$$

Also, if $x_0 > 0$*,*

(ii)

$$c_{t+1} > 0,\ \ s_{t+1} > 0, x_{t+1} > 0,\ \forall t. \tag{4}$$

and

(iii) (Euler-1)

$$\pi u'(c_{t+1}) = \beta'(s_{t+1})V(x_{t+1}), \forall t. \tag{5}$$

(iv) (Euler-2)

$$u'(c_{t+1}) = \beta(s_{t+1})f'(x_{t+1})u'(c_{t+2}),\ \forall t. \tag{6}$$

Proof. (i) Easily follows from the fact that u is increasing.

(ii) First, we will prove that for any t, $x_t > 0$. Assume the contrary. Take the smallest t such that $x_t = 0$ and call it n. Since $x_0 > 0$, we have $x_{n-1} > 0$ for any value of n. This implies that $s_{n-1} > 0$ along the optimal path. Moreover $x_n = 0$ assures that $s_n = 0$ for an optimal path. Hence $c_n = f_{n-1}(x_{n-1})$. Consider

x' such that $x'_n = \varepsilon, c'_n = f_{n-1}(x_{n-1}) - 2\varepsilon,\ s'_n = \varepsilon$, for a sufficiently small ε, and $x_t = x'_t, c_t = c'_t, s_t = s'_t,\ \forall t \neq n$.

We have,

$$U(\mathbf{c}', \mathbf{s}', \mathbf{x}') - U(\mathbf{c}, \mathbf{s}, \mathbf{x}) = \left(\prod_{r=1}^{n-1} \beta(s_r)\right)(u(c_n - 2\varepsilon) + \beta(\frac{\varepsilon}{\pi})V(\varepsilon) - u(c_n)).$$

From Inada Condition on β, for sufficiently small ε, $U(\mathbf{c}', \mathbf{s}', \mathbf{x}') - U(\mathbf{c}, \mathbf{s}, \mathbf{x}) > 0$, which contradicts the optimality of x. Hence, $x_t > 0$, $\forall t$.

Since $x_t > 0$ and $\beta(0) = 0$, we have $s_t > 0, \forall t$.

Now, we will prove that $c_t > 0, \forall t$. Assume the contrary. Clearly zero consumption path after some period can never be optimal, because $x_t > 0$, $\forall t$. Hence, there exists n such that $c_n = 0$, $c_{n+1} > 0$. Consider s'such that $s'_n = s_n - \frac{\varepsilon}{\pi}$, for a sufficiently small ε, and $s_t = s'_t$, $\forall t \neq n$. Let $c'_n = \varepsilon$, $c_t = c'_t$, $\forall t \neq n, t \neq n+1$.

Then, we have:

$$U(\mathbf{c}', \mathbf{s}', \mathbf{x}') - U(\mathbf{c}, \mathbf{s}, \mathbf{x}) = \left(\prod_{r=1}^{n-1} \beta(s_r)\right)\left[u(\varepsilon) + \beta(s_n - \frac{\varepsilon}{\pi})V(x) - \beta(s_n)V(x)\right].$$

From Inada condition on u, along with assumptions on u, f, β, for sufficiently small ε, above expression becomes positive, leading to a contradiction.

(iii) Fix any n. For a sufficiently small $\varepsilon \in \mathbb{R}$, construct $(\mathbf{c}^{\boldsymbol{\varepsilon}}, \mathbf{s}^{\boldsymbol{\varepsilon}}, \mathbf{x}^{\boldsymbol{\varepsilon}})$ as follows:

$c^{\varepsilon}_{n+1} = c_{n+1} - \varepsilon$, $c^{\varepsilon}_{t+1} = c_{t+1}\ \forall t \neq n$,

$s^{\varepsilon}_{n+1} = s_{n+1} + \frac{\varepsilon}{\pi}$, $s^{\varepsilon}_{t+1} = s_{t+1}\ \forall t \neq n$,

$x^{\varepsilon}_{t+1} = x_{t+1}\ \forall t$.

Feasibility of $(\mathbf{c}^{\boldsymbol{\varepsilon}}, \mathbf{s}^{\boldsymbol{\varepsilon}}, \mathbf{x}^{\boldsymbol{\varepsilon}})$ is a result of (ii). Since $(\mathbf{c}, \mathbf{s}, \mathbf{x})$ is optimal, we have

$U(\mathbf{c},\mathbf{s},\mathbf{x}) \geq U(\mathbf{c}^{\boldsymbol{\varepsilon}},\mathbf{s}^{\boldsymbol{\varepsilon}},\mathbf{x}^{\boldsymbol{\varepsilon}})$, $\forall\varepsilon$.

$$
\begin{aligned}
U(\mathbf{c},\mathbf{s},\mathbf{x}) &= \sum_{t=1}^{n}\left(\prod_{r=1}^{t-1}\beta(s_r)\right)u(c_t) + \sum_{t=n+1}^{\infty}\left(\prod_{r=1}^{t-1}\beta(s_r)\right)u(c_t) \\
&= \sum_{t=1}^{n}\left(\prod_{r=1}^{t-1}\beta(s_r)\right)u(c_t) \\
&\quad + \left[\prod_{r=1}^{n}\beta(s_r)\right]\left[u(c_{n+1}) + \beta(s_{n+1})\sum_{t=n+2}^{\infty}\left(\prod_{r=n+2}^{\infty}\beta(s_r)\right)u(c_t)\right] \\
&= \sum_{t=1}^{n}\left(\prod_{r=1}^{t-1}\beta(s_r)\right)u(c_t) + \left[\prod_{r=1}^{n}\beta(s_r)\right]\left[u(c_{n+1}) + \beta(s_{n+1})V(x_{n+1})\right] \\
&\geq U(\mathbf{c}^{\boldsymbol{\varepsilon}},\mathbf{s}^{\boldsymbol{\varepsilon}},\mathbf{x}^{\boldsymbol{\varepsilon}}) \\
&= \sum_{t=1}^{n}\left(\prod_{r=1}^{t-1}\beta(s_r)\right)u(c_t) + \left[\prod_{i=1}^{n}\beta(s_r)\right]\left[u(c_{n+1}-\varepsilon) + \beta(s_{n+1}+\frac{\varepsilon}{\pi})V(x_{n+1})\right]
\end{aligned}
$$

Define $\varpi_1(\varepsilon) := u(c_{n+1}-\varepsilon) + \beta(s_{n+1}+\frac{\varepsilon}{\pi})V(x_{n+1})$. Then,

$$
\varpi_1(0) = u(c_{n+1}) + \beta(s_{n+1})V(x_{n+1}) \geq u(c_{n+1}-\varepsilon) + \beta(s_{n+1}+\frac{\varepsilon}{\pi})V(x_{n+1}) = \varpi_1(\varepsilon),\ \forall\varepsilon,
$$

which implies that $\varpi_1'(0) = 0$. Hence $\pi u'(c_{n+1}) = \beta'(s_{n+1})V(x_{n+1})$.

(iv) Similarly, fix any n. For a sufficiently small $\varepsilon \in \mathbb{R}$, construct $(\mathbf{c}^{\boldsymbol{\varepsilon}},\mathbf{s}^{\boldsymbol{\varepsilon}},\mathbf{x}^{\boldsymbol{\varepsilon}})$ as follows:

$c^{\varepsilon}_{n+1} = c_{n+1}-\varepsilon$, $c^{\varepsilon}_{n+2} = c_{n+2}+f(x_{n+1}+\varepsilon)-f(x_{n+1})$, $c^{\varepsilon}_{t+1} = c_{t+1}\ \forall t \neq n, n+1$,

$s^{\varepsilon}_{t+1} = s_{t+1}\ \forall t$,

$x^{\varepsilon}_{n+1} = x_{n+1}+\varepsilon$, $x^{\varepsilon}_{t+1} = x_{t+1}\ \forall t \neq n$.

Feasibility of $(\mathbf{c}^{\boldsymbol{\varepsilon}},\mathbf{s}^{\boldsymbol{\varepsilon}},\mathbf{x}^{\boldsymbol{\varepsilon}})$ is a result of (ii). Since $(\mathbf{c},\mathbf{s},\mathbf{x})$ is optimal, we have

$U(\mathbf{c}, \mathbf{s}, \mathbf{x}) \geq U(\mathbf{c}^{\boldsymbol{\varepsilon}}, \mathbf{s}^{\boldsymbol{\varepsilon}}, \mathbf{x}^{\boldsymbol{\varepsilon}}), \forall \varepsilon.$

$$
\begin{aligned}
& U(\mathbf{c}, \mathbf{s}, \mathbf{x}) = \sum_{t=1}^{n} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t) + \sum_{t=n+3}^{\infty} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t) \\
& + \left(\prod_{r=1}^{n} \beta(s_r^i) \right) [u(c_{n+1}) + \beta(s_{n+1}) u(c_{n+2})] \\
\geq \; & U(\mathbf{c}^{\boldsymbol{\varepsilon}}, \mathbf{s}^{\boldsymbol{\varepsilon}}, \mathbf{x}^{\boldsymbol{\varepsilon}}) = \sum_{t=1}^{n} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t) + \sum_{t=n+3}^{\infty} \left(\prod_{r=1}^{t-1} \beta(s_r) \right) u(c_t) \\
& + \left(\prod_{i=1}^{n} \beta(s) \right) [u(c_{n+1} - \varepsilon) + \beta(s_{n+1}) u(c_{n+2} + f(x_{n+1} + \varepsilon) - f(x_{n+1}))]
\end{aligned}
$$

Define $\varpi_2(\varepsilon) := u(c_{n+1} - \varepsilon) + \beta(s_{n+1}) u(c_{n+2} + f(x_{n+1} + \varepsilon) - f(x_{n+1}))$. Then,

$$
\begin{aligned}
& \varpi_2(0) = u(c_{n+1}) + \beta(s_{n+1}) u(c_{n+2}) \geq \\
& \qquad u(c_{n+1} - \varepsilon) + \beta(s_{n+1}) u(c_{n+2} + f(x_{n+1} + \varepsilon) - f(x_{n+1})) = \varpi_2(\varepsilon), \ \forall \varepsilon,
\end{aligned}
$$

which implies that $\varpi_2'(0) = 0$. Hence $u'(c_{n+1}) = \beta(s_{n+1}) f'(x_{n+1}) u'(c_{n+2})$. ■

Consider 5. Note that u' and β' are decreasing functions. By (3), $c_{t+1} = f(x_t) - x_{t+1} - s_{t+1}$. Given x_t and x_{t+1}, $u'(f(x_t) - x_{t+1} - s_{t+1})$ increases as we invest more on future-oriented capital s_{t+1}. On the other hand, $\beta'(s_{t+1})$ decreases as we do so. Thus, 5 precisely yields the unique s_{t+1}, given x_t and x_{t+1}. In other words, it allows us to decide how much to share between today's consumption c_{t+1} and the future-oriented capital s_{t+1}, given the path of the capital. Moreover, 6 gives the intertemporal flow of the path.

2.3 Value Function, Bellman Equation, Optimal Policy

We have already defined the value function. It's well defined, non-negative, continuous and strictly increasing. As the recursive structure of the standard optimal growth models is preserved by our model, the satisfaction of Bellman's equation is also straightforward (See Stokey and Lucas, 1989 and Le Van and Dana, 2003):

$$V(x_t) = \max_{\{x_{t+1}\}_{t=0}^{\infty}} \{u(f(x_t) - x_{t+1} - \pi s_{t+1}) + \beta(s_{t+1})V(x_{t+1})\}$$

The optimal policy correspondance, $\mu : R_+ \to R_+$, is defined as follows:

$$\mu(k) := \arg\max_{(c,s,x)} \left\{ \begin{array}{l} u(c) + \beta(s)V(x) \\ \quad \mid c + \pi s + x \in \gamma(k) \end{array} \right\}. \tag{7}$$

Note that, from (3), we can equivalently define the optimal policy correspondence as

$$\mu(k) := \arg\max_{(c,s,x)} \left\{ \begin{array}{l} u(c) + \beta(s)V(x) \mid c + \pi s + x \\ = f(k),\ c \geq 0,\ s \geq 0,\ x \geq 0 \end{array} \right\}.$$

The non-emptiness, upper semi continuity and compact valuedness of the optimal policy correspondence and its equivalance with the optimal path follow easily from the continuity of the value function by a standard application of the theorem of the maximum (see Stern, 2006 and Le Van and Dana, 2003).

In a standard optimal growth model with geometric discounting and the usual concavity assumptions on preferences and technology, the optimal policy correspondence is single valued and the properties of the optimal path is easily found by using the first order conditions together with envelope theorem by differentiating the value function. However, in our model, the objective function includes multiplication of a discount function. This generally destroys the usual concavity argument which is used in the proof of the differentiability of value function and the uniqueness of the optimal paths (see Benveniste and Scheinkman, 1979; Araujo, 1991).

At this point, we need to refer to an important theorem in Stern (2006), concerning the increasingness of the optimal policy correspondence.

Theorem 2.1 (Thm 3.7, Stern 2006) *μ is increasing, i.e. if $k' \geq k$, $(c', s', x') \in \mu(k')$, $(c, s, x) \in \mu(k)$, then $s' + x' \geq s + x$ and $x' \geq x$.*

Indeed, the increasingness of the optimal policy correspondence allows us to claim the monotonicity of the optimal paths, which is crucial in analyzing the dynamic properties of the model.

Bringing together the increasingness and the upper semi-continuity of μ, we will prove that the left and right derivatives of the value function exists, using the methods in Le Van and Dana (2003). Let $\phi(k) := \min \{\pi s + x : (f(k) - x - \pi s, s, x) \in \mu(k)\}$ and $\psi(k) := \max \{\pi s + x : (f(k) - x - \pi s, s, x) \in \mu(k)\}$.

Proposition 2.2 *(i) Left derivative of V exits at every $x_0 > 0$, precisely $V'_{-}(x_0) = u'(f(x_0) - \phi(x_0))f'(x_0)$.*

(ii)Right derivative of V exists at every $x_0 > 0$, precisely $V'_{+}(x_0) = u'(f(x_0) - \psi(x_0))f'(x_0)$.

Proof. (i) Take a sequence of initial capitals x_0^n converging to x_0 from below. Formally, $x_0^n \to x_0$, $x_0^n < x_0$. Let $(c_1, s_1, x_1) \in \mu(x_0)$ be such that $\pi s_1 + x_1 = \phi(x_0)$. Take $(c_1^n, s_1^n, x_1^n) \in \mu(x_0^n)$ for each n. Since $x_0^n < x_0$, increasingness of μ implies that

$x_1^n < x_1$ and $x_1^n + s_1^n < x_1 + s_1$. By (4), $x_1 + \pi s_1 < f(x_0)$. Now recall that $x_0^n \to x_0$, hence $f(x_0^n) \to f(x_0)$. Then for all n large enough, we have $x_1 + \pi s_1 < f(x_0^n) < f(x_0)$. Bringing together all, we have $x_1^n + \pi s_1^n < x_1 + \pi s_1 < f(x_0^n) < f(x_0)$. Particularly, $x_1 + \pi s_1 < f(x_0^n)$ and $x_1^n + \pi s_1^n < f(x_0)$. This means that:

$$\begin{aligned} (f(x_0^n) - \pi s_1 - x_1, s_1, x_1) &\in \gamma(x_0^n), \qquad (8) \\ (f(x_0) - \pi s_1^n - x_1^n, s_1^n, x_1^n) &\in \gamma(x_0). \qquad (9) \end{aligned}$$

By (8), we know that $(f(x_0^n - \pi s_1 - x_1), s_1, x_1)$ is feasible after x_0^n but it need not be optimal. Hence we have:

$$V(x_0^n) \geq u(f(x_0^n) - \pi s_1 - x_1) + \beta(s_1)V(x_1). \qquad (10)$$

Also note that:

$$V(x_0) = u(f(x_0) - \pi s_1 - x_1) + \beta(s_1)V(x_1). \qquad (11)$$

Subtracting (10) from (11), and employing the concavity of u, we obtain:

$$\begin{aligned} V(x_0) - V(x_0^n) &\leq u(f(x_0) - \pi s_1 - x_1) - u(f(x_0^n) - \pi s_1 - x_1) \\ &\leq u'(f(x_0^n) - \pi s_1 - x_1)\left[f(x_0) - f(x_0^n)\right]. \end{aligned}$$

Therefore:

$$\frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} \leq u'(f(x_0^n) - \pi s_1 - x_1)\frac{f(x_0) - f(x_0^n)}{x_0 - x_0^n}.$$

Taking the limit as $x_0^n \to x_0$:

$$\limsup_{x_0^n \to x_0} \frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} \leq u'(f(x_0) - \pi s_1 - x_1)f'(x_0) = u'(f(x_0) - \phi(x_0))f'(x_0). \qquad (12)$$

By (9), similarly, $(f(x_0) - \pi s_1^n - x_1^n, s_1^n, x_1^n)$ is feasible after x_0.Hence:

$$V(x_0) \geq u(f(x_0) - \pi s_1^n - x_1^n) + \beta(s_1^n)V(x_1^n). \tag{13}$$

Also:

$$V(x_0^n) = u(f(x_0^n) - \pi s_1^n - x_1^n) + \beta(s_1^n)V(x_1^n). \tag{14}$$

Subtracting (14) from (13), and employing the concavity of u, we obtain:

$$\begin{aligned} V(x_0) - V(x_0^n) &\geq u(f(x_0) - \pi s_1^n - x_1^n) - u(f(x_0^n) - \pi s_1^n - x_1^n) \\ &\geq u'(f(x_0) - \pi s_1^n - x_1^n)\left[f(x_0) - f(x_0^n)\right]. \end{aligned}$$

Therefore:

$$\frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} \geq u'(f(x_0) - \pi s_1^n - x_1^n)\frac{f(x_0) - f(x_0^n)}{x_0 - x_0^n}.$$

Now recall that μ is upper semi-continuous. Then we may assume (c_1^n, s_1^n, x_1^n) converges to some $(c_1', s_1', x_1') \in \mu(x_0)$. Taking the limit yields:

$$\lim\inf_{x_0^n \to x_0} \frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} \geq u'(f(x_0) - \pi s_1' - x_1')f'(x_0).$$

Note that, since $\pi s_1 + x_1 = \phi(x_0)$, we have $\pi s_1' + x_1' \geq \pi s_1 + x_1$. Then the concavity of u (u' is decrasing) implies:

$$\lim\inf_{x_0^n \to x_0} \frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} \geq u'(f(x_0) - \pi s_1 - x_1)f'(x_0) = u'(f(x_0) - \phi(x_0))f'(x_0). \tag{15}$$

Conjoining (12) and (15), keeping in mind that $\lim\sup \geq \lim\inf$, we obtain:

$$\lim\sup_{x_0^n \to x_0} \frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} = \lim\inf_{x_0^n \to x_0} \frac{V(x_0) - V(x_0^n)}{x_0 - x_0^n} = u'(f(x_0) - \phi(x_0))f'(x_0).$$

Therefore $V'_-(x_0)$ exists and is equal to $u'(f(x_0) - \phi(x_0))f'(x_0)$.

(ii) Now similarly, take a sequence of initial capitals x_0^n converging to x_0 from above. Formally, $x_0^n \to x_0$, $x_0^n > x_0$. Let $(c_1, s_1, x_1) \in \mu(x_0)$ be such that $\pi s_1 + x_1 = \psi(x_0)$. Take $(c_1^n, s_1^n, x_1^n) \in \mu(x_0^n)$ for each n. Since $x_0 < x_0^n$, increasingness of μ implies that $x_1 < x_1^n$ and $x_1 + \pi s_1 < x_1^n + \pi s_1^n$. By (4), $x_1 + \pi s_1 < f(x_0)$.

