CANONICAL INDUCTION FOR TRIVIAL SOURCE RINGS

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By

Yasemin Büyükçolak

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Laurance J. Barker(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Mahmut Kuzucuoğlu

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Ali Sinan Sertöz

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Yasemin Büyükçolak
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Laurance J. Barker
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We discuss the canonical induction formula for some special Mackey functors by following the construction of Boltje. These functors are the ordinary and modular character rings and the trivial source rings. Making use of a natural correspondence between the Mackey algebra and the finite algebra spanned by the three kinds of basic bisets, namely the conjugation, restriction and induction, we investigate the canonical induction formula in terms of the theory of bisets. We focus on the trivial source rings and the canonical induction formula for them. The main aim is to get an explicit formula for the canonical induction of regular bimodules in the trivial source. This gives a first step towards for the canonical induction of blocks.

Keywords: Canonical induction, biset functor, Mackey functor, trivial source ring, monomial ring, regular bimodules.

ÖZET

DEĞERSİZ KAYNAK HALKALARI İÇİN KURALSAL İNDÜKSİYON

Yasemin Büyükçolak
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Boltje'nin yapısını kullanarak bazı özel Mackey izleçleri için kuralsal indüksiyon formulünü ortaya koyduk. Bu izleçler sıradan ve modüler karakter halkaları ve değersiz kaynak halkalarıdır. Mackey cebiri ve üç tür temel ikili setten oluşturulan sonlu cebir arasında doğal bir eşleşme vardır. Bundan dolayı kuralsal indüksiyon formulünü ikili setler teorisi açıından inceledik. Değersiz kaynak halkaları ve onlar için kuralsal indüksiyon formulü üzerine odaklandık. Temel amaç düzenli ikili modüllerin kuralsal indüksiyon formulü için açık bir formül elde etmektir. Bu blokların kuralsal indüksiyonu için ilk adımdır.

Anahtar sözcükler: Kuralsal indüksiyon, iki etki izleci, Mackey izleci, değersiz kaynak halkası, tek terimli halkası, düzenli ikili modüller.

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Contents

Chapter 1

Introduction

A Mackey functor is an algebraic structure having operations which behave like the induction, restriction and conjugation mappings in representation theory. Such operations appear in a variety of diverse contexts. Some important examples of Mackey functors are representation rings, G-algebras, Burnside rings, group cohomology, the algebraic K-theory of group rings and algebraic number theory. It is their widespread occurrence which motivates the study of such operations in abstract. The theory of Mackey functors was first studied by Dress [1], and by Green [2] in the early seventies. In later years, it is well-understood by the works of Thévenaz, Webb, Bouc, Boltje and the others, see the references in [3]. Particularly, Thévenaz-Webb [4] regard Mackey functors as modules of a finite dimensional algebra, which is called the Mackey algebra.

One of the most significant applications of the theory of Mackey functors is the canonical induction formula developed by Boltje [5] and [6]. For a Mackey functor M and a restriction subfunctor $A \subseteq M$, there is a surjective a map

$$\mathrm{lin}_G : A_+(G) \to M(G)$$

which is called the linearization homomorphism. In [5] and [6], Boltje constructed a map

$$\mathrm{can}_G : M(G) \to A_+(G)$$

such that the composition $\text{can}_G \text{lin}_G$ is identity map on M. This map is called the canonical induction formula and it commutes with restriction and conjugation maps.

The theory of bisets was introduced by Bouc [7] and [8] by defining five basic bisets, namely the induction, restriction, conjugation, inflation and deflation bisets. His main result is that all of five basic maps are expressed by certain bisets and conversely any biset is composed by of these five basic bisets. Between the Mackey algebra $\mu_k(G)$ which is defined slightly different from that in Thévenaz-Webb [4] and the finite dimensional algebra $\mathcal{B}_k(G)$ spanned by three kinds of basic bisets, the induction, restriction and conjugation, there exists a natural correspondence given in [4].

Theorem 1.0.1. (Thévenaz-Webb) *The two algebras $\mu_k(G)$ and $\mathcal{B}_k(G)$ are isomorphic.*

The trivial source modules for finite group G over a suitable p-modular system $(K, R, \mathbb{F})$ is firstly given by Broué [9]. An RG-module M is called trivial source RG-module if every indecomposable direct summand of M has the trivial module as a source. The isomorphism classes of trivial source RG-modules generate the group $T_R(G)$ which is closed under multiplication and contains the unit element. In fact, $T_R(G)$ becomes a ring, the so-called trivial source ring of RG. That is,

$$T_R(G) = \bigoplus_{M \in \text{Triv}(RG)} \mathbb{Z}[M]$$

where $\text{Triv}(RG)$ is a set of representatives of isomorphism classes of indecomposable trivial source RG-modules. Since $T_R(G) \cong T_{\mathbb{F}}(G)$, it can be used the isomorphism classes of trivial source $\mathbb{F}G$-modules instead of the isomorphism classes of trivial source RG-modules. The trivial source ring is a Mackey functor with the usual induction, restriction and conjugation maps.

In this thesis, we mainly study the trivial source rings and the canonical induction formula for them. Our principal aim is to get an explicit formula for the canonical induction formula of regular bimodules in the trivial source modules.

Theorem 1.0.2. *For a regular bimodule $\mathbb{F}G$, we have*

$$\mathrm{can}_G(\mathbb{F}G) = \sum_{U \leq G;\, U: p'\text{-}group} [U, \lambda_U]_G$$

$$\textit{where} \qquad \lambda_U = |U| \sum_{U \leq U' \leq G; U, U': p'\text{-}group} \frac{1}{|U'|} \mu(U, U') \mathrm{res}_{U,U'}\Big(\sum_{\varphi \in \hat{U}'(\mathbb{F})} \varphi\Big).$$

Looking the regular bimodules provides a first step towards studying the canonical induction of blocks.

In chapter 2, the necessary basics of some important rings are summarized. First, the theory of ordinary and modular character rings is introduced. Then, the structure and fundamental properties of the monomial ring and the trivial source ring, which are our main objects in this thesis, are explained.

Chapter 3 deals with the canonical induction formula for a Mackey functor M and a restriction subfunctor $A \subset M$. The definitions of three categories, namely Mackey category, restriction category and conjugation category, in terms of bisets are described in detail at the beginning section. The functors $-_+$, $-^+$ and the mark homomorphism between them are explained referring to Boltje [5], [6] and Barker [10]. In the fourth section we define the notion of canonical induction formula which is the section of a certain morphism $\mathrm{lin} : A_+ \to M$. The canonical induction homomorphism is defined as composition of tom Dieck homomorphism and inverse of mark homomorphism. If this homomorphism is a section of linearization homomorphism, it is called the canonical induction formula. The following section examines the case of the invertible group order. We obtain an explicit formula and necessary and sufficient conditions for the canonical induction formula.

Chapter 4 is dedicated to applications of canonical induction formula in the ordinary and modular character rings and the trivial source ring.

In last chapter, which presents our main results, the canonical induction for regular bimodules in the trivial source ring are investigated. The reason of investigating the regular bimodules is that they are a step towards canonical induction on blocks. However unfortunately no general results are obtained about canonical induction of blocks yet.

Chapter 2

Some Important Rings

In this chapter, we shall give a brief summary for some fundamental rings which will be play an important role throughout this thesis. In all sections, we will state necessary definitions, properties and results without proofs since the content of all sections are standard. One can find details in classical books, for instance we used [11], [12], [13] and [14], except the Section 2.2 which can be found in [15] and [16].

2.1 The Character Ring

In this section, we will give a introduction to the ordinary and modular representation theory. For more details, we refer to classical representation theory books [11], [12], [13] and [17].

Throughout, let G be a finite group, $\mathbb{F}$ be a field and $\mathbb{F}G$ be a group algebra of G over $\mathbb{F}$. We understand that all $\mathbb{F}G$-modules are finitely generated and Krull-Schmidt Theorem holds for finitely generated $\mathbb{F}G$-modules, that is for each $\mathbb{F}G$-module a decomposition into a direct sum of finitely many indecomposable $\mathbb{F}G$-modules exists and it is unique up to order and isomorphism.

The ring $\mathbb{F}$ can be viewed as an $\mathbb{F}G$-module via $g \cdot r = r$ for all $g \in G$, $r \in \mathbb{F}$, which is called the *trivial* $\mathbb{F}G$*-module*. Let $H \leq G$ and M be an $\mathbb{F}H$-module. For $g \in G$, the *conjugate* $\mathbb{F}^gH$-module $\mathrm{Con}^g_{{}^gH,H}(M)$ of M is defined to be such that $\mathrm{Con}^g_{{}^gH,H}(M) := M$ as sets and the action is given by $({}^gh) \cdot m := hm$ where $h \in H$, $m \in M$. Note that for $g \in H$, $\mathrm{Con}^g_{{}^gH,H}(M)$ is an $\mathbb{F}H$-module and we have $\mathrm{Con}^g_{{}^gH,H}(M) \cong M$. For an $\mathbb{F}G$-module N, the *restriction* of N to H is defined as an $\mathbb{F}H$-module which is obtained by restricting the representations to H and which is denoted by $\mathrm{Res}_{H,G}(N)$. The *induction* of M to G is defined as the induced $\mathbb{F}G$-module $\mathrm{Ind}_{G,H}(M) := \mathbb{F}G \otimes_{\mathbb{F}H} M$. One can observe that the conjugation, restriction and induction give functors

$$\mathrm{Con}^g_{{}^gH,H} \quad : \quad \mathrm{Mod}(\mathbb{F}H) \quad \to \quad \mathrm{Mod}(\mathbb{F}({}^gH)), \tag{2.1}$$

$$\mathrm{Res}_{H,G} \quad : \quad \mathrm{Mod}(\mathbb{F}G) \quad \to \quad \mathrm{Mod}(\mathbb{F}H), \tag{2.2}$$

$$\mathrm{Ind}_{G,H} \quad : \quad \mathrm{Mod}(\mathbb{F}H) \quad \to \quad \mathrm{Mod}(\mathbb{F}G). \tag{2.3}$$

Some basic properties of conjugation, restriction and induction are as follows:

(i) $\mathrm{Res}_{K,H}(\mathrm{Res}_{H,G}(N)) = \mathrm{Res}_{K,G}(N)$ and $\mathrm{Ind}_{G,H}(\mathrm{Ind}_{H,K}(L)) = \mathrm{Ind}_{G,K}(L)$ and $\mathrm{Con}^{g'}_{{}^{g'}H,{}^gH}(\mathrm{Con}^g_{{}^gH,H}(M)) = \mathrm{Con}^{g'g}_{{}^{g'g}H,H}(M)$,

(ii) $\mathrm{Con}^g_{{}^gK,K}(\mathrm{Res}_{K,H}(M)) = \mathrm{Res}_{{}^gK,{}^gH}(\mathrm{Con}^g_{{}^gH,H}(M))$ and $\mathrm{Con}^g_{{}^gH,H}(\mathrm{Ind}_{H,K}(L)) = \mathrm{Ind}_{{}^gH,{}^gK}(\mathrm{Con}^g_{{}^gK,K}(L))$,

(iii) $N \otimes_{\mathbb{F}} \mathrm{Ind}_{G,H}(M) = \mathrm{Ind}_{G,H}(\mathrm{Res}_{H,G}(N) \otimes_{\mathbb{F}} M)$,

(iv) $\mathrm{Res}_{K,G}(\mathrm{Ind}_{G,H}(M)) = \bigoplus_{t \in [K\backslash G/H]} \mathrm{Ind}_{K,{}^tH\cap K}(\mathrm{Con}^t_{{}^tH\cap K,{}^tK\cap H}(\mathrm{Res}_{{}^tK\cap H,H}(M)))$,

(v) $\mathrm{Res}_{G,G}(N) = N$ and $\mathrm{Ind}_{G,G}(N) = N$ and $\mathrm{Con}^h_{H,H}(M)) = M$

where L is a $\mathbb{F}K$-module, M is a $\mathbb{F}H$-module and N is a $\mathbb{F}G$-module for $K \leq H \leq G$.

A *representation* of G over $\mathbb{F}$ is defined to be a group homomorphism

$$\rho : G \to \mathrm{Aut}_{\mathbb{F}}(V)$$

where V is a vector space over $\mathbb{F}$ and $\mathrm{Aut}_{\mathbb{F}}(V)$ is the group of $\mathbb{F}$-linear automorphisms of V, which is also written as $\mathrm{GL}_{\mathbb{F}}(V)$. If $\dim_{\mathbb{F}}(V) = d$, then

$\mathrm{Aut}_{\mathbb{F}}(V) \cong \mathrm{Aut}_{\mathbb{F}}(\mathbb{F}^d) = \mathrm{GL}_d(\mathbb{F})$ which is described as:

$$\mathrm{GL}_d(\mathbb{F}) = \{A \in \mathrm{Mat}_d(\mathbb{F}) \mid \det(A) \neq 0\}.$$

Also, $d = \dim_{\mathbb{F}}(V)$ is called the *dimension* or *degree* of the representation ρ.

Remark 2.1.1. If $\mathbb{F}$ is a field and G is a finite group, then there is one-to-one correspondence between the finitely generated $\mathbb{F}G$-modules and the representations of G on finite-dimensional $\mathbb{F}$-vector spaces.

Let A be a finite-dimensional algebra over $\mathbb{F}$. An A-module M is called *simple* or *irreducible* if it is non-zero and it has no $\mathbb{F}G$-submodules except 0 and M. If M has an non-zero A-submodule $N \neq M$, then M is called *reducible*. Moreover, an A-module is said to be *completely reducible* or *semisimple* if it is a direct sum of simple A-modules and an $\mathbb{F}$-algebra A is said to be *semisimple* if all A-modules are semisimple.

We now state some important results about $\mathbb{F}G$-modules which can be found in [11] in detail:

Theorem 2.1.2. (Maschke's Theorem) *Let G be a finite group and suppose that the characteristic of $\mathbb{F}$ is either zero or coprime to the order of G. If U is an $\mathbb{F}G$-module and V is an $\mathbb{F}G$-submodule of U, then V is a direct summand of U as $\mathbb{F}G$-module. In other words, there is an $\mathbb{F}G$-module W of U such that $U = V \oplus W$.*

Corollary 2.1.3. *Let G be a finite group, and let $\mathbb{F}$ be field whose characteristic does not divide the order of G. Then, every $\mathbb{F}G$-module is semisimple. In particular, if $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, then every $\mathbb{F}G$-module is semisimple.*

Theorem 2.1.4. (Wedderburn's Structure Theorem) *Let $\mathbb{F}$ be a field. If an algebra A over $\mathbb{F}$ is semisimple, then it is isomorphic to a finite direct sum of matrix algebras over division algebras. That is,*

$$A \cong \bigoplus_{i=1}^{k} \mathrm{Mat}_{n_i}(D_i)$$

where D_i are finite-dimensional division algebras over $\mathbb{F}$. Conversely, every algebra of this form is semisimple.

Corollary 2.1.5. *Suppose that the field $\mathbb{F}$ is algebraically closed. Then any semisimple algebra over $\mathbb{F}$ is isomorphic to a direct sum of finitely many matrix algebras over $\mathbb{F}$.*

For an $\mathbb{F}G$-module M, we denote the isomorphism class of M by $[M]$. *The Representation Ring* $A_{\mathbb{F}}(G)$ is generated by the isomorphism classes of $\mathbb{F}G$-modules with the addition and multiplication given by direct sum and tensor product of $\mathbb{F}G$-modules

$$[M] + [N] = [M \oplus N] \quad \text{and} \quad [M] \cdot [N] = [M \otimes_{\mathbb{F}} N]$$

for $\mathbb{F}G$-modules M, N. Hence, $A_{\mathbb{F}}(G)$ is a commutative ring with identity element $1_{A_{\mathbb{F}}(G)} = [\mathbb{F}]$ where $[\mathbb{F}]$ is the isomorphism class of trivial $\mathbb{F}G$-module $\mathbb{F}$. By the Krull-Schmidt Theorem, $A_{\mathbb{F}}(G)$ is a free abelian group with basis given by the isomorphism classes of indecomposable $\mathbb{F}G$-modules, thus

$$A_{\mathbb{F}}(G) = \bigoplus_i \mathbb{Z}[M_i] \tag{2.4}$$

where the M_i are indecomposable $\mathbb{F}G$-modules. For $\mathbb{F}G$-modules M and N, $[M] = [N]$ in $A_{\mathbb{F}}(G)$ if and only if $M \cong N$.

Let $\rho : G \to GL_{\mathbb{F}}(V)$ be a representation of G over $\mathbb{F}$. The $\mathbb{F}$-*character* of ρ is defined to be the function

$$\chi_\rho : G \to \mathbb{F}, \qquad \text{given by} \;\; \chi_\rho(g) = \mathrm{tr}(\rho(g))$$

where $g \in G$ and tr() indicates the trace. We say that a $\mathbb{F}$-character χ is *trivial* if χ is the $\mathbb{F}$-character of trivial representation. Notice that the $\mathbb{F}$-character of a representation can be regarded as the $\mathbb{F}$-character of the corresponding $\mathbb{F}G$-module, thus $\chi_\rho := \chi_U$ where U is the corresponding $\mathbb{F}G$-module of ρ. A $\mathbb{F}$-character χ is called *irreducible* if χ is the $\mathbb{F}$-character of an irreducible $\mathbb{F}G$-module and a $\mathbb{F}$-character χ is called *reducible* if χ is the $\mathbb{F}$-character of a reducible $\mathbb{F}G$-module. Moreover, an $\mathbb{F}$-character of 1-dimensional $\mathbb{F}G$-module is called *linear* $\mathbb{F}$-*character* of G. Since 1-dimensional $\mathbb{F}G$-modules are simple, all linear $\mathbb{F}$-characters are irreducible. Note that the linear $\mathbb{F}$-characters of G are exactly the same as the group homomorphisms from G to the multiplicative group $\mathbb{F}^\times$ and the set of all linear characters is denoted by $\hat{G}(\mathbb{F}) := \mathrm{Hom}(G, \mathbb{F}^\times)$.

The relation between the $\mathbb{F}$-characters of $\mathbb{F}G$-modules U, V and the operations on these modules is given by

- The $\mathbb{F}$-character of a direct sum $U \oplus V$ is the sum of the $\mathbb{F}$-characters of U and V:
$$\chi_{U\oplus V} := \chi_U + \chi_V,$$
- The $\mathbb{F}$-character of a tensor product $U \otimes V$ is the product of the $\mathbb{F}$-characters of U and V:
$$\chi_{U\otimes V} := \chi_U \cdot \chi_V.$$

We now discuss the case $\mathbb{F} = \mathbb{C}$ where $\mathbb{C}$ is the field of complex numbers, details can be found in [11]. Since $\mathbb{C}$ is an algebraically closed field of characteristic zero, by Maschke's Theorem and Corollary 2.1.5 we get

$$\mathbb{C}G \cong \mathrm{Mat}_1(\mathbb{C}) \oplus \mathrm{Mat}_2(\mathbb{C}) \oplus ... \oplus \mathrm{Mat}_k(\mathbb{C})$$

as $\mathbb{C}G$-algebras. The $\mathbb{C}$-characters are functions $G \to \mathbb{C}$ that are constant on conjugacy classes of G. If $\{S_1, ..., S_k\}$ is a system of representatives of the isomorphism classes simple $\mathbb{C}G$-modules, then we denote the set of irreducible $\mathbb{C}$-characters of G by the $\mathrm{Irr}(\mathbb{C}G) := \{\chi_1, ..., \chi_k\}$ where $\chi_i := \chi_{S_i}$ for $1 \leq i \leq k$. We have an important relation between the number of simple $\mathbb{C}G$-modules and the structure of G:

Proposition 2.1.6. *The number $k := k(G)$ of simple $\mathbb{C}G$-modules is equal to the number of conjugacy classes of G.*

In other words, the number of irreducible $\mathbb{C}$-characters of G is equal to the number of conjugacy classes of G. The following proposition shows another property of irreducible characters:

Proposition 2.1.7. *The irreducible $\mathbb{C}$-characters χ_1, χ_2, ..., χ_k of G comprise a basis for the space of functions $G \to \mathbb{C}$ that are constant on conjugacy classes of G.*

Proof. Since the irreducible $\mathbb{C}$-characters χ_1, χ_2, ..., χ_k are linearly independent, they span a subspace of the space C of functions $G \to \mathbb{C}$ that are constant on conjugacy classes. Then, $\dim_{\mathbb{C}}(C) = k(G)$. Hence, χ_1, χ_2, ..., χ_k span C, and they form a basis of C. □

The *Character Ring* $R(G)$ of G is generated by the $\mathbb{C}$-characters with the addition and multiplication given by direct sum and tensor product of $\mathbb{C}$-characters. Then, $R(G)$ is a commutative ring with trivial $\mathbb{C}$-character $\chi_{\mathbb{C}}$ as identity element. By Proposition 2.1.7, $R(G)$ is a free abelian group with basis given by the set $\mathrm{Irr}(\mathbb{C}G)$ of irreducible $\mathbb{C}$-characters, thus

$$R(G) = \bigoplus_{i=1}^{k} \mathbb{Z}\chi_i \tag{2.5}$$

where the χ_i are irreducible $\mathbb{C}$-characters of G. Notice that $\chi = \psi$ in $R(G)$ if and only if $M \cong N$ for $\mathbb{C}$-characters χ, ψ of $\mathbb{C}G$-modules M and N, respectively. For computing characters, we have two important relations: the row orthogonality given by

$$\sum_{g \in G} \varphi(g)\psi(g^{-1}) = \begin{cases} |G|, & \text{if } \varphi = \psi, \\ 0, & \text{otherwise,} \end{cases}$$

for $\varphi, \psi \in \mathrm{Irr}(\mathbb{C}G)$, and the column orthogonality is given by

$$\sum_{\chi \in \mathrm{Irr}(\mathbb{C}G)} \chi(h)\chi(k^{-1}) = \begin{cases} |C_G(h)|, & \text{if } h =_G k, \\ 0, & \text{otherwise,} \end{cases}$$

for $h, k \in G$.

