POISSON DISORDER PROBLEM WITH CONTROL ON COSTLY OBSERVATIONS

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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A Poisson process X_t changes its rate at an unknown and unobservable time θ from λ_0 to λ_1. Detecting the change time as quickly as possible in an optimal way is described in literature as the Poisson disorder problem. We provide a more realistic generalization of the disorder problem for Poisson process by introducing fixed and continuous costs for being able to observe the arrival process. As a result, in addition to finding the optimal alarm time, we also characterize an optimal way of observing the arrival process. We illustrate the structure of the solution spaces with the help of some numerical examples.

Keywords: Poisson disorder problem; stochastic control; piecewise deterministic Markov processes.

ÖZET

POISSON DISORDER PROBLEM WITH CONTROL ON COSTLY OBSERVATIONS

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X_t Poisson süreci bilinmeyen ve güzlemlenemeyen Θ anında hızını λ_0'dan λ_1'e değiştirmektedir. Bu değişimi mümkün olan en çabuk tespit etmek literatrde Poisson Düzensizlik Problemi olarak tanımlanmaktadır. Bu çalışmada, hız değişiminin tespiti için geçen süre sabit ve sürekli maliyetlerle ilişkilendirilerek Poisson Düzensizlik Problemi daha geniş bir çerçevede ve daha gerçekçi bir bakış açısıyla ele alınmıştır. Sonuç olarak, en iyi alarm zamanının yanısıra, değişimin oluş sürecini gözlemek için en iyi yöntem de ortaya konmuştur. Çözüm uzaylarının yapısının gösterimi için sayısal örneklerden problemlerden faydalanılmıştır.

Anahtar sözcükler: Poisson Düzensizlik Problemi.

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Glossary

J_0 dynamic programming operator. 27, 28

T_1 first arrival of the observed process. 16

$U(\pi)$ minimum Bayes risk. 13

U^{ϵ}_{n+1} $\epsilon-$ optimal control for the problem. 21

$V(\cdot,\cdot)$ value function of the problem. 21

$V_n(\cdot,\cdot)$ successive approximations of $V(\cdot,\cdot)$ which are obtained by terminating the original problem by the n^{th} non-terminating event. 20

X^{δ}_t observed arrival process. 16

Ω collection of all sample paths. 11

Φ_t odds-ratio process. 2

Π_t posterior probability process. 2

$\alpha^{\delta}_{on}(t)$ number of times we have turned on the observation control upto time t. 14

λ_0 rate of the underlying Poisson process before change occurs. 1

λ_1 rate of the underlying Poisson process after change occurs. 1

$\mathbb{P}_0$ reference probability measure. 13

$\mathbb{P}$ probability measure in which our problem is defined in. 11

ρ_n n^{th} non-terminating event. 18

σ_i switching off the observation control for the i^{th} time. 5

τ_i switching on the observation control for the i^{th} time. 5

τ Alarm time. 1

θ unobserved and unknown change time. 1

a fixed cost to switch on the observation control. 5

b cost of continuous observation once the control is switched on. 5

c penalty cost per unit time. 1

$R_\tau(\pi)$ Bayes risk function. 1

non-terminating event either switching on/off the observation control or an observed arrival. x, 18

PDMP piecewise deterministic Markov processes. 15

Chapter 1

Preliminaries

Definition 1.1.1 (Sigma-algebra)**.** *If Ω is a given set, then a $\sigma-$algebra $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:*

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^c \in \mathcal{F}$, *where* $F^c = \Omega \setminus F$ *is the complement of* F *in* Ω

(iii) $A_1, A_2, \ldots \in \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} \Rightarrow A_i \in \mathcal{F}$

Definition 1.1.2 (Probability measure)**.** *Let $(\Omega, \mathcal{F})$ be a measurable space. A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that,*

(i) $\mathbb{P}(\emptyset) = 0$, $\mathbb{P}(\Omega) = 1$

(ii) if $A_1, A_2, \ldots \in \mathcal{F}$ *and* $\{A_i\}_{i=1}^{\infty}$ *is disjoint then*

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

Definition 1.1.3 (Random Variable)**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X is a measurable function from the sample space Ω to $\mathbb{R}$;*

$$X : \Omega \to \mathbb{R},$$

that is, the inverse image of any Borel set is $\mathcal{F}-$ measurable:

$$X^{-1}(A) = \{\omega : X(\omega) \in A\}, \quad \text{for all } A \in \mathbb{B}(\mathbb{R}).$$

Definition 1.1.4 (Stochastic process)**.** *A stochastic process is a parameterized collection of random variables*

$$\{X_t\}_{t \in T}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}^n$.

Definition 1.1.5 (Filtration)**.** *A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}_t\}_{t \geq 0}$ of $\sigma-$algebras $\mathcal{M}_t \subset \mathcal{F}$ such that*

$$0 \leq s < t \Rightarrow \mathcal{M}_s \subset \mathcal{M}_t$$

i.e. $\{\mathcal{M}_t\}$ is increasing.

Definition 1.1.6 (Stopping time)**.** *Let $(I, \leq)$ be an ordered index set, and let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space, i.e., a probability space equipped with a filtration. Then a random variable $\tau : \Omega \to I$ is called a stopping time if*

$$\{\omega : \tau \leq t\} \in \mathcal{F}_t.$$

Definition 1.1.7 (Strong Markov property)**.** *Suppose that $X = (X_t : t \geq 0)$ is a stochastic process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Then X is said to have the strong Markov property if, for each stopping time τ, conditioned on the event $\{\tau < \infty\}$, and for each bounded Borel function $f : \mathbb{R}^n \to \mathbb{R}$ we have,*

$$\mathbb{E}[f(X_{\tau+h})|\mathcal{F}_\tau] = \mathbb{E}[f(X_h)|\sigma(X_\tau)],$$

for all $h \geq 0$.

Chapter 2

Bayesian change-detection problems for Poisson process–Brief review

Change-detection problems involve detecting the point in time (denoted as Θ), when a stochastic process abruptly changes its probability law. Also known as the disorder-problem, it has been studied under various assumptions made on the change-point itself. In this paper, we stick to the Bayesian formulation of the problem, which simply refers to the assumption made on the probability law governing (generally taken to be exponential distribution) the change-point. Historically speaking, such a framework was introduced by Shiryaev (1963), in which detecting the onset of a drift in a Wiener process was the primary object of study.

Later Galtčhuk and Razovskiĭ (1971) formulated a version of this problem for the Poisson process, in which their goal was to detect the change-point when the intensity of the Poisson process changes from a known value (λ_0) to another (λ_1, known). For a particular detection scheme denoted as τ, their Bayes risk measure was,

$$R_\tau(\pi) = \mathbb{P}\left\{\tau < \theta\right\} + c \cdot \mathbb{E}\left[(\tau - \theta)^+\right], \tag{2.1}$$

which has two components, one denoting the frequency of false alarms and the second denoting penalty (\$ c per unit time) for average delay in detection.

In their solution, they however make an assumption ($\lambda + c \geq \lambda_1 > \lambda_0$) that the various constants in the problem are supposed to satisfy. Davis (1976) improved the solution by solving under a less stringent assumption ($\lambda + c \geq \lambda_1 - \lambda_0 > 0$) and also noticed a commonality in the different measures of the Bayes risk.

$$R^1_\tau(\pi) = \mathbb{P}\{\tau < \theta - \epsilon\} + c \cdot \mathbb{E}\left[(\tau - \theta)^+\right], \; R^2_\tau(\pi) = \mathbb{E}\left[(\theta - \tau)^+\right] + c \cdot \mathbb{E}\left[(\tau - \theta)^+\right] \tag{2.2}$$

In essense, he suggested, $R^1_\tau(\pi)$, $R^2_\tau(\pi)$ in (2.2), are special cases of a more general problem,

$$R^D_\tau(\pi) = a + b \int_0^\tau (\Pi_s - k)\mathrm{d}s,$$

where $\Pi_t := \mathbb{P}\left\{\theta \leq t | \mathcal{F}^X_t\right\}$ is the posterior probability process a, b, $k \in \mathbb{R}$ with $b > 0$ and $k \in [0, 1]$ is the only relevant constant to optimizing the Bayes risk. Note also that $R_\tau(\pi)$ in (2.1) is a special case of R^1 with $\epsilon = 0$.

Peskir and Shiryaev (2002) solved the problem by assuming linear penalty for late detection, while Bayraktar and Dayanik (2006), solved the problem assuming (a more general) exponential penalty as in (2.3).

$$R^3_\tau(\pi) = \mathbb{P}\{\tau < \theta\} + c \cdot \mathbb{E}\left[e^{\alpha(\tau-\theta)^+} - 1\right], \tag{2.3}$$

Bayraktar et al. (2005) provided the solution of the problem in its full generality where the authors showed both the linear and exponential penalty forms of cost, $R^i_\tau(\pi)$, $i = 1, 2, 3$; can be expressed in a general form under a reference probability measure, $\mathbb{P}_0$ as

$$\mathfrak{R}(\pi;\, \Phi^{(\alpha)},\, k) = (1-\pi)e^{-\lambda\epsilon} + c(1-\pi)\mathbb{E}_0\left[\int_0^\tau e^{-\lambda t}(\Phi^{(\alpha)}_t - k)\mathrm{d}s\right],$$

where the constants take appropriate values and α takes the same value as in (2.3). Also, to be noted is the use of the odds-ratio process, $\Phi_t := \Pi_t/(1 - \Pi_t)$ instead of Π_t.

One of the initial deviations from the traditional formulation of the problem was studied by Bayraktar et al. (2006), in which the authors solve an adaptive version of the problem in that, not just the change-point is random but also, the intensity after the change-point is assumed random.

Using the theory of optimal stopping for piecewise-deterministic Markov processes (Davis, 1993), Dayanik and Sezer (2006) solved the compound Poisson disorder problem completely in a way which appears more straightforward, unlike the methods used earlier in the literature. This method also forms the basis of our solution technique.

Chapter 3

Introduction

In this study we re-visit the Poisson disorder problem with a different objective in mind. Let us briefly state the classical case– suppose that the rate of a Poisson process X_t changes from one known value to another (known) value at a random and unobservable time θ, which is nonnegative and has exponential distribution

$$\mathbb{P}\{\theta = 0\} = \pi, \qquad \text{and} \qquad \mathbb{P}\{\theta > t\} = (1-\pi)e^{-\lambda t}, \; t \geq 0, \pi \in [0, 1), \lambda > 0.$$

The problem then is to detect the disorder time θ as quickly as possible while minimizing a suitable measure of expected cost,

$$V(\phi) = \inf_{\tau} \left(\mathbb{P}\{\tau < \theta\} + c \cdot \mathbb{E}\left[(\tau - \theta)^+\right] \right). \tag{3.1}$$

In the above optimal stopping problem (3.1), $\mathbb{P}\{\tau < \theta\}$ is understood as the probability of a *'false alarm'*, $\mathbb{E}\left[(\tau - \theta)^+\right]$ as the average delay of detection and c is the penalty cost per unit time for delayed detection. The alarm time τ is a stopping time of the history of the arrival process X_t.

In the classical version of the problem, we have the cushion of continuous, un-interrupted and zero-cost observation of the arrival process, which might be a heavy assumption to make in certain situations. This leads us to the question of what happens in the more realistic case of having to pay to observe the arrival process. This is the question we try to formulate and later, solve.

Formulation of the problem requires us to introduce an observation control. This control enables the user to *switch on* and *switch off* the observation control as and when s/he pleases. When the control is on, user observes the underlying arrival process X_t^δ. We define the Bayes risk as,

$$\begin{aligned} R_\tau^\delta(\pi) := \mathbb{P}\{\tau < \theta\} + c \cdot \mathbb{E}\left[(\tau-\theta)^+\right] + a \cdot \sum_{i=1}^{\infty} \mathbb{P}\{\tau_i \le \tau\} \\ + b \cdot \sum_{i=1}^{\infty} \mathbb{E}\left[(\sigma_i \wedge \tau - \tau_i \wedge \tau)\right] \end{aligned} . \tag{3.2}$$

In the above risk measure (3.2), 'a' denotes the cost to turn the observation control on, $\sum_{i=1}^{\infty} \mathbb{P}\{\tau_i \le \tau\}$ denotes the expected number of times we turn the control on, 'b' denotes the cost incurred per unit time of continuous observation once the control is switched on and lastly, $\sum_{i=1}^{\infty} \mathbb{E}\left[(\sigma_i \wedge \tau - \tau_i \wedge \tau)\right]$ denotes the total average length of time we observe the arrival process X_t^δ. The first two terms have the same meaning as in (3.1). In this framework we attempt to minimize $R_\tau(\pi)$ over the set of all controls and stopping times adapted to the history of X_t^δ, for an optimal control, in addition to the *Bayes-optimal alarm time*. Note, the superscript of X_t^δ is to remind us what is otherwise stated implicitly in our study– we *control* the history of observations.

The solution methodology we adopt in our study is similar to the one presented in Dayanik and Sezer (2006), in which the authors adapted a method of Gugerli (1986) and Davis (1993, Chapter 5) to solve the compound Poisson disorder problem. As in Dayanik and Sezer, we study the sample path behavior of the odds-ratio process Φ_t^δ, which turns out to be the sufficient statistic in our problem. The odds-ratio process also belongs to the family of piecewise deterministic Markov processes (Davis, 1993, Chapter 2). Since only the jump times are random, we are able to slice the time domain to capture these important moments and, use the dynamic programming principle to solve the problem.

In Chapter 5, we define $\{\tau_i, \sigma_i\}$'s as stopping times of the filtrations $\left\{\mathbb{G}_t^{\delta, i-1}, \mathbb{F}_t^{\delta, j}\right\}_{t\ge 0}$, $i \ge 1$, respectively, which are defined appropriately, and τ

as a stopping time with respect to the filtration generated by the observing the arrival process, X_t^δ. We reformulate our expected cost in to the *standard* form, as in Davis (1976). In Chapter 6, we define the successive approximations of the original control problem and show that these approximations converge uniformly at an exponential rate to the original cost function. This section also states the important Theorem 6.1.9, which forms the basis of the numerical scheme which is presented later in Chapter 9. In Chapter 7, we simplify the operators defined in Chapter 6 as deterministic optimization problems which are in turn used for generating the numerical examples. In Chapter 8 we analyze the solution sets in greater detail. The special form of the optimization problems in (7.7) and (7.13) helps us reduce the dimensionality of the problems. We also show that the optimal solutions of these optimization problems admit an alternate characterization which in turn helps us give them a more familiar form in Section 9.1. In Section 8.3, we study the limiting behavior of the value function as a function of costs a and b and show that the classical Poisson disorder problem falls out as a special case when a, $b \searrow 0$ and illustrate this with numerical examples in Section 9.3.

Chapter 4

Understanding the solution

In this thesis we provide a solution to the problem of efficiently deciding when to observe an arrival process in order to detect a change in its probability law. We describe the solution in terms of the odds-ratio process which is simply the ratio of the probability of change already having occured (given all the information upto the current moment) to the probability that the change hasn't occured until now (given all the information upto the current moment).

A low value of the odds-ratio process is an indication that the change hasn't happened yet and in such a scenario it makes logical sense in not observing the underlying arrival process. As one continues with the control switched off, we build up uncertainty in the system and hence we expect the odds-ratio process to increase monotonically which is infact the case. When this odds-ratio process is beyond a certain value, it indicates to us that we are close to the change point and then it would make sense to switch on the observation control. Once the control is turned on, we observe– the only randomness in the system that is the arrivals. Gathering this information one can then update their knowledge of the probability of the change having happened or in otherwords the odds-ratio process. If the odds ratio process falls below a threshold indicating to us that there is a good chance the change hasn't happened, we then turn off the control again.

In order to raise the alarm, we wait until the point when the odds-ratio process is considerably high (although the exact value depends on the values of the constants in the problem). This just indicates to us that we have strong evidence supporting that the change has already happened and it is optimal to raise the alarm.

This description of our solution in terms of the odds-ratio process is fairly intuitive and also easy to implement. Figure 4.1 is a graphical represntation of what we just described.

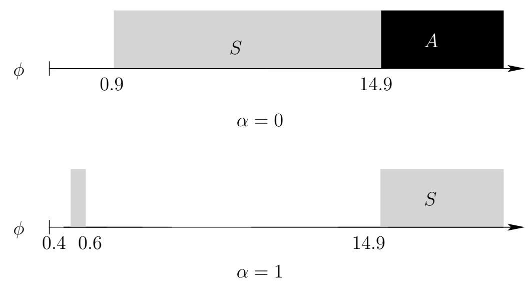


Figure 4.1: In this example $\lambda = 1$, $\lambda_0 = 3$, $c = 0.1$, $\lambda_1 = 2 * \lambda_0$, $a = 0$, $b = 0.01$.