We claim that, for all n large enough, $x_1^n + \pi s_1^n < f(x_0)$. Suppose otherwise: $x_1^n + \pi s_1^n \geq f(x_0)$ for infinitely many n. Then by upper semi-continuity of μ, there exists a subsequence ν_n with $x_1^{\nu_n} + \pi s_1^{\nu_n} \geq f(x_0)$ and a triplet $(\bar{c}_1, \bar{s}_1, \bar{x}_1) \in \mu(x_0)$, such that $(c_1^{\nu_n}, s_1^{\nu_n}, x_1^{\nu_n})$ converges to $(\bar{c}_1, \bar{s}_1, \bar{x}_1)$. Clearly, $x_1^{\nu_n} + \pi s_1^{\nu_n} \geq f(x_0)$ and $(c_1^{\nu_n}, s_1^{\nu_n}, x_1^{\nu_n}) \to (\bar{c}_1, \bar{s}_1, \bar{x}_1)$ implies that $\pi \bar{s}_1 + \bar{x}_1 \geq f(x_0)$. But since $(\bar{c}_1, \bar{s}_1, \bar{x}_1) \in \mu(x_0) \subset \gamma(x_0)$, we obtain $\bar{c}_1 = 0$, which is a contradiction with (4).

Therefore, for all n large enough, we have $x_1 + \pi s_1 < x_1^n + \pi s_1^n < f(x_0) < f(x_0')$. Particularly, $x_1 + \pi s_1 < f(x_0^n)$ and $x_1^n + \pi s_1^n < f(x_0)$. Then same method in (i) applies and we obtain $V'_+(x_0)$ exists and is equal to $u'(f(x_0) - \psi(x_0))f'(x_0)$. ■

Now we will establish the relation between the differentiability of V and the uniqueness of the optimal path. In order to do so, we need the following lemma.

Lemma 2.1 *(i) If $(c', s', x') \in \mu(k)$, $(c, s, x) \in \mu(k)$, $\pi s' + x' = \pi s + x$, then $x' = x$.*

(ii) $\phi(k) = \psi(k)$ if and only if $\mu(k)$ has a single element.

(iii) Given $k > 0$, V is differentiable at k if and only if $\mu(k)$ has a single element.

Proof. (i) By (3), $c' = f(k) - \pi s' - x'$ and $c = f(k) - \pi s - x$. By (5) we have:

$$\begin{aligned} u'(f(k) - \pi s' - x') &= \beta'(s')V(x'), \\ u'(f(k) - \pi s - x) &= \beta'(s)V(x). \end{aligned}$$

Since $\pi s' + x' = \pi s + x$, we have $u'(f(k) - \pi s' - x') = u'(f(k) - \pi s - x)$, i.e. $\beta'(s')V(x') = \beta'(s)V(x)$. W.L.O.G. assume that $x' > x$. Then $s' < s$. Strictly increasingness of V and decreasingness of β' imply that $V(x') > V(x)$ and $\beta'(s') \geq \beta'(s)$. Then $\beta'(s')V(x') > \beta'(s)V(x)$. Contradiction. Hence $x' = x$.

(ii) If $\mu(k)$ is a singleton, it is clear that $\phi(k) = \psi(k)$. If $\phi(k) = \psi(k)$, we will prove that $\mu(k)$ is a singleton. Suppose that there exists at least two different elements in $\mu(x)$, say $(c_\sigma, s_\sigma, x_\sigma)$ and $(c_\varsigma, s_\varsigma, x_\varsigma)$. By definitions of $\phi(x)$ and $\psi(x)$, the equality $\phi(x) = \psi(x)$ implies $\pi s_\sigma + x_\sigma = \pi s_\varsigma + x_\varsigma$. Thus, (i) implies $x_\sigma = x_\varsigma$. Then $s_\sigma = s_\varsigma$ and $c_\sigma = f(k) - \pi s_\sigma - x_\sigma = f(k) - \pi s_\varsigma - x_\varsigma = c_\varsigma$. Hence, $(c_\sigma, s_\sigma, x_\sigma) = (c_\varsigma, s_\varsigma, x_\varsigma)$. Contradiction.

(iii) V is differentiable at $k > 0$ iff $V'_-(k) = V'_+(k)$ iff $u'(f(k) - \phi(k))f'(k) = u'(f(k) - \psi(k))f'(k)$ iff $\phi(k) = \psi(k)$. Then by (ii), V is differentiable at $k > 0$ iff $\mu(k)$ has a single element. ■

Bringing together all of the above, we prove the following proposition concerning the almost everywhere differentiability of V and almost every uniqueness of the optimal path.

Proposition 2.3 *(i) Given x_0, let (c, s, x) be an optimal path from x_0. Then for any $t \geq 1$, $\mu(x_t)$ has a single element.*

(ii) Given x_0, $V'(x_0)$ exists if and only if there exists a unique optimal path from x_0.

(iii) V is differentiable almost everywhere, or equivalently the optimal path is unique for almost every initial capital $x_0 > 0$.

Proof. (i) Assume the contrary. There exists $t \geq 1$ such that $\mu(x_t)$ has at least two different elements, say $(c^{\sigma}_{t+1}, s^{\sigma}_{t+1}, x^{\sigma}_{t+1})$ and $(c^{\varsigma}_{t+1}, s^{\varsigma}_{t+1}, x^{\varsigma}_{t+1})$. Then by (6):

$$\begin{aligned} u'(c_t) &= \beta(s_t) f'(x_t) u'(c^{\sigma}_{t+1}), \\ u'(c_t) &= \beta(s_t) f'(x_t) u'(c^{\varsigma}_{t+1}). \end{aligned}$$

Thus, $c^{\sigma}_{t+1} = c^{\varsigma}_{t+1}$. Along with (3), we obtain $\pi s^{\sigma}_{t+1} + x^{\sigma}_{t+1} = f(x_t) - c^{\sigma}_{t+1} = f(x_t) - c^{\varsigma}_{t+1} = \pi s^{\varsigma}_{t+1} + x^{\varsigma}_{t+1}$. By Lemma 2.1, we have $x^{\sigma}_{t+1} = x^{\varsigma}_{t+1}$. Then $(c^{\sigma}_{t+1}, s^{\sigma}_{t+1}, x^{\sigma}_{t+1}) = (c^{\varsigma}_{t+1}, s^{\varsigma}_{t+1}, x^{\varsigma}_{t+1})$, contradiction.

(ii) From (i), we know that given any $(c_1, s_1, x_1) \in \mu(x_0)$, $\mu(x_1)$ is a singleton, say (c_2, s_2, x_2). Again from (i), we have $\mu(x_2)$ is a singleton say (c_3, s_3, x_3). Then inductively, we know that given any $(c_1, s_1, x_1) \in \mu(x_0)$, the rest of the optimal path is uniquely determined. This means that, the optimal path from x_0 is unique if and only if (c_1, s_1, x_1) is unique, in other words $\mu(x_0)$ is singleton valued. Then by the previous lemma, the optimal path from x_0 is unique if and only if V is differentiable at x_0.

(iii) Since μ is increasing, ϕ and ψ are increasing functions. We know that a bounded increasing function $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere continuos. For the proof, see the appendix of Le Van and Dana 2003. Then ϕ and ψ are almost everywhere continuous. Firstly, we will prove that the points of continuity of ϕ and ψ are exactly the same. Then we will prove that at these continuity points, ϕ and ψ yield equal values.

Now consider a fixed number y, and two variables x, z with $x < y < z$. We have $x^2/y < x < y < z < z^2/y$. Note that by definition, $\phi(k) \leq \psi(k)$ for any k. Then the increasingness of μ, ϕ and ψ imply:

$$\psi(x^2/y) \leq \phi(x) \leq \psi(x) \leq \phi(y) \leq \psi(y) \leq \phi(z) \leq \psi(z) \leq \phi(z^2/y). \tag{16}$$

Let x and z converge to y. Then x^2/y and z^2/y also converge to y.

Now, if ϕ is continuous at y, $\phi(z^2/y) \to \phi(y)$, $\phi(z) \to \phi(y)$. Then $\phi(y) \leq \psi(y) \leq \phi(z) \leq \psi(z) \leq \phi(z^2/y)$ implies that $\psi(z) \to \psi(y)$. Also, $\phi(x) \to \phi(y)$, $\phi(z) \to \phi(y)$. Then $\phi(x) \leq \psi(x) \leq \phi(y) \leq \psi(y) \leq \phi(z)$ implies that $\psi(x) \to \psi(y)$. Hence, ψ is continuous at y.

Conversely, if ψ is continuous at y, $\psi(x^2/y) \to \psi(y)$, $\psi(x) \to \psi(y)$. Then we get $\psi(x^2/y) \leq \phi(x) \leq \psi(x) \leq \phi(y) \leq \psi(y)$ which implies $\phi(x) \to \phi(y)$. Also, $\psi(z) \to \psi(y)$, $\psi(x) \to \psi(y)$. Then $\psi(x) \leq \phi(y) \leq \psi(y) \leq \phi(z) \leq \psi(z)$ implies that $\phi(z) \to \phi(y)$. Hence, ϕ is continuous at y.

Therefore, the continuity points of ϕ and ψ are coincident. Now let y be such a point of continuity. Since $\phi(z) \to \phi(y)$ as $z \to y$, $\phi(y) \leq \psi(y) \leq \phi(z)$ implies that $\phi(y) = \psi(y)$. Now we have proved that ϕ and ψ yield equal values at all of their continuity points. But recall that they are almost everywhere continuous functions. Then, $\phi(k) = \psi(k)$ for almost every k. By the lemma, $\phi(k) = \psi(k)$ implies that $\mu(k)$ is a singleton, which implies that $\mu(k)$ is a singleton for almost every k, hence V is differentiable for almost every k. Then by (ii), V is differentiable almost everywhere, or equivalently the optimal path is unique for almost every initial capital $x_0 > 0$. ■

2.4 Dynamic Properties of the Optimal Paths

We have proved the almost everywhere differentiability of the value function, and the almost everywhere uniqueness of the optimal path, for the so described model, which were left as open questions in Stern (2006). Now as one can easily see, the increasingness of μ implies the monotonicity of the optimal path of capital stock. Formally, for the optimal path (c, s, x) from $x_0 > 0$, x is a monotonic sequence. Moreover, as well known, the existence of the maximum sustainable capital stock along with the monotonicity of the optimal path of capital stock implies that the optimal path of capital converges to a steady state. The monotonicity and convergence results of any optimal path of capital are also available in Stern (2006). On the other hand, the monotonicity or the convergence of c and s are not studied analytically in Stern (2006). In this section, we will investigate the limiting behavior of the optimal path without any specific functional forms. We will prove that c and s are convergent, and provide conditions under which the system does not converge to zero.

Proposition 2.4 *Let (c, s, x) be the optimal path from x_0. Then, seperately c, s, x are convergent sequences. Moreover, if $\bar{c}, \bar{s}, \bar{x}$ are respectively the limits, then either $(\bar{c}, \bar{s}, \bar{x}) = (0, 0, 0)$ or $\bar{c}, \bar{s}, \bar{x} > 0$.*

Proof. The fact that x must be convergent is already established, which also implies that $(f(x_t) - x_{t+1})_t$ converges. If $(f(x_t) - x_{t+1})_t$ converges to zero, clearly the constraint of the problem imply that c and s also converge to zero. Now assume that x converges to $\bar{x}$ and $(f(x_t) - x_{t+1})_t$ coverges to some positive number, i.e. $f(\bar{x}) - \bar{x} > 0$. Notice that this also implies $\bar{x} > 0$. Define $\kappa(x, x', s) := \pi u'(f(x) - x' - \pi s) - \beta'(s)V(x')$. Clearly κ is continuous in all arguments and also strictly increasing in s, which follows from the strict concavity of u and concavity of β. Now there exists $\bar{s}$ such that $\kappa(\bar{x}, \bar{x}, \bar{s}) = 0$. Because, $\kappa(\bar{x}, \bar{x}, 0) = \pi u'(f(\bar{x}) - \bar{x}) - \beta'(0)V(\bar{x}) = -\infty$, $\kappa(\bar{x}, \bar{x}, \frac{f(\bar{x})-\bar{x}}{\pi}) = u'(0) - \beta(\frac{f(\bar{x})-\bar{x}}{\pi})V(\bar{x}) = +\infty$, and κ is strictly increasing in s. Note that since κ is strictly increasing in s, $\bar{s}$ is uniquely determined. Now assume that s is not convergent, in particular s does not converge to $\bar{s}$. Then $\exists \varepsilon > 0$ there exist infinitely many t for which we have $\left| s_{t+1} - \bar{s} \right| > \varepsilon$. W.L.O.G., for infinitely many t, we have $s_{t+1} - \bar{s} > \varepsilon$. Then specifically for these t, we have

$$\kappa(x_t, x_{t+1}, s_{t+1}) > \kappa(x_t, x_{t+1}, \bar{s} + \varepsilon). \tag{17}$$

Note that by (5),

$$\pi u'(f(x_t) - x_{t+1} - \pi s_{t+1}) = \beta'(s_{t+1})V(x_{t+1}),\ \forall t,$$

thus we have

$$\kappa(x_t, x_{t+1}, s_{t+1}) = 0, \forall t. \tag{18}$$

Therefore, as t goes to infinity, the continuity of κ along with (17) and (18) yields $0 \geq \kappa(\bar{x}, \bar{x}, \bar{s} + \varepsilon)$. Recall that $\kappa(\bar{x}, \bar{x}, \bar{s})$, thus $\kappa(\bar{x}, \bar{x}, \bar{s}) \geq \kappa(\bar{x}, \bar{x}, \bar{s} + \varepsilon)$ which is a contradiction with the strictly increasingness of κ. Hence, s converges to $\bar{s}$. Then (3) implies c converges to $\bar{c} := f(\bar{x}) - \bar{x} - \bar{s}$.

Now let the optimal path (c, s, x) from x_0 converge to $(\bar{c}, \bar{s}, \bar{x})$. If $x_0 = 0$, it is clear that $(\bar{c}, \bar{s}, \bar{x}) = (0, 0, 0)$. Consider $x_0 > 0$. Firstly, it is clear that $\bar{x} \leq x_m$. If $\bar{x} = x_m$, the constraint implies that $\bar{c}, \bar{s} = 0$. Note that such a sequence (c, s, x) cannot be optimal because after some period, zero utility will be gained with positive capital accumulation around x_m. If $\bar{x} = 0$, again the constraints imply that $(\bar{c}, \bar{s}, \bar{x}) = (0, 0, 0)$.

Now consider $\bar{x} > 0$. (5) i.e. $\pi u'(c_{t+1}) = \beta'(s_{t+1})V(x_{t+1})$ and the Inada conditions on u and β imply that $\bar{c} = 0$ if and only if $\bar{s} = 0$. However, if $\bar{c} = \bar{s} = 0$ and $\bar{x} > 0$, we obtain a contradiction with (4). Then for the case where $\bar{x} > 0$, we have $\bar{c} > 0$, $\bar{s} > 0$. ■

Formally, define a steady state as any triplet $(\bar{c}, \bar{s}, \bar{x})$ such that $\mu(\bar{x}) = (\bar{c}, \bar{s}, \bar{x})$, i.e. the stationary sequence $(\bar{c}, \bar{s}, \bar{x})$ starting from $\bar{x}$ is optimal. We say $(\bar{c}, \bar{s}, \bar{x})$ is a positive steady state if it is a steady state and $\bar{c}, \bar{s}, \bar{x} > 0$. Clearly from (4), a steady state can either be $(0, 0, 0)$ or a positive steady state.

Now that we have proved the convergence of the optimal path rather than just the optimal path of capital, we will also prove that the optimal path converges to a steady state. In the preceeding part, we will present the condition under which the optimal path converges to a positive steady state.

Theorem 2.2 *The optimal path from any x_0 converges to a steady state.*

Proof. Let the optimal path from x_0 be (c, s, x). It is clear that if $x_0 = 0$, (c, s, x) converges to $(0, 0, 0)$ which is a steady state. Now if $x_0 > 0$, let (c, s, x) converge to $(\bar{c}, \bar{s}, \bar{x}) > 0$, in particular x converges to $\bar{x}$. Recall that μ is upper-semi continuous and singleton valued at each x_t, $t \geq 1$. Then $\mu(x_t) = (c_{t+1}, s_{t+1}, x_{t+1})$ has a convergent subsequence which converges to a point in $\mu(\bar{x})$. We already know that (c, s, x) converges to $(\bar{c}, \bar{s}, \bar{x})$, hence $(\bar{c}, \bar{s}, \bar{x}) \in \mu(\bar{x})$. What is left to prove is that $(\bar{c}, \bar{s}, \bar{x}) = \mu(\bar{x})$ if $(\bar{c}, \bar{s}, \bar{x}) \in \mu(\bar{x})$. However, this is also clear since $(\bar{c}, \bar{s}, \bar{x}) \in \mu(\bar{x})$ implies that the statinoary sequence $(c', s', x') = (\bar{c}, \bar{s}, \bar{x})$ is feasible, and μ is singleton valued at each x'_t, $t \geq 1$, i.e. $(\bar{c}, \bar{s}, \bar{x}) = (c'_2, s'_2, x'_2) = \mu(x'_1) = \mu(\bar{x})$. ■

Proposition 2.5 *If $\beta(0)f'(0) > 1$, the optimal path from $x_0 > 0$ converges to a positive steady state.*

Proof. Assume that $\beta(0)f'(0) > 1$ and $x_0 > 0$. Let the optimal path (c, s, x) from x_0 converge to $(\bar{c}, \bar{s}, \bar{x})$. Suppose that $(\bar{c}, \bar{s}, \bar{x}) = (0, 0, 0)$. As β and f' are continuous functions, there exists N so that for all $n > N$, we have $\beta(s_n)f'(x_n) > 1$. Then consider (6) for $t > N$:

$$u'(c_{t+1}) = \beta(s_{t+1})f'(x_{t+1})u'(c_{t+2}) > u'(c_{t+2})$$

which implies $c_{t+1} < c_{t+2}$. Inductively, we obtain $0 < c_{N+1} < c_{N+2} <$ ...However, c_t also converges to zero, which yields a contradiction. Hence the optimal path converges to a positive steady state. ■

Now if $(\bar{c}, \bar{s}, \bar{x})$ is a positive steady state of the system, then by (3), (5), (6), and (27) we have respectively:

$$\begin{aligned} \bar{c} + \pi\bar{s} + \bar{x} &= f(\bar{x}), \\ \pi u'(\bar{c}) &= \beta'(\bar{s})V(\bar{x}), \\ u'(\bar{c}) &= \beta(\bar{s})f'(\bar{x})u'(\bar{c}), \\ V(\bar{x}) &= u(\bar{c}) + \beta(\bar{s})V(\bar{x}). \end{aligned}$$

∈Note that these are necessary conditions for a positive steady state. Thus, if $\beta(0)f'(0) > 1$, and also if the solution $(\bar{c}, \bar{s}, \bar{x})$ to the above system of equations is unique, then clearly the system will posses global convergence to the prescribed positive steady state. The above equations can be recast as:

$$\bar{c} + \pi\bar{s} + \bar{x} - f(\bar{x}) = 0, \tag{19}$$

$$\frac{\pi u'(\bar{c})}{u(\bar{c})} = \frac{\beta'(\bar{s})}{1 - \beta(\bar{s})}, \tag{20}$$

$$\beta(\bar{s})f'(\bar{x}) = 1. \tag{21}$$

2.5 The Relative Cost of Future Oriented Capital and the Long Run Equilibrium

We show in a numerical example how a small perturbation to the price of future oriented capital qualitatively changes the dynamical properties of the optimal policy. In particular, we demonstrate the occurrence of a saddle-node bifurcation with respect to the price of future oriented capital stock. We use the same functional forms and the parameter set as Stern (2006) did while giving an example for the multiple steady states and divergence and show that, with the same functional forms, only by changing the value of relative cost of future oriented capital, one can also obtain global convergence.

