A MEASURE DISINTEGRATION APPROACH TO SPECTRAL MULTIPLICITY FOR NORMAL OPERATORS

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By
Serdar AY
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Aurelian Gheondea(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Alexander Goncharov

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. M. Özgür Oktel

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

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Serdar AY

M.S. in Mathematics

Supervisor: Assoc. Prof. Dr. Aurelian Gheondea

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In this thesis we studied the notion of direct integral Hilbert spaces, first introduced by J. von Neumann, and the closely related notion of decomposable operators, as defined in Kadison and Ringrose [1997] and Abrahamse and Kriete [1973]. Examples which show that some of the most familiar spaces in analysis are direct integral Hilbert spaces are presented in detail. Then we give a careful treatment of the notion of disintegration of a probability measure on a locally compact separable metric space, and using the machinery we obtain, a proof of the Spectral Multiplicity Theorem for Normal Operators employing the notion of disintegration of measures is given, based on Abrahamse and Kriete [1973], Arveson [1976], Arveson [2002]. In Chapter 5 the notion of essential preimage is presented in the sense of the article Abrahamse and Kriete [1973], and its relation with the spectral multiplicity function is discussed.

Keywords: direct integral Hilbert space, disintegration of measures, normal operators, Spectral Multiplicity Theorem, multiplicity function.

ÖZET

NORMAL OPERATÖRLER İÇİN SPEKTRAL KATLILIĞA ÖLÇÜM ÇÖZÜNÜMÜ YAKLAŞIMI

Serdar AY
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Aurelian Gheondea
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Bu tezde Kadison ve Ringrose [1997] ve Abrahamse ve Kriete [1973]'te tanımlandığı şekilde, J. von Neumann tarafından matematik literatürüne kazandırılan direkt integral Hilbert uzayları ve onunla yakından ilişkili olan ayrıştırılabilir operatörler üzerinde çalıştık. Analizde sıkça çalışılan bazı uzayların direkt integral Hilbert uzayları olduğunu gösteren örnekler ayrıntılı olarak sunuldu. Yerel kompakt ayrılabilir bir metrik uzay üzerinde tanımlı bir olasılık ölçümünün çözünümü kavramı hassas bir şekilde incelendi. Elde edilen araçlar kullanılarak Normal Operatörler için Spektral Katlılık Teoreminin ölçümlerin çözünümü kavramını temel alan, Abrahamse and Kriete [1973], Arveson [1976], ve Arveson [2002]'ye dayanan bir ispatı verildi. Beşinci bölümde Abrahamse and Kriete [1973] makalesinde tanımlandığı şekilde esas önimge kavramı sunuldu ve bu kavramın spektral katlılık fonksiyonu ile ilişkisi tartışıldı.

Anahtar sözcükler: direkt integral Hilbert uzayları, ölçümlerin çözünümü, normal operatörler, Spektral Katlılık Teoremi, katlılık fonksiyonu.

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Contents

Chapter 1

Direct Integral Of Hilbert Spaces

In this section, we recall the definiton and the basic properties of a direct integral of separable Hilbert spaces over a measure space with certain properties, following [13].

Definition 1.1. Let X be a locally compact σ-compact Borel measure space, and let μ be the completion of a positive Borel measure on X, which is taking finite values on compact subsets, so that it is σ-finite. Let $(H_x)_{x\in X}$ be a family of separable Hilbert spaces indexed over X. A separable Hilbert space H is said to be a direct integral of $(H_x)_{x\in X}$ over (X,μ) if it satisfies the following:

D1. For every $h \in H$, there is a function $X \ni x \mapsto h(x)$ defined on X such that $h(x) \in H_x$ for all $x \in X$.

D2. The function $x \mapsto \langle g(x), h(x)\rangle_{H_x}$ on X is μ-integrable for all $g, h \in H$ and such that

$$\langle g, h\rangle_H = \int_X \langle g(x), h(x)\rangle_{H_x}\, d\mu(x).$$

D3. If $f_x \in H_x$ for all $x \in X$ and the function $x \mapsto \langle f_x, g(x)\rangle_{H_x}$ is μ-integrable for every $g \in H$, then there exists $f \in H$ such that $f_x = f(x)$ for μ-almost every $x \in X$.

For H as above we use the notation

$$H = \int_X^{\oplus} H_x \, d\mu(x). \tag{1.1}$$

Remark 1.2. For any $h, g \in H$ consider a linear combination of corresponding functions $ah(x) + g(x)$, $a \in \mathbb{C}$. Clearly, the function $x \mapsto \langle ah(x) + g(x), f(x)\rangle$ is μ-integrable for all $f \in H$. Therefore by D3 in the definition, there is $z \in H$ such that $z(x) = ah(x) + g(x)$ μ-a.e. on X. Then we have $z = ah + g$. This follows by the following calculation:

$$\begin{aligned}
\langle ah + g - z, u\rangle &= \langle ah, u\rangle + \langle g, u\rangle - \langle z, u\rangle \\
&= \int_X \langle ah(x), u(x)\rangle \, d\mu + \int_X \langle g(x), u(x)\rangle \, d\mu - \int_X \langle z(x), u(x)\rangle \, d\mu \\
&= \int_X \langle ah(x) + g(x) - z(x), u(x)\rangle \, d\mu = 0.
\end{aligned}$$

From here it follows that if $h(x) = g(x)$ μ-a.e, then $h = g$, for $h(x) - g(x) = 0$ μ-a.e implies $h - g = 0$ by above.

Proposition 1.3. *Assume that $\{h_a\}_{a\in A}$ is a collection of vectors spanning H for some nonempty set $A \subset X$. Let*

$$H_x^0 := \overline{\operatorname{span}\{h_a(x) \,|\, a \in A\}}$$

be the closure of the linear span of this set. Then $H_x^0 = H_x$ for μ-a.e. $x \in X$.

Proof. Define

$$X_0 := \{x \in X \mid H_x^0 \neq H_x\}.$$

Let $u_x \in H_x \ominus H_x^0$ be a unit vector for $x \in X_0$. For $x \notin X_0$, let $u_x = 0$. Then clearly $\langle u_x, h_a(x)\rangle = 0$ for every $x \in X$.

Fix an arbitrary element $g \in H$. By assumption, there exists a sequence of elements $g_j \in H$ such that $g_j \underset{j\to\infty}{\longrightarrow} g$ in the norm $\|.\|_H$. Here each g_j is a linear combination of elements of $\{h_a\}_{a\in A}$. Let $g_j = \sum_{k=1}^{n_j} b_k h_{a_k}$. Then by the preeceding remark we have that for each j, $g_j(x) = \sum_{k=1}^{n_j} b_k h_{a_k}(x)$ except on a null set N_j. Then we have

$$\langle u_x, g_j(x)\rangle_{H_x} = \langle u_x, \sum_{k=1}^{n_j} b_k h_{a_k}(x)\rangle_{H_x} = 0$$

for any j and $x \notin \bigcup_{j=1}^{\infty} N_j$.

But since
$$\|g - g_j\|_H^2 = \int_X \|(g - g_j)(x)\|_{H_x}^2 d\mu \underset{j\to\infty}{\longrightarrow} 0$$
by a well known property of convergence in L^2 there exists a subsequence g_{j_l} such that
$$\|g(x) - g_{j_l}(x)\|_{H_x} \underset{l\to\infty}{\longrightarrow} 0$$
for all x outside of a null set N_0.

Let $N := \bigcup_{j=0}^{\infty} N_j$. Then $\mu(N) = 0$, and for $x \notin N$ we have
$$\lim_{l\to\infty} \langle u_x, g_{j_l}(x)\rangle_{H_x} = \langle u_x, g(x)\rangle_{H_x} = 0.$$
From here, the function $x \to \langle u_x, g(x)\rangle$ is integrable for any $g \in H$. By D3 in Definition 1.1, there is $u \in H$ such that $u(x) = u_x$ outside of a null set M. Now choosing $g = u$ gives
$$0 = \langle u_x, u(x)\rangle_{H_x} = \langle u_x, u_x\rangle_{H_x}$$
for $x \notin (N \cup M)$, i.e. $u_x = 0$ μ-a.e. Since u_x is a unit vector for $x \in X_0$, it follows that X_0 is a null set. □

Before we go into examples, we need lemmas. The following is Theorem 1 in Chapter IV, §3 of Bourbaki, is given as the countable convexity theorem and is cited here without proof. It can be considered as a generalization of the Minkowski inequality.

Lemma 1.4. *Let X be a locally compact Hausdorff space, and μ be a measure on it as in Definition 1.1. Let $f : X \to \mathbb{C}$ be a numerical function. For every $1 \le p < +\infty$ define $N_p(f) := \big(\int_X |f(x)|^p \, d\mu(x)\big)^{\frac{1}{p}}$. Note that $N_p(f)$ might take the value $+\infty$.*

Now let $(f_n)_{n=1}^{\infty}$ be a sequence of nonnegative functions on X. Then we have
$$N_p\Big(\sum_{n=1}^{\infty} f_n\Big) \le \sum_{n=1}^{\infty} N_p(f_n).$$

Lemma 1.5. *Let (X, Σ, μ) be a σ-finite measure space. Let $1 < p < +\infty$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume that a complex valued function f on X satisfies the following: $fg \in L^1(X, \mu)$ for every $g \in L^q(X, \mu)$. Then $f \in L^p(X, \mu)$.*

Proof. We give two proofs of the lemma.

For the first proof, define $\phi_f : L^q(X, \mu) \longrightarrow \mathbb{C}$ by

$$\phi_f(g) := \int_X fg \, d\mu, \text{ where } g \in L^q(X, \mu).$$

By assumption, ϕ_f is well defined, and clearly it is a linear functional. In order to make use of the Closed Graph Theorem, we show that it has a closed graph. For this, assume that $g_n \in L^q(X, \mu)$ is a sequence such that $\|g_n - g\|_q \longrightarrow 0$, and $\int_X fg_n d\mu \longrightarrow a$, where $a \in \mathbb{C}$. We show that $\int_X fg d\mu = a$.

Since $g_n \xrightarrow{L^q} g$, by a similar property as in the proof of Proposition 1.3 there exists a subsequence g_{n_i} such that $g_{n_i} \longrightarrow g$ pointwise. Passing to such a subsequence if necessary, we can assume that $g_n \longrightarrow g$ pointwise. On the other hand, $|f| < +\infty$ μ-a.e. since $fg \in L^1(X, \mu)$. It follows that $|f| \, |g - g_n| \longrightarrow 0$ pointwise a.e. Now passing to a suitable subsequence of the sequence $|g - g_n|$ if necessary and by Lemma 1.4 we have

$$N_q\Big(\sum_{n=1}^{\infty} |g - g_n|\Big) \leq \sum_{n=1}^{\infty} N_q(|g - g_n|) < +\infty.$$

Therefore it follows that the function $\sum_{n=1}^{\infty} |g - g_n|$ is in $L^q(X, \mu)$ and consequently $f \sum_{n=1}^{\infty} |g - g_n| \in L^1(X, \mu)$. But since $f \sum_{n=1}^{\infty} |g - g_n| \geq |f| \, |g - g_n|$ for each n, by the Lebesgue Dominated Convergence Theorem we have

$$\int_X |f| \, |g - g_n| \, d\mu \underset{n \to \infty}{\longrightarrow} 0.$$

From here, given $\epsilon > 0$ we have

$$\begin{aligned} \left| a - \int_X fg d\mu \right| &\leq \left| a - \int_X fg_n d\mu \right| + \left| \int_X fg_n d\mu - \int_X fg d\mu \right| \\ &\leq \left| a - \int_X fg_n d\mu \right| + \int_X |f| \, |g - g_n| \, d\mu \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

for n sufficiently large, where

$$\left| a - \int_X f g_n d\mu \right| < \frac{\epsilon}{2}$$

follows immeadiately, and

$$\int_X |f| \, |g - g_n| \, d\mu < \frac{\epsilon}{2}$$

follows by the previous paragraph. Hence $\int_X fg d\mu = a$, and the functional has a closed graph. Therefore by the Closed Graph Theorem, ϕ_f is a bounded linear functional.

Now an application of the Riesz Representation Theorem gives

$$\phi_f(g) = \int_X \tilde{f} g \, d\mu$$

for some $\tilde{f} \in L^p(X, \mu)$. It follows that $\tilde{f} = f$, and $f \in L^p(X, \mu)$.

As a second proof, we note that since (X, Σ, μ) is σ-finite, there exists a sequence $X_n \in \Sigma$ such that $X = \bigcup_{n \geq 1} X_n$, $\mu(X_n) < \infty$, and $X_n \subseteq X_{n+1}$ for all $n \geq 1$. Then $\chi_{X_n} \in L^q(X, \mu)$ and hence $f \chi_{X_n} \in L^1(X, \mu)$ for all $n \geq 1$, in particular f is μ-measurable. For each natural number n let $A_n := \{x \in X \mid |f(x)| \leq n\} \in \Sigma$. Then, letting $f_n := f \chi_{X_n \cap A_n}$ we have $f_n \in L^p(X, \mu)$ for all $n \geq 1$. Thus, identifying f_n with the bounded linear functional $L^q(X, \mu) \ni g \mapsto \int_X f_n g \, d\mu =: \Phi_n(g) \in \mathbb{C}$, we have $\|\Phi_n\| = \|f_n\|_p$ for all $n \geq 1$. On the other hand, for any $g \in L^q(X, \mu)$ we have

$$\sup_{n \geq 1} \Big| \int_X f_n g \, d\mu \Big| = \sup_{n \geq 1} \Big| \int_{X_n \cap A_n} f g \, d\mu \Big| \leq \sup_{n \geq 1} \int_{X_n \cap A_n} |fg| \, d\mu \leq \int_X |fg| \, d\mu < \infty,$$

hence, by the Principle of Uniform Boundedness it follows that

$$\sup_{n \geq 1} \|f_n\|_p = \sup_{n \geq 1} \|\Phi_n\| < \infty.$$

Thus, by the Monotone Convergence Theorem we have

$$\int_X |f|^p \, d\mu = \sup_{n \geq 1} \int_X |f_n|^p \, d\mu < \infty,$$

hence $f \in L^p(X, \mu)$. $\square$

Example 1.6. For a measure space (X, Σ, μ) as in Definition 1.1, we have

$$\int_X^{\oplus} C_x \, d\mu(x) = L^2(X, \mu)$$

where $C_x := \mathbb{C}$ with its natural inner product.

