THRESHOLD DYNAMICS
IN ONE-SECTOR
OPTIMAL GROWTH FRAMEWORK

A Ph.D. Dissertation

by
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To Deniz & Damla

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OPTIMAL GROWTH FRAMEWORK

The Graduate School of Economics and Social Sciences
of
İhsan Doğramacı Bilkent University

by

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in

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Doctor of Philosophy in Economics.

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ABSTRACT

THRESHOLD DYNAMICS

IN ONE-SECTOR OPTIMAL GROWTH FRAMEWORK

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This thesis includes three self-contained essays on the threshold dynamics in one-sector optimal growth models. In the first essay, we consider the preferences for wealth-habit in a one-sector optimal growth model. We show that the dynamics may encounter saddle-node bifurcations with respect to the parameters of the preferences. We analytically provide the monotone comparative statics and the continuity of the critical capital stock with respect to these parameters and the discount factor. In the second essay, we analyzed the joint dynamic implications of time-to-build lag in investment and non-convex technologies. We prove the existence of persistent cyclical dynamics even in one-sector optimal growth framework. Finally, in the third essay, considering time-to-build lag in non-classical optimal growth framework inducing threshold dynamics, we analyze the effects of an alternative information structure regarding the initial conditions on the equilibrium dynamics. In particular, we seek to understand the dependence of the cyclical dynamics on the information structure. In all essays, we also support the analytical solutions by numerical methods in order to better understand the underlying dynamics of the optimal path.

Keywords: Threshold dynamics, wealth-dependent preferences, non-convex technologies, time-to-build

ÖZET

TEK SEKTÖRLÜ OPTİMAL BÜYÜME MODELLERİNDE

EŞİK DEĞER DİNAMİKLERİ

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Bu tez, tek sektörlü optimal büyüme modellerinde eşik değer dinamikleri üzerine üç ayrı makale içermektedir. İlk makalede, aktörlerin sadece tüketimden değil servet düzeylerinden de fayda sağlamaları durumunda denge dinamikleri incelenmiştir. Fayda fonksiyonu parametreleri ile zaman tercihindeki küçük değişikliklerin sermaye stoğunun eşik değeri üzerindeki niteliksel etkileri analitik olarak ispatlanmıştır. İkinci makalede, yatırımın operasyonel hale gelmesindeki gecikmeler (time-to-build), teknolojinin dış bükey olmadığı bir model kapsamında ele alınarak, kalıcı

döngüsel dinamiklerin tek sektörlü optimal büyüme modellerinde dahi ortaya çıkabileceği ispatlanmıştır. Üçüncü makalede, yatırımın operasyonel hale gelmesindeki gecikmelerin kalıcı döngüsel dinamiklerin ortaya çıkmasındaki etkisi dikkate alınarak, bu döngüsel dinamiklerin farklı başlangıç şartları altında da ortaya çıkıp çıkmayacağı incelenmiştir. Ayrıca, tüm makalelerde, analitik çözümler sayısal yöntemlerle de desteklenerek, optimal patika dinamiklerinin daha iyi anlaşılması sağlanmaya çalışılmıştır.

Keywords: Eşik değer dinamikleri, servete bağımlı fayda fonksiyonu, dış bükey olmayan teknolojiler, yatırımın operasyonel hale gelmesindeki gecikmeler

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CHAPTER 1

INTRODUCTION

One of the most frequently discussed economic growth and development facts has been the large and the persistent differences in per capita gross domestic products (GDP) across countries. Some countries manage to sustain high levels of per capita GDP over the long periods of time while some others seem to be caught in the poverty or development trap. It is evident that countries at the top of the world income distribution are more than 70 times richer than those at the bottom. For example, in 2000, per capita GDP in the United States was about $35000 while it was about $500 in Tanzania (see Azariadis and Stachursky, 2005).

This gap among the richest and the poorest countries has been increasing over time. Per capita GDP in Tanzania was $478 in 1960 and $457 in 2000. However, per capita GDP in the United States increased from $12,598 in 1960 to $33,523 in 2000 [see Azariadis and Stachursky, 2005].

Until 1960s, the growth rates of the developed countries were only slightly higher than the growth rates of the less developed countries. The developed countries showed the growth rate of 3.2% from 1960s to the 1980s and of 1.5% from 1980 to 1995 while these number were 2.5% and 0.34% for the less developed countries, respectively (see Semmler and Ofori, 2007). This implies that the per capita GDPs have become polarized constituting the existence of convergence clubs and twin-peaks in the world income distribution. This is due to the fact that a substantial fraction of the poor countries has shown very little or no growth and a number of the middle income countries has grown rapidly, in some cases fast enough to catch up with the rich countries (Quah, 1996). The net result of these changes is a movement in the shape of the world income distribution from something that looks like a normal distribution in 1960 to a bi-modal distribution in 1988. Bianchi (1997) and Paap and van Dijk (1998) confirms the existence of twin peaks by rejecting a single peak.

On the other hand, recently, Kremer et al. (2001) suggest that the cross-country income dynamics are characterized by uni-modal distribution asymptotically as it converges to the single peak at high incomes and by bi-modal distribution during this transition.

A growing literature also claims to find evidence of middle income trap across a wide number of countries (Kharas and Kohli, 2011; Felipe et al., 2012; Aiyar et al., 2013; Gabriel Im and Rosenblatt, 2013). It refers to a situation whereby a middle-income country is failing to become a high-income economy and stuck at low growth rates. Instead of steadily moving up over time, its per capita GDP simply fluctuates around a fixed point (e.g. Kharas and Kohli, 2011). Latin America consists of countries that are not so poor but have experienced very little economic growth on average since 1980. For example, Mexico, Brazil, and Argentina had very small amounts of average yearly growth since 1980, ranging from negative growth to 0.6% growth, with a lot of crises, shocks, and collapses.

In this thesis, we analyze to what extent we can capture the above empirical regularities in a one sector optimal growth framework. Recall that the classical optimal growth model suggests that, regardless of the initial conditions, countries with the same technology and preferences monotonically converge to a unique steady state. From the theoretical perspective, we would like to examine how robust the dynamic implications of the classical optimal growth model are against changes in the underlying neo-classical assumptions on technology and preferences. In particular, we advocate wealth dependent preferences and non-convexities in technology together with time-to-build lag.

Recalling that the possibility of cyclical dynamics requires at least two production sectors in a classical optimal growth framework (see Dechert, 1984; Nishimura and Sorger, 1996), we look for a mechanism through which cyclical dynamics become optimal even in a one sector optimal growth framework.

1.1 Related Literature

To account for the development patterns that differ considerably among countries in the long run, a variety of one-sector optimal growth models that incorporate wealth (capital per capita) in utility (e.g., Kurz 1968; Zou 1994; Roy 2010) or some degree of market imperfections based on technological external effects and increasing returns (e.g., Dechert and Nishimura 1983; Kamihigashi and Roy 2007; Akao et al., 2011) have been presented.

This literature demonstrates that the initially underendowed economies may lag permanently behind the otherwise identical economies. The existence of a critical capital stock due to non-convex technologies leads to the long-run dynamics consistent with important economic phenomena such as history dependence and convergence clubs (see Quah 1996; Azariadis, 1996; Azariadis and Stachurski, 2005). Indeed, the economies with low initial capital stock or income converge to a steady state with low per capita income while economies with high initial capital stock or income converge to a steady state with high per capita income. Recently, Akao et al. (2011) show that the critical capital stock is monotone and continuous with respect to the discount factor in a Dechert and Nishimura (1983) framework. These properties are important as they may provide answers to some fundamental questions: "Does a more patient country have a smaller critical capital stock? Do countries with similar levels of patience have similar critical stocks? Could a small change in a discount rate cause an economy locked in poverty trap to grow, or a growing economy to shrink?" However, the properties of critical capital stock with respect to the parameters of the preferences have not been analyzed.

The introduction of increasing returns may not only lead to threshold dynamics but also lead to the multiple equilibrium paths i.e. local indeteminacy due to the complementarity between the private returns to the accumulation of capital stock and the aggregate stock (see among others Benhabib and Perli, 1994; Benhabib and Farmer, 1996; especially Nishimura and Venditti, 2006 for extensive bibliography). In this literature, macroeconomic fluctuations arise as a result of some coordination problems caused by the existence of multiple equilibrium paths (Benhabib and Perli, 1994; Nishimura and Venditti, 2006). In multi-sector models of local indeterminacy, under minor sector-specific externalities, constant aggregate returns at the social level are shown to be compatible with the occurence of macroeconomic fluctuations (see Benhabib and Nishimura, 1998). However, in one sector models, the existence of the macroeconomic fluctuations requires unreasonably high degrees of the external effects or the increasing returns to scale in production (Benhabib and Farmer, 1994).

Considering the role of non-convex technologies in explaining cross country income differences and macroeconomic fluctuations, it is important to state the relevance of it. Sachs (2005) has discussed that there are increasing returns to scale at low per capita capital. Considering the example of a road, half of which is paved and the other half is impassable, he has argued the existence of a threshold effect in which the capital stock becomes useful only when it meets a minimum standard (see Sachs, 2005, p.250). Increasing returns to scale can be due to either the adoption of modern production techniques which involve fixed costs or the non-rivalry property of knowledge intensive technologies. In the presence of fixed costs, non-convexities at the micro level may not be convexified at an efficient scale especially in less developed countries. As the inducement to invest depends on the size of the economy, relatively small market scale would be too small to justify the fixed costs it requires (Rosenstein-Rodan, 1943). In the presence of non-rivalry, Romer (1990) has discussed that once knowledge is created, it spills over the economy at almost zero cost leading to positive externalities and increasing returns to scale. Therefore, an accurate analysis of cross country income differences should take into account that less developed economies experience substantial increasing returns to scale at early stages of their development.

Another strand of the existing literature on cyclical dynamics focuses on time-to-build lag in production or investment. Time-to-build lag can be defined as the elapsed time between the decision and the realization of it. It is in fact an empirically observable fact that is important in explaining macroeconomic fluctuations as Aftalion (1927) claimed that

> *"...the chief responsibility for cyclical fluctuations should be assigned to one of the characteristics of modern technique, namely, the long period required for the production of fixed capital..."*.

The empirical relevance of it was first highlighted by Jevons (1871) with the following lines:

> "*...A vineyard is unproductive for at least 3 years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of 5 or 6 years, before gold is reached...*" (Jevons, 1871, Theory of Capital, Chapter VII, p. 225).

Later, Mayer (1960) has shown the average time between the decision to build an industrial plant and the completion of it to be twenty-one months. Similarly, Lieberman (1987) has found that approximately two years on average are required in the chemical industry from the time of the construction decision of a plant to the time when it becomes operational. Majd and Pindyck (1986) and MacRae (1989) have shown that the construction of a petrochemical plant and a power-generating plant take more than five years to complete, respectively. Recently, Koeva (2000) has confirmed that time-to-build lag is fairly long across industries with the average length of approximately two years in most industries.

On theoretical grounds, the relationship between cyclical dynamics and the time-to-build delay was first analyzed by Kalecki (1935). He introduced a time delay between the investment decision and the realization of it and proved the existence of endogenous cycles. Later by Kydland and Prescott (1982), a discrete-time model in a real business cycle framework was developed to demonstrate the importance of time-to-build lag in explaining the macroeconomic fluctuations. Zak (1999) was the first to show that Kalecki's result holds in a Solowian economy (see also Krawiec and Szydlowski, 2004). Recently, Ferrara et al. (2014) examine the effects of time-to-build delay in the Solow growth model with non-convex technology and show that the result also admits Hopf cycles.

In optimal growth framework, the dynamic implications of time-to-build delay has only been studied in continuous time under the convex technology. The significant implication of such models is that the optimal path is oscillatory in an endogenous way eventually converging to a steady state (Winkler et al., 2003, 2005; Collard et al., 2006, 2008; Ferrara et al. 2013). However, time-to-build lag has been ignored to a large extent in non-classical optimal growth models that induce threshold dynamics.

1.2 Organization of the Thesis

This thesis includes three self-contained essays on the threshold dynamics in one-sector optimal growth models. In the first essay, we analyze the dynamic implications of preferences for wealth habit in a one-sector optimal growth model. The degree of wealth habit serves as a self-assessed reservation or subsistence wealth level as the agent cannot handle a decrease in his wealth below this level (see Bakshi and Chen 1996). Such a formulation with preferences for wealth habit instead of absolute wealth enables to capture not only that wealth is more valuable than its implied consumption rewards, but also that the degree of complementarity between the current and the next period's wealth gets stronger as the weight of wealth in utility increases.

We show that the dynamics may encounter saddle-node bifurcations with respect to the parameters of the preferences: the relative weight of wealth in utility and the degree of wealth habit. We analyze to what extent the existence and the behavior of the critical capital stock in such a framework depend on the parameters of the preferences and the discount factor by analytically providing the monotone comparative statics and the continuity of the critical capital stock with respect to them.

The important feature of the models with threshold dynamics due to either non-convexities or wealth effects is that the monotonic behaviour of the optimal path over time continues to hold as the reduced form utility function has still positive cross partial derivatives (e.g. Benhabib and Nishimura, 1985). Therefore, it is suggested that the possibility of cyclical dynamics requires at least two production sectors (see Dechert, 1984; Nishimura and Sorger, 1996).

In the second essay, we analyzed the effects of time-to-build lag in investment, which is an empirically well-grounded fact, on the implications of models with threshold dynamics. We do this in the familiar Dechert and Nishimura (1983) framework. We show the existence of persistent cyclical dynamics even in a one-sector optimal growth framework when time-to-build lag in investment is introduced in a model with threshold dynamics. Hence, the interaction between the time-to-build lag and the non-convex technology may produce the long-run dynamics consistent with the empirically observed phenomenon of middle income trap. Our results can also be thought as complementary to the explanation of business cycle fluctuations in the sense that we offer an alternative framework that may reduce the dependence of RBC literature on large exogenous shocks in explaining the aggregate data. From a theoretical perspective, we introduce the simplest mechanism in the optimal growth literature generating endogenous economic fluctuations.

In the presence of time to build lag, to form the productive stock requires multiple periods of time and the incomplete productive stock cannot be put in the production process before this time elapses. Accordingly, given a $d-$period time-to-build lag, the information structure describes the initial conditions necessary for enabling the production in the first d periods. Then, depending on these initial conditions, the optimal path can be characterized by non-monotone dynamics. However, in an alternative information structure, the initial conditions may reveal the overall initial stock for the first d periods. Note that given $d-$period time-to-build lag, this overall stock must be allocated optimally over the first d periods. In the last essay, we analyze the effects of such a change in the information structure on the equilibrium dynamics. In particular, we seek to understand the dependence of the cyclical dynamics on the information structure. Do the cyclical dynamics become less pronounced due to the consumption smoothing when the overall stock is allocated optimally over the first d periods? How does the allocation of the total stock over the first d periods change as the total stock increases? Can this be monotone?

CHAPTER 2

SADDLE-NODE BIFURCATIONS IN AN OPTIMAL GROWTH MODEL WITH PREFERENCES FOR WEALTH HABIT[1]

A general tendency in the studies aiming to account for the non-convergent growth paths by means of multiple steady states, indeterminacy or continuum of equilibria is that they are mostly devoted to the analysis of the technology component leaving the preferences essentially unaltered (see Azariadis and Stachurski, 2005, for a recent survey and Nishimura and Venditti, 2006, for extensive bibliography). This paper, focusing on the preference component, formalizes the *capitalist spirit* a la Weber in order to explain theoretically why the differences in per capita output levels among countries persist in the long run (see Quah, 1996; Barro, 1997).

The capitalist spirit refers to the motivation behind the perpetual acquisition of wealth not only for the sake of maximizing long-run consumption but also for the utility from accumulating wealth itself (see Weber, 1958).

[1] This is a joint work with Çağrı Sağlam and Agah Reha Turan. (It is forthcoming in "Studies in Nonlinear Dynamics and Econometrics", Volume 0, Issue 0, Pages 1–12, ISSN (Online) 1558-3708, ISSN (Print) 1081-1826, DOI: 10.1515/snde-2012-0050, June 2013.)

Well before Weber, this view has also been taken by Adam Smith (1937, p 324-325) in which the habitual nature has clearly been seen [2]:

> "...The principle which prompts to save, is the desire of bettering our condition, a desire which...*comes with us from the womb, and never leaves us till we go into the grave*...An augmentation of fortune is the means by which the greater part of men propose and wish to better their condition...and the most likely way of augmenting their fortune, is to save and accumulate some part of what they acquire, either regularly and annually..."