We now consider about some basics of modular representation theory, which are also covered in detail in [13]. Let p be a prime number and $(K, R, \mathbb{F})$ be p-modular system where R is complete discrete valuation ring with quotient field K of characteristic 0 and residue field $\mathbb{F} := R/\wp$ of the characteristic p such that $\wp$ is the maximal ideal of R. Assume here that K is sufficiently large, thus K contains all $|G|^{\text{th}}$ roots of unity and $\mathbb{F}$ is algebraically closed.

Let $f : R \to \mathbb{F} = R/\wp$ be residue class homomorphism. For a primitive $|G|_{p'}^{\text{th}}$ root of unity ξ, we have an isomorphism $f : \langle \xi \rangle \to \langle f(\xi) \rangle$ of cyclic groups. For a

$\mathbb{F}G$-module S and associated representation $\rho : G \to GL_n(\mathbb{F})$, we define a map

$$\phi_S : G_{p'} \to R, \qquad g \mapsto \sum_{i=1}^{n} f^{-1}(\bar{\xi_i}),$$

where $\bar{\xi_1}, \bar{\xi_2}, ..., \bar{\xi_n}$ are the eigenvalues of $\rho(g)$. This mapping is a function $G_{p'} \to K$ which is constant on conjugacy classes of the p'-elements of G and it is called *Brauer* $\mathbb{F}$*-character* of M. If $\{S_1, ..., S_\ell\}$ is a system of representatives of the isomorphism classes of simple $\mathbb{F}G$-modules, then we denote the set of irreducible Brauer $\mathbb{F}$-characters of G by $\mathrm{IBr}(\mathbb{F}G) := \{\phi_1, \phi_2, ..., \phi_\ell\}$ where $\phi_i := \phi_{S_i}$ for $1 \leq i \leq \ell$. Furthermore, we have the following propositions:

Proposition 2.1.8. *The number $\ell := \ell(G)$ of simple $\mathbb{F}G$-modules is equal to the number of conjugacy classes of p'-elements of G.*

Proposition 2.1.9. *The irreducible Brauer $\mathbb{F}$-characters $\phi_1, \phi_2, ..., \phi_\ell$ of G comprise a basis for the space of G-invariant functions $G_{p'} \to K$ that are constant on conjugacy classes of the p'-elements of G.*

The *Brauer Character Ring* $R_\mathbb{F}(G)$ of G is generated by the $\mathbb{F}$-characters with the addition and multiplication given by direct sum and tensor product of Brauer $\mathbb{F}$-characters. Then, $R_\mathbb{F}(G)$ is a commutative ring with trivial $\mathbb{F}$-character $\chi_\mathbb{F}$ as identity element. We can interpret $R_\mathbb{F}(G)$ as a free abelian group with basis given by the set $\mathrm{IBr}(\mathbb{F}G)$ of irreducible Brauer $\mathbb{F}$-characters of G, thus

$$R_\mathbb{F}(G) = \bigoplus_{i=1}^{\ell} \mathbb{Z}\phi_i \tag{2.6}$$

where the ϕ_i are irreducible Brauer $\mathbb{F}$-characters of G. Notice that $\phi = \varphi$ in $R_\mathbb{F}(G)$ if and only if M and N have the same composition factors including multiplicities for Brauer $\mathbb{F}$-characters ϕ, φ of $\mathbb{F}G$-modules M and N, respectively.

Finally, we state some results about conjugation, restriction and induction on $R(G)$ and $R_\mathbb{F}(G)$. Let H be a subgroup of a finite group G and $\mathbb{F}$ be a field. For a $\mathbb{F}$-character ψ of a $\mathbb{F}H$-module W, the $\mathbb{F}$-character $\mathrm{con}^g_{{}^gH,H}(\psi)$ of $\mathrm{con}^g_{{}^gH,H}(W)$ is called the *conjugation* of ψ to G. The character $\mathrm{con}^g_{{}^gH,H}(\psi)$ is obtained by conjugating of the value of ψ on the elements of G. For a $\mathbb{F}$-character $\mathrm{res}_{H,G}(\chi)$

of $\mathrm{res}_{H,G}(U)$, the $\mathbb{F}$-character $\mathrm{res}_{H,G}(\chi)$ of $\mathrm{res}_{H,G}(U)$ is called the *restriction* of χ to G. The character $\mathrm{res}_{H,G}(\chi)$ is obtained from χ by evaluating χ on the elements of H. For a $\mathbb{F}$-character ϕ of a $\mathbb{F}H$-module V, the $\mathbb{F}$-character $\mathrm{ind}_{G,H}(\phi)$ of $\mathrm{ind}_{G,H}(V)$ is called the *induction* of ϕ to G. The values of $\mathbb{F}$-character $\mathrm{ind}_{G,H}(\phi)$ are given by

$$\mathrm{ind}_{G,H}(\phi)(g) = \frac{1}{|H|}\sum_{y \in G} \dot{\phi}(y^{-1}gy)$$

where $\dot{\phi}(g)$ is $\phi(g)$ if $g \in G$ and 0 otherwise, for all $g \in G$. Then, the conjugation, restriction and induction functor on $\mathbb{F}G$-modules give rise to morphisms between the corresponding (Brauer) character rings

$$\begin{array}{rcccc} \mathrm{con}^g_{{}^gH,H} & : & R_{\mathbb{F}}(H) & \to & R_{\mathbb{F}}({}^gH) \\ \mathrm{res}_{H,G} & : & R_{\mathbb{F}}(G) & \to & R_{\mathbb{F}}(H) \\ \mathrm{ind}_{G,H} & : & R_{\mathbb{F}}(H) & \to & R_{\mathbb{F}}(G). \end{array}$$

Notice that $\mathrm{con}^g_{{}^gH,H}$ and $\mathrm{res}_{H,G}$ are ring homomorphisms, however $\mathrm{ind}_{G,H}$ is just an additive group homomorphism.

2.2 The Monomial Ring

In this section, we are concerned with the ring of monomial representations of a finite group which is firstly studied by Dress in [18]. The theory of this ring can also be found in [19], [15] and [16].

Throughout, let R be an arbitrary commutative ring and G be a finite group.

A *monomial representation* of RG is defined to be a finite dimensional RG-module V together with a decomposition $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ into 1-dimensional submodules $V_1, V_2, ..., V_n$, which are called the *lines* of V, and R-linear action of G on V such that $g \in G$ permutes the lines of V. A *morphism* of two monomial representations $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ and $W = W_1 \oplus W_2 \oplus ... \oplus W_m$ of RG is defined to be a homomorphism $f : V \to W$ of RG-modules commuting with the G-action and for each line V_i, $1 \leq i \leq n$, of V there exists a line W_j,

$1 \leq j \leq m$, of W such that $f(V_i) \subseteq W_j$. Morphisms of monomial representations of RG, according to the sequential execution of the corresponding RG-module homomorphisms, are linked. Two monomial representations $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ and $W = W_1 \oplus W_2 \oplus ... \oplus W_m$ of RG are isomorphic, if the corresponding RG-module homomorphism $f : V \to W$ is an isomorphism. Notice that the monomial representations of RG and their morphisms form the *monomial category* which is denoted by $\mathrm{Mon}(RG)$. That is, we can think the objects of $\mathrm{Mon}(RG)$ as RG-modules with some additional structures.

For the monomial representations $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ and $W = W_1 \oplus W_2 \oplus ... \oplus W_m$ of RG, we define

- the *direct sum* $V \oplus W$ of V and W as the direct sum of RG-modules V and W together with the decomposition $V \oplus W = V_1 \oplus V_2 \oplus ... \oplus V_n \oplus W_1 \oplus W_2 \oplus ... \oplus W_m$ and the obvious G-action,
- the *tensor product* $V \otimes W$ of V and W as the tensor product of RG-modules V and W together with the decomposition $V \otimes W = \bigoplus_{i,j} V_i \otimes W_j$ and the diagonal G-action.

Notice that both direct sum and tensor product are in $\mathrm{Mon}(RG)$. Let H be a subgroup of G. For a monomial representation $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ of RH, the *conjugation* of V, denoted by $\mathrm{Con}^g_{{}^gH,H}(V)$, is defined to be the monomial representation of R^gH such that $\mathrm{Con}^g_{{}^gH,H}(V) := V$ as RG-modules and the action is given by $({}^gh) \cdot V_i := hV_i$ for the lines V_i of V, where $h \in H$, $g \in G$. For a monomial representation $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ of RG, the *restriction* of V to H, denoted by $\mathrm{Res}_{H,G}V$, is defined to be the monomial representation of RH such that the underlying module and lines are the same and the H-action is the G-action restricted to H. For a monomial representation $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ of RH, the *induction* of V to G, denoted by $\mathrm{Ind}_{G,H}V$, is defined to be the monomial representation of RG such that the underlying module is $RG \otimes_{RH} V$ with the decomposition $RG \otimes_{RH} V = \bigoplus_{g,i} g \otimes_{RH} V_i$, where g runs through a set of representatives of G/H, and the G-action is given by left multiplication on the factor RG. It is easy to see that $\mathrm{Con}^g_{{}^gH,H}$, $\mathrm{Res}_{H,G}$ and $\mathrm{Ind}_{G,H}$ induces the conjugation, restriction and induction functors

$$\begin{aligned}
\mathrm{Con}^g_{{}^gH,H} &: \mathrm{Mon}(RH) \rightarrow \mathrm{Mon}(R^gH), && (2.7)\\
\mathrm{Res}_{H,G} &: \mathrm{Mon}(RG) \rightarrow \mathrm{Mon}(RH), && (2.8)\\
\mathrm{Ind}_{G,H} &: \mathrm{Mon}(RH) \rightarrow \mathrm{Mon}(RG). && (2.9)
\end{aligned}$$

Furthermore, there is a forgetful functor

$$\mathcal{F}_1 \quad : \quad \mathrm{Set}(G) \quad \rightarrow \quad \mathrm{Mon}(RG)$$

from the category $Set(G)$ of finite G-sets to the category $\mathrm{Mon}(RG)$ of monomial representations of RG which sends each finite G-set X to the RG-module RX with decomposition $\{Rx : x \in X\}$, and there is another forgetful functor

$$\mathcal{F}_2 \quad : \quad \mathrm{Mon}(RG) \quad \rightarrow \quad \mathrm{Mod}(RG)$$

from the category $\mathrm{Mon}(RG)$ of monomial representations of RG to the category $\mathrm{Mod}(RG)$ of RG-modules which commutes with $\oplus$, $\otimes$, $\mathrm{Con}^g_{{}^gH,H}$, $\mathrm{Res}_{H,G}$, $\mathrm{Ind}_{G,H}$ and forgets the decomposition into lines.

The monomial representation V of RG is called *transitive* if the G-action on the lines of V is transitive, thus for any lines V_i, V_j of V there exists a $g \in G$ such that $gV_i = V_j$. That is, any line V_i of V is obviously transitive. Hence, each monomial representation $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ of RG is a direct sum of transitive monomial representations of RG. A decomposition of V corresponds the G-orbits of the set of lines of V and it provides a unique decomposition of V. Thus, each monomial representation V of RG has a unique decomposition into transitive monomial representations of RG.

The *Monomial Ring* $D_R(G)$ is generated by the isomorphism classes of transitive objects in the category $\mathrm{Mon}(RG)$ of monomial representations of RG with the addition and multiplication given by direct sum and tensor product of monomial representations of RG

$$[V] + [W] = [V \oplus W] \quad \text{and} \quad [V][W] = [V \otimes W]$$

for monomial representations V, W of RG. It follows that $D_R(G)$ is a commutative ring with the multiplicative identity $1_{D_R(G)} = [R]$ where R is the trivial

1-dimensional monomial representation of RG. By Krull-Schmidt Theorem, one can say that $D_R(G)$ is a free abelian group with basis given by the isomorphism classes of transitive monomial representations of RG, thus

$$D_R(G) = \bigoplus \mathbb{Z}[S_i] \tag{2.10}$$

where the S_i are transitive monomial representations of RG.

In order to understand the ring $D_R(G)$ more precisely, we need to investigate the isomorphism classes of transitive monomial representations of RG. We define a *monomial pair* of G on R to be (H,φ) where $H \leq G$ and $\varphi \in \hat{H}(R) := \mathrm{Hom}(H, R^{\times})$. Consider the set of all monomial pairs of G on R

$$\mathcal{M}_R(G) := \{ \ (H,\varphi) \mid H \leq G, \varphi \in \hat{H}(R) \ \}.$$

G acts from the left on $\mathcal{M}_R(G)$ by componentwise conjugation ${}^g(H,\varphi) := ({}^gH, {}^g\varphi)$ where ${}^g\varphi({}^gh) = \varphi(h)$. We denote the stabilizer of (H,φ) by

$$N_G(H,\varphi) := \{g \in G | {}^g(H,\varphi) = (H,\varphi)\},$$

so that $H \leq N_G(H,\varphi) \leq N_G(H)$. Also, we denote the G-orbit of (H,φ) by $[H,\varphi]_G$ and the set of G-orbits by $\mathcal{M}_R(G)/G := \{[H,\varphi]_G \mid (H,\varphi) \in \mathcal{M}_G\}$. Then $\mathcal{M}_R(G)$ and $\mathcal{M}_R(G)/G$ become partial order sets with the relations

$$(H,\varphi) \leq (K,\psi) \quad \Leftrightarrow \quad H \leq K \ \text{ and } \ \varphi = \mathrm{res}_{H,G}\psi,$$
$$[H,\varphi]_G \leq [K,\psi]_G \quad \Leftrightarrow \quad (H,\varphi) \leq {}^g(K,\psi) \ \text{ for some } g \in G.$$

Remark 2.2.1. If R is a field of characteristic p, then we define

$$\mathcal{M}_R(G) := \mathcal{M}_R^{(p)}(G) = \{ \ (H,\varphi) \mid H \leq G, \varphi \in \hat{H}(R)_{p'} \ \}$$

where $\hat{H}(R)_{p'}$ denotes the set of p'-elements of the group $\hat{H}(R)$. Obviously, the set of monomial pairs $\mathcal{M}_R^{(p)}(G)$ and the set of G-orbits $\mathcal{M}_R^{(p)}(G)/G$ become partial order sets in a similar way. In the rest of the section, it is understandable that we will use the notation $\mathcal{M}_R(G)$.

Now, we show the bijective correspondence between isomorphism classes of transitive monomial representations of RG and G-orbits of monomial pairs of G on R by the following proposition:

Proposition 2.2.2. [Theorem 2.1.1, [16]] *Let G be a finite group and R be any commutative ring. Then, there is one-to-one correspondence between isomorphism classes of transitive monomial representations of RG and G-orbits of monomial pairs of G on R as in the following way:*

(i) *For each pair $(H, \varphi) \in \mathcal{M}_G(R)$, we have the transitive monomial representation $\text{Ind}_{G,H}(R_\varphi)$ of RG,*

(ii) *Each simple monomial representation V of RG is isomorphic to a $\text{Ind}_{G,H}(R_\varphi)$,*

(iii) *For each pairs $(H, \varphi), (K, \psi) \in \mathcal{M}_G(R)$, we have*

$$\text{Ind}_{G,H}(R_\varphi) \cong \text{Ind}_{G,K}(R_\psi) \Leftrightarrow [H, \varphi]_G = [K, \psi]_G,$$

where R_φ is the corresponding 1-dimensional monomial representation of RH for φ.

Proof. **(i)** The RG-module $\text{Ind}_{G,H}(R_\varphi)$ corresponds to a transitive monomial representation because the G-action on the lines $g \otimes_{RH} R_\varphi$ is the same as G-action on G/H and it is transitive.
(ii) Let $V = V_1 \oplus V_2 \oplus ... \oplus V_n$ be transitive monomial representation of RG. Notice that by letting H be the stabilizer of V_1, $\varphi \in \hat{V}_1(R)$ for RH-module V_1 and $g_i \in G$ such that $g_i V_1 = V_i$ for each i, we observe that the g_i are a set of representatives of G/H. Then, for a non-zero $v \in V_1$ we get $v_i = g_i v \in V_1$ and the correspondence $v_i \leftrightarrow g_i \otimes_{RH} 1$ gives an isomorphism between V and $\text{Ind}_{G,H}(R_\varphi)$.
(iii) If $\text{Ind}_{G,H}(R_\varphi) \cong \text{Ind}_{G,K}(R_\psi)$, then $g \otimes_{RH} R_\varphi \cong g \otimes_{RH} R_\psi$. It follows that $[H, \varphi]_G \leq [K, \psi]_G$ and $[K, \psi]_G \leq [H, \varphi]_G$, hence $[H, \varphi]_G = [K, \psi]_G$. If $[H, \varphi]_G = [K, \psi]_G$, then $H = {}^gK$, $\varphi = \text{Res}_{H,G}({}^g\psi)$. It follows that $\text{Ind}_{G,H}(R_\varphi) = \text{Ind}_{G,{}^gK}(R_{\text{Res}_{H,G}{}^g\psi}) = \text{Ind}_{G,K}(R_\psi)$. □

In the view of the Proposition 2.2.2, we can express the monomial ring $D_R(G)$ as the free abelian group with basis given by the elements of $\mathcal{M}_R(G)/G$, thus

$$D_R(G) = \bigoplus_{[H,\varphi]_G \in \mathcal{M}_R(G)/G} \mathbb{Z}[H, \varphi]_G.$$

Then, $D_R(G)$ is commutative ring with multiplicative identity element $[G, 1_G]_G$. Writing $\mathrm{Ind}_{G,H}(R_\varphi) \otimes \mathrm{Ind}_{G,K}(R_\psi)$ as a direct sum of transitive monomial representations, we can get the multiplication rule in $D_R(G)$ as follows:

Lemma 2.2.3. *Let G be a finite group, R be any commutative ring and $D_R(G)$ be the monomial ring of G. For $[H,\varphi]_G, [K,\psi]_G \in \mathcal{M}_R(G)/G$, we have*

$$[H,\varphi]_G \cdot [K,\psi]_G := \sum_{s \in H/G \backslash K} [H \cap {}^sK, \varphi \cdot {}^s\psi]_G$$

where $\varphi \cdot {}^s\psi = \mathrm{Res}_{H \cap {}^sK, H}\varphi \cdot \mathrm{Res}_{H \cap {}^sK, K}{}^s\psi$.