In order to show this, we begin by noting that elements of $L^2(X, \mu)$ are functions themselves satisfying D1 of Definition 1.1. For D2, it follows from the definition of $L^2(X, \mu)$ and the Hölder inequality that,

$$x \mapsto \langle g(x), h(x) \rangle_{\mathbb{C}} = g(x)\overline{h(x)}$$

is μ-integrable for all $g, h \in H$. For D3, given $g_x \in H_x$ for every x, the condition that the function $x \mapsto \langle g_x, h(x) \rangle$ is μ-integrable for every $h \in H$ implies that the function $X \ni x \mapsto g_x$ is in $L^2(X, \mu)$ by Lemma 1.5. Hence D3 is satisfied, and we have the example.

Example 1.7. The direct sum of separable Hilbert spaces H_n can be expressed as a direct integral over the space $(\mathbb{N}, \mu)$ where μ is the counting measure. Clearly, elements of $H := \bigoplus_{n=1}^{\infty} H_n$ are functions of the form $n \to h(n) \in H_n$. We have

$$\langle h, g \rangle_H = \sum_{n=1}^{\infty} \langle h(n), g(n) \rangle_{H_n} = \int_{\mathbb{N}} \langle h(n), g(n) \rangle_{H_n} d\mu(n).$$

Hence D1 and D2 are satisfied.

For D3, assume $g_n \in H_n$ are such that $\sum_{n=1}^{\infty} \left| \langle g_n, h(n) \rangle \right| < +\infty$ for every $h \in H$. Given any $x \in l^2(\mathbb{N})$ let

$$\tilde{h}(n) := \begin{cases} (g_n \|g_n\|)\overline{x(n)}, & g_n \neq 0 \\ 0, & g_n = 0. \end{cases}$$

Then $\tilde{h}(n) \in H_n$, moreover, $\tilde{h} \in H$ since

$$\sum_{n=1}^{\infty} \|\tilde{h}(n)\|^2 \leq \sum_{n=1}^{\infty} |x(n)|^2 < +\infty.$$

Therefore, by assumption the integral $\int_{\mathbb{N}} \langle g_n, \tilde{h}(n) \rangle_{H_n} \, d\mu(n)$ exists. Moreover we have the following calculation:

$$\int_{\mathbb{N}} \langle g_n, \tilde{h}(n)\rangle_{H_n}\, d\mu(n) = \sum_{n=1}^{\infty} \langle g_n, \tilde{h}(n)\rangle_{H_n}$$
$$= \sum_{n=1}^{\infty} \|g_n\|\, x(n) = \int_{\mathbb{N}} \|g_n\|\, x(n)\, d\mu(n).$$

Applying Lemma 1.5 to the space $l^2(\mathbb{N}) \ni x$, we conclude that $n \to \|g_n\|$ is in $l^2(\mathbb{N})$, and therefore the function $n \to g_n$ is in H. Hence D3 holds, and the example follows.

Example 1.8. (Due to Example 1, p. 217 in [5]) Let (X, Σ, μ) be a measure space as in Definition 1.1 and such that the space $L^2(X, \mu)$ is separable. Let $H_x := H_0$ for all $x \in X$, where H_0 is a fixed separable Hilbert space. Define a measurable set of functions as

$$\tilde{H} := \left\{ h : X \to \bigcup_{x \in X} H_x \,\middle|\, h(x) \in H_x \ \forall x \text{ and } \int_X \|h(x)\|^2 d\mu(x) < +\infty \right\}.$$

Identifying two functions which agree μ-a.e. we get the direct integral H of $(H_x)_x$ over the space (X, Σ, μ) with the inner product

$$\langle h, g\rangle_H = \int_X \langle h(x), g(x)\rangle_{H_0}\, d\mu(x).$$

Moreover we have that H is isometrically isomorphic to the Hilbert space tensor product $G := L^2(X, \mu) \otimes H_0$.

In order to show that the inner product is well defined we note that the function $X \ni x \mapsto \langle h(x), g(x)\rangle_{H_0}$ is μ-measurable. It is also μ-integrable by the Cauchy Schwarz Inequality applied twice, namely

$$\big|\langle h(x), g(x)\rangle\big| \le \|h(x)\|\|g(x)\|$$

and

$$\int_X \|h(x)\|\|g(x)\|\, d\mu(x) \le \left(\int_X \|h(x)\|^2\, d\mu(x)\right)^{\frac{1}{2}} \left(\int_X \|g(x)\|^2\, d\mu(x)\right)^{\frac{1}{2}}.$$

H is complete with respect to this inner product. The following proof comes from [7], Part II. Let $(h_n)_{n=1}^{\infty}$ be a Cauchy sequence in H. It is enough to show that a subsequence $h_{n_k}(x)$ converges to an element $h(x)$ for almost every x, and that
$\|h - h_{n_k}\| \longrightarrow 0$. By passing to a subsequence if necessary, assume that $\sum_{n=1}^{\infty} \|h_{n+1} - h_n\| < +\infty$. Then we have $\sum_{n=1}^{\infty} \|h_{n+1}(x) - h_n(x)\| < +\infty$ except for $x \in N$ with $\mu(N) = 0$, by Lemma 1.4. Then we have that the series $h_1(x) + \sum_{n=1}^{\infty} h_{n+1}(x) - h_n(x)$ converges, since it is Cauchy in H_x, for $x \notin N$. Calling its limit as $h(x)$, we have

$$\|h(x)\| \leq \|h_1(x)\| + \sum_{n=1}^{\infty} \|h_{n+1}(x) - h_n(x)\|.$$

Put $h(x) = 0$ for $x \in N$. Then the function $x \to h(x)$ is μ-measurable since it is the almost everywhere limit of measurable functions. By Lemma 1.4 again, we have $\int_X \|h(x)\|^2 \, d\mu(x) < +\infty$, implying $h \in H$ and

$$\|h - h_M\| \leq \sum_{n=M}^{\infty} \|h_{n+1} - h_n\|$$

showing that h is the limit of h_n, and therefore H is a Hilbert space.

On the other hand, since both $L^2(X, \mu)$ and H_0 are separable Hilbert spaces, G is a separable Hilbert space. The inner product on G is given by

$$\langle h_1 \otimes g_1, h_2 \otimes g_2 \rangle := \langle h_1, h_2 \rangle_{L^2} \langle g_1, g_2 \rangle_{H_0}$$

for the simple tensors and is extended by linearity to the whole G.

Now to show that H and G are isometrically isometric, define a map from G to H as

$$\sum_{i=1}^{\infty} h_i \otimes g_i \to \sum_{j=1}^{\infty} h_j g_j$$

where $\sum_{j=1}^{\infty} h_j g_j$ is the function $\left(x \to \sum_{j=1}^{\infty} h_j(x) g_j\right)$.

We show that this map is well defined. Let us consider the space of finite linear combinations of simple tensors, and call it G_0. Then G_0 is dense in G. Consider the restriction of this map to G_0. We show that this restriction is an onto isometry, and therefore by the completeness of H and G the map is well defined:

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} h_i \otimes g_i \right\|_G^2 &= \left\langle \sum_{i=1}^{n} h_i \otimes g_i, \sum_{i=1}^{n} h_i \otimes g_i \right\rangle_G \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle h_i \otimes g_i, h_j \otimes g_j \rangle_G \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \langle h_i, h_j \rangle_{L^2} \langle g_i, g_j \rangle_{H_0} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\int_X h_i(x) \overline{h_j(x)}\, d\mu(x) \right) \langle g_i, g_j \rangle_{H_0} \\
&= \left\langle \sum_{i=1}^{n} h_i g_i, \sum_{j=1}^{n} h_j g_j \right\rangle_H \\
&= \left\| \sum_{i=1}^{n} h_i g_i \right\|_H^2 .
\end{aligned}
$$

Hence the map is an isometry on G_0. To show that it has a dense image H_0 in H, assume by contradiction that there exists $h \in H$ such that $h \perp \sum_{i=1}^{n} h_i(x) g_i$ for all n. Let $\left(f_j\right)_{j=1}^{\infty}$ be an orthonormal basis for H_0, and let $\left(e_j\right)_{j=1}^{\infty}$ be an orthonormal basis for $L^2(X, \mu)$. Let $h(x) = \sum_{j=1}^{\infty} \lambda_j(x) f_j$ be the Fourier expansion of $h(x)$ for all $H_x = H_0$. We note that for any k, l, $e_k(x) f_l \in H_0$. Then we have

$$0 = \langle h, e_k(x) f_l \rangle = \int_X \langle h(x), e_k(x) f_l \rangle \, d\mu(x) = \int_X \lambda_l(x) \overline{e_k(x)}\, d\mu(x).$$

It follows that $\langle \lambda_l(x), s(x) \rangle = 0$ for any $s(x) \in L^2(X, \mu)$. But we have

$$\int_X \|h(x)\|^2 \, d\mu(x) = \int_X \sum_{l=1}^{\infty} \left| \lambda_l(x) \right|^2 d\mu(x)$$

and it follows that $\lambda_l(x) \in L^2(X, \mu)$ and then clearly $\lambda_l(x) = 0$ μ-a.e. for any l. Hence $h(x) = 0$ μ-a.e. and by Remark 1.2 $h = 0$. Therefore the restriction of the map is an isometry of G_0 onto H_0, and it is an isometric isomorphism between G and H.

Now we verify that H is a direct integral. D1 of Definition 1.1 is clearly satisfied since elements of H are themselves vector valued functions. D2 is clear by above. For D3, assume that a function $x \to h_x \in H_x$ is such that $\langle h_x, g(x)\rangle$ is integrable for all $g \in H$. Let $\left(g_j\right)_{j=1}^{\infty}$ be a set whose linear span has closure H. Since by Proposition 1.3 $\left(g_j(x)\right)_{j=1}^{\infty}$ spans $H_x = H_0$ for μ almost all x, by an application of the Gram-Schmidt orthonormalization process, we can assume that $\left(g_j(x_0)\right)_{j=1}^{\infty}$ is an orthonormal basis of H_0 for some $x_0 \in X$. Let $h_x = \sum_{i=1}^{\infty} \beta_i(x) g_i(x_0)$ be the Fourier expansion of h_x for every x. Consider the element $x \to \sum_{i=1}^{n} \beta_i(x) g_i(x_0)$ of H. Then $\langle h_x, \sum_{i=1}^{n} \beta_i(x) g_i(x_0)\rangle$ is integrable, and we have

$$\int_X \left\langle h_x, \sum_{i=1}^{n} \beta_i(x) g_i(x_0) \right\rangle d\mu(x) = \int_X \sum_{i=1}^{n} \left|\beta_i(x)\right|^2 d\mu(x) < +\infty$$

for every n, from which it follows that $\beta_j \in L^2(X,\mu)$ for every j. Therefore it is clear that the function $x \mapsto \sum_{i=1}^{\infty} \beta_i(x) g_i(x_0) = h_x$ is a representative of the element of H corresponding to the tensor product $\sum_{k=1}^{\infty} \beta_k \otimes g_k(x_0)$, i.e. there is $h \in H$ such that $h_x = h(x)$ μ-a.e. and D3 is satisfied. Hence H is a direct integral.

Proposition 1.9. *Let $H := \int_X^{\oplus} H_x \, d\mu(x)$ be a direct integral Hilbert space. Let*

$$X_n := \left\{ x \in X \,\middle|\, dim\, H_x = n \right\}.$$

Then X_n is measurable for each $n \geq 1$.

Proof. Let $\{h_j\}$ be a countable orthonormal bases for H. Let $\{r_1, r_2, \ldots\}$ be an enumaration of the complex rationals such that $r_1 = 1$. Let $l := (l_1, l_2, \ldots, l_n)$ be an n-tuple of natural numbers with some $l_n = 1$, $k := (k_1, k_2, \ldots, k_n)$ be an n-tuple of distinct natural numbers, and $m \in \mathbb{Z}^+$. Define the set

$$X_{l,k,m} := \left\{ x \in X \,\middle|\, \|r_{l_1} h_{k_1}(x) + \ldots + r_{l_n} h_{k_n}(x)\| < m^{-1} \right\}.$$

Then each $X_{l,k,m}$ is measurable.

By Proposition 1.3, except on a set X_0 of measure zero, we have $\overline{\text{span}\,\{h_j(x)\}} = H_x$. But fixing an $x \in X$ such that the corresponding H_x has dimension strictly less than n, for every $\{h_{k_1}(x), \ldots, h_{k_n}(x)\}$ there exists complex numbers $q_1, \ldots, q_n$, not all zero, such that $\sum_{i=1}^{n} q_i h_{k_i}(x) = 0$. Clearly, this sum can be approximated arbitrarily close by a linear combination of form $\sum_{j=1}^{n} r_{l_j} h_{k_j}(x)$. It follows that for $x \notin X_0$, we have

$$\bigcup_{i=1}^{n-1} X_i = \bigcap_{k,m} \bigcup_{l} X_{l,k,m}$$

and therefore X_n is measurable for any n. $\square$

Chapter 2

Decomposable Operators

In this section we discuss the notion of decomposable operators, following [13].