Assumption 2.5 *$u(c) = c^{\sigma}, \beta(s) = 1 - \delta e^{-s^{\theta}}, f(k) = Ak^{\alpha}$, $A > 0$, and $0 < \{\alpha, \delta, \theta, \sigma\} < 1$.*

Under this functional forms we define the stationary Euler equation as:

$$E = \left\{ k > 0 : \pi = \frac{\theta \left(\ln \frac{\delta}{1 - \frac{1}{\alpha A k^{\alpha-1}}} \right)^{\frac{\theta-1}{\theta}} (Ak^{\alpha} - k)}{\sigma + \theta \ln \frac{\delta}{1 - \frac{1}{\alpha A k^{\alpha-1}}}} \right\}.$$

Consider the following set of parameters:

$$A = 3, \alpha = 3/4, \delta = 0.85, \sigma = 2/3, \theta = 4/5$$

In Figure 1, the bifurcation diagram is presented. It turns out that the critical values for π are $\pi_1 = 0.806637$ and $\pi_2 = 1.64558$. As long as $\pi < 0.806637$, i.e. before bifurcation occurs, there is only one steady state, x_h, which is globally stable. As π slowly increases, the steady state capital stock decreases.

For $\pi = 0.806637$, an additional steady state appears in addition to x_h and the dynamics are now characterized by two steady states, $x_m < x_h$ such that x_h is locally stable and x_m is unstable in the sense that it is stable from the left but unstable from the right.

When π slightly increases from its critical value 0.806637, the unstable steady state splits into one locally stable and one unstable steady state through the saddle-node bifurcation resulting in three steady states, x_l(stable)$<$ x_m(unstable)$<$ x_h(stable), in total. The coexistence of these three steady states is preserved until $\pi = 1.64558$.

As the value of π gets closer to the critical value 1.64558, the highest stable steady state and the unstable steady state approach one another and at the critical value, they merge into a non-hyperbolic steady state. Slightly above the critical value, the non-hyperbolic steady state ceases to exist leaving only the stable steady state, x_l which is now globally stable. Further increases in π only affects the value of the stable steady state.

In sum, two types of the saddle-node bifurcation emerge. The difference lies in the following: In the first one, the saddle-node bifurcation is realized for the pair of steady states x_m and x_h and in the second, it is for the pair of steady states x_l and x_m. Moreover, in the first one, as π increases, a pair of stable and unstable steady states emerges simultaneously from a non-hyperbolic steady state and in the second, the qualitative change is in the form of coalescence of the steady states into a non-hyperbolic steady state.

This result signifies the importance of the price of future oriented capital stock on the dynamic behavior of the system. The possibility of multiple steady states were established numerically for specific functional forms in Stern (2006). However, with the same functional forms, only by changing the value of π, one can also obtain global convergence.

In this chapter, we consider the effects of the relative cost of the future oriented capital on economic variables in the long term. In particular, we demonstrate the occurrence of a saddle-node bifurcation with respect to the price of future oriented capital stock. This is worth to examine optimal growth model with endogenous time preference while relaxing the assumption that the cost of the future-oriented

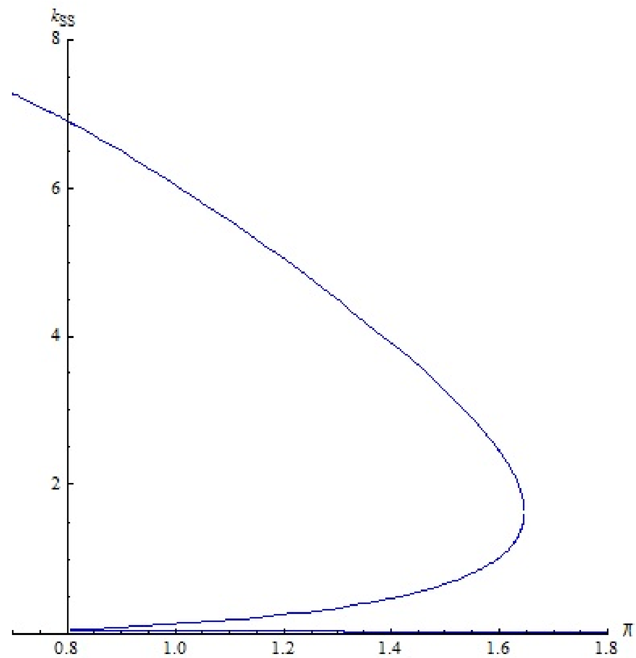

Figure 1: **Bifurcation analysis for variations in** π

capital is constant. Assuming that it changes with wealth would lets us consider the factors influencing the opportunity cost of time and efforts being assigned for the accumulation of future oriented capital stock. This is in our future agenda.

CHAPTER 3

STRATEGIC INTERACTION AND DYNAMICS UNDER ENDOGENOUS TIME PREFERENCE[1]

To account for development patterns that differ considerably among economies in the long run (Quah, 1996; Barro, 1997; Barro and Sala-i-Martin, 1991), a variety of one-sector optimal growth models that incorporate some degree of market imperfections have been presented. These are based on technological external effects and increasing returns (Dechert and Nishimura, 1983; Mitra and Ray, 1984) or the endogeneity of time preference (Becker and Mulligan, 1997; Stern, 2006; Erol et al., 2011). They have characterized the optimal paths and proven the emergence of threshold dynamics according to which the economies with low initial capital stocks or incomes converge to a steady state with low per capita income, while economies with high initial capital stocks converge to a steady state with high per capita income (see Azariadis and Stachurski, 2005, for a survey). However, to what extent these analyses are robust to the considerations of strategic interactions among agents in the economy, still remains as a concern.

[1] This essay is our joint work with Carmen Camacho and Çağrı Sağlam and published in Journal of Mathematical Economics, Volume 49, Issue 4 (August 2013).

This paper presents a strategic growth model with endogenous time preference. Each agent, receiving a share of income which is increasing in her own capital stock and decreasing in her rival's, invests over an infinite horizon to build her stocks. The heterogeneity among agents arises from differences in their initial endowment, their share of aggregate income, and therefore in their subjective discount rates. We adopt the noncooperative open loop Nash equilibrium concept, in which players choose their strategies as simple time functions and they are able to commit themselves to time paths as equilibrium strategies. In this setup, agents choose their strategies simultaneously and each agent is faced with a single criterion optimization problem constrained by the strategies of the rival taken as given. We focus on the qualitative properties of the open-loop Nash equilibria and the dynamic implications of the strategic interaction.

In line with the empirical studies concluding that the rich are more patient than the poor (see Lawrence, 1991, and Samwick, 1998) and in parallel to the idea that the stock of wealth is a key to reach better health services and better insurance markets, we consider that the discount factor is increasing in the stock of wealth. However, this implies that the objective function of each agent's single criterion optimization problem includes a multiplication of the discount function. This generally destroys the usual concavity argument which is used in the proof of the differentiability of value function and the uniqueness of the optimal paths (see Benveniste and Scheinkman, 1979; Araujo, 1991).

Due to this potential lack of concavity and the differentiability of the value functions associated with each agent's problem, we employ the theory of monotone comparative statics and the supermodular games based on order and monotonicity properties on lattices (see Topkis, 1998). The analyses on the properties of supermodular games have been extensively concentrated in static games and to some extent in dynamic games with stationary Markov strategies (see Cooper, 1999; Amir, 2005; Vives, 2005, for a general review). This may stem from the fact that the use of open-loop strategies has been noted for being static in nature, not allowing for genuine strategic interaction between players during the play of the game. There are, however, many situations in which players lack any other information than their own actions and time so that the open-loop strategies can turn out to be unavoidable. The players may be unable to observe the state vector, let alone the actions of their rivals. In this respect, showing how the supermodular game structure can be utilized in the analysis of the dynamic games under open loop strategies is inevitable.

In this paper, we first provide the sufficient conditions of supermodularity for dynamic games with open-loop strategies based on two fundamental elements: the ability to order elements in the strategy space of the agents and the strategic complementarity which implies upward sloping best responses. In our dynamic game the open-loop strategies are vectors instead of simple scalars. Hence, the game requires an additional restriction to guarantee that all components of an agent's best response vector move together. This explains the role of the restriction that the payoff function of each agent has to be supermodular in his own strategy given the strategy of his rival. The supermodular game structure of our model let us provide the existence and the monotonicity results on the greatest and the least equilibria. We sharpen these results by showing the differentiability of the value function and the uniqueness of the best response correspondences almost everywhere. These allow us to derive conclusions on the nature of best responses, the set of equilibria and the long-run dynamics.

In particular, we analyze to what extent the strategic complementarity inherent in agents' strategies can alter the convergence results that could have emerged under a single agent optimal growth model and try to answer the following questions: Can an agent with a larger initial stock credibly maintain this advantage to preempt the rival's investment and reach a better long-run stock of capital? Put differently, is the initial dominance reinforced by the actions of the agents? Can small initial differences be magnified and then propagated through time? Can this kind of initial advantages vanish in the non-cooperative equilibrium of this class of games with strategic complementarity? Can the agent with a low initial capital stock pull the rich to her lower steady state that she would never face while acting by herself? Under what conditions do we have a unique equilibrium with strategic complementarity under open-loop strategies?

The key feature of our analysis is that the stationary state Nash equilibria tend to be symmetric under open-loop strategies. We show that the initially rich can pull the poor out of poverty trap even when sustaining a higher level of steady state capital stock for itself. A remarkable feature of our analysis is that it does not rely on particular parameterization of the exogenous functions involved in the model. Rather, it provides a more flexible framework with regards to the discounting of time, keeps the model analytically tractable and uses only general and plausible qualitative properties.

This chapter is organized as follows. The next section introduces the model. Tools needed while utilizing the supermodularity of the game; equilibrium dynamics and the steady state analysis have been discussed in Section 3.

3.1 The Model

We consider an intertemporal one sector model of a private ownership economy à la Arrow-Debreu with a single good x_t, and two infinitely lived agents, $i = 1, 2$. The single commodity is used as capital, along with labor, to produce output. Labor is presumed to be supplied in fixed amounts, and the capital and consumption are interpreted in per-capita terms. The production function is given by $f(x_t)$. We assume that each agent receives a share of income $\theta^i(x_t^i, x_t^j) = \frac{x_t^i}{x_t^i + x_t^j}$, which is increasing in her own capital stock x_t^i, and decreasing in her rival's, x_t^j. The amount of current resources not consumed is saved individually as capital until the next period. For a given strategy of the rival, each agent chooses a path of consumption $c^i = \{c_t^i\}_{t \geq 0}$ so as to maximize the discounted sum of instantaneous utilities, $\sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i) \right) u^i(c_t^i)$ where the functions u and β denote the instantaneous utility from consumption and the level of discount on future utility, respectively.

In accordance with these, the problem of agent i can be formalized as follows:

$$\max_{\{c_t^i, x_{t+1}^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i) \right) u^i(c_t^i), \qquad (\mathcal{P})$$

subject to

$$c_t^i + x_{t+1}^i \leq \theta^i(x_t^i, x_t^j) f(x_t^i + x_t^j) + (1-\delta) x_t^i, \forall t,$$
$$c_t^i \geq 0, \ x_t^i \geq 0, \forall t,$$
$$(x_0^i, x_0^j) \geq 0, \ \boldsymbol{x}_j = \{x_t^j\}_{t=1}^{\infty} \geq 0, \text{given},$$

where $j \neq i \in \{1, 2\}$, and $\delta \in (0, 1)$ is the depreciation rate of the capital stock. Agents may only differ in their initial endowment, their share of output, and therefore in their subjective discount rates.

We make the following assumptions regarding the properties of the discount, utility and the production functions.

Assumption 3.1 $\beta : R_+ \rightarrow R_{++}$ *is continuous, differentiable, strictly increasing and satisfies* $\sup_{x>0} \beta(x) = \beta_m < 1, \ \sup_{x>0} \beta'(x) < +\infty$.

Assumption 3.2 *$u : R_+ \rightarrow R_+$ is continuous, twice continuously differentiable and satisfies either $u(0) = 0$ or $u(0) = -\infty$. Moreover, u is strictly increasing, strictly concave and $u'(0) = +\infty$ (Inada condition).*

Assumption 3.3 *$f : R_+ \rightarrow R_+$ is continuous, twice continuously differentiable and satisfies $f(0) = 0$. Moreover, f it is strictly increasing and $\lim_{x\rightarrow+\infty} f'(x) < \delta$.*

We say that a path for capital $x_i = (x_1^i, x_2^i, ...)$ is feasible from $(x_0^i, x_0^j) \geq 0$, if for all t and given $x_j \geq 0$, if for any $t \geq 0$, x_i satisfies that $0 \leq x_{t+1}^i \leq g(x_t^i, x_t^j)$ where

$$g^i(x_t^i, x_t^j) = \theta^i(x_t^i, x_t^j) f(x_t^i + x_t^j) + (1 - \delta) x_t^i.$$

$S^i(\boldsymbol{x}_j)$ denotes the set of feasible accumulation paths from (x_0^i, x_0^j). A consumption sequence $c_i = (c_0^i, c_1^i, ...)$ is feasible from $(x_0^i, x_0^j) \geq 0$, when there exists a path for capital, $x_i \in S^i(\boldsymbol{x}_j)$ with $0 \leq c_t^i \leq g^i(x_t^i, x_t^j) - x_{t+1}^i$. As the utility and the discount functions are strictly increasing, we introduce function U defined on the set of feasible sequences as

$$U(\boldsymbol{x}_i \mid \boldsymbol{x}_j) = \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s^i) \right) u\left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right).$$

The preliminary results are summarized in the following lemma which has a standard proof using the Tychonov theorem (see Le Van and Dana, 2003; Stokey and Lucas, 1989).

Lemma 3.1 *Let $\bar{x}$ be the largest point $x \geq 0$ such that $f(x) + (1 - \delta)x = x$. Then, for any x_i in the set of feasible accumulation paths we have $x_t^i \leq A(x_0^i + x_0^j)$ for all t, where $A\left(x_0^i + x_0^j\right) = \max\left\{\left(x_0^i + x_0^j\right), \bar{x}\right\}$. Moreover, the set of feasible accumulation paths is compact in the product topology defined on the space of sequences x_i and U is well defined and upper semicontinuous over this set.*

In a recent paper, Erol et al. (2011) study the dynamic implications of the endogenous rate of time preference depending on the stock of capital in a single consumer one-sector optimal growth model. They prove that even under a convex technology there exists a critical value of initial stock, in the vicinity of which, small differences lead to permanent differences in the optimal path: economies with low initial capital stocks converge to a steady state with low per capita income. On the other hand, economies with high initial capital stocks converge to a steady state with high per capita income. Indeed, it is shown that the critical stock is not an unstable steady state so that if an economy starts at this stock, an indeterminacy will emerge.

In this paper, we propose a capital accumulation game where heterogeneous agents consume strategically. Heterogeneity arises from differences in their initial endowment, their share of aggregate income, and therefore in their subjective discount rates. Our interest focuses on the qualitative properties of the open-loop Nash equilibria and the dynamic implications of the strategic interaction.

3.2 Non-Cooperative Difference Game and Open-Loop Nash Equilibrium

The noncooperative game in consideration is a triplet $(N, \mathbf{S}, \{U^i : i \in N\})$ where $N = \{1, 2\}$ is the set of players, $S = \Pi_{i \in N} S^i$ is the set of joint admissible strategies under open-loop information structure and U^i is the payoff function defined on S for each player $i \in N$, i.e., $U^i = U\left(\boldsymbol{x}_i \mid \boldsymbol{x}_j\right)$.

Any admissible strategy for agent i is an infinite sequence compatible with the information structure of the game which is constant through time and restricted with the initial pair of capital stock in the economy. Accordingly, the set of admissible strategies for agent i, can be written as $S^i = \Pi_{t=1}^{\infty} S_t^i$, where $S_t^i = \left[0, g^i(\widetilde{x}_{t-1}^i, \widetilde{x}_{t-1}^j)\right]$, with $\widetilde{x}_{t-1}^i = \sup S_{t-1}^i$, and $\widetilde{x}_{t-1}^j = \sup S_{t-1}^j$. Indeed, any strategy $x_i \in S^i$ is such that $x_t^i \in S_t^i, \forall t$ where $S_1^i = \left[0, g^i(x_0^i, x_0^j)\right]$, $S_2^i = \left[0, g^i\left(g^i(x_0^i, x_0^j), g^j(x_0^i, x_0^j)\right)\right]$, ..., *etc.*

A few important remarks on the way the set of joint admissible strategies is constructed are in order. Denoting the set of joint feasible strategies by Δ, for any $(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \Delta$, we have

$$\Delta\left(\boldsymbol{x}_i, \boldsymbol{x}_j\right) = \left[\left(\cup_{\boldsymbol{x}_j \in X^j} S^i\left(\boldsymbol{x}_j\right)\right) \times \left(\cup_{\boldsymbol{x}_i \in X^i} S^j\left(\boldsymbol{x}_i\right)\right)\right] \cap \Delta,$$

where $X^j = \{\boldsymbol{x}_j : S^i\left(\boldsymbol{x}_j\right) \neq \varnothing\}$. It is important to recall from Topkis (1998) that $\Delta = \left(\cup_{\boldsymbol{x}_j \in X^j} S^i\left(\boldsymbol{x}_j\right)\right) \times \left(\cup_{\boldsymbol{x}_i \in X^i} S^j\left(\boldsymbol{x}_i\right)\right)$ if and only if $\Delta\left(\boldsymbol{x}_i, \boldsymbol{x}_j\right) = \Delta$ for each $(\boldsymbol{x}_i, \boldsymbol{x}_j)$ in Δ. However, note that under the open-loop information structure of our game, the action spaces of the agents turn out to be dependent on each other converting the game into a "generalized game" in the sense of Debreu (1952). More precisely, $\Delta \neq \left(\cup_{\boldsymbol{x}_j \in X^j} S^i\left(\boldsymbol{x}_j\right)\right) \times \left(\cup_{\boldsymbol{x}_i \in X^i} S^j\left(\boldsymbol{x}_i\right)\right)$ as the set of feasible accumulation paths from (x_0^i, x_0^j) of agent i is constrained by the choices of agent j. This simply prohibits to order elements in the joint feasible strategy space and calls for additional restrictions on the plan of the game in proving the existence of an equilibrium and analyzing the long-run dynamics via order theoretical reasoning. The admissibility condition[2] imposed on the set of feasible strategies of the agents allows to write the set of joint admissible strategies as a simple cross product of the each agent's set of admissible strategies that constitute a complete lattice.[3]

[2]Whenever there exists a positive externality, *i.e.*, $\frac{\partial g^i(x_t^i, x_t^j)}{\partial x_t^j} \geq 0, \forall t$, the admissibility imposed on the set of joint feasible strategies is far from a restriction. Even in case of negative externality, *i.e.*, $\frac{\partial g^i(x_t^i, x_t^j)}{\partial x_t^j} < 0, \forall t$, the long-run implications of our analysis will not rely on such a restriction. Indeed, an admissible strategy of agent i is an infinite sequence of capital stock feasible from (x_0^i, x_0^j) constituted under the consideration of the highest feasible strategy of the rival.

[3]In order to be able to work on a joint strategy space which constitutes a complete lattice, one may also introduce an *ad-hoc* rule that exhausts the available stock of capital at the period where the joint strategies of the agents turn out to be infeasible (see Sundaram, 1989). However, in such a case, showing that the payoff function of each agent exhibits "increasing first differences" on the joint strategies turns out to be unnecessarily complicated.

We adopt the noncooperative open loop Nash equilibrium concept, in which players choose their strategies as simple time functions and they are able to commit themselves to time paths as equilibrium strategies. In this setup, agents choose their strategies simultaneously and each agent is faced with a single criterion optimization problem constrained by the strategies of the rival taken as given.