Keynes (1971, p.12) has described the habitual saving behavior of the capitalist as follows:

> "...They were allowed to call the best part of the cake theirs and were theoretically free to consume it, on the tacit underlying condition that they consumed very little of it in practice...And so *the cake increased*; but to what end was not clearly contemplated...Saving was for old age or for your children; but this was only in theory - the virtue of the cake was that it was never to be consumed, neither by you nor by your children after you..."

In order to formalize the capitalist spirit hypothesis, the early contributions assumed a utility function that depends on the absolute level of wealth in addition to consumption (see Zou, 1994; Olson and Roy, 1996; Francis, 2008; Roy, 2010). However, such a formulation ceases to capture the motive behind the accumulation of wealth for its own sake to the full extent. This stems from the fact that the inclusion of absolute wealth in utility does not alter the degree of complementarity between

[2] See Zou (1994) for the extensive bibliography and for an excellent review of how great economists in the history build the same notion of capitalist spirit.

current and the next period's wealth levels. As such, these models do not necessarily predict that wealthy agents will continue to accumulate more than those with less wealth in a way that is at least roughly consistent with the empirical evidences (see Carroll, 2000).

In this paper we formalize the spirit of capitalism hypothesis by proposing a utility function that expresses preferences not only over consumption but also over the wealth habit and analyze the implications on the long run equilibrium dynamics. Such a formulation enables us to capture the key essences of the capitalist spirit hypothesis that wealth is more valuable than its implied consumption rewards, the utility from the intrinsic wealth accumulation should be increasing with the level of the current wealth and accordingly, the degree of complementarity between current and the next period's wealth should get stronger as the relative weight of wealth in utility increases. The degree of habit in wealth serves as a self-assessed reservation or subsistence wealth level as the agent cannot handle a decrease in his wealth below this level (see Bakshi and Chen, 1996, pp. 136, Model 3). Bakshi and Chen (1996) test such a form of the spirit of capitalism by subjecting the asset pricing-equation under one parameterized preference model to monthly US data and conclude that the estimated values and signs of these preference parameters are confirming the hypothesis. In particular, arguing that self-perception determines happiness and wealth serves as an index of social status, such a preference model does a better job in explaining empirically observed asset prices.

This paper considers the dynamic implications of the spirit of capitalism formalized by the preferences for wealth habit in a one-sector optimal growth model. We show that there exists a unique optimal path from any initial capital stock under convex technology and the optimal paths are monotonic independent of the curvature of production. We prove that the multiplicities of optimal steady states and the history dependent optimal paths arise even under a strictly convex technology. Accordingly, there exists a threshold level of initial capital stock below which the optimal path converges to a low steady state and above which the economy converges to the high steady state. In particular, we show that the dynamics may encounter saddle-node bifurcations with respect to the parameters of the preferences: the relative weight of wealth habit in utility and the degree of wealth habit. Put differently, we show that minor differences in the relative weight of wealth habit in utility and/or the degree of wealth habit lead to permanent differences in the optimal path.

The fact that the presence of wealth effects may lead to multiple stationary states, hence may lead the optimal growth strategy to justify only narrow ranges of development dates back to Kurz (1968). More recently, Roy (2010) and Zou (1994) derive the necessary and sufficient conditions for the possibility of sustained growth under the assumption of wealth in utility. However, in all of these studies, how the critical capital stock varies with respect to the preference parameters and the discount factor receives little or no attention at all. It is of great interest to understand the behavior of the critical capital stock since its existence implies history dependence and drastic changes in the optimal paths[3].

A remarkable feature of our analysis is that our results do not rely on particular parameterization of the exogenous functions involved in the model, rather, it provides a more rigourous framework in regards to the formalization of the capitalist spirit, keeps the model analytically tractable and uses only general and plausible qualitative properties. Indeed, the monotone comparative statics and the continuity of the critical capital stock with respect to the discount factor, the relative weight of wealth habit in utility and the degree of wealth habit have been provided analytically.

[3] Akao, et. al., (2011) analyzes the monotonicity and the continuity of the critical capital stock with respect to the discount factor in a Dechert and Nishimura (1983) framework.

The rest of the paper is organized as follows. Section 2 describes the model and provides the dynamic properties of optimal paths. Section 3 investigates the qualitative implications of the capitalist spirit on the long run dynamics and presents the monotone comparative statics and the continuity of the critical capital stock. The numerical analysis provided in this section complements the theoretical results. Finally, Section 4 concludes.

2.1 The Model

The model differs from the classic one sector optimal growth model by the assumptions on the preferences of the agents. Considering that wealth is more valuable than its implied consumption rewards, we assume that the utility depends not only on current consumption but also on wealth habit at each period. The model is formalized as follows:

$$\max_{\{c_t, z_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t \left[u(c_t) + \eta w(z_t)\right] \qquad (E1-P)$$

subject to

$$\forall t \in \mathbb{Z}_+, c_t + k_{t+1} \leq f(k_t),$$

$$z_t = k_{t+1} - \gamma k_t,$$

$$c_t \geq 0, k_t \geq 0, z_t \geq 0,$$

$$k_0 \geq 0, \text{given},$$

where c_t is the consumption and k_t is the capital stock in period t. z_t refers to the relative change of wealth with respect to the past level of wealth namely the wealth habit at that period. γk_t serves as a self-assessed reservation or subsistence wealth level as the agent cannot handle a decrease in his wealth below this level. $\eta \in \mathbb{R}_+$

measures the relative weight of wealth habit in utility, $\gamma \in (0,1)$ measures the degree of wealth habit and $\beta \in (0,1)$ is the discount factor. We employ an additively separable one-period utility function between consumption and wealth habit not only for analytical convenience but also for being consistent with recent empirical findings[4].

We make the following assumptions.

Assumption 2.1 $u : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous, twice continuously differentiable, and satisfies* $u(0) = 0$. *Moreover, it is strictly increasing, strictly concave and* $u'(0) = +\infty$.

Assumption 2.2 $w : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuously differentiable, strictly increasing, concave and satisfies* $w(0) = 0$.

Assumption 2.3 $f : \mathbb{R}_+ \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing, strictly concave and satisfies* $f(0) = 0$. *Moreover,* $f'(0) > \frac{1}{\beta}$ *and* $\lim_{x\to\infty} f'(x) < 1$.

Assumption 2.3 implies the existence of a maximum sustainable capital stock $A(k_0) = \max\{k_0, \bar{k}\}$, where $\bar{k}$ is such that $f(\bar{k}) = \bar{k}$ and $f(k) < k$ for all $k > \bar{k}$. For any initial condition $k_0 \geq 0$, a sequence of capital stocks $\mathbf{k} = (k_0, k_1, k_2, ...)$ is feasible from k_0 if $\gamma k_t \leq k_{t+1} \leq f(k_t)$ for all t. A sequence of consumption $\boldsymbol{c} = (c_0, c_1, c_2, ...)$ is feasible if there exists a $\mathbf{k}$ feasible from k_0 such that $0 \leq c_t \leq f(k_t) - k_{t+1}$ for all t. The preliminary results are summarized in the following proposition.

Proposition 2.1 *(i) For any* $k_0 \geq 0$, *there exists a unique optimal path* $\mathbf{k}$. *The associated optimal consumption path,* $\mathbf{c}$ *is given by* $c_t = f(k_t) - k_{t+1}, \forall t$. *(ii) If* $k_0 > 0$, *every solution* $(\mathbf{k}, \mathbf{c})$ *to the optimal growth model satisfies* $c_t > 0$, $k_t > 0, \forall t$.

[4]Compared to the multiplicative form, the separable form of the preferences is more consistent with the empirical findings on the behavior of the wealthy households since these preferences do not put any restrictions on either the substitutability or the complementarity between consumption and wealth habit (see Francis (2009) for details about the functional form of the utility function).

Proof. The existence of an optimal growth path follows from the facts that the set of feasible capital stock is compact for the product topology defined on the space of infinite sequences of real numbers and the objective criteria is continuous for this product topology. Uniqueness of the optimal path follows from the strict concavity of u, f and the concavity of w. From the increasingness of u in consumption, we have $c_t = f(k_t) - k_{t+1}$. The interiority folows from the Inada conditions (see Stokey and Lucas, 1989; Le Van and Dana, 2003). ■

Let V denote the value function, i.e.

$$V(k_0) = \max_{\{k_{t+1}\}_{t=0}^{\infty}} \left\{ \sum_{t=0}^{+\infty} \beta^t \left[u(f(k_t) - k_{t+1}) + \eta w(k_{t+1} - \gamma k_t) \right] \right| \quad \forall t, \gamma k_t \leq k_{t+1} \leq f(k_t),\ k_0 \geq 0,\ \text{given} \}.$$

Under Assumptions 1-3, the value function V is non-negative, continuous, strictly increasing, strictly concave, differentiable and satisfies the Bellman equation (e.g., Duran and Le Van, 2003):

$$V(k_0) = \max \left\{ u\left(f(k_0) - k\right) + \eta w(k - \gamma k_0) + \beta V(k) \mid \gamma k_0 \leq k \leq f(k_0) \right\}.$$

The optimal policy function $g : \mathbb{R}_+ \to \mathbb{R}_+$ is defined by:

$$g(k_0) = \arg\max_k \left\{ u\left(f(k_0) - k\right) + \eta w(k - \gamma k_0) + \beta V(k) \mid \gamma k_0 \leq k \leq f(k_0) \right\}.$$

We will now show that the Euler equation begins to hold after some finite period of time and that the optimal path is globally monotonic.

Lemma 2.1 *Let* $\mathbf{k}$ *be an optimal path from* $k_0 > 0$. *Then, there cannot be an integer* T *such that* $\gamma k_t = k_{t+1}$ *for all* $t \geq T$.

Proof. Let $\mathbf{k}$ be an optimal path from $k_0 > 0$. Assume that there exists such T. Since $k_t \to 0$, under the Assumption 2.3 there exists an integer $T' \geq T$ such that $\beta f'(k_{T'+1}) > 1$. The positivity of the optimal consumption implies that $k_{t+1} < f(k_t)$ for all t so that there exists $\varepsilon > 0$ small enough such that

$$\gamma k_t < (1+\varepsilon)k_{t+1} \leq f(k_t), \forall t \geq T'.$$

Define $\tilde{\mathbf{k}}$ as $\tilde{k}_t = k_t$ for $t = 1, ..., T'$ and $\tilde{k}_t = (1+\varepsilon)k_t$ for $t \geq T'+1$. It is feasible as we have:

$$\gamma\tilde{k}_t = \tilde{k}_{t+1} \text{ and } \tilde{k}_{t+1} = (1+\varepsilon)k_{t+1} \leq f(k_t) < f(\tilde{k}_t), \text{ for } t \geq T'+1.$$

Next, we show that $\tilde{\mathbf{k}}$ dominates $\mathbf{k}$ for some ε small enough. Define $\Delta(\varepsilon) = U(\tilde{\mathbf{k}}) - U(\mathbf{k})$. By setting $\hat{f}(k) = f(k) - \gamma k$, we have:

$$\begin{aligned}&\Delta(\varepsilon) = \beta^{T'}\left[u(\hat{f}(k_{T'}) - \gamma\varepsilon k_{T'}) - u(\hat{f}(k_{T'}))\right] + \beta^{T'}\eta w(\gamma\varepsilon k_{T'}) + \\ &\beta^{T+1'}\left[u(\tilde{f}((1+\varepsilon)k_{T'+1})) - u(\tilde{f}(k_{T'+1})\right] + \sum_{\tau > T'+1}^{+\infty}\beta^{\tau}\left[u(\tilde{f}((1+\varepsilon)k_{\tau})) - u(\tilde{f}(k_{\tau})\right]\end{aligned}$$

Since the second and the last terms are positive and u, f are concave and differentiable we get

$$\frac{\Delta(\varepsilon)}{\beta^{T'}} > [u(\hat{f}(k_{T'}) - \gamma\varepsilon k_{T'}) - u(\hat{f}(k_{T'}))] + \beta[u(\hat{f}((1+\varepsilon)k_{T'+1})) - u(\hat{f}(k_{T'+1})].$$

As $\varepsilon \to 0$, we have:

$$\frac{\Delta(\varepsilon)}{\beta^{T'}} > \gamma k_{T'}\left[-u'(\hat{f}(k_{T'})) + \beta u'(\hat{f}(k_{T'+1}))\hat{f}'(k_{T'+1})\right] > 0,$$

which contradicts with the optimality of $\mathbf{k}$. ■

Lemma 2.2 *Let* $\mathbf{k}$ *be an optimal path from* $k_0 > 0$*. Then, there exists an integer* T *such that* $\gamma k_t < k_{t+1}$ *for all* $t \geq T$*.*

Proof. Let $\mathbf{k}$ be an optimal path from $k_0 > 0$. Assume on the contrary that for any T there exists $T' \geq T$ such that $\gamma k_{T'-1} = k_{T'}$. Note that T' can be chosen so that $\gamma k_{T'} < k_{T'+1}$ and $\beta f'(k_{T'}) > 1$ by Lemma 2.1 and by the fact that $k_t \to 0$.

The positivity of the optimal consumption implies that $k_{T'} < f(k_{T'-1})$ for all t so that there is $\varepsilon > 0$ small enough to verify that

$$k_{T'} + \varepsilon < f(k_{T'-1}) \text{ and } \gamma(k_{T'} + \varepsilon) < k_{T'+1}.$$

Let $\tilde{\mathbf{k}}$ be a feasible sequence defines as

$$\tilde{k}_t = k_t \text{ for } t \neq T' \text{ and } \tilde{k}_{T'} = k_{T'} + \varepsilon.$$

Let us define $\Delta : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\Delta(\varepsilon) = u(f(k_{T'-1}) - k_{T'} - \varepsilon) + \eta w(\varepsilon) + \\ \beta u(f(k_{T'} + \varepsilon) - k_{T'+1}) + \beta\eta w(k_{T'+1} - \gamma(k_{T'} + \varepsilon)).$$

Differentiating $\Delta(\varepsilon)$ with respect to ε and evaluating at $\varepsilon = 0$, we obtain that

$$\Delta'(0) > -u'(f(k_{T'-1}) - k_{T'} - \varepsilon) + \beta u'(f(k_{T'} + \varepsilon) - k_{T'+1})f'(k_{T'}) \geq 0,$$

contradicting with the fact that $\Delta(\varepsilon)$ must have a maximum at $\varepsilon = 0$. ■

Proposition 2.2 *(i) Let* $\mathbf{k}$ *be the optimal path from* $k_0 > 0$*. Then, there exists an integer* T *such that* $\gamma k_t < k_{t+1} < f(k_t)$ *for all* $t \geq T$ *and we have the Euler equation:*

$$\forall t \geq T,\ u'\left(f(k_t) - k_{t+1}\right) - \eta w'(k_{t+1} - \gamma k_t) = \\ \beta u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_{t+1}) - \beta\eta\gamma w'(k_{t+2} - \gamma k_{t+1}).$$

(ii) Let $\tilde{k}_0 > k_0$ *and* $\tilde{\mathbf{k}}$ *be the optimal path from* $\tilde{k}_0$*. Then we have:* $\forall t,\ \tilde{k}_t \geq k_t$*.*

Proof. The proof follows from Duran and Le Van (2003).

i) Assume that $k_0 > 0$ and that $\mathbf{k}$ is optimal from k_0. Proposition 2.1-(ii) establishes $k_{t+1} < f(k_t)$ for all t. Lemma 2.2 ensures that there is some T with $\gamma k_t < k_{t+1}$ for all $t \geq T$. This implies that constraints are not binding from time T onwards, and hence the Euler equation begins to hold after T.

ii) Given $k_0' > k_0$, $k_1' \geq k_1$ follows from Benhabib & Nishimura (1985). If $k_1' = k_1$, then $k_t'^* = k_t^*$ for $t \geq 1$ as there is a unique optimal path associated to k_1. By using this argument for $t > 1$, we can conclude that $k_t' \geq k_t$, $\forall t$. ■

The monotonicity of the optimal path stems from the fact that the utility as a function of capital stocks or the reduced form utility function has the positive cross partial derivatives (see Benhabib and Nishimura, 1985). Recall that this condition is also necessary for the existence of the multiple steady states (see Benhabib et al., 1987). The presence of wealth effects in utility has already been shown to lead to multiplicity of steady states (see e.g. Kurz, 1968). In such models where $\gamma = 0$, the emergence of multiplicities solely depends on the relative weight of absolute wealth in utility. However, in our model, the existence of multiple steady states depends on the interplay between the relative weight of wealth effect in utility and the degree of wealth habit.