The figure below is a simulation of the problem and its solution. We generate six different paths of the odds-ratio process to get an idea as to how the optimal alarm time looks like. For a complete solution (for the constants used in this example), refer to the state space partitions given in Figure 9.1(e)-(f). In the Figures 4.2 and 4.3 we assume the change time, $\theta = \frac{1}{\lambda} = 1$.

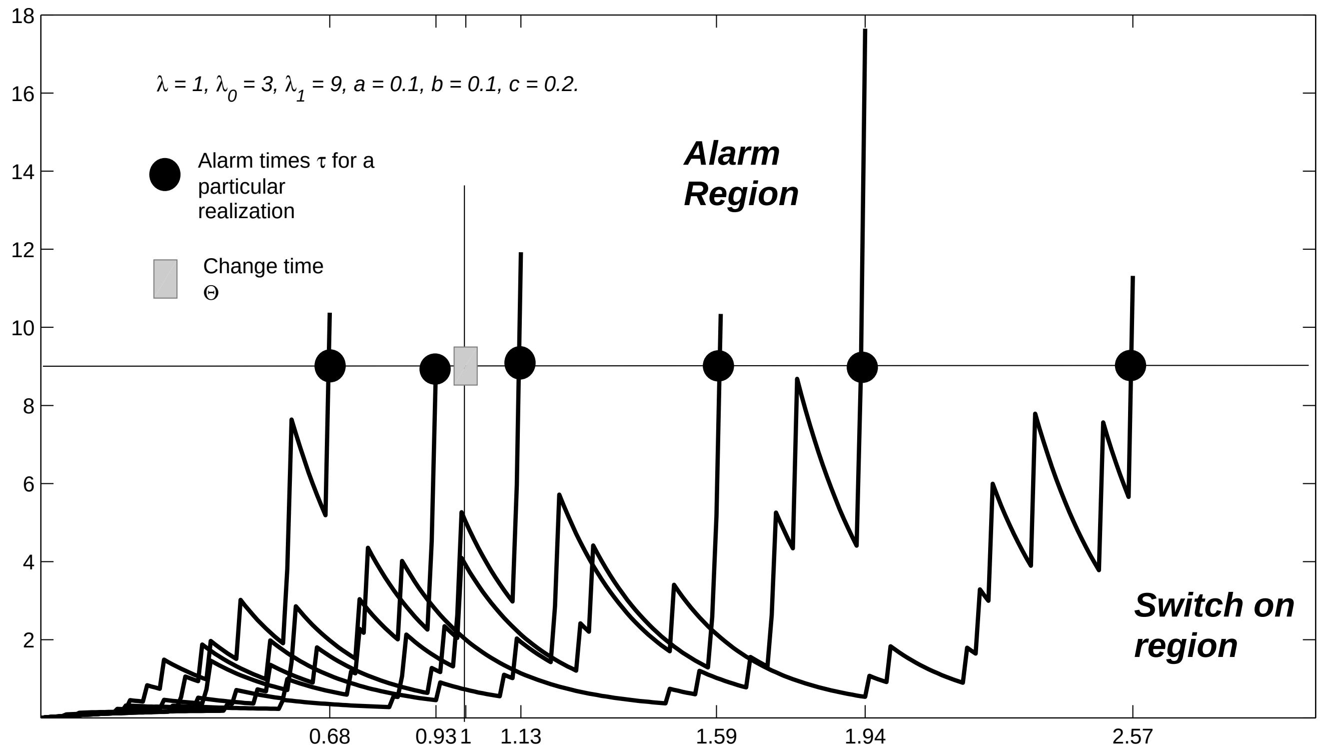


Figure 4.2: Six different sample paths of the odds-ratio process, Φ^{δ} and the corresponding optimal alarm times. In the above paths we start with $\tau_1 = 0$ a.s. and never switch off the control until we hit the alarm region threshold.

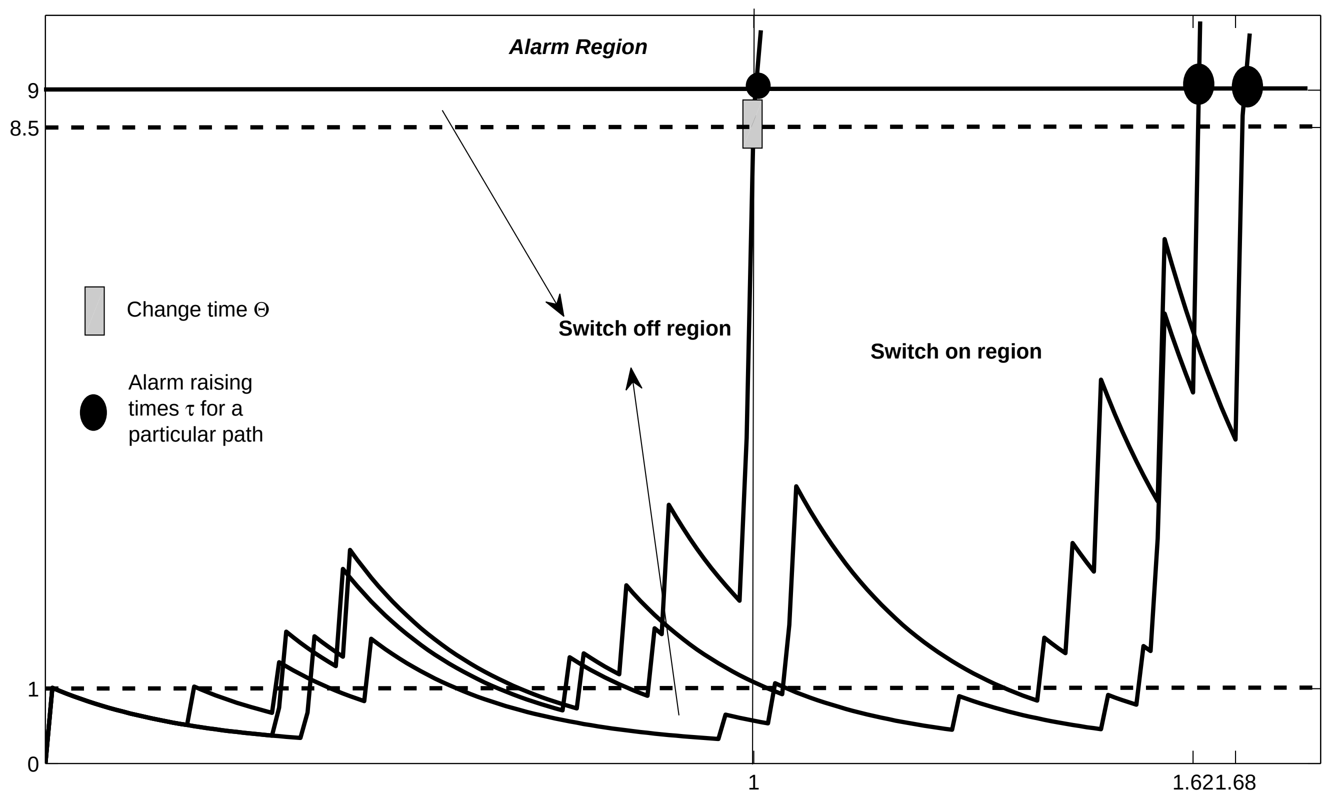


Figure 4.3: Three different sample paths of the odds-ratio process, Φ^{δ} and the corresponding optimal alarm times and switching on/off regions. In the above paths we start with $\tau_1 > 0$ a.s. and optimal control is described in Figure 9.1 (a)-(b).

Chapter 5

Problem Description

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space hosting:

- two independent Poisson processes $(X_t^0)_{t\geq 0}$ and $(X_t^1)_{t\geq 0}$ with rates λ_0 and λ_1.
- a r.v. θ independent of X^0 and X^1 with distribution $\mathbb{P}\{\theta = 0\} = \pi$ and $\mathbb{P}\{\theta > t\} = (1-\pi)e^{-\lambda t}$ for some constants $\pi \in [0,1)$, $\lambda > 0$.

In order to define the process that is observed under a sampling policy $\delta = (\tau_1, \sigma_1, \dots)$, we first define the filtrations on which these stopping times are defined. These filtrations are defined in a successive fashion capturing all the information that is there in order to switch on/off the observation control or raise the alarm.

(i) τ_1 be a stopping time of $\{\mathcal{F}_t^0\}_{t\geq 0}$ where $\mathcal{F}_t^0 = \{\emptyset, \Omega\}$, $\forall t \geq 0$.
Define $\mathcal{F}_t^{\delta,1} \equiv \mathcal{F}_t^{(\tau_1)} = \sigma\left((X(s) - X(\tau_1)) \cdot 1_{(\tau_1,\infty)}(s),\ 1_{(\tau_1,\infty)}(s),\ 0 \leq s \leq t\right)$ and let σ_1 be a stopping time of the filtration $\left\{\mathcal{F}_t^{\delta,1}\right\}_{t\geq 0}$.
Define $\mathcal{G}_t^{\delta,1} \equiv \mathcal{F}_t^{(\tau_1,\sigma_1)} = \sigma\big((X(s) - X(\tau_1)) \cdot 1_{(\tau_1,\sigma_1]}(s),\ 1_{(\tau_1,\infty)}(s),\ 1_{(\sigma_1,\infty)}(s),$
$0 \leq s \leq t\big)$

(ii) Let τ_2 be a stopping time of $\left\{\mathcal{G}_t^{\delta,1}\right\}_{t\geq 0}$.

Define $\mathcal{F}_t^{\delta,2} \equiv \mathcal{F}_t^{(\tau_1,\sigma_1,\tau_2)} = \sigma\Big((X(s)-X(\tau_1))\cdot 1_{(\tau_1,\sigma_1]}(s),\ (X(s)-X(\tau_2))\cdot 1_{(\tau_2,\infty)}(s),\ 1_{(\tau_1,\infty)}(s),\ 1_{(\sigma_1,\infty)}(s),\ 1_{(\tau_2,\infty)}(s),\ 0\leq s\leq t\Big)$. Let σ_2 be a stopping time of the filtration $\left\{\mathcal{F}_t^{\delta,2}\right\}_{t\geq 0}$.

Define $\mathcal{G}_t^{\delta,2} \equiv \mathcal{F}_t^{(\tau_1,\sigma_1,\tau_2,\sigma_2)} = \sigma\Big((X(s)-X(\tau_1))\cdot 1_{(\tau_1,\sigma_1]}(s),\ (X(s)-X(\tau_2))\cdot 1_{(\tau_2,\sigma_2]}(s),\ 1_{(\tau_1,\infty)}(s),\ 1_{(\sigma_1,\infty)}(s),\ 1_{(\tau_2,\infty)}(s),\ 1_{(\sigma_2,\infty)}(s),\ 0\leq s\leq t\Big)$.

$$\vdots$$

(iii) Let τ_n be a stopping time of $\left\{\mathcal{G}_t^{\delta,n-1}\right\}_{t\geq 0}$.

Define $\mathcal{F}_t^{\delta,n} \equiv \mathcal{F}_t^{(\tau_1,\sigma_1,\cdots,\sigma_{n-1},\tau_n)}$ as done previously. Let σ_n be a stopping time of $\left\{\mathcal{F}_t^{\delta,n}\right\}_{t\geq 0}$.

Define $\mathcal{G}_t^{\delta,n} \equiv \mathcal{F}_t^{(\tau_1,\sigma_1,\cdots,\tau_n,\sigma_n)} = \sigma\Bigg((X(s)-X(\tau_1))\cdot 1_{(\tau_1,\sigma_1]}(s),\ \cdots,\ (X(s)-X(\tau_n))\cdot 1_{(\tau_n,\sigma_n]}(s),\ 1_{(\tau_1,\infty)}(s),\ \cdots,\ 1_{(\tau_n,\infty)}(s),\ 1_{(\sigma_1,\infty)}(s),\ \cdots,\ 1_{(\sigma_n,\infty)}(s),\ 0\leq s\leq t\Bigg)$.

Finally, let τ be the stopping time of $\left\{\mathcal{F}_t^{\delta}\right\}_{t\geq 0}$ where $\mathcal{F}_t^{\delta} \equiv \bigcap_{k=1}^{\infty}\left(\mathcal{F}_t^{\delta,k}\cap\mathcal{G}_t^{\delta,k}\right)$ where $\delta = (\tau_1,\sigma_1,\ \tau_2,\sigma_2,\ \cdots)$.

The observed process under sampling policy δ is then given by,

$$X_t^{\delta} := \sum_{i=1}^{\infty}(X_{\sigma_i\wedge t} - X_{\tau_i\wedge t}),\quad t\geq 0, \tag{5.1}$$

where

$$X_t = \int_0^t 1_{\{s\leq\theta\}}\mathrm{d}X_s^0 + \int_0^t 1_{\{s>\theta\}}\mathrm{d}X_s^1.$$

The objective is to detect the disorder time θ as quickly as possible such that the alarm time τ and the observation control δ minimize the Bayes risk which is defined in (3.2) and is restated as,

$$R_\tau^\delta(\pi) = \mathbb{E}\left[1_{\{\tau<\theta\}} + c(\tau-\theta)^+ + \sum_{i=1}^{\infty} a 1_{\{\tau_i \le \tau\}} + \sum_{i=1}^{\infty} b(\sigma_i \wedge \tau - \tau_i \wedge \tau)\right], \quad (5.2)$$

over the set of all start times, end times and alarm times of appropriate filtrations and the minimum Bayes risk obtained for optimal alarm time and observation control is defined as

$$U(\pi) := \inf_{(\tau,\,\delta)\in\,\mathcal{M}} R_\tau^\delta(\pi), \quad (5.3)$$

where $\mathcal{M} = \Big\{(\tau,\ \delta);\ \delta = (\tau_1, \sigma_1, \cdots),\ \tau_i \in \mathbb{G}^{\delta,\,i-1},\ i \ge 1,\ \ \sigma_j \in \mathbb{F}^{\delta,\,j},\ j \ge 1,$

$\tau \in \mathbb{F}^\delta\Big\}$, $\mathbb{F}^{\delta,\,j} = \{\mathcal{F}_t^{\delta,\,j}\}_{t\ge 0}$, $\mathbb{G}^{\delta,\,i} = \{\mathcal{G}_t^{\delta,\,i}\}_{t\ge 0}$ and $\mathbb{F}^\delta = \{\mathcal{F}_t^\delta\}_{t\ge 0}$.

Let us also define a reference probability measure $\mathbb{P}_0$ on the measureable space $(\Omega, \mathcal{F})$ which supports the following independent stochastic elements:

(i) a r.v.θ with distribution $\mathbb{P}_0\{\theta = 0\} = \pi$ and $\mathbb{P}_0\{\theta > t\} = (1-\pi)e^{-\lambda t}$, $t \ge 0$ and

(ii) a homogenous Poisson process $X = \{X(t);\, t \ge 0\}$ with rate λ_0.

We enlarge the filtration generated by the observed process X_t^δ by including the sigma-algebra generated by the random variable θ as follows, $\mathcal{H}_t^\delta = \mathcal{F}_t^\delta \vee \sigma(\theta)$ and we define $\mathcal{F}$ to be the sigma-algebra generated by $\cup_{t\ge 0}\{\mathcal{H}_t^\delta\}$. Thus under the probability measure $\mathbb{P}_0$, we not just have the information generated by the process X_t^δ, we also have the knowledge of the random variable θ. Defining these stochastic elements to be *independent* under $\mathbb{P}_0$, proves to be useful for the calculations done under the measure $\mathbb{P}_0$. We are now left with the task of retrieving the probability measure $\mathbb{P}$ defined on $(\Omega,\ \mathcal{F})$, that we started out with. This we do by defining a stochastic process Z_t^δ which is adapted to the enhanced filtration $\mathcal{H}_t^\delta$ as follows,

$$Z_t^\delta = 1_{\{\theta>t\}} + 1_{\{\theta\le t\}} \cdot \frac{L_t^\delta}{L_\theta^\delta}, \quad (5.4)$$

where

$$L_t^\delta = \exp\left\{\log\left(\frac{\lambda_1}{\lambda_0}\right)\int_0^t \alpha^\delta(s)\mathrm{d}X_s - (\lambda_1-\lambda_0)\int_0^t \alpha^\delta(s)\mathrm{d}s\right\}, \tag{5.5}$$

and

$$\alpha^\delta(s) = \sum_{i=1}^\infty 1_{(\tau_i,\sigma_i]}(s).$$

We then define the Radon-Nikodym process as follows,

$$\left.\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\mathbb{P}_0}\right|_{\mathcal{H}_t^\delta} = Z_t^\delta. \tag{5.6}$$

Since $\mathbb{P}_0$ and $\mathbb{P}$ agree on $\mathcal{H}_0^\delta = \sigma(\theta)$, the random variable θ has the same probability law under both measures. Also *given* θ, the process X_t is Poisson with intensity λ_0 on the event $\{t<\theta\}$ and is Poisson with intesntiy λ_1 on the event $\{t\geq\theta\}$. This verifies the probability laws that X_t and θ were assumed to follow under the measure $\mathbb{P}$.

