NASH EQUILIBRIA IN CLAIM BASED ESTATE DIVISION PROBLEMS

A Master's Thesis

by
ABDULKADİR İNEL

Department of
Economics
İhsan Doğramacı Bilkent University
Ankara
July 2014

NASH EQUILIBRIA IN CLAIM BASED ESTATE DIVISION PROBLEMS

Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

ABDULKADİR İNEL

In Partial Fulfillment of the Requirements For the Degree
of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
İHSAN DOĞRAMACI BİLKENT UNIVERSITY
ANKARA

JULY 2014

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Emin Karagözoğlu

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Rahmi İlkılıç

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Zafer Akın

Examining Committee Member

Approval of the Graduate School of Economics and Social Sciences

---

Prof. Dr. Erdal Erel

Director

ABSTRACT

NASH EQUILIBRIA IN CLAIM BASED ESTATE DIVISION PROBLEMS

İNEL, Abdulkadir

M.A., Department of Economics

Supervisor: Assist. Prof. Emin Karagözoğlu

July 2014

Estate division game is an allocation of an estate between players based on a rule. In this thesis, we consider estate division games and study the necessary and sufficient conditions for division rules under which Nash equilibria induce *equal division.* Ashlagi, Karagözoğlu, Klaus (2012) introduce classes of properties for division rules and show that they are sufficient for all Nash equilibria to induce *equal division.* In this study, we propose a different property, namely *conditional full compensation*, and prove that it is also a sufficient condition for division rules in order for all Nash equilibria outcomes under these rules to be *equal division.* We, then, show that under any rule satisfying *claims boundedness* and *conditional equal division lower bound*, *equal division* is a Nash equilibrium outcome. Finally, we prove that letting at least one player get more than the difference between the whole estate and the sum of other players' claims is a necessary condition for all Nash equilibria to induce *equal division.*

Keywords: Estate division game, division rule, Nash equilibrium, equal division.

ÖZET

TALEBE DAYALI VARLIK PAYLAŞIM PROBLEMLERİNDE NASH DENGELERİ

İNEL, Abdulkadir

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Yard. Doç. Emin Karagözoğlu

Temmuz 2014

Paylaşım oyunu bir varlığın oyuncular arasında bir kurala göre paylaştırılmasıdır. Bu tez çalışmamızda, varlık paylaşım oyunları ele alınmakta ve bu oyunların Nash dengelerinde *eşit paylaşımı* veren bölünme kurallarının gereklilik ve yeterlilik şartları incelenmektedir. Ashlagi, Karagözoğlu, Klaus (2012) bölünme kuralları için özellik sınıfları sunuyorlar ve bu sınıfların tüm Nash dengelerinin *eşit paylaşımı* vermesi için yeterli olduğunu gösteriyorlar. Bu çalışmamızda farklı bir özellik, yani *şartlı tam tazminat*, önerilmekte ve bunun tüm Nash dengelerinde *eşit paylaşımı* veren paylaşım kuralları için yeterli koşul olduğu gösterilmektedir. Daha sonra *taleplerin monotonluğu* ve *şartlı eşit paylaşım alt sınırını* sağlayan herhangi bir paylaşım kuralı altında oynanan oyunda, *eşit paylaşımın* bir Nash dengesi olduğu gösterilmektedir. Son olarak, en az bir oyuncunun varlıktan diğer talepler çıkarıldığında kalan miktardan fazla almasının, tüm Nash dengelerinin *eşit paylaşımı* vermesi için gerekli bir koşul olduğu ispatlanmaktadır.

Anahtar Kelimeler: Varlık paylaşım oyunu, paylaşım kuralı, Nash dengesi, eşit paylaşım.

ACKNOWLEDGEMENTS

I would like to thank my advisors Emin Karagözoğlu and Rahmi İlkılıç for their personal and academic advice and help. I would also like to thank İsmail Özkaraca, Hüsniye Burçin İkizler and Kerim Keskin for their assistance and guidance...

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Estate division game is a situation in which a perfectly divisible good is to be distributed among a group of players. Each player lays a claim on the good and an authority in charge is to divide the good based on players' claims. If the total demand does not exceed the amount of the good, giving each player what they claim might be a reasonable solution. The problem occurs when the total demand exceeds the good, and therefore, cannot be met. The model described here is also known as a *claim game*, and the rule that is used to divide the good is called a *division rule*. The case where the good is insufficient to meet the demand is called a *bankruptcy problem*. In this paper, in general, we deal with claim games and their Nash equilibrium outcomes induced by certain classes of division rules. We study non-cooperative games in which there is an estate with positive value to be allocated among players, who are rewarded with an amount of the estate based on their claims and the division rule.[1] Here, we are interested in the relationship between pure strategy Nash equilibria of claim games and equal division of the estate. We do not work with a specific division rule. Instead, we try to determine the properties that division rules must satisfy in order for equal division to be induced by a Nash equilibrium.

[1]Every claim is basically a nonnegative amount of the estate and does not exceed the estate.