Let $H \leq G$. Then, the restriction functor $\mathrm{Res}_{H,G} : \mathrm{Mon}(RG) \to \mathrm{Mon}(RH)$ induces a ring homomorphism between the corresponding rings

$$\begin{array}{rcccl} \mathrm{res}_{H,G} & : & D_R(G) & \to & D_R(H), \\ & & [K,\psi]_G & \mapsto & \displaystyle\sum_{s \in H/G \backslash K} [H \cap {}^sK, {}^s\psi]_H \end{array}$$

where ${}^s\psi = \mathrm{res}_{H \cap {}^sK, K}{}^s\psi$. On the other hand, for $(K,\psi) \in \mathcal{M}_R(H)$ we have an isomorphism $\mathrm{Ind}_{G,H}(\mathrm{Ind}_{H,K}(R_\psi)) \cong \mathrm{Ind}_{G,K}(R_\psi)$ of monomial representations of RG. Then, the induction functor $\mathrm{Ind}_{G,H} : \mathrm{Mon}(RH) \to \mathrm{Mon}(RG)$ induces an additive group homomorphism between the corresponding groups

$$\begin{array}{rcccl} \mathrm{ind}_{G,H} & : & D_R(H) & \to & D_R(G), \\ & & [K,\psi]_H & \mapsto & [K,\psi]_G. \end{array}$$

Moreover, the forgetful functor $\mathcal{F}_1 : \mathrm{Set}(G) \to \mathrm{Mon}(RG)$ induces an embedding ring homomorphism between the corresponding rings

$$\begin{array}{rcccl} \eta_G & : & B(G) & \to & D_R(G), \\ & & [G/H] & \mapsto & [H,1]_G \end{array}$$

where $B(G)$ denotes the Burnside ring of G, and the forgetful functor $\mathcal{F}_2 : \mathrm{Mon}(RG) \to \mathrm{Mod}(RG)$ induces a ring homomorphism between the corresponding rings

$$\begin{array}{rcccl} \pi_G & : & D_R(G) & \to & R_R(G), \\ & & [H,\varphi]_G & \mapsto & \mathrm{Ind}_{G,H}(R_\psi) \end{array}$$

where $R_R(G)$ is the representation ring of G over R. There is no confusion that the subindex R indicates the ring R and the other R indicates the character ring. Note that the injective ring homomorphism $\eta_G : B(G) \to D_R(G)$ has an inverse

$$\begin{array}{rcccl} \tau_G & : & D_R(G) & \to & B(G), \\ & & [H, \varphi]_G & \mapsto & [G/H], \end{array}$$

that is $\tau_G \cdot \eta_G$ is identity on $B(G)$. Also, by Brauer's induction theorem the ring homomorphism $\pi_G : D_R(G) \to R_R(G)$ is surjective.

Before ending this section we state two more properties of monomial ring $D_R(G)$:

Proposition 2.2.4. (Mackey Formula) *Given$U, V \subseteq G$, then we have*

$$\text{res}_{V,G}\text{ind}_{G,U}(x) = \sum_{s \in V/G\backslash U} \text{ind}_{V,V\cap^s U}\text{res}_{V\cap^s U,^s U}(^s x)$$

for all $x \in D_R(U)$.

Proof. See Proposition 1.29 of [19]. □

Proposition 2.2.5. *Let $H \leq G$, then we have*

$$x \cdot \text{ind}_{G,H}(y) = \text{ind}_{G,H}(\text{res}_{H,G}(x) \cdot y)$$

for all $x \in D_R(G), y \in D_R(H)$.

Proof. See Proposition 1.30 of [19]. □

2.3 The Trivial Source Ring

In this section, we will mention about the trivial source ring which is the object of our main purpose. The general theory for the trivial source rings can found in classical representation theory books such as [12], [13]. For more details, see [20], [21] and [16].

Throughout this section, let G be a finite group and $(K, R, \mathbb{F})$ be a p-modular system where R is a complete discrete valuation ring with residue field $\mathbb{F}$ of characteristic $p > 0$ and quotient field K of characteristic zero which is sufficiently large, thus contains all $|G|^{\text{th}}$ roots of unity. That is, K and $\mathbb{F}$ are splitting fields for G and its all subgroups. Assume that all RG-modules used here are finitely generated and the Krull-Schmidt Theorem holds for finitely generated RG-modules.

Let H be a subgroup of G. An RG-module M is called *projective relative to* H or *H-projective* if every short exact sequence of RG-modules

$$0 \to A \to B \to M \to 0$$

for which the short exact sequence of restrictions to H

$$0 \to \text{Res}_{H,G}(A) \to \text{Res}_{H,G}(B) \to \text{Res}_{H,G}(M) \to 0$$

splits, is also a split exact sequence of RG-modules. Similarly, an RG-module M is called *injective relative to* H or *H-injective* if every short exact sequence of RG-modules

$$0 \to M \to B \to A \to 0$$

for which

$$0 \to \text{Res}_{H,G}(M) \to \text{Res}_{H,G}(B) \to \text{Res}_{H,G}(A) \to 0$$

is a split exact sequence of RH-modules, is also a split exact sequence of RG-modules. Notice that if $H = G$ then there is no any restriction on the RG-module M. If $H = 1$, then every projective RG-module M is 1-projective. Moreover, for a field R, the RG-module M is projective if and only if it is 1-projective.

Theorem 2.3.1. [Theorem 19.2, [13]] *Let M be an finitely generated RG-module. Then the following are equivalent:*

1. *M is H-projective.*
2. *M is H-injective.*
3. *M is a direct summand of* $\text{Ind}_{G,H}(\text{Res}_{H,G}(M))$.

4. M is a direct summand of $\mathrm{Ind}_{G,H}(N)$ *for some RH-module N.*

We now state some observations which will be fundamental in the rest of section:

- If M is H-projective RG-module, then it is K-projective for every subgroup $K \geq_G H$.
- Let L be an RH-module and M be an RG-module such that $M \mid \mathrm{Ind}_{G,H}(L)$. If L is K-projective for a subgroup K of H, then M is K-projective.
- Let M be an RG-lattice with $M = M_1 \oplus M_2$ for RG-lattices M_1 and M_2. Then M is H-projective if and only if both M_1 and M_2 are H-projective.
- If M is an H-projective RG-lattice, then M is also ${}^gH$-projective for all $g \in G$.
- Let $|G : H|$ is a unit in R. Then every RG-module M is H-projective and $M \mid \mathrm{Ind}_{G,H}(\mathrm{Res}_{H,G}(M))$.

Hence, it is easy to obtain the following corollaries:

Corollary 2.3.2. *Let R be a field of characteristic $p > 0$, and let P be a fixed p-Sylow subgroup of G. Then every RG-module M is P-projective.*

Proof. See Theorem 63.7 of [22]. □

Corollary 2.3.3. *Suppose G is invertible in R. Then every RG-module which is projective as an R-module is projective as an RG-module. In particular, every short exact sequence of such modules splits.*

Proof. See Theorem 3.6.11 of [12]. □

In the rest of section, we will concern with RG-lattices, thus finitely generated RG-module which are R-projective. If R is a field of characteristic p, then every finitely generated RG-modules is an RG-lattice.

An RG-lattice is said to be *indecomposable* if it is non-zero and cannot be written as a direct sum of two non-zero sublattices. Also, we denote the set of all indecomposable RG-lattices by Indec(RG). Since Krull-Schmidt Theorem holds for all RG-lattices, any RG-lattice M can be expressed as a direct sum

$$M = \bigoplus_{i=1}^{m} U_i$$

where $U_i \in \text{Indec}(RG)$. Notice that the indecomposable summands $\{U_i\}$ are determined in a unique way up to isomorphism. In the view of this fact, we give our attention to the indecomposable lattices instead of any lattice. The theory of vertices and sources are based on the structure of indecomposable RG-lattices in terms of certain p-subgroups of G and associated RG-lattices.

For an RG-lattice M, we denote by $\mathcal{V}(M)$ the set of all subgroups H of G such that M is H-projective. Note that for each RG-lattice M, $\mathcal{V}(M)$ is a partially ordered set under the relation $\leq_G$ and $\mathcal{V}(M)$ is nonempty since M is G-projective.

Definition 2.3.4. For each $M \in \text{Indec}(RG)$, there exists a minimal subgroup $D \in \mathcal{V}(M)$, thus $D \leq_G H$ for all $H \in \mathcal{V}(M)$. Such a subgroup D is called a *vertex* of M, and the set of all vertices of M is denoted by $\text{vtx}(M)$.

Definition 2.3.5. Let $M \in \text{Indec}(RG)$. For each $D \in \text{vtx}(M)$, there exists an $L \in \text{Indec}(RD)$ such that $M \mid \text{Ind}_{G,H}(L)$. Such a RD-module L is called a *source* of M.

Notice that each $M \in \text{Indec}(RG)$ has a vertex and by the Krull-Schmidt Theorem for each $D \in vtx(M)$ there exists an $L \in \text{Indec}(RD)$ which is a source of M. We have the following characterization of the vertices and sources of indecomposable modules:

Proposition 2.3.6. *Let M be an indecomposable RG-module. Then:*

(i) *The vertices of M are conjugate in G.*

(ii) *The vertices of M are always p-subgroups of G.*

(iii) *If $D \in vtx(M)$ and L, L' are indecomposable RD-lattices which are both sources of M, then $L' \cong {}^{g}L$ for some $g \in N_G(D)$.*

Proof. See Theorem 19.13 of [13]. □

Theorem 2.3.7. *Let M be an indecomposable RG-module with vertex D and H be a subgroup of G such that M is H-projective. Let $\mathrm{Res}_{H,G}(M) = L_1 \oplus \ . \ . \ . \ \oplus L_s$ where each L_i is indecomposable RH-module with vertex D_i for each $1 \leq i \leq s$. Then:*

(i) *$D_i \leq_G D$ for each i.*

(ii) *$M \mid \mathrm{Ind}_{G,H}(L_i)$ for some i, and for this i we have $D_i =_G D$. Moreover, if $D \leq H$ then $D_i =_H D$ for some i.*

(iii) *If $D_i =_G D$ then M and L_i have a common source.*

Proof. See pp. 113, Lemma 4.6 of [14]. □

Corollary 2.3.8. *Let $H \leq G$ and $L \in \mathrm{Indec}(RH)$. Then there exists a $M \in \mathrm{Indec}(RG)$ such that $M \mid \mathrm{Ind}_{G,H}(L)$ and $L \mid \mathrm{Res}_{H,G}(M)$. Moreover, the modules L and M have a vertex and source in common.*

We are now able to state major results on restriction and induction of indecomposable lattices which can be found in [23] in detail:

Theorem 2.3.9. (Green Correspondence) *Let D be a p-subgroup of G and let $H \leq N_G(D)$. Then there is a one-to-one correspondence from the set of all isomorphism classes of indecomposable RG-lattices with vertex D onto the set of all isomorphism classes of indecomposable RH-lattices with vertex D as in the following way:*

(i) *If M is an indecomposable RG-lattice with vertex D, then $\mathrm{Res}_{H,G}(M)$ has a unique indecomposable direct summand $f(M)$ with vertex D, up to isomorphism. Moreover, $f(M)$ has multiplicity 1 in $\mathrm{Res}_{H,G}(M)$, and M and $f(M)$ have a common source.*

(ii) *If N is an indecomposable RH-lattice with vertex D, then* $\mathrm{Ind}_{G,H}(N)$ *has a unique indecomposable direct summand $g(N)$ with vertex D, up to isomorphism. Moreover, $g(N)$ has multiplicity 1 in* $\mathrm{Ind}_{G,H}(N)$*, and N and $g(N)$ have a common source.*

That is, $g(f(M)) = M$ and $f(g(N)) = N$ for all $M \in \mathrm{Indec}(RG)$, $N \in \mathrm{Indec}(RH)$. It follows that the we have $[M] \leftrightarrow [N]$ between isomorphism class of indecomposable RG-lattice M with vertex D and isomorphism class of indecomposable RH-lattice N with vertex D where $N \mid \mathrm{Res}_{H,G}(M)$ and $M \mid \mathrm{Ind}_{G,H}(N)$.

Theorem 2.3.10. (Green Indecomposability) *Let H be a subnormal subgroup of G of index a power of p and let M be an indecomposable RH-module. Then* $\mathrm{Ind}_{G,H}(M)$ *is an indecomposable RG-module. In particular, if G is a p-group and if M is an indecomposable RP-module for some subgroup P of G, then* $\mathrm{Ind}_{G,P}(M)$ *is indecomposable.*

In [21] and [16], letting $H := N_G(P)$ the Green correspondence gives another one-to-one correspondence as in the following way: Let M be an indecomposable RG-lattice with vertex P and trivial source R. Then, M is in Green correspondence to an indecomposable $R[N_G(P)]$-lattice N with vertex P and trivial source. Since P is normal in $N_G(P)$, P acts trivially on N. That is, N can be regarded as a projective indecomposable $R[N_G(P)/P]$-lattice. Conversely, let N be a projective indecomposable $R[N_G(P)/P]$-lattice. The inflation $\mathrm{Inf}_{N_G(P),P}(N)$ is an indecomposable $RN_G(P)$-lattice with vertex P and trivial source. Then $\mathrm{Inf}_{N_G(P),P}(N)$ is in Green correspondence to an indecomposable RG-lattice with vertex P and trivial source. This provides a bijection between the set of isomorphism classes of indecomposable RG-lattices with vertex P and trivial source and the set of isomorphism classes of indecomposable projective $R[N_G(P)/P]$-lattices.

We define a finitely generated RG-module M to be a *trivial source module* if each indecomposable direct summand of M has the trivial module R as its source. Moreover, an RG-lattice is said to be a *permutation* RG-module if it has a G-invariant finite R-basis and an RG-lattice M is said to be a *p-permutation RG-module* if $\mathrm{Res}_{P,G}(M)$ is a permutation RP-module for every Sylow p-subgroup P

of G. We give the characterization of p-permutation modules via the following lemmas which can be found in Section 27 of [24] and [9]:

Lemma 2.3.11. *Let G be a p-group and P be a subgroup of G. Then $\mathrm{Ind}_{P,G}(R)$ is indecomposable. Particularly, P is a vertex of $\mathrm{Ind}_{P,G}(R)$ and the trivial RP-lattice R is a source of $\mathrm{Ind}_{P,G}(R)$.*

Proof. The indecomposability of $\mathrm{Ind}_{P,G}(R)$ comes directly from the Green's indecomposability Theorem. Moreover P is the vertex of the trivial RP-module R and R is a direct summand of $\mathrm{Res}_{P,G}\mathrm{Ind}_{G,P}(R)$. Hence, P is a vertex of $\mathrm{Ind}_{P,G}(R)$ and the trivial RP-lattice R is a source of $\mathrm{Ind}_{P,G}(R)$. □

Lemma 2.3.12. *Let $H \leq G$, M and M' be p-permutation RG-modules, N be a p-permutation RH-module. Then*

1. *The modules $M \oplus M'$ and $M \otimes_R M'$ are p-permutation RG-modules.*
2. *The module $\mathrm{Res}_{H,G}(M)$ is p-permutation RH-module and The module $\mathrm{Ind}_{G,H}(N)$ is p-permutation RG-module.*
3. *Any direct summand of a p-permutation module is also a p-permutation module.*

Proof. The first two assertions are obvious. To prove the third assertion, it suffices to work with the restriction to a Sylow p-subgroup P. If M is a permutation RP-lattice, then M is a direct sum

$$M \cong \bigoplus_{Q_i} \mathrm{Ind}_{Q_i,P}(R)$$

for some subgroups Q_i. By the indecomposibility of $\mathrm{Ind}_{Q_i,P}(R)$ and the Krull-Schmidt Theorem, any direct summand of M is isomorphic to the direct sum of some of these factors. It follows that any direct summand is again a permutation RP-lattice. □

Corollary 2.3.13. *If G is a p-group, any direct summand of a permutation RG-lattice is a permutation RG-lattice.*

Now we are ready to give the connection between the trivial source modules and p-permutation modules:

Proposition 2.3.14. *Let M be an indecomposable RG-lattice and P be a Sylow p-subgroup of G. Then the following are equivalent:*

(i) *M is a trivial source module.*

(ii) *M is a p-permutation RG-module.*

(iii) *M is isomorphic to a direct summand of a permutation RG-module.*

Proof. $(i) \Rightarrow (iii)$ Let M be an indecomposable trivial source RG-lattice with vertex Q. Then, M is isomorphic to a direct summand of $\mathrm{Ind}_{G,Q}(R)$ which is a permutation RG-lattice. $(iii) \Rightarrow (ii)$ By Lemma 2.3.12, it is obvious. $(ii) \Rightarrow (i)$ Let M be an indecomposable p-permutation RG-module and P be a vertex of M. Then, M is isomorphic to a direct summand of $\mathrm{Ind}_{G,P}\mathrm{Res}_{P,G}(M)$. Since $\mathrm{Res}_{P,G}(M)$ is a permutation lattice, it is of the form

$$\mathrm{Res}_{P,G}(M) \cong \bigoplus_{Q_i} \mathrm{Ind}_{P,Q_i}(R)$$

for some subgroups $Q_i \leq P$. Inducing this to G and using the Krull-Schmidt Theorem, we deduce that M is indecomposable and isomorphic to a direct summand of $\mathrm{Ind}_{G,Q_i}(R)$ for some Q_i, say Q. The vertex P is the minimal subgroup with this property, thus $Q = P$. It follows that R is a source of M. $\square$

We shall use the terminology trivial source module rather than p-permutation module, because the important point is the existence of trivial source for this thesis.

The *Trivial Source Ring* $T_R(G)$ is generated by the isomorphism classes of indecomposable trivial source RG-modules with the addition and multiplication operations

$$[M] + [N] = [M \oplus N] \qquad \text{and} \qquad [M][N] = [M \otimes N]$$

where M and N are indecomposable trivial source RG-modules. It follows that $T_R(G)$ is a commutative ring with the multiplicative identity $1_{T_R(G)} = [R]$ where $[R]$ is the class of the trivial RG-module R. Since the Krull-Schmidt Theorem holds for RG-modules, the additive group $T_R(G)$ becomes a free abelian group with basis given by the isomorphism classes of indecomposable trivial source RG-modules

$$T_R(G) = \bigoplus \mathbb{Z}[M_i]$$

where the M_i are indecomposable trivial source RG-modules.

Remark 2.3.15. We may replace R by its residue field $\mathbb{F}$. It is known as *reduction modulo p*, for details see Proposition 81.17 of [25]. It yields an isomorphism $T_R(G) \cong T_{\mathbb{F}}(G)$. Here, the ring $T_{\mathbb{F}}(G)$ is generated from the isomorphism classes of trivial source $\mathbb{F}G$-modules. Because of that, we may use the isomorphism classes of trivial source $\mathbb{F}G$-modules instead of the isomorphism classes of trivial source RG-modules for constructing the trivial source ring $T_R(G)$.