For each vector $x_j \in S^j$, the best response correspondence for agent i is the set of all strategies that are optimal for agent i given x_j :

$$Br^i(\boldsymbol{x}_j) = \arg\max_{\boldsymbol{x}_i \in S^i} U(\boldsymbol{x}_i \mid \boldsymbol{x}_j).$$

A feasible joint strategy (x_i^*, x_j^*) is an open loop Nash equilibrium if

$$U\left(\boldsymbol{x}_i^* \mid \boldsymbol{x}_j^*\right) \geq U\left(\boldsymbol{x}_i \mid \boldsymbol{x}_j^*\right) \text{ for each } \boldsymbol{x}_i \in S^i \text{ and each } i \in N. \tag{22}$$

Given an equilibrium path, there is no feasible way for any agent to strictly improve his life-time discounted utility as the strategies of the other agent remains unchanged. The set of all equilibrium paths for this noncooperative game $(N, \mathbf{S}, \{U^i : i \in N\})$ is then identical to the set of pairs of sequences, (x_i^*, x_j^*) such that

$$\boldsymbol{x}_i^* \in Br^i\left(\boldsymbol{x}_j^*\right) \text{ and } \boldsymbol{x}_j^* \in Br^j\left(\boldsymbol{x}_i^*\right). \tag{23}$$

We will first prove that the best response correspondence of each agent is non-empty so that there exists an optimal solution to problem $\mathcal{P}$. The dynamic properties of the best response correspondence then follows from the standard analysis in optimal growth models (see Stokey and Lucas, 1989; Le Van and Dana, 2003 and Erol et al., 2011) .

3.3 Dynamic Properties of the Best Responce Correspondances

The existence of an optimal path associated with $\mathcal{P}$ follows from the set of feasible accumulation paths being compact in the product topology defined on the space of sequences x_i and U^i being upper semicontinuous for this product topology. Let $x_i \in Br^i(\boldsymbol{x}_j)$ so that x_i solves $\mathcal{P}$ given x_j. We can prove that the associated optimal consumption and capital paths are positive at equilibrium.

Proposition 3.1 *Let $x_i \in Br^i\left(\boldsymbol{x}_j^*\right)$.*

i) The associated optimal consumption path, c_i $(i, j \in N,\ i \neq j)$ is given by

$$c_t^i = g^i(x_t^i, x_t^{j*}) - x_{t+1}^i, \forall t.$$

ii) Given $\{x_0^j, x_j^*\}$, if $x_0^i > 0$, every solution $(\boldsymbol{x}_i, \mathbf{c}_i)$ to $\mathcal{P}$ satisfies

$$c_t^i > 0,\ x_t^i > 0,\ \forall t. \tag{24}$$

Proof. It can be easily checked from the first order conditions and the Inada condition. ■

In accordance with these, let the value function V associated with $\mathcal{P}$ be defined by

$$\forall x_0^i \geq 0,\ V\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) = \max_{\boldsymbol{x}_i \in S^i} U\left(\boldsymbol{x}_i \mid \boldsymbol{x}_j^*\right). \tag{25}$$

The bounds on discounting together with the existence of a maximum sustainable capital stock guarantee a finite value function. Under Assumptions 1 and 2, one can immediately show that the value function is non-negative and strictly increasing. If

$u(0) = 0$ then the continuity of the value function immediately follows as well. If $u(0) = -\infty$ then the value function turns out to be continuous in the generalized sense so that it is continuous at any strictly positive point and it converges to $-\infty$ when the stock of capital converges to zero (see Le Van and Dana, 2003). Given these, the Bellman equation associated with $\mathcal{P}$ follows.

Proposition 3.2 *i) V satisfies the following Bellman equation:*

$$\forall x_0^i \geq 0,\ V\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) = \max\{u\left(g^i(x_0^i, x_0^j) - x^i\right) + \beta(x^i)V\left(x^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) \mid 0 \leq x^i \leq g^i(x_0^i, x_0^j)\}. \quad (26)$$

ii) A sequence $x_i \in S^i(x_j^*)$ is an optimal solution so that $x_i \in Br^i\left(\boldsymbol{x}_j^*\right)$ if and only if it satisfies:

$$\forall t,\ V\left(x_t^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) = u\left(g^i(x_t^i, x_t^{j^*}) - x_{t+1}^i\right) + \beta(x_{t+1}^i)V\left(x_{t+1}^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right). \quad (27)$$

Proof. See Le Van and Dana (2003) or Erol et al. (2011). ■

The optimal policy correspondence associated with $\mathcal{P}$, $\mu^i : \mathbb{R}_+ \to \mathbb{R}_+$ is defined as follows:

$$\mu^i\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) = \arg\max\{u\left(g^i(x_0^i, x_0^j) - x^i\right) + \beta(x^i)V\left(x^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) \mid x^i \in \left[0,\ g^i(x_0^i, x_0^j)\right]\}.$$

It is important to note that although the utility function is strictly concave, the solution to $\mathcal{P}$ may not be unique as the multiplication of a discount function destroys the concave structure needed for uniqueness. We can prove the following properties for the optimal policy correspondence associated with $\mathcal{P}$.

Proposition 3.3 *i)* $\mu^i\left(0 \mid \{x_0^j, \boldsymbol{x}_j^*\}\right) = \{0\}$.

ii) If $x_0^i > 0$ *and* $x_1^i \in \mu^i\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right)$, *then* $0 < x_1^i < g^i(x_0^i, x_0^j)$.

iii) μ^i *is upper semicontinuous.*

iv) $x_i \in Br^i(x_j^*)$ *if and only if* $x_{t+1}^i \in \mu^i\left(x_t^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right), \forall t$.

v) The optimal correspondence μ^i *is increasing: if* $x_0^i < \tilde{x}_0^i$, $x_1^i \in \mu^i\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right)$ *and* $\tilde{x}_1^i \in \mu^i\left(\tilde{x}_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right)$ *then* $x_1^i < \tilde{x}_1^i$.

Proof. ii) Follows easily from (24). iii) See Le Van and Dana (2003). iv) Follows from (27). v) See Dechert and Nishimura (1983) or Amir et al. (1991). ■

The increasingness of μ^i is crucial for the convergence of optimal paths associated with $\mathcal{P}$ and hence for the analysis of the long-run dynamics. Moreover, we have also proven that the optimal correspondence, μ is not only closed but also upper semi-continuous.

With the positivity of the optimal consumption and the stock of capital, the Euler equation associated with $\mathcal{P}$ easily follows.

Proposition 3.4 *When $x_0^i > 0$, any solution $x_i \in Br^i(x_j^*)$ satisfies the Euler equation associated with $\mathcal{P}$ for all t:*

$$u'\left(g^i(x_t^i, x_t^{j^*}) - x_{t+1}^i\right) = \beta\left(x_{t+1}^i\right) u'\left(g^i(x_{t+1}^i, x_{t+1}^{j^*}) - x_{t+2}^i\right) \frac{\partial g^i(x_{t+1}^i, x_{t+1}^{j^*})}{\partial x_{t+1}^i} + \beta'\left(x_{t+1}^i\right) V\left(x_{t+1}^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right). \quad (28)$$

Recall that in a standard optimal growth model with geometric discounting and the usual concavity assumptions on preferences and technology, the optimal policy correspondence is single valued. Furthermore, the properties of the optimal path are easily found using the first order conditions together with the envelope theorem, differentiating the value function. However, in the i'th agent's problem $\mathcal{P}$, although the utility function is strictly concave, the solution, namely $Br^i(x_j^*)$ may not be unique as the objective function includes the multiplication of a discount function. This generally destroys the usual concavity argument in the proof of the differentiability of value function and the uniqueness of the optimal paths (see Benveniste and Scheinkman, 1979; Araujo, 1991). To this end, we show that the value function associated with the i'th agent's problem $\mathcal{P}$ is differentiable almost everywhere so that there exists a unique path from almost everywhere.

Proposition 3.5 *i) If $x_i \in Br^i(x_j^*)$, then V is differentiable at any x_t^i, $t \geq 1$. If $x_i \in Br^i(x_j^*)$, there exists a unique optimal path from x_t^i for any $t \geq 1$.*

ii) V is differentiable almost everywhere, i.e. the optimal path is unique for almost every $x_0 > 0$.

Proof. See Le Van and Dana (2003). ■

We prove in the next proposition that the optimal paths associated with $\mathcal{P}$ are monotonic. As a monotone real valued sequence will either diverge to infinity or converge to some real number, the monotonicity of the optimal capital sequences $x_i \in Br^i(x_j^*)$ will be crucial in the analysis of the dynamic properties and the long-run behavior of the best response correspondences.

Proposition 3.6 *For any initial condition (x_0^i, x_0^j), the optimal path $x_i \in Br^i(x_j^*)$ is monotonic.*

Proof. Since μ^i is increasing, if $x_0^i > x_1^i$, we have $x_1^i > x_2^i$. Then, by induction, it is true that $x_t^i > x_{t+1}^i$, $\forall t$. If $x_1^i > x_0^i$, using the same argument yields $x_{t+1}^i > x_t^i$, $\forall t$. Now if $x_1^i = x_0^i$, then $x_0^i \in \mu^i\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right)$. Recall that there exists a unique equilibrium path from x_t^i for any $t \geq 1$. Since $x_0^i \in \mu^i\left(x_0^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right)$, $x_t^i = x_0^i$, $\forall t$. ■

We will now present the condition under which the convergence to a steady state is guaranteed and concentrate on the behavior of the optimal paths $x_i \in Br^i(x_j^*)$ associated with $\mathcal{P}$.

Proposition 3.7 *i) There exists an $\xi > 0$ such that if $\sup_{x>0} f'(x) < \frac{1-\xi}{\beta_m}$, then any optimal path $x_i \in Br^i(x_j^*)$ converges to zero.*

ii) Assume $x_0^i > 0$. Let $\inf_{x>0} \beta(x) = \underline{\beta}$. If $\frac{\partial g^i(0, x_0^{j^*})}{\partial x_0^i} > \frac{1}{\underline{\beta}}$, then the optimal path $x_i^* \in Br^i(x_j^*)$ converges to a steady state $x^i > 0$.

Proof. See Erol et al. (2011). ■

We will now concentrate on the existence of an open-loop Nash equilibrium to the non-cooperative game $(N, \mathbf{S}, \{U^i : i \in N\})$. To achieve our goal, we will show that the non-cooperative game $(N, \mathbf{S}, \{U^i : i \in N\})$ is a supermodular game under open-loop strategies. Besides, we will prove that under some regularity conditions, the set of equilibria is a non-empty complete lattice.

3.4 Supermodular games and the existence of Nash equilibrium

Let us first outline the fundamental properties of the supermodular games:

Definition 3.1 *(Topkis, 1998) A non-cooperative game* $(N, \mathbf{S}, \{U^i : i \in N\})$ *is a supermodular game if the set* S *of admissible joint strategies is a sublattice of* R^m *(or of* $\prod_{i\in N} R^{m_i}$*), and if for each* $i, j \in N$, $i \neq j$, *the payoff function* U^i *is supermodular in* x_i *on* S^i *for each* x_j *in* S^j *and* U^i *has increasing differences in* (x_i, x_j) *on* $S^i \times S^j$.

These hypotheses on the payoff function for each agent i, imply that any two components of agent i's strategy are complements and each component of i's strategy is complementary with any component of j's strategy. The following theorem provides the existence of extremal equilibria in supermodular games with modest regularity conditions.

Theorem 3.1 *(Topkis, 1998) Consider a supermodular non-cooperative game denoted by* $(N, \mathbf{S}, \{U^i : i \in N\})$ *for which the set* S *of admissible joint strategies is nonempty, compact and for each* $i, j \in N$, $i \neq j$, *the payoff function* U^i *is upper semicontinuous in* x_i *on* $S^i(\boldsymbol{x}_j)$ *for each* x_j *in* S^j, *then the set of equilibria is a nonempty complete lattice and a greatest and a least equilibrium exist.*

There are two fundamental elements in supermodular games: the ability to order elements in the strategy space of the agents and the strategic complementarity, which implies upward sloping best responses. These properties of supermodular games have been extensively used in static games and to some extent in dynamic games with Markov perfect strategies (see Cooper, 1999; Amir, 2005; Vives, 2005, for a general review). We already stated in the introduction that open-loop strategies do not allow for genuine interaction between the players during the game. However, they are most appropriate in situations where the information about the other players is reduced to their initial condition or games with a very large number of players. We can add games in which agents decide whether to enter or not a coalition. Suppose a number of players who evaluate the benefits of entering a coalition for a given period of time. They would set the rules of the coalition at the time the decision is taken. To evaluate the coalition, players would play open-loop strategies. For all these reasons, we believe it is important to show how the supermodular game structure can be utilized in the analysis of dynamic games under open loop strategies.

Here we consider a dynamic game with open-loop strategies and place related restrictions on strategy spaces and payoff functions which lead to ordered strategy sets and monotone best responses: as the other player selects higher strategies, the remaining player will as well. This will allow us to derive conclusions on the nature of best responses and the set of equilibria. To this end, the next proposition is crucial as it establishes the conditions under which our capital accumulation game turns out to be supermodular.

Proposition 3.8 *The non-cooperative game* $(N, \mathbf{S}, \{U^i : i \in N\})$ *is a supermodular game if for each* $i, j \in N$, $i \neq j$, *and for all* t,

$$\frac{\partial^2 \left[\beta(x_t^i) u \left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right)\right]}{\partial x_t^i \partial x_{t+1}^i} \geq 0, \tag{29}$$

$$\frac{\partial^2 \left[\beta(x_t^i) u \left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right)\right]}{\partial x_t^i \partial x_t^j} \geq 0. \tag{30}$$

The set of equilibria for this supermodular game is a nonempty complete lattice and there exist a greatest and a least equilibrium.

Proof. See the Appendix. ■

In this dynamic game with open-loop information structure strategies are vectors instead of simple scalars. Hence, the game requires an additional restriction to guarantee that all components of an agent's best response vector move together. This explains the role of the restriction (29) that the payoff function of each agent has to be supermodular in his own strategy given the strategy of his rival. Put differently, given the choice of the rival, the agent is better off combining high activity in one component of choice with high activity in another (see Cooper, 1999). The restriction (30) ensures that the gains to a higher strategy by one player increase with the strategy taken by the other so that the best responses turn out to be monotone. The key characteristic of a supermodular game, namely the presence of strategic complementarities is ensured by the restriction (30) which essentially implies the monotonicity property of the best responses.

Though the conditions (29) and (30) can be interpreted along general lines regarding the supermodularity of the non-cooperative dynamic games under open loop information structure, a further refinement of the conditions will be useful in providing their limitations and economic interpretations to the full extent. Indeed, under Assumptions (1)-(3), the conditions (29) and (30) can be recast for each $i, j \in N$, $i \neq j$, and for all t as

$$\frac{\beta'(x_t^i)}{\beta(x_t^i)} + \frac{u''(c_t^i)}{u'(c_t^i)} \frac{\partial g^i(x_t^i, x_t^j)}{\partial x_t^i} \leq 0,$$

$$\left(\frac{\beta'(x_t^i)}{\beta(x_t^i)} + \frac{u''(c_t^i)}{u'(c_t^i)} \frac{\partial g^i(x_t^i, x_t^j)}{\partial x_t^i} \right) \frac{\partial g^i(x_t^i, x_t^j)}{\partial x_t^j} + \frac{\partial^2 g^i(x_t^i, x_t^j)}{\partial x_t^i \partial x_t^j} \geq 0,$$

respectively. Accordingly, the supermodularity of the non-cooperative game given by $(N, \mathbf{S}, \{U^i : i \in N\})$ crucially depends on the sensitivity of the agents' time preferences with respect to their stock of wealth and the sensitivity of the gains to a higher strategy by one player with respect to the rival's stock of capital. Note that for sufficiently low values of the marginal rate of patience, if the aggregate income of each player decreases with the rival's stock of capital, a sufficient condition for the supermodularity of the noncooperative game turns out to be the supermodularity of the aggregate income of each agent, i.e. $\frac{\partial^2 g^i(x_t{}^i, x_t{}^j)}{\partial x_t^i \partial x_t^j} \geq 0$, $\forall i, j \in N$, $i \neq j$, and $\forall t$. However, if the aggregate income of each player increases with the rival's stock of capital then the supermodularity of each agent's aggregate income becomes necessary for the game to be supermodular.

Proposition 3.8 ensures the existence of an open loop Nash equilibrium path in a supermodular game. The following proposition provides monotone comparative statics results.

Proposition 3.9 *Let T be a partially ordered set of parameters and $(\Gamma(\tau), \tau \in T)$ with $\Gamma(\tau) = (N, \mathbf{S}^\tau, \{U^{\tau i} : i \in N\})$ be a parameterized family of supermodular games. S^τ and $U^{\tau i}$ denote the dependence of S and U^i on parameter τ. The set S^τ of admissible joint strategies is nonempty and compact for each τ in T and is increasing in τ on T. If for each $i, j \in N$, $i \neq j$, and for all t,*

$$\frac{\partial^2 \left[\beta(x_t^i, \tau) u \left(g^i(x_t^i, x_t^j, \tau) - x_{t+1}^i, \tau\right)\right]}{\partial x_t^i \partial \tau} \geq 0, \tag{31}$$

then the greatest and the least equilibrium of game $\Gamma(\tau)$ are increasing in τ on T.

Proof. See the Appendix. ■

On a parameterized collection of supermodular games, the condition (31) ensures that the best response correspondence of the agents, hence the extremal equilibrium will be increasing in a parameter that may be endogenous or exogenous due to the strategic complementarity inherent in the agents' strategies. As an example, consider that $\beta(x) = \eta - \gamma e^{-(x+\rho)^{\varepsilon}}$ where $0 < \gamma e^{-\rho^{\varepsilon}} < \eta < 1$, $0 < \varepsilon < 1$, and $\rho > 0$ (see Stern, 2006). Proposition 3.9 asserts that an increase in η, a measure of patience, leads to an increase in the extremal open loop Nash equilibria of the supermodular game $(N, \mathbf{S}^{\eta}, \{U^{\eta i} : i \in N\})$. That is, the more patient agents are, the larger the individuals capital stock at every period in the greatest and the least equilibrium trajectories.

In what follows we will concentrate on the dynamic properties of the open-loop Nash equilibria of the supermodular game $(N, \mathbf{S}, \{U^i : i \in N\})$. Since the open loop Nash equilibrium is weakly time consistent, instead of referring to the discrete time Hamiltonian, we will focus on the closed loop representation of the equilibrium strategies and utilize (26), (27), and Proposition 3.7 in determining the dynamic properties of the equilibrium paths.

3.5 Dynamic Properties of the Open-Loop Nash Equilibrium and the Steady State

Let (x_i^*, x_j^*) be an open loop Nash equilibrium from (x_0^i, x_0^j) of the noncooperative game $(N, \mathbf{S}, \{U^i : i \in N\})$. We denote by $\{x_0^i, \boldsymbol{x}_i^*\}$ the i'th agent's trajectory of capital stock at such an equilibrium path, i.e., $\{x_0^i, \boldsymbol{x}_i^*\} = \left(x_0^i, x_1^{i^*}, .., x_t^{i^*}, ...\right)$. It is then clear that (x_i^*, x_j^*) satisfies (23). Hence x_i^* solves $\mathcal{P}$ given x_j^* so that $x_{t+1}^i \in \mu^i\left(x_t^i \mid \{x_0^j, \boldsymbol{x}_j^*\}\right), \forall t$.

