NOISE BENEFITS IN JOINT DETECTION AND ESTIMATION SYSTEMS

A THESIS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

NOISE BENEFITS IN JOINT DETECTION AND ESTIMATION SYSTEMS

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Adding noise to inputs of some suboptimal detectors or estimators can improve their performance under certain conditions. In the literature, noise benefits have been studied for detection and estimation systems separately. In this thesis, noise benefits are investigated for joint detection and estimation systems. The analysis is performed under the Neyman-Pearson (NP) and Bayesian detection frameworks and the Bayesian estimation framework. The maximization of the system performance is formulated as an optimization problem. The optimal additive noise is shown to have a specific form, which is derived under both NP and Bayesian detection frameworks. In addition, the proposed optimization problem is approximated as a linear programming (LP) problem, and conditions under which the performance of the system cannot be improved via additive noise are obtained. With an illustrative numerical example, performance comparison between the noise enhanced system and the original system is presented to support the theoretical analysis.

Keywords: Detection, estimation, linear programming, noise benefits, joint detection and estimation, stochastic resonance.

ÖZET

BİRLİKTE SEZİM VE KESTİRİM SİSTEMLERİNDE GÜRÜLTÜNÜN FAYDALARI

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Belirli koşullar altında, optimal olmayan bazı sezici ve kestiricilerin performansını girdilerine gürültü ekleyerek geliştirmek mümkündür. Literatürde, gürültünün faydaları sezici ve kestirici sistemleri için ayrı ayrı ele alınmıştır. Bu tezde, birlikte sezim ve kestirim sistemleri için gürültünün faydaları incelenmektedir. Analiz, Neyman-Pearson (NP) ve Bayes sezim çerçeveleri ve Bayes kestirim çerçevesi altında gerçekleştirilmektedir. Sistem performansının en yüksek seviyeye çıkarılması, bir optimizasyon problemi olarak tanımlanmaktadır. Optimal gürültü dağılımının belirli bir istatiksel şekle sahip olduğu gösterilmekte ve bu optimal gürültü dağılım şekli NP ve Bayes sezim çerçeveleri için ayrı ayrı elde edilmektedir. Ayrıca önerilen optimizasyon probleminin, bir doğrusal programlama (DP) problemi olarak yaklaşımı sunulmakta ve sistem performansının DP yaklaşımı altında gürültü ile geliştirilemeyeceği koşullar elde edilmektedir. Bir sayısal örnek üzerinde, kuramsal bulguları desteklemek amacıyla, gürültü eklenmiş sistem ile orijinal sistemin performansları karşılaştırılmaktadır.

Anahtar sözcükler: Sezim, kestirim, doğrusal programlama, gürültünün faydaları, birlikte sezim ve kestirim, stokastik rezonans.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Objectives and Contributions of the Thesis

Although an increase in the noise power is generally associated with a performance degradation, addition of noise to a system may introduce performance improvements under certain arrangements and conditions in a number of electrical engineering applications including neural signal processing, biomedical signal processing, lasers, nano-electronics, digital audio and image processing, analog-to-digital converters, control theory, statistical signal processing, and information theory, as exemplified in [1] and references therein. In addition to the electrical engineering applications, in a broader sense, the observation of benefits of increasing the noise level for enhancing a given system is reported in a variety of sciences including biology, climatology, chemistry, network science, mathematics, ecology, finance, and physics [2] (and references therein).

In the literature, this phenomenon is also referred to as "stochastic resonance" (SR) in some contexts. The term SR was firstly used in [3] within the context of stochastically behaving dynamical systems. A detailed and concrete analysis of the response of bistable systems subject to a weak periodic or random signal is presented in [4]. In order to clarify the SR concept, it is valuable to present a famous technique "dithering" which is defined as adding a random signal (which can be regarded as noise) to the analog input signal prior to the quantization operation [5]. In this particular application in digital signal processing, dithered quantization systems demonstrate SR behaviors and perform superior (reduction in the averaged quantization error and improvement in the dynamic range) in comparison to the quantization systems without dithering [6, 7]. Dithering is a commonly employed technique in audio and image processing, and stands as a real life application example of noise-enhanced signal processing.

Noise-enhanced signal processing focuses on revealing and analyzing possible practical benefits of SR phenomenon in the field of signal processing. SR is initially observed in nonlinear bistable systems driven by a periodic input signal in the presence of the noise in the form of maximizing the signal to noise ratio [3, 4, 8, 9]. Later these results are extended into the analysis of the response to arbitrary aperiodic input signals [10–15]. In addition to these single threshold SR studies, array (a network of quantizers) noise benefits are explored in the suprathreshold SR literature [16–22]. Recent studies indicate that noise benefits can be observed in various detection and estimation theory problems in the signal processing field [19–60].

Triggered by this aforementioned repertory, noise enhanced signal processing is still an active area that invokes novel contributions. It is valuable to notice the variety of these contributions by demonstrating some of the interesting results from different fields. In [19], it is shown that an increase in the internal noise levels of the components of a pooling network, which is composed of parallel and independent threshold detectors, may result in improved performance. A recursive systematic convolutional (RSC) coded and unity-rate convolutional (URC) pre-coded SR system is introduced, where the SR phenomenon is observed as a non-monotonicity in the maximum achievable rate curve with respect to the noise power under Gaussian mixture distributed system noise [24]. A novel cooperative spectrum sensing mechanism is introduced based on the network of SR detectors which outperforms (have a larger detection probability under the constant false alarm rate criterion) the conventional energy detection method [25]. A noise enhanced watermark decoding scheme is introduced and its properties are analyzed under various quantizer noise distributions [26].

Some results from the literature of noise enhanced statistical signal processing can be summarized as follows: In [27], it is shown that the detection probability of the optimal detector for a described network with nonlinear elements driven by a weak sinusoidal signal in white Gaussian noise is non-monotonic with respect to the noise power and fixed false alarm probability. For an optimal Bayesian estimator, in a given nonlinear setting, with examples of a quantizer [28] and phase noise on a periodic wave [29], a non-monotonic behavior in the estimation mean-square error is demonstrated as the intrinsic noise level increases. In [30], the proposed simple suboptimal nonlinear detector scheme, in which the detector parameters are chosen according to the system noise level and distribution, outperforms the matched filter under non-Gaussian noise in the Neyman-Pearson (NP) framework. In [31], it is noted that the performance of some optimal detection strategies display a non-monotonic behavior with respect to the noise root-mean square amplitude in a binary hypothesis testing problem with a nonlinear setting, where non-Gaussian noise (two different distributions are examined for numerical purposes: Gaussian mixture and uniform distributions) acts on the phase of a periodic signal. Three very important keywords appear in this context: nonlinear, non-Gaussian and non-monotone. Non-Gaussianity is not a sine qua non; however, noise benefits are more effective and more likely to occur in non-Gaussian background noise.

As mentioned above, the SR effect is marked by a non-monotonic curve of the signal-to-noise ratio, detection probability (under fixed false alarm rate), or estimation risk, which demonstrates a resonance peak with an increasing noise power. However, adjustment of the current noise level of a given system is not a straightforward operation. The control of the system noise process through changing physical parameters may not be always a possibility. Also one cannot simply increase the present system noise power by adding an independent random process with the same statistical properties [34, 35]. To overcome this practical disadvantage and to exploit the potential of stochastic resonance, alternative approaches are proposed and examined in the literature. An important proposal is the inducement of stochastic resonance by tuning the parameters of a nonlinear system [35–39].

An alternative approach is the injection of a random process independent of both the meaningful information signal (transmitted or hidden signal) and the background noise (unwanted signal). It is firstly shown by Kay in [40] that addition of independent randomness may improve suboptimal detectors under certain conditions. Later, it is proved that a suboptimal detector in the Bayesian framework may be improved (i.e., the Bayes risk can be reduced) by adding a constant signal to the observation signal; that is, the optimal probability density function is a single Dirac delta function [41]. This intuition is extended in various directions and it is demonstrated that injection of additive noise to the observation signal at the input of an suboptimal detector can enhance the system performance [43–60]. In this thesis, performance improvements through noise benefits are addressed in the context of joint detection and estimation systems by adding an independent noise component to the observation signal at the input of a suboptimal system. Notice that the most critical keyword in this approach is suboptimality. Under non-Gaussian background noise, optimal detectors/estimators are often nonlinear, difficult to implement, and complex systems [32, 61]. The main target is to improve the performance of a fairly simple and practical system by adding specific randomness at the input.

Chen *et al* revealed that the detection probability of a suboptimal detector in the NP framework can be increased via additive independent noise [43]. They examined the convex geometry of the problem and specified the nature of the optimal probability distribution of additive noise as a probability mass function with at most two point masses. This result is generalized for M-ary composite hypothesis testing problems under NP, restricted NP and restricted Bayes criteria [51, 55, 60]. In estimation problems, additive noise can also be utilized to improve the performance of a given suboptimal estimator [29, 45, 56]. As an example of noise benefits for an estimation system, it is shown that Bayesian estimator performance can be enhanced by adding non-Gaussian noise to the system, and this result is extended to the general parameter estimation problem in [45]. As an alternative example of noise enhancement application, injection of noise to blind multiple error rate estimators in wireless relay networks can also be given [56].

Noise benefits in a joint detection and estimation system, which is presented in [62], are examined in this thesis. Without introducing any modification to the structure of the system, the aim is to improve the performance of the joint system by only adding noise to the observation signal at the input. Therefore, the detector and estimator are assumed to be given and fixed. In [62], optimal detectors and estimators are derived for this joint system. However, the optimal structures may be overcomplicated for an implementation. In this thesis, it is assumed that the given joint detection and estimation system is suboptimal, and the purpose is defined as the examination of the performance improvements through noise benefits under this assumption. If the given system is optimal, it is not possible to improve its performance further with additive noise. Optimal detection and estimation systems are constructed with the sufficient statistics of the information to be revealed, and the addition of independent information cannot further improve the performance [63,64]. A simplified and limited version of this discussion is presented in [65].

1.2 Organization of the Thesis

The organization of the thesis is as follows. In Chapter 2, the properties of the considered joint detection and estimation system are presented in Bayesian and NP detection frameworks. The detection probability, false alarm probability, and the conditional estimation risks of the noise enhanced joint system are derived. Finally, the targeted performance improvement is defined as an optimization problem.

In Chapter 3, the optimal additive noise probability density function is revealed to correspond to a discrete probability mass function. In real life applications, additive noise values cannot have infinite precision due to the memory and processor limitations. Restriction of the additive noise values to a finite sized set leads to a linear programming problem, which is presented in Chapter 3. Also sufficient conditions are derived for the improvability of the given system via noise. For the linear programming approximated problem, a necessary and sufficient condition for the improvability of the system via additive noise is obtained.

Finally, theoretical findings are also illustrated on a numerical example in Chapter 4. The performance of a given joint system that is composed of a correlator detector and a sample mean estimator (which are optimal for disjoint detection and estimation problems in Gaussian background noise) is investigated and the noise enhancement effects are analyzed. Also, the efficiency of the linear programming approximation is discussed.

Conclusion chapter finalizes the thesis by highlighting the main contributions of this study. Possible future research ideas are presented.

Chapter 2

Background and Problem Definition

2.1 Background

In this study, possible noise benefits are investigated for a given joint detection and estimation problem. The joint mechanism is presented in [62] and evaluated in the NP detection and Bayesian estimation frameworks. In addition, the same joint mechanism can also be analyzed in a Bayesian detection and estimation framework. The aim of this study is to enhance the performance of a given joint system by introducing additive noise at the input, without making any modifications to the given detector or estimator. In this section, the structure of the joint system is examined according to both NP and Bayesian criteria. The former is already explained in [62] and is briefly summarized here. The characteristics of the system in the Bayesian framework are derived and presented.

For both cases, the mechanism consists of a detector and an estimator subsequent to it. The detection is based on the following binary composite hypothesis testing problem:

$$\begin{aligned}\mathcal{H}_0 &: \boldsymbol{X} \sim f_0^X(\boldsymbol{x}) \\ \mathcal{H}_1 &: \boldsymbol{X} \sim f_1^X(\boldsymbol{x}|\boldsymbol{\Theta}=\boldsymbol{\theta}),\ \boldsymbol{\Theta} \sim \pi(\boldsymbol{\theta})\end{aligned} \tag{2.1}$$

where $\boldsymbol{X} \in \mathbb{R}^K$ is the observation signal. Under hypothesis $\mathcal{H}_0$, the distribution of the observation signal is completely known as $f_0^X(\boldsymbol{x})$. Under hypothesis $\mathcal{H}_1$, the conditional distribution $f_1^X(\boldsymbol{x}|\boldsymbol{\theta})$ of the observation signal $\boldsymbol{X}$ is given. It is also assumed that the prior distribution of the unknown parameter $\boldsymbol{\Theta}$ is available as $\pi(\boldsymbol{\theta})$ in the parameter space Λ.

As the detector, a generic decision rule $\phi(\boldsymbol{x})$ is considered. In this formulation, the input of the detector is the observation and the output is the decision probability in preference to hypothesis $\mathcal{H}_1$ with $0 \le \phi(\boldsymbol{x}) \le 1$. Notice that if the image of the detector $\phi(\boldsymbol{x})$ is restricted to $\{0,1\}$, this forms a partition in the observation space with two decision regions. Each observation $\boldsymbol{x}$ in the observation space $\mathbb{R}^K$ belongs to one of these two decision regions and the decided hypothesis can be deterministically known. By allowing $\phi(\boldsymbol{x})$ to take values from the unit interval, a more general detector formulation is considered where it may be possible to decide in favor of $\mathcal{H}_0$ or $\mathcal{H}_1$ for a given observation $\boldsymbol{x}$.

In the composite hypothesis-testing problem definition (2.1), the unknown parameter distribution is defined only under $\mathcal{H}_1$. It is assumed that the unknown parameter value is a known value θ_0 under $\mathcal{H}_0$ and this knowledge is already included in $f_0^X(\boldsymbol{x})$. Following the decision, an estimate of parameter $\boldsymbol{\Theta}$ is also provided only if the decision is hypothesis $\mathcal{H}_1$ and it is assumed that estimation function $\hat{\theta}(\boldsymbol{x})$ is also given. In that mechanism, it is implicitly assumed that the estimator output is θ_0 if the decision is $\mathcal{H}_0$. The introduced mechanism is also depicted on Figure 2.1.

With respect to the problem definition, different decision schemes such as Bayesian or NP approaches and estimation functions can be regarded in this context. If the prior probabilities of the hypotheses $P(\mathcal{H}_i)$ are unknown, an NP type hypothesis-testing problem is defined. If the prior probabilities are given, the Bayesian approach could be adopted [63]. The noise enhanced joint detection and

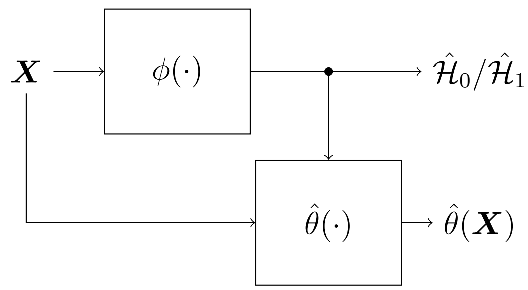


Figure 2.1: Joint detection and estimation scheme: Observation signal $\boldsymbol{X}$ is input to both the detector $\phi(\boldsymbol{X})$ and to the estimator $\hat{\theta}(\boldsymbol{X})$. An estimation is provided if the decision is $\hat{\mathcal{H}}_1$.

estimation system is analyzed in both of these frameworks in parallel throughout the thesis.

2.1.1 Neyman-Pearson (NP) Hypothesis-Testing Framework

In [62], it is assumed that the prior distributions of the hypotheses, $P(\mathcal{H}_i)$, are unknown. As it has been explained, this calls for an NP type detection problem. In this study, the common terminology of detection and estimation theory is used to define events [63]. The "miss" event describes the case when $\hat{\mathcal{H}}_0$ is decided although the true hypothesis is $\mathcal{H}_1$. The "false alarm" event describes the case when $\hat{\mathcal{H}}_1$ is decided although the true hypothesis is $\mathcal{H}_0$. The "detection" event term is used when both true and decided hypotheses are $\mathcal{H}_1$.