For a subgroup H of G, each trivial source $\mathbb{F}H$-module can be regarded as an trivial source $\mathbb{F}^gH$-module by the conjugation action. Since conjugation preserves permutation modules, we have a conjugation functor $\mathrm{Con}^g_{{}^gH,H}$ from the trivial source $\mathbb{F}H$-modules to the trivial source $\mathbb{F}^gH$-modules. The conjugation functor gives rise to ring homomorphism between the corresponding rings

$$\mathrm{con}^g_{{}^gH,H} \quad : \quad T_{\mathbb{F}}(H) \quad \to \quad T_{\mathbb{F}}({}^gH).$$

For a group homomorphism $f : G \to G'$, every trivial source $\mathbb{F}G$-module can be regarded as an trivial source $\mathbb{F}G'$-module using the G'-action given by restriction along f. Since restriction preserves permutation modules, we have a restriction functor $\mathrm{Res}_{G',G}$ from the trivial source $\mathbb{F}G$-modules to the trivial source $\mathbb{F}G'$-modules. For an inclusion $f : G \to H$, $H \leq G$, the restriction functor gives rise to a ring homomorphism between the corresponding rings

$$\mathrm{res}_{H,G} \quad : \quad T_{\mathbb{F}}(G) \quad \to \quad T_{\mathbb{F}}(H).$$

On the other hand, $H \leq G$, for a trivial source $\mathbb{F}H$-module M we can construct a trivial source $\mathbb{F}G$-module $\mathrm{ind}_{G,H}(M) := \mathbb{F}G \otimes M$. Since induction preserves permutation modules, we have an induction functor $\mathrm{Ind}_{G,H}$ from the trivial source $\mathbb{F}H$-modules to the trivial source $\mathbb{F}G$-modules. The induction functor gives rise to a group homomorphism between the corresponding groups

$$\mathrm{ind}_{G,H} \quad : \quad T_{\mathbb{F}}(H) \quad \to \quad T_{\mathbb{F}}(G).$$

Before closing this section, we state an important result about the relation between the Burnside ring $B(G)$ and the trivial source ring $T_{\mathbb{F}}(G)$ and the modular character ring $R_{\mathbb{F}}(G)$. Every finite G-set determines a permutation $\mathbb{F}G$-module, thus a trivial source $\mathbb{F}G$-module. Thus, we have a ring homomorphism $B(G) \to T_{\mathbb{F}}(G)$ where $B(G)$ is the Burnside ring of G. On the other hand, every trivial source $\mathbb{F}G$-module determines a Brauer character. Thus, we have a ring homomorphism $T_{\mathbb{F}}(G) \to R_{\mathbb{F}}(G)$. Moreover, we have

Proposition 2.3.16. *Let G be a finite group, $(K, R, \mathbb{F})$ be a p-modular system. Then,*

1. *If G is a p-group, then $T_{\mathbb{F}}(G) \cong B(G)$.*
2. *If G is a p'-group, then $T_{\mathbb{F}}(G) \cong R_{\mathbb{F}}(G)$.*

Proof. (1) Let M be an indecomposable trivial source $\mathbb{F}G$-module with vertex P for a p-group G. Then $M|\mathrm{Ind}_{G,P}(\mathbb{F})$. By Green's indecomposability we have $[M] = [\mathrm{Ind}_{G,P}(\mathbb{F})]$. Conversely, $\mathrm{Ind}_{G,P}(\mathbb{F})$ is an indecomposable trivial source $\mathbb{F}G$-module with vertex P, for all $P \leq G$. Therefore, $\mathcal{U} := \{[\mathrm{Ind}_{G,P}(\mathbb{F})] : P \leq G\}$ is a $\mathbb{Z}$-basis for $T_{\mathbb{F}}(G)$. Let $P, Q \leq G$ with $[\mathrm{Ind}_{G,P}(\mathbb{F})] = [\mathrm{Ind}_{G,Q}(\mathbb{F})]$. Then

$$P \in \mathrm{vtx}(\mathrm{Ind}_{G,P}(\mathbb{F})) = \mathrm{vtx}(\mathrm{Ind}_{G,Q}(\mathbb{F})) \ni U.$$

Conversely, if $Q = {}^{g}P$ for some $g \in G$, then

$$[\mathrm{Ind}_{G,P}(\mathbb{F})] = \mathrm{con}_{G}^{g}([\mathrm{Ind}_{G,P}(\mathbb{F})]) = [\mathrm{Ind}_{G,{}^{g}P}({}^{g}\mathbb{F})] = [\mathrm{Ind}_{G,Q}(\mathbb{F})].$$

It means that $[\mathrm{Ind}_{G,P}(\mathbb{F})] = [\mathrm{Ind}_{G,Q}(\mathbb{F})]$ if and only if $Q = {}^{g}P$. A well-defined bijection $\alpha : [\mathrm{Ind}_{G,P}(\mathbb{F})] \mapsto [G/P]$ extends linearly to $\mathbb{Z}$-module isomorphism

$T_{\mathbb{F}}(G) \to B(G)$. From Mackey product formula, for $P, Q \leq G$ we have:

$$\alpha([\mathrm{Ind}_{G,P}(\mathbb{F})][\mathrm{Ind}_{G,Q}(\mathbb{F})] = \alpha(\sum_{PgQ \in P\backslash G/Q} [\mathrm{Ind}_{G,({}^gP\cap Q)}(\mathbb{F})]) \tag{2.11}$$

$$= \sum_{PgQ \in P\backslash G/Q} [G/{}^gP \cap Q] \tag{2.12}$$

$$= [G/P][G/Q] \tag{2.13}$$

$$= \alpha([\mathrm{Ind}_{G,P}(\mathbb{F})])\alpha([\mathrm{Ind}_{G,Q}(\mathbb{F})]). \tag{2.14}$$

It follows that α is a ring isomorphism.

(2) For a p'-group G, the indecomposable trivial source $\mathbb{F}G$-modules are precisely the simple $\mathbb{F}G$-modules. Hence, the assertion follows immediately. □

Chapter 3

Canonical Induction Formula

Throughout, G denotes a finite group and k a commutative ring with unity.

3.1 The Categories

In this section, we introduce the notions of Mackey functors, restriction functors and conjugation functors for G in terms of bisets.

We give an introductory review of theory of bisets that can be found in detail in Chapter 2 and 3 of [7] and [8]. For finite groups K, H, an (K, H)-*biset* is defined to be a set equipped with a left K-action and right H-action that commute with each other. Every (K, H)-biset can be regarded as a $K \times H$-set and vice versa. We define the *Burnside group* $B(K, H)$ of (K, H)-bisets as the free abelian group on the set of isomorphism classes of transitive (K, H)-bisets, thus

$$B(K, H) = \bigoplus_{U \leq_{K \times H} K \times H} \mathbb{Z}[\frac{K \times H}{U}]$$

where the sum runs over a set of representatives of conjugacy classes of subgroups of $K \times H$. For the transitive bisets $(\frac{L \times K}{U})$ and $(\frac{K \times H}{V})$, the *Mackey product*, a composition product of bisets, is explicitly given by

$$(\frac{L \times K}{U}) \times_K (\frac{K \times H}{V}) = \sum_{x \in p_2(U) \backslash K / p_1(V)} (\frac{L \times H}{U * {}^{(x,1)}V})$$

where the subgroup $U * V$ of $L \times H$ is defined by

$$U * V = \{(l,h) \in L \times H : (l,k) \in U \text{ and } (k,h) \in V \text{ for some } k \in K\}$$

and the subgroup $p_1(V)$ (resp. $p_2(U)$) of K is the projection of V (resp. of U) to K. Note that the Mackey product induces a bilinear map

$$B(L,K) \times_K B(K,H) \to B(L,H).$$

In [7], Bouc proved that any transitive biset is a Mackey product of five basic bisets in the form of

$$(\frac{K \times H}{U}) = \mathrm{ind}_{K,D} \cdot \mathrm{inf}_{D,D/C} \cdot \mathrm{iso}^{\theta}_{D/C,B/A} \cdot \mathrm{def}_{B/A,B} \cdot \mathrm{res}_{B,H}$$

for suitable $C \trianglelefteq D \leq K$, $A \trianglelefteq B \leq H$ and group isomorphism $\theta : B/A \to D/C$. Here, the five basic bisets $\mathrm{iso}^{\theta}_{G',G}$, $\mathrm{res}_{H,G}$, $\mathrm{ind}_{G,H}$, $\mathrm{inf}_{G,G/N}$ and $\mathrm{def}_{G/N,G}$ are given as follows:

- For an isomorphism $\theta : G \to G'$ of finite groups, the *isogation biset* is defined to be

$$\mathrm{iso}^{\theta}_{G',G} = {}_{G'}G'_G = [\frac{G' \times G}{\triangle(G',\theta,G)}] \qquad \text{where } \triangle(G',\theta,G) = \{(\theta(g),g) : g \in G\},$$

- For $H \leq G \trianglerighteq N$, the *induction, restriction, inflation* and *deflation bisets* are defined to be

$$\mathrm{ind}_{G,H} = {}_GG_H = [\frac{G \times H}{\triangle(G,H)}], \qquad \text{where } \triangle(G,H) = \{(h,h) : h \in H\},$$
$$\mathrm{res}_{H,G} = {}_HG_G = [\frac{H \times G}{\triangle(H,G)}], \qquad \text{where } \triangle(H,G) = \{(h,h) : h \in H\},$$
$$\mathrm{inf}_{G,G/N} = {}_GG_{G/N} = [\frac{G \times G/N}{\triangle(G,G/N)}], \qquad \text{where } \triangle(G,G/N) = \{(g,gN) : g \in G\},$$
$$\mathrm{def}_{G/N,G} = {}_{G/N}G_G = [\frac{G/N \times G}{\triangle(G/N,G)}], \qquad \text{where } \triangle(G/N,G) = \{(gN,g) : g \in G\}.$$

The *Biset Category* $\mathcal{C}$ is defined to be the category whose objects are finite groups, morphisms are $\mathrm{Hom}_{\mathcal{C}}(G,H) = B(H,G)$ for finite groups G, H and composition is the operation $B(K,H)\times_H B(H,G) \to B(K,G)$. The isomorphism classes of transitive (H,G)-bisets are called the *transitive morphisms* from G to H and they form a $\mathbb{Z}$-basis for $B(H,G)$. Also, we define the category $k\mathcal{C}$ to be the category whose objects are finite groups, morphisms are $\mathrm{Hom}_{k\mathcal{C}}(G,H) = k\otimes_{\mathbb{Z}} B(H,G)$ for finite groups G, H and composition is the k-linear extension of composition in $\mathcal{C}$. Furthermore, the *biset functor* over k is defined as an k-linear functor from $k\mathcal{C}$ to k-modules. Notice that biset functors over k form a category together with natural transformations of functors.

The *conjugation morphism* is defined as the isogation morphism

$$\mathrm{con}^g_{{}^gH,H} = [\frac{{}^gH \times H}{\{({}^gh),h) : h \in H\}}] \ : \ H \ \to \ {}^gH.$$

where $H \leq G$ and $g \in G$. Moreover, we define a *proper induction morphism* or a *proper restriction morphism* to be an induction or restriction morphism that is not an conjugation. In the rest of section, we ignore the inflation and deflation morphisms because we mainly deal with the conjugations, restrictions and inductions.

We are now able discuss the Mackey algebra which may be defined in different ways such that as an algebra in terms of bisets or as an algebra by means of axioms. After giving the definition of the Mackey algebra in two ways, we provide a proof of the equivalence of the biset definition with the axiomatic definition. For details, we refer to [4] and [26]. For a fixed G, let $\mathcal{S}(G)$ be the set of all subgroups of G. Note that $\mathcal{S}(G)$ is closed under G- conjugation and subgroups. First, consider a finite dimensional algebra

$$\mathcal{B}_k(G) := \bigoplus_{H,K\leq\mathcal{S}(G)} k \otimes_{\mathbb{Z}} B'(K,H)$$

for G over k, where $B'(K,H)$ is spanned by the bisets having the form $\mathrm{ind}_{K,{}^gU}\mathrm{con}^g_{{}^gU,U}\mathrm{res}_{U,H}$ where $g \in G$, $U \leq H$, ${}^gU \leq K$. The multiplication in $\mathcal{B}_k(G)$ is given by the same as composition in $k\mathcal{C}$. It is clear that $\mathcal{B}_k(G)$ has a basis consisting of the isomorphism classes of three kinds of transitive (K,H)-bisets, namely by conjugation, restriction and induction bisets, where $H, K \in \mathcal{S}(G)$.

Lemma 3.1.1. *Given $L \leq K \leq H \in \mathcal{S}(G)$, the bisets satisfies the following relations:*

1. $\mathrm{con}^x_{{}^gK,K} = \mathrm{res}_{K,K} = \mathrm{ind}_{K,K}$ *where* $x \in C_H(K)K$,
2. $\mathrm{con}^g_{{}^{gh}K,{}^hK} \cdot \mathrm{con}^h_{{}^hK,K} = \mathrm{con}^{gh}_{{}^{gh}K,K}$ *and* $\mathrm{ind}_{G,H} \cdot \mathrm{ind}_{H,L} = \mathrm{ind}_{G,L}$ *and* $\mathrm{res}_{L,H} \cdot \mathrm{res}_{H,G} = \mathrm{res}_{L,G}$,
3. $\mathrm{con}^g_{{}^gH,H} \cdot \mathrm{ind}_{H,K} = \mathrm{ind}_{{}^gH,{}^gK} \cdot \mathrm{con}^g_{{}^gK,K}$ *and* $\mathrm{con}^g_{{}^gK,K} \cdot \mathrm{res}_{K,H} = \mathrm{res}_{{}^gK,{}^gH} \cdot \mathrm{con}^g_{{}^gH,H}$,
4. $\mathrm{res}_{L,H}\mathrm{ind}_{H,K} = \sum_{g\in[L\backslash H/K]} \mathrm{ind}_{L,L\cap {}^gK} \cdot \mathrm{con}^g_{L\cap {}^gK,L^g\cap K}\mathrm{res}_{L^g\cap K,K}$,

Proof. We show the relation (4) explicitly and the others can be checked similarly. Since we have

$$\mathrm{ind}_{H,K} = [\frac{H \times K}{T}] \quad \text{and} \quad \mathrm{res}_{L,H} = [\frac{L \times H}{R}]$$

where $T = \{(k,k)|k \in K\}$ and $R = \{(l,l)|l \in L\}$, the Mackey product formula gives

$$[\frac{L \times H}{R}] \times_H [\frac{H \times K}{T}] = \sum_{x \in p_2(R))\backslash H/p_1(T)} [\frac{L \times K}{R * {}^{(x,1)}T}]$$

where $R * T = \{(l,h) \in L * K|(l,h) \in R \text{ and } (h,k) \in T \text{ for some } h \in H\}$. Because the map $L((l,h)R,_H (h',k)T)K \mapsto p_2(R)h^{-1}h'p_1(T)$ with the inverse map $p_2(R)hp_1(T) \mapsto L((1,1)R,_H (h,1)T)K$ gives a bijection

$$L\backslash([\frac{L \times H}{R}] \times_H [\frac{H \times K}{T}])/K \leftrightarrow p_2(R)\backslash H/p_1(T)$$

, we obtain

$$\sum_{x \in p_2(R))\backslash H/p_1(T)} [\frac{L \times K}{R * {}^(x,1)T}] = \sum_{x \in [L\backslash H/K]} \mathrm{ind}_{L,({}^xK\cap L)}\mathrm{con}^x_{({}^xK\cap L),(K\cap L^x)}\mathrm{res}_{(K\cap L^x),K}$$

as required. □

Consider the algebra $\mathcal{F}_k(G)$ freely generated over k by the elements c^g_L, r^K_H, t^K_H where $L \leq K \leq H \in \mathcal{S}(G)$. The *Mackey algebra* $\mu_k(G)$ for G over k is defined to be the quotient algebra of $\mathcal{F}_k(G)$ by the ideal $\mathcal{J}$ generated by the relations

1. $c_K^x = r_K^K = t_K^K$ when $x \in C_H(K)K$,

2. $c_K^{gh} = c_{{}^hK}^g \cdot c_K^h$ and $r_H^L = r_K^L \cdot r_H^K$ and $t_H^L = t_H^K \cdot t_K^L$,

3. $c_K^g \cdot r_H^K = r_{{}^gH}^{{}^gK} \cdot c_H^g$ and $c_H^g \cdot t_H^K = t_{{}^gH}^{{}^gK} \cdot c_K^g$,

4. $r_H^L \cdot t_H^K = \sum_{x \in [L \backslash H / K]} t_L^{{}^xK \cap L} \cdot c_{K \cap L^x}^x \cdot r_K^{K \cap L^x}$,

5. All other products are zero.

We indicate that the algebra $\mu_k(G)$ just given corresponds to the algebra $\mathcal{B}_k(G)$ given in terms of bisets by the following proposition:

Theorem 3.1.2. *The two algebras $\mu_k(G)$ and $\mathcal{B}_k(G)$ are isomorphic.*

Proof. Let identify $c_K^g \leftrightarrow \mathrm{con}^g_{{}^gK,K}$, $r_H^K \leftrightarrow \mathrm{res}_{K,H}$ and $t_H^K \leftrightarrow \mathrm{ind}_{H,K}$. Then, the assignment

$$t_K^{{}^gU} \cdot c_U^g \cdot r_H^U \quad \to \quad \mathrm{ind}_{K,{}^gU}\mathrm{con}^g_{{}^gU,U}\mathrm{res}_{U,H} = (\frac{K \times H}{A})$$

where $A = \{(k,h) \in K \times H | k = {}^gu \text{ and } h = u \text{ for some } u \in U\}$ extends linearly to an algebra homomorphism $\alpha : \mu_k(G) \to \mathcal{B}_k(G)$ by defining a homomorphism $\alpha' : \mathcal{F}_k(G) \to \mathcal{B}_k(G)$ which is zero on $\mathcal{J}$ by the Lemma 3.1.1.

On the other hand, there is an k-linear homomorphism $\beta : \mathcal{B}_k(G) \to \mu_k(G)$ by defining

$$\mathrm{ind}_{H,{}^gU}\mathrm{con}^g_{{}^gU,U}\mathrm{res}_{U,K} \to t_K^U \cdot c_U^g \cdot r_H^U + J.$$

This definition is independent of the choice of representative of the basis element up to isomorphism. That is, if $t_K^{{}^{g'}L} \cdot c_L^{g'} \cdot r_H^L$ is in the same isomorphism class then we have

$$\begin{aligned} t_H^{{}^{g'}L} \cdot c_L^{g'} \cdot r_K^L + \mathcal{J} &= t_H^{{}^{g'w}U} \cdot c_{{}^wU}^{g'} \cdot r_K^{{}^wU} + \mathcal{J} \\ &= c_H^{h^{-1}} \cdot t_H^{{}^{hg}U} \cdot c_{{}^wU}^{g'} \cdot r_K^{{}^wU} \cdot c_{U^w}^w + \mathcal{J} \\ &= t_H^{{}^gU} \cdot c_{{}^{g'w}U}^{h^{-1}} \cdot c_{{}^wU}^{g'} \cdot c_U^w \cdot r_K^U + \mathcal{J} \\ &= t_H^{{}^gU} \cdot c_U^g \cdot r_K^U + \mathcal{J} \end{aligned}$$

where $w = (g')^{-1}hg$ and $L = {}^wU$. One can immediately observe that α and β are mutually inverse homomorphisms. □

From now on we shall identify the two algebras $\mu_k(G)$ and $\mathcal{B}_k(G)$. We define the *restriction algebra* $\rho_k(G)$ for G over k as the subalgebra of the Mackey algebra $\mu_k(G)$ which is generated by $\mathrm{con}^g_{{}^gH,H}$ and $\mathrm{res}_{K,H}$ where $K \leq H \leq G$ and $g \in G$. We define the *conjugation algebra* $\gamma_k(G)$ for G over k as the subalgebra of the restriction algebra $\rho_k(G)$ which is generated by $\mathrm{con}^g_{{}^gH,H}$ where $H \leq G$ and $g \in G$. Moreover, a *Mackey functor* for G over k is defined to be a $\mu_k(G)$-module, and similarly a *restriction functor* and a *conjugation functor* is defined to be a $\rho_k(G)$-module and a $\gamma_k(G)$-module, respectively.

A *morphism* of Mackey (resp. restriction and conjugation) functors for G over k is k-module homomorphism commuting with its morphisms. Hence, the class of Mackey functors, restriction functors and conjugation functors for G over k with their morphisms form the Mackey category $\mathcal{M}_k(G)$, the restriction category $\mathcal{R}_k(G)$ and the conjugation category $\mathcal{C}_k(G)$ on G, respectively. Notice that the category $\mathcal{M}_k(G)$ has all inductions, restrictions and conjugations, and the category $\mathcal{R}_k(G)$ is the subcategory of $\mathcal{M}_k(G)$ obtained by removing all proper inductions, and the category $\mathcal{C}_k(G)$ is the subcategory of $\mathcal{M}_k(G)$ obtained by removing all proper inductions and proper restrictions, where $\mathrm{Obj}(\mathcal{M}_k) = \mathrm{Obj}(\mathcal{R}_k) = \mathrm{Obj}(\mathcal{C}_k) = \mathcal{S}(G)$.

Before closing this section, we also mention two important functors which can be found in [5] and [6]. First, for a Mackey functor M for G over k and $H \in \mathcal{S}(G)$, we define the k-submodule

$$\mathcal{I}(M)(H) := \sum_{K<H} \mathrm{ind}_{H,K}(M(K)) = \sum_{K<H} \mathrm{im}(\mathrm{ind}_{H,G} : M(K) \to M(H))$$

of $M(H)$. The k-submodules $\mathcal{I}(M)(H)$ form a conjugation subfunctor of M for G over k. Since morphisms of Mackey functors commute with induction morphisms, these submodules are preserved under such morphisms. Hence, we obtain a functor

$$\mathcal{I} : \mathcal{M}_k(G) \to \mathcal{C}_k(G).$$

For $M \in \mathcal{M}_k(G)$, a subgroup H of G is called *primordial* if $\mathcal{I}(M)(H) \neq M(H)$, thus H is not primordial for M, if each element of $M(H)$ can be obtained as a sum of properly induced elements. We denote the set of primordial subgroup for

M by $\mathcal{P}(M)$. Note that for $H \leq G$ we have

$$M(H) = \sum_{K \leq H, K \in \mathcal{P}(M)} \mathrm{ind}_{H,K}(M(K)).$$

Secondly, for a restriction functor A for G over k and $H \leq G$, we define the k-submodule

$$\mathcal{K}(A)(H) := \bigcap_{K<H} \ker(\mathrm{res}_{K,H} : A(H) \to A(K))$$

of $A(H)$. The k-submodules $\mathcal{K}(A)(H)$ form a conjugation subfunctor of A for G over k and they are preserved under morphisms of restriction functors for G over k. Hence, we obtain a functor

$$\mathcal{K} : \mathcal{R}_k(G) \to \mathcal{C}_k(G).$$

For $A \in \mathcal{R}_k(G)$, a subgroup H of G is called *coprimordial* for A if $\mathcal{K}(A)(H) \neq 0$, thus H is not coprimordial for A, if the elements of $A(H)$ are uniquely determined by proper restriction maps. We denote the set of coprimordial subgroups for A by $\mathcal{C}(A)$. Note that for $H \leq G$ two elements $x, y \in A(H)$ are equal if and only if $\mathrm{res}_{K,H}(x) = \mathrm{res}_{K,H}(y)$ for all $K \leq H$ with $K \in \mathcal{C}(A)$.

