SOME CRITERIA OF SELFADJOINTNESS FOR UNBOUNDED OPERATORS IN HILBERT SPACES

A THESIS
SUBMITTED TO THE DEPARTMENT OF MATHEMATICS
AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE
OF BILKENT UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

By
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June, 2013

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

SOME CRITERIA OF SELFADJOINTNESS FOR UNBOUNDED OPERATORS IN HILBERT SPACES

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June, 2013

This is a detailed presentation of some criteria of selfadjointness for unbounded operators in a Hilbert space, through operator Cauchy problems. We also include detailed preliminary results on unbounded linear operators in Hilbert spaces, the spectral theory of selfadjoint operators in Hilbert spaces, as well as the theory of extensions of Hermitian operators. The material of this thesis is classical, it was presented in the Operator Theory Seminar during the last two years, and contains material that can be found scattered through the textbooks cited in the bibliography list.

Keywords: Selfadjointness Criteria, Unbounded Operators.

ÖZET

HİLBERT UZAYLARINDAKİ SINIRSIZ OPERATÖRLERDE BAZI ÖZEŞLENİK ÖLÇÜTLERİ

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Haziran, 2013

Bu tezde, Hilbert uzaylarındaki sınırsız operatörler için özeşlenik olma kriterleri, operatör Cauchy problemleri temel alınarak, ayrıntılı bir şekilde sunulmuştur. Hilbert uzaylarındaki sınırsız doğrusal operatörler ve Hilbert uzaylarındaki özeşlenik operatörlerin spektral teorisi hakkında detaylı bir ön hazırlık çalışması da ekledik. Bu tezdeki bilgiler klasik olup, son iki yıldaki Operatör Teorisi Seminerlerinde sunulmuştur, ve kaynakçada belirtilen kitaplardaki bazı sonuçları içerir.

Anahtar sözcükler: Özeşlenik Kriterleri, Sınırsız Operatörler.

Acknowledgement

I would like to thank my parents and my wife for their encouragement helped me to study. Secondly I would like to thank my advisor Aurelian Gheondea for his personal and academic advice and help. Finally, I would like to thank TÜBİTAK for their scholars ...

Contents

Introduction

This is a detailed presentation of some criteria of selfadjointness for unbounded operators in a Hilbert space, through operator Cauchy problems. We also include detailed preliminary results on unbounded linear operators in Hilbert spaces, the spectral theory of selfadjoint operators in Hilbert spaces, as well as the theory of extensions of Hermitian operators. The material of this thesis is classical and contains material that can be found scattered throughout the textbooks cited in the bibliography list. The contents of this thesis was presented by us in the Operator Theory Seminar during the last two academic years.

In Chapter 1 we briefly recall some results on the geometry of Hilbert spaces and their orthogonal projections, then we prove a characterization of Borel measures through their Fourier Transforms and, finally, we prove, by means of the Sobolev mollification method, the embedding of the space of locally integrable functions in the space of distributions.

The second chapter is dedicated to recalling the basic results of operator theory of unbounded operators in Hilbert space. As recognized more than one hundred years ago, when dealing with unbounded operators defined on subspaces we encounter difficulties from the very beginning, especially concerning the simple algebraic operations as addition and multiplication. On the other hand, the lack of boundedness (continuity) of general linear operators is treated by the weaker but extremely useful notion of closability. In this respect, we briefly recall the approach of J. von Neumann by means of operations on the graphs of operators that provides an elegant approach to the duality, that is, adjoint operators.

Another big difficulty, probably one of the biggest, in the spectral theory of unbounded operators on Hilbert spaces is the gap between Hermitian operators and selfadjoint operators. In the third chapter we consider basic spectral properties of Hermitian operators and we define and prove the basic properties of defect numbers and defect subspaces, which provide an illuminating approach to estimating this gap.

Chapter 4 contains a detailed presentation of the von Neumann's theory of Cayley Transform of Hermitian operators that provides an elegant treatment of the problem of selfadjoint extensions of Hermitian operators through the well-understood geometric method of unitary extensions of isometric operators. From the point of view of functional (operational) calculus, the Cayley Transform is a fractional linear transformation mapping one of the complex half-planes into the unit disc. The details of the extension theory for Hermitian operators are presented in Chapter 5, where we also prove the positive selfadjointness of the operators A^*A, for any densely defined closed operator A in a Hilbert space.

Chapter 6 is dedicated to a careful presentation of the spectral theory of (unbounded) selfadjoint operators on Hilbert spaces, the construction and the basic properties of spectral measures, the functional calculus with unbounded measurable functions, images of spectral measures, products of spectral measures, the Spectral Theorem for selfadjoint operators, and the delicate question of commutation of unbounded selfadjoint operators. As a by-product, we also make a brief but consistent review of the spectral theory of unbounded normal operators on Hilbert spaces.

The last chapter contains the main results that make the topics of this thesis. We start with a careful presentation of Stone's Theorem on the infinitesimal operator associated to a strongly one-parameter continuous group of operators on a Hilbert space and provides, through relevant examples, the main technical tool that we use, namely the operator Cauchy problems. As first main result, we prove Schrödinger Criterion that characterizes the essential selfadjointness of a Hermitian operator A by means of the unique solvability of an operator Cauchy problem associated to the adjoint operator A^*. The second main result is the Hyperbolic Criterion that says that a Hermitian operator A is essentially selfadjoint if and only if a second-order operator Cauchy problem associated to the adjoint operator A^* is uniquely solvable. A similar result, called the Parabolic Criterion, holds in terms of a first-order operator Cauchy problem. Then, we introduce and briefly recall the Denjoy-Carleman Theorem on quasianalytic functions, that we use in order to define the subspace of quasianalytic vectors associated to a Hermitian operator and prove the criterion of selfadjointness of a Hermitian operator by the totality of the set of its quasianalytic vectors. We also briefly discuss analytic and Stieltjes vectors associated to Hermitian operators and correspondingly derive criteria of selfadjointness. Finally we consider the selfadjointness of bounded perturbations of selfadjoint operators by the subordinating method.

Chapter 1

Preliminary Results

Theorem 1.0.1 (The Riesz Representation Theorem)**.** *Let $L : \mathcal{H} \mapsto \mathbb{C}$ be a bounded linear functional. Then there is a unique vector $h_0 \in \mathcal{H}$ such that $L(h) = \langle h, h_0 \rangle$, $\forall h \in \mathcal{H}$. Moreover, $||L|| = ||h_0||$.*

For a proof see [4].

Definition 1.0.2. An idempotent on $\mathcal{H}$ is a bounded linear operator E on $\mathcal{H}$ such that $E^2 = E$. A projection is an idempotent P such that $\ker P = (R(P))^{\perp}$ where $R(P)$ is the range of P.

Proposition 1.0.3. *If E is an idempotent on $\mathcal{H}$ and $E \neq 0$, then the following assertions are equivalent.*

(1) E is a projection.

(2) E is selfadjoint.

For a proof see [4].

Definition 1.0.4 (Fourier Transform of Measures)**.** The characteristic functional of a bounded Borel measure μ on $\mathbb{R}$ is the complex function

$$\widetilde{\mu}(y) = \int_{\mathbb{R}} e^{-iyx} \, \mathrm{d}\, \mu(x). \tag{1.0.1}$$

Theorem 1.0.5 (Uniqueness of Fourier Transform of Measures)**.** *If two bounded Borel measures have equal Fourier transforms, then they coincide.*

Proof. It suffices to prove that any measure with zero Fourier transform equals to zero. Suppose that $\widetilde{\mu}(y) = \int_{\mathbb{R}} e^{-iyx} \, \mathrm{d}\, \mu(x)$ for some bounded Borel measure μ. We will show that for every bounded Borel function f of $\mathbb{R}$, $\int_{\mathbb{R}} f(x) \, \mathrm{d}\, \mu(x) = 0$. Note that once we prove this, then by considering mollification functions on the intervals $[-n-\epsilon, n+\epsilon]$, we can conclude that $\mu \equiv 0$.

Assume W.L.O.G $||\mu|| \leq 1$, and $|f| \leq 1$ be a given bounded continuous function. Let $0 < \epsilon < 1$ be given. Consider a continuous function f_0 with bounded support K such that $|f_0| \leq 1$ and $\int_{\mathbb{R}} |f(x) - f_0(x)| \, \mathrm{d}\, \mu(x) < \epsilon$. Let $k \in \mathbb{N}$ be a sufficiently large number such that $[-\pi k, \pi k]$ contains K and $|\mu|(\mathbb{R} \setminus [-\pi k, \pi k]) < \epsilon$. By Stone-Weierstrass Theorem, there exists g of the form $\sum_{j=1}^{m} c_j e^{-iy_j x}$ such that $|f_0(x) - g(x)| < \epsilon$ on $[-\pi k, \pi k]$. Note that $\int_{\mathbb{R}} g(x) \, \mathrm{d}\, \mu(x) = 0$ by the assumption. Hence

$$\begin{aligned}
|\int_{\mathbb{R}} f(x) \, \mathrm{d}\, \mu(x)| &\leq |\int_{\mathbb{R}} f(x) - f_0(x) \, \mathrm{d}\, \mu(x)| + |\int_{\mathbb{R}} f_0(x) \, \mathrm{d}\, \mu(x)| \\
&< \epsilon + |\int_{\mathbb{R}} f_0(x) \, \mathrm{d}\, \mu(x)| \\
&\leq \epsilon + |\int_{\mathbb{R}} (f_0(x) - g(x)) \, \mathrm{d}\, \mu(x)| + |\int_{\mathbb{R}} g(x) \, \mathrm{d}\, \mu(x)| \\
&= \epsilon + |\int_{\mathbb{R}} (f_0(x) - g(x)) \, \mathrm{d}\, \mu(x)| \\
&< 2\epsilon + |\int_{\mathbb{R} \setminus [-\pi k, \pi k]} |g(x)| \, \mathrm{d}\, \mu(x)|.
\end{aligned}$$

$|f_0(x)| \leq 1$ and by periodicity of g, we have $|g| \leq 1 + \epsilon < 2$ on $\mathbb{R}$.Then

$$\begin{aligned}
|\int_{\mathbb{R}} f(x) \, \mathrm{d}\, \mu(x)| &< 2\epsilon + 2\epsilon \\
&= 4\epsilon.
\end{aligned}$$

□

Lemma 1.0.6. $\forall u \in L_p(\mathbb{R}^n)$, $\forall h > 0$, $\forall 1 \leq p < \infty$

$$||u_h||_p \leq ||u||_p,$$

where $||\cdot||_p$ *is the notation for the norm on* $L_p(\mathbb{R}^n)$ *and* $u_h(x) = \int_{\mathbb{R}^n} w_h(x-y)u(y)\,\mathrm{d}\,y$ *with* $w_h(x) = \frac{1}{h^n}w(\frac{x}{h})$ *and*

$$w(x) = \begin{cases} c\cdot e^{-\frac{1}{1-|x|^2}} & : \quad |x| < 1 \\ 0 & : \quad |x| \geq 1 \end{cases}$$

where c is a constant such that $\int_{\mathbb{R}^n} w(x)\,\mathrm{d}\,x = 1$, $(x, y \in \mathbb{R}^n)$.

Proof. Let $y = hz$ and use the formula of change of variables to get

$$u_h = \int_{\mathbb{R}^n} w_h(x-y)u(y)\,\mathrm{d}\,y = \int_{\mathbb{R}^n} w(z)u(x-hz)\,\mathrm{d}\,z.$$

So

$$\begin{aligned} ||u_h||_p &= (\int_{\mathbb{R}^n} |\int_{\mathbb{R}^n} w(z)u(x-hz)\,\mathrm{d}\,z|^p\,\mathrm{d}\,x)^{1/p} \\ &\leq (\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |w(z)u(x-hz)\,\mathrm{d}\,z|^p\,\mathrm{d}\,x)^{1/p} \\ &= (\int_{\mathbb{R}^n} w(z)\int_{\mathbb{R}^n} |u(x-hz)\,\mathrm{d}\,z|^p\,\mathrm{d}\,x)^{1/p}. \end{aligned}$$

By translation invariance of Lebesgue measure

$$\begin{aligned} ||u_h||_p &\leq \int_{\mathbb{R}^n} w(z)||u||_p\,\mathrm{d}\,z \\ &= ||u||_p. \end{aligned}$$

□

Lemma 1.0.7. *Let* $1 \leq p < \infty$, $u \in L_p(\mathbb{R}^n)$, $\epsilon > 0$. *Then* $\exists\delta(u,\epsilon) > 0$ *such that* $\forall y \in \mathbb{R}^n$, $|y| \leq \delta(u,\epsilon)$, *we have*

$$||u(\cdot + y) - u(\cdot)||_p < \epsilon.$$

Proof. C_0^∞ is dense in $L_p(\mathbb{R}^n)$, so $\exists\psi \in C_0^\infty(\mathbb{R}^n)$ such that $||\psi - u||_p < \frac{\epsilon}{3}$. By translation invariance $||u(\cdot + y) - \psi(\cdot + y)||_p = ||u - \psi||_p$. Thus

$$||\psi(\cdot + y) - u(\cdot)||_p \leq ||u(\cdot + y) - \psi(\cdot + y)||_p + ||\psi(\cdot + y) - \psi(\cdot)||_p + ||\psi(\cdot) - u(\cdot)||_p$$

for sufficiently small y since $\psi \in C_0^\infty$ we can make the middle term as small as we want. So

$$\begin{aligned} ||\psi(\cdot + y) - u(\cdot)||_p &< \epsilon/3 + \epsilon/3 + \epsilon/3 \\ &= \epsilon. \end{aligned}$$

□

Theorem 1.0.8. *Let $f \in L_1^{loc}(\Omega)$, Ω be a domain in $\mathbb{R}^n$ such that $\forall \psi \in C_0^\infty(\Omega)$, $\int_\Omega f(x)\psi(x)\,\mathrm{d}\,x = 0$. Then $f = 0$ almost everywhere on Ω.*

Proof. Let K be bounded domain in Ω such that $K \subseteq \overline{K} \subseteq \Omega$ with distance between K and boundary of Ω is positive. So there exists bounded domain such that $K \subseteq \overline{K} \subseteq G \subseteq \overline{G} \subseteq \Omega$. Let

$$g(x) = \begin{cases} f(x)\chi_G(x) & : \quad x \in \Omega \\ 0 & : \quad x \notin \Omega \end{cases}$$

Clearly, $g \in L_1^{loc}(\mathbb{R}^n)$. $\exists h_0 > 0$ such that $\forall 0 < h < h_0$, $g_h(x) = 0$ $\forall x \in K$. So

$$\begin{aligned} g_h(x) &= \int_{\mathbb{R}^n} g(y) w_h(x-y) dy \\ &= \int_{\mathbb{R}^n} f(y)\chi_G(y) w_h(x-y)\,\mathrm{d}\,y. \end{aligned}$$

$w_h(x-y) = 0$ if $x \in \Omega \setminus (G + B_{h_0}(0))$, $w_h, w_h(\cdot - y) \in C_0^\infty(\Omega)$. Thus

$$g_h(x) = \int_\Omega f(y) w_h(x-y)\,\mathrm{d}\,y.$$

In view of the fact $g_h(x) - g(x) = \int_{\mathbb{R}^n}(g(x-hy) - g(x))w(y)\,\mathrm{d}\,y$ together with Lemma (1.0.6) and Lemma (1.0.7) we get

$$||g_h - g||_1 \leq \int_{\mathbb{R}^n} ||g(\cdot - hy) - g(y)||_1 w(y)\,\mathrm{d}\,y \to 0 \quad as \quad h \to 0^+,$$
$$||g|L_1(K)|| = ||g_h - g|L_1(K)|| \leq ||g_h - g|L_1(\Omega)|| \to 0 \quad as \quad h \to 0^+$$

where $||\cdot|L_1(K)||$ is the notation for the norm on the space $L_1(K)$. So, $g = 0$ almost everywhere on K and this implies $f = 0$ almost everywhere on K where K is arbitrary bounded domain. Hence, $f = 0$ almost everywhere on Ω. $\square$

Lemma 1.0.9. *Let P_{G_1} and P_{G_2} be projections onto the subspaces $G_1, G_2 \subseteq \mathcal{H}$, respectively. Then, $P_{G_1} + P_{G_2}$ is a projection if and only if $P_{G_1}P_{G_2} = P_{G_2}P_{G_1} = 0$. In this case $P = P_1 + P_2$ is a projection onto $G_1 \oplus G_2$.*

Proof. " $\Rightarrow$ " Let $P = P_{G_1} + P_{G_2}$ then $P^2 = P$ gives

$$\begin{aligned} P_{G_1} + P_{G_2} = (P_{G_1} + P_{G_2})^2 &= P_{G_1}^2 + P_{G_1}P_{G_2} + P_{G_2}P_{G_1} + P_{G_2}^2 \\ &= P_{G_1} + P_{G_2} + P_{G_1}P_{G_2} + P_{G_2}P_{G_1}. \end{aligned} \tag{1.0.2}$$

Hence

$$P_{G_1}P_{G_2} + P_{G_2}P_{G_1} = 0. \tag{1.0.3}$$

For given $f \in \mathcal{H}$ set $g = P_{G_2}P_{G_1}f$, then we have

$$P_{G_1}g = P_{G_1}P_{G_2}P_{G_1}f.$$

by using the equality (1.0.3) we get $P_{G_1}P_{G_2} = -P_{G_2}P_{G_1}$. Then

$$\begin{aligned} P_{G_1}g &= -P_{G_2}P_{G_1}^2 f \\ &= -P_{G_2}P_{G_1}f \\ &= -g \end{aligned}$$

which implies $P_{G_1}g = P_{G_1}^2 g = -P_{G_1}g$. That is $g = 0$. Since f is arbitrary, we are done.

" $\Leftarrow$ " In view of the assumption $P_{G_1}P_{G_2} = P_{G_2}P_{G_1} = 0$ and (1.0.2) we get $P^2 = P$. Moreover $\forall f, g \in \mathcal{H}$ we have

$$\begin{aligned} \langle Pf, g\rangle &= \langle P_{G_1}f, g\rangle + \langle P_{G_2}f, g\rangle \\ &= \langle f, P_{G_1}g\rangle + \langle f, P_{G_2}g\rangle \\ &= \langle f, Pf\rangle. \end{aligned}$$

That is, $P^* = P$. Thus, together with the fact $P^2 = P$, P is a projection onto the subspace $G = \{f \in \mathcal{H} |\ Pf = f\}$. But that is, $f = Pf = P_{G_1}f \oplus P_{G_2}f$ whence $f \in G_1 \oplus G_2$. □

Theorem 1.0.10. *For any bounded measurable function $f : R \mapsto \mathbb{R}$ defined in a measurable space $(R, \mathcal{R})$, there exists a sequence $(f_n)_{n=1}^{\infty}$ of simple measurable functions that converges uniformly to f. If $f(x) \geq 0$, then the functions $f_n \geq 0$ can be chosen to make the sequence $(f_n)_{n=1}^{\infty}$ nondecreasing.*

Proof. Assume W.L.O.G. f is nonnegative. Indeed, once we proved the theorem for nonnegative functions, then we can split f into negative and positive parts and do the same calculations to get the desired result. Define

$$f_n(x) = \begin{cases} \frac{k-1}{2^n} & : \quad \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}, \quad k = 1, 2, .., n2^n, \\ n & : \quad f(x) \geq n. \end{cases}$$

The sequence is clearly nonnegative, measurable and consists of simple functions. Since f is bounded, there exists M such that $0 \leq f(x) \leq M$, $\forall x \in R$. Then in view of the construction of $f_n(x)$'s we have $\forall n \geq M$, $\forall x \in R$, $|f_n(x) - f(x)| < \frac{1}{2^n}$. That is, f_n converges uniformly to f. $\square$

Chapter 2

General Theory of Unbounded Operators in Hilbert Spaces

2.1 Definitions

Definition 2.1.1. Let A, B two operators acting on Hilbert Space $\mathcal{H}$ with domain $D(A), D(B)$ and range $R(A), R(B)$ respectively. Then

(a) A and B are equal if $D(A) = D(B)$ and $Af = Bf \quad \forall f \in D(A)$.

(b) A is extension of B if $D(A) \supseteq D(B)$ and $Af = Bf \quad \forall f \in D(B)$.

(c) A is restriction of B if $D(A) \subseteq D(B)$ and $Af = Bf \quad \forall f \in D(A)$.

Throughout the thesis we will use these notations for domain and range of an operator.

Remark 2.1.2. Note that if A is a bounded operator, then A can be extended to a bounded linear operator on $\overline{D(A)}$, and then extended to $\mathcal{H}$ by letting $A = 0$ on $[D(A)]^{\perp}$. Thus, we always assume that bounded linear operators have full domain, i.e. we suppose $D(A) = H$ for all bounded operators A acting on $\mathcal{H}$ unless we explicitly state $D(A)$.

Example 2.1.3. We assume that $\mathcal{H} = L_2((a,b))$, $D(A_k) = C^k([a,b])$, and $f \mapsto f' \in L_2$ $(k \in \mathbb{N})(f \in D(A_k) \subset L_2)$. Note that $\forall k \in \mathbb{N}$, $D(A_k)$ is dense in $\mathcal{H}$ and $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$.

Definition 2.1.4. Let A, B two operators acting on Hilbert Space $\mathcal{H}$ and let $\lambda \in \mathbb{C}$. We set

(a) $(\lambda A)f = \lambda(Af) \ \ (f \in D(\lambda A) = D(A))$.

(b) $(A+B)f = Af + Bf \ \ (f \in D(A+B) = D(A) \cap D(B))$.

(c) $(AB)f = A(Bf) \ \ (f \in D(AB) = \{f \in D(B) \bigm| Bf \in D(A)\})$.

Note that $D(A+B)$ and $D(AB)$ may not be dense in $\mathcal{H}$, even if $D(A)$ and $D(B)$ are dense in $\mathcal{H}$.

Definition 2.1.5. Let A be an operator acting on Hilbert Space $\mathcal{H}$ and establishes a 1-1 correspondence between $D(A)$ and $R(A)$. Then we say that an (algebraically) inverse operator A^{-1} exists with $D(A^{-1}) = R(A)$ and $R(A^{-1}) = D(A)$, where $R(A)$ is the range of the operator A.

Remark 2.1.6. Clearly a criterion for existence of the algebraically inverse operator exists can be formulated as $\ker A := \{f \in D(A) \bigm| Af = 0\} = \{0\}$.

Consider the orthogonal sum $\mathcal{H} \oplus \mathcal{H}$ of pairs (f,g); $f, g \in \mathcal{H}$. Linear operators with these pairs are "coordinatewise" and their inner product is introduced as follows:

$$\langle (f_1, g_1), (f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} = \langle f_1, f_2 \rangle + \langle g_1, g_2 \rangle \quad (f_1, f_2, g_1, g_2 \in \mathcal{H}).$$

Now we define the set

$$\Gamma_A := \{(f, Af) \in \mathcal{H} \oplus \mathcal{H} | \ f \in D(A)\}$$

which is called the graph of the operator A. It is clear by construction of graph of an operator that, $\Gamma_A \subseteq \Gamma_B$ if and only if $A \subseteq B$. Note also that, linearity of A implies linearity of the set Γ_A in $\mathcal{H} \oplus \mathcal{H}$.

Question: Do we have the inverse statement? That is, if a set L is linear in $\mathcal{H} \oplus \mathcal{H}$ then does it follows that L is a graph of an operator? In fact, we have the following proposition.

Corollary 2.1.7. *Linear subset L of $\mathcal{H} \oplus \mathcal{H}$ is a graph of an operator if and only if for any f such that $(f, g) \in L$, g is uniquely determined.*

Proof. " $\Rightarrow$ " Clear.

" $\Leftarrow$ " If g is uniquely determined for given f, then we can define the operator A such that $Af = g$. By assumption this is well defined, hence we are done. $\square$

Remark 2.1.8. In view of the Corollary, a linear set $L \subseteq \mathcal{H} \oplus \mathcal{H}$ is the graph of an operator if $(0, h) \in L$ implies $h = 0$.

If $D(A)$ is dense in $\mathcal{H}$, then we say that A is densely defined operator.

Consider the following two operators acting on $\mathcal{H} \oplus \mathcal{H}$: $\forall (f, g) \in \mathcal{H} \oplus \mathcal{H}$,

$$\begin{aligned} (f, g) &\mapsto U(f, g) = (g, f) \in \mathcal{H} \oplus \mathcal{H}, \\ (f, g) &\mapsto O(f, g) = (-g, f) \in \mathcal{H} \oplus \mathcal{H}. \end{aligned} \tag{2.1.1}$$

Claim: These operators are isometric.

Proof.

$$\begin{aligned} \langle U(f_1, g_1), U(f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} &= \langle (g_1, f_1), (g_2, f_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} \\ &= \langle g_1, g_2 \rangle + \langle f_1, f_2 \rangle \\ &= \langle (f_1, g_1), (f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}}. \end{aligned} \tag{2.1.2}$$

Similarly,

$$\begin{aligned} \langle O(f_1, g_1), O(f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} &= \langle (-g_1, f_1), (-g_2, f_2) \rangle_{\mathcal{H} \oplus \mathcal{H}} \\ &= \langle -g_1, -g_2 \rangle + \langle f_1, f_2 \rangle \\ &= \langle (f_1, g_1), (f_2, g_2) \rangle_{\mathcal{H} \oplus \mathcal{H}}. \end{aligned} \tag{2.1.3}$$

$R(U) = \mathcal{H} \oplus \mathcal{H}$, $R(O) = \mathcal{H} \oplus \mathcal{H}$. Thus, U and O are unitary operators. $\square$

In particular one can easily get, $U^2 = 1$, $O^2 = -1$ and $OU = -UO$.

Theorem 2.1.9. *Let A be an operator with, in general, nondense domain. In order that the algebraically inverse operator A^{-1} exist, it is necessary and sufficient that the set $U\Gamma_A$ be the graph of a certain operator. Furthermore, $\Gamma_{A^{-1}} = U\Gamma_A$.*

Proof. " $\Rightarrow$ " Suppose A^{-1} exists and $(f, g) \in \Gamma_A$, i.e. $f \in D(A)$ and $g = Af$. Then $g \in D(A^{-1})$ and $f = A^{-1}g$, i.e. $U(f, g) = (g, f) \in \Gamma_{A^{-1}}$. In other words $U\Gamma_A \subseteq \Gamma_{A^{-1}}$.

Conversely, Let $(\tilde{g}, \tilde{f}) \in \Gamma_{A^{-1}}$, i.e. $\tilde{g} \in D(A^{-1}) = R(A)$ and $\tilde{f} = A^{-1}\tilde{g}$. So $\tilde{g} = A\tilde{f}$, $\tilde{f} \in D(A)$ and $(\tilde{f}, \tilde{g}) \in \Gamma_A$ or equivalently $(\tilde{g}, \tilde{f}) = U(\tilde{f}, \tilde{g}) \in U\Gamma_A$.