The detection probability, which is the probability of deciding in favor of hypothesis $\mathcal{H}_1$ when true hypothesis is hypothesis $\mathcal{H}_1$, is expressed as

$$P_1^{\boldsymbol{x}}(\hat{\mathcal{H}}_1) \coloneqq P(\hat{\mathcal{H}}_1|\boldsymbol{X}{=}\boldsymbol{x}, \mathcal{H}_1) \ . \tag{2.2}$$

The detection probability is evaluated as the performance metric with respect to the false alarm rate (probability) constraint. The false alarm rate is the probability of deciding $\mathcal{H}_1$ when hypothesis $\mathcal{H}_0$ is true, which is defined as

$$P_0^{\boldsymbol{x}}(\hat{\mathcal{H}}_1) \coloneqq P(\hat{\mathcal{H}}_1|\boldsymbol{X}{=}\boldsymbol{x}, \mathcal{H}_0) \ . \tag{2.3}$$

The miss probability is not additionally defined since it is equal to $1 - P_1^{\boldsymbol{x}}(\hat{\mathcal{H}}_1)$.

For the hypotheses in (2.1), the false alarm and detection probabilities can be expressed as follows [62]:

$$P_0^x(\hat{\mathcal{H}}_1) = \int_{\mathbb{R}^K} \phi(\boldsymbol{x}) f_0^X(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \tag{2.4}$$

$$P_1^x(\hat{\mathcal{H}}_1) = \int_{\Lambda} \int_{\mathbb{R}^K} \phi(\boldsymbol{x}) \pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{x}|\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{x}\, \mathrm{d}\boldsymbol{\theta} \tag{2.5}$$

For a Bayesian type estimation problem in which the prior distribution of the parameter is provided, Bayes estimation risk is employed as the performance criterion. Bayes estimation risk is given by

$$r^x(\hat{\theta}) = E\{C[\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{X})]\} , \tag{2.6}$$

which is the expectation of the cost function $C[\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{X})]$ over the distributions of observation $\boldsymbol{X}$ and parameter $\boldsymbol{\Theta}$. Squared error ($C[\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x})] = [\boldsymbol{\theta} - \hat{\theta}(\boldsymbol{x})]^2$), absolute error ($C[\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x})] = |\boldsymbol{\theta} - \hat{\theta}(\boldsymbol{x})|$) or 0-1 loss functions ($C[\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x})] = 1$ if $|\boldsymbol{\theta} - \hat{\theta}(\boldsymbol{x})| < \Delta$ for some $\Delta > 0$, and equal to zero otherwise) are three most commonly used cost functions in the literature [63]. The choice may depend on the application. For the binary hypothesis-testing problem in (2.1), the Bayes risk in (2.6) can be expressed as

$$r^x(\hat{\theta}) = \sum_{i=0}^{1} \sum_{j=0}^{1} P(\mathcal{H}_i) P_i^x(\hat{\mathcal{H}}_j) E\{C(\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{X})|\mathcal{H}_i, \hat{\mathcal{H}}_j\} . \tag{2.7}$$

In this mechanism, the estimation is dependent on the detection result; hence, the overall Bayes estimation risk is not independent of the detection performance. Due to this dependency, the calculation of the Bayes estimation risk requires the evaluation of the conditional risks for different true hypothesis and decided hypothesis pairs. As it is clear from (2.7), it is not possible to analytically evaluate the overall Bayes estimation risk function $r^x(\hat{\theta})$ in the NP framework since the prior distributions of the hypotheses $P(\mathcal{H}_i)$ are unknown. To avoid this complication, the conditional Bayes estimation risk $J^x(\phi, \hat{\theta})$ which is presented in [62] as the Bayes estimation risk under the true hypothesis $\mathcal{H}_1$ and decision $\hat{\mathcal{H}}_j$ is adopted (2.8). Furthermore, it should be noted that if the decision is not correct, it is expected the estimation error is relatively higher and and may be regarded as useless for specific applications. Therefore, taking into consideration only the estimation error when the decision is correct could be justified as a rational argument. Since a probability distribution for unknown parameter Θ is not defined under true hypothesis $\mathcal{H}_0$, the estimation error conditioned on the true hypothesis testing event is equivalent to the estimation error given true hypothesis $\mathcal{H}_1$ and decision $\hat{\mathcal{H}}_j$. The conditional Bayes estimation risk is expressed as [62]

$$J^x(\phi, \hat{\theta}) = E\{c(\boldsymbol{\Theta}, \theta(\hat{\boldsymbol{X}}))|\mathcal{H}_1, \hat{\mathcal{H}}_1\} = \frac{\int\limits_{\Lambda}\int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x}))\phi(\boldsymbol{x}) f_1^X(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta}}{P_1^x(\hat{\mathcal{H}}_1)}\,. \tag{2.8}$$

In [62], the problem is defined as the minimization of the conditional Bayes risk with respect to the false alarm and detection probability constraints. In this work, the problem and mechanism are altered as it is explained in the next section.

2.1.2 Bayesian Hypothesis Testing Framework

If the prior probabilities of the hypotheses $P(\mathcal{H}_i)$ are known, the Bayesian decision making approach in which the aim is the minimization of the Bayes estimation risk could be undertaken. Bayes detection risk $r^x(\phi)$ is the expectation of the decision cost C_{ij} over true hypothesis $\mathcal{H}_i$ and decision $\hat{\mathcal{H}}_j$ [63].

$$r^x(\phi) = P(\mathcal{H}_0)\sum_{j=0}^{1} C_{0j}P_0^x(\hat{\mathcal{H}}_j) + P(\mathcal{H}_1)\sum_{j=0}^{1} C_{1j}P_1^x(\hat{\mathcal{H}}_j)\,. \tag{2.9}$$

Determining the values of the cost variables C_{ij} generally depends on the application. As a reasonable choice, C_{ij} is chosen as zero when $i = j$ and one when $i \neq j$, which is called uniform cost assignment (UCA). The correct detection cost is not included in the Bayes detection risk, and the same weights are given to different incorrect decision probabilities. Then, Bayes detection risk is calculated as

$$r^x(\phi) = P(\mathcal{H}_0) \int\limits_{\mathbb{R}^K} \phi(\boldsymbol{x}) f_0^X(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + P(\mathcal{H}_1) \int\limits_{\Lambda} \int\limits_{\mathbb{R}^K} (1-\phi(\boldsymbol{x}))\pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{x}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \,. \quad (2.10)$$

Since the prior probabilities of the true hypotheses are known, the overall Bayes estimation risk function given in (2.7) can be evaluated. As it is explained, Θ is assumed to have a deterministic value under $\mathcal{H}_0$ and is equal to $\boldsymbol{\theta}_0$. If the decision is $\hat{\mathcal{H}}_1$, estimate $\hat{\theta}(\boldsymbol{x})$ is produced for given observation $\boldsymbol{x}$. If the decision is $\hat{\mathcal{H}}_0$, the trivial estimation result is $\boldsymbol{\theta}_0$. Notice that if the decision is correct when the true hypothesis is $\hat{\mathcal{H}}_0$, the conditional estimation risk for this case is equal to zero. With this remark, the Bayes estimation risk can be obtained as

$$\begin{aligned} r^x(\hat{\theta}) = P(\mathcal{H}_1) \Bigg[\int\limits_{\Lambda} \int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \boldsymbol{\theta}_0)(1-\phi(\boldsymbol{x})) f_1^X(\boldsymbol{x}|\boldsymbol{\theta}) \pi(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \, + \\ \int\limits_{\Lambda} \int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x}))\phi(\boldsymbol{x}) f_1^X(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \Bigg] + P(\mathcal{H}_0) \int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}_0, \hat{\theta}(\boldsymbol{x}))\phi(\boldsymbol{x}) f_0^X(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \end{aligned} \quad (2.11)$$

For a Bayes type detection and estimation system, the aim is the minimization of the Bayes estimation risk with respect to the Bayes detection risk as it has been explained in the next section.

2.2 Problem Definition

In this study, possible improvements on the performance of the considered joint detection and estimation system are investigated by adding noise $\boldsymbol{N}$ to the original observation $\boldsymbol{X}$.

$$\boldsymbol{Y} = \boldsymbol{X} + \boldsymbol{N} \quad (2.12)$$

The new observation $\boldsymbol{Y}$ in (2.12) is fed into the given joint detection and estimation system. The problem is defined as the determination of the optimum distribution of the additive noise $f^N(\boldsymbol{n})$ without modifying the given joint detection and estimation system; that is, detector $\phi(\cdot)$ and estimator $\hat{\theta}(\cdot)$ are fixed. The additive noise $\boldsymbol{N}$ is independent of the observation signal $\boldsymbol{X}$.

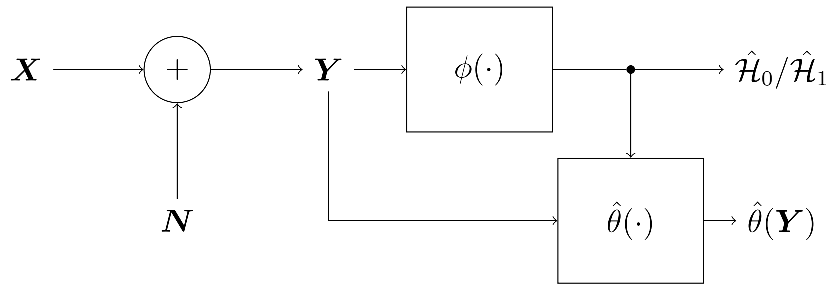


Figure 2.2: Joint detection and estimation scheme with noise enhancement: The only modification on the original system depicted in Figure 2.1 is the introduction of the additive noise $\boldsymbol{N}$.

Notice that all of the introduced performance evaluation functions in Section 2.1, detection probability $P_1^x(\hat{\mathcal{H}}_1)$, false alarm rate $P_0^x(\hat{\mathcal{H}}_1)$, conditional Bayes estimation risk $J^x(\phi, \hat{\theta})$, Bayes detection risk $r^x(\phi)$ and overall Bayes estimation risk $r^x(\hat{\theta})$, are written with a superscript x to emphasize that they are written for the original system without noise enhancement. Introduction of the additive noise at the input of the system will change these expressions. After this modification, they are not notated with the x superscript. These updates are presented in the next two subsections.

In general, the optimality of the detector and estimator to minimize the decision cost and estimation risk is an important goal in the detection and estimation theory. Optimal detectors and estimators for this joint detection and estimation scheme in the NP hypothesis-testing framework are already obtained in [62]. In this work, it should be clarified that the detector and estimator are assumed to be fixed, and are not altered. The only modification is the introduction of the additive noise $\boldsymbol{N}$.

In the literature, the performance criterion for the NP types of problems is determined as the maximization of the detection probability with an upper bound on the false alarm rate. The minimization of the Bayes risk is the main objective in the Bayesian type detection (or estimation) problems where the prior probabilities of the hypotheses (or the distribution of the unknown parameter) are provided [63]. In this study, the aim is determined as the optimization of the estimation performance without causing any detection performance degradation. Depending upon the application, the problem can be defined differently. It is not possible to cover all cases here, and the provided discussion can be considered to construct and solve similar problems.

2.2.1 NP Hypothesis-Testing Framework

In Section 2.1.1, the false alarm rate (2.4), detection probability (2.5) and conditional Bayes estimation risk (2.8) of the joint mechanism are presented. After the introduction of the additive noise, these functions are updated as follows:

$$P_0(\hat{\mathcal{H}}_1) = \int_{\mathbb{R}^K} f^N(\boldsymbol{n}) \int_{\mathbb{R}^K} \phi(\boldsymbol{y}) f_0^X(\boldsymbol{y}-\boldsymbol{n}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{n} \tag{2.13}$$

$$P_1(\hat{\mathcal{H}}_1) = \int_{\mathbb{R}^K} f^N(\boldsymbol{n}) \int_{\Lambda} \int_{\mathbb{R}^K} \phi(\boldsymbol{y}) \pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{n} \tag{2.14}$$

$$J(\phi, \hat{\theta}) = \frac{\int_{\mathbb{R}^K} f^N(\boldsymbol{n}) \int_{\Lambda} \int_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{y})) \phi(\boldsymbol{y}) \pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{n}}{P_1(\hat{\mathcal{H}}_1)}. \tag{2.15}$$

After some manipulation, (2.13), (2.14) and (2.15) can be expressed as the expectations of auxiliary functions over additive noise distribution:

$$P_0(\hat{\mathcal{H}}_1) = E\{T(\boldsymbol{n})\} \text{ where } T(\boldsymbol{n}) = \int_{\mathbb{R}^K} \phi(\boldsymbol{y}) f_0^X(\boldsymbol{y}-\boldsymbol{n}) \, \mathrm{d}\boldsymbol{y} \tag{2.16}$$

$$P_1(\hat{\mathcal{H}}_1) = E\{R(\boldsymbol{n})\} \text{ where } R(\boldsymbol{n}) = \int_{\Lambda} \int_{\mathbb{R}^K} \phi(\boldsymbol{y}) \pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta} \tag{2.17}$$

$$J(\phi, \hat{\theta}) = \frac{E\{G_{11}(\boldsymbol{n})\}}{E\{R(\boldsymbol{n})\}} \text{ where } G_{11}(\boldsymbol{n}) = \int_{\Lambda} \int_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{y})) \phi(\boldsymbol{y}) \pi(\theta) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}. \tag{2.18}$$

Notice that $R(\boldsymbol{n}_0)$,$T(\boldsymbol{n}_0)$ and $G_{11}(\boldsymbol{n}_0)/R(\boldsymbol{n}_0)$ respectively correspond to the detection probability, false alarm rate and estimation risks of the system when the additive noise is equal to $\boldsymbol{n_0}$.

Minimization of the conditional estimation risk (2.15) subject to the false alarm rate (2.13) and the detection probability (2.14) constraints is targeted. The constraints are chosen as the detection probability $P_1^x(\hat{\mathcal{H}}_1)$ and the false alarm rate $P_0^x(\hat{\mathcal{H}}_1)$ of the original system. In other words, no detection performance degradation is allowed. Equivalently, these constraints can also be expressed as $R(\mathbf{0})$ and $T(\mathbf{0})$. The optimization problem is defined as

$$\min_{f^N(\boldsymbol{n})} \frac{E\{G_{11}(\boldsymbol{n})\}}{E\{R(\boldsymbol{n})\}} \text{ subject to } E\{T(\boldsymbol{n})\} \leq T(\mathbf{0}) \text{ and } E\{R(\boldsymbol{n})\} \geq R(\mathbf{0}). \quad (2.19)$$

2.2.2 Bayesian Hypothesis-Testing Framework

In Section 2.1.2; the characteristics of the joint detection and estimation mechanism are investigated in the Bayes framework. Similar to the NP problem definition, after the modification of the joint system by adding noise to the observation, the Bayes detection risk (2.10) and the overall Bayes estimation risk (2.11) expressions need to be revised. Bayes detection risk [63] is given as the following, where $T(\boldsymbol{n})$ and $R(\boldsymbol{n})$ functions are as defined in (2.16) and (2.17).

$$\begin{aligned} r(\phi) = \int_{\mathbb{R}^K} f^N(\boldsymbol{n}) \Big[P(\mathcal{H}_1) \int_{\Lambda} \int_{\mathbb{R}^K} (1-\phi(\boldsymbol{y}))\pi(\boldsymbol{\theta}) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta} \quad + \\ P(\mathcal{H}_0) \int_{\mathbb{R}^K} \phi(\boldsymbol{y}) f_0^X(\boldsymbol{y}-\boldsymbol{n})\mathrm{d}\boldsymbol{y} \Big] \mathrm{d}\boldsymbol{n} = P(\mathcal{H}_1) + E\Big\{ P(\mathcal{H}_0)T(\boldsymbol{n}) - P(\mathcal{H}_1)R(\boldsymbol{n}) \Big\} \end{aligned} \quad (2.20)$$

The overall estimation risk of the joint detection and estimation system after the introduction of additive noise is edited with the introduction of new auxiliary functions $G_{10}(\boldsymbol{n})$ and $G_{01}(\boldsymbol{n})$:

$$\begin{aligned} & r(\hat{\theta}) = E\Big\{ P(\mathcal{H}_0)G_{01}(\boldsymbol{n}) + P(\mathcal{H}_1)\big[G_{11}(\boldsymbol{n}) + G_{10}(\boldsymbol{n})\big] \Big\} \quad \text{where} \\ & G_{01}(\boldsymbol{n}) = \int_{\Lambda} \int_{\mathbb{R}^K} c(\boldsymbol{\theta}, \boldsymbol{\theta}_0)(1-\phi(\boldsymbol{y})) f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta})\pi(\boldsymbol{\theta}) \, \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta} \\ & G_{00}(\boldsymbol{n}) = \int_{\mathbb{R}^K} c(\boldsymbol{\theta}_0, \hat{\theta}(\boldsymbol{y}))\phi(\boldsymbol{y}) f_0^X(\boldsymbol{y}-\boldsymbol{n})\mathrm{d}\boldsymbol{y} \end{aligned} \quad (2.21)$$

As previously mentioned, the purpose is the minimization of the estimation risk with respect to the detection risk. The Bayes detection risk constraint for the noise added system is specified as the Bayes detection risk of the original system, which is $P(\mathcal{H}_1) + P(\mathcal{H}_0)T(\mathbf{0}) - P(\mathcal{H}_1)R(\mathbf{0})$. Then, the optimization problem is given by

$$\begin{aligned} &\min_{f^N(\boldsymbol{n})} E\Big\{P(\mathcal{H}_0)G_{01}(\boldsymbol{n}) + P(\mathcal{H}_1)\big[G_{11}(\boldsymbol{n}) + G_{10}(\boldsymbol{n})\big]\Big\} \\ &\text{subject to } E\Big\{P(\mathcal{H}_0)T(\boldsymbol{n}) - P(\mathcal{H}_1)R(\boldsymbol{n})\Big\} \leq P(\mathcal{H}_0)T(\mathbf{0}) - P(\mathcal{H}_1)R(\mathbf{0})\,. \end{aligned} \tag{2.22}$$

Stated optimization problems (2.19) and (2.22) require a search over all possible probability density distributions and not easy to solve. In the next section, the form of the optimal probability density functions are specified and optimization problems are restated accordingly.