The cost function defined in (5.2) could be rewritten in such a way that the r.v. θ could be eliminated from it (by conditioning on $\mathcal{F}_\tau^\delta$ under the $\mathbb{P}_0$ measure) to obtain the following equivalent formulation

$$\begin{aligned} R_\tau^\delta(\pi) = (1-\pi) + c(1-\pi)\mathbb{E}_0\Bigg\{ &\int_0^\tau e^{-\lambda s}\left[\Phi_s^\delta + \frac{b}{c}\alpha^\delta(s)\left(1+\Phi_s^\delta\right) - \frac{\lambda}{c}\right]\mathrm{d}s \\ &+ \frac{a}{c}\int_0^\tau e^{-\lambda s}\left(1+\Phi_s^\delta\right)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} \end{aligned}, \tag{5.7}$$

where Φ_t^δ, the odds-ratio process and observation on-times $\alpha_{on}^\delta(t)$ are defined as

$$\Phi_t^\delta = \frac{\mathbb{P}\left\{\theta\leq t|\mathcal{F}_t^\delta\right\}}{\mathbb{P}\left\{\theta> t|\mathcal{F}_t^\delta\right\}}, \quad t\geq 0 \quad \text{and} \tag{5.8}$$

$$\alpha_{on}^\delta(t) = \sum_{i=1}^\infty 1_{[\tau_i,\infty)}(t). \tag{5.9}$$

The details of the above formulation and the dynamics of Φ_t^δ are provided in the appendix. The minimum Bayes risk in (5.3) can be written as,

$$U(\pi) = (1-\pi) + c(1-\pi)V\left(\alpha, \frac{\pi}{1-\pi}\right), \quad \pi \in [0,1)$$

in terms of the value function

$$V(\alpha,\phi) := \inf_{(\tau,\delta)\in\mathcal{M}} \mathbb{E}_0\left[\int_0^\tau e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^\tau e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\middle|\alpha_0^\delta = \alpha,\right. ,$$
$$\left.\Phi_0^\delta = \phi\right] \tag{5.10}$$

where

$$\alpha_0^\delta = \alpha \;=\; 1_{\{\tau_1=0\}},$$
$$g(\alpha,\phi) = \phi + \frac{b}{c}\alpha(1+\phi) - \frac{\lambda}{c}, \qquad \alpha \in \{0,1\}\,,\; \phi \in \mathbb{R}_+ \tag{5.11}$$
$$h(\phi) = \frac{a}{c}(1+\phi), \quad \phi \in \mathbb{R}_+. \tag{5.12}$$

The process driving the above value function is the odds-ratio process, Φ_t^δ. This process admits the stochastic differential equation given in (A.7), from which equation it is also clear that Φ_t^δ belongs to the class of piecewise deterministic Markov processes (*PDMP*), first introduced in Davis (1993, Chapter 2). *PDMP* is an important class of non-diffusion processes that have numerous applications in real world and some of these are outlined in Davis. We adapt the theory that is developed by Davis for optimal stopping problems involving *PDMP*s.

Our method of solving the original value function $V(\alpha,\,\phi)$ involves in slicing the time domain in such a way that, the randomness only appears at the end points of these sliced intervals. Between these end points our problem evovles deterministically. This forms the basis of the dynamic programming approach we adopt to solve our value function. The following operators acting on bounded functions $w:\ \{0,1\}\times\mathbb{R}_+\mapsto\mathbb{R}$ help us in formulating and studying the subproblems.

$$(Jw)(t,\,s,\,1,\,\phi)=\mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{t\wedge s\wedge T_1} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s+1_{\{s<t\wedge T_1\}}e^{-\lambda s}w(0,\Phi_s^\delta) + 1_{\{T_1<t\wedge s\}}e^{-\lambda T_1}w(1,\Phi_{T_1}^\delta)\Bigg],$$

$$(Jw)\,(t,\,q,\,0,\phi)=\mathbb{E}_0^{(0,\,\phi)}\left[\int_0^{t\wedge q} e^{-\lambda s}g(\alpha_s^\delta,\Phi_s^\delta)\mathrm{d}s+1_{\{q<t\}}e^{-\lambda q}\Big(h(\Phi_q^\delta)+w(1,\Phi_q^\delta)\Big)\right],$$

$$(J_m w)(1,\,\phi)=\inf_{t,\,s\geq m}(Jw)(t,\,s,\,1,\,\phi),\quad \phi,\,m,\,t,\,s\in\mathbb{R}_+,$$
$$(J_m w)(0,\,\phi)=\inf_{t,\,q\geq m}(Jw)(t,\,q,\,0,\,\phi),\quad \phi,\,m,\,t,\,q\in\mathbb{R}_+.$$

where $T_1,\,T_2,\,\cdots$ are the jump times of the process X_t^δ. Then owing to the characterization of stopping times of a jump process (refer to §7) we can show that,

$$(J_0 w)\,(1,\phi)=\inf_{(\tau,\,\sigma_1)\in\mathcal{M}}\mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{\tau\wedge\sigma_1\wedge T_1} e^{-\lambda s}g(\alpha_s^\delta,\Phi_s^\delta)\mathrm{d}s+1_{\{\sigma_1<\tau\wedge T_1\}}e^{-\lambda\sigma_1}\cdot w(0,\Phi_{\sigma_1}^\delta)+1_{\{T_1<\sigma_1\wedge\tau\}}e^{-\lambda T_1}w(1,\Phi_{T_1}^\delta)\Bigg], \tag{5.13}$$

$$(J_0 w)\,(0,\phi)=\inf_{(\tau,\,\tau_1)\in\mathcal{M}}\mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{\tau\wedge\tau_1} e^{-\lambda s}g(\alpha_s^\delta,\Phi_s^\delta)\mathrm{d}s+1_{\{\tau_1<\tau\}}e^{-\lambda\tau_1}\Big(h(\Phi_{\tau_1}^\delta) + w(1,\Phi_{\tau_1}^\delta)\Big)\Bigg], \tag{5.14}$$

where $\mathbb{E}_0^{(\alpha,\,\phi)}$ is the expectation $\mathbb{E}_0$ under $\mathbb{P}_0$ given that $\alpha_0^\delta=\alpha$ and $\Phi_0^\delta=\phi$. Put simply, if we knew the solution of a subproblem w, the operator J_0 maps it to the optimal solution and control of a larger (in the sense of time domain) subproblem. By repeatedly applying the J_0 operator, we hope to achieve the solution of the original value function $V(\cdot, \cdot)$. This is precisely the goal of the next section wherein we define these subproblems carefully.

Note. If $b > \lambda$ in (5.11), the optimal control has a deterministic structure. We immediately turn off the observation control (if it is initially turned on), never turn it on again and wait until the odds-ratio process hits the level $\frac{\lambda}{c}$ and raise the alarm. That is (δ, τ) is given by

$$\left\{ \begin{array}{ll} \sigma_1 = 0,\, \tau_2 = \infty \text{ and } \tau = \inf\left\{t > 0 \,:\, y(t,\, \phi) > \lambda/c\right\}, & \text{if } \tau_1 = 0, \\ \tau_1 = \infty \text{ and } \tau = \inf\left\{t > 0 \,:\, y(t,\, \phi) > \lambda/c\right\}, & \text{if } \tau_1 > 0, \end{array} \right\}$$

where $y(t, \phi)$ is defined in (7.10).

Chapter 6

Successive approximations

Let us introduce the family of stochastic control problems

$$V_n(\alpha,\phi) := \inf_{(\tau,\delta)\in\mathcal{M}} \mathbb{E}_0^{(\alpha,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right], \tag{6.1}$$

where $g(\cdot,\,\cdot)$, $h(\cdot)$ are as defined in (5.11), (5.12) and ρ_n is the n^{th} non-terminating event that is observed in a particular realization of the problem. We classify events that occur in our problem based on whether they end the problem (the only terminating event is the occurence of τ) or the non-terminating ones (which include observing an arrival T_i, turning on/off (τ_i/σ_i) the observation control). The family of optimal stopping problems in (6.1) are obtained by automatically stopping the odds-ratio process Φ_s^δ at the n^{th} non-terminating event at the latest. $\frac{-1}{c} \le V_n(\alpha,\,\phi) \le 0,\ n \ge 0$, and the sequence $(V_n)_{n\ge 0}$ is decreasing since the random variable ρ_n increases a.s.. Therefore, $\lim_{n\to\infty} V_n(\alpha,\,\phi) = V(\alpha,\,\phi)$ exists everywhere. It is easy to see that $V_n \ge V,\ n \in \mathbb{N}$.

Let us define successively,

$$v_n(\alpha,\,\phi) = \begin{cases} 0, & \text{when } n = 0, \\ (J_0 v_{n-1})(\alpha,\,\phi), & \text{when } n \in \mathbb{N}, \end{cases},\quad \alpha \in \{0,\,1\},\ \phi \ge 0. \tag{6.2}$$

Proposition 6.1.8. *As $n \to \infty$, the sequence $V_n(\alpha,\,\phi)$ converges to $V(\alpha,\,\phi)$. In*

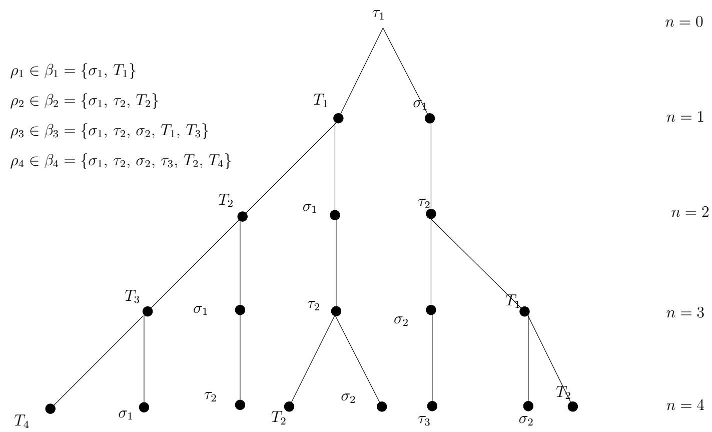


Figure 6.1: Tree of non-terminating events when $\tau_1 = 0$ a.s.

fact, for $n > 2\left\lceil \frac{1}{a} \right\rceil$, $n \in \mathbb{N}$, $\alpha \in \{0,1\}$, $\phi \in \mathbb{R}_+$, *we have,*

$$\frac{-1}{c}\left(\frac{\lambda_0}{\lambda_0+\lambda}\right)^{n-2\left\lceil \frac{1}{a} \right\rceil} \leq V(\alpha,\, \phi) - V_n(\alpha,\, \phi) \leq 0. \tag{6.3}$$

Proof.

$$\begin{aligned}
&\mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right] \\
&= \mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right] \\
&+ \mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}}1_{\{\rho_n<\tau\}}\left\{\int_{\rho_n}^{\tau} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right\}\right] \\
&= \mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right] \\
&+ \mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}}1_{\{\rho_n<\tau\}}\int_{\rho_n}^{\tau} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s\right] \\
&+ \mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}}1_{\{\rho_n<\tau\}}\int_0^{\tau} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right\}\right] \\
&\geq \mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right] \\
&+ \mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}}1_{\{\rho_n<\tau\}}\int_{\rho_n}^{\tau} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s\right] \\
&\geq \mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right]
\end{aligned}$$

$$+ \mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}}\int_{\rho_n}^{\infty} e^{-\lambda s}\left(\frac{-\lambda}{c}\right)\mathrm{d}s\right]$$
$$= \mathbb{E}_0^{(\alpha,\,\phi)}\left[\int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right]$$
$$- \frac{1}{c}\mathbb{E}_0^{(\alpha,\,\phi)}\left[1_{\{\rho_n<\infty\}} e^{-\lambda\rho_n}\right] \tag{6.4}$$

We could note here that an optimal solution to the original problem (5.10) would have an upper bound on the number of times we can turn the observation control on (since $U(\pi) \le (1-\pi) \le 1$). The cost to turn on the observation control is a, hence on the event $\{\rho_n < \infty\}$, $\lceil 1/a \rceil$ is an upper bound on how many times we could turn the control on. If $n > 2\lceil 1/a \rceil$, and we have already switched on the control $\lceil 1/a \rceil$ times without observing an arrival, then any non-terminating event beyond this point can only be caused by an arrival. The n^{th} non-terminating event would be $T_{n-2\lceil 1/a \rceil}$, i.e. $(n-2\lceil 1/a \rceil)^{th}$ arrival of the observed Poisson process X_t^δ. This corresponds to at least the $(n-2\lceil 1/a \rceil)^{th}$ arrival of the original Poisson process X_t, i.e. $T_{n-2\lceil 1/a \rceil} \ge S_{n-2\lceil 1/a \rceil}$ $\mathbb{P}_0-$a.s., where S_i's denote arrival times of the Poisson process X_t. Since $\rho_n \ge T_{n-2\lceil 1/a \rceil} \implies \rho_n \ge S_{n-2\lceil 1/a \rceil}$. We know that under $\mathbb{P}_0$ measure S_i's have Erlang distribution with parameters i and λ_0. Thus,

$$-\frac{1}{c}\mathbb{E}_0^{(\alpha,\,\phi)}\left[e^{-\lambda\rho_n}\right] \ge -\frac{1}{c}\mathbb{E}_0^{(\alpha,\,\phi)}\left[e^{-\lambda S_{n-2\lceil 1/a \rceil}}\right] = \frac{-1}{c}\left(\frac{\lambda_0}{\lambda_0+\lambda}\right)^{n-2\left\lceil \frac{1}{a}\right\rceil}.$$

Putting it back in (6.4) and then by taking infimum on the two sides gives us the first inequality in (6.3). □

The goal hence is to be able to compute these successive control problems. This is where we notice the immediate application of the operators earlier defined. Starting with $v_0 \equiv 0$ (represents in a sense the cost incurred if we are allowed to wait only until the zero-th non-terminating event, $\rho_0 := 0$ a.s.), defining $v_1(\alpha,\,\phi) = (J_0 v_0)(\alpha,\,\phi)$, we compute the optimal cost if we were allowed to wait only until the first non-terminating event in terms of v_0 using the *dynamic programming* principle. If we continue applying this operator repeatedly, we hope to end up with the optimal solution for $V(\cdot,\,\cdot)$.

It is natural to think of a connection between $v_n(\cdot,\,\cdot)$ and $V_n(\cdot,\,\cdot)$ since both essentially represent the same problem of an optimal solution to $V(\cdot,\, \cdot)$ (in the case we are allowed to wait latest until the n^{th} non-terminating event). This is precisely the purpose of the next important theorem which shows that indeed these iterative value functions are equal. The theorem also outlines the solution of the original problem.

Theorem 6.1.9. *For every $v_n(\cdot,\, \cdot)$, $n \in \mathbb{N}$, the functions v_n and V_n coincide. For every $\epsilon \geq 0$, let*

$$A_n^{\epsilon}(\alpha,\, \phi) = \left\{(t,\, s) \in \overline{\mathbb{R}}_+^2 :\ (Jv_n)(t,\, s,\, \alpha,\, \phi) < (J_0 v_n)(\alpha,\, \phi) + \epsilon\right\},\ n \in \mathbb{N}_0,$$

$$\phi \in \mathbb{R}_+,\ \overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{+\infty\} \qquad \textit{and} \qquad \overline{\mathbb{R}}_+^2 := \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+.$$

$$(t_n^{\epsilon}(1,\, \phi),\, s_n^{\epsilon}(1,\, \phi)) \in \underset{(t,\, s) \in A_n^{\epsilon}(1,\, \phi)}{\operatorname{argmin}} \{t \wedge s\},$$
$$(t_n^{\epsilon}(0,\, \phi),\, q_n^{\epsilon}(0,\, \phi)) \in \underset{(t,\, q) \in A_n^{\epsilon}(0,\, \phi)}{\operatorname{argmin}} \{t \wedge q\}.$$

If $A_n^{\epsilon}(1,\, \phi) = \emptyset$, then $t_n^{\epsilon}(1,\, \phi) \wedge s_n^{\epsilon}(1,\, \phi) = +\infty$ and similarly if, $A_n^{\epsilon}(0,\, \phi) = \emptyset$ then $t_n^{\epsilon}(0,\, \phi) \wedge q_n^{\epsilon}(0,\, \phi) = +\infty$ accordingly.