" $\Leftarrow$ " Assume $U\Gamma_A$ be the graph of a certain operator. $U\Gamma_A$ consists of vectors (g, f) with $f \in D(A)$ and $g = Af$. That is, first coordinate g of this vector determines its second coordinates f uniquely. A^{-1} exists. $\square$

2.2 Closed and Closable Operators

First we give three equivalent definitions of a closed operator A acting on $\mathcal{H}$.

(1) A is closed if its graph Γ_A is closed in $\mathcal{H} \oplus \mathcal{H}$.

(2) A is closed if, for any sequence $(f_n)_{n=1}^{\infty} \subseteq D(A)$, the facts that $f_n \to f \in \mathcal{H}$ and $Af_n \to g \in \mathcal{H}$ as $n \to \infty$ imply $f \in D(A)$ and $Af = g$.

(3) In the domain $D(A)$ of an operator A, we introduce graph scalar product

$$\langle f, g \rangle_{\Gamma_A} = \langle f, g \rangle + \langle Af, Ag \rangle \quad (f, g \in D(A)). \tag{2.2.1}$$

Then, A is closed if $D(A)$ is a complete space with respect to the graph scalar product.

The norm corresponding to (2.2.1) is called graph norm.

Theorem 2.2.1. *The definitions above are equivalent.*

Proof. "(1) $\Rightarrow$ (2)"

Suppose $f_n \to f$ and $Af_n \to g$ in $\mathcal{H}$. Then for every $n \in \mathbb{N}$, $(f_n, Af_n) \subseteq \Gamma_A$ and the sequence converges to (f, g) in $\mathcal{H} \oplus \mathcal{H}$ which belongs to Γ_A by the assumption (1). Hence $f \in D(A)$ and $Af = g$.

"(2) $\Rightarrow$ (3)"

Let $(f_n)_{n=1}^{\infty} \subseteq D(A)$ be a Cauchy sequence with respect to graph norm. Then by construction, both $(f_n)_{n=1}^{\infty}$ and $(Af_n)_{n=1}^{\infty}$ are Cauchy sequence in $\mathcal{H}$. Let f and g be their limits. By (2), $f \in D(A)$ and $g = Af$. Thus $f_n \to f$ with respect to graph scalar product.

"(3) $\Rightarrow$ (1)"

Let $\Gamma_A \supseteq (f_n, Af_n) \to (f, g)$ as $n \to \infty$. Then f_n is Cauchy with respect to graph norm, or equivalently Cauchy in "coordinatewise" in $\mathcal{H}$. Thus, $\exists h \in D(A)$, $\exists k \in R(A)$ limits of $f_n's$, $Af_n's$ respectively. $f_n \to f$ in graph norm, so $Ah = k$. In view of uniqueness of limits $f = h$ and $Af = k$. So, $(f, g) \in \Gamma_A$. $\square$

Example 2.2.2. Each A_k, $k \in \mathbb{N}$ appearing in Example(2.1.3) is not closed. Let us prove the case $k = 1$. Consider the following sequence; $f_n(x) = \frac{n}{2}\int_{x-1/n}^{x+1/n} |y| \,\mathrm{d}\, y$. One can easily show that the sequence converges to $f(x) = |x|$, and Af_n converges to $f'(x)$. Since both limits belong to $\mathcal{H}$, A is not closed.

Example 2.2.3. Let A be closed operator and B be a bounded operator on $\mathcal{H}$. Then $A+B$ is closed. Indeed, consider the sequence $(f_n)_{n=1}^{\infty} \subseteq D(A+B) = D(A)$ such that $f_n \to f$ and $(A+B)f_n \to h$. Then using continuity of B we get $f \in D(B) = \mathcal{H}$ and $Bf_n \to Bf$. $Af_n \to h - Bf$, so in view of closedness of A we get $h - Bf = Af$. That is, $f \in D(A) = D(A+B)$ and $(A+B)f = h$.

Example 2.2.4. Let A be a bounded operator on $D(A)$. Then A is closed if and only if $D(A)$ is closed. Indeed, if A is closed and suppose there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq D(A+B) = D(A)$ such that $f_n \to f$. Then by boundedness of A $Af_n \to Af$. In view of A is closed we get $f \in D(A)$. For the inverse implication consider the sequence $(f_n)_{n=1}^{\infty} \subseteq D(A)$ such that $f_n \to f$ and $Af_n \to h$, then since $D(A)$ is closed and A is bounded we get $f \in D(A)$ and $Af_n \to Af$. In view of uniqueness of the limit $h = Af$ and so A is closed.

After recognizing not closed operators, natural question is that, is it possible to add some vectors to their domains to make it closed? In fact, it does not work unless for given f that we want to add to the domain of the operator, the range of the closure operator does not depend on the choice of the convergent sequence to f; i.e for given two different sequences converging to the same vector f, if their ranges converge too then it must be the same vector. In particular following theorem formalizes this idea and its equivalences as definition.

Theorem 2.2.5. *The following assertions are equivalent.*

(1) We say that A admits a closure operator $\widetilde{A}$ if the procedure outlined above is correct.

(2) We say that an operator A is closable if for any given sequence $(f_n)_{n=1}^{\infty} \subseteq D(A)$ with $f_n \to 0$ and $Af_n \to g \in \mathcal{H}$, we have $g = 0$.

(3) A is closable if, the closure $\overline{\Gamma_A}$ of its graph is the graph of some operator.

(4) A is closable if, there exists a closed operator B such that, $A \subseteq B$.

Proof. "(1) $\Rightarrow$ (2)"

Clear by $g = \widetilde{A}0 = 0$.

"(2) $\Rightarrow$ (1)"

Let $f \in \mathcal{H}$ such that $\exists f_n', f_n'' \in D(A)$ for which $f_n' \to f$, $f_n'' \to f$, $Af_n' \to g'$, and $Af_n'' \to g''$ as $n \to \infty$. Set $f_n = f_n' - f_n''$ then by the assumption we get $g' = g''$. That is, (1) holds.

"(3) $\Leftrightarrow$ (2)"

Let $(0, h) \in \overline{\Gamma_A}$. That is, there exists a sequence $(f_n)_{n=1}^{\infty} \subseteq D(A)$ such that $f_n \to 0$ and $Af_n \to h$ as $n \to \infty$. Then

$$\begin{aligned}(2) &\Longleftrightarrow h = 0 \\ &\Longleftrightarrow \overline{\Gamma_A} \ \ is \ \ a \ \ graph.\end{aligned}$$

"(3) $\Leftrightarrow$ (4)"

This is a direct consequence of the fact $A \subseteq B \Longleftrightarrow \Gamma_A \subseteq \Gamma_B$. $\square$

Notice that we do not assume any denseness of the domain in this section.

Example 2.2.6. Let $\mathcal{H} = L_2(0, 1)$, $D(A) = C([0, 1])$, and $(Af)(x) = f(0)$. Now consider the sequence of

$$f_n(x) = \begin{cases} 1 - nx & : \quad 0 \leq x \leq 1/n \\ 0 & : \quad 1 \geq x \geq 1/n \end{cases}$$

Then, for each $n \in \mathbb{N}$, $||Af_n|| = 1$ and $f_n \to 0$ in $\mathcal{H}$. Thus, operator A is non-closable

2.3 The Adjoint Operator

For a bounded operator A, in view of Riesz Theorem we define A^* by $\langle Ax, y\rangle = \langle x, A^*y\rangle$ $\forall x \in \mathcal{H}$. Now suppose A is unbounded densely defined operator acting on $\mathcal{H}$. Then for given $g \in \mathcal{H}$, the functional $F_y(x)$ defined on $D(A)$ by $F_y(x) = \langle Ax, y\rangle$ maybe unbounded, so we cannot use the Riesz Theorem. Thus, consider the following domain for the operator A^*;

$$D(A^*) = \{y \in \mathcal{H} \mid \sup_{0 \neq x \in D(A)} \frac{|\langle Ax, y\rangle|}{||x||} < \infty\}. \tag{2.3.1}$$

So, for $y \in D(A^*)$, $F_y(x)$ is bounded and since A is densely defined there exists unique extension $\widetilde{F_y}$ to $\mathcal{H}$. By Riesz Representation Theorem $\exists! x \in \mathcal{H}$ such that $\widetilde{F_y}(x) = \langle x, z\rangle$, $\forall x \in D(A)$. Define $z = A^*y$ we get $\langle Ax, y\rangle = \langle x, A^*y\rangle$, $\forall x \in D(A), \forall y \in D(A^*)$.

We called A^* the adjoint operator of A. Notice that the assumption of denseness of A is indeed essential to have the uniqueness of the extension of the operator F_y. The denseness condition is indeed sufficient condition to define A^*.

Lemma 2.3.1. *Let $\overline{D(A)} = \mathcal{H}$. Then*

$$\Gamma_{A^*} = (O\Gamma_A)^{\perp} = (\mathcal{H} \oplus \mathcal{H}) \ominus (O\Gamma_A). \tag{2.3.2}$$

where O is the unitary operator defined in (2.1.1). *In particular, if A is closed then $\mathcal{H} \oplus \mathcal{H} = \Gamma_{A^*} \oplus (O\Gamma_A)$.*

Proof. Let $(g, A^*g) \in \Gamma_{A^*}$. That is, $g \in D(A^*)$ and $\langle Af, g\rangle = \langle f, A^*g\rangle$, $(f \in D(A))$. Thus

$$\langle (g, A^*g), O(f, Af)\rangle_{\mathcal{H}\oplus\mathcal{H}} = \langle g, -Af\rangle + \langle A^*g, f\rangle = 0$$

which implies $\Gamma_{A^*} \subseteq (O\Gamma_A)^{\perp}$.

Conversely, let $(g, h) \in (O\Gamma_A)^{\perp}$. Then

$$0 = \langle (g, h), (-Af, f)\rangle_{\mathcal{H}\oplus\mathcal{H}} = -\langle g, Af\rangle + \langle h, f\rangle \quad (\forall f \in D(A)).$$

So, $\forall f \in D(A)$, $\langle f, h\rangle = \langle Af, g\rangle$; i.e, $g \in D(A^*)$ and $h = A^*g$. $(O\Gamma_A)^{\perp} \subseteq \Gamma_{A^*}$. $\square$

Lemma 2.3.2. *If $(O\Gamma_A)^{\perp}$ is a graph of some operator, then $\overline{D(A)} = \mathcal{H}$ and $\Gamma_{A^*} = (O\Gamma_A)^{\perp}$.*

Proof. Let $f \in D(A)$ and $(g, h) \in (O\Gamma_A)^{\perp}$, then $(-Af, f) \in O\Gamma_A$ and

$$0 = \langle (-Af, f), (g, h)\rangle_{\mathcal{H}\oplus\mathcal{H}} = \langle f, h\rangle - \langle Af, g\rangle \quad (f \in D(A)). \tag{2.3.3}$$

Let by contradiction that A is not densely defined. Then $\exists h \neq 0$ such that $h \perp D(A)$. But $(0, h)$ satisfies (2.3.3) clearly and so $(0, h) \in (\Gamma_A)^{\perp}$. This contradicts with the fact that $h \neq 0$. Hence, A^* exists and by Lemma 2.3.1 it satisfies (2.3.2) $\square$

Theorem 2.3.3. *Let A be densely defined operator acting on Hilbert Space $\mathcal{H}$. Then*

(1) A^* *is closed.*

(2) If A *is closable, then* $(\widetilde{A})^* = A^*$.

(3) If $\overline{R(A)} = \mathcal{H}$ *and* A^{-1} *exits, then* $(A^*)^{-1}$ *exists and* $(A^{-1})^* = (A^*)^{-1}$.

(4) If $\overline{D(B)} = \mathcal{H}$*, then* $B \supseteq A \Rightarrow A^* \supseteq B^*$.

(5) If $\overline{D(B)} = \overline{D(A+B)} = \mathcal{H}$*, then* $(A+B)^* \supseteq A^* + B^*$.

(6) If $\overline{D(B)} = \overline{D(BA)} = \mathcal{H}$*, then* $(BA)^* \supseteq A^* B^*$.

Proof. (1) Clear, since $\Gamma_{A^*} = (O\Gamma_A)^\perp$ is closed.

(2) $\Gamma_{(\widetilde{A})^*} = (O_{\overline{\Gamma_A}})^\perp = (\overline{O\Gamma_A})^\perp = (O\Gamma_A)^\perp = \Gamma_{A^*}$.

(3) $(A^{-1})^*$ exists since $\overline{D(A^{-1})} = \overline{R(A)} = \mathcal{H}$. Note also that, $(A^*)^{-1}$ exists too. Indeed,

$$y \in \ker A^* \iff 0 = \langle x, T^* y\rangle = \langle Tx, y\rangle \ (\forall x \in D(T)) \iff y \perp R(T)$$

together with the fact $\overline{R(A)} = \mathcal{H}$ implies $\ker A^* = \{0\}$. So we have

$$\Gamma_{(A^*)^{-1}} = U\Gamma_{A^*} = U(O\Gamma_A)^\perp = (UO\Gamma_A)^\perp = (-OU\Gamma_A)^\perp = (OU\Gamma_A)^\perp = \Gamma_{(A^{-1})^*}.$$

(4) $B \supseteq A \Rightarrow \Gamma_B \supseteq \Gamma_A \Rightarrow O\Gamma_B \supseteq O\Gamma_A \Rightarrow (O\Gamma_B)^\perp \subseteq (O\Gamma_A)^\perp \Rightarrow B^* \subseteq A^*$.

(5) Let $g \in D(A^* + B^*) = D(A^*) + D(B^*)$. Then

$$\langle Af, g\rangle = \langle f, A^* g\rangle \quad (f \in D(A)),$$
$$\langle Bf, g\rangle = \langle f, B^* g\rangle \quad (f \in D(B)).$$

So, $\forall f \in D(A+B)$, by adding these equalities, we get

$$\langle f, (A+B)^* g\rangle = \langle (A+B)f, g\rangle = \langle f, A^* g + B^* g\rangle.$$

That is, $g \in D((A+B)^*)$ and $(A+B)^* g = A^* g + B^* g$. Hence, $A^* + B^* \subseteq (A+B)^*$.

(6) Let $g \in D(A^*B^*)$ and $f \in D(BA)$, then

using the fact $Af \in D(B)$ and $g \in D(B^*)$

$$\langle BAf, g\rangle = \langle Af, B^*g\rangle.$$

In view of $f \in D(A)$ and $B^*g \in D(A^*)$

$$\langle BAf, g\rangle = \langle f, A^*B^*g\rangle. \tag{2.3.4}$$

Hence, $g \in D((BA)^*)$ and $(BA)^*g = A^*B^*g$.

$\square$

Theorem 2.3.4. *Let $\overline{D(A)} = \mathcal{H}$ and $B \in \mathcal{B}(\mathcal{H})$. Then, $(A+B)^* = A^* + B^*$ and $(BA)^* = A^*B^*$.*

Proof. By Theorem 2.3.3 it is enough to prove the inverse inclusions. Let $g \in D((A+B)^*)$ and $f \in D(A+B) = D(A)$, then

$$\langle (A+B)f, g\rangle = \langle Af, g\rangle + \langle Bf, g\rangle.$$

B is bounded operator so we get

$$\langle (A+B)f, g\rangle = \langle Af, g\rangle + \langle f, B^*g\rangle.$$

That is, $\langle Af, g\rangle = \langle (A+B)f, g\rangle - \langle f, B^*g\rangle = \langle f, (A+B)^*g\rangle - \langle f, B^*g\rangle = \langle g, (A+B)^*g - B^*g\rangle$. Hence $g \in D(A^*)$ and $A^*g = (A+B)^*g - B^*g$; consequently $(A+B)^* \subseteq A^* + B^*$.

For the second relation similarly, let $g \in D((BA)^*)$ and $f \in D(BA) = D(A)$. Then by $g \in D(B^*) = \mathcal{H}$; $Af \in D(B) = \mathcal{H}$ we get

$$\langle Af, B^*g\rangle = \langle BAf, g\rangle = \langle f, (BA)^*g\rangle \tag{2.3.5}$$

whence $B^*g \in D(A^*)$ and $A^*B^*g = (BA)^*g$. That is, $g \in D(A^*B^*)$ and $(A^*B^*)g = (BA)^*g$. $\square$

Theorem 2.3.5. *Suppose A is densely defined operator on $\mathcal{H}$ and A is closable, then $(A^*)^*$ exists and satisfies*

$$(A^*)^* = \widetilde{A}. \tag{2.3.6}$$

Conversely, suppose A is densely defined and $(A^*)^*$ exists. Then A is closable and $(A^*)^* = \widetilde{A}$.

Proof. First we suppose A is closed. Then by (2.3.2) we have $\mathcal{H} \oplus \mathcal{H} = \Gamma_{A^*} \oplus O\Gamma_A$. Apply O, unitary operator, to both sides we get; $\mathcal{H} \oplus \mathcal{H} = O\Gamma_{A^*} \oplus \Gamma_A$. Hence

$$(O\Gamma_{A^*})^{\perp} = \Gamma_A. \tag{2.3.7}$$

By Lemma 2.3.2 $(A^*)^*$ exists and by (2.3.2), $\mathcal{H} \oplus \mathcal{H} = \Gamma_{(A^*)^*} \oplus O\Gamma_{A^*}$. Together with (2.3.7) we conclude that $\Gamma_{(A^*)^*} = \Gamma_A$. So we proved the Theorem for the case A is closed.

If A is closable, we do the same calculations to $\tilde{A}$ and by the assumption of (2.3.6) we get, $((\widetilde{A})^*)^* = \widetilde{A}$. At the same time $(\widetilde{A})^* = A^*$. Hence (2.3.6) follows.

Conversely, considering (2.3.2) for the operators A and A^*, we have

$$\begin{aligned} \mathcal{H} \oplus \mathcal{H} &= \Gamma_{A^*} \oplus \overline{O\Gamma_A} \\ &= \Gamma_{A^*} \oplus O\overline{\Gamma_A}. \end{aligned}$$

Apply the unitary operator O to both sides,

$$\mathcal{H} \oplus \mathcal{H} = O\Gamma_{A^*} \oplus \overline{\Gamma_A} \tag{2.3.8}$$

by (2.3.2) for the operator A^* we have

$$= \Gamma_{(A^*)^*} \oplus O\Gamma_{A^*}. \tag{2.3.9}$$

In view of 2.3.8 and 2.3.9, $\overline{\Gamma_A} = \Gamma_{(A^*)^*}$. That is, A is closable. □

Chapter 3

Defect Numbers, Deficient Subspaces

3.1 Defect Numbers

Definition 3.1.1. Let $X \neq \{0\}$ be a complex normed space and $T : D(T) \mapsto X$ a linear operator with domain $D(T) \subseteq X$. A regular value λ of T is a complex number satisfying the following three properties:

(1) $R_\lambda(T) = (T - \lambda I)^{-1}$ exists,

(2) $R_\lambda(T)$ is bounded,

(3) $R_\lambda(T)$ is defined on a dense set in X.

In particular, if we do not state $D(R_\lambda(T))$ explicity, then we can omit property (3). Since we have already know $R_\lambda(T)$ is bounded, we assume $D(T) = \mathcal{H}$ by Remark 2.1.2.

Definition 3.1.2. A point $\lambda \in \mathbb{C}$ is called a point of regular type for the operator A if there exists $c_\lambda > 0$ such that

$$||(A - \lambda I)f|| \geq c_\lambda ||f|| \quad (f \in D(A)). \tag{3.1.1}$$

Clearly (3.1.1) is equivalent that $(A-\lambda I)^{-1}$ exists and continuous. Moreover, if in addition we assume $R(A-\lambda I) = \mathcal{H}$, then λ becomes a regular point.

Some properties:

(1) For a given operator A, the set of points of regular type is open.

Proof. Let λ_0 be a point of regular type. Then $\forall \lambda \in \mathbb{C}$ with $|\lambda - \lambda_0| < \frac{c_{\lambda_0}}{2}$, we have

$$\begin{aligned} ||(A-\lambda I)f|| = ||(A-\lambda_0 I)f - (\lambda-\lambda_0)f|| &\geq ||(A-\lambda_0 I)f|| - |\lambda-\lambda_0|||f|| \\ &\geq c_{\lambda_0}||f|| - |\lambda-\lambda_0|||f|| \\ &\geq \frac{c_{\lambda_0}}{2}||f||. \end{aligned}$$

So, we found open neighborhood around λ_0. □

(2) Let A be closed and let $\lambda \in \mathbb{C}$ be a point of regular type. Then $R(A-\lambda I)$ is a subspace; i.e. $R(A-\lambda I)$ is closed. Conversely, let λ be a point of regular type and $R(A-\lambda I)$ be subspace. Then A is closed. Shortly, let λ be regular type point. Then A is closed if and only if $R(A-\lambda I)$ is closed.

Proof. Let, λ be regular type point then $(A-\lambda I)^{-1}$ is bounded. Since $\mp\lambda I$ are continuous for fixed λ, using Example 2.2.3, $A-\lambda I$ is closed if and only if A is closed. Now, define $(A-\lambda I)^{-1} = T$, T is bounded by assumption. Then in view of Example 2.2.4, $D(T) = \overline{D(T)} \Longleftrightarrow T \;\; is \;\; closed$. Hence, $D(T) = R(A-\lambda I)$ with above observations we have the following assertions:

$$\begin{aligned} R(A-\lambda I) \;\; is \;\; closed &\Longleftrightarrow T \;\; is \;\; closed \\ &\Longleftrightarrow (A-\lambda I) \;\; is \;\; closed \\ &\Longleftrightarrow A \;\; is \;\; closed. \end{aligned}$$ □

(3) Assume that A is closable, and denote its closure with $\widetilde{A}$. Then every point λ of regular type for the operator A is also a point of regular type for $\widetilde{A}$. Furthermore,

$$R(\widetilde{A}-\lambda I) = \overline{R(A-\lambda I)}. \tag{3.1.2}$$

Proof. Clearly by taking limit in $||(A-\lambda I)f|| \geq c_\lambda ||f||$, λ becomes a regular type for $\widetilde{A}$ too. In particular, let $f \in D(\widetilde{A})$, then $\exists (f_n) \subseteq D(A)$ such that $f_n \to f$ and $Af_n \to \widetilde{A}f$. Now, letting $n \to \infty$ in equation(3.1.2) we get the desired conclusion. For the second part, closedness of $\widetilde{A}$ implies closedness of $R(\widetilde{A}-\lambda I)$. Moreover, $A \subseteq \widetilde{A}$ implies $R(A-\lambda I) \subseteq R(\widetilde{A}-\lambda I)$. Taking closure of both sides, we get $\overline{R(A-\lambda I)} \subseteq R(\widetilde{A}-\lambda I)$. For the inverse inclusion, let $g \in R(\widetilde{A}-\lambda I)$ and $g = (\widetilde{A}-\lambda I)f$ for $f \in D(\widetilde{A})$. Thus, $\exists (f_n)_{n=1}^{\infty} \subseteq D(A)$ such that $f_n \to f$ and $Af_n \to \widetilde{A}f$. But then $(A-\lambda I)f_n \to g$ and therefore $g \in \overline{R(A-\lambda I)}$. That is, $R(\widetilde{A}-\lambda I) \subseteq \overline{R(A-\lambda I)}$. □

3.2 Deficient Subspaces

(4) Let $\lambda \in \mathbb{C}$ be a point of regular type for the considered operator A. The subspace $N_\lambda = \mathcal{H} \ominus (R(A-\lambda I)) = (R(A-\lambda I))^\perp$ is called the deficient subspace of the operator A corresponding to λ.

$$\mathcal{H} = \overline{R(A-\lambda I)} \oplus N_\lambda. \tag{3.2.1}$$

In particular, by equation (3.1.2) if $\widetilde{A}$ exists, we can rewrite (3.2.1) as

$$\mathcal{H} = R(\widetilde{A}-\lambda I) \oplus N_\lambda. \tag{3.2.2}$$

(5) We say that ψ is an eigenvector of the operator B with a domain $D(B)$ if $0 \neq \psi \in D(B)$ and $B\psi = \lambda\psi$ with some $\lambda \in \mathbb{C}$, which is called the eigenvalue corresponding to the eigenvector ψ.

(6) The set $\Phi(\lambda)$ which consists of 0 and all eigenvectors corresponding to the same eigenvalue λ is linear. It is clear that if B is closed, then $\Phi(\lambda)$ is closed. We say that $\Phi(\lambda)$ is the corresponding eigenspace to λ. Note that $\Phi(\lambda) = \ker(A-\lambda I)$.

(7) Let $\overline{D(A)} = \mathcal{H}$. Then $N_\lambda = \Phi(\overline{\lambda})$ where $\Phi(\overline{\lambda})$ is for the corresponding operator A^*.

Proof. Let $\psi \in N_\lambda$, then for any given $f \in D(A)$, $\langle (A-\lambda I)f, \psi \rangle = 0$ implies $\langle Af, \psi \rangle = \langle f, \overline{\lambda}\psi \rangle$. That is, $\psi \in D(A^*)$ and $A^*\psi = \overline{\lambda}\psi$, or equivalently $\psi \in \Phi(\overline{\lambda})$

for the operator A^*. For the inverse inclusion, if $A^*\psi = \overline{\lambda}\psi$, then $\forall f \in D(A)$ we have

$$\begin{aligned} \langle \lambda f, \psi \rangle &= \langle f, A^*\psi \rangle = \langle Af, \psi \rangle \\ &\implies \langle (A - \lambda I)f, \psi \rangle = 0 \\ &\implies \psi \perp R(A - \lambda I). \end{aligned}$$

□

Theorem 3.2.1. *Let A be a closed operator in $\mathcal{H}$. Then $n_\lambda = \dim N_\lambda$ is invariant under the changes of λ within a connected component of the set of points λ of regular type for the operator A. Thus, every component G of this sort can be associated with a fixed number n_λ, where $\lambda \in G$. This number is called the defect number of A (in the component G).*

Proof. **Trivial Case**:

Suppose that for each λ_0 of regular type we can find a neighborhood U_{λ_0} that consists of regular points and $\dim N_{\lambda_0} = \dim N_\lambda \;\; \forall \lambda \in U(\lambda_0)$. Now, since path connectedness and connectedness are same, we can construct a closed rectifiable curve $\gamma \subseteq G$ connecting any two points in G. Then select a finite subcovering of any covering of γ. By just passing through this curve, and using the assumption we conclude the result.

Hence, it is enough to prove that for each regular point, we can find a neighborhood satisfying $\dim N_{\lambda_0} = \dim N_\lambda \;\; \forall \lambda \in U(\lambda_0)$.

Suppose to the contrary, then there exists $\{\lambda_n\}_{n=1}^{\infty}$ a sequence of points of regular type such that $\lambda_n \to \lambda$ and $\dim N_{\lambda_n} \neq \dim N_{\lambda_0}$ for all $n \in \mathbb{N}$. Assume W.L.O.G we have the following two cases:

(a) $\dim N_{\lambda_n} < \dim N_{\lambda_0}$ for all $n \in \mathbb{N}$,

(b) $\dim N_{\lambda_n} > \dim N_{\lambda_0}$ for all $n \in \mathbb{N}$.