In the literature, the article closely related to our paper is Ashlagi, Karagözoğlu, Klaus (2012). They study estate division problems and analyze their Nash equilibria. They offer properties for division rules so that all Nash equilibria under these rules induce equal division. They first show that if a division rule satisfies *efficiency*, *equal treatment of equals* and *order preservation of awards* properties, then (i) all agents claiming the largest possible amount is a Nash equilibrium and (ii) all Nash equilibria lead to equal division of the estate. Here, *equal treatment of equals* property requires the rule to reward the agents with identical claims with equal amounts. Moreover, if a rule satisfies *order preservation of awards* property, then an agent who claims more than some other agent cannot get less than the amount the latter agent receives. They, then, prove that changing *order preservation of awards* property with *claims monotonicity* property does not affect the first part of the result found before, but for the second part to hold, the number of players should not be larger than three.That is, (i) under any division rule satisfying these properties, all agents claiming the largest amount possible is a Nash equilibrium, and (ii) if the number of players is less than or equal to three, all Nash equilibria of the estate division game lead to equal division. They give an example to show that the second part does not have to hold with these properties if there are more than three agents (see Ashlagi et al., 2012, Example 1). Finally, they verify that if a division rule satisfies *efficiency*, *equal treatment of equals*, *claims monotonicity* and *nonbossiness* properties, then equal division prevails in all Nash equilibria. Here, *nonbossiness* property implies that if an agent changes his claim and still gets the same reward as before, then all other players' rewards also remain the same.

Cetemen and Karagözoğlu (2014) is also related to our paper in the sense that they consider the division of a dollar among players and examine Nash equilibria of this game. In a standard Divide-the-Dollar game (DD), which is basically a claim game played by players who make claims on one dollar, players get what they claim if the total demand does not exceed one dollar, and get nothing otherwise. The authors introduce an altered version of DD, and call it DD′. Unlike DD, the altered version DD′ is played in two stages: In the first stage, players make claims simultaneously. If the total demand does not exceed one dollar, each player gets their claims. Otherwise, the players move to the second round. In this stage, player with the lower claim at the first stage suggests the amount to be deducted from each player's demand. If other player accepts it, players are awarded with the offer. If the offer is rejected, both players get zero. They show that in DD′, there is unique subgame perfect Nash equilibrium where players make egalitarian demands in the first stage.

Brams and Taylor (1994) try to make the payoffs of DD less severe by making some changes over the rules of the game. They offer a "reasonable" rule scheme which leads to equal division of the dollar if all agents make egalitarian demands, which is *a* Nash equilibrium of this game. The authors prove that there is no "reasonable" payoff scheme for DD that yields egalitarian behavior.

In this thesis, we use a property, *conditional full compensation (CFC)*, and show that all Nash equilibria of a claim game played under *CFC* induce equal division. Properties for division rules that lead to equal division in all Nash equilibria are also studied in Ashlagi et al. (2012). Nevertheless, none of these properties are necessary conditions for a rule to satisfy *CFC*. Therefore, we provide a different sufficient condition for all Nash equilibria outcomes to be equal division. We, then, show that if a rule satisfies *claims boundedness* and *conditional equal division lower bound (CED)*, equal division is a Nash equilibrium outcome. This result might not say much about claim games played by more than two players. But if the game is played with two players, a rule satisfying *claims boundedness* and *CED* also satisfies some well-known properties, which are studied in Ashlagi et al (2012), namely *equal treatment* *of equals*, *order preservation of awards* and *nonbossiness*. Lastly, we present a necessary property for division rules in order for all Nash equilibria outcomes to be equal division under these rules. We, first, require the division rule to satisfy *efficiency* and *claims boundedness* if the total claim does not exceed the estate. Then, we prove that if all Nash equilibria outcomes of a claim game are equal division, there is one player such that the sum of his reward and other players' claims exceed the estate. That is, there exists at least a player who gets more than whatever is left after others' claims are deducted from the estate.

CHAPTER 2

THE CLAIM GAME

2.1 The Model

Let E denote the estate which is to be divided among a group N of agents who have claims over E, with $E > 0$ and $N = \{1, 2, \cdots, n\}$. Here, E is an infinitely divisible resource and $n \geq 2$, $n \in \mathbb{N}$. Let each agent's preference be strictly monotone over the amount of the estate they receive. Moreover, let $C_i = [0, E]$ be the *strategy set* of player $i \in N$. Let a *strategy* for player i be a *claim* c_i with $c_i \in C_i$. Assume that claims are known to all agents and the authority who is responsible for dividing the estate. Let c be a *claims vector* and $C \equiv (C_i)_{i \in N}$ be the *claims space* such that $c \in C \subset \mathbb{R}^n$. Now, for any $c \in C$ and $S \subset N$ with $S \neq \emptyset$, let c_S denote the sum of the claims made by agents who belong to set S, i.e., $c_S = \sum_{i \in S} c_i$. For each i, let C_{-i} denote the cartesian product of claims space of all players except player i, i.e., $C_{-i} \equiv (C_j)_{j \in N/\{i\}}$. Also let $(\bar{c}_i, c_{-i}) \in C$ be the claims vector obtained from $c = (c_i, c_{-i}) \in C$, by changing player i's claim from c_i to $\bar{c}_i$ with $c_{-i} \in C_{-i}$ and $c_i, \bar{c}_i \in C_i$. Let R_i be a real-valued outcome function on the set of claims such that $R_i : C \to \mathbb{R}_+$ with $\sum_{i \in N} R_i(c) \leq E$. Let $R : C \to \mathbb{R}^n_+$ be such that $R(c) \equiv (R_i(c))_{i \in N}$ for any $c \in C$. Here, R is called a *division rule* that rewards player i with $R_i(c)$, when the claim vector of the players is c.