For $\alpha = 1$ we have,

$$U_1^{\epsilon} := t_0^{\epsilon}(1,\, \Phi_0^{\delta}) \wedge s_0^{\epsilon}(1,\, \Phi_0^{\delta}) \wedge T_1,$$

and for $\quad n = 1,\, 2,\, \cdots,$ we have

$$U_{n+1}^{\epsilon} := \begin{Bmatrix} t_n^{\epsilon/3}(1,\, \Phi_0^{\delta}); & \textit{if} \quad t_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) < T_1 \wedge s_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) \\ T_1 + U_n^{\epsilon/3} \circ \theta_{T_1}; & \textit{if} \quad T_1 < s_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) \wedge t_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) \\ s_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) + U_n^{\epsilon/3} \circ \theta_{s_n^{\epsilon/3}(1,\, \Phi_0^{\delta})}; & \textit{if} \quad s_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) < T_1 \wedge t_n^{\epsilon/3}(1,\, \Phi_0^{\delta}) \end{Bmatrix}.$$

For $\alpha = 0$ we have,

$$U_1^{\epsilon} := t_0^{\epsilon}(0,\, \Phi_0^{\delta}) \wedge q_0^{\epsilon}(0,\, \Phi_0^{\delta}),$$

and for $n = 1, 2, \cdots$, *we have*

$$U^{\epsilon}_{n+1} := \begin{Bmatrix} t^{\epsilon/2}_n(0,\, \Phi^{\delta}_0); & \textit{if} \quad t^{\epsilon/2}_n(1,\, \Phi^{\delta}_0) < q^{\epsilon/2}_n(0,\, \Phi^{\delta}_0) \\ q^{\epsilon/2}_n(0,\, \Phi^{\delta}_0) + U^{\epsilon/2}_n \circ \theta_{q^{\epsilon/2}_n(0,\, \Phi^{\delta}_0)}; & \textit{if} \quad q^{\epsilon/2}_n(0,\, \Phi^{\delta}_0) < t^{\epsilon/2}_n(0,\, \Phi^{\delta}_0) \end{Bmatrix}.$$

θ_s *is the shift operator on* Ω. *Then*

$$\mathbb{E}^{(\alpha,\,\phi)}_0 \left[\int_0^{U^{\epsilon}_n} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s) \mathrm{d}s + \int_0^{U^{\epsilon}_n} e^{-\lambda s} h(\Phi^{\delta}_s) \mathrm{d}\alpha^{\delta}_{on}(s) \right] \leq v_n(\alpha,\, \phi) + \epsilon. \quad (6.5)$$

Proof. Refer to appendix §B.1. □

Note. In the above result $t_n(1,\, \phi) \wedge s_n(1,\, \phi) = +\infty$ implies that we do not take any action until the next arrival. We would later see that $t_n(0,\, \phi) \wedge q_n(0,\, \phi) < +\infty$ falls out as a consequence of the optimization problem (Refer to Lemma 8.1.6).

Lemma 6.1.10. *There is a constant* K *such that, for every bounded* $w : \{0,1\} \times \mathbb{R} \mapsto \mathbb{R}$, $K \leq (J_0\, w)(\alpha,\, \phi) \leq 0$, $\alpha \in \{0,1\}$, $\phi \in \mathbb{R}_+$. *If* $w_1(\cdot,\, \cdot)$ *and* $w_2(\cdot,\, \cdot)$ *are bounded functions with* $w_1(\cdot,\, \cdot) \leq w_2(\cdot,\, \cdot)$, *then* $(J_0\, w_1)(\cdot,\, \cdot) \leq (J_0\, w_2)(\cdot,\, \cdot)$.

Proof. Refer to appendix §B.2. □

Lemma 6.1.11. *If* $\phi \mapsto w(\alpha,\, \phi)$ *is increasing and concave for every* $\alpha \in \{0,1\}$ *then so is* $\phi \mapsto (J_0 w)(\alpha,\, \phi)$ *for every* $\alpha \in \{0,\, 1\}$.

Proof. It follows from (7.7), (7.13). □

Lemma 6.1.12. *Every* $v_n(\alpha,\, \phi)$, $n \in \mathbb{N}_0$ *as in (6.2) is bounded and concave, and* $\frac{-1}{c} \leq \cdots \leq v_n \leq \cdots \leq v_1 \leq v_0$. *The limit*

$$v(\alpha,\, \phi) := \lim_{n \to \infty} v_n(\alpha,\, \phi), \quad \alpha \in \{0,1\}\,,\, \phi \in \mathbb{R}_+$$

exists, and is bounded, concave and non decreasing.

Proof. If $v_0(\alpha,\, \phi) \equiv 0$ and $v_n(\alpha,\, \phi) = (J_0\, v_{n-1})(\alpha,\, \phi)$, then $\|v_n(1, \phi)\| \leq \dfrac{1}{c}$, $\forall\, n$ and $\frac{-1}{c} \leq \cdots \leq v_n(1,\, \phi) \leq \cdots \leq v_1(1,\, \phi) \leq v_0(1,\, \phi)$ by using Lemma 6.1.10 and applying J_0 operator successively. Refer to (B.13), (B.15). This result is also available in Dayanik and Sezer (2006, Corollary 3.4, p. 654). □

Chapter 7

Calculating $(J_0 w)$ acting on $w : \{0,1\} \times \mathbb{R}_+ \mapsto \mathbb{R}$

For any stopping rule τ of the filtration $\mathbb{F}$ of a jump process, there is a deterministic time $t_0 \in [0, \infty]$ such that (Davis, 1993, Theorem A2.3)

(i) $\tau 1_{\{\tau < T_1\}} = t_0 1_{\{\tau < T_1\}}$

(ii) $\tau 1_{\{\tau \geq T_1\}} = t_0 1_{\{\tau \geq T_1\}}$

(iii) $\tau \wedge T_1 = t_0 \wedge T_1$.

We can extend this characterization of stoppings times of jump processes to the stopping times $\tau_1 \in \mathbb{G}_0^\delta$ and $\sigma_1 \in \mathbb{F}_1^\delta$ to get,

$$(\tau \wedge \sigma_1 \wedge T_1) = (\tau \wedge T_1) \wedge (\sigma_1 \wedge T_1) = (t_0 \wedge T_1) \wedge (s_0 \wedge T_1) = (t_0 \wedge s_0 \wedge T_1),$$

where the second equality holds since $(\sigma_1 \wedge T_1) = (s_0 \wedge T_1)$ and $(\tau \wedge T_1) = (t_0 \wedge T_1)$,

$$1_{\{\sigma_1 < \tau \wedge T_1\}} = 1_{\{\sigma_1 < \tau\}} \cdot 1_{\{\sigma_1 < T_1\}} = 1_{\{s_0 < \tau\}} \cdot 1_{\{s_0 < T_1\}} = 1_{\{s_0 < \tau \wedge T_1\}} = 1_{\{s_0 < t_0 \wedge T_1\}}$$

where the second equality holds because $\sigma_1 1_{\{\sigma_1 < T_1\}} = s_0 1_{\{\sigma_1 < T_1\}}$ and the last equality holds because $\{\tau \wedge T_1\} = \{t_0 \wedge T_1\}$. These results offer a method of simplifying $J_0(\cdot,\,\cdot)$ operator to a deterministic optimization problem which then reduces the complexity of our study.

Using the above equalities we can rewrite (5.13) as,

$$J_0 w(1,\phi) = \inf_{(t_0,\,s_0)\in\mathcal{M}} \mathbb{E}_0\Bigg[\int_0^{t_0\wedge s_0\wedge T_1} e^{-\lambda s} g(1,\Phi_s^\delta)\mathrm{d}s + 1_{\{s_0<t_0\wedge T_1\}} e^{-\lambda s_0} w(0,\Phi_{s_0}^\delta) + 1_{\{T_1<s_0\wedge t_0\}} e^{-\lambda T_1} w(1,\Phi_{T_1}^\delta)\Bigg]. \tag{7.1}$$

In order to simplify (7.1), we start with studying the process Φ_s^δ whose dynamics are given by (A.7). The process Φ_s^δ does not jump in the interval $s \in [0, t_0 \wedge s_0 \wedge T_1)$ and hence we need only to look at the continuous deterministic part of (A.7) which can solved as follows. Let $x(s,\phi) = \Phi_s^\delta$, $s \in [0, t_0 \wedge s_0 \wedge T_1)$ with $\Phi_0^\delta = \phi$. The deterministic part could be written as,

$$\mathrm{d}x = [\lambda + \widetilde{a}x]\mathrm{d}t,$$

where $\widetilde{a} = \{\lambda - (\lambda_1 - \lambda_0)\}$. Solving the above ODE gives us,

$$x(t,\,\phi) = \begin{cases} \phi_d + (\phi - \phi_d)e^{\widetilde{a}t}, & \widetilde{a} \neq 0, \\ \phi + \lambda t, & \widetilde{a} = 0, \end{cases} \quad t \in [0, t_0 \wedge s_0 \wedge T_1), \tag{7.2}$$

where $\phi_d = -\frac{\lambda}{\widetilde{a}}$. Similarly, at every jump time T_i the process Φ_s^δ follows a pure jump process and the jump size is given by (again using (A.7)),

$$\Delta\Phi_{T_i}^\delta = \left(\frac{\lambda_1}{\lambda_0} - 1\right)\Phi_{T_i^-}^\delta \underbrace{\left(X_{T_i}^\delta - X_{T_i^-}^\delta\right)}_{\text{jump size} = 1}$$

$$\Longrightarrow \left(\Phi_{T_i}^\delta - \Phi_{T_i^-}^\delta\right) = \left(\frac{\lambda_1}{\lambda_0} - 1\right)\Phi_{T_i^-}^\delta.$$

Distributing the expectation in (7.1) we have three terms each of which is simplified as follows,

$$\begin{aligned}
&\int_0^{\infty} e^{-\lambda s} g(1, x(s,\, \phi))\mathbb{E}_0[1_{\{s<t_0\wedge s_0\wedge T_1\}}]\mathrm{d}s \\
&= \int_0^{\infty} 1_{\{s<t_0\wedge s_0\}} e^{-\lambda s} g(1, x(s,\, \phi))\mathbb{P}_0\left\{s < T_1\right\}\mathrm{d}s \\
&= \int_0^{t_0\wedge s_0} e^{-(\lambda+\lambda_0)s} g(1, x(s,\, \phi))\mathrm{d}s \\
&= \int_0^{t_0\wedge s_0} e^{-(\lambda+\lambda_0)s}\left[\phi_d + (\phi-\phi_d)e^{as} + \frac{b}{c}\left(1+\phi_d+(\phi-\phi_d)e^{as}\right) - \frac{\lambda}{c}\right]\mathrm{d}s \\
&\qquad \text{(using (5.11))} \\
&= \left(\frac{1}{\lambda+\lambda_0}\right)\left(\phi_d + \frac{b}{c}(1+\phi_d) - \frac{\lambda}{c}\right)\left(1-e^{-(\lambda+\lambda_0)(t_0\wedge s_0)}\right) \\
&+ \left(\frac{\phi-\phi_d}{\lambda_1}\right)\left(1+\frac{b}{c}\right)\left(1-e^{-\lambda_1(t_0\wedge s_0)}\right),
\end{aligned} \tag{7.3}$$

when $\widetilde{a} \neq 0$ and if $\widetilde{a} = 0$ the integral simplifies to,

$$\begin{aligned}
&\left(\frac{1}{\lambda+\lambda_0}\right)\left(\phi + \frac{b}{c}(1+\phi) - \frac{\lambda}{c}\right)\left(1-e^{-(\lambda+\lambda_0)(t_0\wedge s_0)}\right) \\
&+\lambda\left(1+\frac{b}{c}\right)\left[\frac{1}{(\lambda+\lambda_0)^2} - \frac{e^{-(\lambda+\lambda_0)t_0\wedge s_0}}{\lambda+\lambda_0}\left(t_0\wedge s_0 + \frac{1}{\lambda+\lambda_0}\right)\right].
\end{aligned} \tag{7.4}$$

The second term in (7.1) simplifies to

$$\mathbb{E}_0\left[1_{\{s_0<t_0\wedge T_1\}} e^{-\lambda s_0} w(0, \Phi^{\delta}_{s_0})\right] = 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} w(0, x(s_0,\, \phi)), \tag{7.5}$$

and the last term simplifies to

$$\mathbb{E}_0\left[1_{\{T_1<s_0\wedge t_0\}} e^{-\lambda T_1} w(1, \Phi^{\delta}_{T_1})\right] = \int_0^{s_0\wedge t_0} e^{-(\lambda_0+\lambda)u} w\left(1, \frac{\lambda_1}{\lambda_0} x(u,\, \phi)\right)\lambda_0 \mathrm{d}u. \tag{7.6}$$

By putting together, (7.3), (7.4), (7.5) and (7.6) we have the following,

$$
\begin{aligned}
&(J_0 w)(1,\phi) \\
&= \begin{cases}
\inf\limits_{(t_0,\, s_0)\in\overline{\mathbb{R}}_+^2} \Bigg\{ a_1(t_0,\, s_0) + b_1(t_0,\, s_0,\, \phi) + \lambda\left(1+\dfrac{b}{c}\right)\cdot \\
\left[\dfrac{1}{(\lambda+\lambda_0)^2} - \dfrac{e^{-(\lambda+\lambda_0)t_0\wedge s_0}}{\lambda+\lambda_0}\left(t_0\wedge s_0 + \dfrac{1}{\lambda+\lambda_0}\right)\right]\Bigg\}, & \widetilde{a}=0, \\
\\
\inf\limits_{(t_0,\, s_0)\in\overline{\mathbb{R}}_+^2} \Bigg\{ a_1(t_0,\, s_0) + b_1(t_0,\, s_0,\, \phi) \\
+\left(\dfrac{\phi-\phi_d}{\lambda_1}\right)\left(1+\dfrac{b}{c}\right)\left(1-e^{-\lambda_1(t_0\wedge s_0)}\right)\Bigg\}, & \widetilde{a}\neq 0,
\end{cases}
\end{aligned}
\tag{7.7}
$$

where $a_1(t_0,\, s_0) := \left(\dfrac{1}{\lambda+\lambda_0}\right)\left(\phi + \dfrac{b}{c}(1+\phi) - \dfrac{\lambda}{c}\right)\left(1-e^{-(\lambda+\lambda_0)(t_0\wedge s_0)}\right)$,
$b_1(t_0,\, s_0,\, \phi) := 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} w(0, x(s_0,\, \phi)) + \displaystyle\int_0^{s_0\wedge t_0} e^{-(\lambda_0+\lambda)u}\cdot$
$w\left(1, \dfrac{\lambda_1}{\lambda_0} x(u,\, \phi)\right)\lambda_0 \mathrm{d}u$, $x(\cdot,\cdot)$ is defined as in (7.2) and Φ_t^δ as in (A.7).