Indeed, we can find, if necessary, a proper subsequence which would hold one of these cases.

Case (a):

Denote $P_{N_{\lambda_0}}$ the orthogonal projection onto the subspace N_{λ_0}. Then

$$\dim(P_{N_{\lambda_0}} N_{\lambda_n}) \leq \dim N_{\lambda_n} < \dim N_{\lambda_0} \quad (n \in \mathbb{N}).$$

Thus, $\exists g_n \in N_{\lambda_0} \ominus P_{N_{\lambda_0}} N_{\lambda_n} \quad (n \in \mathbb{N})$.

Claim: $g_n \perp N_{\lambda_n}$ for all $n \in \mathbb{N}$.

Proof of Claim. Let $h \in N_{\lambda_n}$ and $h = h_1 + h_2 \;\; (h_1 = P_{N_{\lambda_0}} h)$. Then

$$\langle g_n, h \rangle = \langle g_n, h_1 \rangle + \langle g_n, h_2 \rangle = 0$$

due to the facts that $g_n \in N_{\lambda_0} \ominus P_{N_{\lambda_0}} N_{\lambda_n}$ and $h_2 \perp N_{\lambda_0}$. □

Since A is closed we have $\mathcal{H} = (R(A - \lambda_n I)) \oplus N_{\lambda_n} \quad (n \in \mathbb{N})$. By Claim, $g_n \in R(A - \lambda_n I)$, i.e. $\exists f_n \in D(A), \quad f_n \neq 0$, such that $g_n = (A - \lambda_n I) f_n$. Assume W.L.O.G $||f_n|| = 1$. In addition, $g_n \in N_{\lambda_0}$ implies $g_n \perp R(A - \lambda_0 I)$. In particular, $g_n \perp (A - \lambda_0 I) f_n$. Hence

$$\begin{aligned} 0 = \langle g_n, (A - \lambda_0 I) f_n) \rangle &= \langle (A - \lambda_n I) f_n, (A - \lambda_0 I) f_n) \rangle \\ &= \langle (A - \lambda_0 I) f_n - (\lambda_n - \lambda_0) f_n, (A - \lambda_0 I) f_n) \rangle \\ &= ||(A - \lambda_0) f_n||^2 - (\lambda_n - \lambda_0) \langle f_n, (A - \lambda_0 I) f_n \rangle. \end{aligned}$$

$$\begin{aligned} \Longrightarrow ||(A - \lambda_0) f_n||^2 &= (\lambda_n - \lambda_0) \langle f_n, (A - \lambda_0 I) f_n \rangle \\ &\leq |\lambda_n - \lambda_0|.||f_n||.||A - \lambda_0 I) f_n||. \end{aligned}$$

In view of $||(A - \lambda_0) f_n|| \geq 0$, we get

$$||(A - \lambda_0) f_n|| \leq |\lambda_n - \lambda_0|.||f_n|| \quad (\forall n \in \mathbb{N}).$$

Letting $n \to \infty$, we get a contradiction with the fact that λ_0 is a point of regular type for A.

Case (b):

Similarly consider the projector $P_{N_{\lambda_n}}$ and the subspace N_{λ_n}. Then

$$\dim(P_{N_{\lambda_n}}, N_{\lambda_0}) \leq \dim N_{\lambda_0} < \dim N_{\lambda_n}.$$

So $\exists g_n \in N_{\lambda_n} \ominus P_{N_{\lambda_n}} N_{\lambda_0}$. That is, $g_n \perp N_{\lambda_0}$ and $g_n \in R(A - \lambda_0 I)$. Thus, there exists $\{f_n\}_{n=1}^{\infty} \in D(A)$ such that $g_n = (A - \lambda_0 I)f_n$. Assume W.L.O.G $||f_n|| = 1$. $g_n \in N_{\lambda_n}$ implies $g_n \perp R(A - \lambda_n I)$. In particular, $g_n \perp (A - \lambda_n I)f_n$. Hence

$$\begin{aligned} 0 = \langle g_n, (A - \lambda_n I)f_n) \rangle &= \langle (A - \lambda_0 I)f_n, (A - \lambda_n I)f_n) \rangle \\ &= \langle (A - \lambda_n I)f_n - (\lambda_0 - \lambda_n)f_n, (A - \lambda_n I)f_n) \rangle \\ &= ||(A - \lambda_n)f_n||^2 - (\lambda_0 - \lambda_n)\langle f_n, (A - \lambda_n I)f_n \rangle. \end{aligned}$$

$$\begin{aligned} \implies ||(A - \lambda_n)f_n||^2 &= (\lambda_0 - \lambda_n)\langle f_n, (A - \lambda_n I)f_n \rangle \\ &\leq |\lambda_0 - \lambda_n| \cdot ||f_n|| \cdot ||A - \lambda_n I)f_n||. \end{aligned}$$

$||(A - \lambda_n)f_n|| \geq 0$, so divide both sides with it, we get

$$||(A - \lambda_n)f_n|| \leq |\lambda_0 - \lambda_n| \cdot ||f_n|| \quad (\forall n \in \mathbb{N}).$$

Letting $n \to \infty$ we get a contradiction with the fact that λ_0 is a point of regular type for A. $\square$

Remark 3.2.2. Note that by Theorem 3.2.1 we can fix a complex number for a defect number of each connected components.

Chapter 4

Cayley and Inverse Cayley Transformation

4.1 Hermitian and Selfadjoint Operators

Let A be an operator with $\overline{D(A)} = \mathcal{H}$. A is called Hermitian if

$$\langle Af, g \rangle = \langle f, Ag \rangle \quad (f, g \in D(A))$$

is called selfadjoint if

$$A^* = A.$$

Proposition 4.1.1. *Let A be densely defined operator on Hilbert space $\mathcal{H}$. Then the followings are equivalent:*

(a) A is Hermitian.

(b) $\langle Af, f \rangle \in \mathbb{R}$, $(f \in D(A))$.

(c) $A \subseteq A^*$.

Proof. "$a \Rightarrow b$" $\langle Af, f \rangle = \langle f, Af \rangle$ and $\langle Af, f \rangle = \overline{\langle f, Af \rangle}$ concludes (b).

"$b \Rightarrow c$" Let $f \in D(A)$, then $\langle Af, f\rangle = \overline{\langle f, Af\rangle} = \langle f, Af\rangle$ implies $f \in D(A^*)$ and $A^*f = Af$ for all $f \in D(A)$. That is $A \subseteq A^*$.

"$c \Rightarrow a$" Trivial by considering the definition of Hermitian. $\square$

Notice that by part (c) all Hermitian operators are closable.

Definition 4.1.2. An operator A is called essentially selfadjoint if its closure $\widetilde{A}$ is selfadjoint.

Lemma 4.1.3. *Any $z \in \mathbb{C}\backslash\mathbb{R}$ is a point of regular type for an arbitrary Hermitian operator.*

Before the proof notice first that, if z is a point of regular type, then z is not an eigenvalue. So, eigenvalues of Hermitian operators are real.

Proof.

$$\begin{aligned}||(A - zI)f||^2 &= ||(A - xI)f - iyf|| \\ &= ||(A - xI)f||^2 + iy\langle (A - xI)f, f\rangle - iy\langle f, (A - xI)f\rangle + y^2||f||^2.\end{aligned}$$

Since A is Hermitian, second and third terms are cancelled.

$$||(A - zI)f||^2 \geq y^2||f||^2.$$ $\square$

Remark 4.1.4. By Lemma 4.1.3, for an arbitrary Hermitian operator A there exists at most two connected components. Thus, there exists (at most) two defect numbers for each component. We will denote them as couples, say (m, n) for the upper and lower half planes respectively.

Theorem 4.1.5. *Let A be closed Hermitian operator acting on Hilbert space $\mathcal{H}$. Then the followings are equivalent.*

(a) A is selfadjoint.

(b) $\sigma(A) \subseteq \mathbb{R}$. Recall that $\sigma(A) :=$ complement of $\rho(A) := \{\lambda \in \mathbb{C} \mid \lambda - AI$ is boundedly invertible$\}$.

(c) $m = n = 0$.

Proof. "$b \Leftrightarrow c$" If $z \in \rho(A)$ then by definition $R(A \mp zI) = \mathcal{H}$ and we have;

$$\sigma(A) \subseteq \mathbb{R} \Longleftrightarrow \ker(A^* \mp zI) = [R(A \pm zI)]^{\perp} = \mathcal{H}^{\perp} = \{0\}. \tag{4.1.1}$$

And we know that $m = \dim \ker(A^* - \overline{z}I)$ and $n = \dim \ker(A^* + \overline{z}I)$. Hence by (4.1.1) we conclude that

$$\sigma(A) \subseteq \mathbb{R} \Longleftrightarrow m = n = 0.$$

"$a \Rightarrow c$" Let $A = A^*$ and fix $z \in \mathbb{C} \setminus \mathbb{R}$. Notice that eigenvalues of A are real numbers. Thus

$$\begin{aligned} m &= \dim(R(A - zI)^{\perp}) \\ &= \dim(\ker(A^* - \overline{z}I)) \\ &= \dim(\ker(A - \overline{z}I)). \\ &= 0. \end{aligned}$$

Similarly $n = 0$.

"$c \Leftarrow a$" It is enough to prove $D(A^*) \subseteq D(A)$. Fix $z \in \mathbb{C} \setminus \mathbb{R}$ and let $g \in D(A^*)$. By assumption, $N_z = 0$, and so $\exists f \in D(A)$ such that $(A - zI)f = (A^* - zI)g \in \mathcal{H}$. In view of the fact $A \subseteq A^*$ we can rewrite the last equality as $(A^* - zI)f = (A^* - zI)g$ or $A^*(f - g) = z(f - g)$. That is, $f - g \in N_z = \{0\}$. Hence $f = g$ and $g \in D(A)$. □

Corollary 4.1.6. *Hermitian operator A is essentially selfadjoint if its defect numbers are zero.*

Let A be a densely defined operator. Assume that there exists $\alpha \in \mathbb{R}$ such that

$$\langle Af, f \rangle \geq \alpha ||f||^2 \quad (f \in D(A)). \tag{4.1.2}$$

The operator A is called semibounded, and the number α is called its vertex.

Remark 4.1.7. $\langle Af, f\rangle$ is real for semibounded operators, thus by Proposition 4.1.1, any semibounded operator is Hermitian.

Remark 4.1.8. Note that by taking limit we can conclude that if A is semibounded operator with $\widetilde{A}$ its closure, then $\widetilde{A}$ also becomes a semibounded operator with the same vertex.

Lemma 4.1.9. *Let A be a semibounded operator with a vertex $\alpha \in \mathbb{R}$, then any $z \in \mathbb{R} \setminus [\alpha, \infty)$ is a point of regular type for this operator.*

Proof. Set $\xi = \alpha - z > 0$. Then $\forall f \in D(A)$ we have

$$\begin{aligned} ||(A - zI)f||^2 &= ||(A - \alpha I)||^2 + \xi\langle (A - \alpha I)f, f\rangle + \xi\langle f, (A - \alpha I)f\rangle + \xi^2||f||^2. \\ &\geq \xi^2||f||^2. \end{aligned} \tag{4.1.3}$$

We used the facts that A is Hermitian and $\langle (A-\alpha I)f, f\rangle = \langle f, (A-\alpha I)f\rangle \geq 0$. □

Remark 4.1.10. In view of Lemma 4.1.9 and Theorem 3.2.1, semibounded operators have equal defect numbers.

Theorem 4.1.11. *Any closed semibounded operator A with a vertex $\alpha \in \mathbb{R}$ has equal defect numbers. In order for this operator to be selfadjoint, it is sufficient that*

$$R(A - zI) = \mathcal{H} \tag{4.1.4}$$

for some $z \in \mathbb{C} \setminus [\alpha, \infty)$.

Proof. Proof follows directly by Theorem 4.1.5, and Remark 4.1.10. □

4.2 Isometric and Unitary Operators

Definition 4.2.1. An operator U acting from $D(U) \subseteq \mathcal{H}$ to $R(U) \subseteq \mathcal{H}$ is called isometric if

$$\langle Uf, Ug\rangle = \langle f, g\rangle \quad (f, g \in D(U)). \tag{4.2.1}$$

This operator is called unitary if, in addition, $D(U) = R(U) = \mathcal{H}$.

Remark 4.2.2. Note that isometric operators are necessarily continuous; therefore, it is always possible to consider $\widetilde{U}$ in $\overline{D(U)} = D(\widetilde{U})$ closing it by continuity. So, we always assume that $\overline{D(U)} = D(U)$; $\overline{R(U)} = R(U)$; and U is closed.

Lemma 4.2.3. *Every $z \in \mathbb{C}$ with $|z| \neq 1$, is a point of regular type of an isometric operator.*

Proof. Let U be an isometric operator and let $|z| < 1$, then

$$||(U - zI)f|| \geq ||Uf|| - |z| \cdot ||f|| = (1 - |z|)||f||.$$

Similarly for $|z| > 1$ we have

$$||(U - zI)f|| \geq |z| \cdot ||f|| - ||Uf|| = (|z| - 1)||f||.$$

□

So, there exists two connected components $\{z \in \mathbb{C}|\ |z| > 1\}$ and $\{z \in \mathbb{C}|\ |z| < 1\}$ so as for Hermitian operators. Denote these defect numbers as m and n.

Theorem 4.2.4. *An isometric operator U is unitary if and only if its defect numbers $m = n = 0$.*

Proof. We will prove that

$$m = \dim(\mathcal{H} \ominus R(U)) \quad \textit{and} \quad n = \dim(\mathcal{H} \ominus D(U)). \tag{4.2.2}$$

Note that once we show these equalities then we are done. Now, we have two cases for m and n. First let $z \in \mathbb{C}$ such that $|z| < 1$. Consider the point $z = 0$, then $n = n_0 = \dim(\mathcal{H} \ominus R(U))$ hence second equality follows. Secondly, let $z \in \mathbb{C}$ such that $|z| > 1$. The algebraically inverse operator of U exists and is isometric. So let n_1 be its second defect number, then according to (4.2.2) second formula applied to U^{-1}, we have $\dim(\mathcal{H} \ominus R(U^{-1} - zI)) = n_1 = \dim(\mathcal{H} \ominus R(U^{-1})) = \dim(\mathcal{H} \ominus D(U)))$. Thus, it remains to show that $R(U^{-1} - zI) = R(U - z^{-1}I)$ $(0 < |z| < 1)$. But note that

$$\begin{aligned} R(U^{-1} - zI) &= (U^{-1} - zI)D(U^{-1} - zI) = (U^{-1} - zI)D(U^{-1}) \\ &= (U^{-1} - zI)R(U) = (1 - zU)D(U). \end{aligned}$$

In view of $f \in D(U)$ if and only if $|z|f \in D(U)$ for any $z \in \mathbb{C}$,

$$\begin{aligned} R(U^{-1} - zI) &= (U - z^{-1}I)D(U) \\ &= R(U - z^{-1}I). \end{aligned}$$

Hence we are done. □

4.3 Direct Cayley Transformation

Let $\mathcal{H}$ be a Hilbert Space and let A be a closed Hermitian operator. Fix $z \in \mathbb{C}$ with Im $z > 0$. Consider $g \in R(A - zI)$, i.e, $g = (A - zI)f$ for some $f \in D(A)$. We construct the mapping $g \mapsto (A - \overline{z}I)f = Ug$.

Since f is uniquely determined for given g by Lemma 4.1.3 and the fact that Im $z > 0$, U is well defined. Moreover, in view of

$$g = (A - zI)f, \quad Ug = (A - \overline{z}I)f, \quad (f \in D(A)) \tag{4.3.1}$$

U is linear with the domain $R(A - zI)$ and the range $R(A - \overline{z}I)$. Since $\ker(A - zI) = \{0\}$ we can rewrite (4.3.1) as

$$Ug = (A - \overline{z}I)(A - zI)^{-1}g. \tag{4.3.2}$$

The operator U above is called Cayley transformation of the operator A. Now consider the following properties of Cayley transformation:

(1) The Cayley transform of a closed Hermitian operator is an isometric operator.

Proof. By (4.3.1) $\forall f_1, f_2 \in D(A)$

$$\begin{aligned} \langle g_1, g_2 \rangle &= \langle (A - zI)f_1, (A - zI)f_2 \rangle \\ &= \langle Af_1, Af_2 \rangle - \overline{z}\langle Af_1, f_2 \rangle - z\langle f_1, Af_2 \rangle + |z|^2 \langle f_1, f_2 \rangle, \end{aligned} \tag{4.3.3}$$

and

$$\begin{aligned} \langle Ug_1, Ug_2 \rangle &= \langle (A - \overline{z}I)f_1, (A - \overline{z}I)f_2 \rangle \\ &= \langle Af_1, Af_2 \rangle - z\langle Af_1, f_2 \rangle - \overline{z}\langle f_1, Af_2 \rangle + |z|^2 \langle f_1, f_2 \rangle. \end{aligned}$$

Since A is Hermitian, U is isometric. $\square$

(2) Let $m(A), n(A)$ and $m(U), n(U)$ ne the defect numbers of the operators A and U, respectively. Then

$$m(A) = m(U) \;\; and \;\; n(A) = n(U). \tag{4.3.4}$$

Proof. By (4.2.2) we have $m(U) = \dim(\mathcal{H} \ominus D(U))$ and $n(U) = \dim(\mathcal{H} \ominus R(U))$. Also by construction of U we have $D(U) = R(A - zI)$ and $R(U) = R(A - \tilde{z}I)$. Thus, in view of closedness of A $m(U) = \dim(\mathcal{H} \ominus R(A - zI)) = m(A)$. Besides, $n(U) = \dim(\mathcal{H} \ominus R(U)) = \dim(\mathcal{H} \ominus R(A - \overline{z}I))$ follows by closedness of A and the fact that $z \in \{z \mid \text{Im } z > 0\}$ implies $\overline{z} \in \{z \mid \text{Im } z < 0\}$. $\square$

(3) Cayley transformation of a selfadjoint operator is a unitary operator.

Proof. This is a direct consequence of (1) and (4.3.4). $\square$

(4) Let $B \supseteq A$ be the closed Hermitian extension of an Hermitian operator A. Then its Cayley transform V is an isometric extension of the Cayley transform of A, say U.

Proof. It follows by (1) and (4.3.1). $\square$

4.4 Inverse Cayley Transformation

For a given closed Hermitian operator by Cayley transformation we can get an isometric operator U. Suppose first that 1 is not an eigenvalue of U. That is;

$$\ker(U - I) = \{0\}. \tag{4.4.1}$$

Then for given $g \in D(U)$ consider the following transformation

$$f = \frac{1}{z - \overline{z}}(U - I)g \mapsto Bf = \frac{1}{z - \overline{z}}(zU - \overline{z}I)g \tag{4.4.2}$$

which is clearly well defined and linear by construction of U and the fact that $\ker(U-I)=\{0\}$. In view of (4.3.1) we have

$$(U-I)g=(z-\overline{z})f \quad (\forall f\in D(A)) \tag{4.4.3}$$

$$(zU-\overline{z}I)g=(z-\overline{z})Af \quad (\forall f\in D(A)) \tag{4.4.4}$$

which the former implies $D(A)=D(B)$ and the latter implies $A=B$. Thus, if (4.4.1) satisfies, then A is called the inverse Cayley transform of the operator U and we have

$$D(A)=R(U-I) \;\; \textit{and} \;\; R(A)=R(zU-\overline{z}I). \tag{4.4.5}$$

Note that we can rewrite (4.3.2) in a similar way, as

$$Af=(zU-\overline{z}I)(U-I)^{-1}f. \tag{4.4.6}$$

Now consider the following properties:

(5) The inverse Cayley transformation of an isometric operator is a closed Hermitian operator.

Proof. In fact, by (4.4.2) $\forall g_1,g_2\in D(U)$ we have

$$\begin{aligned}\langle Af_1,f_2\rangle &= \langle\frac{1}{z-\overline{z}}(zU-\overline{z}I)g_1,\frac{1}{z-\overline{z}}(U-I)g_2\rangle\\ &= \frac{1}{|z-\overline{z}|^2}[z\langle Ug_1,Ug_2\rangle - z\langle Ug_1,g_2\rangle - \overline{z}\langle g_1,Ug_2\rangle + \overline{z}\langle g_1,g_2\rangle],\end{aligned}$$

and similarly

$$\begin{aligned}\langle f_1,Af_2\rangle &= \langle\frac{1}{z-\overline{z}}(U-I)g_1,\frac{1}{z-\overline{z}}(zU-\overline{z}I)g_2\rangle\\ &= \frac{1}{|z-\overline{z}|^2}[\overline{z}\langle Ug_1,Ug_2\rangle - z\langle Ug_1,g_2\rangle - \overline{z}\langle g_1,Ug_2\rangle + z\langle g_1,g_2\rangle].\end{aligned}$$

Since U is isometric, we get $\langle Af_1,f_2\rangle=\langle f_1,Af_2\rangle$ $(f_1,f_2\in D(A))$. That is, A is Hermitian. For closedness part, let $(f_n)_{n=1}^{\infty}\subseteq D(A)=R(U-I)$, and $f_n\to f$, $Af_n\to h$. Then $f_n=(z-\overline{z})^{-1}(U-I)g_n$ and $Af_n=(z-\overline{z})^{-1}(zU-\overline{z}I)g_n$ $(g_n\in D(U))$. By (4.3.1) $g_n=(A-zI))f_n$ and $Ug_n=(A-\overline{z}I)f_n$. Since U is isometric operator, it is closed and $D(U)$ is closed. Thus, $\exists g=\lim g_n=h-zf$

and $Ug = h - \overline{z}f$. Thus, we get $Ug = g + zf - \overline{z}f$ or $(U - I)g = (z - \overline{z})f$. In view of (4.4.2), $f \in D(A)$ and

$$\begin{aligned} zUg - \overline{z}g &= zh - \overline{z}h \\ &= (Af)(z - \overline{z}). \end{aligned}$$

That is, $h = Af$. $\square$

(6) Defect numbers of U and A satisfy (4.3.4).

Proof. In particular almost same proof with property (2) with the equality (4.4.2) proves the desired equalities. $\square$

(7) The inverse Cayley transform of a unitary operator is a selfadjoint operator provided that $D(A) = R(U - I)$ is dense in $\mathcal{H}$.

Proof. Since $D(A) = R(U - I)$ is dense in $\mathcal{H}$, A^* exists and then proof follows by (6). $\square$

Remark 4.4.1. It is evident that a statement similar to (4) is also true, i.e, $V \supseteq U \Longrightarrow B \supseteq A$. However, in this case V should satisfy (4.4.1) so that A^* exists. In particular, by the following lemma we will prove that (4.4.1) is indeed equal to the statement $\overline{D(A)} = \overline{R(U - I)} = \mathcal{H}$. That is, we do not need any extra assumption in order to have inverse Cayley transform.

Lemma 4.4.2. *$\overline{R(U - I)} = \mathcal{H}$ if and only if* $\ker(U - I) = 0$.

Proof. $'' \Rightarrow ''$ Let $h \in \ker(U - I)$; i.e, $Uh = h$ $(h \in D(U))$. $\forall g \in D(U)$, in view of U is isometry,

$$\langle (U - I)g, h \rangle = \langle Ug, h \rangle - \langle g, h \rangle = \langle Ug, Uh \rangle - \langle g, h \rangle = 0.$$

$\overline{R(U - I)}$ is dense in $\mathcal{H}$, so $h = 0$.

$'' \Leftarrow ''$ By contradiction, let $\exists h \neq 0$ with $h \perp R(U-I)$. $\forall f \in D(U)$,

$$\begin{aligned} 0 = \langle Uf - f, h\rangle &= \langle Uf, h\rangle - \langle f, h\rangle \\ &= \langle Uf, h\rangle - \langle Uf, Uh\rangle \\ &= \langle Uf, h - Uh\rangle. \end{aligned}$$

So $\forall f \in D(U)$, $\langle Uf, h - Uh\rangle = 0$; i.e, $h \in \ker(U-I)$. Contradiction. □

(8) Let U be the Cayley transform of the closed Hermitian operator A, and let A_1 be the inverse Cayley transformation of the the isometric operator U. Then we have $A_1 = A$. As a result, $A \mapsto U \mapsto A$. Similarly, $U \mapsto A \mapsto U$.

Proof. Proofs follow directly by constructions of Cayley and Inverse Cayley transformations in view of Lemma 4.4.2. □

Chapter 5

Extensions of Hermitian Operators to Selfadjoint Operators

5.1 Extension Theory

Below we assume that the defect numbers m, n of the operators acting on a Hilbert space $\mathcal{H}$ take the values $0, 1, 2, \ldots$ or ∞. This is true if $\mathcal{H}$ is separable. For general $\mathcal{H}$, the numbers m, n are in fact cardinals.

Theorem 5.1.1. *Let U be an isometric operator in $\mathcal{H}$ with the defect numbers $m = \dim(\mathcal{H} \ominus D(U)) > 0$ and $n = \dim(\mathcal{H} \ominus R(U)) > 0$. Fix $k \leq \min(m, n)$, choose $k-$dimensional subspaces $F \subseteq \mathcal{H} \ominus D(U)$ and $G \subseteq \mathcal{H} \ominus R(U)$, and construct an isometric operator W acting from the whole F to the whole G. The orthogonal sum*

$$V = U \oplus W, \quad D(V) = D(U) \oplus D(W), \quad R(V) = R(U) \oplus R(W)$$

is an isometric extension of the operator U. All possible isometric extensions of this operator can be obtained by using the same procedure for all possible k, F, G, W.

Proof. Proof follows directly by the properties of orthogonal sums. □

Corollary 5.1.2. *If at least one defect numbers of an isometric operator U is zero, then U has no nontrivial isometric extensions.*

Corollary 5.1.3. *In order that U has unitary extensions, it is necessary and sufficient that $m = n$. In order to construct a unitary extension, one must set $F = \mathcal{H} \ominus D(U)$ and $G = \mathcal{H} \ominus R(U)$ and take an isometric operator W with $D(W) = F$ and $R(W) = G$.*

Remark 5.1.4. Let A be a closed Hermitian operator and let B be its closed Hermitian extension. Then

$$A \subseteq B \subseteq B^* \subseteq A^*. \tag{5.1.1}$$

Theorem 5.1.5. *Let A be closed Hermitian operator. In order that A admits nontrivial closed Hermitian extensions it is necessary and sufficient that $m, n > 0$. In order that A admits a selfadjoint extension, it is necessary and sufficient that its defect numbers equal; i.e, $m = n$.*

Proof. In fact, this is a clear consequence of Corollary 5.1.3 and the property (6) of inverse Cayley transform. □

Remark 5.1.6. It is obvious that if we change the point $z \in \mathbb{C} \setminus \mathbb{R}$, then F, G, W would also change in order to get the same extension B. Note also that If $m = 0$ or $n = 0$, then A does not have closed Hermitian extensions in $\mathcal{H}$. In this case it is called maximal.