2.2 Dynamic Properties of Optimal Paths

As a monotone real valued sequence will either diverge to infinity or converge to some real number, the fact that the optimal capital sequences are monotone proves to be crucial in analyzing the dynamic properties and the long-run behavior of our model.

Lemma 2.3 *Let* $\mathbf{k}$ *be an optimal path from* $k_0 > 0$*. Then, there exists an integer* T *such that*

$$\eta w'(k_{t+1} - \gamma k_t) - \beta\eta\gamma w'(k_{t+2} - \gamma k_{t+1}) \geq 0, \ \forall t \geq T.$$

Proof. Let $\mathbf{k}$ be an optimal path from $k_0 > 0$. By Assumption 2.3, $k_t \in [0, A(k_0)]$ for all t and by Proposition 2.2-(ii), $\mathbf{k}$ is monotonic. Therefore, $\mathbf{k}$ must converge to some k^{ss}. Assume on the contrary that for any integer T there exists $\tau \geq T$ such that

$$\eta w'(k_{\tau+1} - \gamma k_\tau) - \beta\eta\gamma w'(k_{\tau+2} - \gamma k_{\tau+1}) < 0.$$

As $\tau \rightarrow +\infty$, we have

$$\eta \left(1 - \beta\gamma\right) w' \left(\left(1 - \gamma\right) k^{ss}\right) > 0,$$

which contradicts with the continuity of w'. ■

Proposition 2.3 *Assume* $f'(0) \geq \frac{1}{\beta}$*. Let* $\mathbf{k}$ *be an optimal path from* $k_0 > 0$*. Then,* $\mathbf{k}$ *can neither converge to* 0 *nor diverge to* $+\infty$*.*

Proof. Let $\mathbf{k}$ be an optimal path from $k_0 > 0$. Recall from Proposition 2.2 that the Euler equation implies for all $t \geq T$:

$$u'\left(f(k_t) - k_{t+1}\right) - \eta w'(k_{t+1} - \gamma k_t) =$$
$$\beta u'\left(f(k_{t+1}) - k_{t+2}\right) f'(k_{t+1}) - \beta\eta\gamma w'(k_{t+2} - \gamma k_{t+1}).$$

Assume first that $\mathbf{k}$ converges to 0. We have for all $t \geq T'$:

$$u'(f(k_t) - k_{t+1}) > \beta u'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \geq u'(f(k_{t+1}) - k_{t+2})$$

where the first inequality follows from Lemma 2.3 and the second from the existence of some $T' \geq T$ with $\beta f'(k_t) \geq 1$ for all $t \geq T'$. This implies that $c_t < c_{t+1}$ for all $t \geq T'$. However, as $k_t \to 0$, we have $c_t \to 0$. This is a contradiction.

Assume now that $\mathbf{k}$ diverges to $+\infty$. This violates the existence of a maximum sustainable capital stock. ■

A point k^s is an optimal steady state if $k^s = g(k^s)$, so that the stationary sequence $\mathbf{k}^s = (k^s, k^s, ..., k^s, ...)$ solves the problem $\mathcal{P}$. Recall that in a concave problem the stationary solution of the Euler equation is optimal since the transversality condition is satisfied. Let the set of stationary solutions to the Euler equation be defined as:

$$E = \left\{ k > 0 : f'(k) + \frac{\eta(1-\beta\gamma)}{\beta} \frac{w'((1-\gamma)k)}{u'(f(k)-k)} = \frac{1}{\beta} \right\}.$$

It is clear that whenever $f'(0) \geq \frac{1}{\beta}$, E will be a non-empty set.

We will now show that our model can support unique optimal steady state with global convergence and the multiplicity of optimal steady states with local convergence. Indeed, we will revisit the results that the presence of wealth effects in utility leads to the multiplicity of optimal steady states even under convex technology (see e.g. Kurz, 1968). However, our concern is to analyze how the critical capital stock behaves in response to the changes in the preference parameters and the discount factor which have been to a large extent left unexplored.

Case 2.1 *(Global Convergence) Assume $k_0 > 0$ and $f'(0) \geq \frac{1}{\beta}$. Consider the case where there exists a unique element k^s in E. By Proposition 2.3, we know that the* *optimal path cannot converge to* 0*. Then, any optimal path converges to the unique optimal steady state* k^s*, irrespective of their initial state.*

Recall that the model reduces to the classic optimal growth model ($\eta = 0$) with ($\gamma = 1 - \delta$) or without irreversible investment ($\gamma = 0$) in which $f'(0) > \frac{1}{\beta}$ is a necessary condition of the existence of a positive steady state implying global convergence (see Duran and Le Van, 2003). However, the presence of wealth effects in utility leads to the multiplicity of the optimal steady states even under convex technology.

Proposition 2.4 *Let* $\hat{k}$, $\tilde{k} \in E$ *be two consecutive steady states. Then, they cannot be both stable.*

Proof. Assume on the contrary that $\hat{k}$ and $\tilde{k}$ are both stable. Without loss of generality, let $\hat{k} < \tilde{k}$. Then, there exists $\hat{k}_0 \in \left(\hat{k}, \tilde{k}\right)$ such that $g(\hat{k}_0)$ converges to $\hat{k}$. Similarly, there exists $\tilde{k}_0 \in \left(\hat{k}, \tilde{k}\right)$ such that $g(\tilde{k}_0)$ converges to $\tilde{k}$. This implies the existence of a critical capital stock, $k_c \in \left(\hat{k}, \tilde{k}\right)$. k_c is then either a genuine critical point at which the optimal policy has a jump or an unstable stationary capital stock which also belongs to the set E. However, the former violates the fact that the optimal policy is continuous and the latter is in contradiction with that $\hat{k}$ and $\tilde{k}$ are two consecutive steady states. $\blacksquare$

Case 2.2 *(Local Convergence) Assume that* $k_0 > 0$ *and* $f'(0) \geq \frac{1}{\beta}$*. Let* $k_l = \min\{k : k \in E\}$ *and* $k_h = \max\{k : k \in E\}$ *denote the lowest and the highest steady states, respectively. Suppose that* k_h *is unstable from the right. Given the existence of a maximum sustainable capital stock, as any optimal path from* $k_0 > k_h$ *has to converge to an optimal steady state, there will be another steady state larger than* k_h*, a contradiction. Hence,* k_h *is stable from the right. It is also impossible to have* k_l *unstable from the left, since the optimal path can not converge to* 0*. Together with*

Proposition 2.4, these already imply the existence of a critical capital stock and the emergence of the threshold dynamics even under convex technology by means of the wealth habit.

It can easily be seen from Proposition 2.3 and the continuity of the optimal policy that the economy can constitute only an odd number of hyperbolic steady states. For the sake of simplicity, consider now that there exist three hyperbolic optimal steady states, $k_l < k_m < k_h$. Proposition 2.3 implies that k_l is stable from the left and k_h is stable from the right. These two results, together with the continuity of the optimal policy, naively suggest the existence of a critical capital stock and the emergence of the threshold dynamics. Moreover, the critical capital stock is equal to the unstable optimal steady state k_m so that the optimal path from $k_0 < k_m$ converges to k_l and the optimal path from $k_0 > k_m$ converges to k_h.

However, there can also be an even number of solutions to the stationary Euler equation which would imply the existence of a non-hyperbolic steady state. Even in such a case, threshold dynamics will emerge. For the sake of simplicity, suppose that there are exactly two optimal steady states, x_l and x_h. This can occur only when the optimal policy has a tangency to $45°$ line either at x_l or at x_h in which case the critical capital stock will be either x_l or x_h, respectively. It is then clear that any small perturbation in one of the parameters would cause a qualitative change in the dynamic properties of the optimal policy leading to a saddle-node bifurcation.

2.3 Wealth Habit and the Long-Run Dynamics

In this section, we analyze the qualitative implications of the wealth habit on the long run dynamics of the model. In order to provide a better exposition of our analysis, we will specify the functional forms and show that our model actually encounters a saddle-node bifurcation:

$$\begin{aligned} f(k) &= Ak^{\alpha} + (1-\delta)k, \\ u(c) &= \frac{c^{1-\sigma}}{1-\sigma}, \\ w(z) &= \frac{z^{1-\theta}}{1-\theta}, \end{aligned}$$

where $\{A, \sigma, \theta, \eta\} > 0$, $\alpha \in (0,1)$, $\delta \in (0,1]$, and $\gamma \in [0,1]$. Check that f, u, and w satisfy the assumption sets. The stationary Euler equation can then be recast as $\Psi(k) = \Phi$ where

$$\begin{aligned} \Psi(k) &= \left(Ak^{\alpha-1} - \delta\right)^{-\sigma} k^{\theta-\sigma}[1 - \beta + \beta\delta - \alpha\beta Ak^{\alpha-1}], \\ \Phi &= \eta(1-\beta\gamma)(1-\gamma)^{-\theta}. \end{aligned}$$

Under these functional forms, we first state the sufficient condition under which the unique steady state exists.

Proposition 2.5 *There exists a unique steady state if either* $\theta \geq \alpha\sigma$ *or* $\frac{\sigma\alpha-\theta}{(\sigma-\theta)\delta} \leq \frac{\alpha\beta}{1-\beta+\beta\delta}$.

Proof. To ensure the existence of a unique steady state, it is sufficient to show that $\Psi(k)$ is non-decreasing. We have

$$\Psi'(k) = \frac{k^{\theta}\left(1-\beta\left(A\alpha k^{\alpha-1}+1-\delta\right)\right)\left(\left(-\sigma\alpha+\theta\right)Ak^{\alpha-1}+\delta\left(\sigma-\theta\right)\right)}{\left(Ak^{\alpha}-\delta k\right)^{\sigma+1}}+ \frac{\left(1-\alpha\right)\beta\alpha Ak^{\alpha-1}\left(Ak^{\alpha-1}-\delta\right)}{\left(Ak^{\alpha}-\delta k\right)^{\sigma+1}}.$$

Note that $\Psi'(k) \geq 0$ whenever $f'(k) \leq 1$, i.e., $k \geq \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}$. Recall that any steady state satisfies $f'(k) < \frac{1}{\beta}$, i.e., $k > \left(\frac{\alpha\beta A}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}$. Let $k^* = \left(\frac{\alpha\beta A}{1-\beta+\beta\delta}\right)^{\frac{1}{1-\alpha}}$ and $k^{**} = \left(\frac{\alpha A}{\delta}\right)^{\frac{1}{1-\alpha}}$. It is then sufficient to show that $\Psi'(k) \geq 0$ when $k \in (k^*, k^{**})$.

Suppose on the contrary that there exists $k \in (k^*, k^{**})$ such that $\Psi'(k) < 0$ so that $(-\alpha\sigma+\theta)Ak^{\alpha-1}+(\sigma-\theta)\delta < 0$.

If $\theta < \alpha\sigma$ then $k < \left(\frac{(\alpha\sigma-\theta)A}{(\sigma-\theta)\delta}\right)^{\frac{1}{1-\alpha}}$ so that $k < k^{**}$ is trivially satisfied. However, $k > k^*$ requires that $\frac{\alpha\sigma-\theta}{(\sigma-\theta)\delta} > \frac{\alpha\beta}{1-\beta+\beta\delta}$, a contradiction.

If $\sigma > \theta \geq \alpha\sigma$ then $k < \left(\frac{(\alpha\sigma-\theta)A}{(\sigma-\theta)\delta}\right)^{\frac{1}{1-\alpha}} \leq 0 < k^*$ contradicts with $k > k^*$.

If $\sigma < \theta$ then $k > \left(\frac{(\alpha\sigma-\theta)A}{(\sigma-\theta)\delta}\right)^{\frac{1}{1-\alpha}}$. To have $k < k^{**}$, $\frac{\alpha\sigma-\theta}{\sigma-\theta} < \alpha$ has to hold, contradicting with $\alpha < 1$. ■

We now show that there can be at most three steady states.

Proposition 2.6 *There can be at most three solutions to the stationary Euler equation.*

Proof. To find the number of solutions to the stationary Euler equation, we prove that there can be at most two local extremum values of $\Psi(k)$. $\Psi'(k) = 0$ implies that

$$\rho_1 k^{2(\alpha-1)} + \rho_2 k^{\alpha-1} + \rho_3 = 0.$$

We have

$$\rho_1 = \beta A^2 \alpha \left(1 - \theta + \alpha \left(\sigma - 1\right)\right),$$

$$\rho_2 = A(1 - \beta(1-\delta)) \left(\theta - \sigma\alpha\right) + \beta A \delta \alpha \left(\theta - \sigma + \alpha - 1\right),$$

$$\rho_3 = \delta \left(\sigma - \theta\right) \left(1 - \beta(1-\delta)\right).$$

This implies that $\Psi(k)$ can have at most two local extremum values, one of which is local maximum and the other is local minimum. As $\Psi(0) < \Phi$ and $\Psi(\bar{k}) > \Phi$, $\Psi(k)$ and Φ can intersect at most three points. ■

The following proposition shows that when there exists a non-hyperbolic steady state there can only be two steady states.

Proposition 2.7 *Let* $\mathbf{k}$ *be an optimal path from* $k_0 > 0$*. Assume* $f'(0) \geq \frac{1}{\beta}$*. If there exist a non-hyperbolic steady state then there can exactly be two steady states.*

Proof. The Jacobian matrix is given by:

$$J = \begin{bmatrix} \Omega & -1 \\ J_1 & J_2 \end{bmatrix}$$

where

$$J_1 = \frac{\beta A\alpha(1-\alpha)k^{\alpha-2}\Delta^{-\sigma}\Omega + \eta\theta((1-\gamma)k)^{-1-\theta}(1-\beta\gamma\Omega)(\Omega-\gamma)}{-\beta\eta\gamma\theta(1-\gamma)^{-1-\theta}k^{-1-\theta} - \beta\sigma\Delta^{-1-\sigma}\Omega},$$

$$J_2 = \frac{-\Delta^{-1-\sigma}(\beta A\alpha(1-\alpha)k^{\alpha-2}\Delta + \sigma) - \eta\theta((1-\gamma)k)^{-1-\theta}(1-\beta\gamma(\Omega-\gamma))}{-\beta\eta\gamma\theta(1-\gamma)^{-1-\theta}k^{-1-\theta} - \beta\sigma\Delta^{-1-\sigma}\Omega},$$

with $\Delta = (Ak^{\alpha} - \delta k)$, and $\Omega = (A\alpha k^{\alpha-1} + 1 - \delta)$.

Let λ_1 and λ_2 denote the eigenvalues of J. Note that $\lambda_1 + \lambda_2 = \Omega + J_2 > 0$ and $\lambda_1\lambda_2 = \frac{1}{\beta} > 0$. At a non-hyperbolic steady state, we have $\lambda_1 + \lambda_2 = 1 + \frac{1}{\beta}$ which reveals that $\Psi'(k) = 0$. To see that a non-hyperbolic steady state is a local extremum point of $\Psi(k)$, note the following: If $\Psi''(k) = 0$ and $\Psi'(k) = 0$ as $\Psi(0) < \Phi$ and $\Psi(\bar{k}) > \Phi$, $\Psi(k)$ and Φ intersect only once. But we know that if there exists a unique steady state, it is a hyperbolic one so that $\Psi''(k) \neq 0$. Moreover, as $\Psi(k)$ and Φ intersect at a local extremum of $\Psi(k)$, there can exactly be two steady states. ■

2.4 Monotone Comparative Statics and the Continuity of the Critical Capital Stock

In this section, we analyze the effects of β, η and γ on the steady states and the critical capital stock. Note that the optimal policy is implicitly dependent on β, η and γ. Hence, in order to facilitate our discussion below, we first make this $\beta-$, $\eta-$ and $\gamma-$ dependence explicit in the notation by writing $g(k, \beta, \eta, \gamma)$, $k_l(\beta, \eta, \gamma)$, $k_c(\beta, \eta, \gamma)$ and $k_h(\beta, \eta, \gamma)$.