We now calculate $(J_0 w)(0,\phi)$ as defined in (5.14). Since in this case the observation control is currently turned off, the stopping time $\tau\wedge\tau_1$ is deterministic. Hence we shall represent them by constants t_0 and r_0 respectively and rewrite (5.14) as follows:

$$
\begin{aligned}
&(J_0 w)(0,\phi) \\
&= \inf_{(t_0,\, r_0)\in\overline{\mathbb{R}}_+^2} \left\{ \int_0^{t_0\wedge r_0} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta)\mathrm{d}s + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(h(\Phi_{r_0}^\delta) + w(1, \Phi_{r_0}^\delta)\right)\right\}.
\end{aligned}
\tag{7.8}
$$

Simplifying the first integral in (7.8), we get

$$
\begin{aligned}
\int_0^{t_0\wedge r_0} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta)\mathrm{d}s &= \int_0^{t_0\wedge r_0} e^{-\lambda s} g(0, \Phi_s^\delta)\mathrm{d}s \\
&= \int_0^{t_0\wedge r_0} e^{-\lambda s}\left(\Phi_s^\delta - \frac{\lambda}{c}\right)\mathrm{d}s, \qquad \text{using (5.11)}.
\end{aligned}
\tag{7.9}
$$

The process Φ_s^δ as defined in (A.7) is continuous deterministic when $s \in (0, t_0 \wedge r_0)$ since the process $\alpha_s^\delta = 0$ in this interval. Therefore, (A.7) simplifies to,

$$\mathrm{d}\Phi_t^\delta = \lambda\left(1+\Phi_t^\delta\right)\mathrm{d}t, \quad \Phi_0^\delta = \phi = \frac{\pi}{1-\pi}.$$

The solution of the above ODE is given by

$$\Phi_t^\delta = y(t,\, \phi) = (1+\phi)e^{\lambda t} - 1. \tag{7.10}$$

Substituting function Φ_t^δ as defined in (7.10) back into (7.9) we get,

$$\begin{aligned}\int_0^{t_0\wedge r_0} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta)\mathrm{d}s &= \int_0^{t_0\wedge r_0} e^{-\lambda s}\left(\Phi_s^\delta - \frac{\lambda}{c}\right)\mathrm{d}s \\ &= \int_0^{t_0\wedge r_0} e^{-\lambda s}\left((1+\phi)e^{\lambda s} - 1 - \frac{\lambda}{c}\right)\mathrm{d}s \\ &= (1+\phi)(t_0\wedge r_0) - (1+\frac{\lambda}{c})\frac{1}{\lambda}\left(1 - e^{-\lambda(t_0\wedge r_0)}\right). \end{aligned} \tag{7.11}$$

Using (5.12) and (7.10), the last term in (7.8) is simplified as follows,

$$\begin{aligned} &1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(h(\Phi_{r_0}^\delta) + w(1, \Phi_{r_0}^\delta)\right) \\ &= 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c}(1+\Phi_{r_0}^\delta) + w(1,\Phi_{r_0}^\delta)\right) \\ &= 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c}(1+(1+\phi)e^{\lambda r_0} - 1) + w(1,\Phi_{r_0}^\delta)\right) \\ &= 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c}(1+\phi)e^{\lambda r_0} + w(1,\Phi_{r_0}^\delta)\right) \\ &= 1_{\{r_0<t_0\}}\left(\frac{a}{c}(1+\phi) + e^{-\lambda r_0} w(1,\Phi_{r_0}^\delta)\right). \end{aligned} \tag{7.12}$$

Using (7.11) and (7.12), (7.8) can be rewritten as,

$$\begin{aligned}(J_0 w)(0,\phi) = \inf_{(t_0,\,r_0)\in\overline{\mathbb{R}}_+^2} &\Bigg\{(1+\phi)(t_0\wedge r_0) - (1+\frac{\lambda}{c})\frac{1}{\lambda}\left(1-e^{-\lambda(t_0\wedge r_0)}\right) \\ &+1_{\{r_0<t_0\}}\left(\frac{a}{c}(1+\phi) + e^{-\lambda r_0} w\left(1, (1+\phi)e^{\lambda r_0} - 1\right)\right)\Bigg\}, \end{aligned} \tag{7.13}$$

since $\Phi_{r_0}^\delta = (1+\phi)e^{\lambda r_0} - 1$ using (7.10).

Note. Let us consider the term inside the indicator event in the above optimization problem:

$$\frac{a}{c}(1+\phi) + e^{-\lambda r_0} w\left(1, (1+\phi)e^{\lambda r_0} - 1\right) \geq \frac{a}{c}(1+\phi) - e^{-\lambda r_0} \cdot \frac{1}{c} \geq \frac{a}{c}(1+\phi) - \frac{1}{c}.$$

Thus if $a > \frac{1}{1+\phi}$, the optimal strategy is to raise the alarm before turning on the observation control.

Chapter 8

Structure and characterization of solution

In this chapter we first (in Section 8.1) focus our attention on the sets where the optimization problems $(Jv_n)(\alpha, \phi, \cdot, \cdot)$ attain their infimums, if they do. We make useful observations in Lemmas 8.1.1 and 8.1.5 that helps us reduce the dimensionality of our optimization problems. We recognize subsets of $\overline{\mathbb{R}}_+^2$ on which it is enough to search for the optimal solutions and in the case the solution set is empty we assign $+\infty$ as the solution. In Section 8.2, we give an alternate characterization of the stopping times which in turn helps us in Chapter 9 in showing that they can be described as the first return times of the odds-ratio process to certain sets. Finally in Section 8.3, we show that the classical Poisson disorder problem falls out as a consequence of the numerical scheme presented in our study.

8.1 Structure of the solution set

The point of the next lemma is to show that $(Jv_n)(t, s, 1, \phi)$ has the same optimal solution on the sets $\overline{\mathbb{R}}_+^2$ and D (as defined in the next Lemma). We show that for every point in $A_n(1, \phi)$ there exists a point in the set $D_n(1, \phi) \subseteq D$ and vice versa. Therefore, it is enough to look for an optimal solution in the region D.

Lemma 8.1.1. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$\inf_{(t,\, s)\in\overline{\mathbb{R}}_+^2} (Jv_n)(t,\, s,\, 1,\, \phi) = \inf_{(t,,\, s)\in D} (Jv_n)(t,\, s,\, 1,\, \phi),$$

where $D := \left\{(t,\, s) \in \overline{\mathbb{R}}_+^2 \;:\; t = s + \epsilon_0\right\}$ *for some arbitrary but fixed* $\epsilon_0 > 0$.

Proof. The set $A_n(1,\, \phi)$ has the following properties:

1. if $(t_1,\, s_1) \in A_n(1,\, \phi)$ such that $t_1 \leq s_1$, then necessarily $v_n(0,\, x(t_1,\, \phi)) = 0$ else we can find better optimal solutions in the set $\{(t,\, s) :\; t > t_1,\, s = t_1\}$. Also since $v_n(0,\, x(t_1,\, \phi)) = 0$, the set of points $\{(t,\, s) : t = t_1$ or $s = t_1\}$ give the same optimal solution and hence also belong to $A_n(1,\, \phi)$.

2. If $(t_1,\, s_1) \in A_n(1,\, \phi)$ and $t_1 > s_1$, there are two possiblities. First, if $v_n(0,\, x(s_1,\, \phi)) = 0$, then all the points in the set $\{(t,\, s) :\; t = s_1$ or $s = s_1\}$ give the same optimal solution and hence belong to $A_n(1,\, \phi)$. If however $v_n(0,\, x(s,\, \phi)) < 0$ then the set of points $\{(t,\, s) :\; t > s_1,\, s = s_1\}$ also give the same optimal solution and hence also belong to $A_n(1,\, \phi)$.

Thus, we have that for every optimal solution obtained in $\overline{\mathbb{R}}_+^2$, there is an element in the set D and we can construct the set $A_n(1,\, \phi)$ from $D_n(1,\, \phi)$, where $D_n(1,\, \phi) := \{(t,\, s) \in D \;:\; (J_0 v_n)(1,\, \phi) = (Jv_n)(t,\, s,\, 1,\, \phi)\}$. □

Corollary 8.1.2. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$\inf_{(t,\, s)\in A_n(1,\, \phi)} \{t \wedge s\} = \inf_{(t,\, s)\in D_n(1,\, \phi)} \{t \wedge s\},$$

and

$$(t_n(1,\, \phi),\, s_n(1,\, \phi)) \;\in\; \arg \inf_{(t,\, s)\in D_n(1,\, \phi)} \{t \wedge s\} \;\subset\; \arg \inf_{(t,\, s)\in A_n(1,\, \phi)} \{t \wedge s\}.$$

Lemma 8.1.3. *The set $D_n(1,\, \phi)$ is closed in $\overline{\mathbb{R}}_+^2$.*

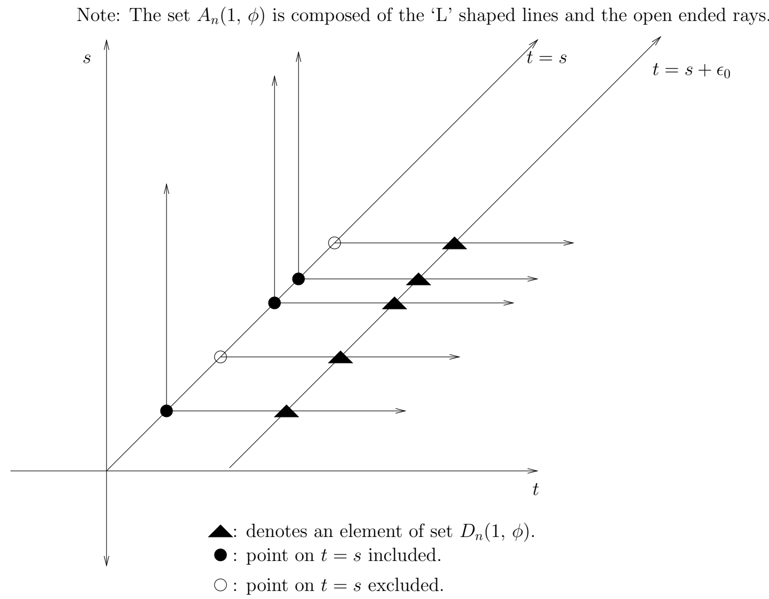


Figure 8.1: Illustration of the regions $A_n(1,\,\phi)$ and $D_n(1,\,\phi)$.

Proof. On the set D, $(Jv_n)(t,\,s,\,1,\,\phi) = (Jv_n)(s+\epsilon,\,s,\,1,\,\phi)$ takes the following form:

$$f(s) := a_1\left(1 - e^{-(\lambda+\lambda_0)s}\right) + a_2(\phi)\left(1 - e^{-\lambda_1 s}\right) + e^{-(\lambda_0+\lambda)s} v_n(0,\,x(s,\,\phi))$$
$$+ \int_0^s e^{-(\lambda+\lambda_0)u} \lambda_0 v_n\left(1,\,\frac{\lambda_1}{\lambda_0} x(u,\,\phi)\right) \mathrm{d}u,$$

since $s < t$ on D. We now have a one-dimensional optimization problem where the objection function $f(s)$ is continuous in s. To see that the set $D_n(1,\,\phi)$ is closed, we take a sequence $(s_m + \epsilon,\,s_m) \subseteq D_n(1,\,\phi)$ that converges to $(s_0 + \epsilon,\,s_0)$. That is we have

$$\begin{aligned}
&(J_0 v_n)(1,\,\phi) \\
&= (Jv_n)(s_m + \epsilon,\,s_m,\,1,\,\phi) \\
&= a_1\left(1 - e^{-(\lambda+\lambda_0)s_m}\right) + a_2(\phi)\left(1 - e^{-\lambda_1 s_m}\right) + e^{-(\lambda_0+\lambda)s_m} v_n(0,\,x(s_m,\,\phi)) \\
&\quad + \int_0^{s_m} e^{-(\lambda+\lambda_0)u} \lambda_0 v_n\left(1,\,\frac{\lambda_1}{\lambda_0} x(u,\,\phi)\right) \mathrm{d}u \\
&\text{(taking limit as } m \to \infty \text{ and owing to the continuity of } f(s)) \\
&= a_1\left(1 - e^{-(\lambda+\lambda_0)s_0}\right) + a_2(\phi)\left(1 - e^{-\lambda_1 s_0}\right) + e^{-(\lambda_0+\lambda)s_0} v_n(0,\,x(s_0,\,\phi))
\end{aligned}$$

$$+ \int_0^{s_0} e^{-(\lambda+\lambda_0)u} \lambda_0 v_n \left(1,\, \frac{\lambda_1}{\lambda_0} x(u,\, \phi)\right) \mathrm{d}u$$
$$= (Jv_n)(s_0 + \epsilon,\, s_0,\, 1,\, \phi).$$

Thus $(s_0 + \epsilon,\, s_0) \in D_n(1,\, \phi)$. □

Corollary 8.1.4. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$(t_n(1,\, \phi),\, s_n(1,\, \phi)) \;\in\; \arg \inf_{(t,\, s) \in D_n(1,\, \phi)} \{t \wedge s\} = \arg \min_{(t,\, s) \in D_n(1,\, \phi)} \{t \wedge s\}.$$

Next we derive similar result for the case when $\alpha = 0$. The point of the next lemma is to show that $(Jv_n)(t,\, r,\, 0,\, \phi)$ has the same optimal solution on the sets $\overline{\mathbb{R}}_+^2$ and $D := D^1 \cup D^2$ (as defined in the next Lemma). We show that for every point in $A_n(0,\, \phi)$ there exists a point in the set $D_n^1(0,\, \phi) \subseteq D^1$ or $D_n^2(0,\, \phi) \subseteq D^2$ or both. Therefore, it is enough to look for an optimal solution in the region D.

Lemma 8.1.5. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$\inf_{(t,\, r) \in \overline{\mathbb{R}}_+^2} (Jv_n)(t,\, r,\, 0,\, \phi) = \inf_{(t,\, r) \in D} (Jv_n)(t,\, r,\, 0,\, \phi)$$

$$= \min \left\{ \inf_{t(t,\, r) \in D^1} (Jv_n)(t,\, r,\, 0,\, \phi),\; \inf_{(t,\, r) \in D^2} (Jv_n)(t,\, r,\, 0,\, \phi) \right\},$$

where $D^1 := \left\{(t,\, r) \in \overline{\mathbb{R}}_+^2 \,:\, r = t + \epsilon_0\right\}$, $D^2 := \left\{(t,\, r) \in \overline{\mathbb{R}}_+^2 \,:\, t = r + \epsilon_0\right\}$ *for some arbitrary but fixed* $\epsilon_0 > 0$.

Proof. The set $A_n(0,\, \phi)$ has the following properties:

1. if $(t_1,\, r_1) \in A_n(0,\, \phi)$ such that $t_1 \le r_1$, then necessarily $y := \frac{a}{c}(1 + \phi) + e^{-\lambda r_1} v_n(1,\, (1+\phi)e^{\lambda r} - 1) \ge 0$. If $y = 0$ then $\left\{(t,\, r) \in \overline{\mathbb{R}}_+^2 \,:\, t = t_1 \text{ or } r = t_1\right\}$ is also a solution set. If $y > 0$ then $\left\{(t,\, r) \in \overline{\mathbb{R}}_+^2 \,:\, t = t_1, r \ge t_1\right\}$ is the solution set.

2. if $(t_1,\, r_1) \in A_n(0,\, \phi)$ such that $t_1 > r_1$, then necessarily $y := \frac{a}{c}(1 + \phi) + e^{-\lambda r_1} v_n(1,\, (1+\phi)e^{\lambda r} - 1) \le 0$. If $y = 0$ then $\left\{(t,\, r) \in \overline{\mathbb{R}}_+^2 \,:\, t = r_1 \text{ or } r = r_1\right\}$

is also a solution set. If $y < 0$ then $\left\{(t,\, r) \in \overline{\mathbb{R}}^2_+ \,:\, r = r_1, t > r_1\right\}$ is the solution set.

Thus, for every point in $\overline{\mathbb{R}}^2_+$ where the optimal solution is obtained, there is an element in the set D and we can construct the set $A_n(0,\, \phi)$ from $D_n(0,\, \phi) := D^1_n(0,\, \phi) \cup D^2_n(0,\, \phi)$, where $D^j_n(0,\, \phi) := \{(t,\, s) \in D^j \,:\, (J_0 v_n)(0,\, \phi) = (Jv_n)(t,\, r,\, 0,\, \phi),\, j \in \{1,\, 2\}\}$. □

Lemma 8.1.6. *For all $\phi \in \mathbb{R}_+$, $n \in \mathbb{N}_0$, $(Jv_n)(t,\, r,\, 0,\, \phi)$ attains infimum inside a bounded set in $\mathbb{R}^2_+ \subset \overline{\mathbb{R}}^2_+$.*

Proof. We have two cases here, (i) when $\phi \geq \frac{\lambda}{c}$ and (ii) when $\phi < \frac{\lambda}{c}$.

(i) In this case, clearly the optimal value on the set D^1 and on the set D^2 is obtained at $t = 0$ and at $r = 0$ respectively. This is because the objective function on both the sets is increasing at 0 and always remains increasing beyond that point. Therefore the optimal decision here is to raise the alarm at $t = 0$ if $\frac{a}{c}(1+\phi)+v_n(1,\, \phi) > 0$ otherwise, turn on the observation control at $r = 0$.

(ii) In this case, if $\frac{a}{c}(1+\phi) + v_n(1,\, \phi) > 0$ then the optimal decision once again is to raise the alarm at $t = t^*$ (as defined below). Otherwise, lets look at the first derivatives of the function on the two sets D^1 and D^2 separately as follows:

   (a) Let $(t,\, r) \in D^1$. On D^1 we have

$$(Jv_n)(t,\, t+\epsilon_0,\, 0,\, \phi) = g(t) := (1+\phi)t - \left(\frac{1}{\lambda} + \frac{1}{c}\right)\left(1 - e^{-\lambda t}\right).$$

   Looking at the first derivate, we have

$$g_t(t) = (1+\phi) - \left(1 + \frac{\lambda}{c}\right) e^{-\lambda t}.$$

   Clearly, $g_t > 0$ when $t > t^* := \frac{-1}{\lambda} \ln \left\{ \frac{1+\phi}{\left(1+\frac{\lambda}{c}\right)} \right\} > 0$. Hence we have that our optimal solution on D^1 is attained at $t = t^*$.