A *claim game*, denoted as $\mathit{\Gamma}(R)$, is the division of the estate E based on R between n players who make their claims simultaneously.

2.2 Some Properties for Division Rules

In this paper, we analyze certain division rule classes under which a Nash equilibrium has a relation to equal division, and study some basic properties that these rule classes satisfy.[2] One of these properties is *claims boundedness*, which is described as follows:

Claims boundedness: For any agent, the reward for any claim made by the agent should not be greater than the claim itself. That is, $\forall i \in N$, we have $R_i(c) \leq c_i$.

This property is usually considered essential for the division rules. There are other properties that are used in Ashlagi et al. (2012). Since this paper is intended to be built mainly on Ashlagi et al. (2012), we also study those properties which are as follows:

Efficiency: Whole estate should be allocated if there are "enough" demand for it. That is, if $c_N \geq E$, then we should have $R_N(c) = E$.

Claims monotonicity: If an agent increases his claim, he should not be rewarded less as long as other players do not make changes in theirs. That is, if $\bar{c}_i > c_i$ with $\bar{c}_i \in C_i$, then we have $R_i(\bar{c}_i, c_{-i}) \geq R_i(c_i, c_{-i})$.

Equal treatment of equals: Any two agents with identical claims get the same amounts of the estate. That is, for any $i, j \in N$, $c_i = c_j$ implies $R_i(c) = R_j(c)$.

Order preservation of awards: Between any two agents, the one with the higher claim should not be rewarded less than the other. That is, if $c_i > c_j$, we should have $R_i(c) \geq R_j(c)$.

Nonbossiness: No agent can change what other agents get without chang-

[2] These properties are obtained from Thomson (2010).

ing his own award. That is, for any $i, j \in N$, if $R_i(\bar{c}_i, c_{-i}) = R_i(c_i, c_{-i})$ with $\bar{c}_i \in C_i$, we should have $R_j(\bar{c}_i, c_{-i}) = R_j(c_i, c_{-i})$.

There are a couple of properties that make two main results in this paper possible, and therefore, are very crucial to us; namely, *conditional equal division lower bounds* and *conditional full compensation*, which are described as below:

Conditional equal division lower bounds, CED: For any $i \in N$, we have $R_i(c) \geq \min\{c_i, \frac{E}{n}\}$.[3]

Conditional full compensation, CFC: For any $i \in N$, if $\sum_{j \in N} \min\{c_i, c_j\} \leq E$, then we have $R_i(c) = c_i$.

Ashlagi et al. (2012) also give a definition of *equal division* of an estate E, which we use throughout this paper.

Equal division: Given an estate E and a set of players N, the vector $(\frac{E}{n}, \cdots, \frac{E}{n}) \in \mathbb{R}^n_{++}$ denotes *equal division* of E.

[3] This bound is proposed by Moulin (2002).

CHAPTER 3

EQUAL DIVISION AND NASH EQUILIBRIUM

3.1 Nash Equilibrium of $\Gamma(\mathbf{R})$ under CFC

We study claim based division games with $n \in \mathbb{N} - \{0, 1\}$ players who are to put forward claims on E. Now, we look for sufficient conditions for the division rules in order for all Nash equilibrium outcome to be equal division. It should be noted that we do not fix any rule for the claim games. We work only with the properties the division rule satisfies. In the following proposition, we show that if a division rule R satisfies *CFC*, then all Nash equilibria of $\mathit{\Gamma}(R)$ give us equal division.

Proposition 1. *Let* R *be a division rule satisfying* CFC*. Then,* $c = (\frac{E}{n}, \dots, \frac{E}{n})$ *is a Nash equilibrium of the claim game* $\Gamma(R)$*. Moreover, all Nash equilibria of* $\Gamma(R)$ *lead to equal division of* E*.*

Proof. Let $c = (\frac{E}{n}, \dots, \frac{E}{n})$. For any player $i \in N$, let c'_i be a claim with $c'_i \neq \frac{E}{n}$ and $c' = (c'_i, c_{-i})$. For all $j \in N - \{i\}$, we have

$$\sum_{k \in N} \min\{c'_j, c'_k\} = \sum_{k \in N} \min\{\frac{E}{n}, c'_k\} \leq \sum_{k \in N} \frac{E}{n} \leq E.$$

This implies that $R_j(c') = \frac{E}{n}$ because R satisfies CFC. Since it is true for all

$j \in N - \{i\}$, we have $R_i(c') \leq E - \sum_{j \in N/\{i\}} R_j(c') = \frac{E}{n}$. That is, c is a Nash equilibrium. In order to prove the second part, assume to the contrary that there exists a Nash equilibrium c^* of $\Gamma(R)$ with $R(c^*) \neq (\frac{E}{n}, \ldots, \frac{E}{n})$. Then, there exists $i \in N$ such that $R_i(c^*) < \frac{E}{n}$.