5.2 Von Neumann Formulas

(1) A linear set $L \subseteq \mathcal{H}$ is called the direct sum of linear sets $L_1, \cdots, L_n \subseteq \mathcal{H}$ if, $\forall f \in L$, there exists unique representation $f = f_1 + \cdots + f_n$, where $f_j \in L_j$, $j = 1, \ldots, n$. In other words, $0 = f_1 + \cdots + f_n$ implies $f_1 = \cdots = f_n = 0$. Denote this direct sum as follows

$$L = L_1 \dotplus L_2 \dotplus \cdots \dotplus L_n. \tag{5.2.1}$$

Now, let A be a closed Hermitian operator in $\mathcal{H}$ and let $z \in \mathbb{C} \setminus \mathbb{R}$ be fixed. Then

$$D(A^*) = D(A) \dotplus N_z \dotplus N_{\overline{z}}. \tag{5.2.2}$$

Thus, according to (5.2.2), $\forall g \in D(A^*)$ there exists unique decomposition such that

$$g = f + h_z + h_{\overline{z}} \ where \ f = f(g) \in \ D(A); \ h_z = h_z(g) \in N_z; \ h_{\overline{z}} = h_{\overline{z}}(g) \in N_{\overline{z}}. \tag{5.2.3}$$

If (5.2.2) is correct, in view of $h_z \in N_z$ and $h_{\overline{z}} \in N_{\overline{z}}$ we get

$$A^*g = Af + \overline{z}h_z + zh_{\overline{z}}. \tag{5.2.4}$$

Proof of equation (5.2.2). It is enough to prove the decomposition (5.2.3) exists and unique.

Existence of (5.2.3):

Let $g \in D(A^*)$. According to the decomposition $\mathcal{H} = R(A - zI) \oplus N_z$, the vector $(A^* - zI)g \in \mathcal{H}$ can be written as

$$(A^* - zI)g = (A - zI)f + (\overline{z} - z)h_z. \tag{5.2.5}$$

Note that $(\overline{z} - z)$ is just constant, here $h_z \in N_z$ and $f \in D(A - zI) = D(A)$. Moreover, $A^*g = Af + \overline{z}h_z + z(g - f - h_z)$. We will show that $g - f - h_z \in N_{\overline{z}} = \Phi(z)$ for the operator A^*. Indeed,

$$A^*(g - f - h_z) = A^*g - A^*f - A^*h_z.$$

Since A is Hermitian and $h_z \in N_z$, we have $A^*f = Af, \ \forall f \in D(A)$ and $A^*h_z = \overline{z}h_z$. Then in view of (5.2.5)

$$\begin{aligned} A^*(g - f - h_z) &= (A^* - zI)g + zg - Af - \overline{z}h_z \\ &= (A - zI)f + (\overline{z} - z)h_z + zg - Af - \overline{z}h_z \\ &= z(g - f - h_z). \end{aligned}$$

That is, $h_{\overline{z}} := g - f - h_z \in N_{\overline{z}}$ and $g = f + h_z + h_{\overline{z}}$ where $f \in D(A)$, $h_z \in N_z$, $h_{\overline{z}} \in N_{\overline{z}}$. Hence existence part is proved.

Uniqueness of (5.2.3):

$$\textit{Suppose}\ \ 0 = f + h_z + h_{\overline{z}}\ \ \textit{where}\ \ f \in D(A),\ h_z \in N_z\ \textit{and}\ h_{\overline{z}} \in N_{\overline{z}}. \tag{5.2.6}$$

Consider A^* for the equality above. Then

$$\begin{aligned} 0 &= A^*f + A^*h_z + A^*h_{\overline{z}} \\ &= Af + \overline{z}h_z + zh_{\overline{z}} \\ &= (A - zI)f + \overline{z}h_z + z(h_{\overline{z}} + f) \\ &= (A - zI)f + \overline{z}h_z + z(-h_z) \\ &= (A - zI)f + (\overline{z} - z)h_z. \end{aligned} \tag{5.2.7}$$

But, $(A - zI)f \in R(A - zI)$, $(\overline{z} - z)h_z \in N_z$ and $R(A - zI) \oplus N_z = \mathcal{H}$. Thus, by (5.2.7) $(A - zI)f = 0$ and $(\overline{z} - z)h_z = 0.z$ is a point of regular type for A, so $\ker(A - zI) = \{0\}$. Hence, $f = h_z = 0$ and by (5.2.6), $h_{\overline{z}} = 0$. □

(2) Fix $z \in \mathbb{C} \setminus \mathbb{R}$. Let W be the operator associated with the extension B according to Theorem 5.1.5, $D(W) = F \subseteq N_z$, and $R(W) = G \subseteq N_{\overline{z}}$. Then the set $D(B)$ admits a decomposition

$$D(B) = D(A) \dotplus (W - I)F, \quad (F = D(W)). \tag{5.2.8}$$

i.e, $\forall g \in D(B) \subseteq D(A^*)$, decomposition (5.2.3) takes the form

$$g = f - h_z + Wh_z \quad (f \in D(A),\ h_z \in F \subseteq N_z,\ Wh_z \in WF \subseteq N_{\overline{z}}). \tag{5.2.9}$$

Since $B \subseteq A^*$; the action of B upon g is defined by (5.2.4), namely

$$Bg = A^*g = Af - \overline{z}h_z + zWh_z. \tag{5.2.10}$$

Proof. Apply (4.4.5) to $V = U \oplus W$, we obtain

$$\begin{aligned} D(B) &= R(U - I) \dotplus R(W - I) \\ &= D(A) \dotplus R(W - I) \\ &= D(A) \dotplus (W - I)F. \end{aligned}$$

□

Theorem 5.2.1. *Let A be a closed densely defined operator acting on $\mathcal{H}$. Then A^*A is selfadjoint and nonnegative. That is, $\langle A^*Af, f\rangle \geq 0 \quad \forall f \in D(A^*A)$.*

Proof. $D(A^*A) = \{f \in D(A) \mid Af \in D(A^*)\}$ implies

$$\langle A^*Af, f\rangle = \langle Af, Af\rangle \geq 0 \quad (\forall f \in D(A^*A)).$$

That is, A^*A is nonnegative. Moreover, since A is closed $(A^*)^* = A$. Consequently, write (2.3.2) for A^*, with the fact that $\Gamma_{A^*} = (O\Gamma_A)^{\perp}$

$$\mathcal{H} \oplus \mathcal{H} = \Gamma_A \oplus O\Gamma_{A^*}. \tag{5.2.11}$$

$\forall h \in \mathcal{H}$, $(h, 0)$ can be decomposed according to (5.2.11) as; there exists $f \in D(A)$ and $g \in D(A^*)$ such that

$$\begin{aligned}(h, 0) &= (f, Af) + O(g, A^*g) \\ &= (f, Af) + (-A^*g, g) \Longleftrightarrow h = f - A^*g \ \ \textit{and} \ \ 0 = Af + g.\end{aligned} \tag{5.2.12}$$

Thus $\forall h \in \mathcal{H}$ we have

$$h = f + A^*Af = (I + A^*A)f, \quad f \in D(A), \ Af = -g \in D(A^*). \tag{5.2.13}$$

We will prove that $\overline{D(A^*A)} = \mathcal{H}$. Suppose to the contrary, then $\exists h \in \mathcal{H}$ such that $0 \neq h \perp D(A^*A)$. By (5.2.13) $\exists f \in D(A^*A)$ for which $f + A^*Af = h$ and so

$$0 = \langle h, f\rangle = \langle f + A^*Af, f\rangle = ||f||^2 + ||Af||^2.$$

Hence $f = 0$ and so $h = 0$. Contradiction, so $\overline{D(A^*A)} = \mathcal{H}$. Now by (5.2.13) $R(A^*A + I) = \mathcal{H}$, and so by Theorem 4.1.11 (with $\alpha = 0$ and $z = -1 \in \mathbb{C}\setminus[0, \infty)$) we get A^*A is self adjoint. $\square$

Remark 5.2.2. Note that the Theorem is also correct for AA^* too. Similar proof can be done by just replacing A with A^*.

Chapter 6

Spectral Theorems for Unbounded Operators

6.1 Spectral Measure and Its Properties

Definition 6.1.1. An operator valued function $E : \mathcal{R} \mapsto \mathcal{B}(\mathcal{H})$ is called a spectral measure on R if it satisfies

(a) $\forall \alpha \in R$, $E(\alpha)$ is a projector in $\mathcal{H}$; $E(\emptyset) = 0$ and $E(R) = 1$,

(b) E is countably additive, i.e, $\forall (\alpha_j)_{j=1}^{\infty} \subseteq \mathcal{R}$ of disjoint sets, we have

$$E(\bigcup_{n=1}^{\infty} \alpha_j) = \sum_{n=1}^{\infty} E(\alpha_j), \tag{6.1.1}$$

where the series converges in the strong sense.

Theorem 6.1.2. *Let E be a spectral measure. Then*

$$E(\alpha)E(\beta) = E(\alpha \cap \beta) \quad (\alpha, \beta \in \mathbb{R}). \tag{6.1.2}$$

Proof. Suppose first $\alpha \cap \beta = \emptyset$. By finitely additivity of E, $E(\alpha \cup \beta) = E(\alpha) + E(\beta)$ which is indeed a projector due to the fact $\alpha \cup \beta \in \mathcal{R}$. By Lemma (1.0.9)

$E(\alpha)E(\beta) = 0$. So

$$E(\alpha)E(\beta) = 0 = E(\emptyset) = E(\alpha \cap \beta).$$

For the general case, let $\eta = \alpha \cap \beta$. Then, $\alpha = (\alpha \setminus \eta) \cup \eta$ and $\beta = (\beta \setminus \eta) \cup \eta$. By what we just proved

$$E(\alpha \setminus \eta)E(\eta) = E(\beta \setminus \eta)E(\eta) = E(\alpha \setminus \eta)E(\beta \setminus \eta) = 0.$$

Hence

$$\begin{aligned} E(\alpha)E(\beta) &= (E(\alpha \setminus \eta) + E(\eta))(E(\beta \setminus \eta) + E(\eta)) \\ &= E(\alpha \setminus \eta)E(\beta \setminus \eta) + E(\alpha \setminus \eta)E(\eta) + E(\eta)E(\beta \setminus \eta) + E^2(\eta) \\ &= E(\eta). \end{aligned}$$

□

Corollary 6.1.3. *$E(\alpha)$, $(\alpha \in \mathcal{R})$ commute.*

Remark 6.1.4. In condition (b) of (6.1.1), strong convergence of (6.1.1) can be replaced by weak convergence. Indeed Theorem (6.1.2) follows by finitely additivity of E, and so it remains true if we consider the sequence for weak convergence. Therefore, for mutually disjoint α_j's we get mutually orthogonal vectors $E(\alpha_j)f$; and for mutually orthogonal vectors weak convergence and strong convergence are equal.

Let E be a spectral measure on R and $f \in \mathcal{H}$. Then

$$\rho_{f,f}(\alpha) = \langle E(\alpha)f, f\rangle = ||E(\alpha)f||^2 \geq 0 \quad (\alpha \in \mathcal{R}) \tag{6.1.3}$$

is clearly a nonnegative finite measure on $\mathcal{R}$. Moreover, for $f, g \in \mathcal{H}$

$$\rho_{f,g}(\alpha) = \langle E(\alpha)f, g\rangle \in \mathbb{C} \quad (\alpha \in \mathcal{R}) \tag{6.1.4}$$

is a complex measure on $\mathcal{R}$.

Remark 6.1.5. Spectral measures are monotone, i.e, for given spectral measure E and $\forall \alpha, \beta \in \mathcal{R}$,

$$\alpha \subseteq \beta \Rightarrow E(\alpha) \leq E(\beta) \tag{6.1.5}$$

where $A \leq B \Leftrightarrow \langle Af, f\rangle \leq \langle Bf, f\rangle$ $(\forall f \in \mathcal{H})$. Indeed, $\beta = \alpha \cup (\beta \setminus \alpha)$ implies $E(\beta) = E(\alpha) + E(\beta \setminus \alpha)$ which yields monotonicity. In view of $\beta \setminus \alpha \in \mathcal{R}$, $E(\beta \setminus \alpha)$ is a projector and $E(\beta \setminus \alpha) \geq 0$.

Theorem 6.1.6. *Let $(\alpha_n)_{n=1}^{\infty}$, $(\beta_n)_{n=1}^{\infty}$ be decreasing and increasing sequences respectively such that, $\alpha_n, \beta_n \in \mathcal{R}$ $(n \in \mathbb{N})$, $\alpha_1 \supseteq \alpha_2 \supseteq \ldots$, $\beta_1 \subseteq \beta_2 \subseteq \ldots$. Then in the sense of strong convergence we have*

$$\lim_{n\to\infty} E(\alpha_n) = E(\cap_{n=1}^{\infty}\alpha_n),$$
$$\lim_{n\to\infty} E(\beta_n) = E(\cup_{n=1}^{\infty}\beta_n). \tag{6.1.6}$$

Proof. We will show the first relation in which second follows similarly. Set $\alpha = \cap_{n=1}^{\infty}\alpha_n$ and $\eta_n = \alpha_n \setminus \alpha$ $(n \in \mathbb{N})$. Then $\eta_1 \supseteq \eta_2 \supseteq \ldots$ and $\cap_{n=1}^{\infty}\eta_n = \emptyset$. Now consider the measure (6.1.3) for fixed $f \in \mathcal{H}$. Then

$$||E(\eta_n)f||^2 = \rho_{f,f}(\eta_n) \to 0.$$

That is, $E(\eta_n) \to 0$ strongly. $E(\eta_n) = E(\alpha_n) - E(\alpha)$, so letting n tends to infinity we conclude the first relation. For the second relation follows similarly by setting $\beta = \cup_{n=1}^{\infty}\beta_n$ and $\eta_n = \beta \setminus \beta_n$. □

Definition 6.1.7. A function $b(x,y) : \mathcal{H} \oplus \mathcal{H} \mapsto \mathbb{C}$ is called a bilinear form if it is linear in the first variable and antilinear in the second variable.

Definition 6.1.8. A bilinear form is called bounded if;

$$(\exists c > 0) \quad (\forall x, y \in \mathcal{H}) : |b(x,y)| \leq c \cdot ||x|| \cdot ||y||. \tag{6.1.7}$$

Theorem 6.1.9. *For every bounded bilinear form b, one can indicate a unique bounded operator A such that $b(x,y) = \langle Ax, y\rangle$.*

Proof. Fixed $x \in \mathcal{H}$. Set $f(y) = \overline{b(x,y)}$, then $f(y)$ is a bounded functional on $\mathcal{H}$. By Riesz Theorem, there exits unique vector $a_x \in \mathcal{H}$ such that $\overline{b(x,y)} = \langle y, a_x\rangle$. Now, define $A : \mathcal{H} \mapsto \mathcal{H}$ by $Ax = a_x$ for given $x \in \mathcal{H}$. A is linear and continuous clearly. Indeed, in view of b is linear in the first variable we have; $\forall a_1, a_2 \in \mathbb{C}$

$$\begin{aligned}\langle A(a_1x_1 + a_2x_2), y\rangle &= b(a_1x_1 + a_2x_2, y)\\ &= a_1 b(x_1, y) + a_2 b(x_2, y)\\ &= a_1\langle Ax_1, y\rangle + a_2\langle Ax_2, y\rangle.\end{aligned}$$

That is, A is linear. In view of (6.1.7), $|\langle Ax, y\rangle| \leq c \cdot ||x|| \cdot ||y||$. Hence $||Ax|| \leq c \cdot ||x||$. That is, A is bounded. Uniqueness of A is clear. □

Theorem 6.1.10. *Let E be a spectral measure on the algebra $\mathcal{R}$. Then there exists unique spectral measure E_σ on the σ-algebra $\mathcal{R}_\sigma$ such that $E_\sigma\big|\mathcal{R} = E$.*

Proof. Consider the complex measure $\rho_{f,g}(\alpha) = \langle E(\alpha)f, g\rangle$, $\alpha \in \mathcal{R}$. Then by the standard theory of extension for scalar measures, we have $\tilde{\rho}_{f,g}(\alpha) = \langle E(\alpha)f, g\rangle$, $\alpha \in \mathcal{R}_\sigma$. Fix $\alpha \in \mathcal{R}_\sigma$, then $\tilde{\rho}_{f,g}(\alpha) = \langle E(\alpha)f, g\rangle$ is a bounded bilinear form. Indeed, we know that $\rho_{f,g}(\alpha) = \langle E(\alpha)f, g\rangle$ is bilinear for $\alpha \in \mathcal{R}$, and by taking extension bilinearity preserves. Boundedness of $\tilde{\rho}_{f,g}(\alpha)$ is clear. Thus, by Theorem (6.1.9) there exists $E_\sigma(\alpha)$ such that

$$\tilde{\rho}_{f,g}(\alpha) = \langle E_\sigma(\alpha)f, g\rangle \quad (f, g \in \mathcal{H}; \alpha \in \mathcal{R}_\sigma). \tag{6.1.8}$$

Note that $E_\sigma(\emptyset) = 0$, $E_\sigma(R) = I$ and $E_\sigma(\alpha)$ is countably additive in the sense of weak convergence by (6.1.8). Thus, according to Remark (6.1.4), E_σ is the required spectral measure. Uniqueness follows by the uniqueness of the extensions of the scalar measures. □

6.2 The Construction of Spectral Integrals

6.2.1 Integrals of Simple Functions

Denote the collection of all simple functions over the measure space $(R, \mathcal{R})$ by $S(R, \mathcal{R}) = S$. S is an algebra with respect to ordinary summation and multiplication.

Definition 6.2.1.

$$\textit{Let} \;\; F(\lambda) = \sum_{k=1}^{n} F_k \chi_{\alpha_k}(\lambda) \;\; \textit{where} \;\; F_k \in \mathbb{C}; \alpha_k \cap \alpha_j = \emptyset,\; k \neq j; \lambda \in \mathbb{R}. \tag{6.2.1}$$

Now define the spectral integral as

$$\int_R F(\lambda)\,\mathrm{d}\,E(\lambda) = \int_R (\sum_{k=1}^{n} F_k \chi_{\alpha_k}(\lambda))\,\mathrm{d}\,E(\lambda) := \sum_{k=1}^{n} F_k E(\alpha_k). \tag{6.2.2}$$

Remark 6.2.2. Notice that (6.2.2) does not depend on the representation (6.2.1). Indeed for two different representations such that

$$F(\lambda) = \sum_{k=1}^{n} F_k \chi_{\alpha_k}(\lambda) = \sum_{k=1}^{m} \tilde{F}_k \chi_{\beta_k}(\lambda).$$

we have $\bigcup_{k=1}^{n} \alpha_k = \bigcup_{k=1}^{m} \beta_k = \bigcup_{k=1}^{n} \bigcup_{j=1}^{m} (\alpha_k \cap \beta_j)$. So

$$F(\lambda) = \sum_{k=1}^{n} F_k \chi_{\alpha_k}(\lambda) = \sum_{k=1}^{m} \tilde{F}_k \chi_{\beta_k}(\lambda) = \sum_{k=1}^{m} \sum_{j=1}^{n} G_{k,j} \chi_{\alpha_k \cap \beta_j}(\lambda). \tag{6.2.3}$$

In view of finite additivity of the spectral measure E, Definition (6.2.1) is well defined.

Properties:

(1) Linearity:

$\forall a, b \in \mathbb{C}; \forall F, G \in S$ we have

$$\int_R (aF(\lambda) + bG(\lambda)) \,\mathrm{d}\, E(\lambda) = a \int_R F(\lambda) \,\mathrm{d}\, E(\lambda) + b \int_R G(\lambda)) \,\mathrm{d}\, E(\lambda). \tag{6.2.4}$$

Proof. Proof follows directly from the linearity of the finite sum in (6.2.2) □

(2) Multiplicativity of an Integral:

$\forall F, G \in S$ we have

$$\int_R F(\lambda) G(\lambda) \,\mathrm{d}\, E(\lambda) = \int_R F(\lambda) \,\mathrm{d}\, E(\lambda) \int_R G(\lambda) \,\mathrm{d}\, E(\lambda). \tag{6.2.5}$$

Proof. Let $F(\lambda) = \sum_{j=1}^{n} F_j \chi_{\alpha_j}(\lambda)$ and $G(\lambda) = \sum_{k=1}^{n} G_k \chi_{\alpha_k}(\lambda)$. Then

$$\begin{aligned}\int_R F(\lambda) \,\mathrm{d}\, E(\lambda) \int_R G(\lambda) \,\mathrm{d}\, E(\lambda) &= (\sum_{j=1}^{n} F_j E(\alpha_j)(\lambda))(\sum_{k=1}^{n} G_k E(\alpha_k)(\lambda)) \\ &= \sum_{k,j=1}^{n} F_j G_k E(\alpha_j) E(\alpha_k).\end{aligned}$$

In view of the fact $E(\alpha_j)E(\alpha_k) = \delta_{jk}E(\alpha_j)$

$$\begin{aligned}\int_R F(\lambda)\,\mathrm{d}\,E(\lambda)\int_R G(\lambda)\,\mathrm{d}\,E(\lambda) &= \sum_{j=1}^{n} F_j G_j E(\alpha_j)E(\alpha_j) \\ &= \int_R F(\lambda)G(\lambda)\,\mathrm{d}\,E(\lambda).\end{aligned}$$ □

(3)

$$(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))^* = \int_R \overline{F(\lambda)}\,\mathrm{d}\,E(\lambda) \quad (F \in S). \tag{6.2.6}$$

Proof.

$$\begin{aligned}(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))^* &= (\sum_{j=1}^{n} F_j E(\alpha_j))^* \\ &= \sum_{j=1}^{n} \overline{F_j} E(\alpha_j) \\ &= \int_R \overline{F(\lambda)}\,\mathrm{d}\,E(\lambda).\end{aligned}$$ □

(4)

$$\langle(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f, g\rangle = \int_R F(\lambda)\,\mathrm{d}(E(\lambda)f, g) \quad (F \in S; f, g \in \mathcal{H}). \tag{6.2.7}$$

Note that integral on the right hand side of (6.2.7) means integration with respect to the complex measure (6.1.4).

Proof.

$$\begin{aligned}\int_R F(\lambda)\,\mathrm{d}(E(\lambda)f, g) &= \sum_{k=1}^{n} F_k \langle E(\alpha_k)f, g\rangle \\ &= \langle\sum_{k=1}^{n} F_k E(\alpha_k)f, g\rangle \\ &= \langle(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f, g\rangle.\end{aligned}$$ □

(5)

$$||(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 = \int_R |F(\lambda)|^2\,\mathrm{d}(E(\lambda)f,f) \quad (F \in S; f \in \mathcal{H}). \tag{6.2.8}$$

Proof.

$$\begin{aligned} ||(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 &= \langle (\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))^*(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f, f\rangle \\ &= \langle (\int_R \overline{F(\lambda)}\,\mathrm{d}\,E(\lambda))(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f, f\rangle \end{aligned}$$

using (6.2.5), we get

$$= \langle (\int_R |F(\lambda)|^2\,\mathrm{d}\,E(\lambda))f, f\rangle$$

in view of (6.2.7),

$$= \int_R |F(\lambda)|^2\,\mathrm{d}(E(\lambda)f,f). \qquad \square$$

(6)

$$||\int_R F(\lambda)\,\mathrm{d}\,E(\lambda)|| \le \sup\{|F(\lambda)| \mid \lambda \in R\} \quad (F \in S). \tag{6.2.9}$$

Proof. Let $f \in \mathcal{H}$, then by (6.2.8)

$$\begin{aligned} ||(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 &= \int_R |F(\lambda)|^2\,\mathrm{d}(E(\lambda)f,f) \\ &\le \sup\{|F(\lambda)|^2 \mid \lambda \in R\} \cdot \langle E(R)f, f\rangle \\ &= \sup\{|F(\lambda)|^2 \mid \lambda \in R\} \cdot ||f||^2 \qquad \square \end{aligned}$$

6.2.2 Integrals of Bounded Measurable Functions

Denote the collection of all bounded measurable functions over the measure space $(R, \mathcal{R})$ by $L_\infty(R, \mathcal{R}) = L_\infty$. Just as S, this collection is also an algebra with respect to standard algebraic operations.

Remark 6.2.3. Recall that by Theorem (1.0.10), $\forall F \in L_\infty$, $\exists (F_n)_{n=1}^{\infty}$ of simple functions, such that F_n converges uniformly to F. That is, $\sup\{|F_n(\lambda) - F(\lambda)| \bigm| \lambda \in R\} \to 0$ as $n \to \infty$.

In particular, we can define the spectral integral for bounded functions with the following definition.

Definition 6.2.4.

$$\int_R F(\lambda)\,\mathrm{d}\,E(\lambda) := \lim_{n\to\infty} \int_R F_n(\lambda)\,\mathrm{d}\,E(\lambda) \tag{6.2.10}$$

where the limit is understood in the operator norm.

Indeed, in view of (6.2.4) and (6.2.9)

$$||\int_R F_n\,\mathrm{d}\,E(\lambda) - \int_R F_m(\lambda)\,\mathrm{d}\,E(\lambda)|| = ||\int_R (F_n(\lambda) - F_m(\lambda))\,\mathrm{d}\,E(\lambda)||.$$

since $\sup\{|F_n(\lambda) - F(\lambda)| \bigm| \lambda \in R\} \to 0$, we have

$$||\int_R F_n\,\mathrm{d}\,E(\lambda) - \int_R F_m(\lambda)\,\mathrm{d}\,E(\lambda)|| \leq \sup\{|F_n(\lambda) - F_m(\lambda)| \bigm| \lambda \in R\} \to 0 \tag{6.2.11}$$

as $m, n \to \infty$. Thus, limit exists due to the fact that $\mathcal{B}(\mathcal{H})$, bounded measurable functions on $\mathcal{H}$, is complete. Further, limit (6.2.10) does not depend on the choice of (F_n) that approximates F. In fact, if (F_n') is another sequence of this sort, then using (6.2.11) they have the same limit.

(7) The integrals of bounded measurable functions $F, G \in L_\infty$ also possess properties (1) to (6).

Proof. Indeed, (1) to (4) follows directly by limit arguments.

$$\begin{aligned} ||(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 &= ||(\lim_{n\to\infty} \int_R F_n(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 \\ &= \lim_{n\to\infty} ||(\int_R F_n(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 \\ &= \lim_{n\to\infty} \int_R |F_n(\lambda)|^2\,\mathrm{d}(E(\lambda)f, f). \end{aligned}$$

by using Lebesgue Dominated Convergence Theorem with the dominating function F,

$$||(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))f||^2 = \int_R |F(\lambda)|^2\,\mathrm{d}(E(\lambda)f,f).$$

Hence we proved (5).

(6) follows similarly by using Lebesgue Dominated Convergence Theorem with the dominating function F. □

6.2.3 Integrals of Unbounded Measurable Functions

Denote the collection of all measurable functions with respect to $\mathcal{R}$ such that

$$E(\{\lambda \in R \mid |F(\lambda)| = \infty\}) = 0 \tag{6.2.12}$$

by $L_0(R, \mathcal{R}, E) = L_0$. Similar as S and L_∞, L_0 forms an algebra with respect to the ordinary operations. The definition of a spectral integral becomes more complicated since we need a correct domain to be well defined. The following lemma describes the domain.