Let $\inf_{x>0} \beta(x) = \underline{\beta}$ and $\frac{\partial g^i(0, x_0^{j^*})}{\partial x_0^i} > \frac{1}{\underline{\beta}}$, for each $i, j \in N, i \neq j$. The optimal paths $x_i^* \in Br^i(x_j^*)$ and $x_j^* \in Br^j(\boldsymbol{x}_i^*)$ converge to the steady state values $x_i > 0$ and

$x_j > 0$, respectively. These steady state values solve the following stationary state Euler equations:

$$u'\left(g^i(x^i,x^j)-x^i\right)=\beta'(x^i)\frac{u\left(g^i(x^i,x^j)-x^i\right)}{1-\beta(x^i)}+ \\ \beta(x^i)u'\left(g^i(x^i,x^j)-x^i\right)\frac{\partial g^i(x^i,x^j)}{\partial x^i}, \forall i,j\in N, i\neq j. \quad (32)$$

However, it is important to note that the stationary sequences associated with each solution of (32) may not induce a steady state open loop Nash equilibrium (see Dockner and Nishimura, 2001) unless they constitute a best reply to each other. A steady state open loop Nash equilibrium (x_i, x_j) is defined as the stationary sequences associated with a solution (x_i, x_j) to (32) such that $\{x_0^i, \boldsymbol{x}_i\} = (x^i, x^i, .., x^i, ...) \in Br^i(x_j)$ and $\left\{x_0^j, \boldsymbol{x}_j\right\} = (x^j, x^j, .., x^j, ...) \in Br^j(x_i)$.

If the game was symmetric, we could have already concluded that the two agents will end up with the same amount of physical capital in the long-run. Indeed, if a supermodular game is symmetric, then a greatest and a least equilibria exist and they are symmetric. Amir et al. (2008) show that monotonicity induces the greatest and the least equilibrium converge to the highest and the lowest symmetric steady states, respectively. A game is symmetric supermodular if on top of the usual conditions for supermodularity, the agents' strategy spaces are identical and

$$U\left(\boldsymbol{x}_i \mid \boldsymbol{x}_j\right) = U\left(\boldsymbol{x}_j \mid \boldsymbol{x}_i\right),\ \forall(\boldsymbol{x}_i, \boldsymbol{x}_j) \in \mathbf{S}.$$

Neither of these latter conditions are met in our game. Strategy spaces coincide only when the initial conditions are identical for both of the agents. Furthermore, the second condition also fails at the first period since we have $u\left(g^i(x_0^i, x_0^j) - x_1^i\right) \neq u\left(g^j(x_0^j, x_0^i) - x_1^i\right)$ for each $i, j \in N, i \neq j$. Despite the lack of symmetry in the game, we prove that our game always leads to symmetric steady states so that the initial wealth differences vanish in the long run.

Proposition 3.10 *All steady state open loop Nash equilibria of the supermodular game* $(N, \mathbf{S}, \{U^i : i \in N\})$ *are symmetric.*

Proof. See the Appendix. ∎

In the following proposition, we show that the individual level of capital stock at the lowest and the highest steady state open loop Nash equilibrium is greater than the lowest and the highest steady states of the associated single agent optimal growth problem, respectively.

Proposition 3.11 *Let* x_L *and* x_H *denote the lowest and the highest steady states of the optimal growth problem and* x_L^o *and* x_H^o *denote the lowest and the highest steady state open loop Nash equilibrium of the noncooperative game* $(N, \mathbf{S}, \{U^i : i \in N\})$*. If the game is supermodular then we have* $x_L \leq x_L^o$ *and* $x_H \leq x_H^o$*.*

Proof. See the Appendix. ∎

Next, we identify the most preferred steady state depending on the game elements.

Corollary 3.1 *Let* $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ *and* $(\check{\mathbf{x}}, \check{\mathbf{x}})$ *denote the highest and the lowest symmetric stationary open loop Nash equilibria of the game* $(N, \mathbf{S}, \{U^i : i \in N\})$*. If*

$$\frac{\partial}{\partial x}\left(\frac{u\left(g(x,x)-x\right)}{1-\beta(x)}\right) > 0,$$

then $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ *is the most preferred steady state open loop Nash equilibrium. Otherwise,* $(\check{\mathbf{x}}, \check{\mathbf{x}})$ *will be the most preferred.*

Proof. See the Appendix. ■

Recall that the stationary sequences associated with each solution of (32) may not constitute a steady state open loop Nash equilibrium. This moves the concern on the number of solutions to the stationary Euler equations and among those that will be induced by a steady state open loop Nash equilibrium. Since one cannot provide an analytical answer to this question, we will move on to the numerical analysis of our problem. Next section is devoted to this end. For given functional forms, we analyze the solutions to (32). In order to determine which of these indeed constitute a steady state open loop Nash equilibrium, we employ the iterations of the Bellman operator (26) for the given stationary strategy of the rival. We compute the dynamics of the equilibrium paths from an initial condition, using the supermodular structure of the game on top of the iterations of the Bellman operator.

3.6 Characterization of the Long-Run Equilibria: Numerical Analysis

The analysis of the solutions to the stationary Euler equations and the determination of solutions induced by a stationary open loop Nash equilibrium can not be carried out without specifying the forms of the utility, discount and the production functions. In what follows, our analysis will be based on the functional forms specified in accordance with Stern (2006). The utility, production and the discount functions are specified as

$$
\begin{aligned}
u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \\
f(x) &= Ax^{\alpha} + (1-\delta)x, \\
\beta(x) &= \eta - \gamma e^{-(x+\rho)^{\varepsilon}},
\end{aligned}
$$

where $0 < \{A, \rho\},\ 0 < \alpha, \sigma, \varepsilon < 1$, and $0 < \gamma e^{-\rho^{\varepsilon}} < \eta < 1$. Under these functional forms, in reference to Proposition 3.8, a sufficient condition for our strategic growth model to become a supermodular game is:

$$\frac{\beta'(x_t^i)}{\beta(x_t^i)} \leq \frac{(\sigma - 1)\left(x_t^j + \alpha x_t^i\right) + x_t^i}{x_t^i\left(x_t^i + x_t^j\right)}, \forall t \geq 1. \tag{33}$$

Condition (33) is trivially checked for $\sigma \geq 1$ when ρ is sufficiently large, since $\frac{\beta'}{\beta}$ is a decreasing function and its supremum equals $\frac{\gamma\varepsilon\rho^{-1+\varepsilon}}{\eta - \gamma e^{-\rho^{\varepsilon}}}$. In accordance with these, we utilize the following set of fairly standard coefficients as our benchmark parameterization:

$$A = 0.75,\ \alpha = 0.4,\ \delta = 0.03,\ \sigma = 1.5,\ \eta = 0.95,\ \gamma = 2.5,\ \rho = 4.5,\ \varepsilon = 0.99,$$

under which the maximum sustainable level of capital stock turns out to be:

$$A\left(x_0^i + x_0^j\right) = \max\left\{\left(x_0^i + x_0^j\right), \bar{x}\right\},\ \text{where } \bar{x} \text{ is } 213.747.$$[4]

We prove in the first subsection of the sequel that strategic interaction removes indeterminacy, in case indeterminacy existed in the single agent optimal growth problem. Then, we continue our numerical analysis with the case in which multiplicity of equilibria persists in the long-run.

[4]Our analysis utilizes a felicity function constrained to a negative domain. Schumacher (2011) shows that if the discount rate is endogeneized via a state variable, the domain of the felicity function should be constrained to a positive domain. In a negative domain, a higher stock of capital would have a negative impact on overall welfare. However, under our parameterization, even with a negative felicity function constrained to a negative domain, a higher stock of capital will have a positive impact on welfare. Indeed, our discount factor attributed to the utility of consumption at period t increases with the level of capital stock, as in Becker and Mulligan (1997) and Stern (2006).

3.6.1 Strategic Interaction Removes Indeterminacy

Under our benchmark parametrization, we prove numerically that strategic growth removes indeterminacy and implies global convergence towards a unique symmetric steady state. In order to provide a better exposition of this point, and to provide a basis of comparison for our strategic growth model, we first recall the analysis of Erol et al. (2011) and we concentrate on the dynamic implications of endogenous discounting in the single agent framework.

Case 3.1 *Optimal growth framework.*

Consider problem $\mathcal{P}$ in which only agent i starts with a positive stock, $x_0^j = 0$ and $x_0^i > 0$, i.e. agent i acts alone. There exist three solutions to (32): $x_l = 0.5953$, $x_m = 2.9155$, and $x_h = 8.4913$. In order to determine which of these are actually the optimal steady states, we analyze the optimal policy using the Bellman operator. Figure 2 indicates that x_l and x_h are stable optimal steady states, but in contrast with Stern (2006), Figure 2 strongly indicates that x_m is not an optimal steady state. Indeed, if it were, the optimal policy would have crossed the $y = x$ line at x_m. The Bellman operator also reveals the existence of a genuine critical point at $x_c \approx 5.5846$: for any initial capital stock level lower than x_c, the economy will face a development trap, enforcing convergence to a very low capital level x_l. On the other hand, for any initial capital level higher than x_c, the optimal path will converge to x_h. However, if an economy starts at x_c, an indeterminacy will emerge.

Case 3.2 *Strategic growth*

Consider now the case where $(x_0^i, x_0^j) > 0$. Let $\inf_{x>0} \beta(x) = \underline{\beta}$. Note that $\frac{\partial g^i(0, x_0^{j*})}{\partial x_0^i} > \frac{1}{\underline{\beta}}$. Under the benchmark parameter values, there exists a unique symmetric solution of the stationary state Euler equations (32): $(x_i^*, x_j^*) = (10.8906, 10.8906)$.

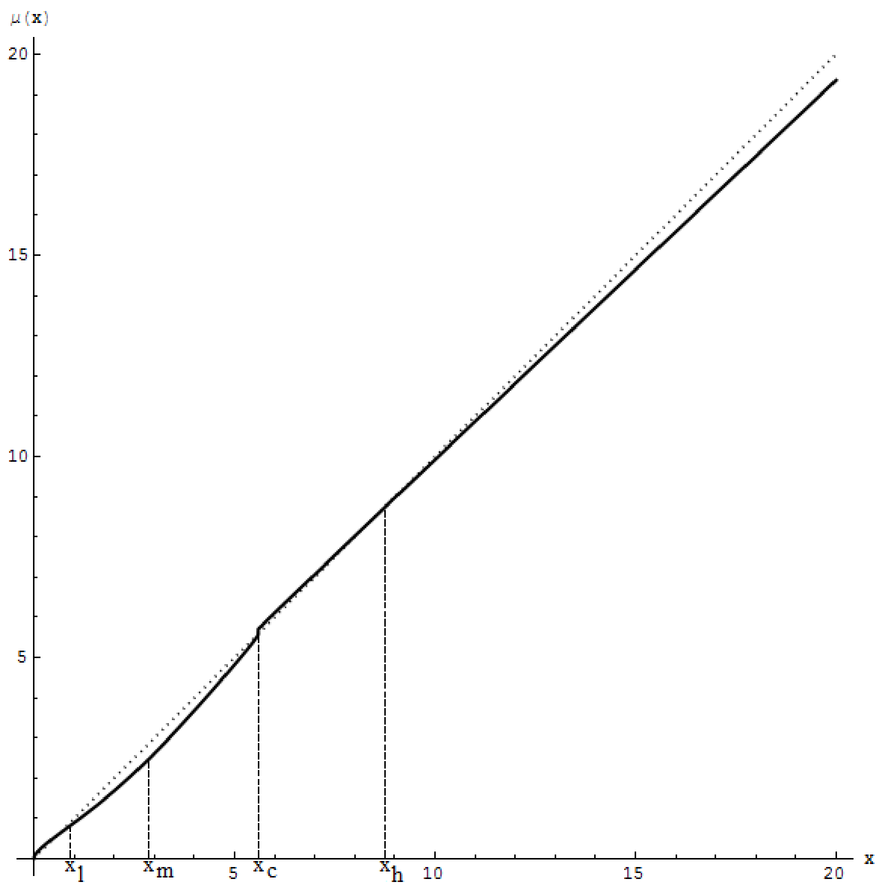

Figure 2: **Optimal policy after** 300 **iterations on the zero function**

Indeed, consider a two region economy, one with a large initial stock of capital and the other with almost none, take for example $x_0^i = 0.1$ and $x_0^j = 10$. Whenever they act independently of each other, the poor region ends up in a development trap, $x_l = 0.5953$, whereas the rich region reaches a steady state level of capital stock $x_h = 8.4913$. However, if there exists a strategic interaction between the two regions, they both reach an identical steady state level of capital $(x_i^*, x_j^*) = (10.8906, 10.8906)$. Thanks to the strategic interaction between regions, the rich region pulls the poor out of poverty trap while sustaining a higher level of steady state capital stock in the rich region.

3.6.2 Multiplicity of Equilibria

As we have already underlined, in a single agent optimal growth framework, the capital dependent time preference rate generates a critical point. In the vicinity of this critical point, small differences lead to permanent differences in the optimal path. Since this result heavily depends on the value of ρ, we would like to deviate from the benchmark to explore how strategic interaction modifies the single agent result. Hence, we assign a lower value to ρ, $\rho = 4$. The supermodular game $(N, \mathbf{S}, \{U^i : i \in N\})$ studied in this section exhibits multiple long-run equilibria although a single agent optimal growth model exhibits global convergence. Indeed, there exist three solutions to the stationary Euler equation (32) of the single agent optimal growth problem: $x_l = 0.3708$, $x_m = 4.0061$, and $x_h = 8.4315$. Among these three solutions, x_l turns out to be the only optimal steady state (see Figure 3). The natural question is then to what extent strategic growth dynamics are affected from such a change? When we consider the dynamic implications of strategic growth we note that there are multiple solutions of the stationary state Euler equations as listed below. The stationary sequences associated with only those in bold are indeed

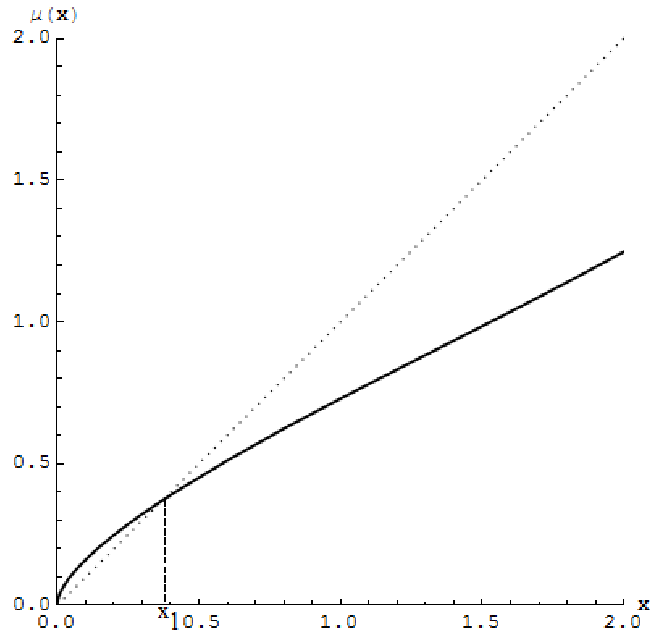

Figure 3: **Low steady state** ($x_l = 0.59$) **is optimal**

constituting a steady state open loop Nash Equilibrium as we show later:

$$(x_i, x_j) = \{(\mathbf{0.864}, \mathbf{0.864}), (2.2404, 2.2404), (\mathbf{10.8863}, \mathbf{10.8863}),$$
$$(2.7941, 1.1354), (2.3869, 10.4927), (1.1353, 2.7941), (10.4927, 2.3869)\}.$$

The stationary sequences associated with the asymmetrical solutions above do not constitute a steady state open loop equilibrium of our game as announced in Proposition 3.10. Among the three symmetric solutions to the stationary Euler equations (32), the stationary sequences associated with the lowest and the highest pairs constitute a steady state open loop equilibrium is shown in Figures 4 and 5 respectively. In other words, when the initial condition is $(x_0^i, x_0^j) = (10.8863, 10.8863)$, then the stationary strategies of the agents sticking to the initial condition constitute a best reply to each other so that $(10.8863, 10.8863)$ turns out to be a steady state open loop equilibrium. The same is true for $(0.864, 0.864)$. As we show in Proposition 3.11, both of them are higher than the unique steady state of the single agent optimal growth problem.

We can easily show that threshold dynamics emerge. There exist critical values of initial capital (x_c, x_c), below which an open loop Nash equilibrium of our supermodular game will converge to the lowest steady state $(0.864, 0.864)$. There exists a second critical point, (x^c, x^c) above which a sequence of an open loop Nash equilibrium will converge to the highest steady state $(10.8863, 10.8863)$. Noteworthy, these critical values are not a solution of the stationary state Euler equations so that the stationary sequences associated with these can not constitute a stationary state open loop equilibrium of the game. As the optimal policy of agent i is upper semi continuous, given the strategy of the rival $x_j^* \to 10.8863$, the graph of $\mu^i(x^i \mid \{x_0^j, x_j^*\})$ jumps over $45°$ line at x^c so that $x^c \notin \mu^i(x^c \mid \{x_0^j, x_j^*\})$. With an analogous reasoning, given the strategy of the rival is $x_j^* \to 0.864$, the graph of $\mu^i(x^i \mid \{x_0^j, x_j^*\})$ jumps over $45°$ line at x_c so that $x_c \notin \mu^i(x_c \mid \{x_0^j, x_j^*\})$ This implies even further that as soon as $x^c = x_c$ an indeterminacy arises so that for a game emanating from such a critical stock of capital, the best responses of the two agents that converge either to low or to the high steady state may both constitute an open loop Nash equilibrium.

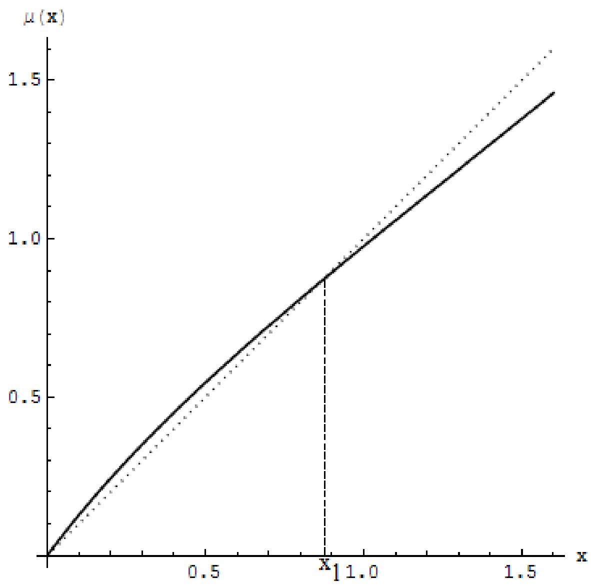

Figure 4: **Stationary sequence associated with** $(x_i = 0.863991)$ **is an open loop Nash equilibrium**

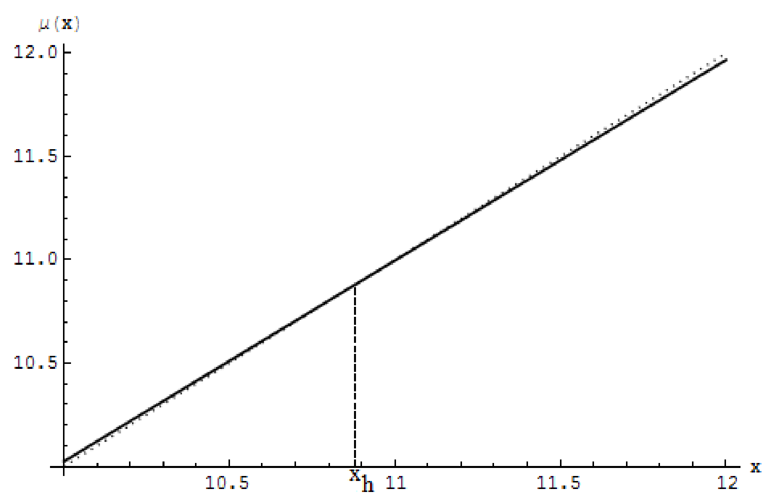

Figure 5: **Stationary sequence associated with** $(x_i = 10.8863)$ **is an open loop Nash equilibrium**

CHAPTER 4

GAMES OF COMMON PROPERTY RESOURCES UNDER ENDOGENOUS DISCOUNTING

The exploitation of the natural resources over time is directly linked with the issue of time preference. While the time preference is one of the major factor on allocation of the resources, these resources can also affect society's time preference. There is a new but growing literature considering the dependence of time preference to the aggregate variables while studying extinction and exploitation of renewable resources. Meng (2006) extends the one sector optimal growth model by considering the effects of social factors on time preference, particularly economy wide avarage consumption and production. Ayong Le Kama and Schubert (2007) study the consequences of endogenous time preference depending on the environmental quality in an optimal growth framework. Tsur and Zemel (2009) consider the social discount rate that includes the occurance hazard under risk of abrupt climate change and study optimal emission policy for a growing economy. However, none of these studies investigate the implications of endogenous discounting under strategic interaction.