Chapter 3

Optimal Noise Distribution and Non-Improvability Conditions

3.1 Optimum Noise Distribution

The optimization problems in (2.19) and (2.22) require a search over all possible probability density functions (PDFs). This complex problem can be simplified by the specification of the optimum noise distribution structure. This problem is solved in [43] using Caratheodory's theorem for noise enhanced binary hypothesis testing structure. It has been proven that the optimum additive noise distribution is a probability mass function (PMF) with at most two point masses under certain conditions in the binary hypothesis testing problem, where the objective function is the detection probability and the constraint function is the false alarm probability. Using the primal-dual concept, [49] reaches PMFs with at most two point masses under certain conditions for binary hypothesis testing problems. In [55] and [51], the proof given in [43] is extended to hypothesis testing problems with $(M - 1)$ constraint functions and the optimum noise distribution is found to have M point masses.

In this study, the objective function is the Bayes estimation risk in both of the defined optimization problems in (2.19) and (2.22), and constraint functions are defined in terms of the detection probability. The structure of the defined problem is again the same as the geometry of the hypothesis testing problems. The same principles can be applied to both of the optimization problems in (2.19) and (2.22) and the optimum noise distribution structure can be specified under certain conditions.

Theorem 1. *Define set* Z *as* $Z = \{\boldsymbol{z} = (z_0, z_1, ..., z_{K-1}) : z_i \in [a_i, b_i], i = 1, 2, ...K\}$ *where* a_i *and* b_i *are finite numbers, and define set* U *as* $U = \{\boldsymbol{u} = (u_0, u_1, u_2) : u_0 = R(\boldsymbol{n}), u_1 = T(\boldsymbol{n}), u_2 = G_{11}(\boldsymbol{n}),\ \text{for}\ \boldsymbol{n} \in Z\}$*. Assume that the support set of the additive noise random variable is set* Z*. If* U *is a compact set in* $\mathbb{R}^K$*, the optimal solution of* (2.19) *can be represented by a discrete probability distribution with at most three point masses; that is,*

$$f_{opt}^{N}(\boldsymbol{n}) = \sum_{i=1}^{3} \lambda_i \delta(\boldsymbol{n} - \boldsymbol{n_i}) \tag{3.1}$$

Proof. U is the set of all possible detection probability, false alarm rate and conditional estimation risk triples for a given additive noise value $\boldsymbol{n}$ where $\boldsymbol{n} \in Z$. U is a closed and bounded set by the assumption; hence, it is compact. (A subset of $\mathbb{R}^K$ is a closed and bounded set if and only if it is a compact set by Heine-Borel theorem.). Define V as the convex hull of set U. In addition, define W as the set of possible values of $E\{R(\boldsymbol{n})\}$, $E\{T(\boldsymbol{n})\}$ and $E\{G_{11}(\boldsymbol{n})\}$ for all possible expectation operators:

$$\begin{aligned} W = \{(w_0, w_1, w_2) : w_0 = E\{R(\boldsymbol{n})\}, w_1 = E\{T(\boldsymbol{n})\}, \\ w_2 = E\{G_{11}(\boldsymbol{n})\};\ \forall f_N(\boldsymbol{n}),\ \boldsymbol{n} \in Z\}. \end{aligned} \tag{3.2}$$

It is already asserted in the literature that set W (the values that the expectation operator can possibly take) and set V (the convex hull of the sample space) are equal [66]; that is, $W = V$. By the corollary of Carathéodory's theorem, V is also a compact set [67]. Since it is assumed that set U is a compact set in the theorem definition; by the corollary, every point on the boundary of the set V is an element of V and V is a bounded set. The optimal point lies on the boundary of V. From Carathéodory's theorem [67], it can be concluded that any point on the boundary of V can be expressed as the convex combination of at most three different points in U. The compactness assumption assures that the set of optimal points constitutes a compact set as a subset of V. The convex combination of three elements of U is equivalent to an expectation operation over additive noise $\boldsymbol{N}$, where its distribution is a probability mass function with three point masses. $\square$

The same approach can be adopted to obtain the optimal solution of the problem (2.22) and it is stated without a proof. Define U as the set of all possible Bayes detection risk (2.20) and Bayes estimation risk (2.21) pairs for a given additive noise value $\boldsymbol{n} \in Z$, where Z is $Z = \{\boldsymbol{z} = (z_0, z_1, ..., z_{K-1}) : z_i \in [a_i, b_i], i = 1, 2, ...K\}$, with a_i and b_i being finite numbers. Assume that the support set of the additive noise random variable is set Z. If U is a compact set in $\mathbb{R}^K$, the optimal solution of the (2.22) is given by a probability mass function with at most two point masses; that is,

$$f_{opt}^{N}(\boldsymbol{n}) = \sum_{i=1}^{2} \lambda_i \delta(\boldsymbol{n} - \boldsymbol{n_i}). \tag{3.3}$$

The results in Theorem 1 and (3.3) can be applied to (2.19) and (2.22) and the optimization problems can be restated as follows:

For the NP detection framework:

$$\begin{aligned} &\min_{\lambda_1,\lambda_2,\lambda_3,\boldsymbol{n}_1,\boldsymbol{n}_2,\boldsymbol{n}_3} \frac{\sum_{i=1}^{3} \lambda_i G_{11}(\boldsymbol{n_i})}{\sum_{i=1}^{3} \lambda_i R(\boldsymbol{n_i})} \\ &\text{subject to } \sum_{i=1}^{3} \lambda_i T(\boldsymbol{n_i}) \leq T(\boldsymbol{0}) \ , \ \sum_{i=1}^{3} \lambda_i R(\boldsymbol{n_i}) \geq R(\boldsymbol{0}) \\ &\lambda_1, \lambda_2, \lambda_3 \geq 0 \text{ and } \lambda_1 + \lambda_2 + \lambda_3 = 1. \end{aligned} \tag{3.4}$$

For the Bayes detection framework:

$$\begin{aligned} &\min_{\lambda_1,\lambda_2,\boldsymbol{n}_1,\boldsymbol{n}_2} \sum_{i=1}^{2} \lambda_i \Big[P(\mathcal{H}_0) G_{01}(\boldsymbol{n_i}) + P(\mathcal{H}_1) \big[G_{11}(\boldsymbol{n_i}) + G_{10}(\boldsymbol{n_i}) \big] \Big] \\ &\text{subject to} \sum_{i=1}^{2} \lambda_i \Big[P(\mathcal{H}_0) T(\boldsymbol{n_i}) - P(\mathcal{H}_1) R(\boldsymbol{n_i}) \Big] \leq P(\mathcal{H}_0) T(\boldsymbol{0}) - P(\mathcal{H}_1) R(\boldsymbol{0}) \\ &\lambda_1, \lambda_2 \geq 0 \text{ and } \lambda_1 + \lambda_2 = 1. \end{aligned} \tag{3.5}$$

3.2 Linear Programming Approximation

The characteristics of the optimization problems in (3.4) and (3.5) are related to the given joint detection and estimation mechanism with the statistics of observation signal $\boldsymbol{X}$ and parameter Θ. The problems may not be convex in general. Different evolutionary computational techniques such as particle swarm optimization techniques can be carried out [68, 69]. Alternatively, the given optimization problems can be approximated as linear programming (LP) problems. LP problems are a special case of convex problems and they have lower computational load (solvable in polynomial time) than the possible global optimization techniques [70].

In order to achieve the LP approximation of the problem (3.4), the support of the additive noise is restricted to a finite set $\mathbb{S} = \{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_M\}$. In real life applications, it is not possible to generate an additive noise component which can take infinitely many different values in an interval; hence, it is a reasonable assumption that additive noise component can only have finite precision. With this approach, the possible values of $R(\mathbf{n})$, $T(\mathbf{n})$ and $G_{11}(\mathbf{n})$ can be expressed as M dimensional column vectors and the expectation operation reduces to a convex combination of the elements of these column vectors with weights $\lambda_1, \lambda_2, \cdots \lambda_M$. The optimal values of the LP approximated problems are worse than or equal to the optimal values of the original optimization problems (3.4) and (3.5). And the gap between these results is dependent upon the number of noise samples, which is denoted by M in this study. For notational convenience, these column vectors are defined as

$$\begin{aligned}
\mathbf{t}^{\mathsf{T}} &= [T(\mathbf{n}_1) \;\; T(\mathbf{n}_2) \;\; \cdots \;\; T(\mathbf{n}_M)] \\
\mathbf{r}^{\mathsf{T}} &= [R(\mathbf{n}_1) \;\; R(\mathbf{n}_2) \;\; \cdots \;\; R(\mathbf{n}_M)] \\
\mathbf{g}^{\mathsf{T}} &= [G_{11}(\mathbf{n}_1) \; G_{11}(\mathbf{n}_2) \cdots G_{11}(\mathbf{n}_M)]
\end{aligned}$$

Then, the optimization problem in (3.4), which considers the minimization of the conditional Bayes estimation risk, can be approximated as the following linear fractional programming (LFP) problem:

$$
\begin{aligned}
\underset{\boldsymbol{\lambda}}{\text{minimize}} \quad & \frac{\mathbf{g}^{\intercal}\boldsymbol{\lambda}}{\mathbf{r}^{\intercal}\boldsymbol{\lambda}} \\
\text{subject to} \quad & \mathbf{r}^{\intercal}\boldsymbol{\lambda} \geq R(\mathbf{0}) \\
& \mathbf{t}^{\intercal}\boldsymbol{\lambda} \leq T(\mathbf{0}) \\
& \mathbf{1}^{\intercal}\boldsymbol{\lambda} = 1 \\
& \boldsymbol{\lambda} \succeq 0\,.
\end{aligned}
\tag{3.6}
$$

An example of transformation from a linear fractional programming (LFP) to linear programming (LP) is presented in [70]. The same approach can be followed to obtain an LP problem as explained in the following. The optimization variable $\mathbf{l}$ in the LP problem, which is presented in (3.8), is expressed as

$$
\mathbf{l} = \frac{\boldsymbol{\lambda}}{\mathbf{r}^{\intercal}\boldsymbol{\lambda}}\ . \tag{3.7}
$$

Notice that $\mathbf{r}$ and $\boldsymbol{\lambda}$ have non-negative components, and $\mathbf{r}^{\intercal}\boldsymbol{\lambda}$ represents the detection probability of the noise added mechanism. Therefore, it can be assumed that $\mathbf{r}^{\intercal}\boldsymbol{\lambda}$ is positive valued and less than or equal to 1. With this assumption, it is straightforward to prove the equivalence of the LP and LFP problem by showing that if $\boldsymbol{\lambda}$ is feasible in (3.6), then $\mathbf{l}$ is also feasible in (3.8) with the same objective value, and vice versa. Hence, the following problem is obtained:

$$
\begin{aligned}
\underset{\mathbf{l}}{\text{minimize}} \quad & \mathbf{g}^{\intercal}\mathbf{l} \\
\text{subject to} \quad & \mathbf{t}^{\intercal}\mathbf{l} \leq T(\mathbf{0})(\mathbf{1}^{T}\mathbf{l}) \\
& \mathbf{1}^{\intercal}\mathbf{l} \leq 1/R(\mathbf{0}) \\
& \mathbf{r}^{\intercal}\mathbf{l} = 1 \\
& \mathbf{l} \succeq 0\,.
\end{aligned}
\tag{3.8}
$$

The LP approximation of the optimization problem (3.5) is also obtained through limiting the possible additive noise values to a finite set $\mathbb{S}' =$

$\{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_{M'}\}$. With that restriction, the LP problem is given as

$$\begin{aligned} \underset{\boldsymbol{\lambda}}{\text{minimize}} \quad & \mathbf{q}^\mathsf{T}\boldsymbol{\lambda} \\ \text{subject to} \quad & \mathbf{p}^\mathsf{T}\boldsymbol{\lambda} \leq P(\mathcal{H}_0)T(\mathbf{0}) - P(\mathcal{H}_1)R(\mathbf{0}) \\ & \mathbf{1}^\mathsf{T}\boldsymbol{\lambda} = 1 \\ & \boldsymbol{\lambda} \succeq 0 \ . \end{aligned} \tag{3.9}$$

where

$$\mathbf{p}^\mathsf{T} = [p_1 \ p_2 \ \cdots \ p_{n_{M'}}] \ , \quad p_i = P(\mathcal{H}_0)T(\boldsymbol{n}_i) - P(\mathcal{H}_1)R(\boldsymbol{n}_i)$$
$$\mathbf{q}^\mathsf{T} = [q_1 \ q_2 \ \cdots \ q_{n_{M'}}] \ , \quad q_i = P(\mathcal{H}_0)G_{01}(\boldsymbol{n}_i) + P(\mathcal{H}_1)\big[G_{11}(\boldsymbol{n}_i) + G_{10}(\boldsymbol{n}_i)\big]$$

3.3 Improvability and Non-improvability Conditions

Before attempting to solve the optimization problems in (3.4) and (3.5), or the LP problems in (3.8) and (3.9); it is worthwhile to investigate the improvability of these problems since the defined optimization problems can be complex in general. Before moving on the proposed conditions, first the terms improvability and non-improvability need to be clarified.

The joint detection and estimation system in the NP framework is called improvable if there exists a probability distribution $f^N(\boldsymbol{n})$ for the additive noise $\boldsymbol{N}$ such that $J(\phi, \hat{\theta}) < J^x(\phi, \hat{\theta})$ satisfying the conditions $P_1(\hat{\mathcal{H}}_1) \geq P_1^x(\hat{\mathcal{H}}_1)$ and $P_0(\hat{\mathcal{H}}_1) \leq P_0^x(\hat{\mathcal{H}}_1)$, and non-improvable if there does not exist such a distribution. Similarly, the joint system in the Bayes detection framework is called improvable if there exists a probability distribution $f^N(\boldsymbol{n})$ such that $r(\hat{\theta}) < r^x(\hat{\theta})$ and $r(\phi) \leq r^x(\phi)$, and non-improvable otherwise. Improvable and non-improvable joint detection and estimation systems under the LP approximation can also be defined in a similar fashion for both detection frameworks.