Lemma 6.2.5. *Let $F \in L_0$. Then, the set*

$$D_F = \{f \in \mathcal{H} \mid \int_R |F(\lambda)|^2\,\mathrm{d}(E(\lambda)f,f) < \infty\} \tag{6.2.13}$$

is linear and everywhere dense in $\mathcal{H}$.

Proof. Let $f, g \in D_F$ and $\alpha \in \mathcal{R}$. Then

$$\begin{aligned}\langle E(\alpha)(f+g), f+g\rangle &= ||E(\alpha)(f+g)||^2 \\ &\le (||E(\alpha)f|| + ||E(\alpha)g||)^2.\end{aligned}$$

using the fact aritmetic mean greater or equal than geometric mean,

$$\begin{aligned}\langle E(\alpha)(f+g), f+g\rangle &\le 2(||E(\alpha)f||^2 + ||E(\alpha)g||^2) \\ &= 2(\langle E(\alpha)f,f\rangle + \langle E(\alpha)g,g\rangle).\end{aligned} \tag{6.2.14}$$

In view of (6.2.14)

$$\int_R |F(\lambda)|^2 \,\mathrm{d}(E(\lambda)(f+g),(f+g)) \leq 2[\int_R |F(\lambda)|^2 \,\mathrm{d}(E(\lambda)f,f) + \int_R |F(\lambda)|^2 \,\mathrm{d}(E(\lambda)g,g)]$$
$$< \infty.$$

Hence $f+g \in D_F$. Since clearly $\alpha f \in D_F$ too, D_F is linear.

In order to prove denseness of D_F, consider the sets $\alpha_n = \{\lambda \in R \mid |F(\lambda)| > n\} \in \mathcal{R}$ $(n \in \mathbb{N})$. Clearly, $\alpha_1 \supseteq \alpha_2 \supseteq \ldots$ and set $\alpha = \bigcap_{n=1}^{\infty}$. Notice that $F \in Ł_0$ implies $E(\alpha) = 0$. In view of

$$\int_R |F(\lambda)|^2 \,\mathrm{d}(E(\lambda)E(R \setminus \alpha_n)f, E(R \setminus \alpha_n)f) \leq n \cdot ||f|| < \infty,$$

$$R(E(R \setminus \alpha_n)) \subseteq D_F \quad (n \in \mathbb{N}). \tag{6.2.15}$$

$\forall f \in \mathcal{H}$, consider the sequence of functions $f_n = E(R \setminus \alpha_n)f \in D_F$; which converges to $E(R)f - E(\alpha)f = f$ as $n \to \infty$. That is, D_F is dense in $\mathcal{H}$. □

We now proceed to the definition of spectral integral for functions in L_0. Indeed, $\forall F \in L_0$ and $N \geq 0$, denote F_N its cutoff function by

$$F_N(\lambda) = \begin{cases} F(\lambda) & : \quad \lambda \in \{\lambda \in R \mid |F(\lambda)| \leq N\} \\ N & : \quad \text{otherwise} \end{cases}$$

Then, by definition we set $\forall f \in D_F$;

$$I_F f = \int_R F(\lambda) \,\mathrm{d}\, E(\lambda) f := \lim_{N \to \infty} \int_R F_N(\lambda) \,\mathrm{d}\, E(\lambda) f \tag{6.2.16}$$

where the convergence is in the sense of convergence in $\mathcal{H}$.

Remark 6.2.6. In fact,$\forall M, N \geq 0$ using the facts (6.2.4) and (6.2.8) for bounded measurable functions we obtain, $\forall f \in D_F$;

$$|| \int_R F_M(\lambda) \,\mathrm{d}\, E(\lambda) f - \int_R F_N(\lambda) \,\mathrm{d}\, E(\lambda) f||^2 = || \int_R (F_M(\lambda) - F_N(\lambda)) \,\mathrm{d}\, E(\lambda) f||^2$$
$$= \int_R |F_M(\lambda) - F_N(\lambda)|^2 \,\mathrm{d}(E(\lambda)f,f).$$

In view of $\int_R |F(\lambda)|^2 \, \mathrm{d}(E(\lambda)f, f) < \infty$,

$$|| \int_R F_M(\lambda) \, \mathrm{d}\, E(\lambda) f - \int_R F_N(\lambda) \, \mathrm{d}\, E(\lambda) f||^2 \to 0 \quad as \ M, N \to \infty.$$

Hence, (6.2.16) is well defined.

Let's describe the properties of I_F. First notice that (6) is meaningless.

(8) For $F \in L_0$ (4) and (5) satisfies for $f \in D_F$.

Proof. Proofs follows by the same limit arguments together with the fact $f \in D_F$ and Lebesgue Dominated Convergence Theorem with the dominating function $F \in L_0$. □

We will consider the first three properties in the following theorems:

Theorem 6.2.7. *Let $F, G \in L_0$ and $a, b \in \mathbb{C}$. Then*

$$\int_{\mathbb{R}} (aF(\lambda) + bG(\lambda)) \, \mathrm{d}\, E(\lambda) = \overline{a \int_{\mathbb{R}} F(\lambda) \, \mathrm{d}\, E(\lambda) + b \int_{\mathbb{R}} G(\lambda)) \, \mathrm{d}\, E(\lambda)}, \tag{6.2.17}$$

$$\int_{\mathbb{R}} F(\lambda) G(\lambda)) \, \mathrm{d}\, E(\lambda) = \overline{\int_{\mathbb{R}} F(\lambda) \, \mathrm{d}\, E(\lambda) \int_{\mathbb{R}} G(\lambda)) \, \mathrm{d}\, E(\lambda)}. \tag{6.2.18}$$

If one of F, G is bounded then it is not necessary to take closure of the right-hand side of (6.2.17) and (6.2.18)

Proof. We first prove (6.2.18).

Claim 1:

$$\langle E(\eta) I_G f, I_G f \rangle = \int_R |G(\lambda)|^2 \chi_\eta(\lambda) \, \mathrm{d}(E(\lambda)f, f) \quad (f \in D(I_G), \eta \in \mathcal{R}). \tag{6.2.19}$$

Proof of Claim 1. Indeed, using (6.2.5) and (6.2.6) for $G \in L_\infty$ we get

$$\langle E(\eta) I_G f, I_G f \rangle = \langle I_{\overline{G}} I_{\chi_\eta} I_G f, f \rangle = \langle I_{|G|^2 \chi_\eta} f, f \rangle = \int_R |G(\lambda)|^2 \chi_\eta(\lambda) \, \mathrm{d}(E(\lambda)f, f).$$

For the case $G \in L_0$, we take cutoff functions G_N and use the identity (6.2.19), we get

$$\begin{aligned}\langle E(\eta)I_G f, I_G f\rangle &= \langle E(\eta) \lim_{N\to\infty} I_{G_N} f, \lim_{N\to\infty} I_{G_N} f\rangle \\ &= \lim_{N\to\infty} \langle I_{|G_N|^2\chi_\eta} f, f\rangle.\end{aligned}$$

By Lebesgue Dominated Convergence Theorem with the dominating function G together with the facts; meausure is finite and $|G_N(\lambda)|^2\chi_\eta \le |G|^2$, we conclude

$$\langle E(\eta)I_G f, I_G f\rangle = \langle I_{|G|^2\chi_\eta} f, f\rangle.$$

□

Claim 2:

$$D(I_F I_G) = D(I_{FG}) \bigcap D(I_G). \tag{6.2.20}$$

Proof of Claim 2.

$$\begin{aligned}D(I_F I_G) &= \{f \in D(I_G) \mid I_G f \in D(I_F)\} \\ &= \{f \in D(I_G) \mid \int_R |F(\lambda)|^2 \,\mathrm{d}(E(\lambda)I_G f, I_G f) < \infty\}\end{aligned}$$

by Claim 1,

$$\begin{aligned}&= \{f \in D(I_G) \mid \int_R |F(\lambda)|^2 |G(\lambda)|^2 \,\mathrm{d}(E(\lambda)f, f) < \infty\} \\ &= D(I_{FG}) \bigcap D(I_G).\end{aligned}$$

□

Let $f \in D(I_F I_G) \subseteq D(I_{FG})$ and F_N, G_M are cutoff functions of F, G respectively. Then $I_{F_N} I_{G_M} f = I_{F_N G_M} f$. Letting $M \to \infty$ we get $I_{G_M} f \to I_G f$ and due to the fact that F_N is continuous we obtain $I_{F_N} I_G f = I_{F_N G} f$. Passing to the limit as $N \to \infty$ with similar arguments, we conclude that $I_F I_G f = I_{FG} f$. That is, $I_F I_G \subseteq I_{FG}$. Next we will show $I_{FG} \subseteq \overline{I_F I_G}$. Then due to the fact that I_{FG} is closed we are done for (6.2.18) Indeed, denote

$$\begin{aligned}\alpha_n &= \{\lambda \in R \mid |F(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}), \\ \beta_n &= \{\lambda \in R \mid |G(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}), \\ \eta_n &= \{\lambda \in R \mid |F(\lambda) + G(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}),\end{aligned}$$

and set $\delta_n = \alpha_n \bigcup \beta_n \bigcup \eta_n$, $(n \in \mathbb{N})$. Then $\delta_1 \supseteq \delta_2 \supseteq \ldots$ and $E(\bigcap_{n=1}^{\infty} \delta_n) = 0$. For $f \in D(I_{FG})$ consider $f_n = E(R \backslash \delta_n) f \in D(I_{FG}) \bigcap D(I_F) \bigcap D(I_G)$. Then as in the proof of Lemma (6.2.5) this sequence gives the required approximation. Hence we proved (6.2.18). Note that if $G \in L_\infty$ then (6.2.20) turns to $D(I_F I_G) = D(I_{FG})$. Since by above calculations their actions on the same domain coincide, we get $I_F I_G = I_{FG}$. Now, we will prove (6.2.17) for special case; if one of F, G is bounded. Let $F \in L_0, G \in L_\infty$. The fact arithmetic mean greater or equal than geometric mean implies

$$|F(\lambda) + G(\lambda)|^2 \leq 2(|F(\lambda)|^2 + |G(\lambda)|^2).$$

That is, $D(I_F + I_G) \subseteq D(I_{F+G})$. Moreover, by virtue of (6.2.4) for bounded functions we have; $\forall f \in D(I_F + I_G)$, $I_{F_N + G_N} f = I_{F_N} f + I_{G_N} f$ for $N \geq 0$. Letting $N \to \infty$ we get $I_{F+G} f = I_F f + I_G f$. So, $I_F + I_G \subseteq I_{F+G}$. Note also that $D(I_{F+G}) \subseteq D(I_F) = D(I_F + I_G)$ due to the facts that G is bounded and $|F(\lambda)|^2 \leq |F(\lambda) + G(\lambda)|^2$. Hence we get $I_{F+G} = I_F + I_G$. That is,we proved (6.2.17) for the special case $F \in L_0, G \in L_\infty$.

For the case $F, G \in L_0$; by doing same calculations as in the special case, one can deduce $I_F + I_G \subseteq I_{F+G}$. So it is enough to prove $I_{F+G} \subseteq \overline{I_F + I_G}$. Indeed denote the sets

$$\alpha_n = \{\lambda \in R \bigm| |F(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}),$$
$$\beta_n = \{\lambda \in R \bigm| |G(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}),$$
$$\eta_n = \{\lambda \in R \bigm| |F(\lambda) + G(\lambda)| > n\} \in \mathcal{R}, \quad (n \in \mathbb{N}),$$

and set $\delta_n = \alpha_n \bigcup \beta_n \bigcup \eta_n$, $(n \in \mathbb{N})$. Then $\delta_1 \supseteq \delta_2 \supseteq \ldots$ and $E(\bigcap_{n=1}^{\infty} \delta_n) = 0$. Let $f \in D(I_{F+G})$, then consider the sequence of functions of the form $f_n = E(R \setminus \delta_n) f \in D(I_{F+G}) \bigcap D(I_F) \bigcap D(I_G)$. Letting $f_n \to f$ implies $I_{F+G} f_n \to I_{F+G} f$ as $n \to \infty$. In fact, last relation follows by

$$\begin{aligned} I_{F+G} f_n &= \int_R (F(\lambda) + G(\lambda)) \,\mathrm{d}\, E(\lambda) E(R \setminus \delta_n) f \\ &= \int_R (F(\lambda) + G(\lambda)) \,\mathrm{d}\, E(\lambda) \int_R \chi_{R \setminus \delta_n}(\lambda) \,\mathrm{d}\, E(\lambda) f \end{aligned}$$

we proved the special case of (6.2.17),

$$= \int_R (F(\lambda) + G(\lambda)) \chi_{R \setminus \delta_n}(\lambda) \,\mathrm{d}\, E(\lambda) f. \tag{6.2.21}$$

Hence

$$||I_{F+G}f_n - I_{F+G}f||^2 = \int_R |F(\lambda) + G(\lambda)|^2 \chi_{\delta_n}(\lambda)\,\mathrm{d}(E(\lambda)f, f).$$

By Lebesgue Dominated Convergence Theorem with the dominating function $F + G$,

$$||I_{F+G}f_n - I_{F+G}f||^2 \to 0 \quad (n \to \infty),$$

whence $f \in D(\overline{I_F + I_G})$ and $I_{F+G} = \overline{(I_F + I_G)}$. □

Theorem 6.2.8. *Let $F \in L_0$. Then I_F in (6.2.16) is closed, $(I_F)^* = I_{\overline{F}}$ and $D((I_F)^*) = D(I_F) = D_F$. More precisely,*

$$(\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))^* = \int_R \overline{F(\lambda)}\,\mathrm{d}\,E(\lambda) \quad (F \in L_0). \tag{6.2.22}$$

Note also that, if $F \in L_0$ is real valued then I_F is selfadjoint.

Proof. Let $f, g \in D_F$. $(\forall N \geq 0)$ in view of (6.2.6) for bounded measurable functions,

$$\langle \int_R F_N(\lambda)\,\mathrm{d}\,E(\lambda)f, g\rangle = \langle f, \int_R \overline{F_N(\lambda)}\,\mathrm{d}\,E(\lambda)g\rangle.$$

Since $D_{\overline{F}} = D_F$ we can let $n \to \infty$ and conclude that $\langle I_F f, g\rangle = \langle f, I_{\overline{F}}g\rangle$. That is, $I_{\overline{F}} \subseteq (I_F)^*$. For the inverse implication we have;

$$\langle I_F, g\rangle = \langle f, (I_F)^* g\rangle \quad (f \in D_F, g \in D((I_F)^*)). \tag{6.2.23}$$

Let $f = E(R \setminus \alpha_n)h$ $(h \in \mathcal{H})$, which is possible by (6.2.15).

$$\begin{aligned} I_F f &= (\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))(E(R \setminus \alpha_n)h) \\ &= (\int_R F(\lambda)\,\mathrm{d}\,E(\lambda))(\int_R \chi_{\setminus \alpha_n}\,\mathrm{d}\,E(\lambda)h). \end{aligned}$$

In view of Theorem 6.2.7

$$I_F f = \int_R F(\lambda)\chi_{R\setminus\alpha_n}\,\mathrm{d}\,E(\lambda)h. \tag{6.2.24}$$

By using the boundedness of $F(\lambda)\chi_{R\setminus\alpha_n}$ we obtain

$$\langle \int_R F(\lambda)\chi_{R\setminus\alpha_n}\,\mathrm{d}\,E(\lambda)h, g\rangle = \langle h, \int_R \overline{F(\lambda)}\chi_{R\setminus\alpha_n}\,\mathrm{d}\,E(\lambda)g\rangle$$

in view of (6.2.6) for bounded functions,

$$\begin{aligned} &= \langle h, (I_{F\chi_{R\setminus\alpha_n}})^* g\rangle \\ &= \langle h, (I_F I_{\chi_{R\setminus\alpha_n}})^* g\rangle \\ &= \langle h, E(R\setminus\alpha_n)(I_F)^* g\rangle. \end{aligned}$$

Whence since $h \in \mathcal{H}$ is arbitrary, we conclude that

$$\int_R \overline{F(\lambda)}\chi_{R\setminus\alpha_n}(\lambda)\,\mathrm{d}\,E(\lambda)g = E(R\setminus\alpha_n)(I_F)^* g.$$

Using (6.2.8) for unbounded measurable functions we have; $\forall n \in \mathbb{N}$,

$$\int_R |F(\lambda)\chi_{R\setminus\alpha_n}(\lambda)|^2\,\mathrm{d}(E(\lambda)g, g) = ||E(R\setminus\alpha_n)(I_F)^* g||^2 \le ||(I_F)^* g|| < \infty. \tag{6.2.25}$$

Letting $n \to \infty$ we conclude that $g \in D_F$ and $(I_F)^* \subseteq I_{\overline{F}}$. Closeness of I_F directly follows by $I_F = (I_{\overline{F}})^*$. □

Definition 6.2.9. A closed densely defined operator A acting on Hilbert space $\mathcal{H}$ is normal if

$$AA^* = A^*A. \tag{6.2.26}$$

Remark 6.2.10. $\forall F \in L_0$, I_F defined in (6.2.16) is normal. Indeed, in view of the facts Theorem 6.2.8, Theorem 5.2.1 and 6.2.18,

$$\begin{aligned} (I_F)^* I_F &= \overline{(I_F)^* I_F} \\ &= I_{|F|^2} \\ &= I_F (I_F)^*. \end{aligned}$$

That is, I_F is normal.

Remark 6.2.11. Note that $F, G \in L_\infty$ (or more generally, if one of them are bounded) the operators I_F and I_G commutes. However, if both $F, G \in L_0$ then, by (6.2.18), the commutation is conventional; that is, $\overline{(I_F I_G)} = \overline{(I_G I_F)}$.

6.3 Image of a Spectral Measure

Let $(X, \mathcal{M})$ be a measurable space, Y be another space and ψ be a fixed bijective mapping from X to Y. Then by given $\mathcal{M}$ and ψ, we can define $\mathcal{N}$ the σ-algebra $\mathcal{N}$ that consists of all sets $\alpha' \subseteq \mathcal{N}$ whose preimage $\psi^{-1}(\alpha') \in \mathbb{R}$. One can deduce easily that $\mathcal{N}$ is indeed a σ-algebra. Thus, we have constructed the measurable space $(Y, \mathcal{N})$.

Let E be a spectral measure defined on $(X, \mathcal{M})$. We now will describe spectral measure on $(Y, \mathcal{N})$. We set

$$\alpha' \to E'(\alpha') = E(\psi^{-1}(\alpha')), \quad \alpha' \in \mathcal{N}. \tag{6.3.1}$$

Claim: (6.3.1) is indeed a spectral measure defined on $(Y, \mathcal{N})$.

Proof. Since by construction $\forall \alpha' \in Y$, $\psi^{-1}(\alpha') \in \mathbb{R}$, so $E(\psi^{-1}(\alpha'))$ is a projector $\forall \alpha' \in \mathcal{N}$. Moreover, $E'(\emptyset) = E(\psi^{-1}(\emptyset)) = E(\emptyset) = 0$ and $E'(Y) = E(\psi^{-1}(Y)) = E(X) = I$. Hence we showed first property. Moreover, let $(\alpha'_j)_{n=1}^{\infty}$ be disjoint sets, then

$$E'(\bigcup_{n=1}^{\infty} \alpha'_j) = E(\psi^{-1}(\bigcup_{n=1}^{\infty} \alpha'_j))$$

by bijectivity of ψ^{-1},

$$= E(\bigcup_{j=1}^{\infty} \psi^{-1}(\alpha'_j))$$

since (α'_j) are disjoint, $\psi^{-1}(\alpha'_j)$ are disjoint too,

$$\begin{aligned} &= \sum_{j \geq 1} E(\psi^{-1}(\alpha'_j)) \\ &= \sum_{j \geq 1} E'(\alpha'_j). \end{aligned} \tag{6.3.2}$$

Hence, we construct the spectral measure. $\square$

Remark 6.3.1. $\psi^{-1}((F')^{-1}(z)) = (F' \circ \psi)^{-1}(z) \quad (z \in \mathbb{C})$. So, using this relation and construction of $\mathcal{N}$, we get

$$(F')^{-1}(\delta) \in \mathcal{N} \Longleftrightarrow \psi^{-1}((F')^{-1}(\delta)) \in \mathcal{M} \Longleftrightarrow (F' \circ \psi)^{-1}(\delta) \in \mathcal{M}.$$

This implies that F' is measurable with respect to $\mathcal{N}$ if and only if $F' \circ \psi$ is measurable with respect to $\mathcal{M}$. In particular following theorem gives the interpretation of change of variables.

Theorem 6.3.2. *Let $L_0(Y, \mathcal{N}, E')$ be the collection of complex valued functions defined on Y, measurable with respect to $\mathcal{N}$ and finite E'-almost everywhere. If $F' \in L_0(Y, \mathcal{N}, E')$, then $F' \circ \psi \in L_0(X, \mathcal{M}, E)$ and followings holds:*

$$\int_X F'(\psi(\lambda)) \,\mathrm{d}\, E(\lambda) = \int_Y F'(\lambda') \,\mathrm{d}\, E'(\lambda') = I_{F'}, \tag{6.3.3}$$

$$\begin{aligned} D(I_{F'}) &= \{f \in \mathcal{H} | \int_X |F'(\psi(\lambda))|^2 \,\mathrm{d}(E(\lambda)f, f) < \infty\} \\ &= \{f \in \mathcal{H} | \int_Y |F'(\lambda')|^2 \,\mathrm{d}(E'(\lambda')f, f) < \infty\}. \end{aligned} \tag{6.3.4}$$

Proof. Let $\alpha' = \{\lambda' \in Y |\ |F'(\lambda')| = \infty\}$, then $\psi^{-1}(\alpha') = \{\lambda \in \mathbb{R} |\ |F'(\psi(\lambda))| = \infty\}$. Thus, $E(\psi^{-1}(\alpha')) = E'(\alpha') = 0$ means that the function $F' \circ \psi$ is finite E-almost everywhere. By the Remark 6.3.1 we know that $E((F' \circ \psi(\lambda))) = 0$, that is $F' \circ \psi$ is finite E-almost everywhere or $F' \circ \psi \in L_0(X, \mathcal{M}, E)$. Hence we proved the first statement of the theorem. Now, Let $F' \in S(Y, \mathcal{N}, E')$, then for any disjoint sets $\alpha'_k \in \mathcal{N}$,

$$\begin{aligned} \int_Y F'(\lambda') \,\mathrm{d}\, E'(\lambda') &= \int_Y (\sum_{k=1}^{n} F'_k \chi_{\alpha'_k}(\lambda')) \,\mathrm{d}\, E'(\lambda') \\ &= \sum_{k=1}^{n} F'_k E'(\alpha'_k) = \sum_{k=1}^{n} F'_k E(\psi^{-1}(\alpha'_k)) \end{aligned}$$

using the fact that $\int_\alpha \mathrm{d}\, E(\lambda) = E(\alpha) \quad (\alpha \in \mathcal{M})$,

$$= \int_X (\sum_{k=1}^{n} F'_k \chi_{\psi^{-1}(\alpha'_k)}(\lambda)) \,\mathrm{d}\, E(\lambda)$$

by $\chi_{\alpha'}(\psi(\lambda)) = \chi_{\psi^{-1}(\alpha')}(\lambda) \quad \forall \alpha' \subseteq Y$,

$$= \int_X F'(\psi(\lambda)) \,\mathrm{d}\, E(\lambda).$$

Hence, for the special case $F' \in S(Y, \mathcal{N}, E')$, we are done. Note that for the case $F' \in L_\infty(Y, \mathcal{N}, E')$ equation (6.3.3) can be observed by the limit transition in the uniform approximation of F' by $F'_n \in S(Y, \mathcal{N})$. In the case where $F'_N(\psi(\lambda)) = (F' \circ \psi)_N(\lambda)$ and the fact that integrals in (6.3.4) are equal into account. This is a clear consequence of standart change of variables with respect to a scalar measure. $\square$

6.4 Product of Spectral Measures

Let $(R_1, \mathcal{R}_1)$, $(R_2, \mathcal{R}_2)$ be two measurable spaces with spectral measures E_1 and E_2 respectively. Suppose E_1 and E_2 commute for all $\alpha_1 \in \mathcal{R}_1$ and $\alpha_2 \in \mathcal{R}_2$. Denote $\mathcal{R}$ the direct product $\mathcal{R}_1 \times \mathcal{R}_2$. More precisely, it is the σ-algebra composed of all subsets of $\mathcal{R}_1 \times \mathcal{R}_2$ that belong to the σ span of all possible rectangles $\alpha_1 \times \alpha_2$ with $\alpha_1 \in \mathcal{R}_1,\ \alpha_2 \in \mathcal{R}_2$.

Now consider the following construction:

$$E(\alpha_1 \times \alpha_2) := E_1(\alpha_1) E_2(\alpha_2) \quad (\alpha_1 \in \mathcal{R}_1,\ \alpha_2 \in \mathcal{R}_2). \tag{6.4.1}$$

E is a spectral measure defined on $\mathcal{R}$:

Proof. $(E_1(\alpha_1)E_2(\alpha_2))^* = E_2^*(\alpha_2)E_1^*(\alpha_1)$. So,

$$\begin{aligned} (E(\alpha_1 \times \alpha_2))^* &= E_2^*(\alpha_2)E_1^*(\alpha_1) \\ &= E_2(\alpha_2)E_1(\alpha_1). \end{aligned}$$

By assumption E_1 and E_2 commute,

$$\begin{aligned} (E(\alpha_1 \times \alpha_2))^* &= E_1(\alpha_1)E_2(\alpha_2) \\ &= E(\alpha_1 \times \alpha_2). \end{aligned} \tag{6.4.2}$$

Thus, it is self adjoint. Moreover,

$$E^2(\alpha_1 \times \alpha_2) = E_1(\alpha_1)E_2(\alpha_2)E_1(\alpha_1)E_2(\alpha_2)$$

E_1 and E_2 commute,

$$
\begin{aligned}
&= E_1^2(\alpha_1)E_2^2(\alpha_2) \\
&= E_1(\alpha_1)E_2(\alpha_2) \\
&= E(\alpha_1 \times \alpha_2).
\end{aligned}
$$

That is, $E^2 = E$. Hence, E is a spectral measure defined on $\mathcal{R}$. This E is called the direct product of E_1 and E_2 and denoted by $E = E_1 \times E_2$. $\square$

Notice that the assumption of commuting spectral measures is essential.

Definition 6.4.1. Let R be a complete metric space and let $\mathcal{R} = \mathcal{B}(\mathbb{R})$ be the σ-algebra of its Borel subsets. The spectral measure defined on $\mathcal{R} = \mathcal{B}(\mathbb{R})$ is called Borel spectral measure.