The fishery model has been used as a metaphor for any kind of renewable resource on which the property rights is not well defined.(see Long, 2010, for a comprehensive survey) In these models, the sets of feasible strategies available to the players are interdependent and in addition, the agents' choices in the current period affect the payoffs and their choice sets in the future.

In this chapter, we focus on the economy wide determinants of the individual rate of time preference, particularly, we consider the socially determined time preference which depends on the level of fish stock and completely characterize the basic fishery model under this setup. Our aim is to incorporate this insight into the theoretical formulations in a way that will increase the accuracy of predictions while not losing any other values like tractability and generality.

Our interest focuses on the qualitative properties of open-loop Nash equilibrium (OLNE). By using a discrete time formulation, we study existence and efficiency of the equilibrium. Under constant discounting, there exists OLNE that is Pareto-efficient has been shown by Chierella et. al. (1984) and Dockner and Kaitala (1989) in continuous time. In discrete time, Amir and Nannerup (2006) obtain this result for a logarithmic utility and cobb douglas production function. We show that this result can not be extended to the endogenous discounting case: We can not rely on symmetric social planner problem while showing existence and qualitative properties of Nash equilibrium. Instead, we use a topological fixed point theorem to show existence of OLNE.

Depending on the return is bounded or unbounded from below, OLNE may result in overexploitation or underexploitation of the resources relative to efficient solution. The OLNE differs from the collusive equilibria in terms of not only efficiency but also equilibrium dynamics. As it is shown by Erol et. al. (2011) under standard preferences and technology, endogenous discounting creates threshold dynamics such that every equilibrium that starts to the left of the threshold level converges to low steady state and every equilibrium that starts to the right of the threshold level converges to the high steady state. We show that open loop information structure can remove indeterminacy that we may face under collusive equilibrium and be a source of multiplicity despite the uniqueness we may face under collusive equilibrium.

This chapter is organized as follows. The next section introduces the model. The existence and efficiency of collusive, and open loop equilibrium are discussed successively in section 3 and 4. Finally, section 5 concludes.

4.1 The Model

Consider a finite number, n, of agents having identical preferences. $N = \{1, 2, ..., n\}$ is the set of players. The real valued function u gives the instantaneous utility from consumption where the ith player consumption at period t is denoted by c_t^i. At each period t, $f(k_t)$ units of the resource are available for consumption. Then, the leftover resource stock, $k_{t+1} = f(k_t) - \sum\limits_{i=1}^{N} c_t^i$, generates the next period resource, $f(k_{t+1})$. In accordance with these, the problem of each agent i can be formalized as follows:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(k_s) \right) u(c_t^i), \tag{34}$$

subject to

$$\forall t, \ \sum_{s=1}^{N} c_t^s + k_{t+1} \leq f(k_t) \tag{35}$$

$$\forall i, c_t^i \geq 0, \ k_{t+1} \geq 0, \tag{36}$$

$$k_0 > 0, \text{given},$$

where the real valued function $\beta(k_s)$ is the level of discount on future utility.

We make the following assumptions regarding the properties of the utility, production and discount functions.

Assumption 4.1 *$u : R_+ \to R_+$ is continuous, twice continuously differentiable and satisfies $u(0) = 0$. Moreover, it is strictly increasing, strictly concave and $u'(0) = +\infty$ (Inada condition).*

Assumption 4.2 *$u : R_+ \to R_+$ is continuous, twice continuously differentiable and $u(0) = -\infty$. Moreover, it is strictly increasing and strictly concave.*

Assumption 4.3 *$f : R_+ \to R_+$ is continuous and strictly increasing. Moreover, there exists an $\bar{k}$ such $f(k) < k$ whenever $k > \bar{k}$ and $f'(0) = +\infty$.*

Assumption 4.4 *$\beta : R_+ \to R_{++}$ is continuous and differentiable. Moreover, it satisfies $\sup_{k>0} \beta(k) = \beta_m < 1,\ \sup\limits_{k>0} \beta'(k) < +\infty$.*

Since the Samuelson's seminal work in 1937, discounted utility model in which an instantenous utility is discounted with a constant rate has been used broadly in spite of the reservations of the Samuelson by himself, on the normative and descriptive validity of the model. We assume that the rate of time preference depends on the resource stock. In particular, the discount factor attributed to the utility of consumption at period t increases with the level of resource stock available at the same period.

4.2 The Collusive Equilibrium

Social planner who is seeking the symmetric pareto optimal solution solves the following problem:

$$\max_{\left\{k_t^i,c_t^i\right\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(k_s) \right) \sum_{s=1}^{N} u(c_t^s)$$

subject to

$$\forall t, \ \sum_{s=1}^{N} c_t^s + k_{t+1} \leq f(k_t)$$

$$k_0 \geq 0, \ given$$

For any initial condition $k_0 \geq 0$, when $k = (k_0, k_1, k_2, ...)$ is such that $0 \leq k_{t+1} \leq f(k_t)$ for all t, we say it is feasible from k_0 and the class of all feasible accumulation paths is denoted by $\Pi(k_0)$.

As the period utility is strictly concave, we can impose symmetry of agent's consumption paths and the problem turns out to be:

$$V(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \prod_{s=1}^{t} \beta(k_s) u(c_t)$$

subject to

$$\forall t, \ nc_t + k_{t+1} \leq f(k_t)$$

$$k_0 \geq 0, given$$

The case with bounded from below returns is analyzed completely by Erol et al.(2011). By generalizing to the case with unbounded from below returns, we provide the results needed:

Proposition 4.1 *Under the assumptions of 4.1 or 4.2 and 4.3 and 4.4, for any* $k_0 \geq 0$,

(i) There exists an optimal accumulation path k *such that*

$$\forall t,\ k_t \leq A(k_0) = \max\{k_0, \bar{k}\}.$$

The associated optimal consumption path c *is given by:*$\forall t \in 1..\infty$ *and* $\forall s \in N$, $c_t^s = \frac{f(k_t)-k_{t+1}}{n}$.

(ii) If $k_0 > 0$, *the optimal path* $(\mathbf{c}, \mathbf{k})$ *satisfies* $k_t > 0$, $c_t^s = c_t^l > 0$ *for all* $s, l \in N$.

(iii) $V(k_0)$ *is strictly increasing and verifies the Bellman equation:*

$$V(k_0) = \max\left\{u\left(\frac{f(k_0) - k_1}{N}\right) + \beta\left(k_1\right)V(k_1) : 0 < k_1 < f(k_0)\right\}. \tag{37}$$

(iv) Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ *be the argmax correspondence of 37. Then,* g *is non-empty, upper semi-continuous and increasing in the sense that if* $k_0 \leq k_0'$ *then* $k_1 \in g\left(k_0\right) < k_1' \in g\left(k_0'\right)$. *Moreover, a sequence* k *feasible from* $k_0 \geq 0$ *is optimal if and only if* $k_{t+1} \in g\left(k_t\right)$ *for all* t.

(v) When $k_0 > 0$ *any solution* $k \in \Pi\left(k_0\right)$ *satisfies the Euler equation:*

$$u'(c_t) = \beta\left(k_{t+1}\right)u'(c_{t+1})f'(k_{t+1}) + N\beta'\left(k_{t+1}\right)V\left(k_{t+1}\right) \tag{38}$$

(vi) An optimal path k *from* k_0 *is monotonic.*

Proof. Under the case with bounded from below returns, the results follows from Erol et. al.(2011) Lemma 1 and Proposition 1,2,3,4,5 and 8.

When we have unbounded from below returns, we have to reconsider part (i) and (iv). As the utility and the discount functions are strictly increasing, we introduce function U defined on the set of feasible sequences as

$$U(\mathbf{k}) = \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(k_s) \right) u(\frac{f(k_t) - k_{t+1}}{N}).$$

Since $u()$ is concave, there exist A$\geq 0, B \geq 0$ such that u$(c) \leq Bc + C$, $\forall c \geq 0$.For every $k \in \Pi(k_0)$, we have $f(k_t) \leq A(k_0)$. This implies that, for a given $\epsilon > 0$, there exist T_0 such that, for any $T \geq T_0$, for any $k \in \Pi(k_0)$, we have

$$\sum_{t=T}^{\infty} \left(\prod_{s=T+1}^{t} \beta(k_s) \right) u^{+}(\frac{f(k_t) - k_{t+1}}{n}) \leq \beta_m^T \frac{BA(k_0) + C}{1 - \beta_m} \leq \epsilon.$$

where u^{+} denotes the positive value of u.

Let $k^{\mathbf{n}} \in \Pi(k_0)$ converge to $k \in \Pi(k_0)$ and $\epsilon > 0$ be given. Then there exists T_0 such that, for any n, for any $T \geq T_0$,

$$U(\mathbf{k^n}) \leq \sum_{t=0}^{T} \left(\prod_{s=1}^{t} \beta(k_s^n) \right) u(\frac{f(k_t^n) - k_{t+1}^n}{N}) + \epsilon.$$

By letting $T \to \infty$, we have

$$\limsup U(\mathbf{k^n}) \leq U(\mathbf{k}) + \epsilon$$

Since ϵ is arbitrarily chosen, we have $\limsup U(\mathbf{k^n}) \leq U(\mathbf{k})$ i.e. $U()$ is upper semi-continuous. The existence of the optimal solution follows from the fact that $U(\mathbf{k})$ is upper semi continuous and $\Pi(k_0)$ is compact in the product topology (see Erol et.

al.(2011), Lemma 1 a,b.)

(iv) Since $V(0) = -\infty$, we can not use the the theorem of the maximum while proving the upper semi-continuity of $g\left(\right)$. Let $\{k_0^n\}$ be a sequence that converges to k_0 and let $y^n \in g\left(k_0^n\right), \forall n$. We have $0 \leq y^n \leq f(k_0^n), \forall n$. Since $f\left(\right)$ is continuous, the sequence $\{y^n\}$ will be uniformly bounded when n is large enough. We can assume that it converges to some $y \in [0, f(k_0)]$. Take any $z \in [0, f(k_0)]$. One can find a sequence $\{z^n\}$ included in $[0, f(k_0^n)]$ for every n large enough, which converges to z. We have:

$$\forall n, V(k_0^n) = u\left(f(k_0^n) - y^n\right) + \beta\left(y^n\right) V(y^n) \geq u\left(f(k_0^n) - z^n\right) + \beta\left(z^n\right) V(z^n).$$

By letting n go to infinity, we have

$$V(k_0) = u\left(f(k_0) - y\right) + \beta\left(y\right) V(y) \geq u\left(f(k_0) - z\right) + \beta\left(z\right) V(z)$$

so that $y \in g\left(k_0\right)$. ■

Next proposition provides the condition under which the solution to the collusive problem is independent of the number of players.

Proposition 4.2 *If the period utility is homogenous in consumption, then the optimal accumulation path for the collusive problem is independent of number of players.*

Proof. Since $u\left(\frac{f(k_t)-k_{t+1}}{N}\right) = h\left(N\right) u\left(f(k_t) - k_{t+1}\right)$, collusive problem can be written as,

$$V(k_0) = h\left(N\right) \max_{\{k_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \prod_{s=1}^{t} \beta(k_s) u\left(f(k_t) - k_{t+1}\right)$$

which assures that the solution does not depend on N. ■

Since the optimal path is monotonic and bounded with $A(k_0) = \max\{k_0, \bar{k}\}$, there exist a steady state satisfying the necessary condition below:

$$u'\left(\frac{f(k)-k}{N}\right) = \beta(k)\, u'\left(\frac{f(k)-k}{N}\right) f'(k) + N\frac{\beta'(k)\, u\left(\frac{f(k)-k}{N}\right)}{1-\beta(k)} \tag{39}$$

Next, we adopt the noncooperative open loop Nash equilibrium concept, in which players choose their strategies as simple time functions and they are able to commit themselves to the time paths as equilibrium strategies. Players choose their strategies simultaneously and best response correspondance of each player is defined with a single criterion optimization problem constrained by the strategies of the rival taken as given.

4.3 The Open-Loop Equilibrium

First, we show that the OLNE is efficient under constant discounting so that existence of OLNE directly follows from the collusive problem. But, this result can not be generalized to social time preference depending on the stock of wealth.

Given the optimal decisions of the rival, each agent chooses a path of consumption $c^i = \{c_t^i\}_{t \geq 0}$ so as to maximize a discounted sum of instantaneous utilities. In accordance with these, the problem of each agent i can be formalized as follows:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(k_s) \right) u(c_t^i), \text{ subject to} \tag{40}$$

$$\forall t, \ \sum_{s=1}^{N} c_t^s + k_{t+1} \leq f(k_t)$$

$$k_0 \geq 0, \ \boldsymbol{c}_j = \{c_t^j\}_{t=1}^{\infty} \geq 0, \text{given, where } j \neq i \in \{1..N\}$$

Given the open loop strategies of the rivals', agent i faces a dynamic optimization problem. For any initial condition $k_0 \geq 0$, and for a given $c_j \geq 0$, when $k = (k_1, k_2, ...)$ is such that $\sum_{s \neq i} c_t^s \leq f(k_t) - k_{t+1}$ for all t, we say it is feasible from k_0 The class of all such feasible accumulation paths from k_0 is denoted by $S^i(\boldsymbol{c}_{-i})$. A consumption sequence $c_i = (c_0^i, c_1^i, ...)$ is feasible from $k_0 \geq 0$, when there exists $k \in S^i(\boldsymbol{c}_{-i})$ with $0 \leq c_t^i \leq f(k_t) - \sum_{s \neq i} c_t^s - k_{t+1}$. As utility is strictly increasing, we introduce the function U defined on the set of feasible sequences as

$$\mathrm{U}(\boldsymbol{k} \mid \boldsymbol{c}_{-i}) = \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(k_s) \right) \mathrm{u} \left(f(k_t) - \sum_{s \neq i} c_t^s - k_{t+1} \right).$$

We can solve the dynamic optimization problem via the discrete time Maximum Principle (for the details, see Dechert 1997). By introducing the discount rate as another constraint, the problem can be rewritten as:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_t u(c_t^i), \text{ subject to}$$

$$\forall t, \qquad \sum_{s=1}^{N} c_t^s + k_{t+1} \leq f(k_t) \text{ and } \beta_{t+1} = \beta(k_{t+1})\beta_t$$

$$k_0 \geq 0, \ \beta_0 = 1, \boldsymbol{c}_j = \{c_t^j\}_{t=1}^{\infty} \geq 0, \text{given, where } j \neq i \in \{1..N\}$$

The state variables are $\{\beta_t, k_t\}$, costate variables are $\{p_t^i, s_t^i\}$ and control variable is $\{c_t^i\}$. The discrete time Hamiltonian for the i'th agent's problem can be written as:

$$H^i\left(\beta_t, k_t, p_{t+1}^i, s_{t+1}^i, c_t^i\right) = \beta_t u(c_t^i) + p_{t+1}^i \left[f(k_t) - \sum_{s=1}^N c_t^s\right] + s_{t+1}^i \left[\beta(f(k_t) - \sum_{s=1}^N c_t^s)\beta_t\right]$$

The first order condition for the optimization is:

$$\beta_t u'(c_t^i) = p_{t+1}^i + s_{t+1}^i \beta'(f(k_t) - \sum_{s=1}^N c_t^s)\beta_t \tag{41}$$

and the canonical equations are:

$$p_t^i = \left(p_{t+1}^i + s_{t+1}^i \left[\beta'(f(k_t) - \sum_{s=1}^N c_t^s)\beta_t\right]\right) f'(k_t) \tag{42}$$

$$\mathrm{s}_t^i = \mathrm{u}(\mathrm{c}_t^i) + \mathrm{s}_{t+1}^i \left[\beta(f(k_t) - \sum_{s=1}^N c_t^s)\right] \tag{43}$$

By iterating (43), we get

$$s_t^i = \frac{1}{\beta_t} \sum_{r=t}^{\infty} u(c_r^i)\beta_r$$

From (41) and (42), one can recast that

$$p_t^i = \beta_t u'(c_t) f'(k_t).$$

Substituting $\left(p_{t+1}^i, s_{t+1}^i\right)$ and rewriting (41);

$$u'(c_t^i) = \beta\left(k_{t+1}\right) u'(c_{t+1}^i) f'(k_{t+1}) + \frac{\beta'\left(k_{t+1}\right)}{\beta_{t+1}} \sum_{r=t+1}^{\infty} u(c_r^i)\beta_r$$

By defining $V^i(k_{t+1}) = u(c^i_{t+1}) + \beta\,(k_{t+2})\,V^i(k_{t+2})$, we obtain that

$$u'(c^i_t) = \beta\,(k_{t+1})\,u'(c^i_{t+1})f'(k_{t+1}) + \beta'\,(k_{t+1})\,V^i(k_{t+1}) \tag{44}$$

Condition (44) is identical to (38) under constant discounting. This assures that there exist an open loop Nash equilibrium which is efficient. However, this is not valid under endogenously determined time preference. We can not rely on collusive problem while showing existence and qualitative properties of open loop Nash equilibrium.

4.4 Existence of OLNE

We first define the best response correspondance on the feasible accumulation paths. Suppose that we are given a sequence of resource stock $k = (k_1, k_2, ...)$ satisfying $k_{t+1} \leq f(k_t), \forall t$.By the symmetry of the problem, we let all players but player i, consume $c_t = \frac{(f(k_t)-k_{t+1})}{n}$. Consider the following problem,

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\prod_{s=1}^{t} \beta(x_s) \right) u \left(f(x_t) - x_{t+1} - \frac{(N-1)\,(f(k_t) - k_{t+1})}{N} \right) \tag{45}$$

$$\text{subject to } \forall t,\ 0 \leq x_{t+1} \leq f(x_t) - \frac{(N-1)\,(f(k_t) - k_{t+1})}{N}$$

$$x_0 \geq 0,\ \boldsymbol{k} = \{k_t\}_{t=1}^{\infty} \geq 0, \text{given.}$$

The value function associated with the problem (45) takes the following form:

$$V\left(x_0,k_1,k_2,k_3..\right)=\max_{\{x_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\beta(x_s)\right)u\left(f(x_t)-x_{t+1}-\frac{(N-1)\left(f(k_t)-k_{t+1}\right)}{N}\right),$$
$$\text{subject to }\forall t,\ 0\leq x_{t+1}\leq f(x_t)-\frac{(N-1)\left(f(k_t)-k_{t+1}\right)}{N}$$
$$x_0\geq 0,\ \boldsymbol{k}=\{k_t\}_{t=1}^{\infty}\geq 0,\text{given}.$$

Let

$$U\left(x_0,\boldsymbol{k},\boldsymbol{x}\right)=\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\beta(x_s)\right)u\left(f(x_t)-x_{t+1}-\frac{(N-1)\left(f(k_t)-k_{t+1}\right)}{N}\right),$$

and

$$\Pi\left(x_0,\boldsymbol{k}\right)=\left\{\boldsymbol{x}\in\Gamma(k_0)\text{ s.t }0\leq x_{t+1}\leq f(x_t)-\frac{(N-1)\left(f(k_t)-k_{t+1}\right)}{N},\forall t\geq 0\right\}.$$

$$V\left(x_0,\boldsymbol{k}\right)=\max\left\{U\left(x_0,\boldsymbol{k},\boldsymbol{x}\right)\mid\boldsymbol{x}\in\Pi\left(x_0,\boldsymbol{k}\right)\right\}.$$

Let φ denote the solution to the 45 defined as below,

$$\varphi(x_0,\boldsymbol{k})=\arg\max\left\{U\left(x_0,\boldsymbol{k},\boldsymbol{x}\right)\mid\boldsymbol{x}\in\Pi\left(x_0,\boldsymbol{k}\right)\right\},$$

Proposition 4.3 *The solution to the 45, $\varphi(x_0,k)$, is upper semi continuous in k and it is ordered so that if $y,\hat{y}\in\varphi(x_0,k)$ either $y\geq\hat{y}$ or $y\leq\hat{y}$.*

Proof. Let $\{\boldsymbol{k}^n\}$ be a sequence that converges to k and let $y^n\in\varphi(x_0,k^n)$, $\forall n$. We can assume that it converges to some $y\in\Pi\left(x_0,\boldsymbol{k}\right)$. Take any $z\in\Pi\left(x_0,\boldsymbol{k}\right)$. One can find a sequence $\{\boldsymbol{z}^n\}$ included in $\Pi\left(x_0,\boldsymbol{k}\right)$ for every n large enough, which converges to z. We have:

$$\forall n,V\left(x_0,\boldsymbol{k^n}\right)=U\left(x_0,\boldsymbol{k^n},\boldsymbol{y}^n\right)\geq U\left(x_0,\boldsymbol{k^n},\boldsymbol{z}^n\right)$$

By letting n go to infinity, we have

$$V\left(x_0,\boldsymbol{k}\right)=U\left(x_0,\boldsymbol{k},\boldsymbol{y}\right)\geq U\left(x_0,\boldsymbol{k},\boldsymbol{z}\right)$$

so that $y\in\varphi(x_0,k)$.