3.2 The Plus Constructions: $-_+$ and $-^+$

In this section, for a group H belonging to the set $\mathcal{S}(G)$ we are going to define two important functors

$$-_+ : \mathcal{R}_k(H) \to \mathcal{M}_k(H)$$

and

$$-^+ : \mathcal{C}_k(H) \to \mathcal{M}_k(H).$$

For more details, we refer to [10], [5] and [6].

For a restriction functor A for H over k, $\bigoplus_{K \leq H} A(K)$ becomes an kH-module such that the action of an element $h \in H$ restricts to conjugation map $\mathrm{con}^h_{{}^hK,K}$: $A(K) \to A({}^hK)$ for each K. We define A_+ as H-cofixed quotient k-module

$$A_+(H) := (\bigoplus_{K \leq H} A(K))_H.$$

For $K \leq H$ and $a \in A(K)$, we write the image of a in $A_+(H)$ as $[K, a]_H$. Then, we can write each element $x \in A_+(H)$ in the form

$$x = \sum_{K \leq_H H} [K, a_K]_H$$

where $a_K \in A(K)$. Note that

$$\sum_{K \leq_H H} [K, a_K]_H = \sum_{K \leq_H H} [K, a'_K]_H \text{ if and only if } a'_K =^{n_K} (a_K)$$

where K runs over a set of representatives for the conjugacy classes of subgroups of H, and $n_K \in N_H(K)$. That is, we can identify A_+ as

$$A_+(H) = \bigoplus_{K \leq_H H} A(K)_{N_H(K)}.$$

Notice that the action of K on $A(K)$ is trivial and $A(K)$ is an $kN_H(K)/K$-module.

For $K \leq H$, the conjugation, restriction and induction morphisms on A_+ are defined as follows:

$$\begin{array}{rcccl}
\mathrm{con}^h_{{}^hK,K} & : & A_+(K) & \to & A_+({}^hK) \\
 & & [V, a_V]_K & \to & [{}^hV, {}^h(a_V)]_{{}^hK}, \\
\mathrm{res}_{K,H} & : & A_+(H) & \to & A_+(K) \\
 & & [U, a_U]_H & \to & \displaystyle\sum_{g \in K \backslash H/U} [K \cap^h U, \mathrm{con}^g_{K \cap^h U, K^h \cap U}(\mathrm{res}_{K^h \cap U, U}(a_U))]_H, \\
\mathrm{ind}_{H,K} & : & A_+(K) & \to & A_+(H) \\
 & & [V, b_V]_K & \to & [V, b_V]_H,
\end{array}$$

where $U \leq H$, $V \leq K$, $a_U \in A(U)$ and $b_V \in A(V)$. After verifications of relations between these morphisms, we get that A_+ is an Mackey functor for H over k. For a morphism $f : A \to B$ of restriction functors for H over k, the map

$$f_{+H} : A_+(H) \to B_+(H), \qquad [U, x_U]_H \mapsto [U, f_U(x_U)]_H$$

is a morphism of Mackey functors for H over k. Hence, we conclude that $-_+$ is a functor from $\mathcal{R}_k(H)$ to $\mathcal{M}_k(H)$.

For a conjugation functor C for H over k, let H act on $\bigoplus_{K\leq H} C(K)$ via the conjugation maps $\mathrm{con}^h_{{}^hK,K} : C(K) \to C({}^hK)$. We define C^+ as H-fixed k-submodule

$$C^+(H) := (\bigoplus_{K\leq H} C(K))^H.$$

We can identify C^+ as

$$C^+(H) = \bigoplus_{K\leq_H H} C(K)^{N_H(K)}.$$

For an element $\xi^K \in C(K)^{N_H(K)}$, we write $[K,\xi^K]^H$ to express ξ^K regarded as an element of $C^+(H)$. That is, any element $\xi \in C^+(H)$ can be written in a unique way

$$\xi = \sum_{K\leq_H H} [K,\xi^K]^H$$

where $\xi^K \in C(K)^{N_H(K)}$. Notice that for an $\rho_k(H)$-module A, the expression $[K,x_K]_H$ is defined for all $x_K \in A(K)$, and the element $[K,x_K]_H \in A(K)_{N_H(K)}$ does not determine x_H, in general. However, the expression $[K,\xi^K]^H$ is defined only for $\xi^K \in C(K)^{N_H(K)}$, and the expression $[K,\xi^K]^H$ does determine ξ^K.

For $K \leq H$, the conjugation, restriction and induction morphisms on A_+ are defined as follows:

$$\begin{array}{rcrcl}
\mathrm{con}^h_{{}^hK,K} & : & C^+(K) & \to & C^+({}^hK) \\
 & & [V,\eta^V]^K & \to & [{}^hV,{}^h(\eta^{{}^hV})]^{{}^hK}, \\
\mathrm{res}_{K,H} & : & C^+(H) & \to & C^+(K) \\
 & & [U,\xi^U]^H & \to & [U,\xi^U]^K, \\
\mathrm{ind}_{H,K} & : & C^+(K) & \to & C^+(H) \\
 & & [V,\eta^V]^K & \to & \displaystyle\sum_{h\in H/K} [{}^hV,\mathrm{con}^h_{{}^hK,K}(\eta^V)]^H,
\end{array}$$

where $U \leq H$, $V \leq K$, $\xi \in C^+(H)$ and $\eta \in C^+(K)$. After verifications of relations between these morphisms, we get that C^+ is an Mackey functor for H over k. Also, for a morphism $f : X \to Y$ of conjugation functors for H over k the map

$$f^+_H : X^+(H) \to Y^+(H), \qquad [U,\xi^U]^H \mapsto [U,f_U(\xi^U)]^H$$

is a morphism of Mackey functors for H over k and then $-^+$ is a functor from $\mathcal{C}_k(H)$ to $\mathcal{M}_k(H)$.

3.3 The Mark Homomorphism

In this section, we will relate the plus constructions $-_+$ and $-^+$ to each other by the mark homomorphism. Details can be found in [10], [5] and [6].

Let H be a group in $\mathcal{S}(G)$. For a $\rho_k(H)$-module A and $K \leq H$,

(i) The inclusion $A(H) \to \bigoplus_{K \leq H} A(K)$ induces the k-linear map

$$\iota_H^A : A(H) \to A_+(H), \quad a_H \mapsto [H, a_H]_H,$$

which is injective and form a morphism $\iota^A : A \to A_+$ of restriction functors for G over k.

(ii) The projection $\bigoplus_{K \leq H} A(K) \to A(H)$ induces the k-linear map

$$\pi_H^A : A_+(H) \to A(H), \quad [K, a_K]_H \mapsto \begin{cases} a_K, & \text{if } K = H, \\ 0, & \text{if } K < H, \end{cases}$$

which is called the *Brauer morphism* on $A_+(H)$. In other words, the Brauer morphism on $A_+(H)$ is given by

$$\pi_H^A(\sum_{K \leq_H H} [K, a_K]_H) = a_H.$$

The maps π_H^A are well-defined because H acts trivially on the k-submodule $A(H)$ of $\bigoplus_{K \leq H} A(K)$ and $[H, a_H]_H$ can be expressed as an element of the k-submodule $A(H) = A(H)_H$ of the k-module $A_+(H) = (\bigoplus_{K \leq H} A(K))_H$. Furthermore, the k-linear maps $\pi_H^A : A_+(H) \to A(H)$ are surjective and form a morphism $\pi^A : A_+ \to A$ of conjugation functors for H over k.

Notice that $\pi^A : A_+ \to A$ is the splitting morphism for $\iota^A : A \to A_+$, i.e. $\pi^A \circ \iota^A = \mathrm{id}_A$.

We are now able to define the mark homomorphism which connects the plus constructions $-_+$ and $-^+$. The *mark homomorphism* is defined to be

$$\rho_H^A \quad := \quad (\pi_K^A \circ \mathrm{res}_{K,H})_{K \leq H} \quad : \quad A_+(H) \quad \to \quad A^+(H).$$

Let $x \in A_+(H)$ and $\xi = \rho_H^A(x)$ with $x = \sum_{K \leq_H H} [K, x_K]_H$ and $\xi = \sum_{U \leq_H H} [U, \xi^U]^H$. Then, we have

$$\begin{aligned} \xi^U \quad = \quad \pi_U^A(\mathrm{res}_{U,H}(x)) \ &= \pi_U^A(\sum_{K \leq_H H} \mathrm{res}_{U,H}([K, x_K]_H)) \\ &= \sum_{K \leq_H H,\, hK \subset H \,:\, U \leq^h K} \mathrm{con}_{U,{}^hU}^h(\mathrm{res}_{{}^hU,K}(x_K)) \\ &= \sum_{K \leq_H H,\, hK \subset H \,:\, U \leq^h K} \mathrm{res}_{U,{}^hK}(\mathrm{con}_{{}^hK,K}^h(x_K)) \\ &= \sum_{K : U \leq K \leq H} \frac{|N_H(K)|}{|K|} \mathrm{res}_{U,K}(x_K). \end{aligned}$$

The first three equations comes directly from applying definitions and properties. To obtain the last equation, we change the indexing and then h runs over coset representatives $gN_H(K) \leq H$ instead of ${}^hK \leq H$. That is, ${}^hK$ runs over all the subgroups of H without repetitions. By using definitions of conjugation, restriction and induction morphisms for A_+ and A^+, we deduce that the mark homomorphisms form a morphism $\rho^A : A_+ \to A^+$ of Mackey functors for H over k.

Finally, we mention that ρ_H^A becomes an isomorphism where $|G|$ is invertible in k. Indeed, the inverse map of ρ_H^A is defined to be

$$\begin{aligned} (\rho_H^A)^{-1} \quad : \quad A^+(H) \quad &\to \quad A_+(H) \\ \xi \quad &\mapsto \quad \frac{1}{|H|} \sum_{U,K \leq H} |U| \mu(U,K) [U, \mathrm{res}_{U,K}(\xi^K)]^H, \end{aligned}$$

where $\xi \in A^+(H)$ with $\xi = \sum_{K \leq_H H} [K, \xi^K]^H$ and $\mu(U,K)$ denotes the Möbius function of the poset of subgroups of H. By letting $x = (\rho_H^A)^{-1}(\xi)$ with $x = \sum_{V \leq_H H} [V, x_V]_H$, we conclude that

$$x_V = \frac{1}{|H|} \sum_{K \leq H,\, N_H(K)h \leq H} |V| \mu(K^h, V) \mathrm{con}_{V,V^h}^h(\mathrm{res}_{V^h,K}(\xi^K)).$$

3.4 The Canonical Induction Formula

In this section, we will introduce the canonical induction homomorphism which sometimes serves as a splitting morphism for the linearization homomorphism. For more details, see [10], [5] and [6].

Before defining the canonical induction homomorphism, we give a generalized notion of the tom Dieck homomorphism and linearization homomorphism. Let H be a group in $\mathcal{S}(G)$, M be an $\mu_k(H)$-module and $A \subseteq M$ be an $\rho_k(H)$-submodule of M. That is, $A(K) \subseteq M(K)$, $K \leq H$, are k-submodules and stable under the conjugation and restriction morphisms of M. Suppose that ν is an embedding $\mathcal{R}_k(H)$-homomorphism $A \hookrightarrow \mathrm{Res}_{\mathcal{R}_k,\mathcal{M}_k}(M)$, where $\mathrm{Res}_{\mathcal{R}_k,\mathcal{M}_k}(M)$ is the restriction of M as an $\rho_k(H)$-module. We define the *linearization homomorphism* associated with ν to be an $\mathcal{M}_k(H)$-homomorphism

$$\begin{aligned} \mathrm{lin}_H \ = \ \mathrm{lin}_H^{\nu} \ : A_+(H) \ &\to \ M(H), \\ x \ &\mapsto \ \sum_{K \leq_H H} \mathrm{ind}_{H,K}(\nu_K(x_K)) \end{aligned}$$

where $x \in A_+(H)$ with $x = \sum_{K \leq_H H} [K, x_K]_H$. The linearization homomorphisms lin_H form a morphism $\mathrm{lin} : A_+ \to M$ of Mackey functors for H over k. Notice that we can also regard M as an $\rho_k(h)$-module and A as an $\gamma_k(H)$-submodule of M. Suppose that p is an projection $\mathcal{C}_k(H)$-homomorphism $\mathrm{Res}_{\mathcal{C}_k,\mathcal{R}_k}(M) \twoheadrightarrow A$, where $\mathrm{Res}_{\mathcal{C}_k,\mathcal{R}_k}(M)$ is the restriction of M as an $\gamma_k(H)$-module. We define the *tom Dieck homomorphism* associated with p to be an $\mathcal{R}_k(H)$-homomorphism

$$\begin{aligned} \mathrm{die}_H \ = \ \mathrm{die}_H^{p} \ : M(H) \ &\to \ \mathrm{Res}_{\mathcal{R}_k,\mathcal{M}_k}(A^+(H)), \\ m \ &\mapsto \ \sum_{K \leq_H H} [K, p_K(\mathrm{res}_{K,H}(m))]^H \end{aligned}$$

where $m \in M(H)$. The tom Dieck homomorphisms die_H form a morphism $\mathrm{die} : M \to \mathrm{Res}_{\mathcal{R}_k,\mathcal{M}_k}(A^+)$ of restriction functors for H over k.

The main purpose of this chapter is to construct a section of linearization homomorphism lin_H, thus a homomorphism $\mathrm{can}_H : M(H) \to A_+(H)$ such that $\mathrm{lin}_H \circ \mathrm{can}_H = \mathrm{id}_{M(H)}$. By Brauer's induction theorem we know that there is always such a section. However, there are many different choices for such a section and it is not unique in general.

Definition 3.4.1. Let M be an $\mu_k(H)$-module, $A \subseteq M$ be an $\rho_k(H)$-submodule of M and let $\mathrm{lin} : A_+ \to M$ be the linearization homomorphism of M from A. A homomorphism $\mathrm{can} : M \to A_+$ with $\mathrm{lin} \circ \mathrm{can} = \mathrm{id}_M$ is called a *canonical induction formula* for $\mathrm{lin} : A_+ \to M$.

3.5 The Case of Invertible Group Order

In this section, we give the canonical induction formula explicitly and the necessary and sufficient condition for the canonical induction formula. Throughout, we understand that $|G|$ is invertible in k. More details can be found in [10], [5] and [6].

Let H be a group in $\mathcal{S}(G)$, M be a $\mu_k(H)$-module, $A \subseteq M$ be a $\rho_k(H)$-submodule M. For module homomorphisms $p : \mathrm{Res}_{\mathcal{C}_k,\mathcal{M}_k}(M) \twoheadrightarrow \mathrm{Res}_{\mathcal{C}_k,\mathcal{R}_k}(A)$ and $\nu : A \hookrightarrow \mathrm{Res}_{\mathcal{R}_k,\mathcal{M}_k}(M)$, we define the *canonical induction homomorphism* associated with p to be an $\mathcal{R}_k(H)$-homomorphism $\mathrm{can}_H : M(H) \to A_+(H)$ such that $\mathrm{can}_H = \rho_H^{-1} \circ \mathrm{die}_H$. That is, we have a commutative diagram of $\mathcal{R}_k(H)$-homomorphisms as follows:

$$\begin{array}{ccc} M(H) & & \\ {\scriptstyle \mathrm{die}_H}\downarrow & \searrow^{\mathrm{can}_H} & \\ A^+(H) & \xleftarrow[\rho_H]{} & A_+(H) \end{array} \tag{3.1}$$

The mark homomorphism $\rho_H : A_+(H) \to A^+(H)$ and its inverse ρ_H^{-1} are $\mathcal{M}_k(H)$-isomorphisms by our assumptions on k and G. Notice that since ρ_H, ρ_H^{-1} form a morphism of Mackey functors and die_H forms a morphism of restriction functors, one can easily see that the canonical induction homomorphisms can_H forms a morphisms of restriction functors. In other words, the diagrams

$$
\begin{array}{ccc}
M(H) & \xrightarrow{\mathrm{can}_H} & A_+(H) \\
\Big\downarrow \mathrm{res}_{K,H} & \circ & \Big\downarrow \mathrm{res}_{K,H} \\
M(K) & \xrightarrow{\mathrm{can}_K} & A_+(K)
\end{array}
\qquad
\begin{array}{ccc}
M(H) & \xrightarrow{\mathrm{can}_H} & A_+(H) \\
\Big\downarrow \mathrm{con}^{g}_{{}^{g}H,H} & \circ & \Big\downarrow \mathrm{con}_{{}^{g}H,H} \\
M({}^{g}H) & \xrightarrow{\mathrm{can}_{{}^{g}H}} & A_+({}^{g}H)
\end{array}
\tag{3.2}
$$

commutes where $K \leq H$ and $g \in G$.

We now want to give an explicit formula for the homomorphism $\mathrm{can} : M \to A_+$. Before that, we need the following result for ρ^{-1}:

Proposition 3.5.1. *For $\xi \in A^+(H)$, let $x = \rho^{-1}(\xi)$. Then, we have*

$$x_K = \frac{|H|}{|N_H(K)|} \sum_{U \leq H} \mu(K, U) \mathrm{res}_{K,U}(\xi^U)$$

as an element of $A(K)_{N_H(K)}$ for all $K \leq H$. Note that the term in this summation becomes zero when $K \nleq U$.

Proof. First, notice that for $K \leq H$, letting $a_K \in A(K)_{N_H(K)}$ and $x = \sum_{K \leq H} [K, a_K]_H$, we have

$$x_K = \sum_{N_H(K)k \leq H} \mathrm{con}^h_{K,K^h}(a_{K^h})$$

as an element of $A(K)_{N_H(K)}$. In particular, $x = \sum_{K \leq H} \frac{|N_H(K)|}{|H|} [K, x_K]_H$. Then, by using this observation and the formula for ρ^{-1} given in the previous section, we get immediately

$$x_K = \frac{|K|}{|N_H(K)|} \sum_{U \leq H} \mu(K, U) \mathrm{res}_{K,U}(\xi^U)$$

as an element of $A(K)_{N_H(K)}$ for all $K \leq H$. □

Then, we have an explicit formula for the homomorphism $\mathrm{can} : M \to A_+$ as in the following way:

Proposition 3.5.2. *For $m \in M(H)$, we have*

$$\mathrm{can}_H(m) = \frac{1}{|H|} \sum_{K,U \leq H} |K|\mu(K,U)[K, \mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m)))]_H.$$

Proof. Let $\xi = \mathrm{die}_H(m)$ and $x = \rho^{-1}(\xi)$. The definition of die_H gives $\xi^U = p_U(\mathrm{res}_{U,H}(m))$. By the Proposition 3.5.1, we get

$$x_K = \frac{|K|}{|N_H(K)|} \sum_{U \leq H} \mu(K,U)\mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m))).$$

It follows that

$$\mathrm{can}_H(m) := \rho_H^{-1}\mathrm{die}_H(m) = \frac{1}{|H|} \sum_{K,U \leq H} |K|\mu(K,U)[K, \mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m)))]_H.$$

□

Furthermore, to get the necessary and sufficient conditions for the homomorphism $\mathrm{can}_H : M(H) \to A_+(H)$ being a canonical induction formula, we need the followings:

Lemma 3.5.3. *Every Mackey functor M for H over k has a $k \otimes B$-module structure in a unique way where the Burnside ring functor B acts via*

$$(k \otimes_{\mathbb{Z}} B(H)) \otimes_k M(H) \to M(H),$$
$$[H/K] \otimes_k m \mapsto \mathrm{ind}_{H,K}(\mathrm{res}_{K,H}(m))$$

for $K \leq H$ and $m \in M(H)$.