Definition 2.1. Let $H = \int_X^{\oplus} H_x \, d\mu(x)$ be a direct integral and let $T \in \mathcal{B}(H)$, with $\mathcal{B}(H)$ denoting the set of bounded linear operators $B : H \to H$. Then T is said to be decomposable if there is a function $x \mapsto T(x)$, called a decomposition of T, such that $T(x) \in \mathcal{B}(H_x)$ for every $x \in X$ and $T(x)h(x) = (Th)(x)$ holds for μ-a.e.-x and for every $h \in H$. By Proposition 2.7 below, we will see that the function $x \mapsto \|T(x)\|$ is measurable and essentially bounded, so as another notation $T = \int_X^{\oplus} T(x) \, d\mu(x)$ is also used, due to, for instance [7], Part II.

If in addition for all $x \in X$ we have that $T(x) = f(x)I_x$ where $f : X \to \mathbb{C}$ is a function such that $f \in L^\infty(X, \mu)$ and I_x is the identity operator on H_x, then T is called a diagonalizable operator.

Example 2.2. Let $(H_n)_{n=1}^\infty$ be separable Hilbert spaces, and let $A_n \in \mathcal{B}(H_n)$ be operators such that $A_n = f_n I_n$ for every n, where $(f_n)_{n=1}^\infty \subset \mathbb{C}$ is a bounded sequence, and I_n is the identity operator of H_n, with the property that $\sup \|A_n\| < +\infty$. By Example 1.7 the direct sum $H := \oplus_{n=1}^\infty H_n$ is a direct integral Hilbert space. It follows that the operator $A := \oplus_{n=1}^\infty A_n \in \mathcal{B}(H)$ and A is diagonalizable.

Example 2.3. Let $H = \int_X^{\oplus} H_x \, d\mu(x)$ be a direct integral, and $f \in L^\infty(X, \mu)$. Then clearly the function $x \mapsto \langle f(x)h(x), g(x)\rangle_{H_x}$ is integrable for every $h, g \in H$. Therefore there exists $z \in H$ such that $f(x)h(x) = z(x)$ μ-a.e. Define $M_f : H \to$

H to be $M_f(h) := z$. Then M_f is a diagonalizable operator with decomposition $x \to f(x)I_x$, where I_x is the identity operator on H_x. In particular, if $f = \chi_{X_0}$ is a characteristic function of some measurable subset $X_0 \subset X$, then M_f is the projection corresponding to X_0, and is diagonalizable.

Proof. Clearly the operator M_f is linear. Also we have

$$\begin{aligned}\|M_f(h)\|^2 = \|z\|^2 &= \int_X \langle f(x)h(x), f(x)h(x)\rangle \, d\mu(x) \\ &= \int_X \left|f(x)\right|^2 \langle h(x), h(x)\rangle \, d\mu(x) \le M\|h\|^2\end{aligned}$$

for some $M \ge 0$. Therefore M_f is bounded. But $M_f(h)(x) = z(x) = f(x)h(x) = (f(x)I_x)\, h(x)$. Hence M_f is diagonalizable with decomposition $x \mapsto f(x)I_x$. $\square$

Proposition 2.4. *Let $x \mapsto T(x)$ and $x \mapsto T'(x)$ be two decompositions of $T \in \mathcal{B}(H)$, where $H = \int_X^\oplus H_x \, d\mu(x)$. Then $T(x) = T'(x)$ for μ-a.e.-x. Conversely if $T(x) = S(x)$ μ-a.e. for two decomposable operators $T, S \in \mathcal{B}(H)$, then $T = S$.*

Proof. By Proposition 1.3, except on a null set N_0, $\{h_j(x)\}_{j=1}^\infty$ spans H_x where $\{h_j\}_{j=1}^\infty$ is an orthonormal basis of H. We have $T(x)\,(h_j(x)) = (Th_j)\,(x)$ except on a null set N_j' for each j. Similarly we have $T'(x)\,(h_j(x)) = (Th_j)\,(x)$ except on a null set N_j''. Let $N_j := N_j' \cup N_j''$. Then letting $N := \bigcup_{j=0}^\infty N_j$, we have that the equality

$$T(x)\,(h(x)) = (Th)\,(x) = T'(x)\,(h(x))$$

holds everywhere except on N, where $\mu(N) = 0$, i.e. $T(x) = T'(x)$ μ-a.e.

For the converse implication we have the equality

$$\begin{aligned}\langle Th, g\rangle_H &= \int_X \langle Th(x), g(x)\rangle_{H_x} \, d\mu(x) = \int_X \langle T(x)h(x), g(x)\rangle \, d\mu(x) \\ &= \int_X \langle S(x)h(x), g(x)\rangle \, d\mu(x) = \int_X \langle Sh(x), g(x)\rangle \, d\mu(x) = \langle Sh, g\rangle\end{aligned}$$

for any $h, g \in H$. Hence $T = S$. $\square$

Next we have some algebraic properties of decomposable operators in the following proposition.

Proposition 2.5. *Let $H = \int_X^{\oplus} H_x \, d\mu(x)$, and $T_1, T_2 \in \mathcal{B}(H)$ be decomposable operators. Then the operators $aT_1 + T_2$ for $a \in \mathbb{C}$, T_1^*, T_2^*, T_1T_2 and the identity operator I are decomposable. Moreover, a decomposition for each of the operators above are given by*

i) $(aT_1 + T_2)(x) = aT_1(x) + T_2(x)$

ii) $(T_1T_2)(x) = T_1(x)T_2(x)$

iii) $T_1^*(x) = T_1(x)^*$

iv) $I(x) = I_x$ *where* I_x *is the identity operator for* H_x

We also have the following property

v) If $T_1(x) \leq T_2(x)$ *μ-a.e. then* $T_1 \leq T_2$.

Proof. i) Define $(aT_1 + T_2)(x) := aT_1(x) + T_2(x)$ for every $x \in X$. Then for every $h \in H$ we have the equalities

$$\begin{aligned}(aT_1 + T_2)(x)h(x) &= aT_1(x)h(x) + T_2(x)h(x) = a(T_1h)(x) + (T_2h)(x) \\ &= (aT_1h + T_2h)(x) = (aT_1 + T_2)(h)(x)\end{aligned}$$

where the second equality holds for μ-a.e.-x by the definition of a decomposition, and the third equality follows by Remark 1.2. Therefore the equality $(aT_1 + T_2)(x)h(x) = (aT_1 + T_2)(h)(x)$ holds μ-a.e. and it follows that $aT_1 + T_2$ is decomposable with decomposition $x \mapsto aT_1(x) + T_2(x)$.

ii) Similar to i), define $(T_1T_2)(x) := T_1(x)T_2(x)$. Then for any $h \in H$ we have

$$\begin{aligned}(T_1T_2)(x)h(x) &= (T_1(x)T_2(x))\,h(x) = T_1(x)\,(T_2(x)h(x)) \\ &= T_1(x)\,((T_2h)(x)) = T_1T_2h(x)\end{aligned}$$

where the second and third equalities are by the definition of a decomposition, and they hold μ-a.e. So we have $(T_1T_2)(x)h(x) = (T_1T_2h)(x)$ μ-a.e. and it follows that T_1T_2 is decomposable with decomposition $x \mapsto T_1(x)T_2(x)$.

iii) Define $T_1^*(x) := T_1(x)^*$. Then we have

$$\langle T_1^*(x)h(x), g(x)\rangle_{H_x} = \langle h(x), T_1(x)g(x)\rangle = \langle h(x), (T_1g)(x)\rangle$$

which holds μ-a.e. for every $h, g \in H$. But the function $x \mapsto \langle h(x), (T_1 g)(x)\rangle_{H_x}$ is μ-integrable by D2 of Definition 1.1, and hence by D3, there exists $z \in H$ such that $T_1^*(x)h(x) = z(x)$ μ-a.e. Now we have

$$\begin{aligned}\langle T_1^* h - z, g\rangle_H &= \langle h, T_1 g\rangle - \langle z, g\rangle \\ &= \int_X \langle h(x), T_1(x)g(x)\rangle_{H_x}\, d\mu(x) - \int_X \langle T_1^*(x)h(x), g(x)\rangle\, d\mu(x) = 0\end{aligned}$$

for every $g \in H$ clearly. Hence $T_1^* h = z$. It follows that

$$(T_1^* h)(x) = z(x) = T_1^*(x)h(x) = T_1(x)^* h(x)$$

μ-a.e. Therefore T_1^* is decomposable with decomposition $x \mapsto T_1(x)^*$.

iv) Defining $I(x) := I_x$ we have $I(x)h(x) = (I_x)h(x) = h(x) = (Ih)(x)$ for every x and therefore I is decomposable with decomposition $x \mapsto I_x$.

v) We have

$$\begin{aligned}\langle T_1 h, h\rangle_H &= \int_X \langle (T_1 h)(x), h(x)\rangle_{H_x}\, d\mu(x) = \int_X \langle T_1(x)h(x), h(x)\rangle\, d\mu(x) \\ &\leq \int_X \langle T_2(x)h(x), h(x)\rangle\, d\mu(x) = \langle T_2 h, h\rangle\end{aligned}$$

μ-a.e. for every $h \in H$. It follows that $T_1 \leq T_2$. □

We also have the converse of item v) in the above proposition.

Proposition 2.6. *Let $H := \int_X^{\oplus} H_x\, d\mu(x)$. Let A_1, A_2 be decomposable self adjoint operators such that $A_1 \leq A_2$. Then $A_1(x) \leq A_2(x)$ μ-a.e.*

Proof. By Proposition 2.5, the operator $A_2 - A_1$ is decomposable with decomposition $A_2(x) - A_1(x)$. Therefore it sufficies to show that if a positive operator $A \geq 0$ is decomposable, then $A(x) \geq 0$.

Since H is a separable Hilbert space, choose a countable dense set in H, and consider its linear span over the rationals, to get a countable linear set $\{h_j\}_{j=1}^{\infty}$. By Proposition 1.3 we have that $\{h_j(x)\}_{j=1}^{\infty}$ spans H_x except for $x \in N_0$ where $\mu(N_0) = 0$.

We now show that the set $\{h_j(x)\}_{j=1}^{\infty}$ is actually dense in H_x except on some set of μ measure zero: Consider a linear combination $r_1h_1 + r_2h_2 + \ldots + r_nh_n$, where $r_1, r_2, \ldots, r_n \in \mathbb{Q}$. It is easy to see that there is h_j which is equal to this sum. By Remark 1.2 $r_1h_1(x)+r_2h_2(x)+\ldots+r_nh_n(x) = h_j(x)$ μ-a.e. But there are at most countably many such rational complex linear combination. Enumarate them, and then let $N_1, N_2, \ldots$ be the sets where this equation does not hold. Then $\mu(N_i) = 0$ for each i, and letting $N := \bigcup_{i=0}^{\infty} N_i$, it follows that $\{h_j\}_{j=1}^{\infty}$ is dense in H_x except for $x \in N$, where $\mu(N) = 0$.

Since $A \geq 0$, we have $0 \leq \langle Ah_j, h_j\rangle = \int_X \langle Ah_j(x), h_j(x)\rangle \, d\mu(x)$. By contradiction, assume that $\langle A(x)h_j(x), h_j(x)\rangle < a < 0$ for $x \in X_0$ for some set $X_0 \subset X$ with $0 < \mu(X_0) < +\infty$. Letting f be the characteristic function of the set X_0, we have that the function $x \mapsto \langle f(x)h_j(x), g(x)\rangle$ is μ-integrable for every $g \in H$. Hence there exists $z_j \in H$ such that $z_j(x) = f(x)h_j(x)$ μ-a.e. for every j. Then we have

$$\begin{aligned}\langle Az_j, z_j\rangle_H &= \int_X \langle A(x)f(x)h_j(x), f(x)h_j(x)\rangle_{H_x} \, d\mu(x) \\ &= \int_{X_0} \langle A(x)h_j(x), h_j(x)\rangle \, d\mu(x) \leq a\mu(X_0) < 0\end{aligned}$$

which is a contradiction. Hence $\mu(X_0) = 0$, and consequently for each j we have $0 \leq \langle A(x)h_j(x), h_j(x)\rangle$ except for $x \in M_j \subset X$, where $\mu(M_j) = 0$. Let $M := \bigcup_{j=1}^{\infty} M_j$.

Now if $x \notin N \cup M$, then we have $0 \leq \langle A(x)h_j(x), h_j(x)\rangle$ for every j and $\{h_j(x)\}_{j=1}^{\infty}$ is dense in H_x. It follows that $0 \leq A(x)$ for $x \notin N \cup M$, where $\mu(N \cup M) = 0$, and the proposition is shown. $\square$

We have another proposition which makes use of the proof of the preceeding proposition.

Proposition 2.7. *Let $H := \int_X^{\oplus} H_x \, d\mu(x)$. If the operator $T \in \mathcal{B}(H)$ is decomposable, then the function $x \mapsto \|T(x)\|$ is in $L^{\infty}(X, \mu)$, and it has essential bound $\|T\|$.*

Proof. Since T is decomposable, by Proposition 2.5 the operator T^* and consequently the positive operator T^*T are also decomposable with decompositions $x \mapsto T^*(x)$ and $T^*(x)T(x)$ respectively. But $\|T(x)\|^2 = \|T^*(x)T(x)\|$. Hence it is enough to show the proposition for a positive decomposable operator S.