Claim 1. *If $c_k = \frac{E}{n}$ for some $k \in N$, then $R_k(c_k, c_{-k}) = \frac{E}{n}$ for any $c_{-k} \in C_{-k}$.*

Proof. For any j, we have $\min\{\frac{E}{n}, c_j\} \leq \frac{E}{n}$. Then,

$$\sum_{j \in N} \min\{c_k, c_j\} = \frac{E}{n} + \sum_{j \in N \setminus \{k\}} \min\{\frac{E}{n}, c_j\} \leq \frac{E}{n} + \sum_{j \in N \setminus \{k\}} \frac{E}{n} = E.$$

Then, $R_k(c_k, c_{-k}) = \frac{E}{n}$ by CFC property. □

Now, let $\bar{c}_i = \frac{E}{n}$ with $\bar{c}_i \in C_i$. Then, we have $R_i(\bar{c}_i, c^*_{-i}) = \frac{E}{n} > R_i(c^*)$, where equality comes from Claim 1, a contradiction. Therefore, c^* is not a Nash equilibrium. □

Remark 1. A division rule R satisfying *CFC* does not have to satisfy *efficiency*, *claims monotonicity*, *equal treatment of equals*, *order preservation of awards* or *nonbossiness.*

Remark 1 implies that a division rule satisfying CFC does not necessarily meet the conditions required in the results established in Ashlagi et al. (2012, Theorem 1, 2, and 3). Therefore, Proposition 1 provides an *alternative* sufficient condition for division rules under which all Nash equilibria outcome are equal division.

3.2 Equal Division under CED

In the following proposition, we prove that if a division rule R satisfies *claims boundedness* and *CED*, and if there is a claim vector $c \in C$ which leads to equal division of the estate among the players under R, then c is a Nash equilibrium of the claim game $\Gamma(R)$.

Proposition 2. *Let* R *be a division rule satisfying* claims boundedness *and* CED*. Then, equal division of* E *is a Nash equilibrium outcome of* $\Gamma(R)$*.*

Proof. Assume there is a $c^* \in C$ such that $R_i(c^*) = \frac{E}{n}$ for all $i \in N$. Suppose, to the contrary, there exists a player j with $j \in N$ who has a better response c'_j to c^*_{-j} than c^*_j, with $c'_j, c^*_j \in C_j$ and $c^*_{-j} \in C_{-j}$. That is, $R_j(c') > R_j(c^*) = \frac{E}{n}$, where $c' = (c'_j, c^*_{-j})$. Then, there exists at least a player k with $k \in N$ who gets less than $\frac{E}{n}$ when $c = c'$. That is, $R_k(c') < \frac{E}{n}$, since $\sum_i R_i(c) \leq E$ for any $c \in C$. Moreover, since R satisfies *claims boundedness* property, we also have $c^*_k \geq R_k(c*) = \frac{E}{n}$ with $c^*_k \in C_k$. So, we have $c^*_k \geq \frac{E}{n}$. However, since player k does not change his claim, i.e., $c^*_k = c'_k$, we must have $R_k(c') \geq \min\{c^*_k, \frac{E}{n}\} = \frac{E}{n}$. A contradiction. Therefore, c^*_j is a best response to c^*_{-j} for player j. That is, c^* is a Nash equilibrium of $\Gamma(R)$. □

We, next, determine the properties that the types of division rules studied in Proposition 2, i.e., the division rules satisfying *claims boundedness* and *CED*, satisfy. We first show with the following proposition that *claims boundedness* and *CED* properties are sufficient for a division rule to satisfy *equal treatment of equals* when there are 2 players in the game.

Proposition 3. *Let* R *be a division rule satisfying* claims boundedness *and* CED*. If* n=2*, then* R *also satisfies* equal treatment of equals*.*

Proof. Let $n = 2$, $c_1 \in C_1$ and $c_2 \in C_2$. Let, also, $c = (c_1, c_2)$ with $c_1 = c_2 = \bar{c}$. If $\bar{c} \leq \frac{E}{2}$, then $\bar{c} \geq R_1(c) \geq \min\{\bar{c}, \frac{E}{2}\} = \bar{c}$, where the first inequality is due to *claims boundedness* property. Therefore, $R_1(c) = \bar{c}$. Using the same argument, we get $R_2(c) = \bar{c}$. So, $c_1 = c_2$ implies $R_1(c) = R_2(c)$. If $\bar{c} > \frac{E}{2}$, then $R_i(c) \geq \min\{\bar{c}, \frac{E}{2}\} = \frac{E}{2}$ for $i = 1, 2$. Together with the fact that $\sum_i R_i(\bar{c}) \leq E$, we have $R_1(c) = R_2(c) = \frac{E}{2}$. Therefore, R satisfies *equal treatment of equals.* □

With the following remarks, we show that if either *claims boundedness* property or $n = 2$ is not satisfied, then Proposition 3 no longer has to hold.