(b) Let $(t, r) \in D^2$. On D^2 we have

$$(Jv_n)(r + \epsilon_0,\, r,\, 0,\, \phi) = f(r) := (1+\phi)r - \left(\frac{1}{\lambda} + \frac{1}{c}\right)\left(1 - e^{-\lambda r}\right)$$

$$+\frac{a}{c}(1+\phi) + e^{-\lambda r} v_n\left(1,\, y(r,\, \phi)\right),$$

where $y(r, \phi)$ is as defined in (7.10). Looking at the first derivate, we have

$$f_r(r) = \underbrace{(1+\phi) - \left(1 + \frac{\lambda}{c}\right) e^{-\lambda r}}_{S:=} - \lambda e^{-\lambda r} v_n(1,\, y(r,\, \phi))$$

$$+ v_n^{(y)}\left(1,\, y(r,\, \phi)\right) \cdot \lambda(1+\phi).$$

The last two terms in the above derivative are always positive owing to non-positive and monotone character (w.r.t. second argument) of $v_n(\cdot, \cdot)$. We can ensure $S > 0$ if we have $r > r^* := \frac{-1}{\lambda} \ln\left\{\frac{1+\phi}{\left(1+\frac{\lambda}{c}\right)}\right\} > 0$. Hence once again we have that, our optimal solution is attained $r \in [0, r^*]$.

Thus we check the better of the two optimal solutions and take the corresponding action.

□

Corollary 8.1.7. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$\inf_{(t,r)\in A_n(0,\phi)} \{t \wedge r\} = \inf_{(t,r)\in D_n(0,\phi)} \{t \wedge r\},$$

and

$$(t_n(0,\, \phi),\, r_n(0,\, \phi)) \in \arg \inf_{(t,r)\in D_n(0,\phi)} \{t \wedge r\} \subset \arg \inf_{(t,r)\in A_n(0,\phi)} \{t \wedge r\}.$$

Lemma 8.1.8. *The set $D_n(0, \phi)$ is closed in $\overline{\mathbb{R}}_+^2$.*

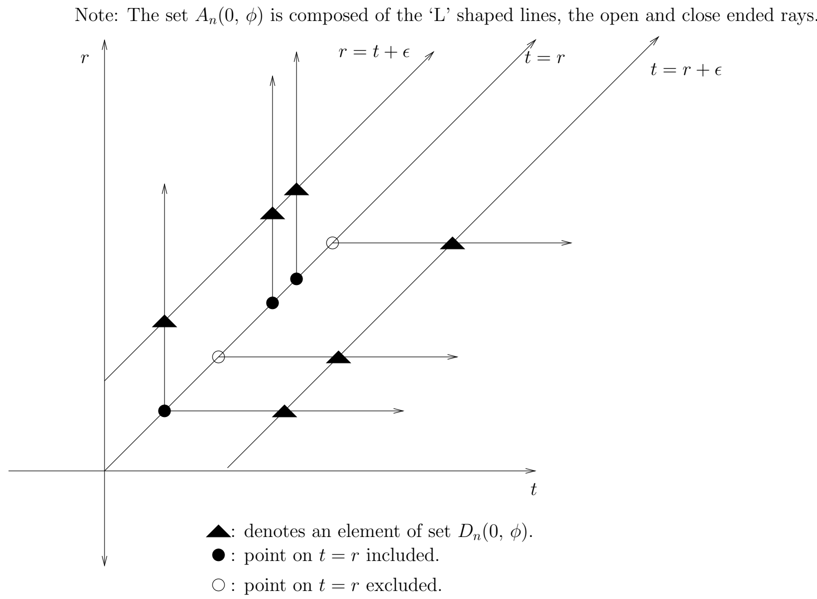


Figure 8.2: Illustration of the regions $A_n(0, \phi)$ and $D_n(0, \phi)$.

Proof. We first show that $D_n^1(0, \phi)$ and $D_n^2(0, \phi)$ are closed. On the set D^1, $(Jv_n)(t, r, 0, \phi) = (Jv_n)(t, t+\epsilon, 0, \phi)$ takes the following form:

$$g(t) := (1+\phi)t - \left(\frac{1}{\lambda} + \frac{1}{c}\right)\left(1 - e^{-\lambda t}\right)$$

since $t \le r$ on D^1. We now have a one-dimensional optimization problem where the objection function $g(t)$ is continuous in t. To see that the set $D_n^1(0, \phi)$ is closed, we take a sequence $(t_m, t_m+\epsilon) \subseteq D_n^1(0, \phi)$ that converges to $(t_0, t_0+\epsilon)$. That is we have

$$\begin{aligned}
(J_0 v_n)(0, \phi) &= (Jv_n)(t_m, t_m+\epsilon, 0, \phi) \\
&= (1+\phi)t_m - \left(\frac{1}{\lambda} + \frac{1}{c}\right)\left(1 - e^{-\lambda t_m}\right) \\
&\quad \text{(taking limit as } m \to \infty \text{ and owing to the continuity of } g(s)) \\
&= (1+\phi)t_0 - \left(\frac{1}{\lambda} + \frac{1}{c}\right)\left(1 - e^{-\lambda t_0}\right) \\
&= (Jv_n)(t_0, t_0+\epsilon, 0, \phi).
\end{aligned}$$

Thus $(t_0, t_0+\epsilon) \in D_n^1(0, \phi)$. Using similar arguments, we can also show that

$D_n^2(0, \phi)$ is closed. Since $D_n(0, \phi) = D_n^1(0, \phi) \cup D_n^2(0, \phi)$ we have the required result. □

Corollary 8.1.9. *For all $n \in \mathbb{N}_0$, $\phi \in \mathbb{R}_+$ we have,*

$$(t_n(0, \phi), r_n(0, \phi)) \in \arg \inf_{(t, r)\in D_n(0, \phi)} \{t \wedge r\} = \arg \min_{(t, r)\in D_n(0, \phi)} \{t \wedge r\}.$$

8.2 Alternate characterization

As described earlier, in this section we show that stopping times $t_n(\alpha, \phi)$, $s_n(1, \phi)$ and $q_n(0, \phi)$ which are described as the smallest minimizers of the optimization problems, admit another characterization.

Proposition 8.2.1. *For any bounded function $w : \mathbb{R}_+ \mapsto \mathbb{R}$, $t \in \mathbb{R}_+$ and $\phi \in \mathbb{R}_+$ we have,*

$$(J_t w)(1, \phi) = (Jw)(t, t + \epsilon, 1, \phi) + e^{-(\lambda+\lambda_0)t} \cdot (J_0 w)(1, x(t, \phi))$$

where ϵ is an arbitrary positive constant.

Proof. Refer to Appendix §B.3. □

Remark 8.2.2. *For every $t \in [0, t_n(1, \phi) \wedge s_n(1, \phi)]$ we have $(J_t v_n)(1, \phi)$ $= \inf_{(u_1 \wedge u_2)>t} (Jv_n)(u_1, u_2, 1, \phi) = (J_0 v_n)(1, \phi) = v_{n+1}(1, \phi)$.*

Let us consider the case where $t_n(1, \phi) \wedge s_n(1, \phi) = t_n(1, \phi)$ and $t \in [0, t_n(1, \phi)]$. From the previous proposition we would have

$$(J_t v_n)(1, \phi) = \mathbb{E}_0^{(1, \phi)} \left[\int_0^{t\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \, ds + 1_{\{T_1<t\}} e^{-\lambda T_1} v_n(1, \Phi_{T_1}^\delta) \right]$$
$$+ e^{-(\lambda+\lambda_0)t} \cdot (J_0 v_n)(1, x(t, \phi))$$

At time $t = t_n(1, \phi)$ Proposition 8.2.1 implies that,

$(J_0 v_n)(1, \phi)$

$$
\begin{aligned}
&= \mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{t_n(1,\,\phi)\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\, \mathrm{d}s + 1_{\{T_1<t_n(1,\,\phi)\}} e^{-\lambda T_1} v_n(1,\, \Phi_{T_1}^\delta)\right] \\
&\quad + e^{-(\lambda+\lambda_0)t_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(t_n(1,\,\phi),\, \phi)) \\
&= (J v_n)(t_n(1,\,\phi),\, s_n(1,\,\phi),\, 1,\, \phi) + e^{-(\lambda+\lambda_0)t_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(t_n(1,\,\phi),\, \phi)) \\
&= (J_0 v_n)(1,\, \phi) + e^{-(\lambda+\lambda_0)t_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(t_n(1,\,\phi),\, \phi))
\end{aligned}
$$

which gives us $v_{n+1}(1,\, x(t_n(1,\,\phi),\, \phi)) = 0$. In the next Lemma we show that $t = t_n(1,\,\phi)$ is in fact the first time $t \mapsto v_{n+1}(1,\, x(t,\,\phi))$ hits level 0.

Lemma 8.2.3. *If $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = t_n(1,\,\phi)$, then for $\phi \in \mathbb{R}_+$ and $(v_n(1,\,\phi))_{n\geq 0}$ as defined in (6.2), we have,*

$$
t_n(1,\,\phi) = \inf\left\{t \geq 0 :\; v_{n+1}(1,\, x(t,\,\phi)) = 0\right\}
$$

and $v_{n+1}(1,\, x(t_n(1,\,\phi),\,\phi)) < v_n(0,\, x(t_n(1,\,\phi),\,\phi))$.

Proof. Since $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = \min\{t \wedge s :\; (t,\,s) \in A_n(1,\,\phi)\}$ and $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = t_n(1,\,\phi) \equiv t_n$, then $0 \leq t < t_n \;\Rightarrow\; (t,\,s) \notin A_n(1,\,\phi),\ \forall\, s$(o/w $t \wedge s < t_n = t_n \wedge s_n$, which contradicts the definition of t_n and s_n) $\Rightarrow\; (Jv_n)(t,\, t+s,\, 1,\, \phi) > (J_0 v_n)(1,\,\phi),\ \forall s \;\Rightarrow\; (J_t v_n)(1,\,\phi) > (J_0 v_n)(1,\,\phi) + e^{-(\lambda+\lambda_0)t} \cdot (J_0 v_n)(1,\, x(t,\,\phi)) \;\Rightarrow\; (J_0 v_n)(1,\,\phi) > (J_0 v_n)(1,\,\phi) + e^{-(\lambda+\lambda_0)t} \cdot (J_0 v_n)(1,\, x(t,\,\phi))$ which follows from the last remark. Thus, $(J_0 v_n)(1,\, x(t,\,\phi)) < 0$. □

Let us now consider $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = s_n(1,\,\phi)$ and $t \in [0,\, s_n(1,\,\phi)]$. By Proposition 8.2.1, at $t = s_n(1,\,\phi)$ we would have,

$$
\begin{aligned}
&(J_0 v_n)(1,\,\phi) \\
&= \mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{s_n(1,\,\phi)\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\, \mathrm{d}s + 1_{\{T_1<s_n(1,\,\phi)\}} e^{-\lambda T_1} v_n(1,\, \Phi_{T_1}^\delta)\right] \\
&\qquad\qquad + e^{-(\lambda+\lambda_0)s_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(s_n(1,\,\phi),\,\phi)) \\
&= (J v_n)(t_n(1,\,\phi),\, s_n(1,\,\phi),\, 1,\, \phi) - \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{s_n(1,\,\phi)<T_1\}} e^{-\lambda s_n(1,\,\phi)} v_n(0,\, \Phi_{s_n(1,\,\phi)}^\delta)\right] \\
&\qquad\qquad + e^{-(\lambda+\lambda_0)s_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(s_n(1,\,\phi),\,\phi)) \\
&= (J_0 v_n)(1,\,\phi) + e^{-(\lambda+\lambda_0)s_n(1,\,\phi)} \cdot (J_0 v_n)(1,\, x(s_n(1,\,\phi),\,\phi))
\end{aligned}
$$

$$- \mathbb{E}_0^{(1,\,\phi)} \left[1_{\{s_n(1,\,\phi) < T_1\}} e^{-\lambda s_n(1,\,\phi)} v_n(0,\, \Phi^{\delta}_{s_n(1,\,\phi)}) \right]$$

which gives us $e^{-(\lambda+\lambda_0)s_n(1,\,\phi)} v_{n+1}(1,\, x(s_n(1,\,\phi),\, \phi)) = \mathbb{E}_0^{(1,\,\phi)} \Big[1_{\{s_n(1,\,\phi) < T_1\}} \cdot$ $e^{-\lambda s_n(1,\,\phi)} v_n(0,\, x(s_n(1,\,\phi),\, \phi)) \Big] = e^{-(\lambda+\lambda_0)s_n(1,\,\phi)} \cdot v_n(0,\, x(s_n(1,\,\phi),\, \phi))$
$\Longrightarrow v_{n+1}(1,\, x(s_n(1,\,\phi),\, \phi)) = v_n(0,\, x(s_n(1,\,\phi),\, \phi))$.

Lemma 8.2.4. *If $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = s_n(1,\,\phi)$, then for $\phi \in \mathbb{R}_+$ and $(v_n(1,\,\phi))_{n\geq 0}$ as defined in (6.2), we have,*

$$s_n(1,\,\phi) = \inf \Big\{ s \geq 0 :\ v_{n+1}(1,\, x(s,\,\phi)) = v_n(0,\, x(s,\,\phi)) \Big\}$$

and $v_{n+1}(1,\, x(s_n(1,\,\phi),\, \phi)) < 0$.

Proof. Since $t_n(1,\,\phi) \wedge s_n(1,\,\phi) = \min\{t \wedge s :\ (t,\, s) \in A_n(1,\,\phi)\}$ and $t_n(1,\,\phi)$ $\wedge\, s_n(1,\,\phi) = s_n(1,\,\phi) \equiv s_n$, then $0 < s < s_n \Rightarrow (t,\, s) \notin A_n(1,\,\phi), \forall\, t \Rightarrow$ $(Jv_n)(t+s,\, s,\, 1,\, \phi) > (J_0 v_n)(1,\,\phi)\ \forall t \Rightarrow (J_s v_n)(1,\,\phi) > (J_0 v_n)(1,\,\phi)$ $-\mathbb{E}_0^{(1,\,\phi)} \left[1_{\{s<T_1\}} e^{-\lambda s} \cdot v_n(0,\, \Phi^{\delta}_s) \right] + e^{-(\lambda+\lambda_0)s} \cdot (J_0 v_n)(1,\, x(s,\,\phi)) \Rightarrow (J_0 v_n)(1,\,\phi)$ $> (J_0 v_n)(1,\,\phi) - e^{-(\lambda+\lambda_0)s} \cdot v_n(0,\, x(s,\,\phi)) + e^{-(\lambda+\lambda_0)s} \cdot (J_0 v_n)(1,\, x(s,\,\phi))$ $\Rightarrow v_{n+1}(1,\, x(s,\,\phi)) < v_n(0,\, x(s,\,\phi))$. □

Remark 8.2.5. *What we observe in Lemma 8.2.3 is that a line of the form $\{(t^*,\, s) :\ t^* < s\}$ belongs to $A_n(1,\,\phi)$ only if the line $\{(t,\, t^*) :\ t \geq t^*\}$ also belongs to $A_n(1,\,\phi)$. Without this additional condition, we get a contradiction that $0 = v_{n+1}(1,\, x(t_n(1,\,\phi),\, \phi)) < v_n(0,\, x(t_n(1,\,\phi),\, \phi)) \leq 0$. If $A_n(1,\,\phi)$ contains lines of both of these kinds: $\{(t^*,\, s) :\ t^* < s\}$ & $\{(t,\, t^*) :\ t \geq t^*\}$, this would imply $A_n(1,\,\phi)$ also contains points of the kind $(t^*,\, t^*)$, i.e., $t_n(1,\,\phi) = s_n(1,\,\phi) = t^*$. This implies, such a $t_n(1,\,\phi)$ is the first time process $v_{n+1}(1,\, x(t,\,\phi))$ hits zero and, it also is the first time $v_{n+1}(1,\, x(t,\,\phi)) = v_n(0,\, x(t,\,\phi))$ (= 0), thus avoiding any contradictions. This agrees with our earlier observation in Lemma 5.8, where we showed, in the case $\alpha = 1$, it is enough to optimally decide when to turn off the observation control (also refer to Figure 8.1). Hence, whenever it is optimal to either raise the alarm or turn off the observation control, we* decide *to* turn off observation control. *This choice can be reflected in Lemma 8.2.4 by changing the additional* *condition* $(v_{n+1}(0,\, y(t_n(0,\,\phi),\,\phi)) < h(y(t_n(0,\,\phi),\,\phi)) + v_n(1,\, y(t_n(0,\,\phi),\,\phi)))$ *to* $v_{n+1}(0,\, y(t_n(0,\,\phi),\,\phi)) \leq h(y(t_n(0,\,\phi),\,\phi)) + v_n(1,\, y(t_n(0,\,\phi),\,\phi))$.