Theorem 6.4.2. *Let E_1, E_2 be two commuting Borel spectral measures in the spaces R_1, R_2 respectively. Then* (6.4.1) *determines a unique spectral measures $E = E_1 \times E_2$ defined on $\mathcal{R} = \mathcal{B}(R_1 \times R_2)$.*

Proof. First recall that every scalar finite measure is automatically regular; i.e.

$$\mu(\alpha) = \inf\{\mu(o) |\ o \supseteq \alpha,\ o \ \ is \ \ open\} \quad (\alpha \in \mathcal{B}(R)) \tag{6.4.3}$$

$$\mu(\alpha) = \sup\{\mu(u) |\ u \supseteq \alpha,\ u \ \ is \ \ compact\} \quad (\alpha \in \mathcal{B}(R)) \tag{6.4.4}$$

Let $\mathcal{R}'$ be the algebra spanned by the collection of all rectangles $\alpha_1 \times \alpha_2$. By Theorem 6.1.10 it is enough to show that E is a spectral measure on $\mathcal{R}'$.

$\forall \alpha \in \mathcal{R}'$, $E(\alpha)$ is a projector, since α can be represented as finite union of disjoint rectangles, that is $E(\alpha)$ is the sum of finite projectors $E(\alpha_1 \times \alpha_2)$ which are mutually orthogonal. Moreover, $E(\emptyset) = 0$, $E(R_1 \times R_2) = 1$. We prove that, for every triangle $\alpha_1 \times \alpha_2$ ($\alpha_1 \in \mathcal{B}(R_1)$, $\alpha_2 \in \mathcal{B}(R_2)$), $f \in \mathcal{H}$ and $\epsilon > 0$, one can find two rectangles $o_1 \times o_2$ and $u_1 \times u_2$ where $o_i \supseteq \alpha_i \supseteq u_i$, o_i are open, u_i are compact, $(i = 1, 2)$, such that

$$
\begin{aligned}
&\langle E(o_1 \times o_2)f, f\rangle - \langle E(\alpha_1 \times \alpha_2)f, f\rangle < \epsilon, \\
&\langle E(\alpha_1 \times \alpha_2)f, f\rangle - \langle E(u_1 \times u_2)f, f\rangle < \epsilon.
\end{aligned} \tag{6.4.5}
$$

We will only show the first relation. Since scalar measures are regular, for given $\delta > 0$ we take open sets $o_1 \supseteq \alpha_1$ and $o_2 \supseteq \alpha_2$ such that $\langle E_1(o_1 \setminus \alpha_1)f, f\rangle < \delta$ and $\langle E_2(o_2 \setminus \alpha_2)f, f\rangle < \delta$. $(o_1 \times o_2) \setminus (\alpha_1 \times \alpha_2) = ((o_1 \setminus \alpha_1) \times o_2) \cup (\alpha_1 \times (o_2 \setminus \alpha_2))$, and notice that these two sets are disjoint. So

$$\begin{aligned}
\langle E(o_1 \times o_2)f, f\rangle &- \langle E(\alpha_1 \times \alpha_2)f, f\rangle \\
&= \langle E((o_1 \setminus \alpha_1) \times o_2)f, f\rangle + \langle E(\alpha_1 \times (o_2 \setminus \alpha_2))f, f\rangle \\
&= \langle E_1(o_1 \setminus \alpha_1)E_2(o_2)f, f\rangle + \langle E_1(\alpha_1)E_2(o_2 \setminus \alpha_2)f, f\rangle \\
&\leq ||f||(||E_1(o_1 \setminus \alpha_1)f|| + ||E_2(o_2 \setminus \alpha_2)f||) \\
&= ||f||(\langle E_1(o_1 \setminus \alpha_1)f, f\rangle^{1/2} + \langle E_2(o_2 \setminus \alpha_2)f, f\rangle^{1/2}) \\
&\leq 2\sqrt{\delta}||f||. \qquad (6.4.6)
\end{aligned}$$

By choosing δ such that $2\sqrt{\delta}||f|| = \epsilon$, we proved (6.4.5). Let $\alpha_1, \ldots, \in \mathcal{R}'$ be mutually disjoint sets such that $\alpha = \bigcup_{k=1}^{\infty} \alpha_k \in \mathcal{R}'$. In order to prove E is countably additive, it suffices to show

$$\langle E(\alpha)f, f\rangle \leq \sum_{k=1}^{\infty} \langle E(\alpha_k)f, f\rangle. \qquad (6.4.7)$$

Indeed, $\forall n \in \mathbb{N}$

$$\begin{aligned}
\sum_{k=1}^{n} \langle E(\alpha_k)f, f\rangle &= \langle E(\bigcup_{k=1}^{n} \alpha_k)f, f\rangle \\
&\leq \langle E(\alpha)f, f\rangle. \qquad (6.4.8)
\end{aligned}$$

Letting $n \to \infty$ we get $\sum_{k=1}^{\infty} \langle E(\alpha)f, f\rangle \leq \langle E(\alpha)f, f\rangle$. Thus, (6.4.7) is enough to prove countably additivity.

Since α is union of finitely many rectangles, we can apply (6.4.5) and find, for given $\epsilon > 0$, a compact set $U \subseteq \alpha$ such that $\langle E(\alpha)f, f\rangle - \langle E(U)f, f\rangle < \epsilon$. Similarly, open sets $O_k \supseteq \alpha_k$ such that $\langle E(O_k)f, f\rangle - \langle E(\alpha_k)f, f\rangle < \frac{\epsilon}{2^k}$. Suppose the family $(O_k)_{k=1}^{\infty}$ covers U. Since U is compact $\exists n \in \mathbb{N}$ such that $\bigcup_{k=1}^{n} O_k \supseteq U$.

Thus

$$\begin{aligned}
\langle E(\alpha)f,f\rangle &< \langle E(U)f,f\rangle + \epsilon \\
&\leq \langle E(\bigcup_{k=1}^{n} O_k)f,f\rangle + \epsilon \\
&\leq \sum_{k=1}^{\infty} \langle E(O_k)f,f\rangle + \epsilon \\
&< \sum_{k=1}^{\infty} \langle E(\alpha_k)f,f\rangle + 2\epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we conclude (6.4.7). Thus, E is indeed a spectral measure on $\mathcal{R}'$. □

6.5 Spectral Theorem for Selfadjoint Operators

In this section we suppose we know the spectral decomposition theorems for bounded cases.

Theorem 6.5.1. *Let A be an arbitrary selfadjoint operator. Then there exists a unique spectral measure E of A such that the following representation is true;*

$$A = \int_{\mathbb{R}} \lambda \, \mathrm{d}\, E(\lambda), \quad D(A) = \{f \in \mathcal{H} \mid \int_{\mathbb{R}} \lambda^2 \, \mathrm{d}(E(\lambda)f,f) < \infty\}. \tag{6.5.1}$$

In particular, $\mathbb{R}$ can be replaced by $\sigma(A)$, spectrum of the operator A.

Proof. Let A' be an arbitrary selfadjoint operator. Set $R' = \mathbb{R} \cup \{\infty\}$ and $R = \mathbb{T}$ (the unit circle). Consider, ψ, the inverse mapping of the following mapping;

$$\lambda' \mapsto \lambda = \frac{\lambda' + i}{\lambda' - i} \in \mathbb{R} \quad (\lambda' \in R'). \tag{6.5.2}$$

That is; $\forall \lambda \in R$,

$$\lambda \mapsto \psi(\lambda) = i\frac{\lambda + 1}{\lambda - 1} \in R'. \tag{6.5.3}$$

Consider the Cayley transformation U of A' for $z = i$,

$$U = (A' + iI)(A' - iI)^{-1}. \tag{6.5.4}$$

We know by properties of Cayley transform, U is unitary. Let E be its spectral measure of U. Denote E' the image of E under the mapping (6.5.3). Therefore, E' becomes spectral measure on R'. Note that $E'(\{\infty\}) = E(\{1\}) = 0$. Last equality follows by the fact that $\ker(U - I) = \{0\}$. So, E' is a spectral measure on R'. In view of (6.3.3) and (6.3.4),

$$\int_{\mathbb{R}} \lambda' \,\mathrm{d}\, E'(\lambda') = \int_{R'} \lambda' \,\mathrm{d}\, E'(\lambda') = \int_{\mathbb{T}} i\frac{\lambda+1}{\lambda-1} \,\mathrm{d}\, E(\lambda) = i(U+I)(U-I)^{-1} = A'. \tag{6.5.5}$$

That is (6.5.1) is proved. The statement, $\mathbb{R}$ can be replaced by $\sigma(A)$, follows by,

$$i(U+I)(U-I)^{-1} = \int_{\sigma(U)} i\frac{\lambda+1}{\lambda-1} \,\mathrm{d}\, E(\lambda) = \int_{\sigma(A)} \lambda' \,\mathrm{d}\, E'(\lambda').$$

Uniqueness of E' follows by the uniqueness of E. □

6.6 Commuting Operators

Definition 6.6.1. Let A be a bounded linear operator acting on $\mathcal{H}$. We say that an operator A commutes with an operator B if $AB \subseteq BA$ (notation: $A \smile B$).

Remark 6.6.2. If both operators are bounded, then commutation becomes in the equality sense, i.e. A, B two bounded operators are commute if $AB = BA$.

Theorem 6.6.3. *If a bounded operator B commutes with a spectral measure E, then it commutes with any $I_F := \int_R F(\lambda) \,\mathrm{d}\, E(\lambda)$ where F is measurable function on $(R, \mathcal{R})$ such that*

$$E(\{\lambda \in R \mid |F(\lambda)| = \infty\}) = 0. \tag{6.6.1}$$

Proof. $\forall f \in D(I_F)$, in view of (6.2.7)

$$\langle I_F Bf, g\rangle = \int_R F(\lambda) \,\mathrm{d}(E(\lambda)Bf, g).$$

By the assumption of commuting B and E (notice both are bounded),

$$\begin{aligned}
\langle I_F B f, g\rangle &= \int_R F(\lambda)\,\mathrm{d}(BE(\lambda)f, g) \\
&= \int_R F(\lambda)\,\mathrm{d}(E(\lambda)f, B^* g) \\
&= \langle I_F f, B^* g\rangle \\
&= \langle B I_F f, g\rangle.
\end{aligned}$$

That is, $BI_F \subseteq I_F B$. □

Theorem 6.6.4. *Let B be a bounded operator and V be a unitary operator. Then they commutes if and only if M commutes with F_V, spectral measure of V.*

Proof. One side follows directly by Theorem (6.6.3). To prove inverse implication, suppose M commutes with V. Notice both operators are bounded, so $MV = VM$. Let $V = \int_{\mathbb{T}} \lambda \,\mathrm{d}\, F_V(\lambda)$ be spectral decomposition of the unitary operator V. In view of $VM = MV$ and (6.2.7) we have; $\forall f, g \in \mathcal{H}$,

$$\langle MVf, g\rangle = \langle VMf, g\rangle = \int_{\mathbb{T}} \lambda\,\mathrm{d}(F_V(\lambda)Mf, g), \tag{6.6.2}$$

$$\langle MVf, g\rangle = \langle Vf, M^* g\rangle = \int_{\mathbb{T}} \lambda\,\mathrm{d}(F_V(\lambda)f, M^* g). \tag{6.6.3}$$

Note that by simple change of variables we get $\int_{\mathbb{T}} \lambda \,\mathrm{d}\,\mu(\lambda) = \int_{\mathbb{R}} e^{i\lambda t}\,\mathrm{d}\,\tilde{\mu}(\lambda)$. Thus, by using the uniqueness of Fourier Transform of measures (Theorem (1.0.5)) together with (6.6.3) and (6.6.2) we conclude $\langle F_V M f, g\rangle = \langle F_V f, M^* g\rangle$. That is $F_V M = M F_V$. □

Theorem 6.6.5. *Let B be a bounded operator. Then B commutes with a selfadjoint operator A if and only if B commutes with E_A, spectral measure of A.*

Proof. One side follows by Theorem (6.6.3).

Denote $R_z(A) = (A - zI)^{-1}$. For sufficiently large $|z|$, $(R_{\overline{z}}(A))^{-1}$ exists and equals to $-\sum_{n=0}^{\infty} (\overline{z})^{-n-1} A^n$. Thus, B commutes with A implies B commutes with $R_{\overline{z}}(A)$. Let U be the Cayley transform of A. Then

$$\begin{aligned}U &= (A+zI)(A-\overline{z}I)^{-1} \\ &= (A-\overline{z}I+(z+\overline{z})I)(A-\overline{z}I)^{-1} \\ &= I+(z+\overline{z})(A-\overline{z}I)^{-1} \\ &= I+(z+\overline{z})R_{\overline{z}}(A),\end{aligned}$$

in which implies B commutes with U. By Theorem (6.6.4), B commutes with F_U, the spectral measure of U. So in view of the Cayley transform, B commutes with E_A. □

6.7 Spectral Theorem for Normal Operators

Theorem 6.7.1. *Let A be nonnegative selfadjoint operator. Then there exists a unique nonnegative selfadjoint operator B such that $B^2 = A$.*

Proof. Let E be the spectral measure of A. A is selfadjoint, so $\sigma(A) \subseteq \mathbb{R}$. For $\lambda \in \mathbb{R}$, the function $\sqrt{\lambda}$ is real and so define;

$$B := \int_{\mathbb{R}^+} \sqrt{\lambda}\, \mathrm{d}\, E(\lambda). \tag{6.7.1}$$

Note that

$$B^* = (\int_{\mathbb{R}^+} \sqrt{\lambda}\, \mathrm{d}\, E(\lambda))^*$$

in view of Theorem (6.2.8) and the fact that $\sqrt{\lambda}$ is real on $\mathbb{R}^+$,

$$\begin{aligned}&= \int_{\mathbb{R}^+} \sqrt{\lambda}\, \mathrm{d}\, E(\lambda) \\ &= B.\end{aligned}$$

That is, B is selfadjoint. Moreover, in view of (6.2.7) for unbounded measurable functions;

$$\langle Bf, f\rangle = \int_{\mathbb{R}^+} \sqrt{\lambda}\, \mathrm{d}(E(\lambda)f, f) \geq 0.$$

Thus, B is nonnegative selfadjoint operator.Further, in view of (6.2.18) and the fact that B is closed we have

$$\begin{aligned} A &= \int_{\mathbb{R}^+} \lambda \, \mathrm{d}\, E(\lambda) \\ &= \overline{\int_{\mathbb{R}^+} \sqrt{\lambda} \, \mathrm{d}\, E(\lambda) \int_{\mathbb{R}^+} \sqrt{\lambda} \, \mathrm{d}\, E(\lambda)} \end{aligned}$$

Since A is closed,

$$\begin{aligned} &= \int_{\mathbb{R}^+} \sqrt{\lambda} \, \mathrm{d}\, E(\lambda) \int_{\mathbb{R}^+} \sqrt{\lambda} \, \mathrm{d}\, E(\lambda) \\ &= B^2. \end{aligned}$$

□

Lemma 6.7.2. *Let T be closed densely defined operator on $\mathcal{H}$. By Theorem* (5.2.1)*, T^*T is selfadjoint and nonnegative on $\mathcal{H}$. Define*

$$K = (T^*T)^{1/2}. \tag{6.7.2}$$

The operator (6.7.2) *is the unique selfadjoint nonnegative operator on $\mathcal{H}$ satisfying,*

$$D(K) = D(T), \quad ||Kf|| = ||Tf||, \quad \forall f \in D(T). \tag{6.7.3}$$

Proof. Proof follows directly by Theorem (6.7.1) and the fact that;

$$||Kf||^2 = \langle K^2 f, f\rangle = \langle T^*Tf, f\rangle = ||Tf||^2.$$

□

The operator (6.7.2) *is called the modulus of T and denoted by $K = |T|$. Note that by* (6.7.3)

$$\ker |T| = \ker T, \tag{6.7.4}$$

which implies

$$\overline{R(|T|)} = [\ker |T|]^{\perp} = [\ker T]^{\perp} = \overline{R(T^*)}. \tag{6.7.5}$$

Definition 6.7.3. A partial isometry is a linear map W between Hilbert spaces H_1 and H_2 such that the restriction of W to the orthogonal complement of its kernel is an isometry. We call the orthogonal complement of the kernel of W the initial subspace of W, and the range of W is called the final subspace of W.

Theorem 6.7.4. *Let T be closed densely defined operator on $\mathcal{H}$ and $K = |T|$ be he operator* (1). *Then there exists a unique partially isometric operator V acting on $\mathcal{H}$ with initial space $\overline{R(T^*)}$ and range $\overline{R(T)}$ such that*

$$T = V|T|. \tag{6.7.6}$$

Proof. Let $x = Kf$ for some $f \in D(T)$. Define $Vx = Tf$. Let $Vx_1 = Vx_2$ then $Tf_1 = Vx_1 = Vx_2 = Tf_2$ implies $f_1 = f_2$, that is $x_1 = Kf_1 = Kf_2 = x_2$. So V is well defined. By (6.7.3) V is isometric, and its clear that (6.7.6) follows by $Tf = VKf$. Note that $D(V) = R(K) \cap D(T)$ and $D(T)$ is dense in $\mathcal{H}$. So we can extend V by continuity to $\overline{R(K)} = \overline{R(T^*)}$. Define V to be zero on $\ker K = \ker T$. Uniqueness of V is now direct consequence of the construction. □

Representation (6.7.6) is called the polar decomposition of T.

Lemma 6.7.5. *Under the hypotheses of Theorem* (6.7.4)*, the following 'antipolar' decomposition hold*

$$T = |T^*|V, \quad T^* = |T|V^*. \tag{6.7.7}$$

Proof. Let $K = (T^*T)^{1/2}$ and $g \in D(T) = D(K)$. Then for $f \in D(T) = D(K) = D(K^*)$,

$$\langle Tg, Vf\rangle = \langle VKg, Vf\rangle = \langle V^*VKg, f\rangle.$$

Since V is isometric on $\overline{R(T^*)}$ and $Kg \in R(K) \subseteq \overline{R(T^*)}$ by (6.7.5),

$$\langle Tg, Vf\rangle = \langle Kg, f\rangle = \langle g, Kf\rangle.$$

That is, $Vf \in D(T^*)$ and $T^*Vf = Kf$, or equivalently $K \subseteq T^*V$. Moreover, let $h \in D(T^*)$ and $g \in D(T)$, then

$$\langle T^*h, g\rangle = \langle h, Tg\rangle = \langle h, VKg\rangle = \langle V^*h, Kg\rangle.$$

That is, $V^*h \in D(K^*) = D(K)$ and $T^*h = KV^*h$. Hence, $T^* \subseteq KV^*$. Using (6.7.5), the facts $VV^* = \mathcal{P}_{\overline{R(T)}} = \mathcal{P}_{[\ker T^*]^\perp}$, $V^*V = \mathcal{P}_{\overline{R(T^*)}} = \mathcal{P}_{[\ker K]^\perp}$ implies that $(KV^*)V = K$. These with together $T^* \subseteq KV^*$ and $K \subseteq T^*V$ implies the followings;

$$KV^* \subseteq (T^*V)V^* = T^* \subseteq KV^*,$$

$$TV^* \subseteq (KV).$$

Hence we are done for the second equality.

In order to prove the first equality one can rewrite Theorem 6.7.4 for T^* and deduce that $U = V^*$ where U is the corresponding partial isometry for T^*. Hence first relation follows by second equality with the fact that $U = V^*$. $\square$

Remark 6.7.6. Suppose T is normal, i.e T is closed densely with $T^*T = TT^*$. Then proving the Lemma (6.7.2) for $K = (TT^*)^{1/2}$ we get

$$K = |T| = |T^*|, \quad D(K) = D(T) = D(T^*). \tag{6.7.8}$$

$$\ker T^* = \ker T = \ker |T|. \tag{6.7.9}$$

$$R(T^*) = R(|T|) = R(|T^*|) = R(T). \tag{6.7.10}$$

Theorem 6.7.7. *Let T be a normal operator on $\mathcal{H}$. Then $V, |T|$ in (6.7.6) commute. The restriction of V to $\overline{R(|T|)} = \overline{R(T)}$ is unitary.*

Proof. By (6.7.10) V is unitary on $\overline{R(T)}$. It follows from (6.7.8), (6.7.6) (6.7.7) for T that $|T|$ commutes with V. $\square$

Theorem 6.7.8. *Let T be a normal operator (not necessarily bounded) on $\mathcal{H}$. Then there exists a unique spectral measure $E = E_T$ defined on the Borel subsets of $\mathbb{C}$ such that*

$$T = \int_{\mathbb{C}} z \, \mathrm{d}\, E_T(z). \tag{6.7.11}$$

Proof. **Simple Case**: $\ker T = \{0\}$

Consider the polar decomposition of T, $T = V|T|$. By Theorem (6.7.7) $K = |T|$ and V commutes. By (6.7.9) we have $\ker T^* = \ker T = \{0\}$. We know that

K is positive selfadjoint, so ,its corresponding spectral measure is supported on positive real axis; and spectral measure of unitary operator V is supported on unit dist $\mathbb{T}$. Since $K \smile V$, the spectral measures E_K and F_V commutes. Recall that this was the only requirement of well defined product spectral measure. Now consider $\widetilde{E}$ of their product spectral measure, which is a Borel measure defined on $\mathbb{R}_+ \times \mathbb{T}$.

Consider the map π defined on $\mathbb{R}_+ \times \mathbb{T}$ to $\mathbb{C}$ by $z = r\eta$ $(r \in \mathbb{R}_+,\ \eta \in \mathbb{T})$. The image of $\widetilde{E}$ under the map π is a spectral measure, denote with E_τ. Note that by construction of π, $E_\tau\{0\} = E_K\{0\} = 0$.

Thus, by using change of variables for spectral measures and the facts that $E_K(\mathbb{R}_+) = F_V(\mathbb{T}) = 1$, we have

$$\int_{\mathbb{C}} |z| \,\mathrm{d}\, E_\tau(z) = \int_{\mathbb{R}_+ \times \mathbb{T}} r \,\mathrm{d}\, \widetilde{E}(r, \eta) = K = |T|, \tag{6.7.12}$$

$$\int_{\mathbb{C}} |z^{-1}| z \,\mathrm{d}\, E_\tau(z) = \int_{\mathbb{R}_+ \times \mathbb{T}} \eta \,\mathrm{d}\, \widetilde{E}(r, \eta) = V. \tag{6.7.13}$$

These equations together with multiplication property for spectral integrals yield representation (6.7.11) for $T = VK$. It is established simultaneously that

$$D(T) = D(T^*) = D(|T|) = \{f \in \mathcal{H} | \int |z|^2 \,\mathrm{d}\, \mu_f(z) < \infty\}, \tag{6.7.14}$$

$$T^* = \int \overline{z} \,\mathrm{d}\, E_\tau(z). \tag{6.7.15}$$

General Case: $\ker T \neq \{0\}$

Let T_0 be the restriction of T to the subspace $[\ker T]^\perp = \overline{R(T)}$. Then by above, T_0 admists representation (6.7.11) with corresponding spectral measure E_{τ_0} on $\overline{R(T)}$. Consider now spectral measure E^0 on $\ker T$ defined by $E^0(\mathbb{C}\backslash\{0\}) = 0$, $E^0(\{0\}) \ker T = \ker T$, and define $E_\tau = E_{\tau_0} \oplus E^0$. It is clear by the properties of the orthogonal sum that E_τ is a spectral measure on $\mathcal{H}$ satisfying (6.7.11).

Uniqueness follows directly by the uniqueness of spectral measures E_K and F_V. □

Chapter 7

Criteria of Selfadjointness

7.1 Stone's Theorem, Operator Differential Equations

Let A be a selfadjoint operator acting on $\mathcal{H}$, and let E be its spectral measure. We construct the operator valued function

$$t \mapsto U(t) = \int_{\mathbb{R}} e^{it\lambda} \, \mathrm{d}\, E(\lambda) := e^{itA} \quad (\lambda, t \in \mathbb{R}). \tag{7.1.1}$$

In view of (6.2.8) for bounded functions, $||U(t)f||^2 = \int_{\mathbb{R}} \mathrm{d}(E(\lambda)f, f) = ||f||^2$. That is, U is unitary for each $t \in \mathbb{R}$. Moreover

$$U(t+s) = U(t)U(s) \quad (t, s \in \mathbb{R}). \tag{7.1.2}$$

Claim 1: The function defined on (7.1.1) is strongly continuous, i.e, $(\forall f \in \mathcal{H})$ $(\forall t \in \mathbb{R}) : U(s)f \to U(t)f$ as $s \to t$.

Proof of Claim 1. By relation (6.2.8) we obtain

$$\begin{aligned} ||U(s)f - U(t)f||^2 &= || \int_{\mathbb{R}} (e^{is\lambda} - e^{it\lambda}) \, \mathrm{d}\, E(\lambda) f||^2 \\ &= \int_{\mathbb{R}} |e^{is\lambda} - e^{it\lambda}|^2 \, \mathrm{d}(E(\lambda)f, f) \to 0 \quad as \quad s \to t. \end{aligned}$$

Last conclusion follows by Lebesgue Dominated Convergence Theorem, with the dominating function "2". □

Claim 2: $U(t)$ is strongly continuously differentiable, i.e. $\forall f \in D(A)$ and $t \in \mathbb{R}$, the strong derivative

$$U'(t)f = \lim_{h\to 0} \frac{(U(t+h) - U(t))f}{h} = iU(t)Af \tag{7.1.3}$$

exists and is a continuous vector function.

proof of Claim 2.

$$\begin{aligned} ||iU(t)Af - \frac{(U(t+h) - U(t))f}{h}||^2 &= ||\int_{\mathbb{R}} (ie^{it\lambda}\lambda - \frac{1}{h}(e^{i(t+h)\lambda} - e^{it\lambda}))\, \mathrm{d}\, E(\lambda)f||^2 \\ &= \int_{\mathbb{R}} |i\lambda - \frac{1}{h}(e^{ih\lambda} - 1)|^2\, \mathrm{d}(E(\lambda f, f). \end{aligned} \tag{7.1.4}$$

We used the facts that $e^{it\lambda}$ is bounded, and multiplication of spectral integrals by virtue of Theorem (6.2.7). Now, consider the functional g, from $\mathbb{R}$ to $\mathbb{C}$ defined by,

$$g(h) = \begin{cases} \frac{e^{ih\lambda}-1}{h} & : \quad h \neq 0 \\ i\lambda & : \quad h = 0 \end{cases}$$

Then clearly, g is continuous function at $h = 0$. Thus, as $h \to 0$, $|i\lambda - \frac{e^{ih\lambda}-1}{h}|^2 \to 0$. Note that for given $f \in D(A)$, $iU(t)Af$ is a continuous vector function. Now (7.1.3) follows by and using Lebesgue Dominated Convergence Theorem. □

Consider an operator differential equation

$$u'(t) = iAu(t) \quad (t \in \mathbb{R}) \tag{7.1.5}$$

where $t \mapsto u(t) \in \mathcal{H}$ is the required solution. The function u is assumed to be strongly continuously differentiable and $u(t) \in D(A) \quad \forall t$. Such solutions are called strong. The strong solution of (7.1.5) satisfying the initial condition $u(0) = u_0 \in D(A)$, i.e., the solution of the corresponding Cauchy problem, exists and is given by the formula

$$u(t) = U(t)u_0 \quad (t \in \mathbb{R}). \tag{7.1.6}$$

This Cauchy problem is uniquely solvable. In more details we will consider this problem later. We say that a function from $\mathbb{R}$ to unitary operators in $\mathcal{H}$ defined by, $t \mapsto U(t)$ defines a one-parameter unitary group.