Remark 2. A division rule R with *CED* does not have to satisfy *equal treatment of equals* when *n=2*. For instance; let *n=2*, *E=1*, $c_1 = c_2 = 0$ and $R_1(c_1, c_2) = 1$ and $R_2(c_1, c_2) = 0$. Here, $R_i(c_i, c_{-i}) \geq \min\{c_i, \frac{1}{2}\}$ for all i, i.e., it satisfies the required property but it does not satisfy *equal treatment of equals.*

Remark 3. A division rule R with *claims boundedness* and *CED* does not have to satisfy *equal treatment of equals.* For instance; let *n=3*, *E=1*, $c_1 = c_2 = \frac{1}{2}$, $c_3 = \frac{1}{4}$, $R_1(c) = \frac{5}{12}$, $R_2(c) = \frac{1}{3}$ and $R_3(c) = \frac{1}{4}$ with $c = (c_1, c_2, c_3)$. Here, $c_i = \frac{1}{2} \geq R_i(c) \geq \frac{1}{3} = \min\{c_i, \frac{1}{3}\}$ for i=1,2 and $c_3 = \frac{1}{4} \geq R_3(c) \geq \frac{1}{4} = \min\{c_3, \frac{1}{3}\}$, i.e., R satisfies *claims boundedness* and *CED*, but $R_1 \neq R_2$, even though we have $c_1 = c_2$.

We then show with the following proposition that *claims boundedness* and *CED* are sufficient for a division rule to satisfy *order preservation of awards* when there are 2 players in the game.

Proposition 4. *Let* R *be a division rule satisfying* claims boundedness *and* CED*. If* n=2*, then* R *also satisfies* order preservation of awards*.*

Proof. Let $N = \{1, 2\}$, $c_1 \in C_1$, $c_2 \in C_2$ and $c = (c_1, c_2)$. Assume, without loss of generality, that $c_1 \leq c_2$. Now, we have three cases to analyze:

(a) If $c_1 \leq c_2 \leq \frac{E}{2}$, we have $R_1(c) = c_1 \leq c_2 = R_2(c)$, where equalities come from CED and claims *boundedness* properties.

(b) If $c_1 \leq \frac{E}{2} \leq c_2$, we have $R_1(c) = c_1 \leq \frac{E}{2} = \min\{c_2, \frac{E}{2}\} \leq R_2(c)$.

(c) If $\frac{E}{2} \leq c_1 \leq c_2$, we have $R_1(c) = R_2(c) = \frac{E}{2}$, since $\sum_i R_i(c) \leq E$.

Therefore, $c_1 \leq c_2$ implies that $R_1(c) \leq R_2(c)$, i.e., R satisfies *order preservation of awards.* $\square$

With the following remarks, we show that if either *claims boundedness* or $n = 2$ is not satisfied, then Proposition 4 no longer has to hold.

Remark 4. A division rule R with *CED* does not have to satisfy *order preservation of awards* when n=*2*. For instance; let n=*2*, E=*1*, $c_1 < c_2 < \frac{1}{2}$ and $R_1(c_1, c_2) = \frac{1}{2}$ and $R_2(c_1, c_2) = c_2$. Here, R satisfies CED clearly. However, $R_1(c) \nleq R_2(c)$ even though we have $c_1 \leq c_2$.

Remark 5. A division rule R with *claims boundedness* and *CED* does not have to satisfy *order preservation of awards.* For instance; let n=*3*, E=*1*, $c_1 = \frac{1}{2}$, $c_2 = \frac{1}{2} - \varepsilon$ with $\varepsilon < \frac{1}{6}$, $c_3 = \frac{1}{6}$. Now, let $c = (c_1, c_2, c_3)$ and let $R_1(c) = \frac{1}{3}$, $R_2(c) = \frac{1}{2} - \varepsilon$ and $R_3(c) = \frac{1}{6}$. Here, we have $c_2 \leq c_1$ and $R_2(c) \nleq R_1(c)$ even though R satisfies *claims boundedness* and *CED.*

We finally show in the following proposition that *claims boundedness* and *CED* are sufficient for a division rule to satisfy *nonbossiness* when there are 2 players in the game.

Proposition 5. *Let* R *be a division rule satisfying* claims boundedness *and* CED. *If* n=2*, then* R *also satisfies* nonbossiness.

Proof. Let $n = 2$; $c_1, \bar{c_1} \in C_1$ with $c_1 \neq \bar{c_1}$; $c_2 \in C_2$, $c = (c_1, c_2)$ and $c' = (\bar{c_1}, c_2)$.

(a) Assume that $c_1 < \frac{E}{2}$. If $\bar{c_1} < \frac{E}{2}$, we have $R_1(c) = c_1 \neq \bar{c_1} = R_1(c')$. If $\bar{c_1} \geq \frac{E}{2}$, we have $R_1(c) = c_1 < \frac{E}{2} \leq \min\{\bar{c_1}, \frac{E}{2}\} \leq R_1(c')$. Therefore, it is not possible for player *1* to keep his reward the same but change his claim, when at least one of his claims is less than $\frac{E}{2}$.

(b) Now assume that $c_1, \bar{c_1} \geq \frac{E}{2}$. Assume further, that $R_1(c) = R_1(c')$. If $c_2 < \frac{E}{2}$, we have $R_2(c) = c_2 = R_2(c')$ because of *claims boundedness* and *CED*, i.e., the second player's reward does not change. If $c_2 \geq \frac{E}{2}$, then $R_2(c) \geq \min\{c_2, \frac{E}{2}\} \geq \frac{E}{2}$. Together with the fact that $R_1(c) \geq \min\{c_1, \frac{E}{2}\} \geq \frac{E}{2}$, we must have $R_1(c) = R_2(c) = \frac{E}{2}$. Similarly, we have $R_2(c') \geq \min\{c_2, \frac{E}{2}\} \geq \frac{E}{2}$. Since $R_1(c') = R_1(c) = \frac{E}{2}$, we have $R_2(c') = \frac{E}{2}$. That is, the second player's reward does not change if player 1 gets the same amount as before.