Proposition 3.5.4. *Let $M \in \mu_k(H)$. Then, there exists an idempotent e_H of the Burnside ring $B(H)$ such that*

$$e_H \cdot M(H) = \mathcal{K}(M)(H) = \bigcap_{K<H} \ker(\mathrm{res}_{K,H} : M(H) \to M(K))$$

and

$$(1 - e_H) \cdot M(H) = \mathcal{I}(M)(H) = \sum_{K<H} \mathrm{im}(\mathrm{ind}_{H,G} : M(K) \to M(H)).$$

Corollary 3.5.5. *Let $M \in \mu_k(H)$ and $m \in M(H)$. Then $m = 0$ if and only if $e_H^{(H)} \cdot m = 0$ and $\mathrm{res}_{K,H}(m) = 0$ for all $K < H$.*

By using the above decomposition, we get the necessary and sufficient condition to ensure that the homomorphism $\mathrm{can}_H : M(H) \to A_+(H)$ is a canonical induction formula:

Theorem 3.5.6. *Let M be an $\mu_k(H)$-module, $A \subseteq M$ be an $\rho_k(H)$-submodule of M and $\mathrm{can} : M \to A_+$ be associated canonical induction homomorphism. Then, the following conditions are equivalent:*

(i) *We have $\mathrm{lin} \circ \mathrm{can} = \mathrm{id}_M$.*

(ii) *For all $H \in \mathcal{S}(G)$ and $m \in M(H)$, we have $\nu_H(p_H(m)) - m \in \mathcal{I}(M)(H)$.*

Proof. Let $x = \mathrm{can}_H(m)$. By the definition of linearization homomorphism

$$\mathrm{lin}_H(\mathrm{can}_H(m)) = \mathrm{lin}_H(x) = \sum_{K \leq_H H} \mathrm{ind}_{H,K}(\nu_K(x_K)).$$

By the proof of Proposition 3.5.2, we have

$$x_K = \frac{|K|}{|N_H(K)|} \sum_{U \leq H} \mu(K, U)\mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m))).$$

Hence, we obtain the formula

$$\mathrm{lin}_H(\mathrm{can}_H(m)) = \sum_{K \leq_H H, U \leq H} \frac{|K|}{|N_H(K)|} \mu(K, U)\mathrm{ind}_{H,K}(\nu_K(\mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m))))).$$

Notice that since ν_K is an embedding $\mathcal{R}_k(G)$-homomorphism we have

$$\nu_K(\mathrm{res}_{K,U}(p_U(\mathrm{res}_{U,H}(m)))) = \mathrm{res}_{K,U}(\nu_U(p_U(\mathrm{res}_{U,H}(m)))).$$

By Proposition 3.5.4, since $e_H^{(H)}$ annihilates $\mathcal{I}(M)(H)$ we get $e_H^{(H)}\mathrm{lin}_H(\mathrm{can}_H(m)) = e_H^{(H)}\nu_H(p_H(m))$.

If the first condition (i) holds, then we have $e_H^{(H)} m = e_H^{(H)}\nu_H(p_H(m))$ which means that the condition (ii) holds. Conversely, suppose that the condition (ii) holds. Then, by Corollary 3.5.5 it is sufficient to show that

$$e_H^{(H)}\mathrm{lin}_H(\mathrm{can}_H(m)) = e_H^{(H)} m \quad \text{and} \quad \mathrm{res}_{K,H}(\mathrm{lin}_H(\mathrm{can}_H(m))) = \mathrm{res}_{K,H}(m)$$

for all $K < H$. First equation follows from $e_H^{(H)}\text{lin}_H(\text{can}_H(m)) = e_H^{(H)}\nu_H(p_H(m))$. For second equation, inductively, assume that the condition (i) holds for all $K < H$. Since lin_H and can_H commute with restrictions, we have

$$\text{res}_{K,H}(\text{lin}_H(\text{can}_H(m))) = \text{lin}_K(\text{can}_K(\text{res}_{K,H}(m))) = \text{res}_{K,H}(m).$$

Then, we conclude that the condition (i) holds. □

It means that can : $M \to A_+$ is a canonical induction formula if the equivalent conditions (i) and (ii) on the Theorem 3.5.6 hold.

Remark 3.5.7. If can : $M \to A_+$ is a canonical induction formula for lin : $A_+ \to M$, then we obtain the formula

$$m = \frac{1}{|H|} \sum_{K,U \leq H} |K|\mu(K,U)\text{ind}_{H,K}(\text{res}_{K,U}(\nu_U p_U(\text{res}_{U,H}(m))))$$

where $m \in M(H)$ and $\text{res}_{K,U}(\nu_U \pi_U(\text{res}_{U,H}(m))) \in A(K)$.

Chapter 4

Applications of Canonical Induction Formula

4.1 Canonical Induction for the Character Ring

Throughout this section, let G be a finite group and K be a field of characteristic 0 which is is sufficiently large, thus contains all $|G|^{\text{th}}$ roots of unity. That is, K is a splitting field for G and its subgroups. Let $\mathbb{F}$ be an algebraically closed field of characteristic p for prime p.

Remember that $A_K(G)$ is generated by the isomorphism classes of KG-modules. One can observe that $A_K(G)$ can be identified with the character ring $R_K(G)$ which is free abelian group with basis given by the set $\operatorname{Irr}(KG)$ of irreducible K-characters of KG-modules, thus

$$A_K(G) := R_K(G) = \bigoplus_{\chi \in \operatorname{Irr}(KG)} \mathbb{Z}\chi.$$

Moreover, we denote the set of 1-dimensional K-characters by $\hat{G}(K) := \operatorname{Hom}(G, K^\times)$ which is $\hat{G}(K) \subseteq \operatorname{Irr}(KG)$. We define $R_K^{\text{ab}}(G)$ as the subring of

$R_K(G)$ which is spanned by the elements of $\hat{G}(K)$, thus

$$R_K^{\text{ab}}(G) = \bigoplus_{\phi \in \hat{G}(K)} \mathbb{Z}\phi.$$

Notice that none of $R_K(G)$, $R_K^{\text{ab}}(G)$, $\text{Irr}(KG)$ and $\hat{G}(K)$ depend on K. For any splitting field K' for G of characteristic 0, the corresponding rings and sets can be identified. Because of this reason, we work with the field of complex numbers by omitting the subindex $\mathbb{C}$, thus $R_{\mathbb{C}}(G) := R(G)$.

For $H \in \mathcal{S}(G)$, the character rings $R(H)$ are Mackey functors for G over $\mathbb{Z}$ with the usual conjugation, restriction, induction morphisms and multiplication. Then one can regard $R^{\text{ab}}(H)$, $H \leq G$, as the restriction subfunctors of $R(H)$ over $\mathbb{Z}$ with the inherited conjugation and restriction morphisms of $R(H)$. Since the induction of a linear $\mathbb{C}$-character may have non-linear $\mathbb{C}$-characters, R^{ab} is not a Mackey subfunctor of R over $\mathbb{Z}$. Furthermore, the set $\mathcal{C}(R)$ of coprimordial subgroups for R consists of the set of cyclic subgroups of G and the set $\mathcal{P}(R)$ of primordial subgroups for R consists of the set of elementary subgroups of G.

By Section 3.2, any element x of $R_+^{\text{ab}}(H)$ can be written as

$$x = \sum_{K \leq_H H} [K, \phi_K]_H \qquad\qquad \phi_K \in R^{\text{ab}}(K)$$

with

$$\sum_{K \leq_H H} [K, \phi_K]_H = \sum_{K \leq_H H} [K, \vartheta_K]_H \quad \Leftrightarrow \quad \phi_K = {}^{n_K}(\vartheta_K)$$

where $n_K \in N_H(K)$. Then, we can express $R_+^{\text{ab}}(H)$ as a free abelian group with basis given by the set of H-orbits $[K, \phi_K]_H$ of the elements $(K, \phi_K) \in \mathcal{M}_{\mathbb{C}}(H)$ where $\mathcal{M}_{\mathbb{C}}(H)$ is the set of monomial pairs of H on $\mathbb{C}$, thus

$$R_+^{\text{ab}}(H) := \bigoplus_{(K,\phi_K) \in \mathcal{M}_{\mathbb{C}}(H)} \mathbb{Z}[K, \phi_K]_H$$

where $\mathcal{M}_{\mathbb{C}}(H) := \{(K, \phi_K) \mid K \leq H, \phi_K \in \hat{K}(\mathbb{C})\}$. It is easy to see that $R_+^{\text{ab}}(H)$ corresponds to the monomial ring $D_{\mathbb{C}}(H) := D(H)$. Note that for any $H \in \mathcal{S}(G)$ the monomial rings $D(H)$ are also Mackey functors for G over $\mathbb{Z}$ with the associated conjugation, restriction and induction morphisms. Hence, we have the linearization homomorphism

$$\begin{array}{rcl} \mathrm{lin}_H & : & D(H) \quad \rightarrow \quad R(H) \\ & & \sum_{K \leq_H H} [K, \phi_K]_H \mapsto \sum_{K \leq_H H} \mathrm{ind}_{H,K}(\phi_K) \end{array}$$

and the canonical induction homomorphism

$$\mathrm{can}_H \quad : \quad R(H) \quad \rightarrow \quad D(H)$$

given by the explicit formula

$$\mathrm{can}_H(\chi) = \frac{1}{|H|} \sum_{(K,\phi_K) \leq (K',\phi'_{K'}) \in \mathcal{M}_{\mathbb{C}}(H)} |K| \cdot \mu((K, \phi_K), (K', \phi'_{K'})) \cdot$$
$$\cdot [K, \mathrm{res}_{K,K'}(p_{K'}(\mathrm{res}_{K',H}(\chi)))]_H$$

where $\chi \in R(H)$. Here, we have the projection homomorphism p : $\mathrm{Res}_{\mathcal{C}_{\mathbb{Z}},\mathcal{M}_{\mathbb{Z}}}(R) \rightarrow \mathrm{Res}_{\mathcal{C}_{\mathbb{Z}}\mathcal{R}_{\mathbb{Z}}}(R^{\mathrm{ab}})$ is given by

$$p_H(\chi) := \begin{cases} \chi, & \text{if } \dim_{\mathbb{C}}(\chi) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

Recall also that $A_{\mathbb{F}}(G)$ is generated by the isomorphism classes of $\mathbb{F}G$-modules. One can see that $A_{\mathbb{F}}(G)$ can be identified with the Brauer character ring $R_{\mathbb{F}}(G)$ which is free abelian group with basis given by the set $\mathrm{IBr}(\mathbb{F}G)$ of irreducible Brauer $\mathbb{F}$-characters of G, thus

$$A_{\mathbb{F}}(G) := R_{\mathbb{F}}(G) = \bigoplus_{\varphi \in \mathrm{IBr}(\mathbb{F}G)} \mathbb{Z}\varphi.$$

We denote by $\hat{G}(\mathbb{F}) := \mathrm{Hom}(G, \mathbb{F}^{\times})$ the set of 1-dimensional Brauer $\mathbb{F}$-characters and $R_{\mathbb{F}}^{\mathrm{ab}}(G)$ is defined as the subring of $R_{\mathbb{F}}(G)$ which is spanned by the elements of $\hat{G}(\mathbb{F})$, thus

$$R_{\mathbb{F}}^{\mathrm{ab}}(G) = \bigoplus_{\psi \in \hat{G}(\mathbb{F})} \mathbb{Z}\psi.$$

Note that none of $R_{\mathbb{F}}(G)$, $R_{\mathbb{F}}^{\mathrm{ab}}(G)$, $\mathrm{IBr}(\mathbb{F}G)$ and $\hat{G}(\mathbb{F})$ depend on the choice of $\mathbb{F}$, however only on p.

For $H \in \mathcal{S}(G)$, the Brauer character rings $R_{\mathbb{F}}(H)$ are Mackey functors for G over $\mathbb{Z}$ with the usual conjugation, restriction, induction morphisms and multiplication. Then the rings $R_{\mathbb{F}}^{\text{ab}}(H)$, $H \leq G$, are the restriction subfunctors of $R(H)$ over $\mathbb{Z}$ with the inherited conjugation and restriction morphisms of $R(H)$. In addition, the set $\mathcal{C}(R_{\mathbb{F}})$ of coprimordial subgroups for $R_{\mathbb{F}}$ consists of the set of cyclic p'-subgroups of G.

Similar to the preceding, $R_{\mathbb{F}+}^{\text{ab}}(H)$ can be expressed as a free abelian group with basis given by the set of H-orbits $[K, \psi_K]_H$ of the elements $(K, \psi_K) \in \mathcal{M}_{\mathbb{F}}(H)$ where $\mathcal{M}_{\mathbb{F}}(H)$ is the set of monomial pairs of H on $\mathbb{F}$, thus

$$R_{\mathbb{F}+}^{\text{ab}}(H) := \bigoplus_{(K,\psi_H)\in\mathcal{M}_{\mathbb{F}}(H)} \mathbb{Z} \cdot [K, \psi_K]_H$$

where $\mathcal{M}_{\mathbb{F}}(H) = \{(K, \psi_K) \mid K \leq H, \psi_K \in \hat{K}(\mathbb{F})_{p'}\}$. It follows that $R_{\mathbb{F}+}^{\text{ab}}(H)$ is identified with the monomial ring $D_{\mathbb{F}}(G)$. Therefore, we have the linearization homomorphism

$$\begin{array}{rcccl} \text{lin}_H & : & D_{\mathbb{F}}(H) & \to & R_{\mathbb{F}}(H) \\ & & \displaystyle\sum_{H\leq_G G} [K, \phi_K]_H & \mapsto & \displaystyle\sum_{K\leq_H H} \text{ind}_{H,K}(\phi_K) \end{array}$$

and the canonical induction homomorphism

$$\text{can}_H \quad : \quad R_{\mathbb{F}}(H) \quad \to \quad D_{\mathbb{F}}(H)$$

given by the explicit formula

$$\text{can}_H(\varphi) = \frac{1}{|H|} \sum_{(K,\psi_K)\leq(K',\psi'_{K'})\in\mathcal{M}_{\mathbb{F}}(H)} |K| \cdot \mu((K, \psi_K), (K', \psi'_{K'}))\cdot$$
$$\cdot[K, \text{res}_{K,K'}(p_{K'}(\text{res}_{K',H}(\varphi)))]_H$$

where $\varphi \in R_{\mathbb{F}}(H)$. Here, we have the projection homomorphism p : $\text{Res}_{\mathcal{C}_{\mathbb{Z}},\mathcal{M}_{\mathbb{Z}}}(R_{\mathbb{F}}) \to \text{Res}_{\mathcal{C}_{\mathbb{Z}}\mathcal{R}_{\mathbb{Z}}}(R_{\mathbb{F}}^{\text{ab}})$ is given by

$$p_H(\varphi) := \begin{cases} \varphi, & \text{if } \dim_{\mathbb{F}}(\varphi) = 1, \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Canonical Induction for the Trivial Source Ring

Throughout this section, G denotes a finite group and $(K, R, \mathbb{F})$ denotes a p-modular system where R is a complete discrete valuation ring with algebraically closed residue field $\mathbb{F}$ of characteristic p and with a quotient field K of characteristic zero which is a splitting field for G and its all subgroups.

Recall that the trivial source ring $T_R(G)$ is a free abelian group on the set of isomorphism classes of indecomposable trivial source RG-modules, thus

$$T_R(G) = \bigoplus_{M \in \mathrm{Triv}(RG)} \mathbb{Z}[M]$$

where $\mathrm{Triv}(RG)$ denotes the set of indecomposable trivial source RG-modules. Notice that the isomorphism classes of indecomposable RG-modules whose sources are 1-dimensional can be parametrized by $\hat{G}(R) := \mathrm{Hom}(G, R^\times)$ where $R_\varphi = R$ as R-module for $\varphi \in \hat{G}(R)$ and $g \cdot \alpha = \varphi(g)\alpha$ gives the G-action for $\alpha \in R$, $g \in G$. For $\varphi \in \hat{G}(R)$, each Sylow p-subgroup P of G is a vertex of R_φ and $R_{\mathrm{res}_{P,G}(\varphi)}$ is a source of R_φ. It means that R_φ is a trivial source module if and only if $\mathrm{res}_{P,G}(\varphi) = 1$. It is equivalent to say that $\varphi \in \hat{G}(R)_{p'}$, thus the p'-part of the group $\hat{G}(R)$. Therefore, the isomorphism classes of indecomposable trivial source RG-modules can be parametrized by $\hat{G}(R)_{p'} := \mathrm{Hom}(G, R^\times)_{p'}$.

We define $T_R^{\mathrm{ab}}(G)$ as the subring of $T_R(G)$ spanned by the elements $[R_\varphi]$ for $\varphi \in \hat{G}(R)_{p'}$. Using φ instead of $[\mathbb{F}_\varphi]$ for $\varphi \in \hat{G}(\mathbb{F})_{p'}$, one can regard $\hat{G}(\mathbb{F})_{p'}$ as a $\mathbb{Z}$-basis of $T_R^{\mathrm{ab}}(G)$, thus

$$T_R^{\mathrm{ab}}(G) = \bigoplus_{\varphi \in \hat{G}(\mathbb{F})_{p'}} \mathbb{Z}\varphi.$$

Note that since the reduction modulo p induces a bijection between the set of isomorphism classes of trivial source RG-modules and the set isomorphism classes of trivial source $\mathbb{F}G$-modules, we have $T_R(G) \cong T_{\mathbb{F}}(G)$. Because of this reason, we work with the residue field $\mathbb{F}$ instead of R.

For $H \in \mathcal{S}(G)$, the trivial source rings $T_{\mathbb{F}}(H)$ are Mackey functor for G over $\mathbb{Z}$ with the usual conjugation, restriction, induction morphisms and multiplication. Then one can see $T_{\mathbb{F}}^{\text{ab}}(G)$, $H \leq G$, as the restriction subfunctors of $T_{\mathbb{F}}(H)$ with the inherited conjugation and restriction morphisms of $T_{\mathbb{F}}(G)$. Moreover, the set $\mathcal{C}(T_{\mathbb{F}})$ of coprimordial subgroups for $T_{\mathbb{F}}$ consists of the set of p-hypo-elementary subgroups of G, thus subgroups H such that $H/O_p(H)$ is cyclic p'-group.

One can regard $T_{\mathbb{F}}^{\text{ab}}(H)$ as a free abelian group with basis given by the set of H-orbits $[K, \varphi_K]_H$ of the elements $(K, \varphi_K) \in \mathcal{M}_{\mathbb{F}}^T(H)$ where $\mathcal{M}_{\mathbb{F}}^T(H)$ is the set of monomial pairs (K, φ) of H on $\mathbb{F}$ such that $K/\ker(\varphi_K)$ being an p'-group, or equivalently $K \leq H$ and $\varphi_K \in \hat{K}(\mathbb{F})_{p'}$, thus

$$T_{\mathbb{F}+}^{\text{ab}}(H) := \bigoplus_{(K \in \varphi) \in \mathcal{M}_{\mathbb{F}}^T(H)} \mathbb{Z}[K, \varphi]_H$$

where $\mathcal{M}_{\mathbb{F}}^T(H) := \mathcal{M}_{\mathbb{F}}(H) = \{(K, \varphi) | K \leq H, \varphi \in \hat{H}(\mathbb{F})_{p'}\}$. Then, $T_{\mathbb{F}+}^{\text{ab}}(H)$ can be identified with the monomial ring $D_{\mathbb{F}}(G)$. Hence, we have the linearization homomorphism

$$\begin{array}{rcccc} \text{lin}_H & : & D_{\mathbb{F}}(H) & \to & T_{\mathbb{F}}(H) \\ & & \displaystyle\sum_{H \leq_G G} [K, \phi_K]_H & \mapsto & \displaystyle\sum_{K \leq_H H} \text{ind}_{H,K}(\phi_K) \end{array}$$

and the canonical induction homomorphism

$$\text{can}_H \quad : \quad T_{\mathbb{F}}(H) \quad \to \quad D_{\mathbb{F}}(H)$$

given by the explicit formula

$$\text{can}_H([M]) = \frac{1}{|H|} \sum_{(K, \tau_K) \leq (K', \tau'_{K'}) \in \mathcal{M}_{\mathbb{F}}^T(H)} |K| \cdot \mu((K, \tau_K), (K', \tau'_{K'})) \cdot$$
$$\cdot [K, \text{res}_{K,K'}(p_{K'}(\text{res}_{K',H}([M])))]_H$$

where $M \in T_{\mathbb{F}}(H)$. Here, we have the projection homomorphism p : $\text{Res}_{\mathcal{C}_{\mathbb{Z}}, \mathcal{M}_{\mathbb{Z}}}(T_{\mathbb{F}}) \to \text{Res}_{\mathcal{C}_{\mathbb{Z}} \mathcal{R}_{\mathbb{Z}}}(T_{\mathbb{F}}^{\text{ab}})$ is given by

$$p_H(M) := \begin{cases} \phi_M, & \text{if } \dim_{\mathbb{F}}(M) = 1, \\ 0, & \text{otherwise} \end{cases}$$

where ϕ_M is the corresponding linear character for 1-dim trivial source module M.