Proposition 8.2.6. *For any bounded function* $w : \mathbb{R}_+ \mapsto \mathbb{R}$, $t \in \mathbb{R}_+$ *and* $\phi \in \mathbb{R}_+$ *we have,*

$$(J_t w)(0,\,\phi) = (Jw)(t,\, t+\epsilon,\, 0,\,\phi) + e^{-\lambda t} \cdot (J_0 w)(0,\, y(t,\,\phi)) \tag{8.1}$$

where ϵ *is an arbitrary positive constant.*

Proof. Refer to Appendix §B.4. □

Remark 8.2.7. *For* $t \in [0,\, t_n(0,\,\phi) \wedge q_n(0,\,\phi)]$ *we have* $(J_t v_n)(0,\,\phi) = \inf_{(u_1 \wedge u_2) \geq t} (J v_n)(u_1,\, u_2,\, 0,\,\phi) = (J_0 v_n)(0,\,\phi) = v_{n+1}(0,\,\phi)$.

Following the approach as in Lemmas 8.2.3 and 8.2.4 we have,

Lemma 8.2.8. *If* $t_n(0,\,\phi) \wedge q_n(0,\,\phi) = t_n(0,\,\phi)$, *then for* $\phi \in \mathbb{R}_+$ *and* $(v_n(0,\,\phi))_{n\geq 0}$ *as defined in (6.2), we have,*

$$t_n(0,\,\phi) = \inf\left\{t > 0 :\; v_{n+1}(0,\, y(t,\,\phi)) = 0\right\}$$

and $v_{n+1}(0,\, y(t_n(0,\,\phi),\,\phi)) < h(y(t_n(0,\,\phi),\,\phi)) + v_n(1,\, y(t_n(0,\,\phi),\,\phi))$.

Lemma 8.2.9. *If* $t_n(0,\,\phi) \wedge q_n(0,\,\phi) = q_n(0,\,\phi)$, *then for* $\phi \in \mathbb{R}_+$ *and* $(v_n(0,\,\phi))_{n\geq 0}$ *as defined in (6.2), we have,*

$$q_n(0,\,\phi) = \inf\left\{q > 0 :\; v_{n+1}(0,\, y(q,\,\phi)) = h(y(q,\,\phi)) + v_n(1,\, y(q,\,\phi))\right\}$$

and $v_{n+1}(0,\, y(q_n(0,\,\phi),\,\phi)) < 0$.

Proofs of above two Lemmas follow by using arguments similar to those used in Lemmas 8.2.3, 8.2.4.

Remark 8.2.10. *This remark extends last remark to the case when* $\alpha = 0$. *We noted in proof of Lemma 8.1.5 that it is possible to have* $t_n(0,\phi) = q_n(0,\phi)$, *when*

$y = 0$ (y as defined in Lemma 8.1.5). We now use characterizations given in Lemmas 8.2.8, 8.2.9 to see if they match earlier observations made in Lemma 8.1.5. We look for conditions under which there can exist $t_n(0, \phi)$, $q_n(0, \phi)$ such that they are equal. This would imply $t_n(0, \phi)$ would have to satisfy condition in Lemma 8.2.8, i.e. $t = t_n(0, \phi)$ is the first time $v_{n+1}(0, y(t, \phi))$ hits zero and, also have to satisfy condition in Lemma 8.2.9, i.e., $t = t_n(0, \phi)$ is the first time $v_{n+1}(0, y(t, \phi)) = h(y(t, \phi)) + v_n(0, y(t, \phi))$. These two conditions together imply that $t_n(0, \phi) = q_n(0, \phi)$ only when $0 = v_{n+1}(0, y(t_n(0, \phi), \phi)) = h(y(t_n(0, \phi), \phi)) + v_n(0, y(t_n(0, \phi), \phi)) \implies \frac{a}{c}(1 + \phi) + e^{-\lambda t_n(0,\phi)} v_n(1, (1 + \phi)e^{\lambda t_n(0,\phi)} - 1) = 0 = y$. Thus it agrees with observations made on structure of the solution space in Lemma 8.1.5 (also refer to Figure 8.2). Whenever it is optimal to either raise the alarm or turn on the observation control, we decide *to* raise the alarm. *This choice can be reflected in Lemma 8.2.8 by changing the additional condition ($v_{n+1}(0, y(t_n(0, \phi), \phi)) < h(y(t_n(0, \phi), \phi)) + v_n(1, y(t_n(0, \phi), \phi))$) to $v_{n+1}(0, y(t_n(0, \phi), \phi)) \leq h(y(t_n(0, \phi), \phi)) + v_n(1, y(t_n(0, \phi), \phi))$ without altering Lemma 8.2.9.*

In the next proposition, we show the threshold $\bar{\xi}$ for ϕ (as defined in (41) in Dayanik and Sezer (2006)) beyond which value function (for notational purposes we denote it with $V^{0,0}(\phi)$) of the compound Poisson disorder problem becomes zero, also serves the same purpose for the value function defined in our study.

Proposition 8.2.11. *We have*

$$\{\phi \in \mathbb{R}_+ : V(\alpha, \phi) = 0, \alpha \in \{0, 1\}\} \supseteq \left\{\phi \in \mathbb{R}_+ : V^{0,0}(\phi) = 0\right\} \supseteq [\bar{\xi}, \infty),$$

where

$$\bar{\xi} := \max\left\{\frac{\lambda + \lambda_0}{c}, \left[\frac{\lambda + \lambda_0}{c} - \phi_d\right]\left(\frac{\lambda_1}{\lambda + \lambda_0}\right) + \phi_d\right\}.$$

Proof. $U(\pi) = (1 - \pi) + c(1 - \pi) \cdot V\left(\alpha, \dfrac{\pi}{1 - \pi}\right) = \inf_{(\tau,\delta)} \mathbb{E}\Big[1_{\{\tau<\theta\}} + c(\tau - \theta)^+ + \sum_{i=1}^{\infty} a 1_{\{\tau_i \leq \tau\}} + \sum_{i=1}^{\infty} b(\sigma_i \wedge \tau$

$- \tau_i \wedge \tau)\Big] \geq \inf_{\tau} \mathbb{E}\left[1_{\{\tau<\theta\}} + c(\tau - \theta)^+\right] = (1 - \pi) + c(1 - \pi) \cdot V_0\left(\dfrac{\pi}{1 - \pi}\right) \implies$

$V(\alpha, \phi) \geq V^{0,0}(\phi), \forall \alpha \in \{0, 1\}, \forall \phi \in \mathbb{R}_+$. Also $V(\alpha, \phi) \leq 0, \forall \alpha \in \{0, 1\}, \forall \phi \in \mathbb{R}_+$, and the first inclusion follows immediately. For the second inclusion refer to Dayanik and Sezer (2006, Proposition 4.2). □

8.3 Limiting behavior of expected cost as a function of a and b.

We now refer back to Proposition 8.2.11, where we compared $V^{a,b}(\alpha, \phi)$ with $V^{0,0}(\phi)$. The purpose of next proposition is to show $V^{0,0}(\phi)$ is indeed the limiting value function that is obtained by reducing the switching on cost ($\$\,a$) and continuous observation cost ($\$\,b$) to 0. In other words, we recover the standard Poisson disorder problem. By also showing that successive approximations $V_n^{0,0}(\alpha, \phi)$ converge to $V^{0,0}(\phi)$ as $n \to \infty$, we show that the numerical scheme presented in this study also holds good for the standard case of the Poisson disorder problem.

Proposition 8.3.1. *For $\alpha \in \{0, 1\}$ and $\phi \in \mathbb{R}_+$, the following hold,*

(i) $\lim_{a,b\downarrow 0} V^{a,b}(\alpha, \phi) = V^{0,0}(\phi)$.

(ii) $\inf_n V_n^{0,0}(\alpha, \phi) = V^{0,0}(\phi)$.

Proof. (i) $\lim_{a,b\downarrow 0} V^{a,b}(\alpha, \phi) = \inf_{a,b\downarrow 0} V^{a,b}(\alpha, \phi) = \inf_{a,b\downarrow 0} \inf_{(\tau,\delta)\in\mathcal{M}} \Big\{ F(\tau, \delta) + b \cdot G(\tau, \delta) + a \cdot H(\tau, \delta) \Big\} = \inf_{(\tau,\delta)\in\mathcal{M}} \inf_{a,b\downarrow 0} \Big\{ F(\tau, \delta) + b \cdot G(\tau, \delta) + a \cdot H(\tau, \delta) \Big\} = \inf_{(\tau,\delta)\in\mathcal{M}} \mathbb{E}_0^{\alpha,\phi} \left[\int_0^\tau e^{-\lambda s} \left(\phi - \frac{\lambda}{c} \right) \mathrm{d}s \right]$. We can recall that the last infimum is in fact the value function (refer to eq. (17)) in Dayanik and Sezer (2006), denoted in our study as $V^{0,0}(\phi)$.

(ii) From part (i) we have, $V^{0,0}(\phi) = \lim_{a,b\downarrow 0} V^{a,b}(\alpha, \phi) = \lim_{a,b\downarrow 0} \inf_n V_n^{a,b}(\alpha, \phi) = \inf_{a,b\downarrow 0} \inf_n V_n^{a,b}(\alpha, \phi)$

$= \inf_n \inf_{a,\,b\,\downarrow 0} V_n^{a,\,b}(\alpha,\,\phi) = \inf_n \inf_{a,\,b\,\downarrow 0} V_n^{a,\,b}(\alpha,\,\phi) = \inf_n V_n^{0,\,0}(\alpha,\,\phi)$ (arguments for the last step are similar to those presented in item (i)).

□

It follows from the above proposition that for $\alpha \in \{0,\,1\}$ and $\phi \in \mathbb{R}_+$, we have,

$$\lim_{a\downarrow 0} V^{a,b}(\alpha,\,\phi) = V^{0,b}(\alpha,\,\phi),\ \forall\, b>0 \quad \text{and} \quad \lim_{b\downarrow 0} V^{a,b}(\alpha,\,\phi) = V^{a,0}(\alpha,\,\phi),\ \forall\, a>0.$$

In addition for $n = 0,\,1,\,2,\,\ldots$, we have,

$$\lim_{a\downarrow 0} V_n^{a,b}(\alpha,\,\phi) = V_n^{0,b}(\alpha,\,\phi),\ \forall\, b>0 \quad \text{and} \quad \lim_{b\downarrow 0} V_n^{a,b}(\alpha,\,\phi) = V_n^{a,0}(\alpha,\,\phi),\ \forall\, a>0.$$

The uniform bound in (6.1.8) also holds i.e., as $n \to \infty$, $\alpha \in \{0,1\}$, $\phi \in \mathbb{R}_+$,

$$\frac{-1}{c}\left(\frac{\lambda_0}{\lambda_0+\lambda}\right)^{n-2\left\lceil \frac{1}{a}\right\rceil} \le V^{a,0}(\alpha,\,\phi) - V_n^{a,0}(\alpha,\,\phi) \le 0,\ \forall\, a>0.$$

Moreover, $(a,b) \mapsto V^{a,b}$ is increasing and concave. Concavity implies continuity of the map $(a,b) \mapsto V^{a,b}$ in the interior of $(a,b) \in [0,\infty)\times[0,\infty)$. Because continuity at boundaries ($a=0$ or $b=0$) is established in the above paragraph, we now know that $(a,b) \mapsto V^{a,b}$ is continuous everywhere on $[0,\infty)\times[0,\infty)$.

In subsection §9.3, we see the successive approximations $V_n^{0,\,0}(\alpha,\,\phi)$ converge to the value function (denoted as $V^{0,\,0}(\phi)$ in our study), in Bayraktar et al. (2005)(check Figure 9.5 and references therein for a comparison).

Chapter 9

Solution and illustrations

9.1 Solution structure

In section §7, we introduced a family of optimal stopping problems (6.1) that *uniformly* converge (refer to Proposition 6.1.8) to (5.10). Theorem 6.1.9 provides an algorithm for computing $\epsilon-$optimal rule (U_n^ϵ) for the original problem in (5.10). Clearly, for any $\epsilon > 0$, we can always find an $n \in \mathbb{N}$ such that $V_n(\alpha, \phi) \approx V(\alpha, \phi)$. For such an n, U_n^ϵ is determined by non-terminating events i.e., switching on/off observation control and observable jump times as follows:

1. if $\alpha = 1$ initially, we look for $t_{n-1}(1, \phi) \wedge s_{n-1}(1, \phi) \wedge T_1$. If $t_{n-1}(1, \phi)$ happens first the we raise the alarm at $t_{n-1}(1, \phi)$. If $s_{n-1}(1, \phi)$ happens first, we turn off observation control at $s_{n-1}(1, \phi)$, and re-start the problem with different initial conditions (since Φ_t^δ is now at $x(s_{n-1}(1, \phi), \phi)$) to determine U_{n-1}^ϵ (then, refer to item 2). Finally if T_1 happens first, i.e. we have an arrival, we update our odds-ratio process to $\frac{\lambda_1}{\lambda_0} x(T_1, \phi)$ and re-start our problem with this new initial condition to determine U_{n-1}^ϵ.

2. if $\alpha = 0$ initially, we look for $t_{n-1}(0, \phi) \wedge q_{n-1}(1, \phi)$. If $t_{n-1}(0, \phi)$ happens first the we raise the alarm at $t_{n-1}(0, \phi)$. If $q_{n-1}(0, \phi)$ happens first, we turn on observation control at $q_{n-1}(0, \phi)$, and re-start the problem with different initial conditions (since Φ_t^δ is now at $y(q_{n-1}(0, \phi), \phi)$) to determine U_{n-1}^ϵ

(then, refer to item 1).

In brief, we switch between items 1 and 2 when a non-terminating event (NT) happens before a terminating (T) one (i.e. raising alarm) and stop at the n^{th} stage. It is also useful to revisit Lemma 8.1.1 here, due to which we have that, in the case $\alpha = 1$, one need not bother with raising the alarm and this helps reduce dimensionality of the optimization problem in (7.7).

In section §8 (refer to Lemmas 8.2.3-8.2.9), we show that the above description of solution leads to an alternate characterization (that is common in theory of optimal stopping) of turning on/off and raising alarm times. We expand this idea further and illustrate it through concrete examples. In order to do that, let us begin by defining switching ($\mathcal{S}_n(\alpha)$), alarm ($\mathcal{A}_n(\alpha)$) and stationary (null) regions ($\mathcal{N}_n(\alpha)$).

(i) When $\alpha = 1$, $n = 0, 1, 2, \ldots,$

$$\mathcal{S}_n(1) := \Big\{ \phi \in \mathbb{R}^+ :\ v_{n+1}(1, \phi) \geq v_n(0, \phi),\, v_{n+1}(1, \phi) \leq 0 \Big\}$$

$$= \Big\{ \phi \in \mathbb{R}^+ :\ v_{n+1}(1, \phi) = v_n(0, \phi) \Big\},$$

$$\mathcal{N}_n(1) := \mathbb{R}^+ \setminus \mathcal{S}_n(1).$$

Note. $\mathcal{A}_n(1) = \emptyset$.

(ii) When $\alpha = 0$, $n = 0, 1, 2, \ldots,$

$$\mathcal{S}_n(0) := \Big\{ \phi \in \mathbb{R}^+ :\ v_{n+1}(0, \phi) \geq h(\phi) + v_n(1, \phi),\, v_{n+1}(0, \phi) < 0 \Big\},$$

$$\mathcal{A}_n(0) := \Big\{ \phi \in \mathbb{R}^+ :\ v_{n+1}(0, \phi) = 0,\, v_{n+1}(0, \phi) \leq h(\phi) + v_n(1, \phi) \Big\},$$

and

$$\mathcal{N}_n(0) := \mathbb{R}^+ \setminus \left(\mathcal{S}_n(0) \cup \mathcal{A}_n(0) \right).$$

The following corollary is immediate owing to above definitions,

Corollary 9.1.1. *For* $n = 0, 1, 2, \ldots,$ $\mathcal{C}_n(\alpha)$, $\mathcal{A}_n(\alpha)$ *and* $\mathcal{N}_n(\alpha)$ *are mutually disjoint where* $\alpha = 0$ *or* 1.