Therefore, $R_i(c) = R_i(c')$ implies $R_j(c) = R_j(c')$ for all $c, c' \in C$ and all $i, j \in N$. $\square$

With the following remarks, we show that if either *claims boundedness* or $n = 2$ is not satisfied, then Proposition 5 no longer has to hold.

Remark 6. A division rule R with *CED* does not have to satisfy *nonbossiness* when *n=2*. For instance; let *n=2*, *E=1*, $c_1 = 0$, $\bar{c_1} = \frac{1}{4}$, $c_2 = \frac{1}{2}$, $c = (c_1, c_2)$ and $c' = (\bar{c_1}, c_2)$. Now, let $R_1(c) = R_1(c') = \frac{1}{4}$, $R_2(c) = \frac{3}{4}$ and $R_2(c') = \frac{1}{2}$. Clearly, R satisfies *CED*. However, we $R_2(c) \neq R_2(c')$ even though we have $R_1(c) = R_1(c')$.

Remark 7. A division rule R with *claims boundedness* and *CED* does not have to satisfy *nonbossiness*. For instance; let *n=3*, *E=1*, $c_1 = c_2 = \frac{4}{9}$, $c_3 = \frac{1}{9}$ and $\bar{c_1} = \frac{1}{2}$. Let, also, $c = (c_1, c_2, c_3)$ and $c' = (\bar{c_1}, c_2, c_3)$. Now, let $R_1(c) = R_1(c') = R_2(c) = \frac{4}{9}$, $R_2(c') = \frac{1}{3}$, $R_3(c) = R_3(c') = \frac{1}{9}$. Here, we have $R_1(c) = R_1(c')$ and $R_2(c) \neq R_2(c')$. That is, R satisfies *claims boundedness* and *CED* but it does not satisfy *nonbossiness*.

Note that, *claims boundedness* and *CED* are not sufficient for a division rule to satisfy *efficiency* or *claims monotonicty* (see Appendix 4.1).

3.3 Necessary Properties for Equal Division at Nash Equilibrium

Ashlagi et al. (2012) first prove that for any claim game based on a division rule satisfying *efficiency*, *equal treatment of equals* and *order preservation of awards*, all Nash equilibria induce equal division. They then show that a division rule satisfying *efficiency*, *equal treatment of equals* and *claims monotonicity*, all Nash equilibria induce equal division if the number of players is less than or equal to 3. Finally, they establish that a division rule satisfying *efficiency*, *equal treatment of equals*, *claims monotonicity* and *nonbossiness*, all

Nash equilibria induce equal division. However, none of these properties are necessary for division rules in order for all Nash equilibria of a claim game to induce equal division (see Appendix 4.2). Here, we introduce a property that is necessary for a division rule to achieve equal division at all Nash equilibria.

Let $\Gamma(R)$ be a claim game played by n players, with R satisfying *claims boundedness* and *efficiency* when sum of the claims does not exceed E. Let C^* be the set of all Nash equilibria of $\Gamma(R)$. Let us have $R(c^*) = (\frac{E}{n}, \ldots, \frac{E}{n}) \in \mathbb{R}^n$ for all $c^* \in C^*$. We show in the following proposition that, if the total claim is greater than E, there exists at least one player who gets more than whatever is leftover from E after sum of other agents' claims deducted.

Proposition 6. *Let* R *be a division rule satisfying* claims boundedness *and* efficiency *when* $c_N \leq E$. *Now, if all Nash equilibria of the claim game* $\Gamma(R)$ *lead to equal division of* E, *then* R *satisfies the following property:*
There exists $i \in N$ *such that* $R_i(c) > E - c_{N\setminus\{i\}}$ *whenever* $c_N > E$.

Proof. Assume, to the contrary, that for all $i \in N$, we have $R_i(c) \leq E - c_{N\setminus\{i\}}$ when $c_N > E$. Now, let $c = (c_1, \ldots, c_n)$ be such that $c_N = E$ and $c_j > c_{j'}$ for some $j, j' \in N$ with $j \neq j'$. Then there exists $k \in N$ such that $c_k \neq \frac{E}{n}$.

Claim 2. R(c)=c.

Proof. Since $c_N = E$, R satisfies *claims boundedness* and *efficiency*, i.e., we have $R_i(c) \leq c_i$ (1) for all $i \in N$ and $R_N = E$ (2). Now, (1) and (2) imply that $R_i(c) = c_i$ for all $i \in N$. □

Claim 3. c *is a Nash equilibrium of* $\Gamma(R)$.

Proof. Assume, to the contrary, that there exists $i' \in N$ and $\bar{c}_{i'} \in C_{i'}$ such that $R_{i'}(\bar{c}_{i'}, c_{-i'}) > R_{i'}(c)$. Let $\bar{c} = (\bar{c}_{i'}, c_{-i'})$.

1. If $\bar{c}_{i'} < c_{i'}$, then $\bar{c}_N = \bar{c}_{i'} + \bar{c}_{N\setminus\{i'\}} = \bar{c}_{i'} + c_{N\setminus\{i'\}} < c_N = E$. Therefore, R satisfies *claims boundedness* and *efficiency* for the set of claims, $\bar{c}$. In particular, $R_{i'}(\bar{c}) \leq \bar{c}_{i'} < c_{i'} = R_{i'}(c)$, a contradiction. Here, the last equality comes from Claim 2.