In order to show that $x \mapsto \|S(x)\|$ is a measurable function, let $s \in \mathbb{Q}$ with $s > 0$. Let $\{h_j\}_{j=1}^{\infty}$ and the set $N \subset X$ be as in Proposition 2.6. Define the set

$$X_s := \left\{x \in X \,\middle|\, x \notin N,\ H(x) \leq sI_x\right\}.$$

Then we have

$$X_s := \bigcap_{j=1}^{\infty} \left\{x \in X \,\middle|\, x \notin N, \langle S(x)h_j(x), h_j(x)\rangle \leq s\|h_j(x)\|^2\right\}$$

which follows by the density of $\{h_j(x)\}_{j=1}^{\infty}$ in H_x for $x \notin N$.

Since $\langle S(x)h_j(x), h_j(x)\rangle$ is a Borel function of x for each j, each set appearing in the intersection is Borel. Therefore X_s is Borel. Now we observe that $\|S(x)\| \in (a,b)$ for any $0 \leq a < b \leq +\infty$ if and only if there are $q, r \in \mathbb{Q}$ such that $a \leq q < r \leq b$ with $S(x) \nleq qI_x$ and $S(x) \leq rI_x$. That is, we have $\|S(x)\| \in (a,b)$ if and only if there exists $a \leq q < r \leq b$ with $x \in X_r \setminus X_q$. Since there are at most countable number of pairs (q,r), it follows that the set $\left\{x \in X \middle| x \notin N, \|S(x)\| \in (a,b)\right\}$ is Borel and hence $x \mapsto \|S(x)\|$ is measurable.

For essentially boundedness, note that $0 \leq S \leq \|S\|I$, therefore it follows by Proposition 2.6 that $0 \leq S(x) \leq \|S\|I_x$ μ-a.e. Conversely, if $0 \leq S(x) \leq aI_x$ μ-a.e, then by Proposition 2.5 $0 \leq S \leq aI$, so that $\|S\| \leq a$. Hence the function $x \mapsto \|S(x)\|$ has essential bound $\|S\|$. □

The following two theorems give information on further properties of decomposable operators on a direct integral.

Theorem 2.8. *Let $H := \int_X^{\oplus} H_x \, d\mu(x)$. Then the set $\mathcal{R}$ of decomposable operators on H is a von Neumann algebra.*

Proof. By Proposition 2.5, $\mathcal{R}$ is a C^*-subalgebra containing the identity operator I. In particular it is convex. Therefore it is strongly operator closed if and only if weakly operator closed, a well known fact which can be found for instance in [12], Theorem 5.1.2. Hence it is enough to show that $\mathcal{R}$ is strongly operator closed to conclude that it is a von Neumann algebra.

Let $\|T\| = 1$ with $T \in \overline{\mathcal{R}}$ in the strong operator topology. By the Kaplansky Density Theorem, there exists a sequence of operators $T_n \in \mathcal{R}$ such that $\|T_n\| = 1$ with $T_n \longrightarrow T$ in the strong operator topology. In particular, let $G := \{h_j\}_{j=1}^{\infty} \subset H$ be a countable set whose linear span has closure H. Then $T_n h_j \longrightarrow Th_j$ for every j. Therefore we have

$$\|T_n h_j - Th_j\|_H^2 = \int_X \|T_n(x)h_j(x) - (Th_j)(x)\|_{H_x}^2 \, d\mu(x) \underset{n\to\infty}{\longrightarrow} 0$$

using a decomposition $x \mapsto T_n(x)$.

Since the above is L^2-convergence in particular, there exists a subsequence $T_{n1} \subset T_n$ such that $\|T_{n1}(x)h_1(x) - (Th_1)(x)\| \longrightarrow 0$ μ-a.e. by the L^2 space property used in the previous chapter. By applying the same to the subsequence T_{n1} in place of T_n, there exists a subsequence $T_{n2} \subset T_{n1}$ such that $\|T_{n2}(x)h_2(x) - (Th_2)(x)\| \longrightarrow 0$. Continuing in this manner, we get a sequence of subsequences. By the Cantor diagonalization argument choose the elements T_{nn} from each subsequence to obtain a sequence $T_{11}, T_{22}, \ldots$. Then we have

$$\|T_{nn}(x)h_j(x) - (Th_j)(x)\| \underset{n\to\infty}{\longrightarrow} 0$$

μ-a.e. for each j.

Now we see that there is a null set N such that the following hold for $x \notin N$:

i) Closure of linear span of the set $G_x := \{h_j(x)\}_{j=1}^{\infty} \subset H$ equals to H_x by Proposition 1.3.

ii) $\|T_{nn}(x)\| \leq 1$ noting that a decomposition $x \mapsto \|T_n(x)\|$ is essentially bounded with $\|T_n\| \leq 1$ by Proposition 2.7.

iii) $\|T_{nn}(x)h_j(x) - (Th_j)(x)\| \underset{n\to\infty}{\longrightarrow} 0$ for every j.

It follows that there is an operator $T(x) \in \mathcal{B}(H_x)$ with $\|T(x)\| \leq 1$ such that

$A(x)h_j(x) = (Ah_j)(x)$ for $x \notin N$ by the following arguments: Define

$$T(x)h_j(x) := \lim_{n\to\infty} T_{nn}(x)h_j(x) = (Th_j)(x)$$

and extend the definition of $T(x)$ by linearity. Then extend it continuously to whole H, which can be done by the following: Begin by choosing a linearly independent subset of G_x whose linear span has closure H_x. This can be done by a typical application of Zorn's Lemma. Then by applying the Gram-Schmidt orthogonalization process, without loss of generality we can assume that the set G_x is a countable orthonormal basis for H_x. Let now $u_x \in H_x$ be any element, and $u_x = \sum_{j=1}^{\infty} e_j h_j(x)$ be its Fourier expansion. Then given $\epsilon > 0$ we have

$$\begin{aligned}\left\| T(x)\left(\sum_{j=m_1}^{m_2} e_j h_j(x)\right)\right\|_{H_x} &= \left\| \lim_{n\to\infty} T_{nn}(x)\left(\sum_{j=m_1}^{m_2} e_j h_j(x)\right)\right\| \\ &\leq \left\| \sum_{j=m_1}^{m_2} e_j h_j(x)\right\| \\ &\leq \left(\sum_{j=m_1}^{\infty} \left|e_j\right|^2\right)^{\frac{1}{2}} < \epsilon\end{aligned}$$

for each linear combination $\sum_{j=m_1}^{m_2} e_j h_j(x)$ with sufficiently big m_1 and any $m_2 > m_1$, which follows by $\|T_{nn}(x)\| \leq 1$ and the Bessel's inequality. Therefore the sequence

$$\left\{ T(x)\left(\sum_{j=1}^{m} e_j h_j(x)\right)\right\}_{m=1}^{\infty}$$

is Cauchy in H_x and it has a limit. Define

$$T(x)u_x := \lim_{m\to\infty} T(x)\left(\sum_{j=1}^{m} e_j h_j(x)\right).$$

Therefore the operator $T(x)$ is defined on whole H_x. Now we show that $T(x)$ is in the closed unit ball of $\mathcal{B}(H_x)$. Note that

$$\|T(x)u_x\|_{H_x} = \left\| \lim_{m\to\infty}\lim_{n\to\infty} T_{nn}(x)\left(\sum_{j=1}^{m} e_j h_j(x)\right)\right\|_{H_x} \leq \left(\sum_{j=1}^{\infty} \left|e_j\right|^2\right)^{\frac{1}{2}} = \|u_x\|$$

since

$$\left\| T_{nn}(x)\left(\sum_{j=1}^{m} e_j h_j(x)\right)\right\|_{H_x} \leq \left\| \sum_{j=1}^{m} e_j h_j(x)\right\| = \left(\sum_{j=1}^{m} \left|e_j\right|^2\right)^{\frac{1}{2}} \leq \left(\sum_{j=1}^{m} \left|e_j\right|^2\right)^{\frac{1}{2}}$$

for every m, n.

Now let $F := \{g_i\}_{i=1}^{\infty} \subset H$ be the set of rational linear combinations of elements of G. Then F is countable and dense. Let $u \in H$ be any element. Then there is a subsequence g_{i_1} in F which tends to u in the norm of H. By the L^2-subsequence argument we have a subsubsequence g_{i_2} such that $\|g_{i_2}(x) - u(x)\|_{H_x} \longrightarrow 0$, except on a null set M_1. But then we have $Tg_{i_2} \longrightarrow Tu$. Applying the same argument we have a third subsequence g_{i_3} such that $\|(Tg_{i_3})(x) - (Tu)(x)\| \longrightarrow 0$ except on a null set M_2. Let $M := M_1 \cup M_2$. Then we have $(Tg_{i_3})(x) \longrightarrow (Tu)(x)$ and $T(x)g_{i_3}(x) \longrightarrow T(x)u(x)$. But $(Tg_{i_3})(x) = T(x)g_{i_3}(x)$. It follows that $T(x)u(x) = (Tu)(x)$ for $x \notin M \cup N$. Hence T is decomposable, and $\mathcal{R}$ is a von Neumann algebra. $\square$

Remark 2.9. The same arguments as in the proof of Theorem 2.8 shows that the algebra $\mathcal{C}$ of diagonalizable operators is a von Neumann algebra.

Theorem 2.10. *Let $H := \int_X^{\oplus} H_x \, d\mu(x)$. Then the von Neumann algebra $\mathcal{R}$ of decomposable operators has abelian commutant $\mathcal{R}'$ which coincides with the family $\mathcal{C}$ of diagonalizable operators.*

Proof. Assume that an operator $S \in \mathcal{B}(H)$ is diagonalizable with decomposition $f(x)I_x$ and $T \in \mathcal{B}(H)$ is decomposable with $x \mapsto T(x)$. Then by Proposition 2.5 the operators ST and TS have decompositions $f(x)I_xT(x)$ and $T(x)f(x)I_x$ respectively, i.e. they have the same decompositions. It follows by Proposition 2.5 that $ST = TS$ and $T \in \mathcal{C}'$. Hence $\mathcal{R} \subset \mathcal{C}'$.

To obtain $\mathcal{R}' = \mathcal{C}$, it will be shown that $\mathcal{R} = \mathcal{C}'$. By above we only need to have $\mathcal{C}' \subset \mathcal{R}$. Since by the remark above $\mathcal{C}$ is a von Neumann algebra, then we will have $\mathcal{R}' = \mathcal{C}'' = \mathcal{C}$ by von Neumann Double Commutant Theorem. By the remark it also follows that $\mathcal{C}'$ is a von Neumann algebra. But $\mathcal{R}$ is a von Neumann algebra by Theorem 2.8. Hence it suffices to show that every projection $P \in \mathcal{C}'$ belongs to $\mathcal{R}$.

Let $U := \{u_j\}$ and $V := \{v_j\}$ be orthonormal bases for $P(H)$ and $(I - P)(H)$ respectively. Let $G := \{h_i\}$ be the set of all finite rational linear combinations of the elements of U, V. As in the proof of Proposition 2.6, it follows that the set $G_x := \{h_i(x)\}$ is dense in H_x except for $x \in N$ where $\mu(N) = 0$. Define projection $P(x)$ with range the closure of the linear span of elements of $U_x := \{u_j(x)\}$ for $x \notin N$. Then letting $u \in G$ be such that u is a rational linear combination of elements of U, we have $(Pu)(x) = u(x) = P(x)u(x)$ for $x \notin N$ clearly.

Now let $v \in G$ be a rational linear combination of elements of V. Then $Pv = 0$. We show that $\langle u(x), v(x) \rangle = 0$ for μ-a.e-x. Let R be the diagonalizable projection which corresponds to a measurable set $X_0 \subset X$, as in Example 2.3, i.e. a multiplication operator M_f with $f = \chi_{X_0}$. In this case we have $PR = RP$ since $R \in \mathcal{C}$ and $P \in \mathcal{C}'$ by assumption. Then we have

$$\begin{aligned} 0 = \langle Ru, Pv \rangle_H &= \langle PRu, v \rangle = \langle RPu, v \rangle \\ &= \langle Ru, v \rangle = \int_X \langle f(x)u(x), v(x) \rangle_{H_x} \, d\mu(x) \\ &= \int_{X_0} \langle u(x), v(x) \rangle \, d\mu(x). \end{aligned}$$

Since this holds for every X_0, we have $\langle u(x), v(x) \rangle = 0$ μ-a.e.

To complete the proof, note that by above we have $\langle u_j(x), v(x) \rangle = 0$ except for $x \in M$ for some null set M. Then $0 = P(x)v(x) = (Pv)(x)$ for $x \notin N \cup M$. Hence $Ph_i(x) = P(x)h_i(x)$ for $x \notin N \cup M$. Now if $h \in H$ is any element, then there is a sequence $\{h_{i'}\} \subset \{h_{i'}\}$ such that $h_{i'} \longrightarrow h$. By the L^2 subsequence argument $(Ph)(x) = P(x)h(x)$ for $x \notin N_0 \cup N \cup M$ where N_0 is a null set. Hence $P \in \mathcal{R}$, and the proof is finished. $\square$

Chapter 3

Disintegration of Measures

In this section, the concept of disintegration of probability measures will be reviewed, following [2].