S_3	1	(1,2)	(1,2,3)
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

Table 4.1: Irreducible $\mathbb{C}$-Characters

S_3	1	(1,2)
φ_0	1	1
φ_1	1	-1
φ_2	2	0

Table 4.2: Irreducible $\mathbb{F}_3$-Characters

Example 4.2.1. Let $G = S_3$ and $\mathbb{F}$ be a field of character $p = 3$. First, we find the isomorphism classes of indecomposable $\mathbb{F}_3S_3$-modules with trivial source. Since 3-subgroups of S_3 are C_3 and C_1, these are the only possibilities for vertex P of indecomposable $\mathbb{F}_3S_3$-modules with trivial source.

C_3 : Note that $N_{S_3}(C_3) = S_3$ and then $N_{S_3}(C_3)/C_3 = C_2$. The indecomposable projective $\mathbb{F}_3C_2$-modules are the trivial and the sign modules, which are also simple modules. These projective modules are in Green correspondence to the indecomposable $\mathbb{F}_3S_3$-modules with vertex C_3 and trivial source which are the trivial and the sign $\mathbb{F}_3S_3$-modules. Say, N_1 and N_2 respectively. Notice that both are 1-dimensional $\mathbb{F}_3S_3$-modules.

C_1 **:** Note that $N_{S_3}(C_1) = S_3$ and then $N_{S_3}(C_1)/C_1 = S_3$. We need the indecomposable projective $\mathbb{F}S_3$-modules. By the Tables 4.1, 4.2, we have the decomposition map d, the decomposition matrix D and the Cartan matrix C as follows:

$$d : \begin{cases} \chi_0 \mapsto \varphi_0, \\ \chi_1 \mapsto \varphi_1, \\ \chi_2 \mapsto \varphi_0 + \varphi_1, \end{cases} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

We do not give detailed explanation for decomposition map and Cartan matrix, however it can be found explicitly in Chapter 2 of [13]. Then, we get that

$$\mathbb{F}_3S_3 = \dim(N_1)P_1 + \dim(N_2)P_2 = P_1 + P_2.$$

where P_1, P_2 are the indecomposable projective $\mathbb{F}_3S_3$-modules such that

$$P_1 = \begin{matrix} N_1 \\ N_2 \\ N_1 \end{matrix} \quad \text{and} \quad P_2 = \begin{matrix} N_2 \\ N_1 \\ N_2 \end{matrix}.$$

Note that P_1 and P_2 are projective covers of N_1 and N_2, respectively. These projective modules are in green correspondence to the indecomposable $\mathbb{F}_3 S_3$-modules with vertex C_1 and trivial source which are themselves. Say, $N_3 = P_1$ and $N_4 = P_2$. Notice that both are 3-dimensional $\mathbb{F}_3 S_3$-modules.

Therefore, we have

$$\begin{aligned} T_{\mathbb{F}_3}(S_3) &= \bigoplus_{N \in \{N_1, N_2, N_3, N_4\}} \mathbb{Z}[N] \\ &= \mathbb{Z}[N_1] + \mathbb{Z}[N_2] + \mathbb{Z}[N_3] + \mathbb{Z}[N_4] \end{aligned} \tag{4.1}$$

where $N_1 \sim 1$, $N_2 \sim 1_-$, $N_3 \sim \begin{matrix} 1 \\ 1_- \\ 1 \end{matrix}$ and $N_4 \sim \begin{matrix} 1_- \\ 1 \\ 1_- \end{matrix}$. Remark that $T_{\mathbb{F}_3}(C_3) = \mathbb{Z}[V_1] + \mathbb{Z}[V_2]$ where $V_1 \sim 1$, $V_2 \sim \begin{matrix} 1 \\ 1 \\ 1 \end{matrix}$ and $T_{\mathbb{F}_3}(C_2) = \mathbb{Z}[U_1] + \mathbb{Z}[U_2]$ where $U_1 \sim 1$, $U_2 \sim 1_-$.

Second, we find the isomorphism classes for the monomial Ring $D_{\mathbb{F}_3}(S_3)$. The subgroups of S_3 are C_3, C_2 and C_1. Since the only possible monomial pairs for $D_{\mathbb{F}_3}(S_3)$ are $(1,1), (C_2,1), (C_2,1_-), (C_3,1), (S_3,1)$ and $(S_3,1_-)$, we have

$$\begin{aligned} D_{\mathbb{F}_3}(S_3) &= \bigoplus_{(H,\varphi) \in \mathcal{M}_{\mathbb{F}_3}(S_3)/S_3} \mathbb{Z}[H,\varphi]_{S_3} \\ &= \mathbb{Z}[1,1]_{S_3} + \mathbb{Z}[C_2,1]_{S_3} + \mathbb{Z}[C_2,1_-]_{S_3} + \mathbb{Z}[C_3,1]_{S_3} \\ &\qquad + \mathbb{Z}[S_3,1]_{S_3} + \mathbb{Z}[S_3,1_-]_{S_3}. \end{aligned} \tag{4.2}$$

where $\mathcal{M}_{\mathbb{F}_3}(G) = \{(1,1), (C_2,1), (C_2,1_-), (C_3,1), (S_3,1), (S_3,1_-)\}$. Remark that $D_{\mathbb{F}_3}(C_3) = \mathbb{Z}[1,1]_{C_3} + \mathbb{Z}[C,1]_{C_3}$ and $D_{\mathbb{F}_3}(C_2) = \mathbb{Z}[1,1]_{C_2} + \mathbb{Z}[C_2,1]_{C_2} + \mathbb{Z}[C_2,1_-]_{C_2}$.

Now, we want to construct two maps between the rings $D_{\mathbb{F}_3}(S_3)$ and $T_{\mathbb{F}_3}(S_3)$

$$D_{\mathbb{F}_3}(S_3) \underset{\mathrm{can}_{S_3}}{\overset{\mathrm{lin}_{S_3}}{\rightleftarrows}} T_{\mathbb{F}_3}(S_3) \tag{4.3}$$

such that lin_{S_3} is the linearization morphism and can_{S_3} is the canonical induction formula, thus $\mathrm{lin}_{S_3} \cdot \mathrm{can}_{S_3} = \mathrm{id}_{T_{\mathbb{F}_3}(S_3)}$. By the definition of lin_{S_3}, it is easy to verify that

$$\begin{aligned} \mathrm{lin}_{S_3} \quad : \quad [1,1]_{S_3} &\longmapsto \mathrm{ind}_{C_1}^{S_3}[1_{C_1}] = [\mathbb{F}S_3] = N_3 + N_4, \\ [C_2,1]_{S_3} &\longmapsto \mathrm{ind}_{C_2}^{S_3}[1_{C_2}] = N_3, \\ [C_2,1_-]_{S_3} &\longmapsto \mathrm{ind}_{C_2}^{S_3}[1_{-C_2}] = N_4, \\ [C_3,1]_{S_3} &\longmapsto \mathrm{ind}_{C_3}^{S_3}[1_{C_3}] = [\mathbb{F}C_2] = N_1 + N_2, \\ [S_3,1]_{S_3} &\longmapsto [1_{S_3}] = N_1, \\ [S_3,1_-]_{S_3} &\longmapsto [1_{-S_3}] = N_2. \end{aligned}$$

Moreover, for 1-dimensional modules N_1 and N_2 we have

$$\mathrm{can}_{S_3}[N_1] = [S_3, 1]_{S_3} \quad \text{and} \quad \mathrm{can}_{S_3}[N_2] = [S_3, 1_-]_{S_3}.$$

However, for N_3 and N_4 it is not straightforward. Let $t_1 = [1,1]_{S_3}$, $t_2 = [C_2, 1]_{S_3}$, $t_3 = [C_2, 1_-]_{S_3}$, $t_4 = [C_3, 1]_{S_3}$ $t_5 = [S_3, 1]_{S_3}$, $t_6 = [S_3, 1]_{S_3}$ and say

$$\begin{aligned} \mathrm{can}_{S_3}[N_3] &= \alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 + \alpha_4 t_4 + \alpha_5 t_5 + \alpha_6 t_6 \\ \mathrm{can}_{S_3}[N_4] &= \beta_1 t_1 + \beta_2 t_2 + \beta_3 t_3 + \beta_4 t_4 + \beta_5 t_5 + \beta_6 t_6. \end{aligned} \tag{4.4}$$

There is two fundamental properties of canonical induction formula such that

$$\mathrm{lin}_{S_3}\mathrm{can}_{S_3} = \mathrm{id}_{T_{\mathbb{F}_3}(S_3)} \quad \text{and} \quad \mathrm{res}_H^{S_3}\mathrm{can}_{S_3} = \mathrm{can}_H \mathrm{res}_H^{S_3}$$

where $H \leq S_3$. The property $\mathrm{lin}_{S_3}\mathrm{can}_{S_3} = \mathrm{id}_{T_{\mathbb{F}_3}(S_3)}$ says that

$$\begin{aligned} \mathrm{lin}_{S_3}(\mathrm{can}_{S_3}[N_3]) = \mathrm{lin}_{S_3}(\sum_{i=1}^{6} \alpha_i t_i) &= \sum_{i=1}^{6} \alpha_i \mathrm{lin}_{S_3}(t_i) \\ N_3 &= \alpha_1(N_3 + N_4) + \alpha_2 N_3 + \alpha_3 N_4 + \alpha_4(N_1 + N_2) + \alpha_5 N_1 + \alpha_6 N_2, \end{aligned}$$

and then the first equation system we found is

$$\alpha_1 + \alpha_2 = 1 \qquad \text{and} \qquad \alpha_1 + \alpha_3 = \alpha_4 + \alpha_5 = \alpha_4 + \alpha_6 = 0. \tag{4.5}$$

The following commutative diagram

$$\begin{array}{ccc} T_{\mathbb{F}_3}(S_3) & \xrightarrow{\mathrm{can}_{S_3}} & D_{\mathbb{F}_3}(S_3) \\ \Big\downarrow \mathrm{res}_{C_3}^{S_3} & \circ & \Big\downarrow \mathrm{res}_{C_3}^{S_3} \\ T_{\mathbb{F}_3}(C_3) & \xrightarrow{\mathrm{can}_{C_3}} & D_{\mathbb{F}_3}(C_3) \end{array} \tag{4.6}$$

gives us

$$\begin{aligned}\operatorname{res}^{S_3}_{C_3}(\operatorname{can}_{S_3}[N_3]) &= (2\alpha_1+\alpha_2+\alpha_3)[1,1]_{C_3} + (2\alpha_4+\alpha_5+\alpha_6)[C_3,1]_{C_3},\\ \operatorname{can}_{C_3}(\operatorname{res}^{S_3}_{C_3}[N_3]) &= [1,1]_{C_3}\end{aligned} \tag{4.7}$$

It follows by the second equation system

$$2\alpha_1+\alpha_2+\alpha_3 = 1 \qquad \text{and} \qquad 2\alpha_4+\alpha_5+\alpha_6 = 0. \tag{4.8}$$

Remark that the canonical induction formula for C_3 is given by $\operatorname{can}_{C_3}[V_1] = [C_3,1]_{C_3}$ and $\operatorname{can}_{C_3}[V_2] = [1,1]_{C_3}$ where $V_1 \sim 1$ and $V_2 = \begin{matrix}1\\1\\1\end{matrix}$. In a similar way, by the commutative diagrams

$$\begin{array}{ccc} T_{\mathbb{F}_3}(S_3) & \xrightarrow{\operatorname{can}_{S_3}} & D_{\mathbb{F}_3}(S_3)\\ \Big\downarrow \operatorname{res}^{S_3}_{C_2} & \circ & \Big\downarrow \operatorname{res}^{S_3}_{C_2}\\ T_{\mathbb{F}_3}(C_2) & \xrightarrow{\operatorname{can}_{C_2}} & D_{\mathbb{F}_3}(C_2)\end{array} \qquad \begin{array}{ccc} T_{\mathbb{F}_3}(S_3) & \xrightarrow{\operatorname{can}_{S_3}} & D_{\mathbb{F}_3}(S_3)\\ \Big\downarrow \operatorname{res}^{S_3}_{C_1} & \circ & \Big\downarrow \operatorname{res}^{S_3}_{C_1}\\ T_{\mathbb{F}_3}(C_1) & \xrightarrow{\operatorname{can}_{C_1}} & D_{\mathbb{F}_3}(C_1)\end{array} \tag{4.9}$$

we also obtain

$$\begin{aligned}\operatorname{res}^{S_3}_{C_2}(\operatorname{can}_{S_3}[N_3]) &= (3\alpha_1+\alpha_2+\alpha_3+\alpha_4)[1,1]_{C_2}+(\alpha_2+\alpha_5)[C_2,1]_{C_2}+(\alpha_3+\alpha_6)[C_2,1_-]_{C_2},\\ \operatorname{can}_{C_2}(\operatorname{res}^{S_3}_{C_2}[N_3]) &= 2[C_2,1]_{C_2} + [C_2,1_-]_{C_2}.\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\operatorname{res}^{S_3}_{C_1}(\operatorname{can}_{S_3}[N_3]) &= (6\alpha_1+3\alpha_2+3\alpha_3+3\alpha_4+\alpha_5+\alpha_6)[1,1]_{C_1},\\ \operatorname{can}_{C_1}(\operatorname{res}^{S_3}_{C_1}[N_3]) &= 3[C_1,1]_{C_1}.\end{aligned} \tag{4.11}$$

Remark that the canonical induction formula for C_2 is given by $\operatorname{can}_{C_2}[U_1] = [C_2,1]_{C_2}$ and $\operatorname{can}_{C_2}[U_2] = [C_2,1_-]_{C_2}$ where $U_1 \sim 1$ and $U_1 \sim 1_-$. Hence, the third and fourth equation systems are

$$\begin{aligned}3\alpha_1+\alpha_2+\alpha_3+\alpha_4 = 0, \qquad \alpha_2+\alpha_5 = 2, \qquad \alpha_3+\alpha_6 = 1\\ \text{and} \quad 6\alpha_1+3\alpha_2+3\alpha_3+3\alpha_4+\alpha_5+\alpha_6 = 3.\end{aligned} \tag{4.12}$$

can_{S_3}	N_1	N_2	N_3	N_4
t_1	0	0	-1	-1
t_2	0	0	2	1
t_3	0	0	1	2
t_4	0	0	0	0
t_5	1	0	0	0
t_6	0	1	0	0

Table 4.3: Canonical Induction

lin_{S_3}	t_1	t_2	t_3	t_4	t_5	t_6
N_1	0	0	0	1	1	0
N_2	0	0	0	1	0	0
N_3	1	1	0	0	0	0
N_4	1	0	1	0	0	0

Table 4.4: Linearization

After solving these four equation systems in 4.5, 4.8 and 4.12, we find a general solution as follows:

$$\alpha_1 = A, \quad \alpha_2 = 1-A, \quad \alpha_3 = -A, \quad \alpha_4 = -1-A, \quad \alpha_5 = 1+A \text{ and } \alpha_6 = 1+A \tag{4.13}$$

where A is a variable which could not determined yet.

Similarly, for N_4 we get four equation systems

$$\beta_1 + \beta_3 = 1, \qquad \beta_1 + \beta_2 = \beta_4 + \beta_5 = \beta_4 + \beta_6 = 0; \tag{4.14}$$

$$2\beta_1 + \beta_2 + \beta_3 = 1 \qquad 2\beta_4 + \beta_5 + \beta_6 = 0; \tag{4.15}$$

$$3\beta_1 + \beta_2 + \beta_3 + \beta_4 = 0, \qquad \beta_2 + \beta_5 = 1, \qquad \beta_3 + \beta_6 = 2; \tag{4.16}$$

$$6\beta_1 + 3\beta_2 + 3\beta_3 + 3\beta_4 + \beta_5 + \beta_6 = 3. \tag{4.17}$$

It follows the general solution

$$\beta_1 = B, \quad \beta_2 = -B, \quad \beta_3 = 1-B, \quad \beta_4 = -1-B, \quad \beta_5 = 1+B \text{ and } \beta_6 = 1+B \tag{4.18}$$

where B is a variable which could not determined yet.

On the other hand, we have the canonical induction formula in explicit form by Theorem 3.5.2. Then, we have

$$
\begin{aligned}
\mathrm{can}_{S_3}(N_3) &= \frac{1}{|S_3|}\sum_{H,U\le S_3}|H|\mu(H,U)[H,\mathrm{res}_{H,U}(p_U(\mathrm{res}_{U,S_3}(N_3)))]_{S_3}\\
&= \sum_{H\le S_3}\frac{|H|}{|S_3|}(\sum_{H\le U\le S_3}\mu(H,U)[H,\mathrm{res}_{H,U}(p_U(\mathrm{res}_{U,S_3}(N_3)))]_{S_3})\\
&= \frac{|S_3|}{|S_3|}\mu(S_3,S_3)p_{S_3}(N_3)+\\
&\frac{|C_3|}{|S_3|}\mu(C_3,S_3)\mathrm{res}_{C_3,S_3}(p_{S_3}(N_3))+\frac{|C_3|}{|S_3|}\mu(C_3,C_3)p_{c_3}(\mathrm{res}_{C_3,S_3}(N_3))+\\
&3(\frac{|C_2|}{|S_3|}\mu(C_2,S_3)\mathrm{res}_{C_2,S_3}(p_{S_3}(N_3))+\frac{|C_2|}{|S_3|}\mu(C_2,C_2)p_{C_2}(\mathrm{res}_{C_2,S_3}(N_3)))+\\
&\frac{|C_1|}{|S_3|}\mu(C_1,S_3)\mathrm{res}_{C_1,S_3}(p_{S_3}(N_3))+\frac{|C_1|}{|S_3|}\mu(C_1,C_3)\mathrm{res}_{C_1,C_3}(p_{C_3}(\mathrm{res}_{C_3,S_3}(N_3)))+\\
&3(\frac{|C_1|}{|S_3|}\mu(C_1,C_2)\mathrm{res}_{C_1,C_2}(p_{C_2}(\mathrm{res}_{C_2,S_3}(N_3))))+\frac{|C_1|}{|S_3|}\mu(C_1,C_1)p_{C_1}(\mathrm{res}_{C_1,S_3}(N_3))\\
&= 0+0+0+0+2[C_2,1_+]_{S_3}+[C_2,1_-]+0+0-\frac{3}{2}[C_1,1]_{S_3}+\frac{1}{2}[C_1,1]_{S_3}\\
&= 2[C_2,1_+]_{S_3}+[C_2,1_-]-[C_1,1]_{S_3}
\end{aligned}
$$

and similarly

$$\mathrm{can}_{S_3}(N_4) = [C_2,1_+]_{S_3}+2[C_2,1_-]-[C_1,1]_{S_3}.$$

It is compatible with 4.13 and 4.18 by assigning -1 to A and B. Therefore, we complete the linearization and canonical induction matrices as in the Tables 4.4 and 4.3, and find the unique canonical induction formula for the trivial source ring $T_{\mathbb{F}_3}(S_3)$ from the monomial ring $D_{\mathbb{F}_3}(S_3)$.

Chapter 5

Canonical Induction of Regular bimodules

Throughout, $\mathbb{F}$ denotes an algebraically closed field of characteristic p and A a finite dimensional algebra over field F.

First, we give an introductory review of block theory that can be found in [27] and classical block theory books. An *idempotent* e of A is defined to be an element in A satisfying $e^2 = e$. Two idempotents e and f are called *orthogonal* if $ef = 0 = fe$. An idempotent in A is called *primitive* if it is nonzero and cannot be written as the sum of two nonzero orthogonal idempotents in A. Given idempotent e in A, $1 - e$ is an idempotent in A which is orthogonal to e. This implies that $A = Ae \oplus A(1-e)$ and then Ae is a projective A-module. That is, every idempotent in A defines a projective summand of A. An idempotent $e \in A$ is primitive in A if and only if the projective A-module Ae is indecomposable. In this way, there is a one-to-one correspondence between expressions $1 = e_1 + ... + e_n$ with the orthogonal idempotents e_i and direct sum decompositions ${}_AA_A = A_1 \oplus ... \oplus A_n$ of the regular bimodule, which is given by $A_i = Ae_i$. Hence, the decomposition $1_A = e_1 + ... + e_n$ of the identity in A as a sum of primitive orthogonal idempotents correspond to the decomposition $A = A1_A = Ae_1 \oplus ... \oplus Ae_r$ of ${}_AA_A$ as a direct sum of projective indecomposable modules (PIMs).