2. If $\bar{c}_{i'} > c_{i'}$, then $\bar{c}_N = \bar{c}_{i'} + c_{N\setminus\{i'\}} > c_N = E$. Therefore, $R_{i'}(\bar{c}) \leq E - \bar{c}_{N\setminus\{i'\}} = E - c_{N\setminus\{i'\}} = c_{i'} = R_{i'}(c)$, a contradiction. Here, the last inequality comes from the assumption made at the beginning.

Hence c is a Nash equilibrium of $\Gamma(R)$. □

Therefore, we find a Nash equilibrium at which agents receive unequal payoffs, in particular, $R_k(c) = c_k \neq \frac{E}{n}$, a contradiction. □

CHAPTER 4

CONCLUSION

In this thesis, we study n-player division games in which an estate is to be allocated among players. Instead of fixing a rule for the division games, we determine the properties that division rules have to satisfy in order for Nash equilibria of these games to meet certain criteria. First, we require rules to satisfy *CFC* and show that giving each agent the equal division of the estate is a Nash equilibrium outcome. We, then, show that all Nash equilibria of the games that are played based on these rules lead to equal division. Ashlagi et al. (2012), also, introduce properties that are sufficient for division rules under which all Nash equilibria induce equal division. However, *CFC* does not necessarily imply those properties. Therefore, our result provides an alternative sufficient condition to those presented in Ashlagi et al. (2012). We, next, show that if *claims boundedness* and *CED* are imposed on a division rule, equal division of the estate is a Nash equilibrium outcome. This result is not very significant for division games with $n > 2$ players. When the game is played with 2 players, however, these two conditions imply that the rule satisfies some other intuitive properties studied in Ashlagi et al. (2012).

Finally, we work with necessary conditions for a division rule, under which all Nash equilibria outcomes of a game are equal division. We, first, require the rule to satisfy *efficiency* and *claims boundedness*, when sum of all claims does not exceed the estate. We, then, prove that if all Nash equilibria of a game induce equal division, there exists at least one agent who gets more than the amount which is left after other players' claims are deduced from the estate. Although this gives us a necessary condition for all Nash equilibria to induce equal division, it is not the converse part of the result obtained in Proposition 1. Therefore, it might be interesting to analyze the additional properties beside *CFC* in future works in order to transform Proposition 1 into an if-and-only-if statement.

BIBLIOGRAPHY

Ashlagi, I., Karagözoğlu, E., & Klaus, B. (2012). A non-cooperative support for equal division in estate division problems. *Mathematical Social Sciences 63*, 228-233.

Brams, S., & Taylor, A. (1994). Divide the Dollar: Three Solutions and Extensions. *Theory and Decision* 37: 211-231.

Cetemen, E. D., & Karagözoğlu, E. (2014). Implementing Equal Division with an Ultimatum Threat. *Theory and Decision* 77: 223-236.

Moulin, H. (2002). The proportional random allocation of indivisible units. *Social Choice and Welfare 19*, 381-413.

Thomson, W. (2010). How to divide when there isn't enough: From the Talmud to game theory. *Manuscript, University of Rochester.*

APPENDICES

4.1 Sufficiency of CED

The following remarks show that even if the requirements *claims boundedness* and *CED* are satisfied, a division rule does not have to satisfy *efficiency* or *claims monotonicty.*

Remark 8. A division rule R with *claims boundedness* and *CED* does not have to satisfy *efficiency.* For instance; let *n=2*, *E=1*, $c_1 = 0$, $c_2 = 1$ and $c = (c_1, c_2)$. Let $R_1(c) = 0$ and $R_2(c) = \frac{1}{2}$. Here, we have $\sum_i c_i \geq 1$ but $\sum_i R_i(c) = \frac{1}{2} \neq 1$. That is, *efficiency* is not satisfied.

Remark 9. A division rule R with *claims boundedness* and *CED* does not have to satisfy *claims monotonicty.* For instance; let *n=2*, *E=1*, $c_1 = \frac{3}{4}$, $\bar{c_1} = 1$, $c_2 = \frac{1}{4}$, $c = (c_1, c_2)$ and $c' = (\bar{c_1}, c_2)$. Now, let $R_1(c) = \frac{3}{4}$, $R_2(c) = \frac{1}{4}$, $R_1(c') = \frac{1}{2}$ and $R_2(c') = \frac{1}{4}$. Clearly, R satisfies *claims boundedness* and *CED.* However, $R_1(c') \ngeq R_1(c)$ even though we have $\bar{c_1} > c_1$. That is, R does not satisfy *claims monotonicty.*

4.2 Necessity of Properties

We introduce two division rules and show that none of the properties, *efficiency*, *claims monotonicity*, *equal treatment of equals*, *order preservation of awards* and *nonbossiness*, are necessary conditions for rules in order for all Nash equilibria of a claim game to induce equal division.