Let X be a locally compact separable metric space, considered with its Borel σ-algebra $\mathfrak{a}$. The topological assumptions on X imply the following:

i) X is second countable, i.e. it has a countable topological basis, which follows by the separability of X as a metric space.

ii) Let $C_0(X)$ be the space of continuous complex valued functions on X vanishing at infinity, with the uniform norm topology. Then $C_0(X)$ is separable. This follows by the following more general fact, which can be found in Ch.6, §3, No.1, Lemma 2 of [6]: Assume that the space Y is locally compact, Hausdorff and has countable basis. Then $C_0(Y)$ is separable.

iii) The dual of $C_0(X)$ is the space of complex Borel measures on $(X, \mathfrak{a})$, by the Riesz-Markov theorem (For instance Theorem 7.17 in [8]). Here no regularity assumption is needed since on a locally compact, Hausdorff and second countable space every complex Borel measure is a Radon measure (Theorem 7.8 in [8]). We also note that a Borel probability measure on a metric space is always regular (Theorem 1.2, p.27 in [14]).

With this setting, let μ be a probability measure on $\mathfrak{a}$. The notations $L^\infty(X, \mathfrak{a}, \mu)$, $L^1(X, \mathfrak{a}, \mu)$, $L^2(X, \mathfrak{a}, \mu)$ denote the equivalence classes of the corresponding μ-a.e. equivalent functions, and $\mathcal{L}^\infty(X, \mathfrak{a}, \mu)$, $\mathcal{L}^1(X, \mathfrak{a}, \mu)$, $\mathcal{L}^2(X, \mathfrak{a}, \mu)$ stand for the functions themselves, although members of all of these spaces will be referred as functions as is convenient. Let $\phi \in L^\infty(\mu)$ be a function. Let Y be the essential range of ϕ, and by a well known property we note that $Y \subset \mathbb{C}$ is compact. Define $\nu := \mu \circ \phi^{-1}$ on Y. Then ν is a Borel measure on Y clearly.

In the sense of [2], the expectation operator $E : \mathcal{L}^1(X, \mathfrak{a}, \mu) \to L^1(\nu)$ is defined by the equation

$$\int_X (\psi \circ \phi)\, f \, d\mu = \int_Y \psi E(f) \, d\nu \tag{3.1}$$

where $\psi \in L^\infty(\nu)$ and $f \in \mathcal{L}^1(X, \mathfrak{a}, \mu)$ are any functions. The existence of $E(f)$ is justified by the following arguments: As ψ varies over the characteristic functions of subsets of $Y \subset \mathbb{C}$, the left hand side defines a measure on Y which is absolutely continuous with respect to ν. By the Radon-Nikodym theorem, $E(f) \in L^1(\nu)$ exists as the derivative given by

$$E(f) := \left[\frac{\phi f \, d\mu}{d\nu}\right].$$

From the formula 3.1 it follows that $\|E(f)\|_1 \leq \|f\|_1$, by the following simple observation: Take function ψ_0 as follows:

$$\psi_0(y) := \begin{cases} \overline{E(f)(y)} \,/\, |E(f)(y)|\,, & E(f)(y) \neq 0 \\ 1, & E(f)(y) = 0 \end{cases}$$

Then $|\psi_0| = 1$ everywhere on Y and $\psi_0 \in L^\infty(\nu)$. By 3.1, we have

$$\left| \int_Y \psi E(f) \, d\nu \right| \leq \int_X |\psi \circ \phi| \, |f| \, d\mu \tag{3.2}$$

Inserting ψ_0 into 3.2, we get the desired inequality.

If $f \in L^\infty(\mu)$, we also have the ineqaulity

$$\left| \int_Y \psi E(f) \, d\nu \right| \leq \|f\|_\infty \|\psi\|_1$$

clearly. Hence in this case $E(f) \in L^\infty(\mu)$ and $\|E(f)\|_\infty \leq \|f\|_\infty$. From these inequalities it follows that E is a contraction from $\mathcal{L}^1(X, \mathfrak{a}, \mu)$ to $L^1(\nu)$ and its restriction is also a contraction from $\mathcal{L}^\infty(X, \mathfrak{a}, \mu)$ to $L^\infty(\nu)$. These two inequalities will be useful in what follows.

Now we define a disintegration of the measure μ above, as is given in [2]:

Definition 3.1. A disintegration of the measure μ with respect to the function $\phi \in L^\infty(\mu)$ is a function $y \mapsto \mu_y$ where μ_y is a probability measure on $\mathfrak{a}$ such that

$$E(f)(y) = \int_X f \, d\mu_y \tag{3.3}$$

holds for every $f \in \mathcal{L}^1(X, \mathfrak{a}, \mu)$, for ν-a.e. y.

Before we go into the theorem of existence and uniqueness of disintegration of measures as above, we have three topological lemmas which appear in the proof of the theorem:

Lemma 3.2. *Let X be a locally compact and separable metric space. Then there exists a countable collection of compact sets K_n which exhaust X, that is, $\cup_{n=1}^\infty K_n = X$ and such that $K_n \subseteq K_{n+1}$ for every n. In particular, X is σ-compact.*

Proof. Since X is locally compact, find a compact neighbourhood $\tilde{K}_x$ for each $x \in X$. By i) on p.5, X is second countable and we have a countable basis, call it $(B_i)_{i=1}^\infty$. Hence for every x, there is $B_{i(x)}$ such that $B_{i(x)} \subseteq \tilde{K}_x$. Since a closed subset of a compact set is compact, the closures $\overline{B_{i(x)}}$ are all compact. But $X = \bigcup_{i=1}^\infty B_i$, therefore $X = \bigcup_{i=1}^\infty \overline{B_i}$. Now the sets $K_n := \bigcup_{i=1}^n B_i$ clearly have the desired properties. □

Remark 3.3. The conclusion of Lemma 3.2 remains valid for a locally compact second countable Hausdorff space, with the same arguments.

Definition 3.4. (Definition on p.220 of [10]) The Baire sets of X are the elements of the σ-ring generated by the compact G_δ sets of X.

Lemma 3.5. *On a locally compact separable metric space the Baire sets S and the Borel sets C coincide.*

Proof. $C \supset S$: By Theorem E in Chapter X of [10], if locally compact Hausdorff space X is separable, then every compact subset of X is G_δ. Hence every compact set is Borel and the inclusion follows.

$S \supset C$: Consider any open set $U \subset X$. Since X is second countable, there exists a countable basis $(B_n)_{n=1}^{\infty}$ of open sets. For each $x \in U$, find B_n such that $x \in B_n$ with $\overline{B_n} \subset U$ with $\overline{B_n}$ compact, which can be done since B_n are basis and by the proof of Lemma 3.2 above. Taking the union of such $\overline{B_n}$ gives U clearly. Therefore U is a countable union of compact sets, and the inclusion follows. □

The following is the Urysohn's Lemma for locally compact spaces, which can be found for instance in [8], 4.32.

Lemma 3.6. *For a locally compact Hausdorff space X, if $K \subset V \subset X$ where K is compact and V is open, then there exists a function $h \in C(X)$ which takes values in $[0,1]$ such that $h = 1$ on K and $h = 0$ outside of a compact subset of V.*

Theorem 3.7. *With the setting as in Definition 3.1, there exists a disintegration $y \mapsto \mu_y$ of μ with respect to ϕ which is essentially unique in the sense that if $y \mapsto \mu_y^{'}$ is another disintegration, then $\mu_y = \mu_y^{'}$ for ν-a.e.-y.*

Proof. Let $\mathcal{D} \subset C_0(X)$ be a countable dense and rational linear manifold. The existence of $\mathcal{D}$ follows by ii) of topological properties of X, namely, let $\mathcal{D}$ be the set of all rational linear combinations of a countable dense set in $C_0(X)$.

Taking any $f \in \mathcal{D} \subset C_0(X)$, $f \in \mathcal{L}^{\infty}(X, \mathfrak{a}, \mu)$ clearly. Hence by the above inequalities, $\|E(f)\|_\infty \leq \|f\|_\infty$. Choose a Borel measurable representative of the class $E(f)$ and call it $\tilde{f}$, then $\|\tilde{f}\|_\infty \leq \|f\|_\infty$.

Now for any fixed $\alpha, \beta \in \mathbb{Q}$ and $f, g \in \mathcal{D}$ define the set

$$D(\alpha, \beta, f, g) := \left\{ y \in Y \,\middle|\, \alpha\tilde{f}(y) + \beta\tilde{g}(y) \neq (\alpha f + \beta g)\tilde{\ }(y) \right\}.$$

But we have that $\alpha\tilde{f}+\beta\tilde{g}$ has equivalence class $\alpha E(f)+\beta E(g)$, and $(\alpha f+\beta g)\tilde{}$ has equivalence class $E(\alpha f+\beta g)=\alpha E(f)+\beta E(g)$. Therefore the set $D(\alpha,\beta,f,g)$ has ν measure zero. Taking the union

$$D := \bigcup_{\substack{\alpha,\beta\in\mathbb{Q}\\ f,g\in\mathcal{D}}} D(\alpha,\beta,f,g)$$

we have that $\nu(D)=0$. By the definitions, for $y\in Y\setminus D$, the function $f\mapsto\tilde{f}(y)$ is a bounded rational linear functional on $\mathcal{D}$. By continuity arguments, this functional is uniquely extended first to a bounded complex linear functional on $\mathcal{D}$, and then to a bounded complex linear functional on $C_0(X)$. So we have a bounded linear functional $f\mapsto\tilde{f}(y)$ on $C_0(X)$ for all $y\in Y$ except on a set of ν measure zero. By the Riesz Representation Theorem, there exists a (regular) complex Borel measure μ_y such that $\tilde{f}(y)=\int_X f\,d\mu_y$ ν-a.e. Letting $I_X(x)=1$ on X and $I_Y(y)=1$ on Y, it is not difficult to see that

$$E(I_X)(y)=I_Y(y)=\int_X I_X\,d\mu_y \quad \nu\text{-a.e.}$$

Therefore the measures μ_y are probability measures ν-a.e. For the points $y\in Y\setminus D$ such that μ_y is not a probability measure, redefine μ_y as μ, and for $y\in D$, let $\mu_y:=\mu$. Then 3.3 holds for $f\in\mathcal{D}$. Now the aim is to show that 3.3 holds for all $f\in\mathcal{L}^1(X,\mathfrak{a},\mu)$.

For that, let $\mathcal{K}\subseteq\mathcal{L}^1(X,\mathfrak{a},\mu)$ be the set of functions for which 3.3 holds. Three observations about the set $\mathcal{K}$ are in order, and these turn out to be enough to prove that $\mathcal{K}=\mathcal{L}^1(X,\mathfrak{a},\mu)$.

i) $\mathcal{K}$ is closed under bounded pointwise limits of sequences, that is, if $f_n\in\mathcal{K}$ are such that $|f_n(x)|\le M$ for every $x\in X$ and $n\in\mathbb{N}$, and $f_n\xrightarrow{p.w.}f$ μ-a.e, then $f\in\mathcal{K}$. We note that this also shows that $\mathcal{K}\cap C_0(X)$ is a closed set in $C_0(X)$ with the uniform norm, since if $f_n\longrightarrow f$ in $C_0(X)$, then clearly f is a bounded pointwise limit.

i$'$) For $f_n\in\mathcal{K}$ such that $|f_n|\le f$ and $f_n\xrightarrow{p.w.}f$ where $f\in\mathcal{L}^1(X,\mathfrak{a},\mu)$, we have that $f\in\mathcal{K}$.

ii) $f\in\mathcal{K}$ with $f\ge 0$ form a closed set under monotone pointwise dominated convergence, that is, if $f_n \geq 0$, $f_n \in \mathcal{K}$ with $f_n \leq f_{n+1}$ μ-a.e. for every n, and $f_n \overset{p.w.}{\longrightarrow} f$ μ-a.e, and if $f \in \mathcal{L}^1(X, \mathfrak{a}, \mu)$, then $f \in \mathcal{K}$.

To show i) applications of the Bounded Convergence Theorem and the Lebesgue Dominated Convergence Theorem will be used, as follows: Firstly,

$$\int_X f_n \, d\mu_y \longrightarrow \int_X f \, d\mu_y \tag{3.4}$$

for every y by the Bounded Convergence Theorem. Secondly,

$$\int_Y \psi E(f_n) \, d\nu = \int_Y \psi(y) \left(\int_X f_n \, d\mu_y \right) \, d\nu(y) \longrightarrow \int_Y \psi(y) \left(\int_X f \, d\mu_y \right) \, d\nu(y) \tag{3.5}$$

for every $\psi \in L^\infty(\nu)$. Here Equation 3.3 is used for the equality and Equation 3.4 and the Lebesgue Dominated Convergence Theorem are used to obtain the limit. On the other hand we have

$$\int_Y \psi E(f_n) \, d\nu = \int_X (\psi \circ \phi) \, f_n \, d\mu \longrightarrow \int_X (\psi \circ \phi) \, f \, d\mu = \int_Y \psi E(f) \, d\nu \tag{3.6}$$

using Equation 3.1 for the first and last equalities and the Lebesgue Dominated Convergence Theorem to obtain the limit. By 3.5 and 3.6, we have

$$\int_Y \psi E(f) \, d\nu = \int_Y \psi(y) \left(\int_X f \, d\mu_y \right) \, d\nu(y)$$

for every $\psi \in L^\infty(\nu)$. It follows that $E(f)(y) = \left(\int_X f \, d\mu_y \right)$ ν-a.e. and hence $f \in \mathcal{K}$.

For i$'$) we modify the first line of the above argument:

$$\int_X f_n \, d\mu_y \longrightarrow \int_X f \, d\mu_y$$

for every y by the Lebesgue Dominated Convergence Theorem. Then following the same steps as above gives i$'$).