Note that, (7.1.2) implies $U(t)$ are pairwise commuting and $U(0) = 1$, $U(-t) = [U(t)]^{-1}$. Thus, (7.1.1) is an example of a one-parameter unitary group which is, in addition, strongly continuous.

Theorem 7.1.1 (Stone's Theorem)**.** *A strongly continuous one-parameter unitary group $U(t)$, $(t \in \mathbb{R})$ always admits representation* (7.1.1) *with a certain spectral measure E uniquely determined for a given group. The corresponding operator A is called the infinitesimal generator of this group.*

Proof. Let us construct a linear dense set $\mathcal{D} \subset \mathcal{H}$ and such that

$$t \mapsto U(t)f \quad (t \in \mathbb{R}) \tag{7.1.7}$$

is strongly continuously differentiable $\forall f \in \mathcal{D}$. Let $\mathcal{D}$ be the set of collection of all linear combinations of vectors of the form;

$$g_F := \int_{\mathbb{R}} F(s)U(s)g \, \mathrm{d}\, s \quad F \in C_0^\infty \quad \textit{and} \quad g \in \mathcal{H}. \tag{7.1.8}$$

(7.1.8) indeed exists since $F(.)U(.)f$ is continuous and F has compact support.

Claim 1: $\mathcal{D}$ is dense in $\mathcal{H}$.

Proof of Claim 1. Let $h \perp \mathcal{D}$. Then $\forall F \in C_0^\infty(\mathbb{R})$,

$$0 = \langle g_F, h\rangle = \int_{\mathbb{R}} F(s)\langle U(s)g, h\rangle \, \mathrm{d}\, s.$$

Define $\psi(s) = \langle U(s)g, h\rangle$. It is continuous function and so define the distribution $T_\psi(F) = \int_{\mathbb{R}} F(s)\langle U(s)g, h\rangle \, \mathrm{d}\, s$. We get $T_\psi(F) = 0$ $\forall F \in C_0^\infty$ that is, $T_\psi = 0$. Hence by Theorem (1.0.8) $\psi = 0$ almost everywhere on $\mathbb{R}$. Since ψ is continuous $\psi \equiv 0$ on $\mathbb{R}$. Let $s = 0$, then $U(0) = 1$ by (7.1.2) and we obtain, $\langle g, h\rangle = 0$ $(\forall g \in \mathcal{H})$, that is, $h = 0$. Therefore, $\mathcal{D}$ is dense in $\mathcal{H}$. $\square$

Claim 2: (7.1.7) is strongly continuously differentiable for all $f \in \mathcal{D}$.

Proof of Claim 2. It suffices to consider f of the form (7.1.8). By using (7.1.2) and change of variables we get ($\forall t, h \in \mathbb{R}$)

$$\frac{1}{h}(U(t+h)g_F - U(t)g_F) = \frac{U(t+h)}{h}\int_{\mathbb{R}} F(s)U(s)g\,\mathrm{d}\,s - \frac{U(t)}{h}\int_{\mathbb{R}} F(s)U(s)g\,\mathrm{d}\,s.$$

Both $U(t+h)$ and $U(t)$ are independent for the variable s. So

$$\begin{aligned}
\frac{1}{h}(U(t+h)g_F - U(t)g_F) &= \frac{1}{h}\int_{\mathbb{R}} F(s)(U(t+h)U(s) - U(t)U(s))g\,\mathrm{d}\,s \\
&= \frac{1}{h}\int_{\mathbb{R}} F(s)(U(t+h+s)g\,\mathrm{d}\,s - \frac{1}{h}\int_{\mathbb{R}} F(s)U(t+s))g\,\mathrm{d}\,s \\
&= \int_{\mathbb{R}} \frac{1}{h}(F(s-t-h) - F(s-t))U(s))g\,\mathrm{d}\,s \\
&\to -\int_{\mathbb{R}} F'(s-t)U(s)g\,\mathrm{d}\,s \quad (as \quad h \to 0) \qquad (7.1.9) \\
&= g_{-F'(.-t)} = U'(t)g_F. \qquad (7.1.10)
\end{aligned}$$

Note that (7.1.9), $\frac{1}{h}(F(s-t-h) - F(s-t))U(s))g$ converges pointwisely to $F'(s-t)U(s)g$. So (7.1.9) follows by Lebesgue Dominated Convergence Theorem with the following inequality

$$||\int_{\mathbb{R}} F'(s-t)U(s)g\,\mathrm{d}\,s|| \leq \sup_{s\in\mathbb{R}} |F'(s)|.\int_{\mathbb{R}} ||U(s)g||\,\mathrm{d}\,s \quad (F' \in C_0(\mathbb{R});\ g \in \mathcal{H}), \qquad (7.1.11)$$

and (7.1.10) follows by simple change of variables. □

Claim 3: $U'(t)g_F$ on the right side of (7.1.10) is strongly continuous.

Proof of Claim 3. This is a clear consequence of Lebesgue Dominated Convergence Theorem with the equation (7.1.11). □

Construct an operator A on the space $\mathcal{D}$ by the formula

$$f \mapsto Af = \frac{1}{i}U'(0)f \quad (f \in \mathcal{D}). \qquad (7.1.12)$$

Claim 4: A is Hermitian.

Proof of Claim 4. It suffices to show that $\langle Ag_F, m_K\rangle = \langle g_F, Am_K\rangle$ where $F, K \in C_0^\infty(\mathbb{R})$ and $g, m \in \mathcal{H}$. By (7.1.10),

$$\langle Ag_F, m_K\rangle = \lim_{h\to 0}\langle \frac{1}{ih}(U(h)-1)g_F, m_K\rangle.$$

Since A is Hermitian and $U^*(t) = U(-t)$,

$$\begin{aligned}\langle Ag_F, m_K\rangle &= \lim_{h\to 0}\langle g_F, -\frac{1}{ih}(U(-h)-1)m_K\rangle \\ &= \lim_{H\to 0}\langle g_F, \frac{1}{iH}(U(H)-1)m_K\rangle \\ &= \langle g_F, Am_K\rangle. \end{aligned}$$

□

Claim 5: A is essentially selfadjoint.

Proof of Claim 5. In order to prove $\widetilde{A}$ is selfadjoint we will use Theorem 4.1.5 and so its enough to show that $\ker(A^* \mp zI) = \{0\}$ for some $z \in \mathbb{C}\setminus\mathbb{R}$. To show this we first show that

$$U'(t)g_F = iAU(t)g_F \quad (F \in C_0^\infty(\mathbb{R}),\ g \in \mathcal{H}, t \in \mathbb{R}). \tag{7.1.13}$$

Initially, note that using multiplication of spectral integrals and the fact that $U(t)$ is unitary we can say that $U(t)g_F$ is of the form (7.1.8) and so $AU(t)g_F$ makes sense. Now compute $AU(t)g_F = g_{F(\cdot - t)}$ according to (7.1.12),

$$iAU(t)g_F = U'(0)U(t)g_F.$$

In the view of (7.1.10) for $t = 0$,

$$\begin{aligned} iAU(t)g_F &= \lim_{h\to 0}\frac{1}{h}(U(h)-1)U(t)g_F \\ &= \lim_{h\to 0}\frac{1}{h}(U(t+h)-U(t))g_F \\ &= U'(t)g_F. \end{aligned} \tag{7.1.14}$$

For any $F \in C_0^\infty$, $g \in \mathcal{H}$ and $\psi \in \ker(A^* - zI)$ in the view of (7.1.13),

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\,t}\langle U(t)g_F, \psi\rangle &= \langle U'(t)g_F, \psi\rangle \\ &= i\langle AU(t)g_F, \psi\rangle \\ &= i\langle U(t)g_F, A^*\psi\rangle. \end{aligned}$$

Using the fact $\psi \in \ker(A^* - zI)$

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\langle U(t)g_F, \psi\rangle = i\bar{z}\langle U(t)g_F, \psi\rangle \quad (t \in \mathbb{R}).$$

Denoting $\alpha(t) := \langle U(t)g_F, \psi\rangle$ and solving this first order differential equation we get

$$\alpha(t) = \langle U(t)g_F, \psi\rangle = e^{i\bar{z}t}\alpha(0) \quad (t \in \mathbb{R}).$$

Since $\mathrm{Im}z \neq 0$, $e^{i\bar{z}t}$ is unbounded, thus in order to be $\alpha(t)$ bounded it is necessary and sufficient that $\alpha(0) = 0$. That is, $\langle g_F, \psi\rangle = 0$ or equivalently $\psi \perp \mathcal{D}$. So, $\psi = 0$ and $\ker(A^* - zI) = \{0\}$. Similarly, one can deduce by doing same calculations, $\ker(A^* + zI) = \{0\}$. □

Since $\widetilde{A}$ is selfadjoint, by the formula (7.1.1), we can construct $V(t) = e^{it\widetilde{A}}$ $(t \in \mathbb{R})$. We will show that $U(t) = V(t)$, $\forall t \in \mathbb{R}$. Indeed, by (7.1.5) and (7.1.6), $v(t) = V(t)v_0$, $v_0 \in D(\widetilde{A})$ is a strong solution for

$$v'(t) = i\widetilde{A}v(t) \quad (t \in \mathbb{R}, v(0) = v_0). \tag{7.1.15}$$

But on the other hand, by (7.1.13), $u(t) = U(t)g_F$ (for some $F \in C_0^\infty(\mathbb{R}), g \in \mathcal{H}$) is also a strong solution for (7.1.15). Later in Theorem (7.2.1) we will show that for essentially selfadjoint operators (7.1.15) is uniquely solvable. Thus, $U(t) = V(t)$, $t \in \mathbb{R}$. In order to prove the uniqueness of the spectral measure E it is enough to consider the uniqueness of the Fourier transform of the measure $\rho(\alpha) = \langle E(\alpha)f, g\rangle$. Indeed $\forall f, g \in \mathcal{H}$,

$$\langle U(t)f, g\rangle = \int_{\mathbb{R}} e^{it\lambda}\,\mathrm{d}(E(\lambda)f, g) \quad (t \in \mathbb{R}). \tag{7.1.16}$$

Left hand side is the Fourier transform of $\rho(\alpha) = \langle E(\alpha)f, g\rangle$, which is unique by Theorem (1.0.5). Thus, E is uniquely determined. □

Remark 7.1.2. The strong continuity of a one-parameter unitary group is equivalent its weak continuity. Indeed it is enough to prove only weak continuity implies strong continuity. Precisely, for $f \in \mathcal{H}$,

$$\begin{aligned} ||U(t)f - U(s)f||^2 &= \langle U(t)f - U(s)f, U(t)f - U(s)f\rangle \\ &= \langle (U(t) - U(s))^*(U(t) - U(s))f, f\rangle. \\ &= \langle (2I - U(t-s) - U(s-t))f, f\rangle \to 0 \;\; as \;\; s \to t. \end{aligned}$$

Remark 7.1.3. Throughout the chapter, If we do not specify the boundary conditions of a Cauchy problem, we mean the trivial Boundary Value Cauchy problem; that is $u(0) = u'(0) = \ldots = u^{r-1}(0) = 0$.

We say that a vector function $u(t)$ is a strong solution of the equation

$$(\frac{\mathrm{d}^r u}{\mathrm{d}\, t^r})(t) + Bu(t) = 0 \quad (t \in I) \tag{7.1.17}$$

on I if it is r times strongly continuously differentiable (i.e., has r strong derivatives on I, the last of which is continuous), $u(t) \in D(B)$ for all $t \in I$, and equation (7.1.17) is satisfied.

An r times continuously differentiable vector function $u(t)$ is a strong solution of the equation;

$$(\frac{\mathrm{d}^r u}{\mathrm{d}\, t^r})(t) + B^* u(t) = 0 \quad (t \in I) \tag{7.1.18}$$

if and only if

$$\langle (\frac{\mathrm{d}^r u}{\mathrm{d}\, t^r})(t), f \rangle + \langle u(t), Bf \rangle = 0 \quad (f \in D(B); t \in I) \tag{7.1.19}$$

holds. Indeed by definition of strong solution and the fact that $u(t) \in D(B^*)$, $\forall t \in I$ these equations are equivalent.

We say that the Cauchy problem for the equation (7.1.17) on $I = [0, b)$ $(0 < b \leq \infty)$ is uniquely solvable in the strong sense if each strong solution of this equation on $[0, b)$ such that $u(0) = \ldots = u^{(r-1)}(0) = 0$ vanishes for all $t \in (0, b)$.

Remark 7.1.4. If the Cauchy problem is uniquely solvable on $I = [0, b)$ $(0 < b \leq \infty)$, then it is uniquely solvable on $[0, \infty)$.

Proof. Let $u(t)$ be strong solution of (7.1.17) on $[0, \infty)$ such that $u(0) = \ldots = u^{r-1}(0) = 0$. By assumption of unique solvability on $[0, b)$ we have $u(t) = 0$ on $[0, b)$. Consider the point $c = b/2$. $u(c) = \ldots = u'(c) = 0$ and define $u_1(t) = u(t + c)$. Note that $u_1(t)$ satisfies (7.1.17) since $u(t)$ satisfies (7.1.17) on

$[0,\infty)$. Moreover, $u_1(t) \in D(B)$ and

$$\begin{aligned}\lim_{h\to 0} \frac{(u_1(t+h) - u_1(t))f}{h} &= \lim_{h\to 0} \frac{(u(t+h+c) - u(t+c))f}{h} \\ &= iu(t+c)Af \\ &= iu_1(t)Af.\end{aligned}$$

That is, $u_1(t)$ is strongly continuously differentiable. Hence $u_1(t)$ is a strong solution of (7.1.17) on $[0,\infty)$. So by uniquely solvability on $[0,b)$, $u_1(t) = 0$ on $[0,b)$. That is, $u(t) = 0$ on $[0, b+c)$. Repeating this argument we conclude that $u(t) = 0$ on $[0,\infty)$ as desired. $\square$

We consider two examples in which we assume A is selfadjoint operator in $\mathcal{H}$. Moroever, for the second example we assume further A is positive.

Example 7.1.5. The formal solution of the Cauchy problem

$$(\frac{\mathrm{d}\, u}{\mathrm{d}\, t})(t) + \epsilon A u(t) = 0 \quad (t \in [0,\infty); u(0) = u_0) \tag{7.1.20}$$

has the form

$$u(t) = \int_{\mathbb{R}} e^{-\epsilon t \lambda} \,\mathrm{d}\, E(\lambda) u_0 = e^{-\epsilon t A} u_0 \quad (t \in [0,\infty)). \tag{7.1.21}$$

Just as in the case of (7.1.1) (or as in the case of Stone's Theorem) (7.1.21) is a strong solution of the Cauchy problem (7.1.20) if $u_0 \in D(Ae^{-\epsilon t A})$. Indeed, for $f \in D(A)$

$$\langle \frac{\mathrm{d}\, u}{\mathrm{d}\, t}(t), f\rangle = \frac{\mathrm{d}}{\mathrm{d}\, t} \int_{\mathbb{R}} e^{-\epsilon t \lambda} \,\mathrm{d}(E(\lambda)u_0, f).$$

We can change the order of integration and differentiaon using Fubini's Theorem and the fact that $u_0 \in D(Ae^{-\epsilon t A})$.

$$\begin{aligned}\langle \frac{\mathrm{d}\, u}{\mathrm{d}\, t}(t), f\rangle &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\, t} e^{-\epsilon t \lambda} \,\mathrm{d}(E(\lambda)u_0, f) \\ &= \int_{\mathbb{R}} -\epsilon \lambda e^{-\epsilon t \lambda} \,\mathrm{d}(E(\lambda)u_0, f) \\ &= \int_{\mathbb{R}} e^{-\epsilon t \lambda} \,\mathrm{d}(E(\lambda)u_0, -\bar{\epsilon} A f) \\ &= -\langle u(t), (\epsilon A)^* f\rangle.\end{aligned}$$

Hence by (7.1.19) we show that $u(t)$ is indeed a strong solution (7.1.20) if $u_0 \in D(Ae^{-\epsilon tA})$.

Example 7.1.6. The formal solution of the Cauchy problem

$$(\frac{\mathrm{d}^2 u}{\mathrm{d}\,t^2})(t) + Au(t) = 0 \quad (t \in [0,\infty); \quad u(0) = u_0, \quad u'(0) = u_1) \tag{7.1.22}$$

has the form

$$\begin{aligned} u(t) &= \int_{\mathbb{R}} \cos\sqrt{\lambda}t \,\mathrm{d}\,E(\lambda)u_0 + \int_{\mathbb{R}} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}} \,\mathrm{d}\,E(\lambda)u_1 \\ &= (\cos\sqrt{A}t)u_0 + (\frac{\sin\sqrt{A}t}{\sqrt{A}})u_1 \quad (t \in [0,\infty)). \end{aligned} \tag{7.1.23}$$

Indeed for $f \in D(A)$,

$$\langle \frac{\mathrm{d}\,u}{\mathrm{d}\,t}(t), f\rangle = \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{R}} \cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{R}} \frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,\mathrm{d}(E(\lambda)u_1, f).$$

We can change the order of integration and differentiaon using Fubini's Theorem and the facts that first integrant is bounded and $u_1 \in D(\sqrt{|A|})$

$$\begin{aligned} \langle \frac{\mathrm{d}\,u}{\mathrm{d}\,t}(t), f\rangle &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,t}\cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,t}\frac{\sin\sqrt{\lambda}t}{\sqrt{\lambda}}\,\mathrm{d}(E(\lambda)u_1, f) \\ &= \int_{\mathbb{R}} -\sqrt{\lambda}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \int_{\mathbb{R}} \cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, f). \end{aligned}$$

$$\Rightarrow \langle \frac{\mathrm{d}^2 u}{\mathrm{d}\,t^2}(t), f\rangle = \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{R}} -\sqrt{\lambda}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{R}} \cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, f).$$

We can change the order of integration and differentiaon using Fubini's Theorem and the facts that second integrant is bounded and $u_0 \in D(A)$.

$$\begin{aligned} \langle \frac{\mathrm{d}^2 u}{\mathrm{d}\,t^2}(t), f\rangle &= \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,t} - \sqrt{\lambda}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \int_{\mathbb{R}} \frac{\mathrm{d}}{\mathrm{d}\,t}\cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, f) \\ &= \int_{\mathbb{R}} -\lambda\cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \int_{\mathbb{R}} -\sqrt{\lambda}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, f) \\ &= -[\int_{\mathbb{R}} \lambda\cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, f) + \int_{\mathbb{R}} \frac{\lambda}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, f)] \\ &= -[\int_{\mathbb{R}} \cos\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_0, Af) + \int_{\mathbb{R}} \frac{1}{\sqrt{\lambda}}\sin\sqrt{\lambda}t\,\mathrm{d}(E(\lambda)u_1, Af)] \\ &= -\langle u(t), Af\rangle. \end{aligned}$$

That is, by (7.1.19) $u(t)$ is a strong solution of (7.1.22) if $u_0 \in D(A)$ and $u_1 \in D(\sqrt{|A|})$.

7.2 Schrödinger Criterion of Selfadjointness

Theorem 7.2.1 (The Schrödinger Criterion of Selfadjointness)**.** *Let A be an Hermitian operator acting on $\mathcal{H}$. For its essential selfadjointness, it is necessary that the Cauchy problem for the equation*

$$(\frac{\mathrm{d}\,u}{\mathrm{d}\,t})(t) \pm (iA^*)u(t) = 0 \quad (t \in [0, b)) \tag{7.2.1}$$

be uniquely solvable on $[0, b)$ for all $b \in (0, \infty]$ (in the sense of strong solutions) and it is sufficient that A be semibounded below and that the indicated Cauchy problem be uniquely solvable in the same sense for some $b > 0$.

Proof. Before the proof first we shall prove two important lemmas:

Lemma 7.2.2. *Let C be an operator acting on $\mathcal{H} \oplus \mathcal{H}$ with domain $D(C) = D(A) \oplus D(A)$ according to the formula $Cf = (Af_1, -Af_2)$, where $f = (f_1, f_2) \in D(C)$. Suppose further C satisfies the following equation for some ($b \in (0, \infty]$);*

$$(\frac{\mathrm{d}\,u}{\mathrm{d}\,t})(t) + (iC)^*u(t) = 0 \quad (t \in [0, b)) \tag{7.2.2}$$

for the vector functions with values in $\mathcal{H} \oplus \mathcal{H}$. Then, the Cauchy problems for both equations (7.2.1) *are uniquely solvable in the strong sense on $[0, b)$ if and only if the Cauchy problem for equation* (7.2.2) *is also uniquely solvable in the strong sense.*

Proof. Let $u(t) = \langle u_1(t), u_2(t) \rangle$ $t \in [0, b)$ be a strong solution for the Cauchy problem in (7.2.2). Since $C^*f = (A^*f_1, -A^*f_2) \quad (f \in D(C^*) = D(A^*) \oplus D(A^*))$, the functions $u_1(t), u_2(t)$ are strong solutions of (7.2.1) with signs "$+$" and "$-$" respectively. In the view of the assumption of uniquely solvability for (7.2.1) implies uniqueness for (7.2.2). Converse statement can be proven similarly. □

Lemma 7.2.3. *Suppose that there exists a dense set Φ in $\mathcal{H}$ such that the Cauchy problem*

$$(\frac{\mathrm{d}^r\,\psi}{\mathrm{d}\,t^r})(t) + (-1)^r B\psi(t) = 0 \quad (t \in [0, T]);$$
$$\psi(T) = \psi_0, \psi^{r-1}(T) = \psi_{r-1} \quad (r = 1, 2) \tag{7.2.3}$$

has a strong solution for all $T \in (0,b)$ *and* $\psi_0, \psi_{r-1} \in \Phi$*. Then the Cauchy problem for* (7.1.18) *is uniquely solvable on* $[0,b)$ *in the sense of strong solutions.*

Proof. First we shall prove the case $r = 2$. Let $\alpha(t), \beta(t) \in \mathcal{H} \quad (t \in [0,T])$ be twice strongly continuously differentiable functions. Then

$$\int_0^T \langle \alpha''(t), \beta(t)\rangle \, \mathrm{d}\, t = (\langle \alpha'(t), \beta(t)\rangle)|_0^T - \int_0^T \langle \alpha'(t), \beta'(t)\rangle \, \mathrm{d}\, t.$$

Since $\int_0^T \langle \alpha'(t), \beta'(t)\rangle \, \mathrm{d}\, t = (\langle \alpha(t), \beta'(t)\rangle)|_0^T - \int_0^T \langle \alpha(t), \beta''(t)\rangle \, \mathrm{d}\, t$,

$$\int_0^T \langle \alpha''(t), \beta(t)\rangle \, \mathrm{d}\, t = \int_0^T \langle \alpha(t), \beta''(t)\rangle \, \mathrm{d}\, t - (\langle \alpha'(t), \beta(t)\rangle - \langle \alpha(t), \beta'(t)\rangle)|_0^T. \tag{7.2.4}$$

Let $u(t)$ be a strong solution of (7.1.18) with $r = 2$ on $[0,b)$ such that $u(0) = u'(0) = 0$. Let ψ be a strong solution in (7.2.3). In view of (7.2.4) and the assumption $u(0) = u'(0) = 0$,

$$\int_0^T (\langle u''(t), \psi(t)\rangle - \langle u(t), \psi''(t)\rangle) \, \mathrm{d}\, t = \langle u'(T), \psi_0\rangle - \langle u(T), \psi_1\rangle. \tag{7.2.5}$$

Write the "weak" equality (7.1.19) for $f = \psi(s) \in D(B) \;\; \forall s \in [0,T]$ we get

$$\langle u''(t), \psi(s)\rangle + \langle u(t), B\psi(s)\rangle = 0 \quad (t \in [0,b)).$$

Setting $t = s$ yields,

$$\langle u''(s), \psi(s)\rangle = -\langle u(s), B\psi(s)\rangle \quad (s \in [0,b) \supseteq [0,T]). \tag{7.2.6}$$

Also, by virtue of (7.2.3) with $r = 2$ we have

$$\langle u(t), \psi''(t)\rangle = -\langle u(t), B\psi(t)\rangle = 0 \quad (t \in [0,T]). \tag{7.2.7}$$

(7.2.6), (7.2.7) together with (7.2.5) implies that

$$\langle u'(T), \psi_0\rangle - \langle u(T), \psi_1\rangle = 0 \quad (\psi_0, \psi_1 \in \Phi).$$

Since, Φ is dense $u(T) = u'(T) = 0$, which yields desired assertion. In the case of $r = 1$, we do similar calculations, instead of (7.2.4) we use following integration by parts;

$$\int_0^T \langle \alpha'(t), \beta(t)\rangle \, \mathrm{d}\, t = -\int_0^T \langle \alpha(t), \beta'(t)\rangle \, \mathrm{d}\, t + [\langle \alpha(t), \beta(t)\rangle]|_0^T, \tag{7.2.8}$$

which is well defined since α, β are assumed to be strong solutions, so continuously differentiable functions. $\square$

Now back to the proof of the Theorem, we split it into 3 cases.

(1) Sufficiency part with A has equal defect numbers:

Suppose to the contrary, $\widetilde{A}$ is not selfadjoint, then since A has equal defect numbers (and they are not zero) we can find two different selfadjoint extensions A_1 and A_2 in $\mathcal{H}$. Let E_1, E_2 be their spectral measures respectively. In the view of (7.1.3) for $g \in \mathcal{H}$

$$u_1(t) = \int_{\mathbb{R}} e^{i\lambda t} \,\mathrm{d}\, E_1(\lambda) g \tag{7.2.9}$$

is strongly continuously differentiable and $u_1'(t) = i \int_{\mathbb{R}} \lambda e^{i\lambda t} \,\mathrm{d}\, E_1(\lambda) g$. Note that by (7.1.3) this is indeed a strong solution of (7.2.1) with $+$ sign on $[0, \infty)$. Similarly construct

$$u_2(t) = \int_{\mathbb{R}} e^{i\lambda t} \,\mathrm{d}\, E_2(\lambda) g \tag{7.2.10}$$

and consider $u(t) = u_1(t) - u_2(t)$. Clearly $u(t)$ is also a strong solution, and $u(0) = g - g = 0$. By assumptipn of uniquely solvability and Remark (7.1.4), $u(t) = 0$ on $[0, \infty)$, that is;

$$\int_{\mathbb{R}} e^{i\lambda t} \,\mathrm{d}((E_1(\lambda) - E_2(\lambda))g, h) = 0 \quad (g \in D(A), h \in \mathcal{H}, t \in [0, \infty)). \tag{7.2.11}$$

Note that $u(t)$ is indeed Fourier Transform of $\mathrm{d}((E_1(\lambda) - E_2(\lambda))g, h)$, and so unique. Thus, we get $((E_1(\lambda) - E_2(\lambda))g, h) = 0$, $t \in \mathbb{R}$. Since $g \in D(A)$ which is a dense set and $h \in \mathcal{H}$ is arbitrary we conclude that $E_1 = E_2$. Contradiction.