Theorem 2. *Let p^* denotes the optimal basic feasible solution of the linear fractional problem in* (3.6)*, where the objective is to minimize the conditional Bayes estimation risk with noise enhancement under the condition that the possible values of the additive noise are restricted to a finite set $\mathbb{S} = \{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_M\}$. For* *the LP approximation, the performance of the joint detection and estimation system in the NP framework cannot be improved, that is, $p* \geq G_{11}(\mathbf{0})/R(\mathbf{0})$, if and only if there exist $\gamma_1, \gamma_2,\, \nu \in \mathbb{R}$; $\gamma_1, \gamma_2 \geq 0$ and $\nu \leq -[G_{11}(\mathbf{0}) + \gamma_2]/R(\mathbf{0})$ satisfying the following set of inequalities:*

$$G_{11}(\boldsymbol{n}_i) + \gamma_1(T(\boldsymbol{n}_i) - T(\mathbf{0})) + \gamma_2 + \nu R(\boldsymbol{n}_i) \geq 0 \ , \quad \forall i \in \{1, 2, \cdots, M\} \ . \tag{3.10}$$

Proof. In equation (3.8), the equivalent LP problem of the linear fractional programming (LFP) problem (3.6) is given. The dual problem of the LP problem is found as the following:

$$\begin{aligned} \underset{\nu,\gamma_1,\gamma_2,\boldsymbol{u}}{\text{maximize}} \quad & -\nu - \gamma_2/R(\mathbf{0}) \\ \text{subject to} \quad & G_{11}(\boldsymbol{n}_i) + \gamma_1(T(\boldsymbol{n}_i) - T(\mathbf{0})) + \gamma_2 + \nu R(\boldsymbol{n}_i) = u_i, \ \forall i \in \{1, 2, \cdots, M\} \\ & \gamma_1, \gamma_2, u_1, u_2, \cdots, u_M \geq 0. \end{aligned} \tag{3.11}$$

where $\boldsymbol{u}^\intercal = [u_1 \ u_2 \ \cdots \ u_M]$.

Let $\mathbb{P}$ and $\mathbb{D}$ be the feasible sets of the primal (3.6) and dual (3.11) problems, respectively. The objective functions of the primal and dual problems are denoted as $f^P_{obj}(p)$ and $f^D_{obj}(d)$, where $p \in \mathbb{P}$ and $d \in \mathbb{D}$. p^* and d^* are the optimal solutions of the primal and dual problems. By the strong duality property of the linear programming problems, $p^* = d^*$ [70] .

Sufficient condition for non-improvability: Assume that $\exists\ \gamma_1, \gamma_2, \nu \in \mathbb{R};\ \boldsymbol{u} \in \mathbb{R}^K$ such that $\gamma_1, \gamma_2 \geq 0;\ \boldsymbol{u} \succeq 0;\ \nu \leq -[G_{11}(\mathbf{0}) + \gamma_2]/R(\mathbf{0})$, and $\gamma_1, \gamma_2, \nu, \boldsymbol{u}$ satisfy the following set of equations: $G_{11}(\boldsymbol{n}_i) + \gamma_1(T(\boldsymbol{n}_i) - T(\mathbf{0})) + \gamma_2 + \nu R(\boldsymbol{n}_i) = u_i \geq 0,\ \forall i \in \{1, 2, \cdots, M\}$. These variables describe an element of the dual feasible set $d^o = (\gamma_1, \gamma_2, \nu, \boldsymbol{u}) \in \mathbb{D}$. $f^D_{obj}(d^o) = -\nu - \gamma_2/R(\mathbf{0}) \geq G_{11}(\mathbf{0})/R(\mathbf{0})$ by the assumption. This implies that $G_{11}(\mathbf{0})/R(\mathbf{0}) \leq f^D_{obj}(d^o) \leq d^* = p^*$, and therefore, the conditional Bayes risk of the system in the NP framework cannot be reduced from its original value.

Necessary condition for non-improvability: To prove the necessary condition, it is equivalent to show that the system performance can be improved if $\forall\, \gamma_1, \gamma_2, \nu \in \mathbb{R};\ \boldsymbol{u} \in \mathbb{R}^K$ such that $\gamma_1, \gamma_2 \geq 0;\ \boldsymbol{u} \succeq 0;\ \nu \geq -[G_{11}(\mathbf{0}) + \gamma_2]/R(\mathbf{0})$, the following set of equations are satisfied: $G_{11}(\boldsymbol{n}_i)+\gamma_1(T(\boldsymbol{n}_i)-T(\mathbf{0}))+\gamma_2+\nu R(\boldsymbol{n}_i) = u_i \geq 0$, $\forall i \in \{1, 2, \cdots, M\}$. Observe that γ_2 or ν can always be picked arbitrarily large to satisfy the equality constraints given in (3.11), since $1 \geq R(\boldsymbol{n}_i) \geq 0$, $1 \geq T(\boldsymbol{n}_i) \geq 0$ and $G_{11}(\boldsymbol{n}_i) \geq 0$. Therefore, the feasible set of the dual problem cannot be empty, $\mathbb{D} \neq \emptyset$. Notice that the assumption implies $\forall d \in \mathbb{D}$, $f_{obj}^D(d) < G_{11}(\mathbf{0})/R(\mathbf{0})$. For this reason and with the strong duality property it can be asserted that $d^* = p^* < G_{11}(\mathbf{0})/R(\mathbf{0})$ since $d^* = f_{obj}^D(d^{opt})$, $d^{opt} \in \mathbb{D}$. □

Theorem 3. *Let p^* denote the optimal basic feasible solution of the linear programming problem in (3.9), where the objective is to minimize the Bayes estimation risk with noise enhancement under the condition that the possible values of the additive noise are restricted to a finite set $\mathbb{S}' = \{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}'_M\}$. For the LP approximation, the performance of the joint detection and estimation system in the Bayes detection framework cannot be improved, that is, $p* \geq P(\mathcal{H}_0)G_{01}(\mathbf{0}) + P(\mathcal{H}_1)\big[G_{11}(\mathbf{0}) + G_{10}(\mathbf{0})\big]$, if and only if there exist $\gamma, \nu \in \mathbb{R}$; $\gamma \geq 0$ and satisfying the following set of inequalities:*

$$P(\mathcal{H}_0)\big[\gamma T(\mathbf{0}) + G_{01}(\mathbf{0})\big] \ + P(\mathcal{H}_1)\big[G_{11}(\mathbf{0}) + G_{10}(\mathbf{0}) - \gamma R(\mathbf{0})\big] \ + \nu \leq 0 \ , \quad (3.12)$$

$$P(\mathcal{H}_0)\big[\gamma T(\mathbf{n}_i) + G_{01}(\mathbf{n}_i)\big] + P(\mathcal{H}_1)\big[G_{11}(\mathbf{n}_i) + G_{10}(\mathbf{n}_i) - \gamma R(\mathbf{n}_i)\big] + \nu \geq 0 \quad \forall i \in \{1, 2, \cdots, M'\} \ . \quad (3.13)$$

Notice that if $\mathbf{0} \in \mathbb{S}' = \{\mathbf{n}_1, \mathbf{n}_2, \cdots, \mathbf{n}_{M'}\}$, then the inequality (3.12) must be satisfied with equality. With this, (3.13) can be expressed as

$$\begin{aligned} &P(\mathcal{H}_1)\big[G_{11}(\mathbf{n}_i) + G_{10}(\mathbf{n}_i) - \gamma R(\mathbf{n}_i) - G_{11}(\mathbf{0}) - G_{10}(\mathbf{0}) + \gamma R(\mathbf{0})\big] \\ &\quad + P(\mathcal{H}_0)\big[G_{01}(\mathbf{n}_i) + \gamma T(\mathbf{n}_i) - G_{01}(\mathbf{0}) - \gamma T(\mathbf{0})\big] \geq 0 \end{aligned} \quad (3.14)$$

Notice that the LP approximation is based on sampling the objective and constraint functions. Therefore, the presented sufficient and necessary conditions in Theorems 2 and 3 demonstrate the convex geometry of the optimization problems in (2.19) and (2.22). For similar problem formulations, different necessary or sufficient improvability or nonimprovability conditions are stated in the literature [43,49–51,54]. In [54], firstly, a necessary and sufficient condition is presented for a similar single inequality constrained problem with a continuous support set.

It should be noted that (2.22) is a single inequality constrained problem and its necessary and sufficient non-improvability condition under LP approximation (3.14) share the same structure with the inequality (10) in [54] under a certain condition. Theorem 2 extends this result to the problems with multiple inequality constraints and finite noise random variable support set from a completely different perspective. The merit of this approach, which is presented in the proof of Theorem 2, is that it is generic and can easily be adapted into different problems. In this thesis, the main focus is on the justification of the LP approximation for noise enhancement problems in joint detection and estimation systems. A natural extension of Theorem 2 which is the formulation for a continuous support set is omitted.

Chapter 4

Numerical Examples

4.1 Analysis of a Given Joint Detection Estimation System

In this section, a binary hypothesis testing example is analyzed to demonstrate the noise enhancement effect on the described joint detection and estimation system.

In the hypothesis testing problem presented in (4.1), $\boldsymbol{X}$ is the observation signal and $\boldsymbol{X} = [X_1\ X_2\ \cdots\ X_K]^\intercal$. $\boldsymbol{\Theta}$ is the parameter signal and $\boldsymbol{\Theta} = \Theta\,\mathbf{1}$, where $\mathbf{1} = [1\ 1\ \cdots\ 1]^\intercal$. Θ is taken to be Gaussian distributed in this example and its value is to be estimated. $\boldsymbol{\epsilon} = [\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_K]^\intercal$ is the system noise. ϵ_k's are identically and independent distributed according to a known Gaussian mixture distribution. It is assumed that both of these distributions are known. K signals ($X_k = \Theta + \epsilon_k$) are employed for each decision and for each parameter (Θ) estimation.

$$\begin{aligned} \mathcal{H}_0 &: \boldsymbol{X} = \boldsymbol{\epsilon} \\ \mathcal{H}_1 &: \boldsymbol{X} = \boldsymbol{\epsilon} + \boldsymbol{\Theta} \end{aligned} \tag{4.1}$$

Decision rule $\phi(x)$ is a threshold detector and it gives the probability of deciding in favor of $\mathcal{H}_1$. The subscript PF is written under threshold τ to emphasize that threshold τ_{PF} is determined according to the predetermined probability of false alarm (false alarm rate). The decision rule is a simple and reasonable rule which compares the sample mean of the observations against the threshold; that is,

$$\phi(\boldsymbol{x}) = \begin{cases} 1 & \text{if } \frac{1}{K}\sum_{i=1}^{K} x_i > \tau_{PF} \\ 0 & \text{if } \frac{1}{K}\sum_{i=1}^{K} x_i \leq \tau_{PF} \end{cases} \tag{4.2}$$

The estimation rule is a sample mean estimator, specified by

$$\hat{\theta}(\boldsymbol{x}) = \frac{1}{K}\sum_{i=1}^{K} x_i \ . \tag{4.3}$$

In addition, the estimation cost function, which is presented in (2.8), is a 0-1 loss function:

$$c(\theta, \hat{\theta}(\boldsymbol{x})) = \begin{cases} 1 & \text{if } |\hat{\theta}(\boldsymbol{x}) - \theta| > \triangle \\ 0 & \text{otherwise} \end{cases} \tag{4.4}$$

The components of the system noise $\boldsymbol{\epsilon}$ are identical, independent and Gaussian mixture distributed:

$$f_{\epsilon_k}(\varepsilon) = \sum_{i=1}^{N_m} \frac{\nu_i}{\sqrt{2\pi\sigma^2}} \exp\Big\{ -\frac{(\varepsilon-\mu_i)^2}{2\sigma^2} \Big\} \tag{4.5}$$

Notice that each element of the Gaussian mixture has different mean μ_i and weight ν_i with the same standard deviation σ. The mixture background noise is encountered in a variety of contexts [71] (and references therein) such as co-channel interference [72], ultra-wideband synthetic aperture radar (UWB SAR) imaging [73], and underwater noise modelling [74]. As discussed in Introduction and Background chapters, noise benefits are commonly observed in nonlinear systems and/or under non-Gaussian noise. The standard deviation values are taken equal for all the mixture components just to simplify the analytical evaluation of this problem for $K > 1$. The standard deviation values can also be taken to be different for each mixture component.

Notice that the introduced detector (4.2) is a matched filter and the optimal detector for the NP type coherent detection problems, where the signal is deterministic and the background noise is white Gaussian noise. For notational simplicity, the deterministic signal is taken as all ones vector $\mathbf{1} = [1\ 1\ \cdots\ 1]^{\mathsf{T}}$ in this problem and matched filtering is reduced to a sample mean operator. Similarly, the introduced estimator (4.3) is the optimal maximum a-posteriori probability (MAP) estimator under zero mean white Gaussian noise with all ones signal vector [63].

Finally, the parameter Θ is taken as Gaussian distributed, $\Theta \sim \mathcal{N}(a, b^2)$, that is,

$$\pi(\theta) = \frac{1}{\sqrt{2\pi b^2}} \exp\Big\{ -\frac{(\theta - a)^2}{2b^2} \Big\} \tag{4.6}$$

4.1.1 Scalar Case, $K = 1$

For $K = 1$, with the inclusion of the additive noise N to the system, $Y = X + N$; the detector and estimator mechanisms become the following:

$$\text{Detector } \phi(y) :\ y \underset{\hat{\mathcal{H}}_0}{\overset{\hat{\mathcal{H}}_1}{\gtrless}} \tau_{PF}\ , \quad \text{Estimator } \hat{\theta}(y) = y$$

For this specific example, the $R(n)$, $T(n)$ G_{01}, G_{10} and $G_{11}(n)$ functions defined in equations (2.16), (2.17), (2.18) and (2.21) are derived in the following, where $\mathrm{Q}(\cdot)$ and $\Phi(\cdot)$ are respectively the tail probability function and the cumulative distribution function of the standard Gaussian random variable.

$$T(n) \ = \ \sum_{i=1}^{N_m} \nu_i \mathrm{Q}\Big(\frac{\tau - \mu_i - n}{\sigma_i}\Big) \tag{4.7}$$

$$R(n) \ = \ \sum_{i=1}^{N_m} \nu_i \mathrm{Q}\Big(\frac{\tau - \mu_i - a - n}{\sqrt{\sigma_i^2 + b^2}}\Big) \tag{4.8}$$

$$\begin{aligned} G_{11}(n) = \sum_{i=1}^{N_m} \nu_i \Big\{ \mathcal{Q}\Big(\frac{\tau - \mu_i - n - a}{\sqrt{b^2 + \sigma_i^2}}\Big) \ - \int_{\tau-\triangle}^{\tau+\triangle} \pi(\theta)\ \mathcal{Q}\Big(\frac{\tau - \mu_i - n - \theta}{\sigma_i}\Big)\, \mathrm{d}\theta \\ +\ \mathrm{Q}\Big(\frac{\triangle - \mu_i - n}{\sigma_i}\Big) \mathcal{Q}\Big(\frac{\tau - \triangle - a}{b}\Big) \ +\ \mathrm{Q}\Big(\frac{-\triangle - \mu_i - n}{\sigma_i}\Big) \mathcal{Q}\Big(\frac{\tau + \triangle - a}{b}\Big) \Big\} \end{aligned} \tag{4.9}$$

$$G_{10}(n) = \sum_{i=1}^{N_m} \nu_i \Bigg\{ \Phi\Big(\frac{\tau - \mu_i - n - a}{\sqrt{b^2 + \sigma_i^2}}\Big) - \int_{-\triangle}^{\triangle} \pi(\theta)\, \Phi\Big(\frac{\tau - \mu_i - n - \theta}{\sigma_i}\Big)\, \mathrm{d}\theta \Bigg\} \tag{4.10}$$

$$G_{01}(n) = \begin{cases} \sum_{i=1}^{N_m} \nu_i \Big\{ \mathrm{Q}\big(\frac{\tau-\mu_i-n}{\sigma_i}\big) \Big\} & \text{if } \tau > \triangle, \\ \sum_{i=1}^{N_m} \nu_i \Big\{ \mathrm{Q}\big(\frac{\triangle-\mu_i-n}{\sigma_i}\big) \Big\} & \text{if } \triangle \geq \tau > -\triangle, \\ \sum_{i=1}^{N_m} \nu_i \Big\{ \mathrm{Q}\big(\frac{\tau-\mu_i-n}{\sigma_i}\big) + \mathrm{Q}\big(\frac{\triangle-\mu_i-n}{\sigma_i}\big) - \mathrm{Q}\big(\frac{-\triangle-\mu_i-n}{\sigma_i}\big) \Big\} & \text{if } -\triangle \geq \tau. \end{cases} \tag{4.11}$$