Proposition 2.8 *(i) Let* $\beta_0 = \frac{1}{\lim_{k \to 0} f'(k)}$. *The optimal policy,* $g(k, \beta, \eta, \gamma)$*, is strictly increasing in* $\beta \in [\beta_0, 1)$.
(ii) Let k^s *be a hyperbolic steady state. Then* k^s *is saddle path stable if and only if it is increasing in* η.
(iii) Let k^s *be a hyperbolic steady state. When* $\gamma > \frac{\beta - \theta}{\beta - \theta\beta}$ $\left(\gamma < \frac{\beta - \theta}{\beta - \theta\beta}\right)$, k^s *is saddle path stable if and only if it is increasing (decreasing) in* γ.
(iv) The optimal policy $g(k, \beta, \eta, \gamma)$ *is continuous in* β, η *and* γ.

Proof. (i) Follows from Amir et al. (1991; Theorem 5.5.d)

(ii) From the linearization around the steady state, we have $\lambda_1 + \lambda_2 > 0$ and $\lambda_1 \lambda_2 = \frac{1}{\beta}$. By setting $\lambda_1 = \frac{1}{\beta \lambda_2}$, it can be seen that $\frac{1}{\beta \lambda_2} + \lambda_2$ is increasing in λ_2 when $\lambda_2 \geq \frac{1}{\beta}$. If k^s is saddle path stable then $\lambda_2 > \frac{1}{\beta} > \frac{1}{\sqrt{\beta}}$ and hence $\lambda_1 + \lambda_2 > 1 + \frac{1}{\beta}$ so that

$\Psi'(k) > 0$. Since Φ is increasing in η, a saddle path stable steady state is increasing in η. Conversely, if a hyperbolic steady state is unstable then it is decreasing in η. First, note in such a case that both $\lambda_1 > 1$ and $\lambda_2 > 1$. Since $\lambda_1\lambda_2 = \frac{1}{\beta}$, we must have $\lambda_2 < \frac{1}{\beta}$. By solving the following problem:

$$\max_{1<\lambda_2<\frac{1}{\beta}} \frac{1}{\beta\lambda_2} + \lambda_2,$$

one can show that $\lambda_1 + \lambda_2 < 1 + \frac{1}{\beta}$ which implies $\Psi'(k) < 0$ so that the hyperbolic steady state which is unstable is decreasing in η.

(iii) Φ is increasing (decreasing) in γ when $\gamma \geq \frac{\beta-\theta}{\beta-\theta\beta}$ $\left(\gamma < \frac{\beta-\theta}{\beta-\theta\beta}\right)$. The proof is the same as of (ii).

(iv) See Le Van and Dana (2003), page 34-35. ■

Note that the effect of an increase in γ on the marginal utility of the future capital stock k_{t+1} has two components. The first component is positive since an increase in the subsistence wealth level γk_t triggers the desire to increase the future capital stock:

$$\frac{\partial^2 \left(u(f(k_t) - k_{t+1}) + \eta w(k_{t+1} - \gamma k_t)\right)}{\partial\gamma\partial k_{t+1}} = -\eta w''(k_{t+1} - \gamma k_t)k_t > 0. \tag{1}$$

The second component is negative since it will increase the effect of k_{t+1} on the next period's subsistence level, γk_{t+1}, and make it harder to obtain further utility from the change in k_{t+2} with respect to k_{t+1}:

$$\frac{\partial^2 \left(\beta u(f(k_{t+1}) - k_{t+2}) + \beta\eta w(k_{t+2} - \gamma k_{t+1})\right)}{\partial\gamma\partial k_{t+1}} = \\ -\beta\eta w'(k_{t+2} - \gamma k_{t+1}) + \beta\eta\gamma w''(k_{t+2} - \gamma k_{t+1})k_{t+1} < 0. \tag{2}$$

As long as w is linear, the total effect of the degree of the wealth habit γ on the choice of the future capital stock becomes negative, hence the optimal policy is decreasing in γ. When w is strictly concave, the net effect of a change in γ on the behavior of the optimal policy is ambiguous. However, the behavior of the steady states in response to a change in γ can still be determined. Letting $k_t = k_{t+1} = k_{t+2} = k$, the net change in the marginal utility of k_{t+1} with respect to γ turns out to be $\frac{\eta w'(z)}{1-\gamma}\left[\theta - \beta + \gamma(\beta - \theta\beta)\right]$. Note that if $\gamma \geq \frac{\beta-\theta}{\beta-\theta\beta}$, the marginal utility of k_{t+1} at a stable steady state will increase with γ.

A similar analysis can be undertaken to see the effect of η on capital accumulation in the long-run. As we show in Proposition 2.8-(ii), the values of the stable steady states increase with η.

Corollary 2.1 *Let* $\beta_0 = \frac{1}{\lim\limits_{k\to 0} f'(k)}$. *Then,* $k_l(\beta,\eta,\gamma)$ *and* $k_h(\beta,\eta,\gamma)$ *are strictly increasing,* $k_c(\beta,\eta,\gamma)$ *is strictly decreasing in* $\beta \in [\beta_0, 1)$ *and* $\eta > 0$. *When* $\gamma > \frac{\beta-\theta}{\beta-\theta\beta}$ $\left(\gamma < \frac{\beta-\theta}{\beta-\theta\beta}\right)$, $k_c(\beta,\eta,\gamma)$ *is strictly decreasing (increasing) in* γ, $k_l(\beta,\eta,\gamma)$ *and* $k_h(\beta,\eta,\gamma)$ *are strictly increasing (decreasing). All of them are continuous in* $\beta \in [\beta_0, 1)$, $\gamma \in (0,1)$ *and* $\eta > 0$.

Proof. The proof follows from Proposition 2.8. ■

Akao, et al. (2011) analyzes the monotonicity and the continuity of the critical capital stock in the discount factor in a Dechert and Nishimura (1983) framework. Dechert and Nishimura (1983) concentrates on the effects of non-convex technology on the long-run growth paths. In this essay, our focus is rather on the preference component and we analytically provide the monotone comparative statics and the continuity of the critical capital stock with respect to the discount factor, the relative weight of wealth habit in utility and the degree of wealth habit. The behavior of the critical capital stock with respect to these parameters is important in explaining the persistent differences in per capita capital stocks among countries since minor differences in the relative weight of wealth habit in utility and/or the degree of wealth habit lead to permanent differences in the optimal path.

In the next section, we provide a numerical example in order to illustrate the results presented above.

2.5 Numerical Analysis

We will numerically illustrate an example showing how a small perturbation in η or γ changes the long run dynamics. In particular, we demonstrate the emergence of a saddle-node bifurcation with respect to these parameters. We consider the following set of fairly standard parameterization:

$$A = 1, \alpha = 0.4, \delta = 0.01, \sigma = 0.95, \theta = 0.01, \beta = 0.94.$$

Setting $\gamma = 0.75$, we have multiplicity of steady states when η is between 0.0327486 and 0.0329705 which are the critical values for the emergence of a saddle-node bifurcation (see Figure 1). As long as $\eta < 0.0327486$, i.e., before the bifurcation occurs, there is only one steady state, k_l, which is globally stable. As η increases, the steady state capital stock increases too. For $\eta = 0.0327486$, an additional steady state appears in addition to k_l and the dynamics are now characterized by two steady states, $k_l < k_m$ where k_l is locally stable and k_m is unstable in the sense that it is stable from right but unstable from left. The optimal policy is tangent to $45°$ line at k_m and the corresponding eigenvalue at k_m equals to unity indicating the possible emergence of a saddle-node bifurcation. When η slightly increases from its critical value 0.0327486, the unstable steady state splits into one locally stable and one unstable steady state through a saddle-node bifurcation resulting in three steady states, k_l (stable) $< k_m$

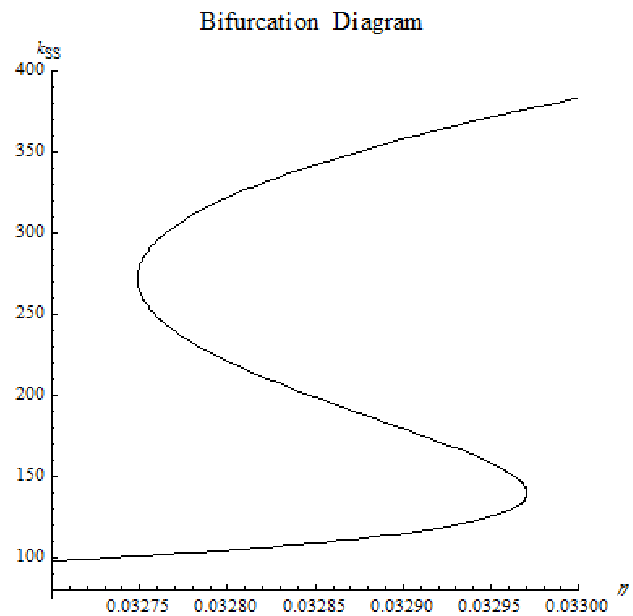

Figure 1: **Bifurcation analysis for variations in η for a fixed value of γ.**

(unstable) $< k_h$ (stable). The coexistence of these three steady states is preserved until $\eta = 0.0329705$.

As the value of η gets closer to the critical value 0.0329705, the lowest stable steady state and the unstable steady state approach one another and at the critical value they merge into a non-hyperbolic steady state through the reverse saddle node bifurcation. Slightly above this critical value of the saddle-node bifurcation, the non-hyperbolic steady state ceases to exist leaving only the stable steady state k_h which is now globally stable. Further increases in η only affects the value of the stable steady state.

In sum, two types of the saddle-node bifurcations emerge. The difference lies in the following: In the first one, the saddle-node bifurcation is realized for the pair of steady states k_m and k_h and in the second, it is for the pair of steady states k_l and k_m. Moreover, in the first one, as η decreases, coalescence of the steady states into

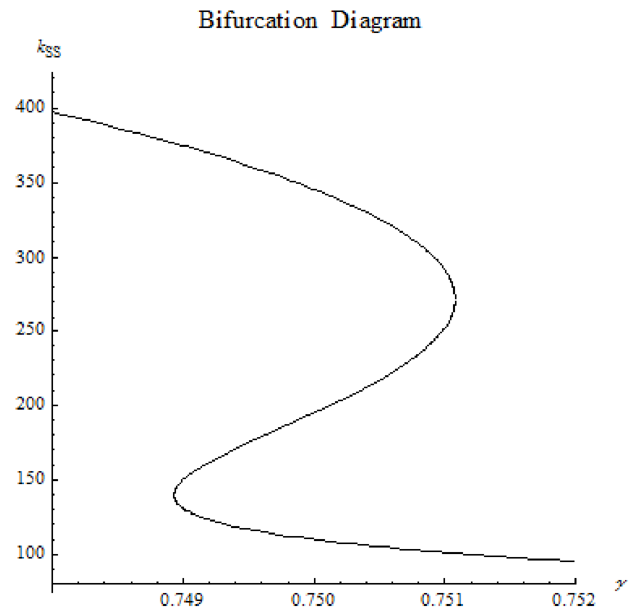

Figure 2: **Bifurcation analysis for variations in γ for a fixed value of η.**

a non-hyperbolic steady state is observed and in the second, the qualitative change is in the form of creating a pair of stable and unstable steady states simultaneously from a non-hyperbolic steady state.

A similar analysis can be done for the variations in γ for a fixed value of η. The resulting bifurcation diagram is given in Figure 2. Note the reverse effect of γ on the values of the steady states compared to the effect of η: An increase in γ decreases the level of the steady state while an increase in η increases it.

Figure 3 plots the steady states against the values of both the relative weight of wealth in utility, η, and the degree of wealth habit, γ. Note that the critical values of η specifying the region of the multiplicity shifts to the right as γ increases. This implies that the economies even with the same η and k_0 can have different per capita capital stocks in the long-run depending on their degree of wealth habit.

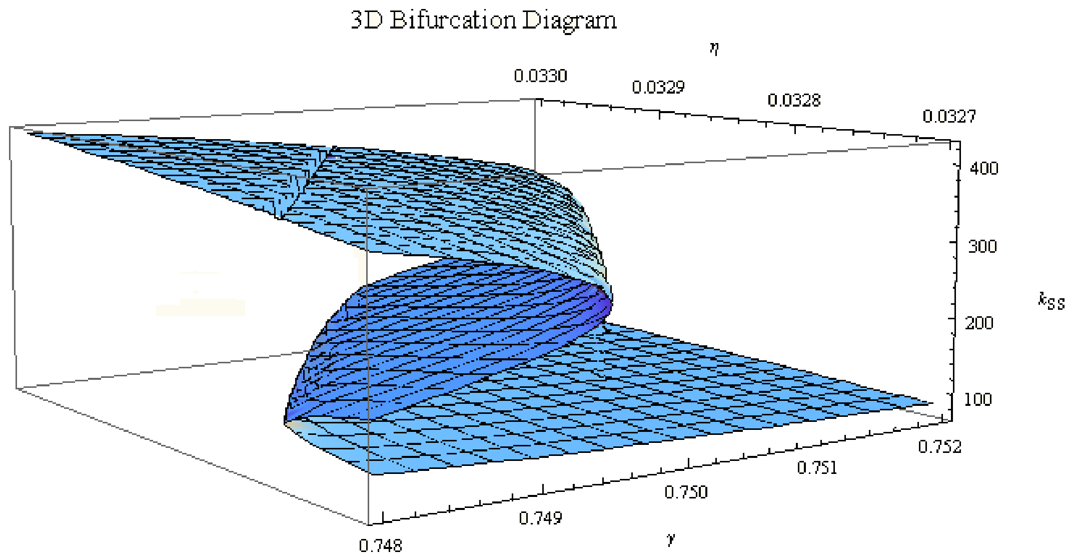

Figure 3: **Bifurcation analysis for variations in η and γ.**

CHAPTER 3

TIME-TO-BUILD AND CYCLICAL DYNAMICS UNDER NON-CONVEXITIES

The existence of a critical capital stock due to non-convex technologies leads to the long-run dynamics consistent with important economic phenomena such as history dependence and convergence clubs (see Quah 1996; Azariadis, 1996; Azariadis and Stachurski, 2005). Indeed, in one-sector optimal growth models with a convex-concave production function, it is shown that the optimal path from an initial capital stock below the critical stock converge to zero, while the optimal path from an initial capital stock above the critical stock converge to a positive steady state suggesting that the initial differences persist in the long-run (see Dechert and Nishimura, 1983; its extensions for the existence of the critical capital stock and Akao et. al. 2011 for the qualitative properties of the critical capital stock).

The important feature in these models is that monotonic behaviour of the optimal path over time continues to hold as the reduced form utility function has still positive cross partial derivatives (e.g. Benhabib and Nishimura, 1985). Therefore, it is suggested that the possibility of cyclical dynamics requires at least two production sectors (see Dechert, 1984; Nishimura and Sorger, 1996).

Dynamic implications of non-convex technologies on optimal growth have extensively been analyzed in the literature under various specifications such as irreversible investment (Majumdar and Nermuth, 1982; Kamihigashi and Roy, 2006, 2007), linear utility function (Majumdar and Mitra, 1983; Kamihigashi and Roy, 2006), nonsmooth and discontinuous production function (Mitra and Ray, 1984; Kamihigashi and Roy, 2006, 2007), wealth-dependent preferences (Majumdar and Mitra, 1984; Olson and Roy, 1996) and under stochastic set-up (Majumdar, Mitra and Nyarko, 1989; Nishimura and Stachurski, 2004).

However, the time dimension of capital, often referred to as time-to-build after Kydland and Prescott (1983), has been ignored to a large extent in non-classical optimal growth models that induce threshold dynamics. This ignorance does not stem from either the empirical irrelevance or the theoretical unfoundedness of time-to-build. In fact, it has been recognized that oscillations are relevant in the classical optimal growth models when time-to-build is taken into account (Collard & Licandro & Puch, 2003; Ferrara et al. 2013; Bambi and Gori, 2013). The major implication of these models is that dynamics are characterized by oscillations along the transition path but the convergence to a steady state is still preserved in the long-run.

In this essay, we extend the familiar Dechert and Nishimura (1983) framework by introducing an investment lag. The cyclical dynamics are shown to persist in the long-run even in a one-sector optimal growth framework when non-convex technologies and time-to-build are taken into account simultaneously. In particular, even though models with non-convex technologies give rise to monotone dynamics, the mechanism that generates such monotone dynamics can also be the driving force for cyclical dynamics if one considers the time-to-build lag in investment.