Lemma 5.0.2. *For primitive idempotents $e, f \in A$, the following are equivalent:*

(i) *e and f are conjugate in A, thus $e = u^{-1}fu$ for a unit u in A.*

(ii) *e and f are associate in A, thus there exist x, y in A such that $xy = e$ and $yx = f$.*

(iii) *The A-modules Ae and Af are isomorphic, thus $Ae \cong_A Af$ as indecomposable modules.*

When any of these conditions holds, we say that e and f are equivalent, denoted by $e \sim f$.

Note that any two primitive idempotents of A are either equivalent or orthogonal. It follows that the map $e \mapsto Ae$ induces a bijection between the equivalence classes of primitive idempotents in A and the isomorphism classes of indecomposable projective A-modules.

Theorem 5.0.3. (Krull-Schmidt) *Let $e_1, e_2, ..., e_r$ and $f_1, f_2, ..., f_s$ be mutually orthogonal primitive idempotent decompositions such that*

$$1 = \sum_{i=1}^{r} e_i = \sum_{j=1}^{s} f_j.$$

Then $r = s$ and after renumbering $e_k \sim f_k$ for $1 \leq k \leq r$. Moreover, given an A-module M with decomposition

$$M = M_1 \oplus ... \oplus M_r = N_1 \oplus ... \oplus N_s$$

for some indecomposable M_i and N_j for $1 \leq i \leq r$, $1 \leq j \leq s$. Then $r = s$ and after renumbering $M_k \cong N_k$ for $1 \leq k \leq r$.

For a finite-dimensional algebra A over $\mathbb{F}$, a *central idempotent* in A is defined to be an idempotent in the centre $Z(A)$ of A. We define a *block idempotent* to be a central idempotent which is primitive. The algebra A contains only finitely many block idempotents $e_1, ..., e_r$ which are pairwise orthogonal and of the form

$$1_A = 1_{Z(A)} = e_1 + ... + e_r.$$

Then, there is a one-to-one correspondence between the decomposition $1_A = e_1 + ... + e_r$ of orthogonal block idempotents e_i and the decomposition $A = B_1 \oplus ... \oplus B_r$ of indecomposable two-sided ideals given by $B_i = Ae_i = e_iA$. The indecomposable two-sided ideals $B_i = Ae_i = e_iA$ in the decomposition $A = B_1 \oplus ... \oplus B_r$ are called the *blocks* of A. Note that each block B_i becomes an $\mathbb{F}$-algebra, which is called a *block algebra*, in its own right with identity element e_i.

Lemma 5.0.4. *Given a finite-dimensional algebra A over $\mathbb{F}$, the block decomposition of A is unique up to isomorphism. Thus, if*

$$A = B_1 \oplus ... \oplus B_r = B'_1 \oplus ... \oplus B'_s$$

where $B_1, ..., B_r, B'_1, ..., B'_s$ are blocks of A, then $r = s$ and after renumbering $B_i \cong_A B'_i$ for $1 \leq i \leq r$.

For an indecomposable A-module M, the decomposition $M = M1_A = Me_1 \oplus ... \oplus Me_r$ shows that $Me_i = M$ for some i and $Me_j = 0$ for $j \neq i$. Furthermore, the decomposition $M = MA = MB_1 \oplus ... \oplus MB_r$ with submodules $MB_1, ..., MB_r$ shows that $MB_i \neq 0$ for a unique $i \in \{1, ..., r\}$. It follows that $M = MB_i$ can be viewed as a B_i-module, and $MB_j = 0$ for $j \neq i$. In this case, we say that M belongs to the block B_i. In the view of preceding way, we get a partition of the projective indecomposable A-modules in terms of the blocks $B_1, ..., B_r$ of A as follows:

$$\{\text{isomorphism classes of indecomposable } A\text{-modules}\} = \coprod_{i=1}^{r} \{\text{isomorphism classes of indecomposable } B_i\text{-modules}\}.$$

We also get a partition of the simple A-modules in terms of the blocks $B_1, ..., B_r$ of A in a similar way:

$$\{\text{isomorphism classes of simple } A\text{-modules}\} = \coprod_{i=1}^{r} \{\text{isomorphism classes of simple } B_i\text{-modules}\}.$$

Consider the group algebra $\mathbb{F}G$. We already know that for each indecomposable $\mathbb{F}G$-module we have a corresponding irreducible $\mathbb{F}$-character. We say that the $\mathbb{F}$-character χ belongs to the block B if the associated $\mathbb{F}G$-module M belongs to B. Moreover, we obtain the block idempotent e associated to B by these characters as follows:

$$e := \sum_{\chi_t} \frac{\chi_t(1)}{|G|} \sum_{g \in G} \chi_t(g)(g^{-1})$$

where χ_t runs over the set of $\mathbb{F}$-characters belonging to e. Notice that an $\mathbb{F}$-character belongs precisely to one block. Moreover, the *principal block* is defined to be the block containing the trivial $\mathbb{F}$-character and the case of block B of *defect* 0 is described as that χ belonging to B is the only irreducible $\mathbb{F}$-character in B.

Remark that for p-modular system $(K, R, \mathbb{F})$ the canonical map $R \to \mathbb{F}$ induces a bijection between the blocks of RG and the blocks of $\mathbb{F}G$. Then, we can use the the blocks of $\mathbb{F}G$ instead of the blocks of RG.

We now discuss the blocks on canonical induction formula for the trivial source rings. Recall that the monomial ring $D_{\mathbb{F}}(G)$ is a free abelian group on the set of G-orbits of monomial pairs of G on $\mathbb{F}$, thus

$$D_{\mathbb{F}}(G) = \bigoplus_{[H,\varphi]_G \in \mathcal{M}_{\mathbb{F}}(G)/G} \mathbb{Z}[H, \varphi]_G.$$

Also, the trivial source ring $T_{\mathbb{F}}(G)$ is a free abelian group on the set of isomorphism classes of indecomposable trivial source modules, thus

$$T_R(G) = \bigoplus_{M \in \mathrm{Triv}(RG)} \mathbb{Z}[M]$$

where $\mathrm{Triv}(RG)$ denotes the set of indecomposable trivial source RG-modules. Note that each pair (P, V) defines an indecomposable $\mathbb{F}G$-module $M_{P,V}$ with vertex P and trivial source where P is a p-subgroup of G and V is indecomposable projective $\mathbb{F}[N_G(P)/P]$-module. Then, for a regular bimodule $\mathbb{F}G$ we have

$$[\mathbb{F}G] = \bigoplus_{(1,V)} \dim(V)[M_{1,V}]$$

where V is indecomposable projective $\mathbb{F}G$-module and $M_{1,V}$ projective cover of V. For a block idempotent b of $\mathbb{F}G$, we define $\tau(b)$ to be the element of $T_{\mathbb{F}}(G)$

associated with a block idempotent b of $\mathbb{F}G$ given by

$$\tau(b) := [\mathbb{F}G \cdot b] = \bigoplus_V \dim(V) \cdot [M_{1,V}]$$

where V is indecomposable projective $\mathbb{F}G \cdot b$-module. It follows that

$$\begin{aligned} \mathrm{can}_G(\tau(1)) = \mathrm{can}_G([\mathbb{F}G \cdot 1]) &= \mathrm{can}_G(\bigoplus_V \dim(V) \cdot [M_{1,V}]) \\ &= \mathrm{can}_G(\tau(b_0 + b_1 + ... + b_n)) = \sum_{i=0}^{n} \mathrm{can}_G(\tau(b_i)). \end{aligned}$$

where the sum runs over the block idempotents of $\mathbb{F}G$. That is, we have a block decomposition of the regular bimodule $\mathbb{F}G$ for the canonical induction as in the following way:

$$\sum_b \mathrm{can}_G(\tau(b)) = \mathrm{can}_G([\mathbb{F}G]) = \mathrm{can}_G(\bigoplus_V \dim(V) \cdot [M_{1,V}]).$$

Example 5.0.5. Let $G = S_3$ and $\mathbb{F}$ be field of characteristic $p = 3$. By example 4.2.1, we already know that $T_{\mathbb{F}_3}(S_3) = \mathbb{Z}[N_1] + \mathbb{Z}[N_2] + \mathbb{Z}[N_3] + \mathbb{Z}[N_4]$ where $N_1 \sim 1$, $N_2 \sim 1_-$, $N_3 \sim \begin{matrix} 1 \\ 1_- \\ 1 \end{matrix}$ and $N_4 \sim \begin{matrix} 1_- \\ 1 \\ 1_- \end{matrix}$, and also $\mathbb{F}_3 S_3 = N_3 + N_4$. By tables 4.1 and 4.2, we observe that there is only one block since there are two simple modules, trivial and sign modules, and the sign representation appears as a composition factor of the projective cover of the trivial module. It means that the sign module belongs to the principal block. Then all three irreducible characters of $\mathbb{C}S_3$ are in the same block of $\mathbb{F}_3 S_3$. That is, $\mathbb{F}_3 S_3$ has a unique block $1 = b_0 \in Z(\mathbb{F}_3 S_3)$ which is the principal block. Note that $\mathbb{F}_3 S_3$ is only block algebra with defect group C_3. Hence, we get

$$\begin{aligned} \mathrm{can}_{S_3}(\tau(1)) = \mathrm{can}_{S_3}([\mathbb{F}_3 S_3]) &= \mathrm{can}_{S_3}(N_3 + N_4) \\ &= 3[C_2, 1_+]_{S_3} + 3[C_2, 1_-]_{S_3} - 2[C_1, 1]_{S_3}. \end{aligned}$$

Example 5.0.6. Let $G = S_3$ and $\mathbb{F}$ be field of characteristic $p = 2$. In a similar way of example 4.2.1, we obtain that $T_{\mathbb{F}_2}(S_3) = \mathbb{Z}[M_1] + \mathbb{Z}[M_2] + \mathbb{Z}[M_3]$ where $M_1 \sim 1$, $M_2 \sim 2$ and $M_3 \sim \begin{matrix} 1 \\ 1 \end{matrix}$, and also $\mathbb{F}_2 S_3 = 2M_2 + M_3$. Then, we observe that there are two blocks since simple module of degree 2 is a block of defect zero and the other simple module is trivial which lies in principal block. The projective cover of trivial module as an $\mathbb{F}_2S_3$-module has character equal to the sum of characters of trivial and sign representations. So principal block idempotent acts as identity on this, meaning that they are in principal block. The other character of degree 2 lies in the other block. That is, $\mathbb{F}_2S_3$ has two blocks b_0 and b_1 which are principal and defect zero blocks, respectively. Notice that

$$b_0 = \varepsilon_1 + \varepsilon_2 = \frac{1}{3}\sum_{g\in C_3} g = \frac{1}{3}([a]^+ + [1]) = 1 + a + a^2$$
$$b_1 = \varepsilon_3 = \frac{1}{3}(2[1] - [a]^+) = a + a^2.$$

Then, $\mathbb{F}_2S_3b_0 \cong \begin{matrix}1\\1\end{matrix} \cong M_3$ and $\mathbb{F}_2S_3b_1 \cong 2 \oplus 2 \cong 2M_2$. That is, as block algebras we have

$$\begin{aligned}\mathbb{F}_2S_3 &\cong \mathbb{F}_2S_3b_0 \oplus \mathbb{F}_2S_3b_1\\ &\cong M_3 \oplus 2M_2\end{aligned}$$

with defect groups C_2 and $\{1\}$, respectively. Hence, we get

$$\begin{aligned}\mathrm{can}_{S_3}(\tau(b_0)) + \mathrm{can}_{S_3}(\tau(b_1)) &= \mathrm{can}_{S_3}([\mathbb{F}_2S_3]) = \mathrm{can}_{S_3}([2M_2 + M_3])\\ &= \mathrm{can}_{S_3}([\mathbb{F}_2S_3 \cdot b_0]) + \mathrm{can}_{S_3}([\mathbb{F}_2S_3 \cdot b_1])\\ &= \mathrm{can}_{S_3}([M_3]) + \mathrm{can}_{S_3}([2M_2])\\ &= [C_3, 1]_{S_3} + 2[C_3, 1_w]_{S_3}\\ &= \sum_{\varphi\in\hat{C}_3(\mathbb{F}_2)} [C_3, \varphi]_{S_3}.\end{aligned}$$

Proceeding in a similar way, we observe the following results:

- For $G = C_2$, we have

$$\mathrm{can}_{C_2}(\mathbb{F}_2C_2) = [1, 1]_{C_2} \text{ and } \mathrm{can}_{C_2}(\mathbb{F}_3C_2) = \sum_{\varphi\in\hat{C}_2(\mathbb{F}_3)} [C_2, \varphi]_{C_2};$$

- For $G = C_3$,

$$\mathrm{can}_{C_3}(\mathbb{F}_2 C_3) = \sum_{\varphi \in \hat{C}_3(\mathbb{F}_2)} [C_3, \varphi]_{C_3} \text{ and } \mathrm{can}_{C_3}(\mathbb{F}_3 C_3) = [1, 1]_{C_3};$$

- For $G = S_3$,

$$\mathrm{can}_{S_3}(\mathbb{F}_2 S_3) = \sum_{\varphi \in \hat{C}_3(\mathbb{F}_2)} [C_3, \varphi]_{C_3} \text{ and}$$
$$\mathrm{can}_{S_3}(\mathbb{F}_3 S_3) = 3[C_2, 1_+]_{S_3} + 3[C_2, 1_-]_{S_3} - 2[C_1, 1]_{S_3}.$$

We now investigate the explicit formula for canonical induction formula in order to get the preceding results in theoretical way. Remember that for a regular bimodule $\mathbb{F}G$, we have

$$\mathrm{can}_G([\mathbb{F}G]) = \frac{1}{|G|} \sum_{U, U' \leq G} |U| \cdot \mu(U, U') \cdot [U, \mathrm{res}_{U,U'}(p_{U'}(\mathrm{res}_{U',G}[\mathbb{F}G]))]$$

where p_G projects 1-dim trivial source $\mathbb{F}G$-modules to the corresponding linear characters and annihilates all others. We define:

$$\Lambda_{\mathbb{F}}(G) := \mathrm{can}_G([\mathbb{F}G]) = \sum_{U \leq G} [U, \lambda_U]_G \tag{5.1}$$

$$\text{where} \qquad \lambda_U = |U| \sum_{U \leq U' \leq G} \frac{1}{|U'|} \mu(U, U') \mathrm{res}_{U,U'}(p_{U'}(\mathrm{res}_{u',G}[\mathbb{F}G])). \tag{5.2}$$

Since $\mathrm{res}_{U',G}[\mathbb{F}G] = |G : U'|[\mathbb{F}U']$, we get

$$\lambda_U = |U| \sum_{U \leq U' \leq G} \frac{1}{|U'|} \mu(U, U') \mathrm{res}_{U,U'}(p_{U'}[\mathbb{F}U']). \tag{5.3}$$

Notice that since $p_{U'}[\mathbb{F}U']$ vanishes except when $\mathbb{F}U'$ has a projective 1-dim module as a component and all projective $\mathbb{F}U'$-modules have dimensions divisible by $|U'|_p$, we get that $p_{U'}[\mathbb{F}U'] = 0$ unless U' is a p'-group. It follows that $\lambda_U = 0$ when $p||U|$. Also, we have $p_{U'}[\mathbb{F}U'] = \sum_{\varphi \in \hat{U}'(\mathbb{F})} \varphi$ for a p'-group U'. Hence, we obtain the following theorem as our main result in this thesis:

Theorem 5.0.7. *For a regular bimodule $\mathbb{F}G$, we have*

$$\Lambda_{\mathbb{F}}(G) = \sum_{U \leq G;\, U: p'\text{-group}} [U, \lambda_U]_G \tag{5.4}$$

$$\textit{where} \qquad \lambda_U = |U| \sum_{U \leq U' \leq G; U': p'\text{-group}} \frac{1}{|U'|} \mu(U, U') \mathrm{res}_{U,U'}\Big(\sum_{\varphi \in \hat{U}'(\mathbb{F})} \varphi\Big). \tag{5.5}$$

We now look at what this theorem implies for specific cases. Let $G = C_q$ where C_q is cyclic group with prime q and $\mathbb{F}$ be a field of characteristic p for prime p. If $p = q$, then the only p'-group of C_q is $C_1 = \{1\}$. Hence,

$$\lambda_{C_1} = |C_1|\frac{1}{|C_1|}\mu(C_1, C_1)1_{C_1} = 1_{C_1}.$$

If $p \neq q$, then p'-groups of C_q are C_1 and C_q. Hence, we have

$$\begin{aligned}\lambda_{C_q} &= |C_q|\frac{1}{|C_q|}\mu(C_q, C_q)(\sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}\varphi) = \sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}\varphi,\\ \lambda_{C_1} &= |C_1|\frac{1}{|C_1|}\mu(C_1, C_1)1_{C_1} + |C_1|\frac{1}{|C_q|}\mu(C_1, C_q)\mathrm{res}_{C_1,C_q}(\sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}\varphi)\\ &= 1_{C_1} - 1_{C_1} = 0.\end{aligned}$$

Proposition 5.0.8. *Let $G = C_q$ where C_q is cyclic group with prime order q. Then,*

$$\Lambda_{\mathbb{F}_p}(C_q) = \begin{cases} [C_1, 1_{C_1}]_{C_q} & \text{if } p = q\\ \sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}[C_q, \varphi]_G & \text{if } p \neq q. \end{cases}$$

Let $G = C_p \rtimes C_q$ where C_p and C_q are cyclic groups with different prime orders p and q, respectively. The p'-groups of G are C_1 and C_q, and then

$$\begin{aligned}\lambda_{C_q} &= |C_q|\frac{1}{|C_q|}\mu(C_q, C_q)(\sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}\varphi) = \sum_{\varphi\in\hat{C}_q}\varphi,\\ \lambda_{C_1} &= |C_1|\frac{1}{|C_1|}\mu(C_1, C_1)1_{C_1} + |C_1|\sum_{C_q\leq G}\frac{1}{|C_q|}\mu(C_1, C_q)\mathrm{res}_{C_1,C_q}(\sum_{\varphi\in\hat{C}_q(\mathbb{F}_p)}\varphi)\\ &= 1_{C_1} - n_q\cdot 1_{C_1}\end{aligned}$$

where n_q is the number of the elements of conjugacy class of C_q in G. Hence, we conclude that

$$\Lambda_{\mathbb{F}_p}(G) = \sum_{U:C_1,C_q}[U, \lambda_U]_G \tag{5.6}$$

$$= n_q([C_q, \sum_{\varphi\in\hat{C}_q}\varphi]_G) + [C_1, (1-n_q)1_{C_1}]_G. \tag{5.7}$$

Notice that C_q is Sylow q-subgroup of G so that n_q equals to the number of Sylow q-subgroups of G. Using Sylow's theorem we obtain some remarkable results about $\Lambda_{\mathbb{F}_p}(G)$ as follows:

Proposition 5.0.9. *Let $G = C_p \rtimes C_q$ where C_p and C_q are cyclic groups with prime orders $p > q$. Then, we have*

$$\Lambda_{\mathbb{F}_p}(G) = n_q([C_q, \sum_{\varphi \in \hat{C}_q} \varphi]_G) + [C_1, (1 - n_q)1_{C_1}]_G. \tag{5.8}$$

In particular, if $p \neq 1(\mod q)$ then

$$\Lambda_{\mathbb{F}_p}(G) = \sum_{\varphi \in \hat{C}_q(\mathbb{F}_p)} [C_q, \varphi]_G. \tag{5.9}$$

Proof. First part is already established in 5.6. We shall show the second part. By Sylow's theorem, we have that (i) $n_q | p$, thus $n_q = 1$ or p, (ii) $n_q = 1 \mod q$ and (iii) $n_q = |G : N_G(Q)|$ for any Sylow q-subgroup Q, thus $n_q = 1, p, q$ or pq. There are two possibilities such that $n_q = p$ where $p = 1(\mod q)$ or $n_q = 1$. By the assumption on p, we obtain $n_q = 1$ and then

$$\Lambda_{\mathbb{F}_p}(G) = \sum_{\varphi \in \hat{C}_q(\mathbb{F}_p)} [C_q, \varphi]_G. \tag{5.10}$$

□

Proposition 5.0.10. *Let $G = C_q \rtimes C_p$ where C_p and C_q are cyclic groups with prime orders $p < q$. Then,*

$$\Lambda_{\mathbb{F}_p}(G) = \sum_{\varphi \in \hat{C}_q(\mathbb{F}_p)} [C_q, \varphi]_G.$$

Proof. If $p < q$, then C_q is normal Sylow q-subgroups of G. That is, $n_q = 1$. By 5.6, we get

$$\Lambda_{\mathbb{F}_p}(G) = \sum_{\varphi \in \hat{C}_q(\mathbb{F}_p)} [C_q, \varphi]_G.$$

□

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