Let us define two division rules for $N = \{1, 2\}$ and an estate E; namely, R^* and R^{**}. By making use of these two rules, the following remarks show that a division rule R with *claims boundedness* does not have to satisfy *efficiency*, *claims monotonicity*, *equal treatment of equals*, *order preservation of awards* or *nonbossiness*, even if the rule rewards all n agents with equal amount, $\frac{E}{n}$, at all Nash equilibria of the claim game. Let R^* be such that

$$R_i^*(c) = \begin{cases} c_i & c_N \leq E \\ \min\{c_i, \frac{E}{2}\} & c_N > E \text{ and } c_i = \min\{c_1, c_2\} \\ 0 & c_N > E \text{ and } c_i \neq \min\{c_1, c_2\} \end{cases}$$

and let R^{**} be such that

$$R_i^{**}(c) = \begin{cases} c_i & c_N \leq E \\ \min\{c_i, E\} & c_N > E \text{ and } i = \min\{j \in N : c_j = \min\{c_1, c_2\}\} \\ 0 & c_N > E \text{ and } i \neq \min\{j \in N : c_j = \min\{c_1, c_2\}\} \end{cases}$$

with $c \in C$ and $c_i \in C_i$ for $i \in N$. It is important to note that R^* and R^{**} *are* division rules and both satisfy *claims boundedness* property. Moreover, $\Gamma(R^*)$ and $\Gamma(R^{**})$ have exactly one Nash equilibrium in which each player claims half of the estate and is rewarded with what he claims. That is, if $\bar{c} = (\bar{c_1}, \bar{c_2}) \in C$ is a Nash equilibrium in $\Gamma(R^*)$ or $\Gamma(R^{**})$, we have $\bar{c_1} = \bar{c_2} = \frac{E}{2}$ and $R^*(\bar{c}) = R^{**}(\bar{c}) = (\frac{E}{2}, \frac{E}{2})$.

Remark 10. A division rule R satisfying *claims boundedness* with all Nash equilibria of the claim game $\Gamma(R)$ inducing equal division among all players does not have to satisfy *efficiency*. For instance; let *n=2*, *E=1* and $R = R^*$. Let, also, $c_1 = \frac{1}{2}$, $c_2 = 1$ with $c_i \in C_i$. So, $R_1(c) = \frac{1}{2}$ and $R_2(c) = 0$. Now, we have $c_N \geq 1$ but $R_N(c) \neq 1$, that is, R does not satisfy *efficiency* although it satisfies the other required properties.

Remark 11. A division rule R satisfying *claims boundedness* with all Nash equilibria of the claim game $\Gamma(R)$ inducing equal division among all players does not have to satisfy *claims monotonicty*. For instance; let $n=2$, $E=1$ and $R = R^*$. Let, also, $c_1 = c_2 = \frac{1}{2}$ and $\bar{c_1} = 1$ with $c_i \in C_i$, $\bar{c_1} \in C_1$. So, $R_1(c_1, c_2) = \frac{1}{2}$ and $R_1(\bar{c_1}, c_2) = 0$. Now, we have $\bar{c_1} > c_1$ but $R_1(\bar{c_1}, c_2) \ngeq R_1(c_1, c_2)$, that is, R does not satisfy *claims monotonicty* although it satisfies the other required properties.

Remark 12. A division rule R satisfying *claims boundedness* with all Nash equilibria of the claim game $\Gamma(R)$ inducing equal division among all players does not have to satisfy *equal treatment of equals*. For instance; let $n=2$, $E=1$ and $R = R^{**}$. Let, also, $c_1 = c_2 = \frac{2}{3}$ with $c_i \in C_i$. So, $R_1(c) = \frac{2}{3}$ and $R_2(c) = 0$. Now, we have $c_1 = c_2$ but $R_1(c) \neq R_2(c)$, that is, R does not satisfy *equal treatment of equals* although it satisfies the other required properties.

Remark 13. A division rule R satisfying *claims boundedness* with all Nash equilibria of the claim game $\Gamma(R)$ inducing equal division among all players does not have to satisfy *order preservation of awards*. For instance; let $n=2$, $E=1$ and $R = R^{**}$. Let, also, $c_1 = \frac{2}{3}$ and $c_2 = 1$ with $c_i \in C_i$. So, $R_1(c) = \frac{2}{3}$ and $R_2(c) = 0$. Now, we have $c_2 > c_1$ but $R_2(c) \ngeq R_1(c)$, that is, R does not satisfy *order preservation of awards* although it satisfies the other required properties.

Remark 14. A division rule R satisfying *claims boundedness* with all Nash equilibria of the claim game $\Gamma(R)$ inducing equal division among all players does not have to satisfy *nonbossiness*. For instance; let $n=2$, $E=1$ and $R = R^*$. Let, also, $c_1 = c_2 = \frac{2}{3}$ and $\bar{c_1} = \frac{1}{2}$ with $c_i \in C_i$, $\bar{c_1} \in C_1$. So, $R_1(c_1, c_2) = R_1(\bar{c_1}, c_2) = \frac{1}{2}$, $R_2(c_1, c_2) = \frac{1}{2}$ and $R_2(\bar{c_1}, c_2) = 0$. Now, we have $R_1(c_1, c_2) = R_1(\bar{c_1}, c_2)$ but $R_2(c_1, c_2) \neq R_2(\bar{c_1}, c_2)$, that is, R does not satisfy *nonbossiness* although it satisfies the other required properties.