We start with a benchmark model of one-sector optimal growth with nonconvex technologies and a two-period investment lag by assuming full depreciation of capital stock. The optimal path belonging to the benchmark model can be separated into two independent sub-sequences. Although each of these sub-sequences is monotone, the optimal path of overall stock of capital need not exhibit monotonic behaviour. The optimal path converges to a two-period limit cycle alternating between zero and a positive steady state depending on the value of the critical capital stock. Using the continuity argument on the depreciation rate, we conclude that the cyclical characteristic of the optimal path remain valid to some degree when the depreciation rate is close to one. In particular, we show the existence of a lower bound for the depreciation rate above which cyclical fluctuations of per capita capital stock continue to occur. While a complete analytical solution is possible under full depreciation, we resort to numerical methods to show that our results under full depreciation are also relevant to some extent in the case that the depreciation rate is less than one.

When non-convex technologies are considered under time-to-build, they together give rise to the existence of cyclical dynamics. One implication of these dynamics is that some economies fail to converge either to zero or a positive steady state depending on the initial conditions. This theoretical result in a qualitative sense can be associated with the empirically observed phenomenon of middle income trap in which a middle-income country fails to catch up with high-income countries and its GDP per capita fluctuates around a fixed point instead of steadily growing over time (Kharas and Kohli, 2011; Felipe et. al., 2012; Aiyar et. al., 2013; Gabriel Im and Rosenblatt, 2013). Actually, such a result may arise in any model that induce threshold dynamics if time-to-build is taken into account.

The rest of the paper is organized as follows. In section 2, we present an elementary result which is crucial in the proof of our main result. In Section 3, we set-up the model and provide the long-run dynamics of the optimal path under full depreciaiton. In section 4, we state our main result and we prove it in Section 5.

3.1 The Model

The model differs from the familiar Dechert and Nishimura (1983) model by the introduction of time-to-build lag. Considering the fact that investment takes time, we formalize the model as follows:

$$\max_{\{c_t, i_t, k_{t+2}\}_{t-0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t) \qquad (E2-P)$$

subject to

$$\begin{aligned} &\forall t \in \mathbb{Z}_+, c_t + i_t = f(k_t), \\ &c_t, i_t \geq 0, \\ &k_{t+2} = i_t + (1-\delta) k_{t+1}, \\ &k_0, k_1 \geq 0, \text{ given} \end{aligned}$$

where c_t is the consumption, i_t is the investment and k_t is the capital stock in period t. Investment made in any period t requires two periods to form the capital stock ready for use in production. Accordingly, the capital stock at time $t+2$ is composed of the investment made in period t and the undepreciated capital stock from period $t+1$ so that

$$k_{t+2} = i_t + (1-\delta) k_{t+1}$$

where $\delta \in (0,1)$ denotes the depreciation rate of the capital stock. We also let the investment in any period t satisfy $i_t \geq 0$.

We make the following assumptions regarding the properties of the utility function u and the production function f. $\beta \in (0,1)$ is the discount factor.

Assumption 3.1 *$u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $(0, +\infty)$, continuous, strictly increasing, strictly concave, and satisfies $u(0) = 0$ and $\lim_{c \to 0} u'(c) = +\infty$.*

Assumption 3.2 *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $(0, +\infty)$, continuous, strictly increasing, and satisfies $f(0) = 0$.*

Assumption 3.3 *There exists a unique $\bar{k} > 0$ such that $f(\bar{k}) + (1-\delta)\bar{k} = \bar{k}$ and $f(k) + (1-\delta)k < k$ for all $k > \bar{k}$.*

Assumption 3.4 *There exists $k^I > 0$ such that f is strictly convex on $[0, k^I]$ and strictly concave on $[k^I, +\infty)$.*

For any initial conditions $(k_0, k_1) \geq 0$, a sequence of capital stocks $\boldsymbol{k} = (k_0, k_1, ...)$ is feasible from (k_0, k_1) if $(1-\delta)k_{t+1} \leq k_{t+2} \leq f(k_t) + (1-\delta)k_{t+1}$ for all t. The set of feasible sequences from (k_0, k_1) is denoted by $\Pi(k_0, k_1)$. A sequence of consumption $\boldsymbol{c} = (c_0, c_1, c_2, ...)$ is feasible from (k_0, k_1) if there exists $\boldsymbol{k} \in \Pi(k_0, k_1)$ such that $0 \leq c_t \leq f(k_t) + (1-\delta)k_{t+1} - k_{t+2}$ for all t.

Assumption 3.3 implies the existence of a maximum sustainable capital stock $A(k_0, k_1) = \max\{k_0, k_1, \bar{k}\}$. Note that if $k_0, k_1 \in [0, \bar{k}]$, then $k_t \in [0, \bar{k}]$ for all $t \in \mathbb{Z}_+$. For the rest of the essay, we restrict attention to the case in which all feasible paths stay in $[0, \bar{k}]$:

Assumption 3.5 *$k_0, k_1 \in [0, \bar{k}]$.*

If $\delta = 0$, then capital never depreciates; we rule out this case by assuming that there exists a positive lower bound on the depreciation rate. In particular, letting $\underline{\delta} \in (0, 1)$, we assume the following for the rest of the essay.

Assumption 3.6 $\delta \in [\underline{\delta}, 1]$.

It is clear that when there is no time-to-build lag, the model is the standard *non-classical* optimal growth model (see Dechert and Nishimura, 1983). As our aim is to analyze the qualitative implications of time-to-build lag on the long-run equilibrium dynamics of an optimal growth model with non-convex technology, in particular, we introduce a two-period time-to-build lag.

3.2 Dynamic Properties of Optimal Paths under Full Depreciation

We will first consider the model under full depreciation and later we will extend our analysis for the partial depreciation case. In particular, we show that the optimal path can exhibit an oscillating behaviour and we prove the existence of a two-period limit cycle.

Let $\delta = 1$. Any feasible path, $\boldsymbol{k}$, satisfies:

$$\begin{aligned} \forall t &\in \mathbb{Z}_+, c_{2t} + k_{2t+2} \leq f\left(k_{2t}\right), \\ \forall t &\in \mathbb{Z}_+, c_{2t+1} + k_{2t+3} \leq f\left(k_{2t+1}\right). \end{aligned}$$

It can be observed that $\boldsymbol{k}$ is comprised of two alternating sub-sequences defined by ${}_0\boldsymbol{k} = \{k_{2t}\}_{t=0}^{+\infty}$ and ${}_1\boldsymbol{k} = \{k_{2t+1}\}_{t=0}^{+\infty}$ from k_0 and k_1, respectively. It should be noted that the optimal choices of these two sub-sequences depend only on their initial conditions. Hence, these sub-sequences are pairwise independent. This implies that the original model inherently includes two submodels.

We define ${}_{\tau}k_t = k_{2t+\tau}$ for any $\tau \in \{0,1\}$. Considering the augmented discount rate $\rho = \beta^2 \in (0,1)$, the submodels for $\tau \in \{0,1\}$ can be recast as follows:

$$W\left({}_{\tau}k_0\right) = \max_{\{{}_{\tau}c_t, {}_{\tau}k_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \rho^t u\left({}_{\tau}c_t\right) \qquad (E2 - SubModel)$$

subject to

$$\forall t \in \mathbb{Z}_+, {}_{\tau}c_t + {}_{\tau}k_{t+1} \leq f\left({}_{\tau}k_t\right),$$

$${}_{\tau}c_t \geq 0, {}_{\tau}k_t \geq 0,$$

$${}_{\tau}k_0 \geq 0, \text{ given,}$$

It is straightforward that $W : \mathbb{R}_+ \to \mathbb{R}_+$, is non-negative, strictly increasing, continuous, and W verifies the Bellman equation:

$$\forall \tau \in \{0,1\}, W\left({}_{\tau}k_0\right) = \max\left\{u\left(f\left({}_{\tau}k_0\right) - {}_{\tau}k_1\right) + \rho W\left({}_{\tau}k_1\right) \mid 0 \leq {}_{\tau}k_1 \leq f\left({}_{\tau}k_0\right)\right\}.$$

The solution to this Bellman equation, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-empty, upper semi-continuous and strictly increasing.

In accordance with these, let $\boldsymbol{k} \in \Pi\left(k_0, k_1\right)$ be a solution to $E2-P$. Note that, for any $\tau \in \{0,1\}$, ${}_{\tau}\mathbf{k}$ solves the corresponding submodel $E2 - SubModel$ and satisfies ${}_{\tau}k_{t+1} = \mu\left({}_{\tau}k_t\right)$, for all $t \in \mathbb{Z}_+$.

Note that the existence of a critical capital stock is important for the existence of a limit cycle in this model. Hence, we simply focus on the c̈ase in which the critical capital stock exists. For the rest of the essay, we maintain the following assumption.

Assumption 3.7 $f'(0) < \frac{1}{\beta^2} < \max\left\{\frac{f(k)}{k}\right\}$.

This assumption corresponds to the intermediate discounting case in Dechert and Nishimura (1983).

In the following proposition, we state the existence of a critical capital stock.

Proposition 3.1 *Assume that $\delta = 1$. Let $\boldsymbol{k} \in \Pi(k_0, k_1)$ be an optimal path. There exists a critical capital stock $k_c > 0$ such that, for any $\tau \in \{0, 1\}$, if ${}_\tau k_0 \in (k_c, +\infty)$, ${}_\tau\boldsymbol{k}$ converges to a strictly positive steady state k^* and if ${}_\tau k_0 \in (0, k_c)$, ${}_\tau\boldsymbol{k}$ converges to 0.*

Proof. See Dechert and Nishimura (1983). ■

Under Assumption 3.7, there exist two strictly positive steady states, say $k_* < k^*$. It is clear from Dechert and Nishimura (1983) that k^* is a locally stable optimal steady state.

When it comes to k_*, the optimal path does not exhibit a convergent behaviour toward k_* even in the case that k_* is an optimal steady state. Then, the monotonicity of the sub-sequences, ${}_\tau\mathbf{k}$ for any $\tau \in \{0, 1\}$, implies that if k_* is an optimal steady state then it is the critical capital stock, i.e. $k_c = k_*$ and the optimal policy is continuous at k_c.

On the other hand, when $k_c \neq k_*$, the critical capital stock, k_c, is merely a point at which the optimal policy has a jump. This follows from the existence of an optimal path converging to 0 and of another one converging to k^*. Notice that k_* is not an optimal steady state[5].

In addition, the optimal policy has a jump at k_c when "$k_c = k_*$ but k_* is not an optimal steady state".

It is important to note that discontinuities in the optimal policy appearing at the critical capital stock leads to indeterminacy. As we focus on the existence of a limit cycle and indeterminacy does not affect our results, we do not analyze indeterminacy results here.

Proposition 3.2 *Assume that $\delta = 1$. Let $\boldsymbol{k} \in \Pi(k_0, k_1)$ be an optimal path. For any $\tau \in \{0, 1\}$,*

(i) If ${}_\tau k_0 \neq 0$ or ${}_\tau k_0 \neq k^*$ then ${}_\tau\, \boldsymbol{k} = \{k_{2t+\tau}\}_{t=0}^{+\infty}$ is monotone in the sense that either ${}_\tau k_{t+1} < {}_\tau k_t$ or ${}_\tau k_{t+1} > {}_\tau k_t$ for all $t \in \mathbb{Z}_+$.

(ii) If ${}_\tau k_0 \in (0, k_c)$ then ${}_\tau\, \boldsymbol{k} = \{k_{2t+\tau}\}_{t=0}^{+\infty}$ monotonically converges to 0.

(iii) If ${}_\tau k_0 \in \left(k_c, \bar{k}\right)$ then ${}_\tau\, \boldsymbol{k} = \{k_{2t+\tau}\}_{t=0}^{+\infty}$ monotonically converges to k^*.

(iiii) If ${}_\tau k_0 = k^*$ then ${}_\tau\, \boldsymbol{k} = \{k_{2t+\tau}\}_{t=0}^{+\infty} = k^*$ for all $t \in \mathbb{Z}_+$.

Proof. See Dechert and Nishimura (1983). ■

Note that Proposition 3.2 constructs the monotonicity of the sub-sequences ${}_\tau \boldsymbol{k}$ for any $\tau \in \{0, 1\}$ which is due to the increasingness of the optimal policy μ. Hence, for any $\tau \in \{0, 1\}$, ${}_\tau \boldsymbol{k}$ forms an optimal path monotonically converging to either 0 or k^* depending on the position of the initial condition, ${}_\tau k_0$, relative to the critical capital stock, k_c. However, it is important to notice here that the monotonic convergence of

[5]The detailed discussion and formal proofs of these arguments can be found in Dechert and Nishimura (1983).

${}_{\tau}\boldsymbol{k}$ for any $\tau \in \{0,1\}$ does not always imply the monotonicity of the overall capital stock, $\boldsymbol{k}$ even if these two sub-sequences converge to the same point. This is the concern of the following proposition.

Proposition 3.3 *Assume that $\delta = 1$. Let $\boldsymbol{k} \in \Pi(k_0, k_1)$ be an optimal path. Denote the maximum and minimum selections of μ by $\overline{\mu}$ and $\underline{\mu}$, respectively. If either $\left(k_0 > k_1 \text{ and } \underline{\mu}(k_0) > k_1\right)$ or $(k_0 < k_1 \text{ and } \overline{\mu}(k_0) < k_1)$, then $\boldsymbol{k}$ follows a non-monotonic path. Under these cases; if $\min\{k_0, k_1\} > k_c$ then the optimal path oscillates toward k^* and if $\max\{k_0, k_1\} < k_c$ then the optimal path oscillates toward 0.*

Proof. For the first part of the Proposition, assume without loss of generality that $k_0 > k_1$ and $\underline{\mu}(k_0) > k_1$. We have $k_0 > k_1 < k_2$. This implies by the increasingness of μ that, for any $k_2 \in \mu(k_0)$ and any $k_3 \in \mu(k_1)$, we have $k_2 > k_3$ and, similarly, we have $k_3 < k_4$. Hence, $k_{2t} > k_{2t+1} < k_{2t+2}$ for all $t \in \mathbb{Z}_+$.

In case that $k_0 < k_1$ and $\overline{\mu}(k_0) < k_1$, similar arguments apply and we obtain that $k_{2t} < k_{2t+1} > k_{2t+2}$ for all $t \in \mathbb{Z}_+$. Hence, $\boldsymbol{k}$ follows a non-monotonic path.

As for the second part of the Proposition, if $\min\{k_0, k_1\} > k_c$ then $\{k_{2t+1}\}_{t=0}^{+\infty} \to k^*$ and $\{k_{2t+1}\}_{t=0}^{+\infty} \to k^*$. From the first part of the Proposition, we conclude that $\boldsymbol{k}$ oscillates toward k^*. Similarly, we prove that $\boldsymbol{k}$ oscillates toward 0 when $\max\{k_0, k_1\} < k_c$.

■

Proposition 3.4 *Assume that $\delta = 1$. Let $\boldsymbol{k} \in \Pi(k_0, k_1)$ be an optimal path. $\boldsymbol{k}$ oscillates toward a two-period limit cycle alternating between 0 and k^* whenever $\max\{k_0, k_1\} > k_c > \min\{k_0, k_1\}$.*

Proof. Consider without loss of generality that $k_0 > k_c > k_1$. Since $k_0 > k_c$, Proposition 3.1 implies for all $t \in \mathbb{Z}_+$ that $k_c < k_0 < k_2 < ... < k_{2t}$ with $\{k_{2t}\}_{t=0}^{+\infty}$

converging to k^*. Similarly, for all $t \in \mathbb{Z}_+$ we have $k_c > k_1 > k_3 > ... > k_{2t+1}$ with $\{k_{2t+1}\}_{t=0}^{+\infty}$ converging to 0. Hence, the overall capital stock $\boldsymbol{k} = \{k_0, k_1, k_2, , ..., k_t, ...\}$ evolves by oscillating and converges to a two-period limit cycle alternating between 0 and k^*. In the case that $k_0 < k_c < k_1$, similar arguments apply. ■

3.3 Dynamic Properties of Optimal Paths under Partial Depreciation

We will now show that dynamic properties of the optimal path remain valid to some extent when δ close to 1.