For ii) nearly the same steps are followed, but the Monotone Convergence Theorem is used. For completeness, we include them here. Firstly,

$$\int_X f_n \, d\mu_y \longrightarrow \int_X f \, d\mu_y \tag{3.7}$$

for every y by the Monotone Convergence Theorem. Secondly,

$$\int_Y \psi E(f_n)\,d\nu = \int_Y \psi(y)\left(\int_X f_n\,d\mu_y\right)\,d\nu(y) \longrightarrow \int_Y \psi(y)\left(\int_X f\,d\mu_y\right)\,d\nu(y) \tag{3.8}$$

for every $\psi \in L^\infty(\nu)$. As in i), Equation 3.3 is used for the equality and Equation 3.7 and the Monotone Convergence Theorem are used to obtain the limit. On the other hand we have

$$\int_Y \psi E(f_n)\,d\nu = \int_X (\psi\circ\phi)\,f_n\,d\mu \longrightarrow \int_X (\psi\circ\phi)\,f\,d\mu = \int_Y \psi E(f)\,d\nu \tag{3.9}$$

using Equation 3.1 for the first and last equalities and the Monotone Convergence Theorem to obtain the limit. By 3.8 and 3.9, we have

$$\int_Y \psi E(f)\,d\nu = \int_Y \psi(y)\left(\int_X f\,d\mu_y\right)\,d\nu(y)$$

for every $\psi \in L^\infty(\nu)$. Since $f \in \mathcal{L}^1(X,\mathfrak{a},\mu)$, we have $E(f)(y) = \left(\int_X f\,d\mu_y\right)$ ν-a.e. and hence $f \in \mathcal{K}$.

Now using i), $\mathcal{K}$ contains $C_0(X)$ since $\mathcal{K}\cap C_0(X)$ is closed in $C_0(X)$ and is containing $\mathcal{D}$, and $\mathcal{D}$ is dense in $C_0(X)$. Moreover, $\mathcal{K}$ contains $C(X)$ of all continuous functions by the following arguments: By Lemma 3.2 there exists an exhaustion of X by compact sets $(K_n)_{n=1}^\infty$ with $K_n \subseteq K_{n+1}$. Therefore for every function $f \in \mathcal{L}^1(X,\mathfrak{a},\mu)$ which is continuous, there exists $f_n \in C_0(X)$ such that $f_n \xrightarrow{p.w.} f$, which can be taken to be dominated by f, i.e. $|f_n| \le f$, by the following: By Lemma 3.6 for each K_n there is a function h_n with $h_n = 1$ on K_n and $h_n = 0$ outside of a compact subset of X. Then letting $f_n := h_n f$ we obtain the desired f_n. Hence by i$'$) we have $f \in \mathcal{K}$, and $\mathcal{K} \supset C(X)$. By i) again, $\mathcal{K}$ contains all bounded Baire functions. But by Lemma 3.5 the Baire sets coincide with the Borel sets, hence the bounded Baire functions are the same with the bounded Borel functions, i.e. with the class $\mathcal{L}^\infty(X,\mathfrak{a},\mu)$.

It is well known that any nonnegative function in $\mathcal{L}^1(X,\mathfrak{a},\mu)$ is a monotone pointwise limit of nonnegative functions from $\mathcal{L}^\infty(X,\mathfrak{a},\mu)$. Therefore by ii) $\mathcal{K}$ contains all nonnegative functions in $\mathcal{L}^1(X,\mathfrak{a},\mu)$, and consequently it contains all $\mathcal{L}^1(X,\mathfrak{a},\mu)$, showing the existence part of the proof.

To show that the disintegration $y \longrightarrow \mu_y$ is essentially unique, suppose that $y \to \mu_y^{'}$ is another disintegration. Let $f \in \mathcal{D}$, and for this f define the set

$$K_f := \left\{ y \in Y \middle| \int_X f \, d\mu_y \neq \int_X f \, d\mu_y^{'} \right\}.$$

By the assumption, $\int_X f \, d\mu_y = \int_X f \, d\mu_y^{'} = E(f)$, as classes in $L^\infty(\nu)$. Hence $\nu(K_f) = 0$, and therefore $\nu(K) = 0$ where $K := \bigcup_{f \in \mathcal{D}} K_f$. It follows that μ_y agrees with $\mu_y^{'}$ except on K for all $f \in \mathcal{D}$. But since $\mathcal{D}$ is dense in $C_0(X)$, μ_y and $\mu_y^{'}$ define the same linear functional on $C_0(X)$ except for $y \in K$. By the Riesz Representation Theorem, they are the same measures, except on a null set, finishing the proof. □

Remark 3.8. In the above proof, once we get $\mathcal{K} \supset C_0(X)$, we could have alternatively argued as follows to conclude that $\mathcal{K} = \mathcal{L}^1(X, \mathfrak{a}, \mu)$. Firstly, we observe that $\mathcal{K}$ is closed under convergence in $\mathcal{L}^1(\mu)$, i.e. if $f_n \in \mathcal{K}$ and $f_n \xrightarrow{\mathcal{L}^1} f$,then $f \in \mathcal{K}$. This follows by an argument similar to the proof of i) in the proof above. Now by Proposition 7.9. in [8], if μ is a Radon measure on a locally compact Hausdorff space X, then $C_c(X)$ of compactly supported continuous functions is dense in $L^p(\mu)$ for $1 \leq p < \infty$. We also know that in a locally compact Hausdorff space, $C_0(X)$ is the closure of $C_c(X)$ in the uniform norm topology, for instance by Proposition 4.35 in [8]. Hence $\mathcal{K}$ contains the closure of $C_0(X)$ in $\mathcal{L}^1(X, \mathfrak{a}, \mu)$, and therefore is equal to $\mathcal{L}^1(X, \mathfrak{a}, \mu)$.

The following theorem gives information about the support of a disintegration of a measure.

Theorem 3.9. *Let π be an $\mathfrak{a}$ measurable representative of ϕ, and assume that $y \mapsto \mu_y$ is a disintegration of μ with respect to ϕ. Then for ν-a.e.-y the measure μ_y is supported by $\pi^{-1}(y)$, i.e. $\mu_y(X \setminus \pi^{-1}(y)) = 0$.*

Proof. Let $\mathcal{U}$ be a countable basis for $Y \subset \mathbb{C}$. For every $u \subset \mathcal{U}$, let $W_U := \pi^{-1}(Y \setminus U)$ and $N_U := \{ y \in U \big| \mu_y(W_U > 0 \}$. Then we have

$$\begin{aligned}\int_U \mu_y(W_U) \, d\nu &= \int_U \int_X \chi_{W_U} \, d\mu_y d\nu = \int_U E(\chi_{W_U}) \, d\nu \\ &= \int_Y \chi_U E(\chi_{W_U}) \, d\nu = \int_X (\chi_U \circ \phi) \chi_{W_U} \, d\mu = 0.\end{aligned}$$

where χ denotes the characteristic function. These equations follow from equations 3.1 and 3.3, and the fact that the function in the last integral is identically zero on X. It follows that $\nu(N_U) = 0$. Taking countable union over all $U \in \mathcal{U}$, let $\bigcup_{U\in\mathcal{U}} N_U = N$. Then $\nu(N) = 0$.

If $y \in Y \setminus N$, then $\mu_y(W_U) = 0$ for every $U \in \mathcal{U}$ which contain y. Taking the union of such W_U we get the set $X \setminus \pi^{-1}(y)$ clearly and thus $\mu_y(X \setminus \pi^{-1}(y)) = 0$ out of a set of measure zero, as desired. $\square$

Chapter 4

Spectral Multiplicity Theorem

In this part we discuss the Spectral Multiplicity Theorem in Direct Integral Representation and give a proof of it based on the arguments in [2], using the machinery developed in the previous part.

Definition 4.1. Let H be a separable Hilbert space. An operator $A \in \mathcal{B}(H)$ is diagonalizable if there exists a σ-finite measure space (X, μ) with $X \subset \mathbb{C}$, a function $\phi \in L^\infty(X, \mu)$ and a unitary operator $W : L^2(X, \mu) \to H$ such that $WM_\phi = AW$, where $M_\phi : L^2(X, \mu) \to L^2(X, \mu)$ defined by $M_\phi(f)(x) := (\phi f)(x)$ is the multiplication operator on $L^2(X, \mu)$.

Remark 4.2. It is not difficult to see that in the case that $H = \int_X^\oplus H_x \, d\mu(x)$, Definition 2.1 and Definition 4.1 agrees.

The following theorem is well known, and can be found for instance in [3], Chapter 2.4.

Theorem 4.3. *(Spectral Theorem for Normal Operators) Let $N \in \mathcal{B}(H)$ be a normal operator on a separable Hilbert space H. Then N is diagonalizable.*

Remark 4.4. Without loss of generality, the measure space (X, μ) appearing in Theorem 4.3 can be taken to be probability measure space, for instance by Exercise 2 in Chapter 2.4. of [3].

For the purposes of this part, we adopt another definition of direct integral Hilbert spaces, as in [2], which is in fact only superficially different from Definition 1.1 that we had in Chapter 1. For completeness, we show their equivalence below.

Definition 4.5. Let $Y \in \mathbb{C}$ be a compact subset of the complex plane, and let ν be a probability measure on it. Let $\mathcal{F}$ be the set of functions $y \mapsto \bigcup_{y\in Y} H_y$ where $\{H_y\}_{y\in Y}$ are nonzero separable Hilbert spaces, such that $f(y) \in H_y$ for every $y \in Y$. Let $\mathcal{N}$ be the set of functions as above which are zero ν-almost everywhere, and identify functions which are the same ν-a.e. Equivalently, we consider the space $\mathcal{F}/\mathcal{N}$. Then a linear subspace H of $\mathcal{F}/\mathcal{N}$ is a direct integral of $\{H_y\}_{y\in Y}$ with respect to the measure ν if the following hold:

E1.For every $f, g \in H$, the scalar function $y \mapsto \langle f(y), g(y)\rangle_{H_y}$ is in $L^1(\nu)$.

E2.H is a Hilbert space with respect to the inner product

$$\langle f, g\rangle := \int_Y \langle f(y), g(y)\rangle \, d\nu(y).$$

E3.There exists a countable subset $P \subset H$ such that the set $\{f(y) \big| f \in P\}$ spans H_y for ν-a.e. y.

E4.H is an $L^\infty(\nu)$-module, i.e. for every $\psi \in L^\infty(\nu)$ and $f \in H$, we have $\psi f \in H$.

Remark 4.6. Definition 1.1 and Definition 4.5 are equivalent for a measure space (Y, ν) where $Y \subset \mathbb{C}$ is compact and ν is a probability measure on it.

Proof. Definition 1.1 $\Rightarrow$ Definition 4.5: Assume H is a direct integral of $\{H_y\}_{y\in Y}$ over (Y, ν) in the sense of Definition 1.1. Then D1 of Definition 1.1 provides the function space as described in Definition 4.5. D2 provides E1 and E2 clearly. By Proposition 1.3, we have a countable set P as in E3. For E4, let M be the essential supremum of the function $|\psi|$ and observe that since $\left|\langle \psi(y)f(y), g(y)\rangle_{H_y}\right| \leq M\left|\langle f(y), g(y)\rangle_{H_y}\right|$ ν-a.e, we have that $y \mapsto \langle \psi(y)f(y), g(y)\rangle$ is integrable for every $g \in H$. Therefore by D3 it follows that $\psi f \in H$.

Definition 4.5 $\Rightarrow$ Definition 1.1: Assume H is a direct integral of $\{H_y\}_{y\in Y}$ over (Y, ν) in the sense of Definition 4.5. Then D1 of Definition 1.1 is satisfied by definition. D2 is satisfied by E1 and E2 clearly. For D3, assume that a function $g : y \mapsto g_y \in H_y$ is such that $\langle g_y, f(y)\rangle$ is ν integrable for all $f \in H$. Firstly, by a similar argument used in Example 1.7 we show that the scalar valued function $y \mapsto \|g_y\|^2_{H_y}$ is ν-integrable. Namely, let $k \in L^2(Y, \nu)$ be a function. Let

$$\tilde{h}(y) := \begin{cases} (g_y/\|g_y\|)\overline{k(y)}, & g_y \neq 0 \\ 0, & g_y = 0 \end{cases}$$

Then for each $y \in Y$, $\tilde{h}(y) \in H_y$.

$$\int_Y \|\tilde{h}(y)\|^2_{H_y}\, d\nu(y) = \int_Y |k(y)|^2\, d\nu(y) < +\infty.$$

Assuming for a while that $\tilde{h} \in H$, by the assumption above we have that the integral $\int_Y \langle g_y, \tilde{h}(y)\rangle_{H_y}\, d\nu(y)$ exists. Then we have

$$\int_Y \langle g_y, \tilde{h}(y)\rangle_{H_y}\, d\nu(y) = \int_Y \|g_y\|_{H_y} k(y)\, d\nu(y).$$

By Lemma 1.5, the function $\tilde{g} : y \mapsto \|g_y\|^2_{H_y}$ is ν-integrable.

By E3, there is a countable set $P := \left(p_i\right)_{i=1}^{\infty} \subset H$ such that closure of the linear span P_y of the set $\left(p_i(y)\right)_{i=1}^{\infty}$ is H_y for ν almost every y. Hence the function g can be put into the form $y \mapsto \sum_{i=1}^{\infty} e_i(y)p_i(y)$, and by choosing suitable $e_i(y)$, we can assume that the partial sums above approach g monotone and pointwise.