Let $y,\hat{y}\in\varphi(x_0,k)$.

Case 1: Suppose that $y_1=\hat{y}_1$. Since the y and $\hat{y}$ satisfy 44,we can conclude that $y_t=\hat{y}_t,\forall t$.

Case 2: Let $y_1\neq\hat{y}_1$. Without loss of generality, consider the case of $y_1>\hat{y}_1$. Define the correspondance,

$$\mu(x\mid\boldsymbol{k})=\operatorname{argmax}_{0\leq y\leq f(x)-\frac{(N-1)(f(k_t)-k_{t+1})}{N}}\left[\begin{array}{c}u\left(f(x)-y-\frac{(N-1)(f(k_t)-k_{t+1})}{N}\right)\\+\beta(y)V\left(y,k_{t+1},k_{t+2}..\right)\end{array}\right]$$

By the monotonicity theorem (Amir, 1996), $\mu(x\mid k)$ is increasing in x. So that $y_2>\hat{y}_2$. Then by induction, $y>\hat{y}$. ■

Proposition 4.4 *Let $f()$ be concave. Then there exists an OLNE.*

Proof. We first show that $\Gamma(k_0)$ is compact and convex set. Take any two elements k,k' in $\Gamma(k_0)$ so that $f(k_t)\leq k_{t+1}$ and $f(k_t')\leq k_{t+1}'$ for all t. We have to show that $\left(\lambda k_{t+1}+(1-\lambda)k_{t+1}'\right)\leq f(\lambda k_t+(1-\lambda)k_t')$ where $\lambda\in[0,1]$. For $t=0$, it is trivially satisfied. For $t\neq0$, this inequality follows from the concavity of production function. Continuity of f assures the compactness of $\Gamma(k_0)$ under product topology. By the proposition 4.3, $\varphi(x_0,k)$ is uppersemi continuous and ordered so that it admits an upper semi continuous selection, $\bar{\varphi}(x_0,k)=\max\varphi(x_0,k)$ where

$\bar{\varphi}(x_0, k) \in \varphi(x_0, k)$. Moreover $\bar{\varphi}$ is closed as it is uppersemicontinuous and closed valued. From the Kakutani-Fan-Glicksberg Theorem[5], we can conclude that $\bar{\varphi}(k)$ has a fixed point in $\Gamma(k_0)$ which coincide with the symmetric OLNE of the game. ■

We get the following equation satisfied by the stationary symmetric OLNE

$$\mathrm{u}'\left(\frac{f(k)-k}{N}\right) = \beta(k)\,\mathrm{u}'\left(\frac{f(k)-k}{N}\right)\mathrm{f}'(\mathrm{k}) + \frac{\beta'(k)\,u\left(\frac{f(k)-k}{N}\right)}{1-\beta(k)} \tag{46}$$

Schumacher (2011) shows that if the discount rate is endogeneized via a state variable, the domain of the felicity function should be constrained to a positive domain. In a negative domain, a higher stock of capital would have a negative impact on overall welfare. However, under our parameterization, even with a negative felicity function constrained to a negative domain, a higher stock of capital will have a positive impact on welfare. Indeed, our discount factor attributed to the utility of consumption at period t increases with the level of capital stock, as in Becker and Mulligan (1997) and Stern (2006). Note that when the period utility is homegenous in consumption, it can be recast as

$$H(k) = \frac{u'(f(k)-k)\,(1-\beta(k)\,f'(k))\,(1-\beta(k))}{\beta'(k)\,u(f(k)-k)} = \frac{1}{N} \tag{47}$$

If $H(k)$ is monotone then there is a unique stationary symmetric OLNE and the equation 46 is sufficient. In that case, the following proposition states that the comparative statics of statioanary symmetric OLNE depends on if the returns are bounded or unbounded from below.

[5]Kakutani - Fan - Glicksberg: Let K be a nonempty, compact, convex subset of locally convex Hausdorf space, and let the correspondance $\varphi : K \twoheadrightarrow K$ have closed graph and nonempty convex values. Then the set of fixed points of φ is compact and nonempty.

Proposition 4.5 *Let us assume that the period utility is homogenous in consumption and $H(k)$ is monotone. If the return is bounded (unbounded) from below, then the unique stationary symmetric OLNE increases (decreases) with the number of players.*

Proof. If the return is bounded (unbounded) from below we have $H(0) = +\infty$ $(H(0) = -\infty)$. Since $H(k)$ is monotone and there exist unique stationary symmetric OLNE, 47 is a sufficient condition. Then the result follows from the fact that $\frac{1}{N}$ is decreasing in N. ■

Corollary 4.1 *Let us assume that the period utility is homogenous in consumption and $H(k)$ is monotone. If the return is bounded (unbounded) from below, then the unique stationary symmetric OLNE is higher (lower) than the unique collusive equilibrium.*

Proof. Under collusive equilibrium we have $H(k) = 1 > \frac{1}{N}$. The result is a direct corollary to 4.5. ■

If $H(k)$ is not monotone, there might be multiple solution to (46). Since equation 46 is just a necessary condition, they may not constitute a steady state open loop Nash equilibrium(see Dockner and Nishimura, 2001). This moves the concern on the number of solutions to the stationary Euler equations and among those that will be induced by a steady state open loop Nash equilibrium. The analysis of the solutions to the stationary Euler equations and the determination of which of these solutions constitute a steady-state can not be carried out without specifying the forms of the utility, discount and the production functions.

4.5 Numerical Analysis

In what follows, our analysis will be based on the functional forms specified in accordance with Stern (2006). The utility, production and the discount functions are specified as

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$
$$f(k) = Ak^{\alpha} + (1-\delta)k,$$
$$\beta(k) = \eta - \gamma e^{-(k+\rho)^{\varepsilon}},$$

where $0 < \{A, \rho\},\ 0 < \alpha, \sigma, \varepsilon < 1$, and $0 < \gamma e^{-\rho^{\varepsilon}} < \eta < 1$.

In accordance with these, we utilize the following set of fairly standard coefficients as our benchmark parameterization:

$$A = 0.75,\ \alpha = 0.4,\ \delta = 0.03,\ \sigma = 1.5,\ \eta = 0.95, \rho = 4.5, \sigma = 1.5, \gamma = 2.5, \varepsilon = 0.99$$

under which the maximum sustainable level of capital stock turns out to be $A(k) = \max\{(k_0, \bar{k}\}$, where $\bar{k}$ is 213.747.

As it is shown by Erol et. al. (2011) under standard preferences and technology, endogenous discounting creates threshold dynamics First we borrow an example from Erol et .al. (2011) showing that endogenous discounting creates threshold dynamics such that every equilibrium that starts to the left of the threshold level converges to low steady state and every equilibrium that starts to the right of the threshold level converges to the high steady state.

4.5.1 Emergence of Threshold Dynamics under Collusive Equilibria

There exist three solutions to (39): $k_l = 0.5953$, $k_m = 2.9155$, and $k_h = 8.4913$. In order to determine which of these are actually the optimal steady states, we analyze the optimal policy using the Bellman operator. Here k_l and k_h are stable optimal steady states and there exists a genuine critical point at $k_c \approx 5.5846$ which is not an unstable steady state. Therefore, for any initial stock lower than k_c, the economy will face a development trap, enforcing convergence to a very low capital level k_l. On the other hand, for any initial capital level higher than k_c, the optimal path will converge to k_h. However, if an economy starts at k_c, an indeterminacy will emerge. Hence even optimally managed renewable resources are sensitive to initial conditions under standard preferences and technology.

OLNE differs from cooperative solution as it is not efficient. We will see that it differs from the cooperative solution also in terms of equilibrium dynamics.

4.5.2 Equilibrium Dynamics under OLNE

Comparison with Collusive Problem

The open loop information structure can remove indeterminacy that we may face under collusive equilibrium and be a source of multiplicity despite the uniqueness we may face under collusive equilibrium.

Case 4.1 *Open Loop information structure removing indeterminacy*

Consider now the case with 2 agents. Under the benchmark parameter values, there exists a unique solution of the stationary state Euler equation (46): (x^*) =

(8.53449) . Open Loop information structure remove indeterminacy that we face under collusive problem.

Case 4.2 *Open Loop information structure as a source of multiplicity*

As we have already underlined, in a single agent optimal growth framework, the capital dependent time preference rate generates a critical point. In the vicinity of this critical point, small differences lead to permanent differences in the optimal path. Since this result heavily depends on the value of ρ, we would like to deviate from the benchmark to explore how the Open Loop information structure modifies the results under collusive problem. Hence, we assign a lower value to ρ, $\rho = 4$. We show that Open Loop information structure creates multiplicity while the capital accumulation path for the collusive problem converges to a unique globally stable steady state. Indeed, there exist three solutions to the stationary state Euler equation (39) of the single agent optimal growth problem: $x_l = 0.3708$, $x_m = 4.0061$, and $x_h = 8.4315$. However, among these three solutions, only x_l turns out to be an optimal steady state implying a global convergence result. When we consider the noncooperative problem under open loop information setting we find $x_l = 0.953053$, $x_m = 2.30966$, and $x_h = 8.50707$ as the solutions of the stationary state Euler equations where x_l and x_h are the steady state OLNE.

Comparative Statics with respect to the Number of Players

Recall that the optimal capital path in collusive problem is independent of the number of players. Under open loop information setting, we show numerically that proposition 4.5 is valid for the extremal equilibria in the case of multiple steady states.

	$\rho = 4.5$		$\rho = 4$	
Number of Players	$\sigma = 1.5$	$\sigma = 0.5$	$\sigma = 1.5$	$\sigma = 0.5$
1	0.59531 8.49127	8.64795	0.37076	8.68861
2	8.53449	8.61245	0.95305 8.50707	8.63394
3	8.54821	8.60014	8.53019	8.61457
4	8.55496	8.59389	8.54142	8.60465
100	8.57396	8.57552	8.57261	8.57513
1000	8.57466	8.57482	8.57375	8.57400

Table 1: How does the steady state OLNE change with the number of players?

Remark 4.1 *Note that number of player may change the equilibrium dynamics. For the case with $\rho = 4$ and $\sigma = 1.5$ we have multiplicity when there is two player and unique steady state OLNE when there is more than two player.*

4.6 Conclusion

In spite of the mathematical convenience that the exogenous time preference bring us, it is not an innocent assumption. As we summarize below, many results we have under exogenously discounted dynamic fishery models can not be extended to endogenous discounting case.

- Endogenous discounting creates threshold dynamics so that even optimally managed renewable resources are sensitive to initial conditions under standard preferences and technology.
- OLNE is not efficient. Depending on whether the return is bounded or unbounded from below, it may cause overexploitation or underexploitation of the

resources relative to efficient solution.

- The OLNE differs from the collusive equilibria in terms of not only efficiency but also equilibrium dynamics. Open loop information structure can remove indeterminacy that we may face under collusive equilibrium and be a source of multiplicity despite the uniqueness we may face under collusive equilibrium.

CHAPTER 5

CONCLUDING REMARKS

In this thesis, we study the existence and qualitative properties of equilibrium dynamics under endogenous time preference. In chapter two, we consider future oriented capital stock people allocate to increase anticipation of future benefits. This idea has been introduced formally by Becker and Mulligan (1997). Stern (2006) adopted this idea into the optimal growth framework to provide a more flexible framework regarding discounting of time. We contribute his efforts by focusing on questions that were left as open and studying how sensitive the equilibrium dynamics are with respect to the cost of future oriented capital stock. To our knowledge, almost none of the studies on endogenous time preference consider the strategic interaction among the agents. In chapter three, we let the discount factor to be increasing in stock of wealth and show that the strategic complementarity among the agents would be a remedy of poverty trap. Lastly, we consider socially determined time preference where the discount factor depends on aggregate resources in the economy. We show that, unlike the case under constant discounting, open loop Nash equilibrium is not efficient and depending on the return is bounded or unbounded from below, it may result in over exploitation or under exploitation of the resources relative to efficient solution.

While considering the strategic interaction we adopt the noncooperative open loop Nash equilibrium concept, in which players choose their strategies as simple time functions and they are able to commit themselves to time paths as equilibrium strategies. While open loop information setting provides a base point, the characterization of the Markov Perfect Nash equilibrium(MPNE) is important to consider the effects of making use of information. In feedback models, MPNE is subgame perfect which requires that players respond optimally to the realizations of random variables as well as unexpected deviations.

In feedback models, studies mostly rely on specific functional forms that allow closed form equilibrium strategies and clear cut comparative statics conclusions. It is desirable to reduce the present interdependence on specific functional forms. Endogenous time preference makes it harder to get close form solutions if it does not make it impossible. Hence, the existence of and algorithms for feedback equilibrium is valuable. In basic fishery models, for the exogenously fixed discount factor, the existence of stationary symmetric Markovian equilibrium (SSME) has originally provided by Sundaram (1989) and Amir (1989). Both of the approaches use Schauder's fixed point theorem, while the former use topological arguments and the latter use lattice theoretic ones. Recently, Datta et al.(2009) use partial ordering methods for characterization of SSME of a finite period game. We will utilize their approach under endogenous time preference and provide a constructive proof for the existence of a Markov equilibria and a monotonicity of a state trajectory being implied.

There is economic literature on addiction and self-control that draws upon a psychological finding that optimal saving plans for the present self would not be optimal for the later self. Strotz (1955), by formally modeling this idea, state that an individual has a myopic tendency to resist delaying consumption in the near future. Considering the fishery problem in chapter four, we can ask what will happen if one agent has time inconsistent preferences. Does it cause the other agents with time consistent preferences act like her? In other words, are the time inconsistent preferences contagious if the property rights are not well defined?

Hopefully, these studies incorporate the insights behind the time preference into theoretical formulations in a way that will increase the accuracy of predictions while not losing any other values like tractability and generality.

SELECT BIBLIOGRAPHY

Amir, R. 1989. "A Lattice-Theoretic Approach to a Class of Dynamic Games," *Computers and Mathematical Applications* 17:1345-1349.

—–. 2005. "Supermodularity and Complementarity in Economics: An Elementary Survey," *Southern Economic Journal* 71(3):636-660.

Amir, R., Mirman, L. J. and Perkins W. R. 1991. "One-Sector Nonclassical Optimal Growth: Optimality Conditions and Comparative Dynamics," *International Economic Review* 32(3): 625-644.

Amir, R., Jakubczyk M. and Knauff M. 2008. "Symmetric versus asymmetric equilibria in symmetric supermodular games," *International Journal of Game Theory* 37(3): 307-320.

Amir, R., Nannerup N. 2006. "Information Structure and the Tragedy of the Commons in Resource Extraction," *Journal of Bioeconomics* 8(2): 147-165.

Araujo A. 1991. "The once but not twice differentiability of the policy function," *Econometrica* 59(5): 1383-93.

Arrow, K. J. and Debreu, G. 1954. "Existence of an Equilibrium for a Competitive Economy," *Econometrica* 22: 265-290.

Azariadis, C. 1996. "The Economics of Poverty Traps Part One: Complete Markets," *Journal of Economic Growth* 1: 449-486.

Azariadis, C. and J. Stachurski 2005. "Poverty Traps", in P. Aghion and S. Durlauf, eds., *Handbook of Economic Growth* Elsevier, Amsterdam.

Barro, R. J. 1997. *Determinants of Economic Growth*, MIT Press, Cambridge, MA.

Barro, R.J., and X. Sala-i-Martin (1991), “Convergence across states and regions,” *Brookings Papers on Economic Activity* 1: 107-182.

Becker, G. S. and Mulligan, C. B. 1997. "The endogenous determination of time preference”, *The Quarterly Journal of Economics* 112(3): 729-758.

Benveniste, L. M. and Scheinkman, J. A. 1979. "On the differentiability of the value function in dynamic models of economies”, *Econometrica* 47: 727-732.

Chiarrella, C., Murray K., Long N. and Okuguchi, K. 1984. "On the Economics of International Fisheries", *International Economic Review* 25: 85-92.

Clemhout, S., Wan, H. Y., 1979. "Survey Paper: Interactive Economic Dynamics and Differential Games", *Journal of Optimization Theory and Applications* 27: 7-30.

Cooper, R. 1999. *Coordination Games: Complementarities and Macroeconomics*, Cambridge University Press, Cambridge

Datta, M., Mirman, L., Morand, O. and Reffett, K. 2009. "Partial Ordering Methods for Constructing Symmetric Markovian Equilibrium in a Class of Dynamic Games”, Unpublished Manuscript.

Debreu, G. 1952. "A social equilibrium existence theorem.”, *Proceedings of the National Academy of Sciences* 38: 886-893.

Dechert, D. 1997. "Non Cooperative Dynamic Games: A Control Theoretic Approach”, Unpublished Manuscript.

Dechert, W. D. and Nishimura, K. 1983. "A complete characterization of optimal growth paths in an aggregated model with non-concave production function”, *Journal of Economic Theory* 31: 332-354.

Dockner, E. and Kaitala. V. 1989. "On Efficient Equilibrium Solutions in Dynamic Games of Resource Management", *Resources and Energy* 11: 23-34.

Dockner, E. and Nishimura, K. 2001. "Characterization of equilibrium strategies in a class of difference games", *Journal of Difference Equations and Applications* 7: 915-926.

Erol S., Le Van, C. and Saglam C. 2011. "Existence, Optimality and Dynamics of Equilibria with Endogenous Time Preference”, *Journal of Mathematical Economics* 47(2): 170-179.

Lawrence, E. C. 1991. "Poverty and the rate of time preference: evidence from panel data”, *Journal of Political Economy* 99: 54-75.

Le Kama, A. and Schubert K. 2007. "A Note On The Consequences of an Endogenous Discounting Depending on the Environmental Quality”, *Macroeconomic Dynamics* 11: 272-289.

Le Van, C. and Dana R. A. 2003. *Dynamic Programming in Economics*, Kluwer Academic Publishers.

Long N. 2011. "Dynamic Games in the Economics of Natural Resources: A Survey", *Dynamic Games and Applications* 1(1): 115-148.

Matsuyama, K. 2008. "poverty traps.", *The New Palgrave Dictionary of Economics.*, Second Edition. Eds. Steven N. Durlauf and Lawrence E. Blume.Palgrave Macmillan, 2008.

Meng, Q. 2006. "Impatience and equilibrium indeterminacy", *Journal of Economic Dynamics and Control* 47(2): 170-179.