Recall from Proposition 3.1 that k_c is the critical capital stock when $\delta = 1$. Let $V : [0, \bar{k}] \times [0, \bar{k}] \times [\underline{\delta}, 1] \to \mathbb{R}_+$ denote the value function of $E2 - P$ as a function of k_0, k_1, and δ:

$$V(k_0, k_1, \delta) = \max_{(1-\delta)k_1 \le k_2 \le f(k_0) + (1-\delta)k_1} \{u(f(k_0) + (1-\delta)k_1) + \beta V(k_1, k_2, \delta)\}.$$

Let $G(k_0, k_1, \delta) : [0, \bar{k}] \times [0, \bar{k}] \times [\underline{\delta}, 1] \to [0, \bar{k}]$ be the optimal policy correspondance. The preliminary results regarding the properties of V and G are summarized in the following proposition.

Proposition 3.5 *V is continuous and G is u.h.c.*

Proof. Let $X = [0, \bar{k}] \times [0, \bar{k}] \times [\underline{\delta}, 1)$. For $(k_0, k_1, \delta) \in X$, define the feasible set as follows:

$$\Gamma(k_0, k_1, \delta) = \{(k_0', k_1', \delta') : k_0' = k_1, \delta' = \delta, (1-\delta)k_1 \le k_1' \le f(k_0) + (1-\delta)k_1\}.$$

Let $A \subset X \times X$ be the graph of Γ. Since A is closed and bounded, it follows by Stokey and Lucas (1989, Theorem 3.4) that Γ is u.h.c. To see that Γ is l.h.c., let $(k_0, k_1, \delta) \in X$ and $(k_0', k_1', \delta') \in \Gamma(k_0, k_1, \delta)$. Let $\{(k_0^n, k_1^n, \delta^n)\}_{n\in\mathbb{N}}$ be a sequence in X with $\lim_{n\to+\infty}(k_0^n, k_1^n, \delta^n) = (k_0, k_1, \delta)$. For $n \in \mathbb{N}$, let $k_0'^n = k_1^n$ and $\delta'^n = \delta$. Then, $\lim_{n\to+\infty} k_0'^n = k_1 = k_0^n$ and $\lim_{n\to+\infty} \delta'^n = \delta = \delta'$. For $n \in \mathbb{N}$, let

$$k_1'^n = \arg\min_{k\in[0,\bar{k}]} \{|k - k_1'| : (1-\delta^n) k_1^n \leq k \leq f(k_0^n) + (1-\delta^n) k_1^n\}.$$

Then, $\left(k_0'^n, k_1'^n, \delta'^n\right) \in \Gamma(k_0^n, k_1^n, \delta^n)$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} k_1'^n = k_1'$. It follows that Γ is l.h.c. Hence, Γ is continuous.

For $((k_0, k_1, \delta), (k_0', k_1', \delta')) \in A$, $u(f(k_0) + (1-\delta) k_1 - k_1')$ is both continuous and bounded. By Stokey and Lucas (1989, Theorem 4.3), V is continuous and G is u.h.c. (see Stokey and Lucas, 1989, exercise 3.11(b)). ■

The following lemma will be crucial in proving the main result of this essay.

Lemma 3.1 *Let $F : [0, \bar{k}] \times [0, \bar{k}] \times [0, \bar{k}] \times [\underline{\delta}, 1) \to \mathbb{R}$ be a continuous function. Let*

$$\forall k_0 \in [0, \bar{k}], \forall k_1 \in [0, \bar{k}], \forall k_2 \in G(k_0, k_1, \delta) \text{ we have } F(k_0, k_1, k_2, \delta) < 0.$$

Then, there exists $\epsilon > 0$ such that

$$\forall \delta' \in B_\epsilon(\delta), \forall k_0 \in [0, \bar{k}], \forall k_1 \in [0, \bar{k}], \forall k_2 \in G(k_0, k_1, \delta')$$
$$\text{we have } F(k_0, k_1, k_2, \delta') < 0$$

where $B_\epsilon(\delta) = \{\delta \in [\underline{\delta}, 1) : |\delta' - \delta| < \epsilon\}$.

Proof. Note first that $\left[0,\bar{k}\right]\times\left[0,\bar{k}\right]$ is a compact set and G is u.h.c by 3.5. Assume on the contrary that there exists no such $\epsilon>0$. Then, there exist sequences $\{\delta^n\}_{n\in\mathbb{N}}\subset[\underline{\delta},1)$, $\{k_0^n\}_{n\in\mathbb{N}}\subset\left[0,\bar{k}\right]$, $\{k_1^n\}_{n\in\mathbb{N}}\subset\left[0,\bar{k}\right]$ and $\{k_2^n\}_{n\in\mathbb{N}}\subset\left[0,\bar{k}\right]$ such that

$$\lim_{n\to+\infty}\delta^n=\delta \text{ and } \forall n\in\mathbb{N} \text{ we have } k_2^n\in G\left(k_0^n,k_1^n,\delta^n\right) \text{ and } F\left(k_0^n,k_1^n,k_2^n,\delta^n\right)\geq 0.$$

By taking sub-sequences, we can assume that as $n\to+\infty$, $\{k_0^n\}_{n\in\mathbb{N}}$ converges to some k_0 and $\{k_1^n\}_{n\in\mathbb{N}}$ converges to some k_1. Since G is u.s.c., we can assume that $\{k_2^n\}_{n\in\mathbb{N}}$ converges to some $k_2\in G\left(k_0,k_1,\delta\right)$. Then, by continuity of F, we have $F\left(k_0,k_1,k_2,\delta\right)\geq 0$, a contradiction. ■

We can restate the results provided in Proposition 3.2 in terms of the optimal policy correspondance as the following:

$$\begin{aligned}
\forall k_0 &\in \left(0,k_c\right),\forall k_1\in\left[0,\bar{k}\right] \text{ we have } k_2\in G\left(k_0,k_1,1\right) \text{ such that } k_2<k_0, && (3)\\
\forall k_0 &\in \left(k_c,k^*\right),\forall k_1\in\left[0,\bar{k}\right] \text{ we have } k_2\in G\left(k_0,k_1,1\right) \text{ such that } k_0<k_2<k^*, && (4)\\
\forall k_0 &= k^*,\forall k_1\in\left[0,\bar{k}\right] \text{ we have } k_2\in G\left(k^*,k_1,1\right) \text{ such that } k_2=k^*, && (5)\\
\forall k_0 &\in \left(k^*,\bar{k}\right],\forall k_1\in\left[0,\bar{k}\right] \text{ we have } k_2\in G\left(k_0,k_1,1\right) \text{ such that } k^*<k_2<k_0. && (6)
\end{aligned}$$

Now, the main result is provided in the following proposition.

Proposition 3.6 *Let $\eta>0$ satisfy*

$$0<k^c-\eta \text{ and } k^c+\eta<k^*.$$

There exists $d\in[\underline{\delta},1)$ such that for any $\delta\in(d,1)$, any optimal path k from (k_0,k_1) satisfies the following:

(i) If $k_0 \leq k^c - \eta$ and $k^c + \eta \leq k_1$, then $k_t < k^c - \eta$ for all $t \in \{0, 2, 4, \ldots\}$ and $k^c + \eta < k_t$ for all $t \in \{1, 3, 5, \ldots\}$.

(ii) If $k^c + \eta \leq k_0$ and $k_1 \leq k^c - \eta$, then $k^c + \eta < k_t$ for all $t \in \{0, 2, 4, \ldots\}$ and $k_t < k^c - \eta$ for all $t \in \{1, 3, 5, \ldots\}$.

Proof. Let $\eta, \theta > 0$ satisfy

$$0 < \theta < k^c - \eta \text{ and } k^c + \eta < k^* - \theta.$$

We have

$$\begin{aligned}
\forall k_0 &\in [0, \theta], \forall k_1 \in \left[0, \bar{k}\right] \text{ we have } k_2 \in G(k_0, k_1, 1) \text{ such that } k_2 < k_c - \eta, \\
\forall k_0 &\in [\theta, k_c - \eta], \forall k_1 \in \left[0, \bar{k}\right] \text{ we have } k_2 \in G(k_0, k_1, 1) \text{ such that } k_2 < k_0, \\
\forall k_0 &\in [k_c + \eta, k^* - \theta], \forall k_1 \in \left[0, \bar{k}\right] \text{ we have } k_2 \in G(k^*, k_1, 1) \text{ such that } k_2 > k_0, \\
\forall k_0 &\in \left[k^* - \theta, \bar{k}\right], \forall k_1 \in \left[0, \bar{k}\right] \text{ we have } k_2 \in G(k_0, k_1, 1) \text{ such that } k_2 > k_c + \eta.
\end{aligned}$$

The second and the third lines directly follow from 3 and 4. To see the first line, let $k_0 \in [0, \theta]$ and $k_1 \in \left[0, \bar{k}\right]$ so that $k_2 \in G(k_0, k_1, 1)$. If $k_0 = 0$ then $k_2 = 0 < k_c - \eta$. If $k_0 > 0$ then $k_2 < k_0 \leq \theta < k_c - \eta$. To see the last line, let $k_0 \in \left[k^* - \theta, \bar{k}\right]$ and $k_1 \in \left[0, \bar{k}\right]$ so that $k_2 \in G(k_0, k_1, 1)$. If $k_0 \in \left[k^* - \theta, \bar{k}\right]$ then $k_2 > k_0 \geq k^* - \theta > k_c + \eta$. If $k_0 > k^*$ then $k_2 > k^* > k_c + \eta$.

From Lemma 3.1, there exists $d \in [\underline{\delta}, 1)$ such that if $\delta \in (d, 1)$, we have

$$\begin{aligned}
\forall k_0 &\in [0, \theta], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k_0, k_1, \delta) \text{ such that } k_2 < k_c - \eta, \\
\forall k_0 &\in [\theta, k_c - \eta], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k_0, k_1, \delta) \text{ such that } k_2 < k_0, \\
\forall k_0 &\in [k_c + \eta, k^* - \theta], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k^*, k_1, \delta) \text{ such that } k_2 > k_0, \\
\forall k_0 &\in [k^* - \theta, \bar{k}], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k_0, k_1, \delta) \text{ such that } k_2 > k_c + \eta.
\end{aligned}$$

We will now show that there exists $d \in [\underline{\delta}, 1)$ such that for any $\delta \in (d, 1)$, we have

$$\begin{aligned}
\forall k_0 &\in [0, k_c - \eta], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k_0, k_1, \delta) \text{ such that } k_2 < k_c - \eta, \\
\forall k_0 &\in [k_c + \eta, \bar{k}], \forall k_1 \in [0, \bar{k}] \text{ we have } k_2 \in G(k_0, k_1, \delta) \text{ such that } k_2 > k_c + \eta.
\end{aligned}$$

Let $\delta \in (d, 1)$. To see the first line, let $k_0 \in [0, k_c - \eta]$ and $k_1 \in [0, \bar{k}]$ so that $k_2 \in G(k_0, k_1, \delta)$. If $k_0 \leq \theta$ then $k_2 < k_c - \eta$. If $k_0 > \theta$ then $k_2 < k_0 \leq k_c - \eta$. To see the second line, let $k_0 \in [k_c + \eta, \bar{k}]$ and $k_1 \in [0, \bar{k}]$ so that $k_2 \in G(k_0, k_1, \delta)$. If $k_0 \geq k^* - \theta$ then $k_2 > k_c + \eta$. If $k_0 < k^* - \theta$ then $k_2 > k_0 \geq k_c + \eta$. ■

3.4 Numerical Analysis

In this section, we use numerical methods to show that our results under full depreciation are relevant to some extent under partial depreciation. In particular, we illustrate cyclical dynamics of the optimal path in the long-run when depreciation rate is close to one.

We consider the following functional forms:

$$f(k) = \frac{\alpha k^p}{\eta + \gamma k^p},$$
$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

under the following set of parameterization:

$$\beta = 0.98, \sigma = 0.5, \alpha = 3, \gamma = 1.5, p = 2, \eta = 1, \delta = 0.9.$$

Note that f and u satisfy the assumptions set.

Under full depreciation, there exists a critical capital stock, $k_c \approx 0.43$ leading to threshold dynamics. The positive steady state is seen to occur at $k^* = 0.941$. When initial capital stocks are given such that $k_0 = 0.4$ and $k_1 = 0.45$ satisfying $k_0 < k_c < k_1$, the optimal path converges to a two-period cycle alternating between 0 and $k^* = 0.941$ as shown in Proposition 3.4. Under partial depreciation, the long-run dynamics of the optimal path may change. In particular, when $\delta = 0.9$, it oscillates towards a positive steady state ($k^*_{\delta=0.9} = 1.095$) given the same initial capital stocks (see Figure 4).

Under partial depreciation, the optimal path may still exhibit cyclical behaviour in the long-run. In accordance with Proposition 3.6, for any $\delta \in (0.9, 1)$, there exists some $\eta > 0$ satisfying $k_0 \leq k^c - \eta$ and $k^c + \eta \leq k_1$ such that the optimal path exhibits cyclical behaviour in the long-run (see Figure 5).

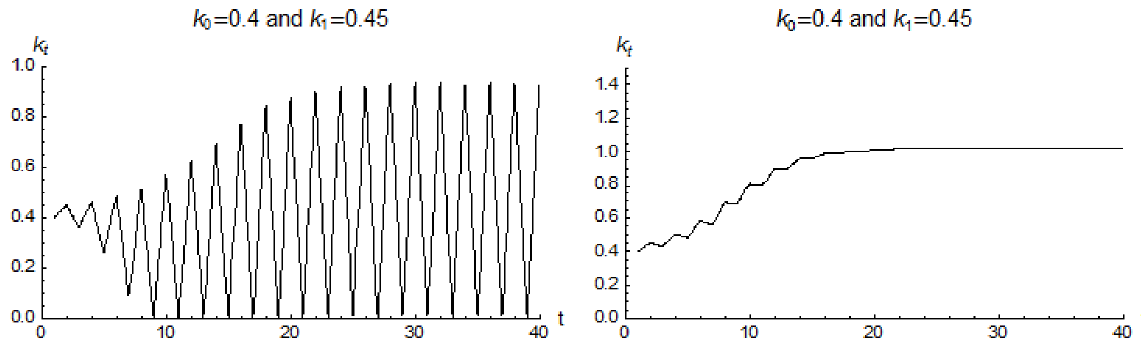

Figure 4: **Dynamics of the optimal path when $\delta = 1$ and when $\delta = 0.9$, respectively.**

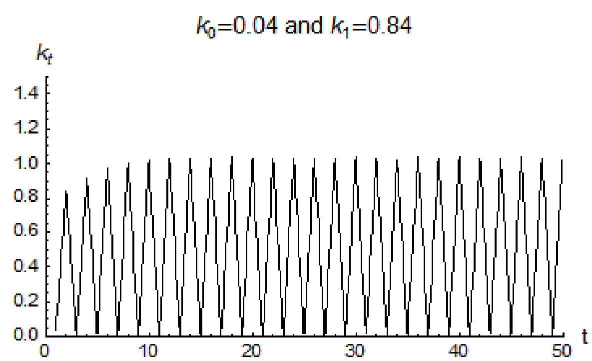

Figure 5: **Cyclical dynamics under partial depreciation for $\eta = 0.39$.**

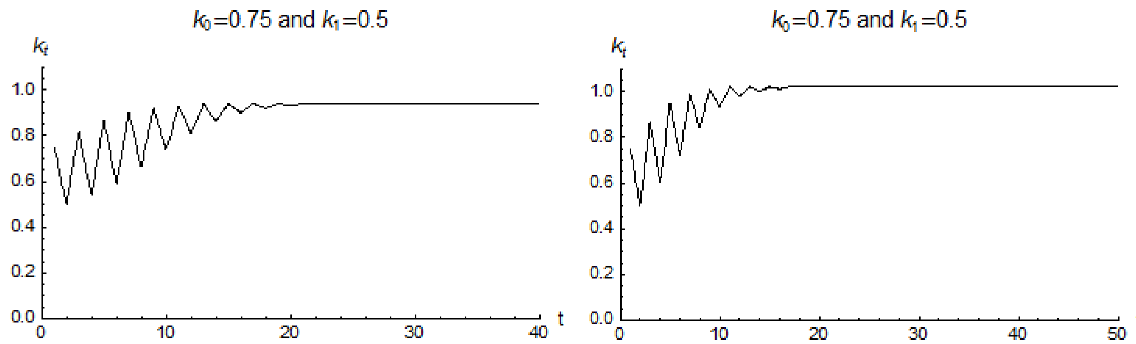

Figure 6: **Oscillations when $\delta = 1$ and when $\delta = 0.9$, respectively.**

Under full depreciation, when initial capital stocks are given such that $k_0 = 0.75$ and $k_1 = 0.5$, the optimal path converges to the positive steady state in an oscillating way as shown in Proposition 3.3. When $\delta = 0.9$, given the same initial capital stocks, the optimal path oscillates towards a positive steady state ($k^*_{\delta=0.9} = 1.095$) in a shorter period of time (see Figure 6).