Now find functions $(d_i^{(m)})_{m=1}^{\infty} \in L^\infty(Y, \nu)$ such that $d_i^{(m)} \underset{m\to\infty}{\longrightarrow} e_i$ monotone and pointwise. Then we have

$$\sum_{i=1}^{n} d_i^{(m)}(y)p_i(y) \underset{m\to\infty}{\longrightarrow} \sum_{i=1}^{n} e_i(y)p_i(y)$$

monotone and pointwise for each n. But by E4, the functions $d_i^{(m)}p_i : y \mapsto d_i^{(m)}(y)p_i(y)$ are in H for each m. By the Monotone Convergence Theorem we have

$$\int_Y \|e_i(y)p_i(y) - d_i^{(m)}(y)p_i(y)\|^2_{H_y}\, d\nu(y) \underset{m\to\infty}{\longrightarrow} 0.$$

for each i. It follows that the sequence $\left(d_i^{(m)}p_i)_{m=1}^{\infty}\right)$ is a Cauchy sequence in H for each i and it has limit e_ip_i. Therefore $e_ip_i \in H$ for every i. Consequently, $\sum_{i=1}^{n} e_ip_i \in H$ for every n.

By another application of the Monotone Convergence Theorem we have

$$\int_Y \|g_y - \sum_{i=1}^{n} e_i(y)p_i(y)\|_{H_y}^2 \, d\nu(y) \underset{n\to\infty}{\longrightarrow} 0.$$

where the functions $y \mapsto \|g_y - \sum_{i=1}^{n} e_i(y)p_i(y)\|_{H_y}^2$ are ν-integrable since the function $\tilde{g}$ is integrable and $\sum_{i=1}^{n} e_i p_i \in H$ for each n. It follows that the sequence $(\sum_{i=1}^{n} e_i p_i)_{n=1}^{\infty}$ is a Cauchy sequence in H, and it has limit g. Therefore $g \in H$. We note that the same arguments for proving $g \in H$ can be used to show that $\tilde{h} \in H$, since it is norm square integrable, and the proof is finished. □

For the rest of the thesis, we will use Definition 4.5 for a direct integral Hilbert space.

With a direct integral H, there is an associated operator Z_H acting on H defined by $Z_H(f) := zf$ where $f \in H$ and $z \in L^\infty(\nu)$ is the function $z(y) := y$ on Y. This operator is well defined by E4 of Definition 4.5.

Before we go into the Spectral Multiplicity Theorem, we have a definition.

Definition 4.7. Let H be a separable Hilbert space and $N \in \mathcal{B}(H)$ be a normal operator. Then there exists a unique spectral measure P associated with N with compact support $\sigma(N)$, the spectrum of N, and $N = \int_{\sigma(N)} y \, dP(y)$ holds, which can be found in for instance Chapter 2.7 of [3], and is also referred to as the Spectral Theorem for Normal Operators by some authors, as in Abrahamse [1]. Then a scalar spectral measure of N is a probability measure ν on $\sigma(N)$ such that for any Borel set $B \subset \sigma(N)$, $P(B) = 0$ if and only if $\nu(B) = 0$. For a normal operator there always exist a scalar spectral measure, for instance by Section 4.4 of [9].

Theorem 4.8. *Let $N \in \mathcal{B}(H)$ be a normal operator on a separable Hilbert space H. Then there exists ν, a scalar spectral measure of N on $Y := \sigma(N)$ and a ν-measurable set of separable Hilbert spaces $\{H_y\}_{y\in Y}$ such that, modulo a unitary identification of H with $\tilde{H} := \int_Y^{\oplus} H_y \, d\nu(y)$ we have $(Nf)(y) = yf(y) = Z_{\tilde{H}}(f)$ for every $f \in \tilde{H}$ and $y \in Y$.*

Define $m : \sigma(N) \to \mathbb{N} \cup \{\infty\}$ by $m(y) := \dim H_y$. Then m is a measurable function, and it is called the multiplicity function of the normal operator N.

As a second part of the theorem we have that the triple $(\sigma(N), [\nu], [m]_\nu)$ *is a complete set of unitary invariants for* N*. Here* $[\nu]$ *denotes probability measures on* $\sigma(N)$ *which are mutually absolutely continuous with respect to* ν *and* $[m]_\nu$ *denotes the functions* $n : \sigma(N) \to \mathbb{N} \cup \{\infty\}$ *which coincide with* m ν*-a.e. Therefore assume that we have* $\tilde{H} := \int_Y^\oplus H_y \, d\nu(y)$ *as above and another direct integral* $H' := \int_{Y'}^\oplus H'_y \, d\nu'(y)$ *with the associated operators* $Z_{\tilde{H}}$ *and* $Z_{H'}$*, where by the first part* $Z_{\tilde{H}}$ *is unitarily equivalent with* N*. Then these operators are unitarily equivalent, and hence unitarily equivalent to* N *if and only if* $Y = Y' = \sigma(N)$*,* ν *is mutually absolutely continuous with* ν' *and* $\dim H_y = \dim H'_y$ ν*-a.e.*

Proof. For the proof of the first part, note that by Theorem 4.3 the operator N is unitarily equivalent to a multiplication operator M_ϕ acting on $L^2(X, \mu)$ with $\phi \in L^\infty(X, \mu)$ where (X, μ) is a probability measure space which is separable and locally compact since $X \subset \mathbb{C}$. Hence by Theorem 3.7 there exists a disintegration $y \mapsto \mu_y$ of μ with respect to $\phi \in L^\infty(X, \mu)$.

Let $f \in \mathcal{L}^2(X, \mu)$. In Equation 3.1 use $|f|^2 \in \mathcal{L}^1(X, \mu)$ with $\psi = 1$. Together with Equation 3.3 we get

$$\int_Y \int_X |f|^2 \, d\mu_y d\nu(y) = \int_Y E(|f|^2) \, d\nu = \int_X |f|^2 \, d\mu = \|f\|_2^2 \tag{4.1}$$

where $\nu = \mu \circ \phi^{-1}$. Then ν is a scalar spectral measure: For any $B \subset \sigma(N)$, if $\nu(B) = \mu \circ \phi^{-1}(B) = 0$, it follows that the function $\phi_1 : X \to \sigma(N)$ defined by

$$\phi_1(x) := \begin{cases} \phi(x), & \text{if } x \in \phi^{-1}(B) \\ 0, & \text{else} \end{cases}$$

is the same function with $\phi_2 = 0$ as elements in $L^\infty(X, \mu)$. Hence for the corresponding multiplication operator we have $M_{\phi_1} = 0$, and consequently $\int_B y \, dP(y) = 0$. Therefore $P(B) = 0$. On the other hand, if $P(B) = 0$, then $\int_B y \, dP(y) = 0$, and by reversing the argument above we get that the μ measure of $\phi^{-1}(B)$ should be zero, i.e. $\nu(B) = 0$.

From Equation 4.1 it follows that $\int_X |f|^2 \, d\mu_y < +\infty$ for ν-a.e.-y. Therefore $f \in \mathcal{L}^2(X, \mu_y)$ for ν-a.e.-y. Let f_y denote the equivalence class of f in $L^2(X, \mu_y)$ for such y. For other $y \in Y$, let $f_y = 0$. Then $f_y \in L^2(X, \mu_y)$ for all $y \in Y$.

Now define a set of functions H on Y as

$$H := \left\{ y \mapsto f_y \middle| f \in \mathcal{L}^2(X, \mu) \right\}.$$

We claim that $H = \int_Y^{\oplus} L^2(\mu_y)\, d\nu(y)$ with the inner product

$$\begin{aligned} \langle \tilde{f}, \tilde{g} \rangle_H &:= \int_Y \langle f_y, g_y \rangle_{L^2(\mu_y)}\, d\nu(y) \\ &= \int_Y \int_X f_y g_y \, d\mu_y \, d\nu(y). \end{aligned}$$

For E1 of Definition 4.5, $y \mapsto \langle f_y, g_y \rangle$ is ν integrable since $f_y, g_y \in L^2(\mu_y)$ and by Equation 4.1. For E2, the inner product above defines a complete metric on H by the following: Suppose $(\tilde{f}_n)_{n \geq 1}$ is a Cauchy sequence in H. Then for any $\epsilon > 0$ and $n, m > N_\epsilon$ we have

$$\langle \tilde{f}_n - \tilde{f}_m, \tilde{f}_n - \tilde{f}_m \rangle = \int_Y \int_X \left| f_{n_y} - f_{m_y} \right|^2 d\mu_y \, d\nu = \int_X \left| f_n - f_m \right|^2 d\mu < \epsilon$$

where the last equation follows by Equation 4.1. But since $L^2(X, \mu)$ is complete, there is f such that $f_n \longrightarrow f$. It follows that $\int_Y \int_X \left| f_{n_y} - f_y \right|^2 d\mu_y \, d\nu \underset{n \to \infty}{\longrightarrow} 0$. Hence H is a Hilbert space with the given inner product. For E3, let $\mathcal{D}$ be a countable dense subset of $C_0(X)$. Clearly $C_0(X) \subset \mathcal{L}^2(X, \mu)$ since μ is a finite measure. Let $P := \left\{ y \mapsto f_y \middle| f \in \mathcal{D} \right\}$. By Equation 4.1 $\mathcal{D}$ is also dense in $\mathcal{L}^2(X, \mu_y)$ for ν-a.e.-y. Hence E3 holds with set P. For E4, if $\psi \in L^\infty(\nu)$, then clearly

$$\left(\psi \tilde{f} \right)(y) = \psi(y) \tilde{f}(y) = \psi(y) f_y \in L^2(\mu_y)$$

and the claim is shown.

Let Z_H be the associated operator on H. Then we have

$$Z_H(\tilde{f})(y) := \left(z \tilde{f} \right)(y) = z(y) \tilde{f}(y) = y f_y \in L^2(\mu_y).$$

Define another operator $\tilde{V} : \mathcal{L}^2(X, \mu) \to H$ by $\tilde{V}(f)(y) = f_y$. By Equation 4.1 $\tilde{V}$ is an isometry, and hence the induced map $V : L^2(X, \mu) \to H$ is also an isometry. It is clearly onto by the definition of H. Hence V is a unitary operator.

By Theorem 3.9, for ν-a.e.-y the measure μ_y is supported by $\pi^{-1}(y)$, where π is a representative of the class ϕ. Therefore $\phi = y$ μ_y-a.e. It follows that

$VM_\phi V^{-1} = Z_H$. Hence N is unitarily equivalent with Z_H and the first part of the theorem is proved.

By Proposition 1.9 we have that the function m is measurable.

The second part of the theorem will not be essential for the remainder of the thesis. For the sake of completeness, we present a proof due to [4], 2.2. Let $Y \subset \mathbb{C}$ be a compact subset. By the first part of the theorem, classifying normal operators having spectrum Y up to unitary equivalence is sufficient. For that, consider the C^* algebra generated by a normal operator N with spectrum Y, and the identity. As in Definition 4.7, let $N = \int_{\sigma(N)} y\, dP(y)$, where P is the corresponding spectral measure on $\sigma(N) = Y$. If M is another normal operator with spectrum $\sigma(M) = Y$, with corresponding spectral measure R, we have $M = \int_{\sigma(M)} y\, dR(y)$. Then N and M are unitarily equivalent if and only if for every $\phi \in C(Y)$ we have

$$\int_{\sigma(N)} \phi(y)\, dP(y) = \int_{\sigma(M)} \phi(y)\, dR(y)$$

i.e. the representations of $C(Y)$ via the spectral measures P and R are equivalent, which implies that P and R are mutually absolutely continuous measures. Let the corresponding scalar spectral measures be ν_N and ν_M as defined in the first part of the theorem, which are also mutually absolutely continuous.

Now let $Y_k := m^{-1}(k)$, $k = \infty, 1, 2, \ldots$ where m is the multiplicity function of N. Then we have the representation

$$\begin{aligned}\int_Y \phi(y)\, dP(y) &= \int_{Y_\infty} \phi(y)\, dP(y) \oplus \bigoplus_{k=1}^{\infty} \int_{Y_k} \phi(y)\, dP(y) \\ &= \infty \int_Y \phi(y)\, dP_\infty(y) \oplus \bigoplus_{k=1}^{\infty} k \int_Y \phi(y)\, dP_k(y)\end{aligned}$$

for any $\phi \in C(Y)$ where P_k is the spectral measure corresponding to

$$k \int_Y y\, dP_k(y) = \int_{Y_k} y\, dP(y)$$

for $k = \infty, 1, 2, \ldots$. Here $k \int_Y \phi(y)\, dP_k(y)$ corresponds to a certain restriction of N, which is unitarily equivalent to a multiplication operator: Namely, let

$L^2(Y_k, \nu, G_k)$ be the vector space of all Borel functions $f : Y_k \to G_k$ such that $\int_{Y_k} \|f(y)\|_{G_k} \, d\nu(y) < +\infty$. Here G_k is a k dimensional Hilbert space; in particular, for $k = \infty$ let G_∞ be a countably infinite dimensional Hilbert space. The functions are Borel in the sense that the functions $y \mapsto \langle f(y), g \rangle_{G_k}$ are Borel for every $g \in G_k$. Identifying functions which agree ν-a.e, we see that $L^2(Y_k, \nu, G_k)$ is a Hilbert space for every k. Now the restriction ϕ_{Y_k} of ϕ to Y_k produces a multiplication operator acting on the space $L^2(Y_k, \nu, G_k)$ in the obvious way, and this operator is unitarily equivalent to $k \int_Y \phi(y) \, dP_k(y)$.