4.1.2 Vector Case, $K > 1$

To evaluate the performance of this system (with and without noise enhancement), the statistics of $\frac{1}{K}\sum_{i=1}^{K} x_i$ needs to be revealed. Additive noise and modified observation signals, which are introduced in 2.2, are represented as $\boldsymbol{N} = [N_1\ N_2\ \cdots\ N_K]^\intercal$ and $\boldsymbol{Y} = [Y_1\ Y_2\ \cdots\ Y_K]^\intercal$. Denote $\frac{1}{K}\sum_{i=1}^{K} N_i$ with $\tilde{N}$ and $\frac{1}{K}\sum_{i=1}^{K} \epsilon_i$ with $\tilde{\epsilon}_K$. Under $\mathcal{H}_1$ and with additive noise; this vector joint detection and estimation problem can be reexpressed as a scalar problem (4.12).

$$\text{Under } \mathcal{H}_1: \ \frac{1}{K}\sum_{i=1}^{K} Y_i = \frac{1}{K}\sum_{i=1}^{K} (\Theta + N_i + \epsilon_i) = \Theta + \tilde{N} + \tilde{\epsilon}_K \tag{4.12}$$

In Appendix C, it is shown that $\tilde{\epsilon}$ also has a Gaussian mixture distribution (C.7). With this result, which is stated in (4.13) below, the vector case reduces to the scalar case. The derived expressions in the $K = 1$ case for $T(n)$ (4.7), $R(n)$ (4.8), $G_{11}(n)$, G_{10} (4.10) and G_{01} (4.11) functions do also apply to the $K > 1$ case, where the only necessary modification is the usage of new mean $\tilde{\mu}_j$, weight $\tilde{\nu}_j$ and standard deviation $\tilde{\sigma}$ values. With this approach, the optimal statistics for the design of $\tilde{N}$ random variable is revealed. The mapping from $\tilde{N}$ to $\boldsymbol{N}$ is left to the designer. A very straightforward choice can be $\boldsymbol{N} = [\tilde{N}K\ \ 0\ \cdots\ 0]^\intercal$.

$$f_{\tilde{\epsilon}_K}(\varepsilon) = \sum_{j=1}^{\tilde{N}_m} \frac{\tilde{\nu}_i}{\sqrt{2\pi\tilde{\sigma}^2}} \exp\Big\{ -\frac{(\varepsilon - \tilde{\mu}_i)^2}{2\tilde{\sigma}^2} \Big\} , \tag{4.13}$$

where

$$\tilde{N}_m = \binom{K + N_m - 1}{N_m - 1},\ \tilde{\sigma}^2 = \frac{\sigma^2}{K},\ \ \tilde{\nu}_j = \left(\frac{K!}{l_1!\ l_2!\ \cdots\ l_{N_m}!}\right)\left(\prod_{i=1}^{N_m} \nu_i^{l_i}\right),\ \ \tilde{\mu}_j = \frac{1}{K}\sum_{i=1}^{N_m} \mu_i l_i$$

for each distinct $\{l_1, l_2, \cdots, l_{N_m}\}$ set satisfying $l_1 + l_2 + \cdots + l_{N_m} = K, \quad l_i \in \{1, 2, \cdots, K\}, \ i \in \{1, 2, \cdots, N_m\}$.

In this joint detection and estimation problem, the components of the system noise $\boldsymbol{\epsilon}$ are independent and identically distributed Gaussian mixture distributed random variables. A similar analysis can also be carried out for a system noise with components being generalized Gaussian distributed. However, in general, it is not possible to express the density of the sum of the independent and identically distributed generalized Gaussian random variables with an exact analytical expression. The distribution of the sum is not generalized Gaussian (only exception is the Gaussian distribution) [75]. However, functions $T(n)$ (4.7), $R(n)$ (4.8), $G_{11}(n)$, G_{10} (4.10) and G_{01} (4.11) can be evaluated numerically and the LP approximation can be applied.

4.1.3 Asymptotic Behaviour of the System, Large K Values

As K goes to infinity ($K \to \infty$), by Lindeberg Levy Central Limit Theorem, $\sqrt{K}\left(\left(\frac{1}{K}\sum_{i=1}^{K}\epsilon_i\right) - \mu_{\epsilon_i}\right)$ converges in distribution to a Gaussian random variable $\mathcal{N}(0, \sigma^2_{\epsilon_i})$ given that $\{\epsilon_1, \epsilon_2, \cdots, \epsilon_K\}$ is a sequence of independent and identically distributed random variables with $E\{\epsilon_i\} = \mu_{\epsilon_i}$, $var\{\epsilon_i\} = \sigma^2_{\epsilon_i} < \infty$. This general result applies to the analysis of the given joint detection and estimation problem in this section. For large K values, the probability density function of $\tilde{\epsilon}_K = \frac{1}{K}\sum_{i=1}^{K}\epsilon_i$ can be approximated with the distribution of a Gaussian random variable $\mathcal{N}(\mu_{\epsilon_i}, \sigma^2_{\epsilon_i}/K)$.

4.2 Numerical Results for the Joint Detection and Estimation System

For the numerical examples, the parameter values are set as follows: The means of the components of the Gaussian mixture distribution $f_\epsilon(e)$ are taken as $\boldsymbol{\nu} =$

[0.40 0.15 0.45]. The weights of the components are $\boldsymbol{\mu} = [5.25\ \ -0.5\ \ -4.5]$. The standard deviations of the components are equal to each other, and this value is considered as the variable to evaluate the performance of noise enhancement for different signal-to-noise (SNR) values. The defined decision rule (4.2) is a threshold detector, where τ_{PF} is set for different standard deviation values such that the false alarm rate of the given system is equal to 0.15 (constant false alarm rate). Regarding (4.6), the mean parameter a is 4.5 and the standard deviation b is 1.25. The estimator is unbiased for the considered scenario. 0-1 Estimation Cost Function parameter $\triangle$ is taken as 0.75.

The optimization problem in (2.19) aims to minimize the conditional estimation risk with respect to the false alarm rate and the detection probability. In this example, the false alarm constraint α and the detection probability constraint β are taken respectively as the false alarm rate and the detection probability of the original system. The support of the additive noise is considered as $[-10, 10]$. In this study, a closed form expression of this optimization problem cannot be presented. However, based on Theorem 1, numerical optimization tools can be used to find the global solution. Similarly, the LP approximation is also applied on this example and the results are displayed in the figures and tables. To solve the global optimization problem and the LP problems, CVX (a package for specifying and solving convex programs) is used [76, 77].

The conditional estimation risk values are plotted versus σ in Figure 4.1 for K = 1 and in Figure 4.2 for K = 4, in the absence (original) and in the presence of additive noise. In these figures, it is observed that the system performance improvement is reduced as the standard deviation σ increases. In other words, the noise enhancement improvement effect is more effective in the high SNR region. The improvement is mainly caused by the multimodal nature of the observation statistics and increasing the standard deviation σ reduces this effect. In both of the figures, the performances of the LP approximations are also illustrated in comparison with the global solution of the optimization problem. Noise samples are taken uniformly from the support of the additive noise with different step size values.

As it is clearly observed from the figure, the accuracy of the LP approximation is heavily dependent on the step size between the noise samples. The solutions of the LP approximation (3.8) to the minimization of the conditional estimation risk optimization problem (3.4) is sufficiently close to the global optimization solution for step sizes 0.4 ($K = 1$) and 0.1 ($K = 4$), which are reasonable values. As it is clear from Figure 4.2, the performance of the given joint detection system is superior for $K = 4$ in comparison to the scalar observation case $K = 1$. In this numerical example, the vector case corresponds to taking more samples and an increase in the signal-to-noise ratio effectively. Similarly, the performances of the LP approximations are depicted with the original curve and the global solution. As the number of samples in the LP approximation increases (equivalently, as the step size interval decreases), it is expected to observe that the LP solution becomes more similar to the global optimal solution. This is the main intuition behind the performance degradation in the LP approximation with higher step size values. The numerical results presented in Figures 4.1 and 4.2 confirm this assessment. Some numerical values of the conditional estimation risk, detection probability and false alarm rate of this noise enhanced system are given in Table 4.1.

In Figure 4.3, the probability density functions of ϵ_k and $\tilde{\epsilon}_K$ are drawn. As it is indicated, ϵ_k has a Gaussian mixture density with three components. Also, $\tilde{\epsilon}_K$ is the sample mean random variable of K independent and identically distributed ϵ_k random variables and it is shown that it has a Gaussian mixture density. In Figures 4.4, 4.5 and 4.6 the solutions of the optimization problem in (3.4) are presented for the solutions of the LP problem in (3.8) with different step size intervals, where the standard deviations of the components of the Gaussian mixture system noise are equal to 0.3 and 0.4. According to Theorem 1, the optimal solution to the optimization problems (2.19) is a probability mass function with at most three point masses. The experimental results confirm this proposal. In these figures, the locations and weights of these point masses are presented.

Notice that Theorem 1 only describes the form of the global optimum solution to the optimization problem in (2.19). It does not directly apply to the LP problem in (3.8). Therefore, it can be expected that the optimal λ^* solutions of the

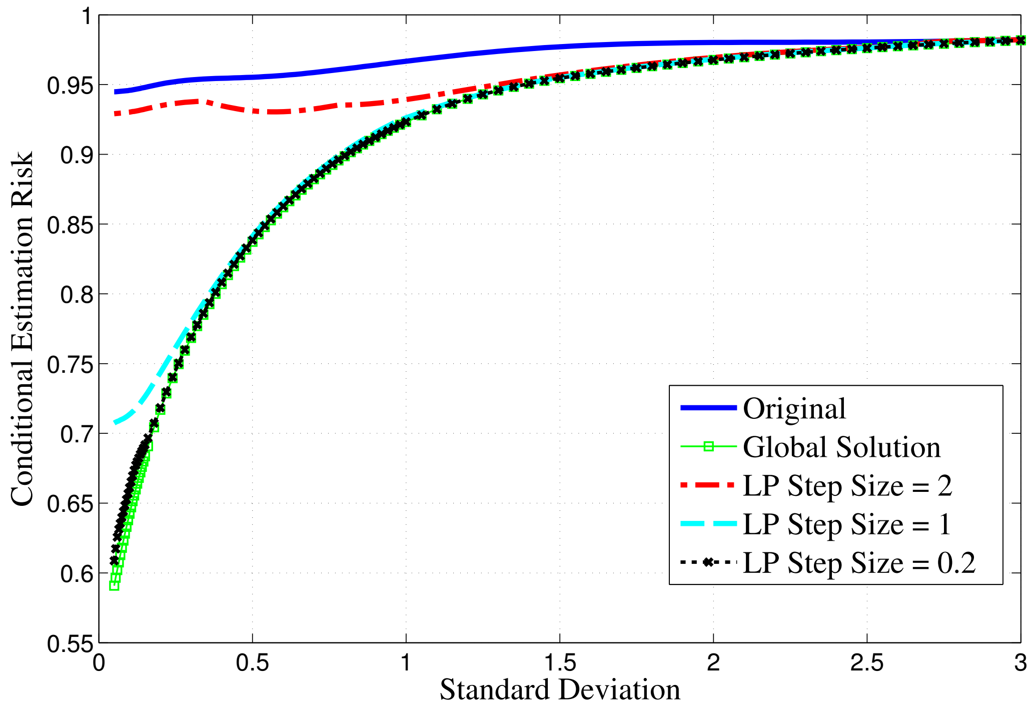


Figure 4.1: Noise enhancement effects for the minimization of the conditional estimation risk, $K = 1$ (NP framework).

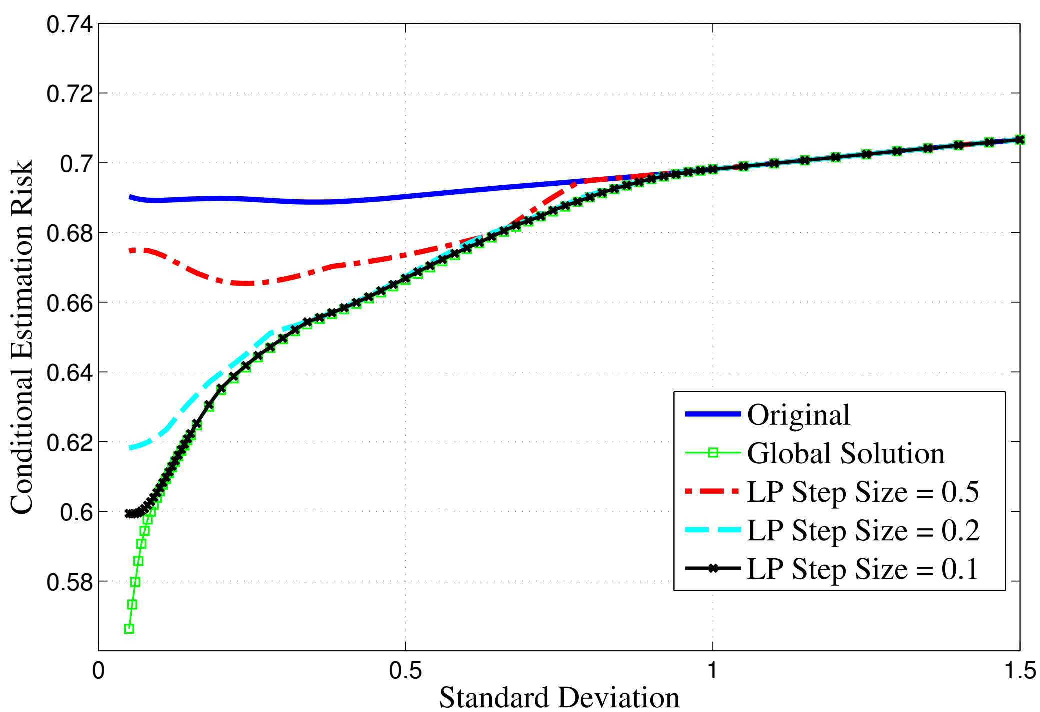
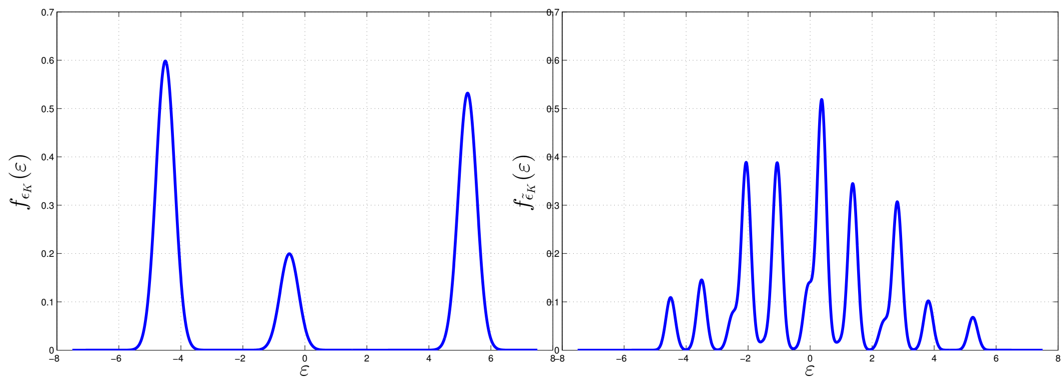


Figure 4.2: Noise enhancement effects for the minimization of the conditional estimation risk, $K = 4$ (NP framework).



(a) $f_{\epsilon_k}(\varepsilon)$ density, K = 1 (b) $f_{\tilde{\epsilon}_K}(\varepsilon)$ density, K = 4

Figure 4.3: Probability density functions of ϵ_k and $\tilde{\epsilon}_K$.