CHAPTER 4

TIME-TO-BUILD AND OPTIMAL RESOURCE ALLOCATION UNDER NON-CONVEXITIES

The empirical relevance of time-to-build lag, a term originally coined by Kydland and Prescott (1982), was first stated by Jevons (1871) who had given an insight into the nature of the capital:

> "A vineyard is unproductive for at least 3 years before it is thoroughly fit for use. In gold mining there is often a long delay, sometimes even of 5 or 6 years, before gold is reached" (Jevons, 1871, Theory of Capital, Chapter VII, p.225).

On theoretical grounds, this ubiquitous relationship between cyclical dynamics and time-to-build delay was first analyzed by Kalecki (1935). He introduced a time delay between the investment decision and the realization of it and proved the existence of endogenous cycles. El-Hodiri et al. (1972) was the first to introduce time-to-build delay into an optimal growth framework in continuous-time but they did not provide any results regarding the cyclical dynamics. Later by Kydland and Prescott (1982), a discrete-time model in a real business cycle framework was developed to demonstrate the importance of time-to-build lag in explaining the cyclical fluctuations.

Zak (1999) showed that Kalecki's result holds in a Solowian economy (see also Krawiec and Szydlowski, 2004). Recently, Ferrara et al. (2014) proved that the solution also admits Hopf cycles in a Solowian economy under non-convex technology and time-to-build delay.

The dynamic implications of time-to-build delay in one-sector classical optimal growth models in continuous-time is that the optimal path oscillates towards a steady state (Winkler et al., 2003, 2005; Collard et al., 2006, 2008; Ferrara et al. 2013). When time-to-build lag is considered in a non-classical optimal growth model inducing threshold dynamics, the optimal path exhibits cyclical dynamics even in the long-run (Chapter 3).

In all these studies, in the presence of time to build lag, to form the productive stock requires multiple periods of time and the incomplete productive stock cannot be put in the production process before this time elapses. Accordingly, given a $d-$period time-to-build lag, the information structure describes the initial conditions necessary for enabling the production in the first d periods so that $k_\tau = \zeta(\tau)$ for $\tau \in (-d, 0]$ for the continuous-time and $k_\tau = \zeta(\tau)$ for $\tau \in \{-d+1, ..., -1, 0\}$ for the discrete-time set-up are provided initially. Then, depending on these initial conditions, the optimal path can be characterized by non-monotone dynamics. However, in an alternative information structure, the initial conditions may reveal the overall initial stock for the first d periods. Note that given $d-$period time-to-build lag, this overall stock must be allocated optimally over the first d periods. In this essay, we analyze the effects of such a change in the information structure on the equilibrium dynamics. In particular, we seek to understand the dependence of the cyclical dynamics on the information structure. Do the cyclical dynamics become less pronounced due to the consumption smoothing when the overall stock is allocated optimally over the first d periods? How does the allocation of the total stock over the first d periods change as the total stock increases? Can this be monotone?

In order to answer these questions, we define a discrete-time dynamic optimization problem in which it takes d periods to regenerate the resource stock with a convex-concave regeneration function. Observing only the overall resource stock, the social planner decides how to allocate this overall stock among k_τ for each $\tau \in \{0, 1, ..., d-1\}$. The unconsumed part of k_τ is invested in period τ to generate the future resource stock, given by $f(k_\tau - c_\tau)$, that will be ready for both consumption and reinvestment in period $\tau + d$. It is important to note here that once k_τ is allocated, k_τ and the future resource stock regenerated from it can only be utilized in periods $\tau, \tau + d, \tau + 2d, ...$ Put differently, if the overall resouce stock is completely allocated to, for example k_0, there will be no resource stock available to consume and to invest in periods other than $0, d, 2d, ...$ This cannot be optimal due to the utility function satisfying the Inada conditions.

It is also important to note that given k_τ for $\tau \in \{0, 1, ..., d-1\}$, the problem can be reformulated as d independent sub-problems each of which is in line with a non-classical optimal growth model (e.g. Dechert and Nishimura, 1983). Therefore, the optimal solution in each sub-problem admits threshold dynamics due to the non-convex structure of the technology. There exists a critical resource stock below which the optimal path converges to zero and above which the optimal path converges to a high steady state suggesting that the initial differences persist in the long-run. It is important to note that the optimal solution in these sub-models is monotonic because of the positive cross partial derivatives (e.g. Benhabib and Nishimura, 1985). However, this need not imply the monotonicity of the overall resource stock in the economy.

We prove that, for small values of the overall resource stock, the optimal path converges to zero in the long-run. This convergence can be oscillatory depending on how the initial resource stock is allocated among the first d periods. We determine the existence of the cyclical dynamics depending on the value of the overall resource stock. We show that due to the non-convexities in the regeneration function, the optimal resource allocation is not necessarily monotone in the level of the initial stock. To provide a better exposition of our theoretical results, we also perform a numerical analysis and demonstrate how the periodicity of the limit cycle changes as time-to-build lag increases.

The rest of the essay is organized as follows. In section 2, we set-up the model and state the existence of the critical stock. In section 3, we present the dynamic properties of the optimal path. In Section 4, we provide the numerical analysis. We conclude in section 5.

4.1 The Model

Given an initial total amount of resource stock, K, the social planner solves the following problem:

$$\max_{\{c_t,k_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t) \qquad (E3-P)$$

subject to

$$\begin{aligned} \forall t &\in \mathbb{Z}_+, k_{t+d} \leq f(k_t - c_t), \\ k_t &\geq 0, c_t \geq 0, \\ k_0 + k_1 + k_2 + ... + k_{d-1} &\leq K, \\ K &\geq 0, \text{ given} \end{aligned}$$

where k_t is the amount of resource stock in period t. In each period,.k_t is both planted and consumed. After the consumption of c_t in period t, the amount of planted resource stock is given by $k_t - c_t$ and the amount of resource stock to be harvested in the future is determined by a regeneration function, given by $f(k_t - c_t)$. d denotes the time-to-build lag so that it requires d periods to regenerate the resource stock before harvesting. K is allocated over the first d periods to determine how much resource stock will be planted in each period $t \in \{0, 1, 2, ..., d-1\}$. $\beta \in (0, 1)$ is the discount factor.

The following assumptions are maintained throughout the essay.

Assumption 4.1 *$u : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing, strictly concave, and satisfies $u(0) = 0$ and $\lim_{c \to 0} u'(c) = +\infty$.*

Assumption 4.2 *$f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing and satisfies $f(0) = 0$.*

Assumption 4.3 *There exists $k^I > 0$ such that f is strictly convex on $\left[0, k^I\right]$ and strictly concave on $\left[k^I, \infty\right)$.*

Assumption 4.4 *There exists $\bar{k} > k^I$ such that $f\left(\bar{k}\right) = \bar{k}$ and $f(k) < k$ for all $k > \bar{k}$.*

Let ϕ be the inverse of f. ϕ is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing and satisfies $\phi(0) = 0$. Moreover, ϕ is strictly concave on $\left[0, k^I\right]$, strictly convex on $\left[k^I, +\infty\right)$ and $\phi(k) > k$ for all $k > \bar{k}$.

For any initial condition $K \geq 0$, a sequence of resource stocks $\boldsymbol{k} = (k_0, k_1, k_2, ...)$ is feasible from K if $0 \leq \phi(k_{t+d}) \leq k_t$ and $\sum_{\tau=0}^{d-1} k_\tau \leq K$ for all $t \in \mathbb{Z}_+$. A sequence of consumption $\boldsymbol{c} = (c_0, c_1, c_2, ...)$ is feasible from K if there exists a $\boldsymbol{k}$ feasible from K such that $0 \leq c_t \leq k_t - \phi(k_{t+1})$ for all $t \in \mathbb{Z}_+$. We denote the set of feasible sequences from K by $\Pi(K)$.

Let $D \equiv \{0, 1, ..., d-1\}$ and $\boldsymbol{k}$ be a solution to $E3 - P$. There exist d sub-sequences of $\boldsymbol{k}$ defined by ${}_\tau\boldsymbol{k} = \{k_{dt+\tau}\}_{t=0}^{+\infty} = \{k_\tau, k_{d+\tau}, k_{2d+\tau}, ...\}$ for any $\tau \in D$. We define ${}_\tau k_t = k_{dt+\tau}$ for any $\tau \in D$. Each of these sub-sequences ${}_\tau\boldsymbol{k}$ solves the following non-convex optimization problem for any given initial resource stock $k_\tau \in \{k_0, k_1, k_2, ..., k_{d-1}\}$:

$$W\left({}_\tau k_0\right) = \max_{\{{}_\tau c_t, {}_\tau k_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \rho^t u\left(c_t\right) \qquad (E3 - R)$$

subject to

$$\forall t \in \mathbb{Z}_+, {}_\tau c_t + \phi\left({}_\tau k_{t+1}\right) \leq {}_\tau k_t,$$

$${}_\tau c_t \geq 0, {}_\tau k_t \geq 0,$$

$${}_\tau k_0 > 0, \text{ given}$$

with $\rho = \beta^d$. Note that $E3 - R$ is in line with Dechert and Nishimura (1983). It is straightforward that W is non-negative, strictly increasing, continuous, differentiable almost everywhere and W verifies the Bellman equation:

$$\forall \tau \in D, W\left({}_\tau k_0\right) = \max\left\{u\left({}_\tau k_0 - \phi\left({}_\tau k_1\right)\right) + \rho W\left({}_\tau k_1\right) \mid 0 \leq \phi\left({}_\tau k_1\right) \leq {}_\tau k_0\right\}.$$

The solution to this Bellman equation, $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is non-empty, upper semi-continuous and strictly increasing. In accordance with these, ${}_\tau\boldsymbol{k}$ satisfies ${}_\tau k_{t+1} = \mu\left({}_\tau k_t\right)$, for all $t \in \mathbb{Z}_+$.

Any solution satisfying $\phi'(k) = \rho$ is a steady state of $E3 - R$. There may be no, only one or two solutions satisfying this equation. For the rest of the essay, we maintain the following assumption under which there exist two steady states, k_*(low steady state) and k^*(high steady state), which is relevant for our analysis.

Assumption 4.5 $\phi'(0) > \rho > \min\left\{\frac{\phi(k)}{k}\right\}$.

We recall the existence of a critical resource stock, k_c, in the following proposition (Dechert and Nishimura, 1983). As the existence of a critical resource stock is important in our set-up for an optimal path to exhibit a cyclical behaviour in $E3-P$, it will be helpful in the rest of the essay.

Proposition 4.1 *Let* $\boldsymbol{k}$ *be an optimal path from* K*. There exists a critical capital stock* $k_c > 0$ *such that, for any* $\tau \in D$*, if* ${}_\tau k_0 \in (k_c, +\infty)$*,* ${}_\tau\boldsymbol{k}$ *converges to a strictly positive steady state* k^* *and if* ${}_\tau k_0 \in (0, k_c)$*,* ${}_\tau\boldsymbol{k}$ *converges to* 0*.*

Then, the problem $E3-P$ can equivalently be recast as

$$\max_{k_0,k_1,\dots,k_{d-1}} W(k_0) + \beta W(k_1) + \beta^2 W(k_2) + \dots + \beta^{d-1} W(k_{d-1}) \qquad (E3-\mathcal{P}')$$

subject to

$$\begin{aligned} k_0 + k_1 + k_2 + \dots + k_{d-1} &= K, \\ k_0, k_1, k_2, \dots, k_{d-1} &\geq 0, \\ K &\geq 0, \text{ given.} \end{aligned}$$

We define the optimal policy correspondance $G : \mathbb{R}_+ \to \mathbb{R}_+^d$ of $E3-\mathcal{P}'$ as

$$G(K) \in \arg \max_{0 \leq k_0+k_1+k_2+\dots+k_{d-1} \leq K} \sum_{\tau \in D} \beta^\tau W(k_\tau).$$

G is upper semi-continuous by the Theorem of the Maximum.

4.2 Dynamic Properties of Optimal Paths

In what follows, we analyze the qualitative implications of time-to-build lag on the resource dynamics given an initial total resource stock under non-convex technology. In particular, we show that the monotonicity of the optimal path associated with $E3-R$ need not imply the monotonic convergence of the optimal path associated with $E3-P$ and we prove the existence of a limit cycle.

We first introduce two lemmas which are critical in proving the main results of the essay. In the first lemma, we show that the optimal consumption is strictly decreasing over the first d periods.

Lemma 4.1 *Let* $\mathbf{k}$ *be an optimal path from* K *and* $\mathbf{c}$ *be the associated consumption path. We have* $c_0 > c_1 > c_2 > ... > c_{d-1}$.

Proof. For any $\tau \in D$, choose some $\epsilon > 0$ such that $k_\tau - \epsilon - \phi(k_{d+\tau}) > 0$. Define the sequence $\mathbf{k}'$ as follows:

$$k'_{\tau-1} = k_{\tau-1} + \epsilon,\ k'_\tau = k_\tau - \epsilon \text{ and } k'_t = k_t,\ \forall t \notin \{\tau-1, \tau\}.$$

The sequence $\mathbf{k}'$ is feasible. From the optimality of $\mathbf{k}$, we obtain the following:

$$\Delta_\epsilon = \sum_{\tau\in D} \beta^\tau W(k'_\tau) - \sum_{\tau\in D} \beta^\tau W(k_\tau) \leq 0.$$

Then,

$$\lim_{\epsilon\to 0} \frac{\Delta_\epsilon}{\epsilon} = u'(k_{\tau-1} - \phi(k_{\tau-1+d})) - \beta u'(k_\tau - \phi(k_{\tau+d})) \leq 0$$

implies $c_{\tau-1} > c_\tau$. ∎

In the second lemma, we show that the optimal allocation of K among the resource stocks of the first d periods has a decreasing order.

Lemma 4.2 *Let* $\mathbf{k}$ *be an optimal path from* K. *We have* $k_0 > k_1 > k_2 > ... > k_{d-1} > 0$.

Proof. Assume on the contrary that $\mathbf{k}$ is an optimal path from K such that $k_\tau < k_{\tau'}$ for any $\tau, \tau' \in D$ with $\tau < \tau'$. Since W is increasing, $\beta^\tau W(k_{\tau'}) + \beta^{\tau'} W(k_\tau) > \beta^\tau W(k_\tau) + \beta^{\tau'} W(k_{\tau'})$ contradicts the optimality of $\mathbf{k}$. Thus, $k_\tau \geq k_{\tau'}$. It cannot be the case that $k_\tau = k_{\tau'}$ since choosing $c_\tau = c_{\tau'}$ violates Lemma 4.1. Hence, combining with the fact that k_τ for any $\tau \in D$ should be strictly positive by the Inada condition, the correct order must be $k_\tau > k_{\tau'} > 0$ for any τ, τ'. ■

In what follows, we prove that the optimal path associated with $E3-P$ can exhibit oscillating behaviour though the monotonicity of the optimal path associated with the problem $E3-R$ and we prove the existence of a limit cycle. In particular, we show the behaviour of the optimal path depending on the value of K. In the following proposition, we demonstrate that the optimal path are oscillatory and converge to a limit cycle when $K = dk_c$. This characteristic of the optimal path remain valid also in the neighborhood of dk_c.