Mitra, T. and D. Ray 1984. "Dynamic optimization on non-convex feasible set: some general results for non-smooth technologies", *Zeitschrift fur Nationaokonomie* 44: 151-175.

Quah, D. T. 1996. "Convergence Empirics Across Economies with (Some) Capital Mobility", *Journal of Economic Growth* 1: 95-124.

Samuelson, P. 1937. "A note on measurement of utility", *Review of Economic Studies* 4: 155-161.

Samwick, A. 1998. "Discount rate homogeneity and social security reform", *Journal of Development Economics* 57: 117-146.

Schumacher, I. 2011. "Endogenous discounting and the domain of the felicity function", *Economic Modelling* 28: 574-581.

Stern, M. L. 2006. "Endogenous time preference and optimal growth", *Economic Theory* 29: 49-70.

Stokey, N. L. and R. Lucas with E. Prescott 1989. *Recursive methods in Economic Dynamics*, Harvard University Press.

Strotz, R. H. 1955. "Myopia and Inconsistency in dynamic utility maximitation", *Review of Economic Studies* 23: 165-180.

Sundaram, R.K. 1989. "Perfect equilibrium in non-randomized strategies in a class of symmetric dynamic games", *Journal of Economic Theory* 47: 153-177.

Topkis, D. M. 1998. *Supermodularity and Complementarity*, Princeton University Press.

Tsur, Y. and Zemel A. 2009. "Endogenous Discounting and Climate Policy", *Environmental and Resource Economics* 44: 507-520.

Vives, X. 2005. "Games with strategic complementarities: New applications to industrial organization", *International Journal of Industrial Organization* 23: 625-637.

APPENDIX

Proof of Proposition 3.8

The non-cooperative game $(N = \{1, 2\}, \mathbf{S}, \{U^i : i \in \{1, 2\}\})$ is supermodular if S is a sublattice of $\left(\prod_{i\in N} \mathbb{R}^{m_i}\right)$, U^i is supermodular in x_i, for any x_j and U^i has increasing differences in $(\boldsymbol{x}_i, \boldsymbol{x}_j)$.

$i)$ The lattice S is a subset of R^∞ and it is a sublattice of it.

$ii)$ Let us start proving that each individual payoff function is supermodular in its own strategy. By definition, a function f is supermodular if and only if

$$f(x \vee y) + f(x \wedge y) \geq f(x) + f(y).$$

Consider two different strategies for agent i, x_i and x_i' who differ from each other at time t and $t+1$ so that ${}_{t+1}x =_{t+1} x_i' = (x^i_{t+2}, x^i_{t+3}, ...)$. Recall that

$$U\left({}_{t+1}\boldsymbol{x_i} \mid \boldsymbol{x}_j\right) = \sum_{\tau=t+1}^{\infty} \left(\prod_{s=t+2}^{\tau} \beta(x^i_s)\right) u\left(g^i(x^i_\tau, x^j_\tau) - x^i_{\tau+1}\right).$$

We need to show that:

$$\beta(x_t'^i)u\left(g^i(x_t'^i, x_t^j) - x_{t+1}'^i\right) + \beta(x_t'^i)\beta(x_{t+1}'^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}}' \mid \boldsymbol{x}_j\right)$$
$$+ \beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right) + \beta(x_t^i)\beta(x_{t+1}^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}} \mid \boldsymbol{x}_j\right)$$
$$\leq \beta(x_t'^i)u\left(g^i(x_t'^i, x_t^j) - x_{t+1}^i\right) + \beta(x_t'^i)\beta(x_{t+1}^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}} \mid \boldsymbol{x}_j\right)$$
$$+ \beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}'^i\right) + \beta(x_t^i)\beta(x_{t+1}'^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}}' \mid \boldsymbol{x}_j\right)$$

Since

$$\left(\beta(x_t^i) - \beta(x_t'^i)\right)\beta(x_{t+1}'^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}}' \mid \boldsymbol{x}_j\right) \geq$$
$$\left(\beta(x_t^i) - \beta(x_t'^i)\right)\beta(x_{t+1}^i)U\left({}_{t+1}\boldsymbol{x}_{\boldsymbol{i}} \mid \boldsymbol{x}_j\right),$$

it is sufficient to show that

$$\beta(x_t'^i)u\left(g^i(x_t'^i, x_t^j) - x_{t+1}'^i\right) + \beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right) \leq$$
$$\beta(x_t'^i)u\left(g^i(x_t'^i, x_t^j) - x_{t+1}^i\right) + \beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}'^i\right).$$

This is equivalent to showing that $\beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right)$ is supermodular in (x_t^i, x_{t+1}^i). As $\beta(x_t^i)u\left(g^i(x_t^i, x_t^j) - x_{t+1}^i\right)$ is differentiable, it holds if

$$\frac{\partial^2 \beta(x_t^i)u\left[g^i(x_t^i, x_t^j) - x_{t+1}^i\right]}{\partial x_t^i \partial x_{t+1}^i} \geq 0.$$

iii) Let us now prove the increasing differences. A function f has increasing differences if and only if

$$f(x, t'') - f(x, t'), \quad t'' > t',$$

is increasing in x.

Let $x_i \in Br^i(x_j)$ so that $x_{t+1}^i \in \mu^i\left(x_t^i \mid \boldsymbol{x}_j\right), \forall t$. As open-loop strategies are only dependent on the initial conditions, one can easily write that

$$x_t^i = \underbrace{\mu^i\left(\mu^i\left(\mu^i\left(...\mu^i\left(x_0^i\right)\right)\right) \mid \boldsymbol{x}_j\right)}_{t-times} \equiv \mu_t^i\left(x_0^i \mid \boldsymbol{x}_j\right), \forall t = 1, 2, ...$$

Now we fix $\hat{x}_j > x_j$ (that is $\hat{x}_t^j > x_t^j$ for all t), and consider a best response of agent i, $\hat{x}_i \equiv \{\hat{x}_t^i\}_{t=1}^{\infty}$ so that $\hat{x}_{t+1}^i \in \hat{\mu}^i\left(\hat{x}_t^i \mid \hat{\mathbf{x}}_j\right), \forall t$. Accordingly,

$$\hat{x}_t^i = \underbrace{\hat{\mu}^i\left(\hat{\mu}^i\left(\hat{\mu}^i\left(...\hat{\mu}^i\left(x_0^i\right)\right)\right) \mid \hat{\mathbf{x}}_j\right)}_{t-times} \equiv \hat{\mu}_t^i\left(x_0^i \mid \hat{\mathbf{x}}_j\right), \forall t = 1, 2, ...$$

We have to check whether

$$U(\hat{\mathbf{x}}_i, \hat{\mathbf{x}}_j) - U(\hat{\mathbf{x}}_i, \boldsymbol{x}_j) \geq U(\boldsymbol{x}_i, \hat{\mathbf{x}}_j) - U(\boldsymbol{x}_i, \boldsymbol{x}_j) \tag{48}$$

for $\hat{\mu}^i\left(x \mid \hat{\mathbf{x}}_j\right) > \mu^i\left(x \mid \boldsymbol{x}_j\right), \forall x$. We can substitute U:

$$\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\beta(\hat{\mu}_s^i\left(x_0^i\right))\right)[u\left(g^i(\hat{\mu}_t^i\left(x_0^i\right), \hat{x}_t^j) - \hat{\mu}_{t+1}^i\left(x_0^i\right)\right) -$$

$$u\left(g^i(\hat{\mu}_t^i\left(x_0^i\right), x_t^j) - \hat{\mu}_{t+1}^i\left(x_0^i\right)\right)] \geq$$

$$\sum_{t=0}^{\infty}\left(\prod_{s=1}^{t}\beta(\mu_s^i\left(x_0^i\right))\right)[u\left(g^i(\mu_t^i\left(x_0^i\right), \hat{x}_t^j) - \mu_{t+1}^i\left(x_0^i\right)\right) -$$

$$u\left(g^i(\mu_t^i\left(x_0^i\right), x_t^j) - \mu_{t+1}^i\left(x_0^i\right)\right)].$$

Since $\hat{\mu}_s^i(x_0^i) \geq \mu_s^i(x_0^i)$ for all s, we have that the above inequality is equivalent to check whether:

$$\beta(\hat{\mu}_t^i(x_0^i))u\left(g^i(\hat{\mu}_t^i(x_0^i), \hat{x}_t^j) - \hat{\mu}_{t+1}^i(x_0^i)\right) - \beta(\hat{\mu}_t^i(x_0^i))u\left(g^i(\hat{\mu}_t^i(x_0^i), x_t^j) - \hat{\mu}_{t+1}^i(x_0^i)\right) \geq$$

$$\beta(\mu_t^i(x_0^i))u\left(g^i(\mu_t^i(x_0^i), \hat{x}_t^j) - \mu_{t+1}^i(x_0^i)\right) - \beta(\mu_t^i(x_0^i))u\left(g^i(\mu_t^i(x_0^i), x_t^j) - \mu_{t+1}^i(x_0^i)\right).$$

We shall proceed by dividing both sides by $\left(\hat{x}_t^j - x_t^j\right)$ and taking the limit when $\left(\hat{x}_t^j - x_t^j\right) \rightarrow 0$. We get that (48) holds if and only if

$$\frac{\partial\left[\beta(\hat{\mu}_t^i(x_0^i))u\left(g^i(\hat{\mu}_t^i(x_0^i), \hat{x}_t^j) - \hat{\mu}_{t+1}^i(x_0^i)\right)\right]}{\partial x_t^j} - \frac{\partial\left[\beta(\mu_t^i(x_0^i))u\left(g^i(\mu_t^i(x_0^i), x_t^j) - \mu_{t+1}^i(x_0^i)\right)\right]}{\partial x_t^j} \geq 0.$$

Note that when $\left(\hat{\mu}^i(x_0^i) - \mu^i(x_0^i)\right) \rightarrow 0$, then $\left(\hat{\mu}_s^i(x_0^i) - \mu_s^i(x_0^i)\right) \rightarrow 0$, $\forall s$. Dividing both sides by $\left(\hat{\mu}_t^i(x_0^i) - \mu_t^i(x_0^i)\right)$ and taking the limit when $\left(\hat{\mu}_t^i(x_0^i) - \mu_t^i(x_0^i)\right) \rightarrow 0$, we obtain that

$$\frac{\partial^2\left[\beta(\mu_t^i(x_0^i))u\left(g^i(\mu_t^i(x_0^i), x_t^j) - \mu_{t+1}^i(x_0^i)\right)\right]}{\partial\left(\mu_t^i(x_0^i)\right)\partial x_t^j} \geq 0.$$

Put differently, in terms of period utilities,

$$\frac{\partial^2\left[\beta({x_t}^i)u\left(g^i({x_t}^i, {x_t}^j) - x_{t+1}^i\right)\right]}{\partial x_t^i \partial x_t^j} \geq 0,$$

together with

$$\frac{\partial^2\left[\beta({x_t}^i)u\left(g^i({x_t}^i,{x_t}^j)-x^i_{t+1}\right)\right]}{\partial x^i_t\partial x^i_{t+1}}\geq 0,$$

ensures the supermodularity of the game. Since S is compact under product topology and U^i is upper semicontinuous in x_i on $S^i\left(\boldsymbol{x}_j\right)$, the result follows from Theorem 3.1.

Proof of Proposition 3.9

By the same reasoning that we use in the proof of proposition 3.8-(iii), condition 31 assures that for each player, $U^{\tau i}$ has increasing differences in $\left(\boldsymbol{x}_i,\tau\right)$. All the rest follows from Topkis (1998) Theorem 4.2.2.

Proof of Proposition 3.10

We proceed in three steps. First, we show that the game $(N,\mathbf{S},\{U^i:i\in N\})$, only admits symmetric and antisymmetric steady states. Then, we prove that antisymmetric steady states are linearly ordered with the highest and the lowest symmetric steady states. Finally, we prove that our game only admits symmetric steady state open loop Nash equilibria.

Lemma 7.1 *If (x_*,x^*) is a steady state open loop Nash equilibrium of the game $(N,\mathbf{S},\{U^i:i\in N\})$, then (x^*,x_*) is also a steady state open loop Nash equilibrium.*

Proof. If (x_*,x^*) is a steady state open loop Nash equilibrium of the game given by $(N,\mathbf{S},\{U^i:i\in N\})$, then (x_*,x^*) satisfies 32 for both agents. (x^*,x_*) is also a steady state since for each $i,j\in N,i\neq j$, $g^i(x^i,x^j)=g^j(x^j,x^i)$ and $\frac{\partial g^i}{\partial x^i}=\frac{\partial g^j}{\partial x^j}$. ■

Lemma 7.1 shows that the set of steady states only contains symmetric and antisymmetric steady states. As the next lemma shows, the antisymmetric steady states are linearly ordered with the highest and the lowest symmetric steady states.

Lemma 7.2 *Let* $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ *and* $(\check{\mathbf{x}}, \check{\mathbf{x}})$ *denote the highest and the lowest symmetric steady steady state open loop Nash equilibria of the supermodular game* $(N, \mathbf{S}, \{U^i : i \in N\})$*. If there exists an asymmetric steady state open loop Nash equilibrium, say* (x_*, x^*) *with* $x^* \neq x_*$*, then* $(\check{\mathbf{x}}, \check{\mathbf{x}}) \leq (x_*, x^*) \leq (\hat{\mathbf{x}}, \hat{\mathbf{x}})$*.*

Proof. Let $x_* < x^*$, without loss of generality. Let x_0 be such that $g^i(x_{0,}x_0) = min\{g^i(x^*, x_*), g^j(x^*, x_*)\}$ for $i \in \{1, 2\}$.

Assume now on the contrary that $x_* < \check{x}^*$. We have, by construction,

$$g^i(x_{0,}x_0) < g^i(x_*, x_*) < g^i(\check{x}^*, \check{x}^*).$$

Let $\tau = (g^i(x_{0,}x_0), g^j(x_{0,}x_0))$ and $\breve{\tau} = (g^i(x^*, x_*), g^j(x^*, x_*))$, where $\tau < \breve{\tau}$. Consider the corresponding supermodular games $\Gamma(\tau)$ and $\Gamma(\breve{\tau})$. By Proposition 3.9, the greatest and the least equilibrium of the game are increasing in τ on T, i.e. steady state levels that the greatest and the least open loop Nash equilibrium of the game converge to will be higher for $\breve{\tau}$. Since $\Gamma(\tau)$ is a symmetric supermodular game, the least equilibrium converge to a symmetric steady state, i.e. $\check{x}^* \leq x_*$,leading to a contradiction.. The case of $x^* \leq \hat{x}^*$ can be shown similarly. ■

Let us prove now that any steady state open loop Nash equilibrium of our game is indeed symmetric:

Let $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ and $(\check{\mathbf{x}}, \check{\mathbf{x}})$ denote the highest and the lowest symmetric steady state open loop equilibria of the game, respectively. First and as a corollary to Lemma 7.2, if the highest and the lowest symmetric steady states coincide, i.e., $\hat{x} = \check{x}$, there does not exist any asymmetric stationary state open loop equilibria of the game. Now consider the case where $\hat{x} \neq \check{x}$ and assume that (x_*, x^*) is an asymmetric stationary state open loop Nash equilibrium with $x_* < x^*$ without loss of generality. Then, by

Lemma 7.1, (x^*, x_*) constitutes an asymmetric steady state open loop Nash equilibrium as well. Consider now the game $(N, \mathbf{S}, \{U^i : i \in N\})$ under the following three initial endowments: (x_*, x^*), (x^*, x_*) and $\left(\frac{x^*+x_*}{2}, \frac{x^*+x_*}{2}\right)$. Recall that the game with a symmetric initial endowment turns out to be symmetric supermodular so that due to the monotonicity of the best replies, the greatest and the least equilibrium converge to the highest and the lowest symmetric steady state, respectively. Accordingly, assume that the equilibrium of the symmetric supermodular game that starts with the initial endowment of $\left(\frac{x^*+x_*}{2}, \frac{x^*+x_*}{2}\right)$ converges to the highest symmetric steady state $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ in the long run, without loss of generality. In comparison with the asymmetric stationary state open loop equilibrium (x_*, x^*), the open loop Nash equilibrium of the game that emanates from $\left(\frac{x^*+x_*}{2}, \frac{x^*+x_*}{2}\right)$ and converging to $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ in the long run reveals that an increase in agent's own initial capital stock and a decrease in the rival's implies the convergence of the agent's stock towards a higher steady state. This then implies that the equilibrium of the supermodular game that emanates from (x^*, x_*) has to monotonically converge to the highest steady state $(\hat{\mathbf{x}}, \hat{\mathbf{x}})$ as well. However, this contradicts with the fact that (x^*, x_*) has to be a steady state open loop Nash equilibrium.

Proof of Proposition 3.11

Let x_0 be such that

$$g^i\left(x_0,x_0\right) = min\left\{f\left(x_L\right), \frac{f\left(2x_L^o\right)}{2}\right\}, \ i \in \{1, 2\},$$

and consider the corresponding symmetric supermodular game. Let $\left\{x_0^j, \boldsymbol{x}_j\right\} = (x_0, 0, 0.., 0, ...)$. Note that $Br^i\left(\boldsymbol{x}_j\right)$ coincides with the single agent optimal growth problem for the given initial capital stock, $f^{-1}\left(g^i\left(x_0,x_0\right)\right)$. As x_L is stable from left, $\exists$ $\boldsymbol{x_i} \in Br^i\left(\boldsymbol{x}_j\right)$ such that $\boldsymbol{x_i} \to x_L$. Since x_L is the lowest steady state and the optimal policy correspondence of a single agent optimal growth problem is ordered (see Erol et al., 2011), we can conclude that $x_{\boldsymbol{i}}$ is the least element of the best response correspondence. Let (x^{*}, x^{*}) constitute the least open loop Nash equilibrium of the game, i.e. $x^{*} \in Br^{i}(\boldsymbol{x}^{*})$ and $x^{*} \in Br^{j}(\boldsymbol{x}^{*})$. Note that the supermodular game structure implies that the least and the greatest elements of the best response correspondence are increasing in opponent's strategy. Having $x_j < x^{*}$ by construction, we conclude that $x^{*} \geq x_{\boldsymbol{i}}$ and $x_L^o \geq x_L$.

For the second case, let x_0 be such that

$$g^{i}(x_{0,}x_0) = max\left\{\frac{f(2x_L^o)}{2}, f(x_H)\right\},\ i \in \{1,2\},$$

and consider the corresponding symmetric supermodular game. Let $\left\{x_0^j, \boldsymbol{x}_j\right\} = (x_0, 0, 0.., 0, ...)$. $Br^{i}(\boldsymbol{x}_j)$ coincides with the single agent optimal growth problem for given initial capital stock, $f^{-1}(g^{i}(x_{0,}x_0))$. As x_H is stable from the right, $\exists$ $x_{\boldsymbol{i}} \in Br^{i}(\boldsymbol{x}_j)$ such that $x_{\boldsymbol{i}} \to x_H$. By the same reasoning as above, $x_{\boldsymbol{i}}$ is the highest element of the $Br^{i}(\boldsymbol{x}_j)$. Having $x_j < x^{*}$ by construction, we get $x^{*} \geq x_{\boldsymbol{i}}$ and $x_L^o \geq x_H$.

Proof of Corollary 3.1

As proven in Proposition 3.10, there only exist symmetric steady state open loop Nash equlibria. At any steady state open loop Nash equilibrium (x, x), agents' payoff can be recast as

$$U(\mathbf{x} \mid \mathbf{x}) = u(g(x,x) - x)\frac{1}{1-\beta(x)}.$$

Note that if

$$\frac{\partial}{\partial x}\left(\frac{u(g(x,x)-x)}{1-\beta(x)}\right) > 0,$$

then $U(\hat{\mathbf{x}} \mid \hat{\mathbf{x}}) \geq U(\mathbf{x} \mid \mathbf{x}) \geq U(\check{\mathbf{x}} \mid \check{\mathbf{x}})$ holds where at least one of the inequalities will be strict.