Letting the corresponding scalar spectral measures be $\nu_N(k)$, we have $\nu_N(k) \perp \nu_N(l)$ for $k \neq l$. It follows that

$$\begin{aligned} N &= \infty \int_{Y_\infty} y \, d\nu_N(\infty) \oplus \bigoplus_{k=1}^{\infty} k \int_{Y_k} y \, d\nu_N(k) \\ &= \int_Y^{\oplus} m(y) y \, d\nu_N. \end{aligned}$$

The same steps can be done for n, the multiplicity function of M, to get

$$\begin{aligned} M &= \infty \int_{Y_\infty} y \, d\nu_M(\infty) \oplus \bigoplus_{i=1}^{\infty} i \int_{Y_k} y \, d\nu_M(i) \\ &= \int_Y^{\oplus} n(y) y \, d\nu_M \end{aligned}$$

and it follows that N and M are unitarily equivalent if and only if ν_N and ν_M are mutually absolutely continuous and their multiplicity functions m and n agree ν_N (hence ν_M) a.e, giving the theorem. $\square$

Chapter 5

The Essential Pre-Image and the Pre-Image

We follow [2] to see what information can be extracted about the unitary invariants $(\sigma(M_\phi), [\nu], [m]_\nu)$ of a multiplication operator M_ϕ in terms of ϕ and μ, using the notion of essential pre-image. The setting will be as in the third part of the thesis.

Definition 5.1. For $y \in Y$ let $B_\delta(y)$ the closed ball of radius $\delta > 0$, where Y is the essential range of the function $\phi \in L^\infty(X, \mathfrak{a}, \mu)$ as before. Let $S \in \mathfrak{a}$. Define a function D_S on Y by

$$D_S(y) := \underline{\lim}_{\delta \to 0} \frac{\mu\left(S \cap \phi^{-1}(B_\delta(y))\right)}{\mu\left(\phi^{-1}(B_\delta(y))\right)} \tag{5.1}$$

When the limit in Equation 5.1 exists, it can be interpreted as the probability that a solution to the equation $\phi(x) = y$ lies in S.

Now we give the definition of essential pre-image in the sense of [2].

Definition 5.2. The essential pre-image of ϕ at y is

$$\phi_\mu^{-1}(y) := \left\{x \in X \middle| D_V(y) > 0 \text{ for every open set } V \ni x\right\}. \tag{5.2}$$

By the previous paragraph, $\phi_\mu^{-1}(y)$ contains points x such that there is a positive probability that a solution to $\phi(x) = y$ is in V, for every open set V containing x.

By a theorem of Besicovitch (a reference can be found at [2], p.851) for ν-a.e-y the limit in Equation 5.1 exists. Moreover D_S is a representative of the Radon-Nikodym derivative $\left[\frac{d\nu_S}{d\nu}\right]$ where $\nu_S(F) := \mu(S \cap \phi^{-1}(F))$.

Let $\psi \in L^\infty(\mu)$. Then we have

$$\int_Y \psi D_S \, d\nu(y) = \int_Y \psi \, d\nu_S = \int_X (\psi \circ \phi)\chi_S \, d\mu(x) \tag{5.3}$$

where χ_S is the characteristic function of S. It follows by Equation 3.1 that D_S is a representative of $E(\chi_S)$.

The following theorem describes the essential pre-image in terms of a disintegration of measure μ.

Theorem 5.3. *If $y \mapsto \mu_y$ is a disintegration of μ with respect to ϕ, then for ν-a.e y, $\phi_\mu^{-1}(y)$ is the closed support $\overline{\operatorname{supp} \mu_y}$ of measure μ_y.*

Proof. Let $\mathcal{U}$ be a countable bases of the topology of X. We have $E(f)(y) = \int_X f \, d\mu_y$ for ν-a.e. y, for every $f \in \mathcal{L}(X, \mathfrak{a}, \mu)$. Let $U \in \mathcal{U}$ and $f := \chi_U$. Then by above, $E(\chi_U) = D_U$ ν-a.e. Hence we have $D_U(y) = E(\chi_U)(y) = \int_X \chi_U \, d\mu_y = \mu_y(U)$ and $D_U(y) = \mu_y(U)$ except on a set $N_U \subset Y$ with $\nu(N_U) = 0$. Let $N := \bigcup_{U \in \mathcal{U}} N_U$. Then $\mu(N) = 0$. Fix any $y \in Y \setminus N$.

To see that $\phi_\mu^{-1}(y) \subset \overline{\operatorname{supp} \mu_y}$ let $x \in \phi_\mu^{-1}(y)$. Then $D_V(y) > 0$ for every open set V containing x and in particular for all $U \in \mathcal{U}$. By above it follows that $D_U(y) = \mu_y(U) > 0$ for every $U \ni x$, and the inclusion follows.

For the converse inclusion, let $x \in \overline{\operatorname{supp} \mu_y}$. Let $V \ni x$ be an open set. Pick $U \in \mathcal{U}$ such that $x \in U \subset V$. Then we have

$$0 < \mu_y(U) = D_U(y) = \varliminf_{\delta \to 0} \frac{\mu\left(U \cap \phi^{-1}(B_\delta(y))\right)}{\mu\left(\phi^{-1}(B_\delta(y))\right)}$$

$$\begin{aligned} &\leq \varliminf_{\delta \to 0} \frac{\mu\left(V \cap \phi^{-1}(B_\delta(y))\right)}{\mu\left(\phi^{-1}(B_\delta(y))\right)} \\ &= D_V(y). \end{aligned}$$

Therefore $x \in \phi_\mu^{-1}(y)$, and the inclusion follows. □

Theorem 5.4. *The multiplicity function for the operator M_ϕ is $m(y) = \#\phi_\mu^{-1}(y)$, where $\#E$ denotes the number of the elements in set E if the cardinality of E is finite, and it is the symbol ∞ if E has infinite cardinality.*

Proof. Let $y \mapsto \mu_y$ be a disintegration of μ with respect to ϕ. Note that the dimension of $L^2(X, \mu_y)$ is exactly the number of points in the closed support of μ_y if this dimension is finite, and hence the result follows from Theorem 4.8 which gives $m(y) = \dim L^2(X, \mu_y)$ and Theorem 5.3. If the dimension of $L^2(X, \mu_y)$ is infinite, then the closed support of $L^2(X, \mu_y)$ contains infinitely many points. Hence the same argument produces the result. □

Comparing the pre-image and the essential pre-image seems natural. It turns out that even for ϕ continuous we only have one inclusion in general.

Theorem 5.5. *If ϕ is continuous, then $\phi_\mu^{-1}(y) \subseteq \phi^{-1}(y)$ for every $y \in Y$.*

Proof. If $x \notin \phi^{-1}(y)$ then by continuity of ϕ there is $\delta > 0$ and an open set $V \ni x$ such that $V \cap \phi^{-1}(B_\delta(y)) = \emptyset$. Hence it is clear that

$$D_V(y) = \varliminf_{\delta \to 0} \frac{\mu\left(V \cap \phi^{-1}(B_\delta(y))\right)}{\mu\left(\phi^{-1}(B_\delta(y))\right)} = 0.$$

It follows that $x \notin \phi_\mu^{-1}(y)$, and the inclusion follows. □

By the definition of the essential pre-image, we always have $\phi_\mu^{-1}(y) \subseteq \overline{\operatorname{supp} \mu_y}$. For if not, there is $x \in \phi_\mu^{-1}(y)$ and $\epsilon > 0$ such that $\mu(B_\epsilon(x)) = 0$, implying $x \notin \phi_\mu^{-1}(y)$. However, in general $\phi_\mu^{-1}(y) \neq \phi^{-1}(y)$. But even if they are not equal it is possible that both of them are infinite sets, i.e. $\#\phi_\mu^{-1}(y) = \#\phi^{-1}(y)$ so that $\phi^{-1}(y)$ can be used to compute the multiplicity function m of M_ϕ. Therefore here we have two questions:

i) When $\phi_\mu^{-1}(y) = \phi^{-1}(y)$?

ii)When $\#\phi_\mu^{-1}(y) = \#\phi^{-1}(y)$ if i) is not satisfied?

In [2], Abrahamse and Kriete give partial answers to these questions and provide illuminating examples. In his article [11] Howland gives a positive answer to the first question in the case X is a complete separable metric space, by showing that it is possible to remove a set of measure zero from X so that the multiplicity function of M_ϕ equals to the cardinality of the pre-image. We present the results and examples in [2].

For the remainder of this part, let $X := [0, 1]$ and μ be the Lebesgue measure on X. For the following two theorems ϕ is real valued on X which has continuous derivative, and let $Z := \{x \in X \big| \phi'(x) = 0\}$.

Theorem 5.6. *If $y \notin \phi(Z)$ then $\phi_\mu^{-1}(y) = \phi^{-1}(y)$.*

Proof. Since $y \notin \phi(Z)$, ϕ is one-to-one in a neighbourhood of every point of $\phi^{-1}(y)$. For if not, $\phi'(y) = 0$ would imply $y \in \phi(Z)$ by the continuity of the derivative. Hence $\phi^{-1}(y)$ is a closed discrete subset of X, and therefore it is finite. Since if not, there exists a limit point x contained in X, and by continuity of ϕ, $x \in \phi^{-1}(y)$ and $\phi^{-1}(y)$ is not a discrete set. Let $\phi^{-1}(y) := \{x_1, x_2, \ldots, x_n\}$. We show that for any x_i and any open set $V \ni x_i$ we have $D_V(y) > 0$. Firstly without loss of generality $\overline{V} \not\ni x_j$ for $i \neq j$, since $V \subset W$ implies $D_V \leq D_W$. Choose $x_i \in W_i$ where W_i is open such that ϕ restricted to W_i is strictly monotone. Then we have

$$\begin{aligned} D_V(y) &= \varliminf_{\delta \to 0} \frac{\mu\left(V \cap \phi^{-1}(B_\delta(y))\right)}{\mu\left(\phi^{-1}(B_\delta(y))\right)} \\ &= \varliminf_{\delta \to 0} \frac{\mu(W_i \cap \phi^{-1}(B_\delta(y)))}{2\delta} \frac{2\delta}{\mu\Big(\bigcup_{i=1}^n W_i \cap \phi^{-1}(B_\delta(y))\Big)} \\ &= \frac{1}{\left|\phi'(x_i)\right|} \left(\sum_{i=1}^n \frac{1}{\left|\phi'(x_i)\right|} \right)^{-1} > 0. \end{aligned}$$

Therefore $x_i \in \phi_\mu^{-1}(y)$. By Theorem 5.5, $\phi_\mu^{-1}(y) = \phi^{-1}(y)$. $\square$

Theorem 5.7. *Assume that the boundary of Z, which we will denote by ∂Z has μ measure zero. If $y \in \phi(Z)$ and $\mu(\phi^{-1}(y)) > 0$, then $\phi_\mu^{-1}(y)$ and $\phi^{-1}(y)$ are both infinite sets.*

Proof. The set $\phi^{-1}(y) \cap (X \setminus Z)$ has μ measure zero since it discrete by the proof of the previous theorem. Since $\mu(\partial Z) = 0$, there is an open interval I with $I \subset Z$ such that I contains a point of $\phi^{-1}(y)$, since by the assumption $\mu(\phi^{-1}(y)) > 0$. But $\phi'(x) = 0$ for all $x \in I \subset Z$, and therefore $\phi = y$ identically on I. It is not difficult to verify that $I \subset \phi_\mu^{-1}(y)$. Hence both $\phi^{-1}(y)$ and $\phi_\mu^{-1}(y)$ are infinite sets. $\square$

Theorems 5.6 and 5.7 give information about every $y \in Y$ except for which are contained in $\phi(Z)$ with having ν measure 0. Therefore, if the set of all such y has ν measure zero, then the multiplicity function can be calculated in terms of the pre-image $\phi^{-1}(y)$. As an illustration, if $\nu(\phi(Z)) = 0$, then by Theorem 5.6 $\phi_\mu^{-1}(y) = \phi^{-1}(y)$ for ν-a.e y. Hence by Theorem 5.4 the function $y \to \#\phi^{-1}(y)$ is the same with the multiplicity function m of M_ϕ ν-a.e, i.e. $y \to \#\phi^{-1}(y)$ is a representative of the class $[m]_\nu$.

We give two of the examples in [2], to illustrate Theorem 5.7, and to see how it may fail. In these examples, $X := [0, 1]$, μ is the Lebesgue measure, and ϕ is a real valued continuous function.

Example 5.8. Let ϕ be zero everywhere on $[0, 1/3]$, strictly increasing on $[1/3, 1/2]$, strictly decreasing on $[1/2, 1]$ and such that $\phi(x) = 2/3 - x$ on some small neighbourhood of $x = 2/3$. Then for any open set $V \subset X$, $D_V(0) = 3\mu(V \cap [0, 1/3])$ and hence $\phi_\mu^{-1}(0) = [0, 1/3]$. Clearly $\phi^{-1}(0) = [0, 1/3] \cup \{2/3\}$. Here the conclusion of Theorem 5.7 is valid.

Example 5.9. Let ϕ be strictly increasing on $[0, 1/2)$, strictly decreasing on $(1/2, 1]$. Also, assume there are open intervals I and J containing $x = 1/3$ and $x = 2/3$ respectively such that $\phi(x) = (x - 1/3)^3$ on I and $\phi(x) = 2/3 - x$ on J. Then if U is any open set containing $1/3$, we have $D_U(0) = 1$, and if $V := (1/2, 1)$, then $D_V(0) = 0$. Hence $\phi_\mu^{-1}(0) = \{1/3\}$. Clearly $\phi^{-1}(0) = \{1/3, 2/3\}$. Therefore here the conclusion of Theorem 5.7 fails.

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