σ=0.3, K=1	τ_{PF}	$E\{T(0)\}$	$E\{R(0)\}$	$\frac{E\{G_{11}(0)\}}{E\{R(0)\}}$	$E\{T(\boldsymbol{n})\}$	$E\{R(\boldsymbol{n})\}$	$\frac{E\{G_{11}(\boldsymbol{n})\}}{E\{R(\boldsymbol{n})\}}$
LP (2.0)	5.3456	0.1500	0.4220	0.9533	0.1500	0.4220	0.9376
LP (1.0)	5.3456	0.1500	0.4220	0.9533	0.1500	0.4220	0.7796
LP (0.2)	5.3456	0.1500	0.4220	0.9533	0.1500	0.4220	0.7694
Opt. Sol.	5.3456	0.1500	0.4220	0.9533	0.1500	0.4220	0.7684
σ=0.3, K=4	τ_{PF}	$E\{T(0)\}$	$E\{R(0)\}$	$\frac{E\{G_{11}(0)\}}{E\{R(0)\}}$	$E\{T(\boldsymbol{n})\}$	$E\{R(\boldsymbol{n})\}$	$\frac{E\{G_{11}(\boldsymbol{n})\}}{E\{R(\boldsymbol{n})\}}$
LP (0.5)	2.7140	0.1500	0.7474	0.6890	0.1500	0.7474	0.6651
LP (0.2)	2.7140	0.1500	0.7474	0.6890	0.1500	0.7474	0.6522
LP (0.1)	2.7140	0.1500	0.7474	0.6890	0.1500	0.7474	0.6496
Opt. Sol.	2.7140	0.1500	0.7474	0.6890	0.1500	0.7474	0.6494
σ=0.4, K=4	τ_{PF}	$E\{T(0)\}$	$E\{R(0)\}$	$\frac{E\{G_{11}(0)\}}{E\{R(0)\}}$	$E\{T(\boldsymbol{n})\}$	$E\{R(\boldsymbol{n})\}$	$\frac{E\{G_{11}(\boldsymbol{n})\}}{E\{R(\boldsymbol{n})\}}$
LP (0.5)	2.6867	0.1500	0.7505	0.6889	0.1500	0.7505	0.6707
LP (0.2)	2.6867	0.1500	0.7505	0.6889	0.1500	0.7505	0.6584
LP (0.1)	2.6867	0.1500	0.7505	0.6889	0.1500	0.7505	0.6584
Opt. Sol.	2.6867	0.1500	0.7505	0.6889	0.1500	0.7505	0.6579

Table 4.1: Optimal solutions for the NP based problem (3.4) and the solutions of the linear fractional programming problem defined in (3.6).

problem (3.8) can have non-zero components other than the three components which are given in Figures 4.4 and 4.5. However, it is observed for this numerical example that these non-zero elements are negligible in general and the LP solutions reflect the three mass structure of the global solution. Theorem 1 expresses that the optimal global solution PMF has at most three distinct point masses. For this numerical example with $K = 4$ and $\sigma = 0.4$, the global optimal solution for additive noise probability distribution is a PMF with two point masses and it is depicted in Figure 4.6. The LP solutions yield similar distributions with two or three point masses.

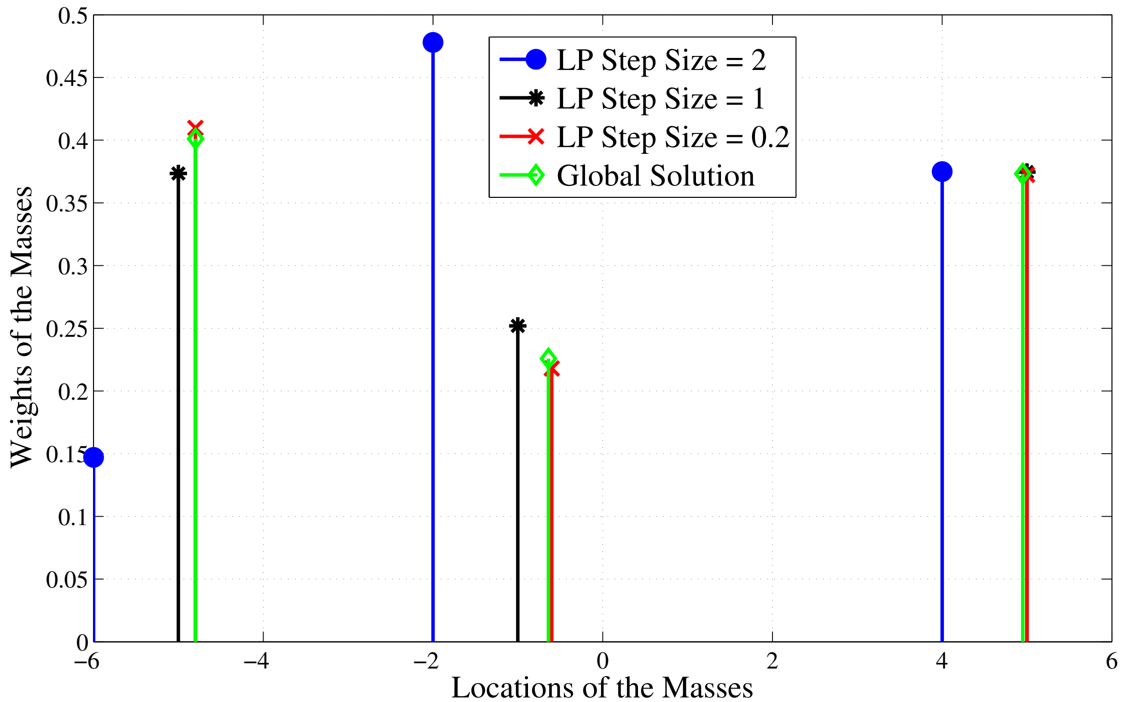


Figure 4.4: Optimal solutions of the NP problem (3.4) and the solutions of the linear fractional programming problem defined in (3.6) for $K = 1$ and $\sigma = 0.3$.

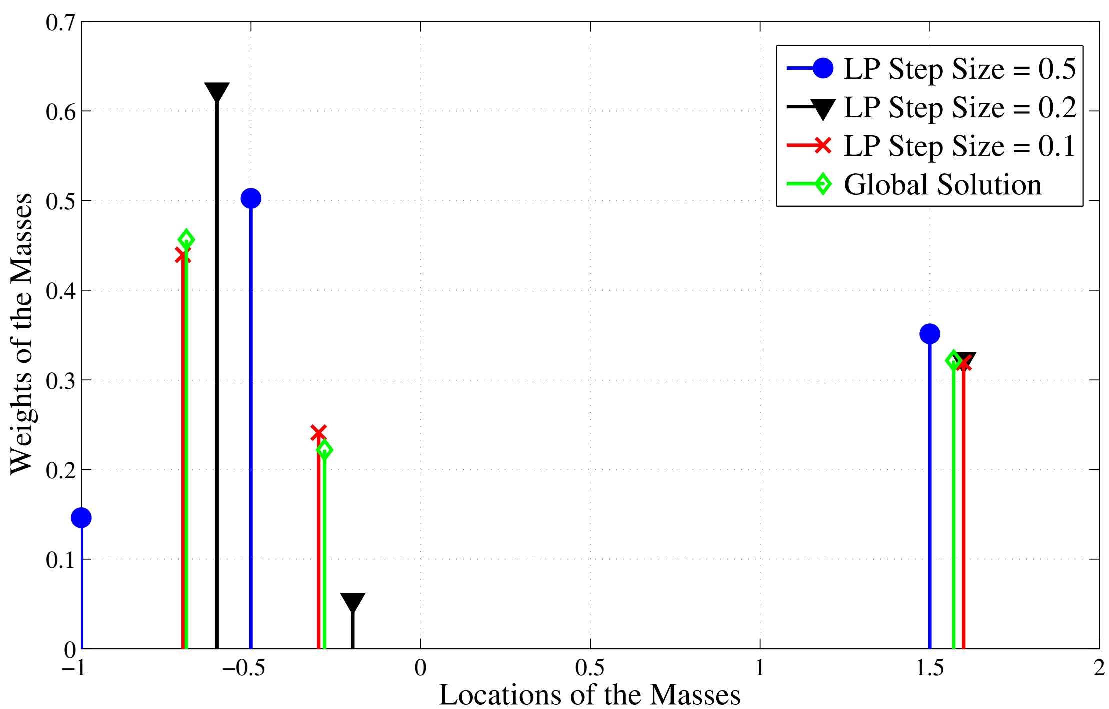


Figure 4.5: Optimal solutions of the NP problem (3.4) and the solutions of the linear fractional programming problem defined in (3.6) for $K = 4$ and $\sigma = 0.3$.

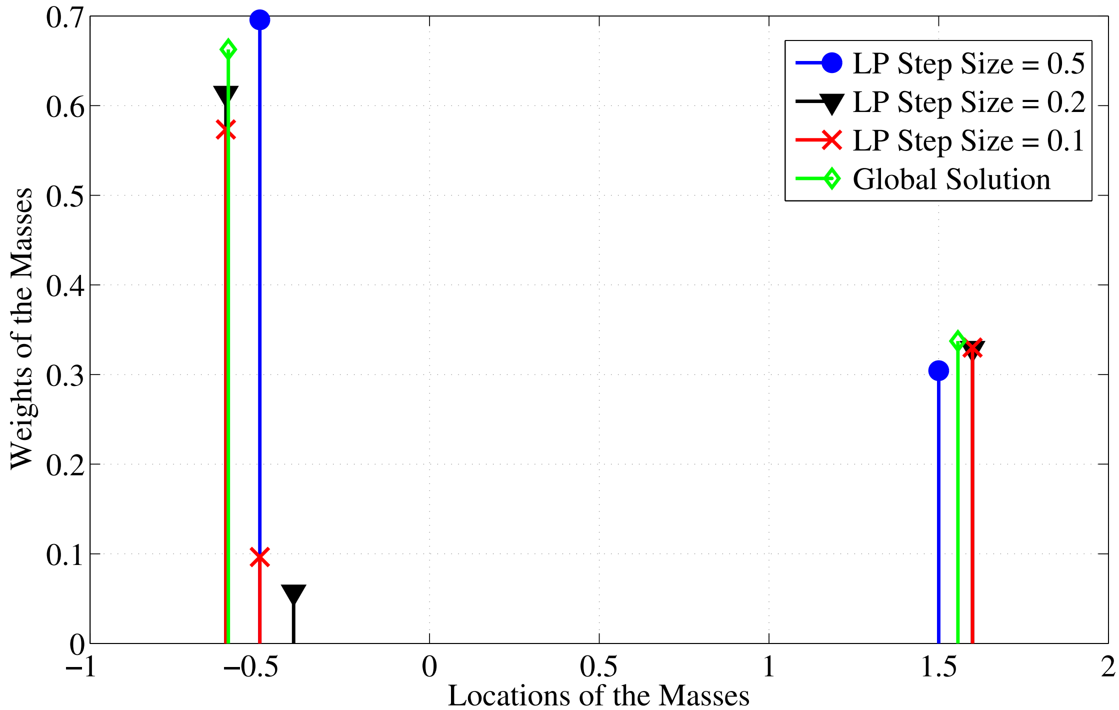


Figure 4.6: Optimal solutions of the NP problem (3.4) and the solutions of the linear fractional programming problem defined in (3.6) for $K = 4$ and $\sigma = 0.4$.

For the same system noise distribution $f_{\epsilon_k}(\varepsilon)$, the problem in the Bayes detection framework is also evaluated for $P(\mathcal{H}_0) = 0.5$ and $\tau = a/2$ in the absence (original) and presence of additive noise in Figures 4.7 and 4.8 . The Bayes estimation risk curves of the LP approximations are also illustrated in comparison with the global solution. The behavior of the curves are very similar to the results for the NP detection framework. The noise enhancement improvement effect is again more effective in the high SNR region. Some numerical values of the Bayes estimation and Bayes detection risks of both original given and noise enhanced joint systems are given in Table 4.2. In Bayes problem, optimal noise probability mass functions have one or two mass points. In Figures 4.9 and 4.10, global solution is a single mass point . In Figure 4.11, global solution has two mass points. Notice that the LP solution with step size 1.0 is a single mass function at location 0 in this figure. As it can be observed from Table 4.2, LP problem with step size 1.0 is non-improvable and the LP solution is adding zero noise.

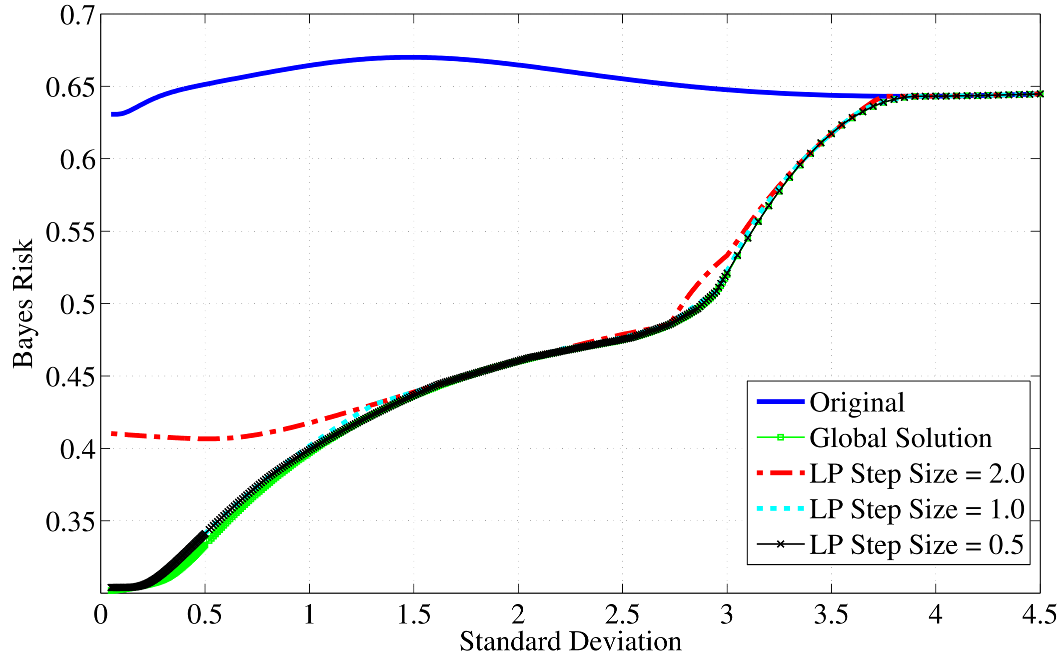


Figure 4.7: Noise enhancement effects for the minimization of the Bayes estimation risk, $K = 1$ (Bayes detection framework).

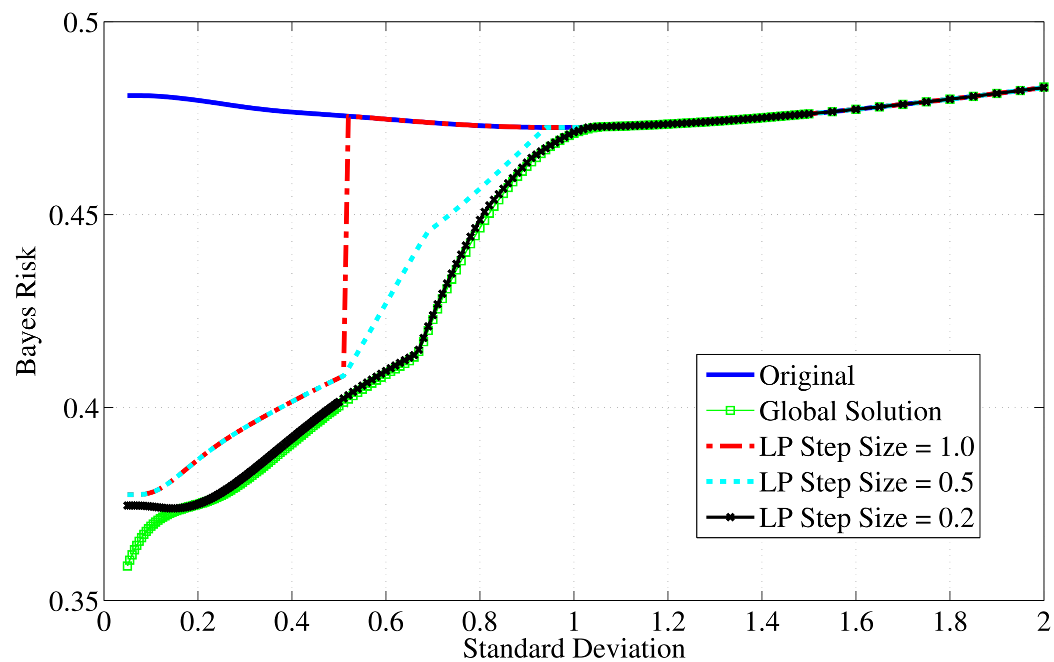


Figure 4.8: Noise enhancement effects for the minimization of the Bayes estimation risk, $K = 4$ (Bayes detection framework).