(2) Sufficiency part with A has deficiency indices (m, n):

Consider the operator C on Lemma (7.2.2). Denote its deficiency indices as M, N where $M = \dim(R(C - zI))^{\perp}$, $N = \dim(R(C - \overline{z}I))^{\perp}$ where $z = (z_1, z_2)$

and $\overline{z} = (\overline{z_1}, \overline{z_2})$ with $z_1, z_2 \in \mathbb{R}^+$. Then

$$\begin{aligned} M &= \dim(R(C - zI))^\perp \\ &= \dim(R(A - z_1 I) \oplus R(A + z_2 I))^\perp && (7.2.12) \\ &= \dim(R(A - z_1 I))^\perp + \dim(R(A + z_2 I))^\perp && (7.2.13) \\ &= m + n. \end{aligned}$$

Similarly $N = m + n$. Note that the orthogonal complements of (7.2.12) and (7.2.13) have different meanings. (7.2.12) is orthogonal complement on $\mathcal{H} \oplus \mathcal{H}$, while (7.2.13) are orthogonal complements on $\mathcal{H}$.

By virtue of Lemma (7.2.2) and the assumption of the sufficiency part, we get (7.2.2) is uniquely solvable on $[0, b)$. Replacing A with $-A$ we get (7.2.2) with $-$ instead of $+$ sign. Thus, our problem turns into the same problem with Case (1), replacing A with C. Hence we conclude that C is essentially selfadjoint, that is $m + n = 0$ or equivalently $m = n = 0$; whence A is essentially selfadjoint.

(3) Necessity part:

Let $\widetilde{A}$ be selfadjoint and E be its spectral measure. Apply Lemma (7.2.3) by setting $r = 1$, $B = (iA)^* = -i\widetilde{A}$. A strong solution of (7.2.3) with these settings exists and equal to

$$\psi(t) = \int_{\mathbb{R}} e^{-i\lambda(t-T)} \,\mathrm{d}\, E(\lambda)\psi_0 \quad (t \in [0, T]). \tag{7.2.14}$$

Note that $\psi(t)$ is clearly continuously differentiable by the same argument as (7.1.3), $\psi(T) \in D(A)$ and it satisfies the Cauchy problem (7.2.3). Thus, by Lemma (7.2.3) equation (7.2.1) with $+$ sign is uniquely solvable on $[0, b)$. For the equation (7.2.1) with $-$ sign we do similar calculations by changing $B = i\widetilde{A}$. □

7.3 Hyperbolic Criterion of Selfadjointness

Theorem 7.3.1 (The Hyperbolic Criterion of Selfadjointness)**.** *Let A be an Hermitian operator acting on $\mathcal{H}$. For its essential selfadjointness, it is necessary that* *the Cauchy problem for the equation*

$$(\frac{\mathrm{d}^2 u}{\mathrm{d}\, t^2})(t) + A^* u(t) = 0 \quad (t \in [0, b)) \tag{7.3.1}$$

be uniquely solvable on $[0, b)$ *for all* $b \in (0, \infty]$ *(in the sense of strong solutions) and it is sufficient that* A *be semibounded below and that the indicated Cauchy problem be uniquely solvable in the same sense for some* $b > 0$.

Proof. **Sufficiency**:

Suppose to the contrary, $\widetilde{A}$ is not selfadjoint. Then A has two different selfadjoint extensions A_1, A_2 in $\mathcal{H}$ bounded below by a number $c > -\infty$. Let E_1, E_2 be corresponding spectral measures respectively. For $g \in D(A) \subseteq D(A_1)$,

$$u_1(t) = \int_c^\infty e^{\sqrt{\lambda} i t} \, \mathrm{d}\, E_1(\lambda) g \tag{7.3.2}$$

is strongly continuously differentiable(we use the fact that $\int_{\mathbb{R}} \lambda^2 \, \mathrm{d}(E_1(\lambda)g, g) < \infty$). In the view of the "weak" equality of (7.1.19) and by Stone's theorem,

$$\begin{aligned} \langle -A \int_c^\infty e^{\sqrt{\lambda} t} \, \mathrm{d}\, E_1(\lambda) g, f \rangle + \langle u(t), Af \rangle = \\ \langle -Au(t), f \rangle + \langle u(t), Af \rangle = 0. \end{aligned}$$

Last equality follows by the fact that A is Hermitian.

Thus, (7.3.2) is a strong solution of (7.3.1) on $[0, \infty)$. We can construct similarly $u_2(t)$ and then consider $u(t) = u_1(t) - u_2(t)$. Then it is easy to see that $u(0) = u'(0) = 0$. That is $u(t)$ is a strong solution of (7.3.1) with trivial boundary conditions. In the view of the assumed uniqueness of strong solutions of the Cauchy problem $u(t) = 0$ for $t \geq 0$. Multiplying this equality scalarly by $h \in \mathcal{H}$, we obtain

$$\int_c^\infty e^{\sqrt{\lambda} i t} \, \mathrm{d}((E_1(\lambda) - E_2(\lambda))g, h) = 0 \quad (t \geq 0).$$

Since the measure is uniquely determined in terms of its Fourier transform by (1.0.5). This concludes that $E_1 = E_2$, contradiction.

Necessity:

Let A be selfadjoint operator and E be its corresponding spectral measure. Consider Lemma (7.2.3) with $r = 2, B = A^* = \widetilde{A}$, and $\Phi = \bigcup_{n=1}^{\infty} E((-n, n))\mathcal{H}$. A strong solution of (7.2.3) exists and is equal to

$$\psi(t) = \int_{\mathbb{R}} \cos(\sqrt{\lambda}(t-T)) \, \mathrm{d}\, E(\lambda)\psi_0 + \int_{\mathbb{R}} \frac{\sin(\sqrt{\lambda}(t-T))}{\sqrt{\lambda}} \, \mathrm{d}\, E(\lambda)\psi_1.$$

Thus, the Cauchy problem for (7.3.1) is uniquely solvable on $[0, b)$ for all $b \in (0, \infty]$ in the sense of strong solutions. □

Theorem 7.3.2. *Let A be an Hermitian operator acting on $\mathcal{H}$ and semibounded below. Assume that there exists a linear set $\Phi \subseteq \mathcal{H}$ dense in $\mathcal{H}$ and such that the Cauchy problem*

$$(\frac{\mathrm{d}^2\, \psi}{\mathrm{d}\, t^2})(t) + A\psi(t) = 0 \quad ((t \in [0, T]);\ \psi(T) = \psi_0,\ \psi'(T) = \psi_1) \tag{7.3.3}$$

has a strong solution for some $b > 0$ and all $T \in (0, b)$ and $\psi_0, \psi_1 \in \Phi$. Then the operator A is essentially selfadjoint.

Proof. By virtue of Lemma (7.2.3), it follows by assumption of the theorem that the Cauchy problem for (7.3.1) has a unique solution on $[0, b)$. Then by Theorem (7.3.1), the operator $\widetilde{A}$ is selfadjoint. □

7.4 Parabolic Criterion of Selfadjointness

Theorem 7.4.1 (The Parabolic Criterion of Selfadjointness)**.** *Let A be an Hermitian operator acting on $\mathcal{H}$. For its essential selfadjointness, it is necessary that the Cauchy problem for the equation*

$$(\frac{\mathrm{d}\, u}{\mathrm{d}\, t})(t) + A^* u(t) = 0 \quad (t \in [0, \infty)) \tag{7.4.1}$$

be uniquely solvable in the sense of strong solutions. For an operator semibounded below, this is also sufficient condition.

Proof. **Sufficiency**:

Suppose to the contrary $\widetilde{A}$ is not selfadjoint. Let A_1, A_2 be different selfadjoint extensions of A bounded below by $c > -\infty$ and let E_1, E_2 be the corresponding spectral measures. Its clear by (7.1.1) that the vector function

$$u_1(t) = \int_c^\infty e^{-\lambda t} \,\mathrm{d}\, E_1(\lambda) g \quad (t \in [0, \infty))\ (g \in D(A) \subseteq D(A_1)) \tag{7.4.2}$$

is strongly continuously differentiable and $u_1(t) \in D(A_1) \subseteq D(A^*)$. The derivative $u_1'(t) = \int_c^\infty e^{-\lambda t} \,\mathrm{d}\, E_1(\lambda) g \quad (t \in [0, \infty))\ (g \in D(A) \subseteq D(A_1))$. The expression $A^* u_1(t) = A_1 u_1(t)$ also of the same form. Thus, (7.4.2) is a strong solution of equation (7.4.1) with $u_1(0) = g$. Further, similarly we construct $u(t) = u_1(t) - u_2(t)$ and then $u(t) = 0$, whence by totally same argument with the proof of (7.3.1) we arrive a contradiction to $E_1 = E_2$.

Necessity:

Consider Lemma (7.2.3) with $r = 1$, $B = \widetilde{A}$, and $\Phi = \bigcup_{n=1}^\infty E((-n, n))\mathcal{H}$, where E is the corresponding spectral measure of $\widetilde{A}$. A strong solution of the corresponding Cauchy problem exists and is equal to

$$\psi(t) = \int_{\mathbb{R}} e^{\lambda(t-T)} \,\mathrm{d}\, E(\lambda) \psi_0 \quad (t \in [0, T]). \qquad \square$$

7.5 Quasianalytic Criterion of Selfadjointness

Definition 7.5.1. Let $[a, b]$ be a finite interval and let $(m_n)_{n=1}^\infty$ be a sequence of positive numbers. Then, the class $C\{m_n\}$ is defined as the linear set of all functions $f \in C^\infty([a, b])$ satisfying the estimates

$$|(D^n(f)(t))| \leq K_f^n m_n \quad (t \in [a, b]; n \in \mathbb{N}) \tag{7.5.1}$$

where K_f is a constant depending on f.

For example $m_n = n!$ gives the class of analytic functions defined on $[a, b]$.

Definition 7.5.2. The class $C\{m_n\}$ is called quasianalytic if $f \in C\{m_n\}$ such that $(D^n f)(t_0) = 0 \ \ (n \in \mathbb{N})$ and $f(t_0) = 0$ at a fixed point $t_0 \in [a, b]$ implies that $f(t) = 0 \ (t \in [a, b])$.

Theorem 7.5.3 (Denjoy-Carleman Criterion)**.** *The class $C\{m_n\}$ is quasianalytic if and only if*

$$\sum_{n=1}^{\infty} (\inf\{m_k^{1/k} |\ k \geq n\})^{-1} = \infty \tag{7.5.2}$$

For a proof see [5] or [9].

Example 7.5.4. The class of analytic functions is quasianalytic.

Proof. First we will show that $k!^{1/k}$ is an increasing sequence. Indeed for $k \geq 1$,

$$\begin{aligned} (k+1)!^{1/k+1} \geq k!^{1/k} &\Leftrightarrow (k+1)!^k \geq k!^{k+1} \\ &\Leftrightarrow (k+1)^k k!^k \geq k!^k . k! \\ &\Leftrightarrow (k+1)^k \geq k!. \end{aligned}$$

$k + 1 > i, \quad i = 1, \ldots, k$, so $k!^{1/k}$ is indeed an increasing sequence. Thus, for $m_n = n!$

$$\sum_{n=1}^{\infty} (\inf\{k!^{1/k} |\ k \geq n\})^{-1} = \sum_{n=1}^{\infty} (n!^{1/n})^{-1}.$$

$n^n > n!$ implies $(n!^{1/n})^{-1} > 1/n$, so

$$\sum_{n=1}^{\infty} (\inf\{k!^{1/k} |\ k \geq n\})^{-1} > \sum_{n=1}^{\infty} (1/n) = \infty.$$

That is, the class of analytic functions is quasianalytic. □

Example 7.5.5. The class $C\{n^{pn}\}$ is quasianalytic if and only if $p \leq 1$.

Proof. Suppose $p \leq 1$. Then k^p, $(k \in \mathbb{N})$ is a decreasing sequence and $\inf\{k^p |\ k \geq n\} = n^p$. By Denjoy-Carleman Criterion $C\{n^{pn}\}$ is quasianalytic if and only if $\sum_{n=1}^{\infty} (\inf\{k^p |\ k \geq n\})^{-1} = \infty$.

$$\sum_{n=1}^{\infty} (\inf\{k^p |\ k \geq n\})^{-1} = \sum_{n=1}^{\infty} \frac{1}{n^p} = \infty.$$

Inverse part can be deduced similarly. □

Definition 7.5.6. Let $\mathcal{H}$ be Hilbert space and let A be an Hermitian operator on it. Then, a vector $f \in \mathcal{H}$ is called quasianalytic (with respect to A) if $f \in \bigcap_{n=1}^{\infty} D(A^n)$ and the class $C\{||A^n f||\}$ is quasianalytic.

Lemma 7.5.7. *A vector $f \in \bigcap_{n=1}^{\infty} D(A^n)$ is quasianalytic if and only if*

$$\sum_{n=1}^{\infty} ||A^n f||^{-1/n} = \infty. \tag{7.5.3}$$

Proof. Assume W.L.O.G $||f|| = 1$. We will show that

$$(||A^n f||^{1/n})_{n=1}^{\infty} \tag{7.5.4}$$

is nondecreasing. Indeed, since A is Hermitian,

$$||Af||^2 = \langle Af, Af\rangle = \langle A^2 f, f\rangle \leq ||A^2 f||||f|| = ||A^2 f||. \tag{7.5.5}$$

i.e, $||Af|| \leq ||A^2 f||^{1/2}$. Now suppose $||A^n f||^{1/n} \leq ||A^{n+1} f||^{1/n+1}$ is proved.

$$\begin{aligned} ||A^{n+1} f||^2 &= \langle A^{n+1} f, A^{n+1} f\rangle \\ &= \langle A^{n+2} f, A^n f\rangle \\ &\leq ||A^{n+2} f|| \cdot ||A^n f|| \end{aligned}$$

by using the assumption we get

$$\leq ||A^{n+2} f|| \cdot ||A^{n+1} f||^{n/n+1}. \tag{7.5.6}$$

Hence $||A^{n+1} f||^{n+2/n+1} \leq ||A^{n+2} f||$ and (7.5.4) is nondecreasing. So

$$\inf\{||A^k f||^{1/k} |\ k \geq n\} = ||A^n f||^{1/n}. \tag{7.5.7}$$

Hence, proof follows by the Denjoy-Carleman criterion. □

Theorem 7.5.8. *Let A be closed Hermitian operator acting on $\mathcal{H}$. It is selfadjoint if and only if $\mathcal{H}$ contains a total set that consists of quasianalytic vectors.*

Proof. Let A be selfadjoint. Then it suffices to prove that each quasianalytic vector ψ has the form $E(a,b)f$ where E is the spectral measure corresponds to the selfadjoint operator A, a, b are some real numbers and $f \in \mathcal{H}$. Since a, b are finite it is clear that $\psi \in \bigcap_{n=1}^{\infty} D(A^n)$. Indeed,

$$\int_a^b \lambda^n \,\mathrm{d}(E(\lambda)f, f) \leq (\max(|a|, |b|, 1))^n \cdot ||f||^2 < \infty.$$

Similarly,

$$\begin{aligned}
||A^n\psi||^2 &= \int_a^b \lambda^{2n} \,\mathrm{d}(E(\lambda)f, f) \\
&\leq (\max(|a|, |b|, 1))^{2n} \cdot ||f||^2 \\
&\leq M^{2n} \cdot ||f||^2 \\
&\Rightarrow ||A^n\psi||^{-1/n} \geq M \cdot ||f||^{-1/n}.
\end{aligned} \tag{7.5.8}$$

That is, (7.5.3) diverges, and so the vector ψ is quasianalytic. Take $a = n$, $b = -n$, $(n \in \mathbb{N})$ we get total set consists of quasianalytic vectors.

For the inverse implication, suppose that A has a total set M of quasianalytic vectors ψ. By using closedness of A and Theorem 7.2.1 it is enough to prove the uniqueness of strong solutions of the Cauchy problem for the equations (7.2.1) if $b = \infty$. Let $u(t)$ be the strong solutions of the Cauchy problems

$$\frac{\mathrm{d}\,u}{dt}(t) - (\phi A)^* u(t) = 0 \quad (t \in [0, \infty),\ u(0) = 0\ \phi = \mp i). \tag{7.5.9}$$

Let ψ be quasianalytic and fix $T > 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\langle u(t), \psi\rangle = \langle (\frac{\mathrm{d}\,u}{\mathrm{d}\,t})(t), \psi\rangle$$

using the "weak" equality (7.1.19) for (7.5.9)

$$= \langle u(t), (\phi A)\psi\rangle \quad (t \in [0, T]).$$

But $(\phi A)\psi \in \bigcap_{n=1}^{\infty} D(A^n)$ and so change ψ to $(\phi A)\psi$ at above equality we get

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\langle u(t), (\phi A)\psi\rangle = \langle u(t), (\phi A)^2\psi\rangle \quad (t \in [0, T]).$$

We can keep on the process, so $\langle u(t), \psi \rangle \in C^\infty([0,T]))$ and by induction

$$D^n \langle u(t), \psi \rangle = \langle u(t), (\phi A)^n \psi \rangle \quad (t \in [0,T] \; n \in \mathbb{N}). \tag{7.5.10}$$

$u(t)$ is strong solution, that is it is bounded; and $[0,T]$ is also bounded, so

$$|D^n \langle u(t), \psi \rangle| \leq M \cdot ||(\phi A)^n \psi|| = M \cdot ||A^n \psi|| \quad (t \in [0,T]; \; n \in \mathbb{N}). \tag{7.5.11}$$

Thus, $f(t) = \langle u(t), \psi \rangle$ belongs to the class $C\{||A^n\psi||\}$. By (7.5.10) and the assumption $u(0) = 0$ implies that $(\forall n \in \mathbb{N})$, $(D^n f)(0) = 0$. By definition of quasianalytic vector ψ, the class $C\{||A^n\psi||\}$ is quasianalytic. Thus, we get $f(t) = \langle u(t), \psi \rangle = 0$ $(t \in [0,T])$. Since M is total and ψ is arbitrary on M, we get $u(t) = 0$, $(t \in [0,T])$. T is arbitrary, so we are done. □

7.6 Other Criteria of Selfadjointness

Definition 7.6.1. Let A be a Hermitian operator acting on a Hilbert space $\mathcal{H}$. Then, a vector $f \in \mathcal{H}$ is called analytic (with respect to A) if $f \in \bigcap_{n=1}^\infty D(A^n)$ and the power series,

$$\sum_{n=0}^\infty \frac{||A^n f||}{n!} z^n \tag{7.6.1}$$

has a nonzero radius of convergence. It is called entire if the radius is infinity.

It is clear that analytic vectors are quasianalytic. Indeed, suppose the series has nonzero radius of convergence. Then

$$\begin{aligned}
\sum_{n=0}^\infty \frac{||A^n f||}{n!} z^n < \infty \Rightarrow \frac{||A^n f||}{n!} &\to 0. \\
&\Rightarrow ||A^n f||^{-1/n} \to \infty \;\; faster \;\; than \;\; n!^{-1/n}. \\
&\Rightarrow \sum_{n=1}^\infty ||A^n f||^{-1/n} \geq \sum_{n=1}^\infty n!^{-1/n} \geq \sum_{n=1}^\infty \frac{1}{n}. \\
&\Rightarrow \sum_{n=1}^\infty ||A^n f||^{-1/n} = \infty.
\end{aligned}$$

Remark 7.6.2. If A is selfadjoint, then it possesses a total set of entire vectors. Indeed, this follows by (7.5.8) and the fact that $A^n\psi \leq M^n||\psi||$.

Definition 7.6.3. Let A be an Hermitian operator on $\mathcal{H}$. A vector $f \in \mathcal{H}$ is called a Stieljies vector (with respect to A) if $f \in \bigcap_{n=1}^{\infty} D(A^n)$ and the class $C\{||A^n f||^{1/2}\}$ is quasianalytic. It is clear by definition that every quasianalytic vector is a Stieljies vector. Thus, if we denote entire, analytic, quasianalytic, and Stieljies vectors (with respect to the operator A) by $\epsilon(A), \mathcal{A}(A), \mathcal{Q}(A), S(A)$ respectively, then we clearly have

$$\epsilon(A) \subseteq \mathcal{A}(A) \subseteq \mathcal{Q}(A) \subseteq S(A). \tag{7.6.2}$$

Theorem 7.6.4. *Let A be a closed Hermitian operator semibounded below. Then, it is selfadjoint if and only if $\mathcal{H}$ contains a total set that consists of Stieljies vectors.*

Proof. By (7.6.2) one side is trivial. For the nontrivial part, by Theorem (7.3.1) it suffices to prove the uniqueness of strong solutions of the Cauchy problem for equation (7.3.1) with $b = \infty$. let $u(t)$ be strong solution of the Cauchy problem with $u(0) = u'(0) = 0$. As in the proof of Theorem (7.5.8), suppose M is a total set of Stieljies vectors, $T > 0$ is fixed and set $f(t) = \langle u(t), \psi\rangle$ for some $\psi \in M$. Then by the "weak" equality (7.1.19) written for (7.3.1) with $f \in C^2([0,T])$ we get

$$(\frac{\mathrm{d}^2 f}{\mathrm{d}\, t^2})(t) = -\langle u(t), A\psi\rangle \quad (t \in [0.T]). \tag{7.6.3}$$

Since $A\psi \in \bigcap_{n=1}^{\infty} D(A^n)$, by the same reason, we conclude as in the proof of Theorem (7.5.8) that, $f \in C^\infty([0,T])$ and

$$\begin{aligned}(D^{2k} f)(t) &= D^{2k}\langle u(t), \psi\rangle \\ &= -D^{2(k-1)}\langle u(t), A\psi\rangle = \dots \\ &= (-1)^k\langle u(t), A^k\psi\rangle \quad (t \in [0,T]\ k \in \mathbb{N}).\end{aligned} \tag{7.6.4}$$

Similarly,

$$(D^{2k+1} f)(t) = (-1)^k\langle u'(t), A^k\psi\rangle \quad (t \in [0,T]\ k \in \mathbb{N}). \tag{7.6.5}$$

$u(t)$ is strong solution, so $u(t), u'(t)$ are bounded on $[0.T]$. Therefore by doing the same calculations as in Theorem (7.5.8) we are done. □

7.7 Selfadjointness of Perturbed Operators

Remark 7.7.1. Let A be a selfadjoint operator on $\mathcal{H}$ and let B be a bounded selfadjoint operator. Then $A+B$ is selfadjoint. Indeed, by Theorem (2.3.4), $(A+B)^* = A^* + B^* = A + B$.

However, in the case of both unbounded operators situation is more complicated. First we introduce the following definition:

Definition 7.7.2. Let A, B acts on $\mathcal{H}$ with $D(B) \supseteq D(A)$. We say B is subordinated to A if

$$||Bf|| \leq p||Af|| + q||f|| \quad (f \in D(A)) \tag{7.7.1}$$

where $p, q \geq 0$ are constants(constants of subordination). If, $\forall p > 0$ there exists q such that (7.7.1) satisfies, then B is called infinitely small as compared to A.

Theorem 7.7.3. *Let A be selfadjoint and B be Hermitian on $\mathcal{H}$ such that $D(B) \supseteq D(A)$. If B is subordinated to A with a constant subordination $p \in [0,1)$, then $A+B$ is selfadjoint.*

Proof. Notice first that by Theorem (2.3.3), $(A+B)^* \supseteq A^* + B^* \supseteq A + B$; that is $A+B$ is Hermitian. Thus, by Theorem (4.1.5), it is enough to prove that $R(A+B-iyI) = \mathcal{H}$ and $R(A+B+iyI) = \mathcal{H}$ for some $y > 0$. We will prove the first relation in which the second one can prove with changing signs of $+$ and $-$ in front of iy. In other words, we will prove that

$$(A+B-iyI)f = g \quad (f \in D(A)) \tag{7.7.2}$$

is solvable for all $g \in \mathcal{H}$. Since A is selfadjoint iy is a point of regular type for A, thus, $(A-iyI)^{-1}$ is exists and bounded, and thus we can transform (7.7.2) to:

$$(I + B(A-iyI)^{-1})(A-iyI)f = g \quad (f \in D(A)). \tag{7.7.3}$$

We used the fact $D(A) \subseteq D(B)$, so that (7.7.3) is indeed well defined and equals to (7.7.2). Denote $C(y) = B(A-iyI)^{-1}$. Since $(A-iyI)^{-1}$ is bounded, in the view of Remark 2.1.2, $D(C) = D(B) \supseteq D(A)$. We will show that for sufficiently large $y > 0$, $||C(y)|| < 1$. First we will establish some inequalities;

$$\begin{aligned}||(A - iyI)f||^2 &= \langle (A - iyI)f, (A - iyI)f\rangle \\ &= \langle Af, Af\rangle + iy\langle Af, f\rangle - iy\langle f, Af\rangle + y^2\langle f, f\rangle.\end{aligned}$$

Since A is selfadjoint,

$$||(A - iyI)f||^2 = ||Af||^2 + y^2||f||^2.$$

Set $(A - iyI)f = g$, then in the view of Remark 2.1.2, last equality turns to

$$||g||^2 = ||A(A - iyI)^{-1}g||^2 + y^2||(A - iyI)^{-1}g||^2 \quad (g \in D((A - iyI)^{-1}) = \mathcal{H})).$$

Thus, we get

$$||A(A - iyI)^{-1}g|| \leq ||g||, \quad \textit{and} \quad ||(A - iyI)^{-1}g|| \leq \frac{1}{y}||g|| \quad (g \in \mathcal{H}). \tag{7.7.4}$$

By (7.7.1) and (7.7.3) together with the fact that $(A - iyI)^{-1}f \in D(A)$ we have

$$||C(y)f|| = ||B(A - iyI)^{-1}f||$$

recall that $D(C) \supseteq D(A)$,

$$\begin{aligned}&\leq p||A(A - iyI)^{-1}f|| + q||(A - iyI)^{-1}f|| \\ &\leq p||f|| + \frac{q}{y}||f|| \quad (f \in D(A)).\end{aligned} \tag{7.7.5}$$

Hence by taking limit for the last inequality we conclude that

$$||C(y)f|| \leq p||f|| + \frac{q}{y}||f|| \quad (f \in D(C)). \tag{7.7.6}$$

Since $p \in [0, 1)$ and y is arbitrary, by picking y sufficiently large we can conclude that $||C(y)|| < 1$. Thus, $(1+C(y))^{-1}$ exists. But that is, (7.7.3) can be solved. $\square$

Remark 7.7.4. Notice that Theorem fails for $p = 1$. Indeed, for A is unbounded selfadjoint and $B = -A$ we get all conditions satisfied, but $A + B = 0|_{D(A)}$ which is not even a closed operator.

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