Proposition 4.2 *Let* $\mathbf{k}$ *be an optimal path from* K. *Let* $K = dk_c$. *We have*

i) There exist $\tau, \tau' \in D$ *with* $\tau < \tau'$ *such that* $k_\tau > k_c > k_{\tau'}$.

ii) The optimal path oscillates towards a cycle alternating between k^* *and* 0.

ii) There exist $\epsilon > 0$ *such that* $\forall \tilde{K} \in B_\epsilon(K) = \{X \in \mathbb{R} : |X - K| < \epsilon\}$ *and* $\forall \left\{\tilde{k}_0, \tilde{k}_1, ..\tilde{k}_{d-1}\right\} \in G\left(\tilde{K}\right)$

$$\tilde{k}_\tau > k_c > \tilde{k}_{\tau'}.$$

Proof. i) We know from Lemma 4.2 that $k_0 > k_1 > ... > k_{d-1} > 0$. Then, there exists at least one $\tau, \tau' \in D$ such that $k_\tau > k_c > k_{\tau'} > 0$ since $K = dk_c$.

ii) Proposition 4.2 (i) implies that $\{k_{2t+\tau}\}_{t=0}^{+\infty} \to k^*$ and $\{k_{2t+\tau'}\}_{t=0}^{+\infty} \to 0$ by Proposition 4.1. Hence, the optimal path oscillates towards a cycle alternating between k^* and 0.

iii) Assume that there is no such $\epsilon > 0$. There exist sequences $\{K^n\}_{n\in\mathbb{N}}$ and $\left\{k_0^n, k_1^n, .., k_{d-1}^n\right\}_{n\in\mathbb{N}}$ such that

$$\{K^n\}_{n\in\mathbb{N}} \to K,$$

$$\forall n \in \mathbb{N},\ \left\{k_0^n, k_1^n, .., k_{d-1}^n\right\} \in G\left(K^n\right) \text{ and either } k_{\tau'}^n \geq k_c \text{ or } k_c \geq k_\tau^n.$$

Since G is upper semi-continuous, there exists a sub-sequence $\left\{k_0^{n^k}, k_1^{n^k}, .., k_{d-1}^{n^k}\right\}_{n^k\in\mathbb{N}}$ converging to $\{k_0, k_1, ..k_{d-1}\} \in G\left(K\right)$.

First, consider the case such that $K^n > K$ for all $n \in \mathbb{N}$. From Lemma 4.2, we have $k_c < k_\tau^n$ for all $n \in \mathbb{N}$. Moreover, if we have $k_{\tau'}^n \geq k_c$ for all $n \in \mathbb{N}$, then as $n \to +\infty$, it is the case that $k_{\tau'} \geq k_c$ which contradicts Proposition 4.2 (i).

Second, consider the case such that $K^n < K$ for all $n \in \mathbb{N}$. From Lemma 4.2, we have $k_{\tau'}^n < k_c$ for all $n \in \mathbb{N}$. Moreover, if we have $k_c \geq k_\tau^n$ for all $n \in \mathbb{N}$, then as $n \to +\infty$, it is the case that $k_c \geq k_\tau$ which contradicts Proposition 4.2 (i). ■

The following proposition shows that for the small values of K, any optimal path eventually converges to 0.

Proposition 4.3 *Let* $\mathbf{k}$ *be an optimal path from* K *and* $K \leq k_c$. *We have*

i) $k_c > k_0 > k_1 > k_2 > ... > k_{d-1}$.

ii) If $\exists\ \tau, \tau' \in \{0, 1, 2, ..., d-1\}$ *and* $\tau > \tau'$ *s.t.* $k_\tau < G(k_{\tau'})$, $\mathbf{k}$ *oscillates towards* 0. *Otherwise, it monotonically converges to* 0.

Proof. i) By Lemma 4.2 and the assumption that $K \leq k_c$, we have $k_c > k_0 > k_1 > k_2 > ... > k_{d-1} > 0$.

ii) Note that after the allocation of K among $k_0, k_1, ..k_{d-1}$, there exist d independent sub-sequences of $\mathbf{k}$ defined by $\left\{\{k_{dt+\tau}\}_{t=0}^{+\infty}\right\}_{\tau \in D}$. $\{k_{dt+\tau}\}_{t=0}^{+\infty}$ solves $E3-R$ given k_τ and $\{k_{dt+\tau'}\}_{t=0}^{+\infty}$ solves $E3-R$ given $k_{\tau'}$. By Proposition 4.3 (i), we have $k_c > k_{\tau'} > k_\tau > 0$. If $k_\tau < G(k_{\tau'})$, increasingness of G implies $k_\tau < k_{\tau'+d} = G(k_{\tau'}) > G(k_\tau) = k_{\tau+d}$. Similarly, we have $k_{\tau+d} < k_{\tau'+2d}$. Hence, $k_{\tau'+dt} > k_{\tau+dt} < k_{\tau'+d(t+1)} > k_{\tau+d(t+1)}$ for all $t \in \mathbb{Z}_+$. If there does not exist any $\tau, \tau' \in \{0, 1, 2, ..., d-1\}$ and $\tau > \tau'$ s.t. $k_\tau < G(k_{\tau'})$, we have $k_t \geq k_{t+1}$ for all $t \in \mathbb{Z}_+$. $\blacksquare$

It should be noted that the periodicity of the existing limit changes since the number of periods for optimal path to return to at least one of the periodic points (k^* or 0) changes depending on d and K.

4.3 Numerical Analysis

In this section, we give a numerical example to provide a better exposition of our analysis. In particular, we illustrate how the allocation of K over the first d periods changes as K increases. We also demonstrate how the periodicity of the limit cycle changes as time-to-build lag increases.

Before we present the numerical results, it would be helpful to gain some insight into the structure of the problem $E3-P$ and to provide the details of the algoritm. The optimization problem $E3-P$ can be represented as a two-stage problem. First, the social planner solves the problem $E3-R$ in which there is no time-to-build lag. Second, he determines the optimal allocation of the initially given total resource stock over the first d periods. Accordingly, what we have done at the first stage is to obtain the value function of the problem $E3-R$. After that, for any $d \geq 2$, we use the backward induction method to solve the problem $E3-P$ given by:

$$\max_{k_0,k_1,\ldots,k_{d-1}} W(k_0) + \beta W(k_1) + \beta^2 W(k_2) + \ldots + +\beta^{d-1} W(k_{d-1}).$$

In numerical analysis, we consider the following functional forms:

$$f(k) = \frac{\alpha k^p}{\eta + \gamma k^p},$$
$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

under the following parameterization

$$\beta = 0.98, \rho = \beta^d, \sigma = 0.5, \alpha = 1.5, \gamma = 0.25, p = 1.75, \eta = 1.$$

Note that f and u satisfy the assumptions set.

It turns out that the maximum sustainable resource stock $\bar{k}$ is 4.758. There exist two solutions to $E3-R$: for $d = 2$, they are the low steady state denoted by $k_* = 0.196$ and the high steady state denoted by $k^* = 3.317$. As d increases, we see

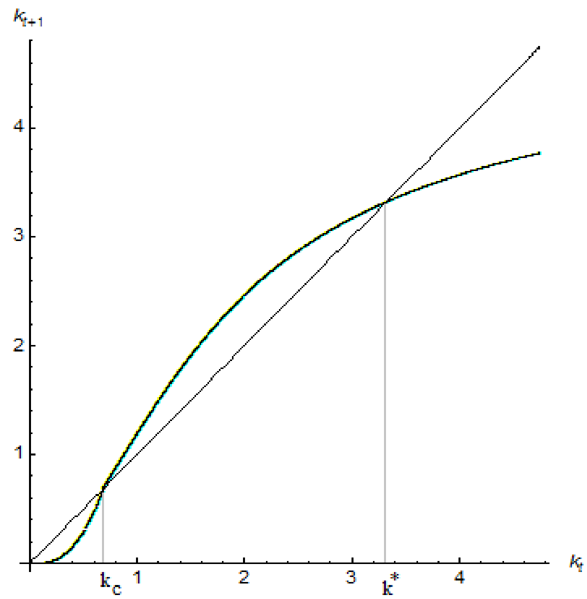

Figure 7: **The optimal policy for iterations of the Bellman operator on the initial zero value function.**

that k_* increases and k^* decreases.

In order to determine which of them are optimal steady states, we analyze the optimal policy associated to $E3 - R$ which is obtained from the iterations of the Bellman operator on the zero function (see Figure 7). The optimal policy strongly indicates that k^* is saddle path stable while k_* is not an optimal steady state. Indeed, if k_* were optimal, the optimal policy would cross the 45° line at k_*.

It is also important to note that, in Figure 8, we see that there is a genuine critical point at around $k_c \approx 0.68$ leading to the threshold dynamics. This implies that the low steady state k_* is clearly not an unstable optimal steady state or non-optimal steady state. The value of this critical capital stock k_c will be crucial for the allocation of K over the first d periods.

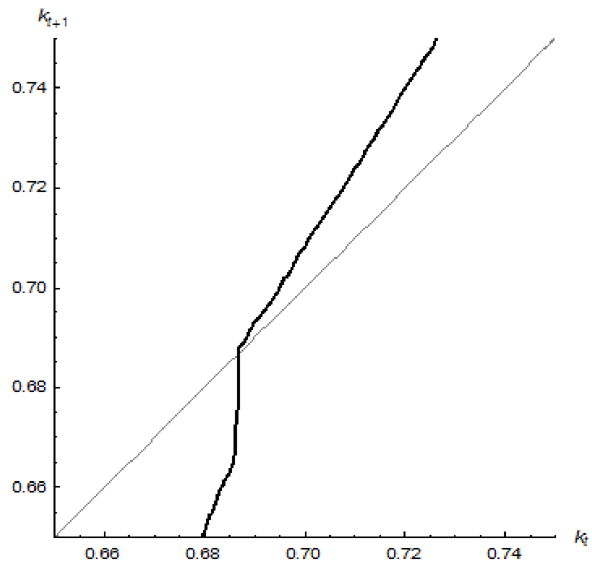

Figure 8: **The critical capital stock $k_c = 0.687$ which is neither an unstable optimal steady state nor non-optimal steady state.**

Consistent to Proposition 4.3, for small values of K, we have $k_\tau < k_c$ for any $\tau \in D$ thus the optimal path converges to zero in the long-run. The behaviour of the optimal path along the transition path is monotone if $k_d > k_\tau$ for any $\tau \in D$.

As K gets larger than $(\tau + 1)\, k_c$ for any $\tau \in D$, k_τ jumps above k_c, k_i for every $i < \tau$ decreases markedly though still remaining above k_c and k_j for $j > \tau$ takes a value very close to 0 (see Proposition 4.2).

For further increases in K, every k_τ for $\tau \in D$ continue to increase maintaining the relationship that $k_0 > k_1 > ... > k_{d-1} > k_c$.

Figure 10 shows a periodic cycle of the resource stock for $K = 2.5$ as the time-to-build lag increases.

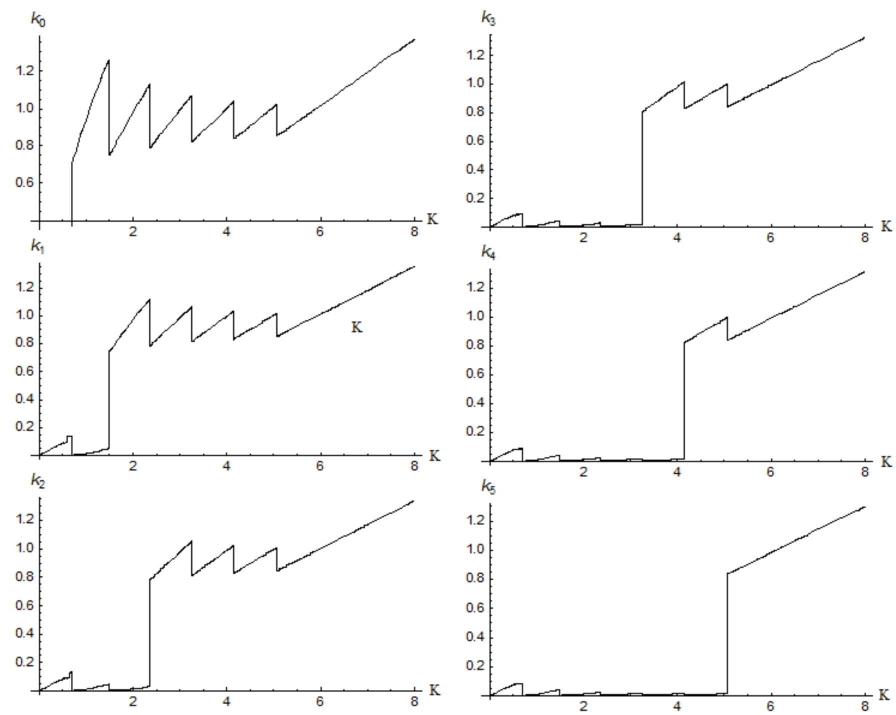

Figure 9: **Allocation of the overall resource stock over the first d periods for $d = 6$.**

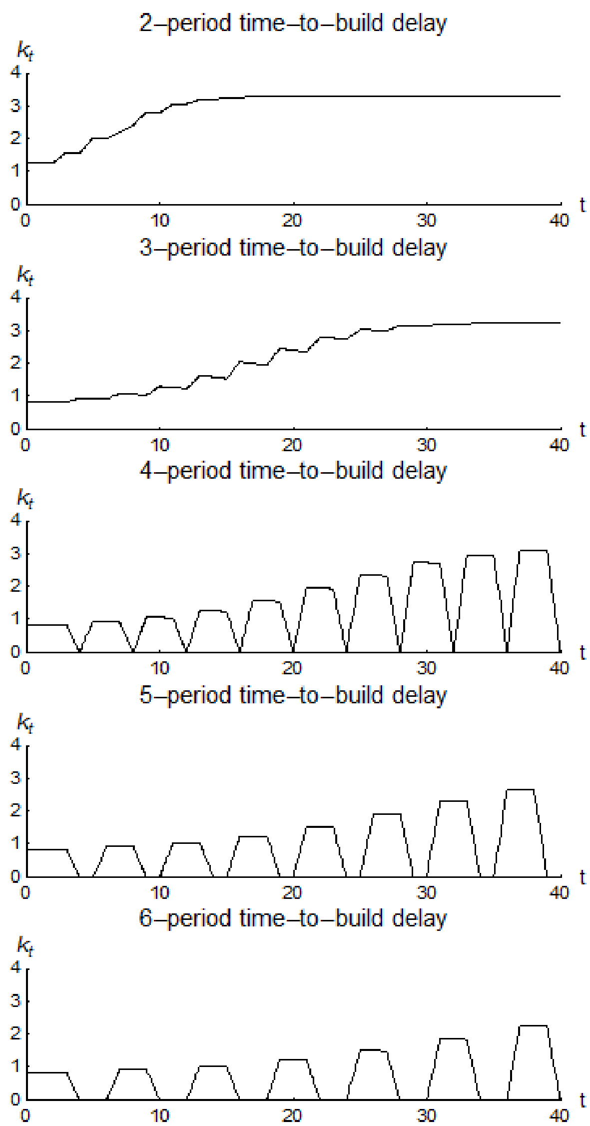

Figure 10: **Periodic cycles of the optimal path for** $K = 2.5$ **for various values of the time-to-build lag.**

CHAPTER 5

CONCLUDING REMARKS

In this thesis, we have focused on the extent to which the one-sector optimal growth models inducing threshold dynamics can explain both the persistent differences in per capita GDP across countries and the fluctuations observed in the aggregate output. These concerns have been treated in three essays. In the first essay, we have analyzed the implications of the preferences for wealth-habit in the formation of poverty trap and in escaping from such trap. In the second essay, we have examined the joint dynamic implications of non-convex technologies and time-to-build lag in investment. The resulting long-run dynamics have been found to be consistent with the empirically observed phenomenon of middle income trap. In the third essay, we have focused on the information structure regarding the initial conditions in the models under time-to-build lag and have explored the effects of a change in the information structure on the equilibrium dynamics.

This thesis should be complemented with the following future work:

- Recently, wealth-dependent preferences (Kurz, 1968; Zou, 1994) and the endogeneity of the time preference (Mantel, 1998; Stern, 2006; Erol et al., 2011) have been put forward to explain theoretically why the differences in per capita GDP across countries persist in the long run. These models show that either the presence of wealth effects or the endogeneity of the time preference can lead to the existence of multiple steady states even under a strictly convex technology. It is important to analyze the effects of time-to-build lag on equilibrium dynamics in such models that induce threshold dynamics even under convex technology.

- Considering that the infrastructure investment takes relatively more time compared to investment in physical capital stock, one also has to examine the effects of time-to-build lag on equilibrium dynamics in such multi-sector growth models.

- It has been shown that introducing discrete choice can be regarded as introducing non-convexities in classical optimal growth models. The existence of complex dynamics in the presence of the discrete choices in consumption has recently been analyzed by Kamihigashi (2012). It is important to show to what extent such discrete choices would intervene the dynamic implications of time-to-build lag both in the short-run and in the long-run.

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