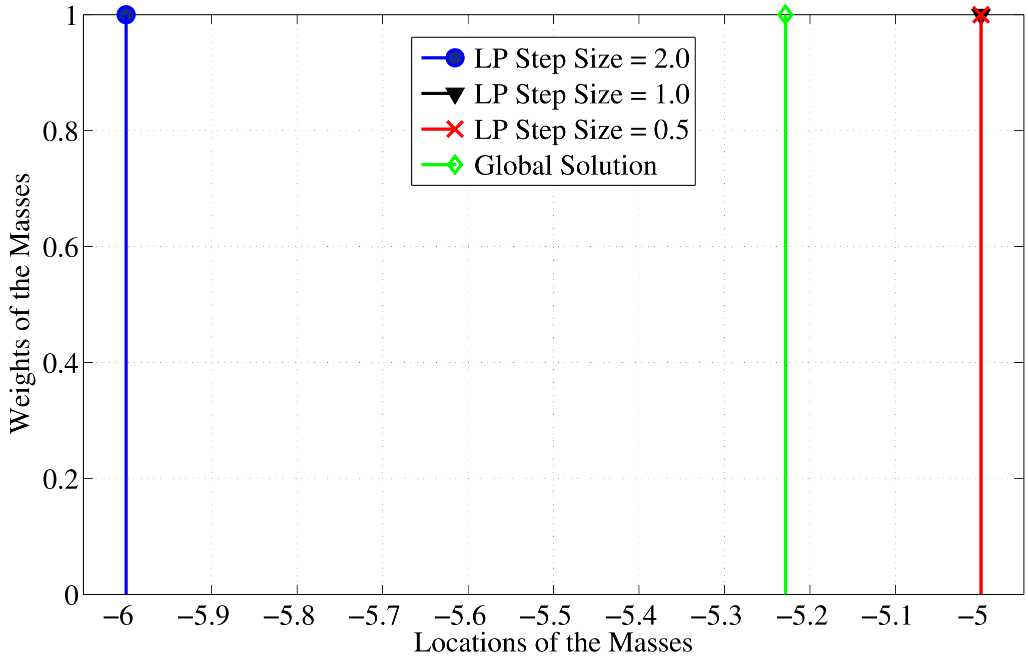


Figure 4.9: Optimal solutions of Bayes Detection Problem (3.5) and the solutions of the linear programming problem defined in (3.9). K $= 1$. $\sigma = 0.5$

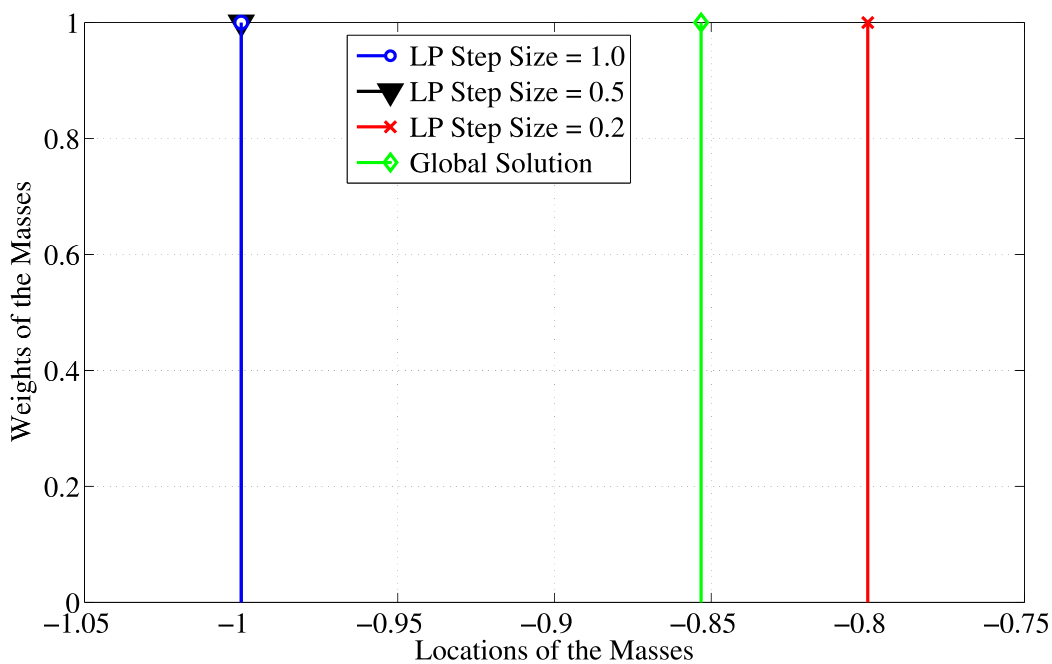


Figure 4.10: Optimal solutions of Bayes Detection Problem (3.5) and the solutions of the linear programming problem defined in (3.9). K $= 4$. $\sigma = 0.5$

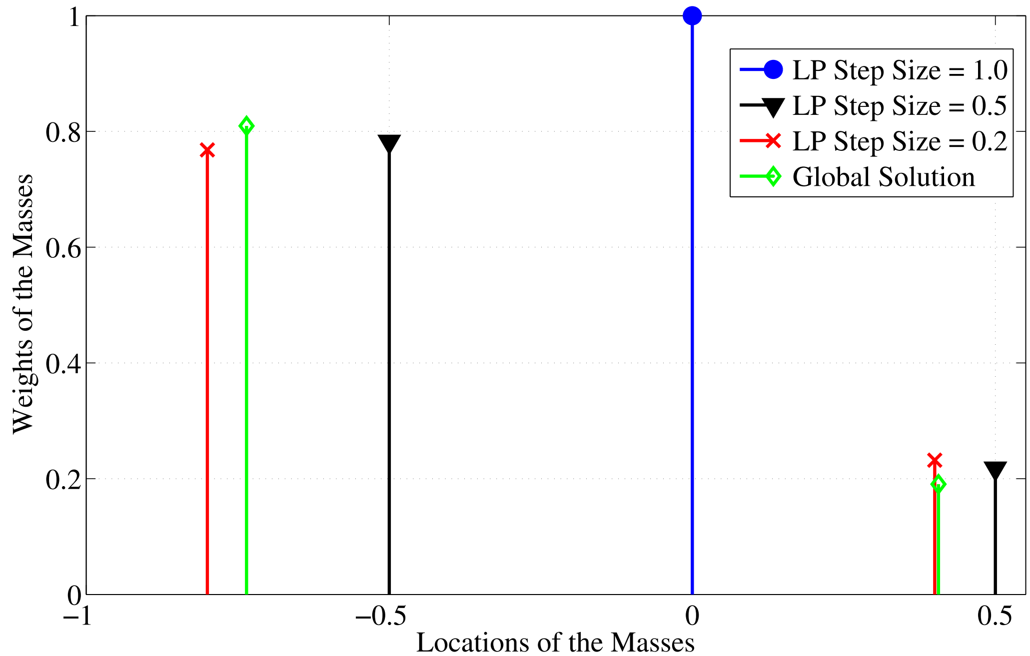


Figure 4.11: Optimal solutions of Bayes Detection Problem (3.5) and the solutions of the linear programming problem defined in (3.9). K = 4. $\sigma = 0.75$

σ=0.5, K=1	$r^x(\phi)$	$r^x(\theta)$	$r^y(\phi)$	$r^y(\theta)$
LP (1.0)	0,1959	0,4757	0.3266	0,4061
LP (0.5)	0.4216	0,6514	0.3057	0,3411
LP (0.2)	0.4216	0,6514	0.3057	0,3411
Opt. Sol.	0.4216	0,6514	0.3088	0,3333
σ=0.5, K=4	$r^x(\phi)$	$r^x(\theta)$	$r^y(\phi)$	$r^y(\theta)$
LP (1.0)	0,1959	0,4757	0.1956	0,4076
LP (0.5)	0,1959	0,4757	0.1956	0,4076
LP (0.2)	0,1959	0,4757	0.1899	0,4015
Opt. Sol.	0,1959	0,4757	0.1906	0,4005
σ=0.75, K=4	$r^x(\phi)$	$r^x(\theta)$	$r^y(\phi)$	$r^y(\theta)$
LP (1.0)	0.1933	0,4734	0.1933	0,4734
LP (0.5)	0.1933	0,4734	0.1933	0,4514
LP (0.2)	0.1933	0,4734	0.1933	0,4372
Opt. Sol.	0.1933	0,4734	0.1933	0,4357

Table 4.2: Optimal solutions of Bayes detection framework problem (3.5) and the solutions of the linear programming problem defined in (3.9).

A different NP detection framework is also analyzed for $\boldsymbol{\mu} = [-3.8 \;\; -1.6 \;\; -0.54 \;\; 0.50 \;\; 2.42 \;\; 4.3]$, $\boldsymbol{\nu} = [0.33 \;\; 0.13 \;\; 0.08 \;\; 0.07 \;\; 0.11 \;\; 0.28]$, $a = 5$, $b = 1$, $\triangle = 1$, and a constant false alarm rate of 0.1. The results are depicted in Figure 4.12. Another different Bayes detection framework problem is also analyzed for $\boldsymbol{\nu} = [0.25 \;\; 0.50 \;\; 0.25]$, $\boldsymbol{\mu} = [3 \;\; -0.5 \;\; -2]$, $a = 3$, $b = 1$, $\tau = a/2$, $\triangle = 0.5$, and $P(\mathcal{H}_0) = 0.5$ in Figure 4.13 for $K = 1$.

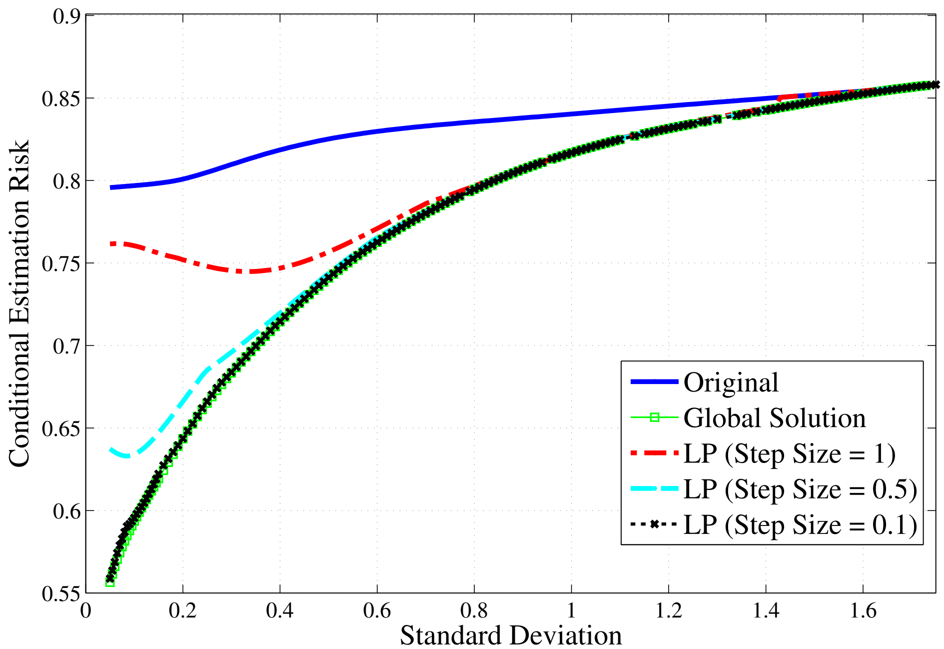


Figure 4.12: Noise enhancement effects for the minimization of the conditional estimation risk, $K = 1$ (NP detection framework).

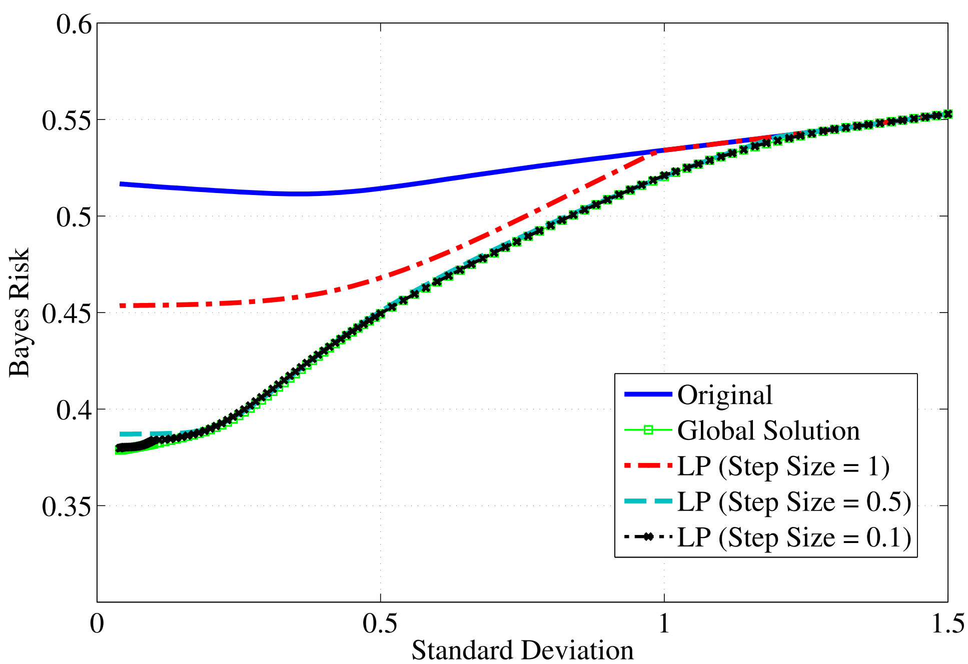


Figure 4.13: Noise enhancement effects for the minimization of the Bayes estimation risk, $K = 1$ (Bayes detection framework).

Chapter 5

Conclusion

In this thesis, to improve the performance of a given joint detection and estimation system, the injection of an independent random variable (additive noise) to the system has been considered. The statistical properties of the joint detection and estimation system have been analyzed in two different detection frameworks: Bayesian and NP. In non-Gaussian environments, the optimal detection, estimation, or joint detection-estimation systems are generally complicated (possibly cannot be expressed in closed form). This may motivate the implementation of suboptimal systems and performance loss. This study shows that noise benefits in joint detection and estimation systems can be realized to improve the performances of given suboptimal and relatively simple joint detection and estimation systems, which are for example optimal in Gaussian background noise and experience heavy performance degradation in a non-Gaussian environment.

For both NP and Bayesian detection frameworks, the optimal additive noise probability distributions have been found to be discrete (i.e., PMFs). With this result, the search for the optimal additive noise has been considerably simplified and can be solved by using global optimization techniques. However, the optimization problem is still not convex in general. In order to overcome possible complexities regarding the solution of the proposed problem, an LP problem approximation has been proposed. This sampling operation in additive noise random variable support set results in a performance degradation in comparison to the global optimal solution. The evaluation of numerical examples has indicated that this performance loss is negligible if the step size between the samples is chosen adequately. With this observation, under both NP and Bayes detection frameworks, it can be expressed that the LP approximation is well-suited for realistic scenarios. This should be noted as an important observation to justify the applicability of the noise enhanced systems in real life scenarios since it is less complex than finding a global solution to the presented optimization problems.

The noise enhancement idea may draw some criticism as the solution of the proposed optimization problems in (3.4) and (3.5) can require heavy computation. However, the LP approximation idea significantly reduces the computational load. One can assume that when the probability distributions of the observations are fixed, the LP solutions can be computed beforehand and stored. However, pre-computation is not always efficient and an adaptive methodology is needed especially if the background noise is time varying. Future work can focus on developing an adaptive algorithm to find a near optimal additive noise distribution under LP approximation while the joint detection and estimation is running in real time.

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Appendix A

Derivation of Conditional Estimation Risks

The equation (2.8) is already stated in [62] as the relation (7) without any derivation. In this section, it is demonstrated that the estimation performance criterion is the average Bayes risk conditioned on true hypothesis $\mathcal{H}_1$ and decision $\hat{\mathcal{H}}_1$. The estimation performance criterion of the joint detection and estimation system is proposed as

$$E\{c(\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{X}))|\mathcal{H}_1, \hat{\mathcal{H}}_1\} = \int\limits_{\Lambda} \int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x})) f^X(\boldsymbol{\theta}, \boldsymbol{x}|\mathcal{H}_1, \hat{\mathcal{H}}_1)\ \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta} \tag{A.1}$$

where $f(\boldsymbol{\theta}, \boldsymbol{x}|\mathcal{H}_1, \hat{\mathcal{H}}_1)$ is the joint probability density function of the observation $\boldsymbol{X}$ and estimated parameter $\boldsymbol{\Theta}$. It can be expressed as

$$f(\boldsymbol{\theta}, \boldsymbol{x}|\mathcal{H}_1, \hat{\mathcal{H}}_1) = f(\boldsymbol{x}|\boldsymbol{\theta}, \mathcal{H}_1, \hat{\mathcal{H}}_1)\pi(\boldsymbol{\theta}|\mathcal{H}_1, \hat{\mathcal{H}}_1) \tag{A.2}$$

Also,

$$f^X(\boldsymbol{x}|\boldsymbol{\theta}, \mathcal{H}_1, \hat{\mathcal{H}}_1) = \frac{f^X(\boldsymbol{x}, \hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)}{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)} = \frac{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1, \boldsymbol{x}) f(\boldsymbol{x}|\boldsymbol{\theta}, \mathcal{H}_1)}{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)} = \frac{\phi(\boldsymbol{x}) f_1^X(\boldsymbol{x}|\boldsymbol{\theta})}{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)} \tag{A.3}$$

Note that $f^X(\boldsymbol{x}|\boldsymbol{\theta}, \mathcal{H}_1)$ is previously denoted as $f_1^X(\boldsymbol{x}|\boldsymbol{\theta})$ in (2.1). The probability of deciding in favor of $\hat{\mathcal{H}}_1$ is the detector function when the observation is given. Providing additional information does not change this probability; hence $P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1, \boldsymbol{x}) = \phi(\boldsymbol{x})$. In addition,

$$\pi(\boldsymbol{\theta}|\mathcal{H}_1, \hat{\mathcal{H}}_1) = \frac{\pi(\boldsymbol{\theta}, \hat{\mathcal{H}}_1|\mathcal{H}_1)}{P(\hat{\mathcal{H}}_1|\mathcal{H}_1)} = \frac{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)\pi(\boldsymbol{\theta}|\mathcal{H}_1)}{P(\hat{\mathcal{H}}_1|\mathcal{H}_1)} \ . \tag{A.4}$$

Inserting (A.2), (A.3), and (A.4) into (A.1) gives the conditional Bayes risk (Equation (7) in [62]):

$$E\{C(\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{X}))|\mathcal{H}_1, \hat{\mathcal{H}}_1\} = \frac{\int\limits_{\Lambda}\int\limits_{\mathbb{R}^K} C(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{x}))\phi(\boldsymbol{x})f_1^X(\boldsymbol{x}|\boldsymbol{\theta})\pi(\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{\theta}}{P(\hat{\mathcal{H}}_1|\mathcal{H}_1)} \tag{A.5}$$

Similarly, the derivation of the conditional estimation risk after noise addition (2.15) is also presented below:

$$E\{c(\boldsymbol{\Theta}, \hat{\theta}(\boldsymbol{Y}))|\mathcal{H}_1, \hat{\mathcal{H}}_1\} = \int\limits_{\Lambda}\int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{y}))f^Y(\boldsymbol{\theta}, \boldsymbol{y}|\mathcal{H}_1, \hat{\mathcal{H}}_1)\,\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}. \tag{A.6}$$

Applying the same approach, the probability density of $\boldsymbol{Y}$ conditioned on true hypothesis $\mathcal{H}_1$, decision $\hat{\mathcal{H}}_1$ and parameter Θ can be obtained as follows:

$$f^Y(\boldsymbol{y}|\boldsymbol{\theta}, \mathcal{H}_1, \hat{\mathcal{H}}_1) = \frac{\phi(\boldsymbol{y})f_1^Y(\boldsymbol{y}|\boldsymbol{\theta})}{P(\hat{\mathcal{H}}_1|\boldsymbol{\theta}, \mathcal{H}_1)} \ . \tag{A.7}$$

Since $\boldsymbol{Y}$ is the summation of two independent random variables, its probability distribution conditioned on true hypothesis $\mathcal{H}_1$ and parameter Θ is given by

$$f_1^Y(\boldsymbol{y}|\boldsymbol{\theta}) = \int\limits_{\mathbb{R}^K} f^N(\boldsymbol{n})f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta})\,\mathrm{d}\boldsymbol{n} \ . \tag{A.8}$$

Inserting equations (A.4), (A.7), and (A.8) into (A.6) gives the conditional estimation risk (2.15) in the presence of additive noise:

$$J(\phi, \hat{\theta}) = \frac{\int\limits_{\mathbb{R}^K} f^N(\boldsymbol{n})\int\limits_{\Lambda}\int\limits_{\mathbb{R}^K} c(\boldsymbol{\theta}, \hat{\theta}(\boldsymbol{y}))\phi(\boldsymbol{y})\pi(\boldsymbol{\theta})f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta})\ \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{n}}{\int\limits_{\mathbb{R}^K} f^N(\boldsymbol{n})\int\limits_{\Lambda}\int\limits_{\mathbb{R}^K} \phi(\boldsymbol{y})\pi(\boldsymbol{\theta})f_1^X(\boldsymbol{y}-\boldsymbol{n}|\boldsymbol{\theta})\ \mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{\theta}\mathrm{d}\boldsymbol{n}} \ . \tag{A.9}$$

Appendix B

Derivation of Auxiliary Functions

In this chapter, the derivation of the auxiliary function $G_{11}(n)$ is presented for the scalar case $K = 1$. G_{01} (4.11) can also be derived in a similar way, and therefore its derivation is not included. As defined in Section 4.1, system noise ϵ is Gaussian mixture distributed and parameter θ is Gaussian distributed. For notational simplicity introduce $f_{1,i}^X(y-n|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\Big(-\frac{(y-n-\mu_i-\theta)^2}{2\sigma^2}\Big)$. Then, $G_{11}(n)$ is calculated as follows:

$$
\begin{aligned}
G_{11}(n) &= \int\limits_{\Lambda}\int\limits_{\mathbb{R}} c(\theta,\hat{\theta}(y))\phi(y)\pi(\theta)f_1^X(y-n|\theta)\ \mathrm{d}y\mathrm{d}\theta \\
&= \sum_{i=1}^{N_m}\nu_i\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} c(\theta,\hat{\theta}(y))\phi(y)\pi(\theta)f_{1,i}^X(y-n|\theta)\mathrm{d}y\mathrm{d}\theta \\
&= \sum_{i=1}^{N_m}\nu_i\Bigg\{\int\limits_{-\infty}^{\tau-\triangle}\pi(\theta)\int\limits_{\tau}^{+\infty} f_{1,i}^X(y-n|\theta)\mathrm{d}y\mathrm{d}\theta + \int\limits_{\tau-\triangle}^{\tau+\triangle}\pi(\theta)\int\limits_{\theta+\triangle}^{+\infty} f_{1,i}^X(y-n|\theta)\mathrm{d}y\mathrm{d}\theta \\
&\quad + \int\limits_{\tau+\triangle}^{+\infty}\pi(\theta)\int\limits_{\tau}^{\theta-\triangle} f_{1,i}^X(y-n|\theta)\mathrm{d}y\mathrm{d}\theta + \int\limits_{\tau+\triangle}^{+\infty}\pi(\theta)\int\limits_{\theta+\triangle}^{+\infty} f_{1,i}^X(y-n|\theta)\mathrm{d}y\mathrm{d}\theta\Bigg\}
\end{aligned}
$$

which can be expressed as

$$G_{11}(n) = \sum_{i=1}^{N_m} \nu_i \Bigg\{ \int\limits_{-\infty}^{\tau-\triangle} \pi(\theta) \mathrm{Q}\Big(\frac{\tau - n - \mu_i - \theta}{\sigma}\Big) \mathrm{d}\theta + \int\limits_{\tau-\triangle}^{\tau+\triangle} \pi(\theta) \mathrm{Q}\Big(\frac{\triangle - n - \mu_i}{\sigma}\Big) \mathrm{d}\theta$$

$$+ \int\limits_{\tau+\triangle}^{+\infty} \pi(\theta) \Big[\mathrm{Q}\Big(\frac{\tau - n - \mu_i - \theta}{\sigma}\Big) - \mathrm{Q}\Big(\frac{\triangle - n - \mu_i}{\sigma}\Big) \Big] \mathrm{d}\theta + \int\limits_{\tau+\triangle}^{+\infty} \pi(\theta) \mathrm{Q}\Big(\frac{\triangle - n - \mu_i}{\sigma}\Big) \mathrm{d}\theta \Bigg\}$$

where $\mathrm{Q}(\cdot)$ is the tail probability function of the standard Gaussian random variable.

The following identity in (B.1) can be used for further simplification [78].

$$\int\limits_{-\infty}^{\infty} f(x)\ \mathrm{Q}(\gamma x + \delta)\ \mathrm{d}x \;=\; \mathrm{Q}\left(\frac{\delta}{\sqrt{1+\gamma^2}}\right) \tag{B.1}$$

In this identity, $f(\cdot)$ is the probability density function of a standard Gaussian random variable. Then, $G_{11}(n)$ becomes

$$G_{11}(n) = \sum_{i=1}^{N_m} \nu_i \Bigg\{ \mathrm{Q}\Big(\frac{\triangle - n - \mu_i}{\sigma}\Big) \mathrm{Q}\Big(\frac{\tau - \triangle - a}{b}\Big) - \mathrm{Q}\Big(\frac{-\triangle - n - \mu_i}{\sigma}\Big) \mathrm{Q}\Big(\frac{\tau + \triangle - a}{b}\Big)$$

$$+ \mathrm{Q}\Big(\frac{\tau - n - \mu_i - a}{\sqrt{b^2 + \sigma^2}}\Big) - \int\limits_{\tau-\triangle}^{\tau+\triangle} \pi(\theta) \mathrm{Q}\Big(\frac{\tau - n - \mu_i - \theta}{\sigma}\Big) \mathrm{d}\theta \Bigg\} \tag{B.2}$$

Appendix C

Derivation for Distribution of $\tilde{\epsilon}_K$

In this section, the probability density function of random variable $\tilde{\epsilon}_K$, which is presented in Section 4.1.2 and Equation (4.12), is derived.

$$\tilde{\epsilon}_K = \frac{1}{K}\sum_{i=1}^{K}\epsilon_i \tag{C.1}$$

Remember that ϵ_i is a Gaussian mixture distributed random variable and the standard deviation values are equal for all the mixture components.

First, define $\epsilon_d = \frac{1}{K}\epsilon$, which is also Gaussian mixture distributed with scaled mixture component mean and variance values:

$$f_{\epsilon_d}(\varepsilon) = K f_{\epsilon}(\varepsilon K) = \sum_{i=1}^{N_m}\nu_i \frac{1}{\sqrt{2\pi\frac{\sigma^2}{K^2}}}\exp\Big\{-\frac{(\varepsilon-\mu_i/K)^2}{2\sigma^2/K^2}\Big\} \tag{C.2}$$

The characteristic function of ϵ_d (note that $\Phi_X(\omega) = \exp(j\mu\omega - \sigma^2\omega^2/2)$ where $X \sim \mathcal{N}(\mu, \sigma^2)$) is given by [79]

$$\begin{aligned}
\Phi_{\epsilon_d}(\omega) = E\big\{\exp(j\omega\epsilon_d)\big\} &= \int_{-\infty}^{+\infty} f_{\epsilon_d}(\varepsilon)\exp(j\omega\varepsilon)\mathrm{d}\varepsilon \\
&= \sum_{i=1}^{N_m}\nu_i\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi\frac{\sigma^2}{K^2}}}\exp\Big(j\omega\varepsilon - \frac{(\varepsilon-\mu_i/K)^2}{2\sigma^2/K^2}\Big)\mathrm{d}\varepsilon \\
&= \sum_{i=1}^{N_m}\nu_i\exp\Big(-\frac{1}{2}\frac{\sigma^2}{K^2}\omega^2 + j\omega\frac{\mu_i}{K}\Big)
\end{aligned} \tag{C.3}$$

Since ϵ_i's are independent and identical distributed random variables, the characteristic function of $\tilde{\epsilon}_K$ can be written as

$$\Phi_{\tilde{\epsilon}_K}(\omega) = \bigg(\Phi_{\epsilon_d}(\omega)\bigg)^K = \exp\Big(-\frac{1}{2K}\sigma^2\omega^2\Big)\bigg(\sum_{i=1}^{N_m}\nu_i \exp\Big(j\omega\frac{\mu_i}{K}\Big)\bigg)^K \tag{C.4}$$

The multinomial theorem states the following identity [80]:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1+k_2+\cdots+k_m=n}\left(\frac{n!}{k_1!\ k_2!\ \cdots\ k_m!}\right)\prod_{i=1}^{m} x_i^{k_i} \tag{C.5}$$

which can be applied to Equation (C.4) to simplify the expression for the haracteristic function of $\tilde{\epsilon}_K$:

$$\Phi_{\tilde{\epsilon}_K}(\omega) = \sum_{\substack{l_1+l_2+\cdots\\+l_{N_m}=K}}\left(\frac{K!}{l_1!\ l_2!\ \cdots\ l_{N_m}!}\right)\left(\prod_{i=1}^{N_m}\nu_i^{l_i}\right)\exp\left(\frac{\sigma^2\omega^2}{2K}\right)\exp\left(j\frac{\omega}{K}\sum_{i=1}^{N_m}\mu_i l_i\right) \tag{C.6}$$

From equation (C.6), it can be seen that $\tilde{\epsilon}$ does also have a Gaussian mixture distribution with new mixture component means, weights and a new standard deviation value. Variance values are again equal for all the mixture components and $\tilde{\sigma}^2 = \frac{\sigma^2}{K}$. Denote the new mixture component mean and weight variables as $\tilde{\boldsymbol{\nu}} = [\tilde{\nu}_1\ \tilde{\nu}_2\ \cdots\ \tilde{\nu}_{\tilde{N}_m}]^\intercal$ and $\tilde{\boldsymbol{\mu}} = [\tilde{\mu}_1\ \tilde{\mu}_2\ \cdots\ \tilde{\mu}_{\tilde{N}_m}]^\intercal$. Each distinct $\{l_1, l_2, \cdots, l_{N_m}\}$ set satisfying $l_1 + l_2 + \cdots + l_{N_m} = K$ corresponds to a distinct mixture component with μ_j and ν_i. The number of distinct $\{l_1, l_2, \cdots, l_{N_m}\}$ sets, equivalently the number of new mixture components $\tilde{N}_m$ in the summation (C.6) is $\binom{K+N_m-1}{N_m-1}$.

$$\tilde{\nu}_j = \left(\frac{K!}{l_1!\ l_2!\ \cdots\ l_{N_m}!}\right)\left(\prod_{i=1}^{N_m}\nu_i^{l_i}\right)$$
$$\tilde{\mu}_j = \frac{1}{K}\sum_{i=1}^{N_m}\mu_i l_i$$

After inverse transformation, the probability density function of $\tilde{\epsilon}$ can be obtained as

$$f_{\tilde{\epsilon}_K}(\varepsilon) = \sum_{\substack{l_1+l_2+\cdots\\+l_{N_m}=K}}\left(\frac{K!}{l_1!\ l_2!\ \cdots\ l_{N_m}!}\right)\left(\prod_{i=1}^{N_m}\nu_i^{l_i}\right)\frac{\exp\left(-\frac{\left(\varepsilon-\frac{1}{K}\sum_{i=1}^{N_m}\mu_i l_i\right)^2}{2\sigma^2/K}\right)}{\sqrt{2\pi\sigma^2/K}} \tag{C.7}$$