The odds-ratio process, Φ_t^δ evolves deterministically between observable jump times and this behavior is described in (7.2) and (7.10). This helps us reformulate turning on/off and alarm times as the first time stochastic process, Φ_t^δ returns to $\mathcal{S}_n(\cdot)$ and $\mathcal{A}_n(\cdot)$ respectively. Mathematically put, we have,

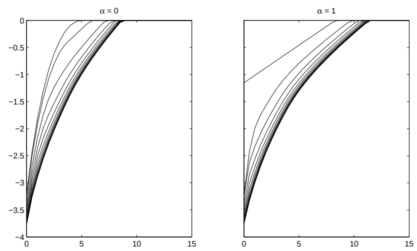
$$t_n(\alpha, \phi) = \inf\left\{t > 0 : \Phi_t^\delta \in \mathcal{A}_{n+1}(\alpha)\right\}$$

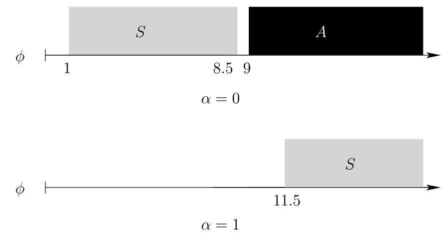
$$= \begin{cases} \inf\left\{t > 0 : x(t, \phi) \in \mathcal{A}_{n+1}(\alpha)\right\}, & \alpha = 1 \\ \inf\left\{t > 0 : y(t, \phi) \in \mathcal{A}_{n+1}(\alpha)\right\}, & \alpha = 0 \end{cases},$$

$$\begin{bmatrix} s_n(1, \phi) = \inf\left\{t > 0 : \Phi_t^\delta \in \mathcal{S}_{n+1}(1)\right\} = \inf\left\{t > 0 : x(t, \phi) \in \mathcal{S}_{n+1}(1)\right\} \\ q_n(0, \phi) = \inf\left\{t > 0 : \Phi_t^\delta \in \mathcal{S}_{n+1}(0)\right\} = \inf\left\{t > 0 : y(t, \phi) \in \mathcal{S}_{n+1}(0)\right\} \end{bmatrix}.$$

9.2 Numerical examples

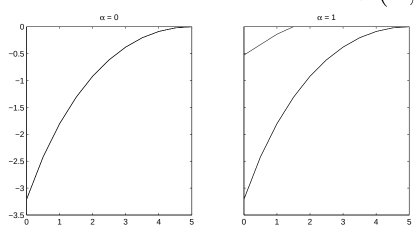
We now illustrate the solution described above with the help of some numerical examples.

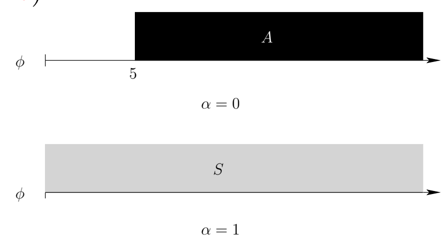




(a) (b)

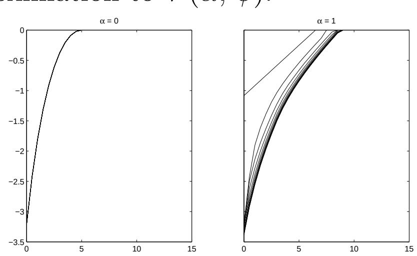
Note. In figures (a) and (b), $a = 0.01$, $b = 0.05$ and, $n = 13$ gives a sufficiently good approximation to value function $V(\alpha, \phi)$ in (5.10).

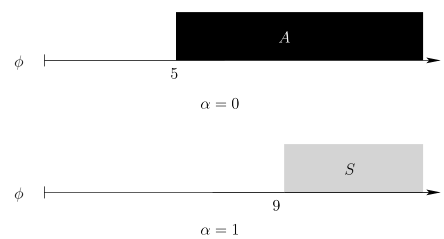




(c) (d)

Note. In figures (c) and (d), $a = 0.01$, $b = 0.5$ and, $n = 3$ gives a sufficiently good approximation to $V(\alpha, \phi)$.





(e) (f)

Note. In figures (e) and (f), $a = 0.1$, $b = 0.1$ and, $n = 12$ gives a sufficiently good approximation to $V(\alpha, \phi)$.

Figure 9.1: In all the above subfigures, $\lambda = 1$, $\lambda_0 = 3$, $c = 0.2$, $\lambda_1 = 3 \cdot \lambda_0$. In subfigures (b), (d), (f)– the two stacked figures illustrate action spaces. S and A denote switching and alarm regions, respectively. The alarm region when $\alpha = 0$ and switching region when $\alpha = 1$, extend all the way to $+\infty$.

Let us compare the different examples in Figure 9.1. In (a) and (b) the cost to turn on control is $a = 0.01$ and cost of continuous observation is $b = 0.05$, which is slightly higher than a. Although we have $b > a$, we see the optimal control is turned on for the most part. This could be due to the fact that a, b are small compared to c and the false detection cost.

In figures (c) and (d) we observe that b is considerably larger than a and infact b is greater than c. As one would expect in this case, although it is cheap to turn the control on, it happens to be too expensive to leave it on even for a very short period.

In figures (e) and (f), we increase both a and b to a value that is comparable with, but less than c. Unlike in the earlier examples where b was either considerably larger or considerably smaller than a (in which case, the solution seems quite intuitive), we now have them equal. This as we might expect would shrink the switching region in the case $\alpha = 1$, since now we can afford to observe over a larger set than compared to the case when b is very large. However, this observable region would be smaller compared to the observable region in the case when b is the smallest.

Next we consider a special case (refer to figure below) where we associate zero cost to turning on the control ($a = 0$) and a non-zero cost for continuous observation ($b > 0$). A quick observation reveals that by increasing the value of b, the switching region for $\alpha = 0$ shrinks. This is intuitive since by increasing the cost of continuous observation one would expect shorter observation intervals. By further increasing the value of b we see the there is no switching region for the case $\alpha = 0$ and the switching region for the case $\alpha = 1$ is $\mathbb{R}_+$, which is again intuitive.

(a) $b = 0.05$ and $n = 10$. (b) $b = 0.08$ and $n = 11$.

Figure 9.2: In both figures, $\lambda = 1$, $\lambda_0 = 3$, $c = 0.1$, $\lambda_1 = 2 \cdot \lambda_0$, $a = 0$.

Some observations. What we observe from our numerical results is that the value of c affects alarm region in the following way– a high value of c implies a smaller alarm region and vice versa. This agrees with our intuition since with a low c value we pay much smaller for a delayed detection and hence we do not mind *not* raising the alarm for a longer period of time.

Switching regions are more directly affected by a and b rather than c. However, proximity of a and b to c also plays a critical role. a, b being close to c indicates a *relatively* high turn on and continuous observation cost. This also can be noted from the fact that the cost function for continuous observation $g(\alpha, \phi)$ in (5.11) has a coefficient of the form $\frac{b}{c}$; and cost function for turning on the observation control $h(\phi)$ in (5.12) has a coefficient of the form $\frac{a}{c}$. We illustrate this in Figure 9.3.

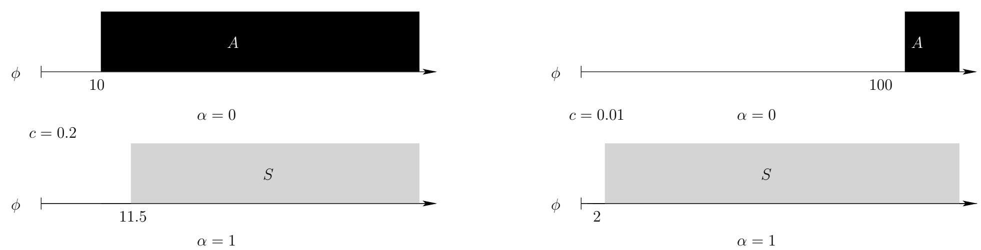
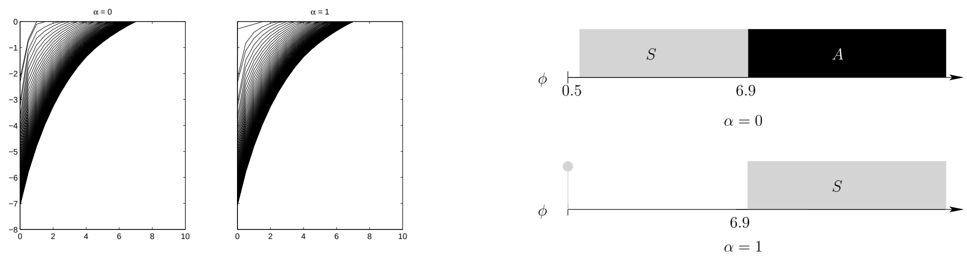


Figure 9.3: In the above figures we have $\lambda = 2$, $\lambda_0 = 3$, $c = 0.2$, $\lambda_1 = 0.5 \cdot \lambda_0$, $a = 0.01$, $b = 0.01$.

In the Figure 9.3, by decreasing c, we observe that the cost to turn on and continuously observe become *relatively* high since the ratio a/c and b/c becomes high. Also to be noted is the shrinking of alarm region in the case when c is smaller.

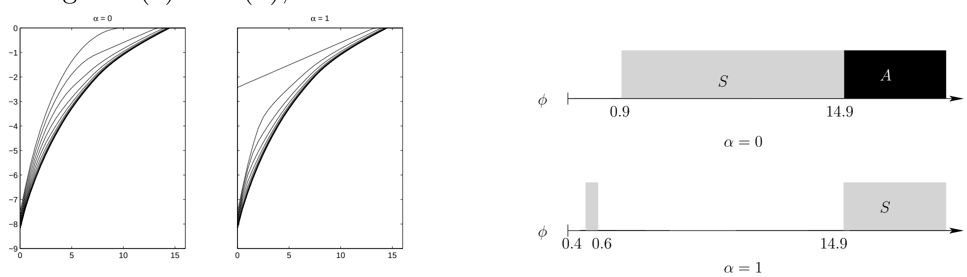
Next, we fix the value of c and then look at how a and b affect the switching regions. a, b together affect the switching regions $\mathcal{S}(0)$ (switching region in the case $\alpha = 0$) and $\mathcal{S}(1)$ (switching region in the case $\alpha = 1$) in a slightly complicated way. It is difficult to see their individual effects unless we restrict ourselves to some special cases. Namely, we consider cases where we assign large or small values to a and b. In the following discussion, note that *large* indicates a value of the order of c and *small* indicates proximity to zero. In the examples, our goal is to observe two things, firstly how the maximum of a and b dominates the other in determining the switching region $\mathcal{S}(0)$; and secondly, how b dominates a in determining the switching region, $\mathcal{S}(1)$.

When a and b are large, the optimal strategy involves in *never* turning on the



(a) (b)

Note. In figures (a) and (b), $\lambda = 0.1$ and $n = 127$.



(c) (d)

Note. In figures (c) and (d), $\lambda = 1$ and $n = 16$.

Figure 9.4: $\lambda_0 = 3$, $c = 0.1$, $\lambda_1 = 2 * \lambda_0$, $a = 0$, $b = 0.01$.

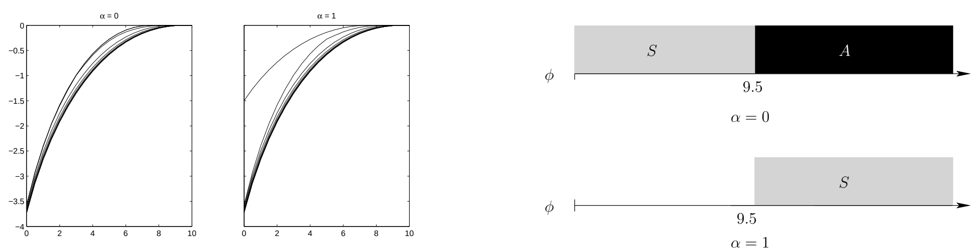
observation control. This implies that the switching region for the case $\alpha = 1$ and the switching region for the case $\alpha = 0$, is the empty set. However once we reduce b significantly while still keeping a relatively high, we see that the optimal stragety would be to never turn off the control if it is initially turned on; and if it is turned off initially, we never turn it on, even though the cost of continuous observation is very low. This happens due to the fact that the value of a is so high that it overpowers the cost of continuous observation and if we are lucky enough to have the control initially switched on, we never turn off since the cost of continuously observing is very low. For the case when we fix b at large value and vary a we get similar results i.e., $\mathcal{S}(0) = \emptyset$. One of the reasons for this behavior could be that, switching in the case $\alpha = 0$ would be expensive if either of the costs i.e., a or b is high.

To observe how b dominates a in determining $\mathcal{S}(1)$ in some cases, we first fix b at a large value, and vary a from large to small values. As expected in both cases the switching region $\mathcal{S}(1)$ is $\mathbb{R}_+$. This is again intuitive since, irrespective of what a is, one would have to pay b continuously if the control were to be left on. Hence, we switch immediately no matter what a is. We now set b to a small value and once again vary a between small and large values. When a is small as well, this is the classical Poisson disorder case and as expected the $\mathcal{S}(1) = \emptyset$ since its very cheap to continuously observe. When a is increased to a large value, we again see that if the observation control was initially switched on, it is optimal to leave it switched on since cost of continuous observation is very low. Thus we see how b has more effect on $\mathcal{S}(1)$ in some cases.

In the other cases (when a , b are not very large or small), it is hard to see how a and b affect the switching regions individually and the action spaces are not trivial.

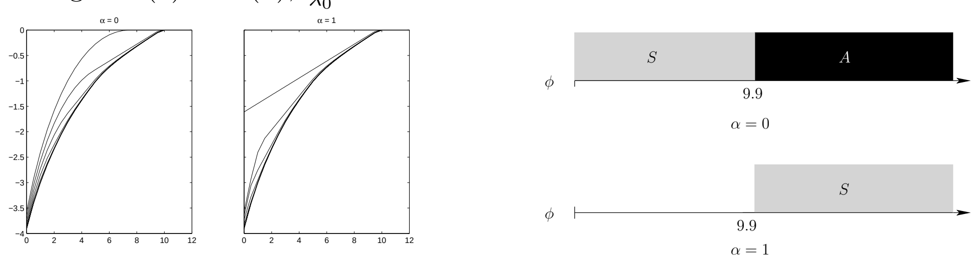
9.3 The standard Poisson disorder problem (when $a = 0$ and $b = 0$).

As expected (refer to Figure 9.5), we always leave the observation control turned on since that would not cost anything to us. We either immediately switch or raise an alarm in the case when $\alpha = 0$ depending on the value of ϕ. Once we turn the control on, we only switch back to $\alpha = 0$ to raise the alarm immediately, as can be noted from the coincidence of the switching region for $\alpha = 1$ and the alarm region for $\alpha = 0$. We can also note that critical thresholds of alarm regions (as defined in Dayanik and Sezer (2006)) for the above two examples coincide with those presented in Figure 6(a) in Dayanik and Sezer (2006). Thus the standard Poisson disorder problem turns out to be a special case of the more general problem solved in our study (refer to Proposition 8.3.1 for a proof of why the numerical scheme presented in this study works in the standard case as well).



(a) (b)

Note. In figures (a) and (b), $\frac{\lambda_1}{\lambda_0} = 0.5$ and $n = 10$.



(c) (d)

Note. In figures (c) and (d), $\frac{\lambda_1}{\lambda_0} = 2$ and $n = 9$.

Figure 9.5: $\lambda = 1.5$, $\lambda_0 = 3$, $c = 0.2$, $a = 0$, $b = 0$ as in Bayraktar et al. (2005). For a comparison refer to Dayanik and Sezer (2006, Figure 5(a)).

Appendix A

Calculations

A.1 Re-formulation of R^δ_τ

Following the results from Bayraktar et al. (2005) we can rewrite

$$\mathbb{E}[(\tau-\theta)^+] = (1-\pi)\mathbb{E}_0\int_0^\tau e^{-\lambda s}\Phi^\delta_s \mathrm{d}s \tag{A.1}$$

$$\mathbb{P}\left\{\tau<\theta\right\} = (1-\pi) - (1-\pi)\lambda\mathbb{E}_0\int_0^\tau e^{-\lambda s}\mathrm{d}s \tag{A.2}$$

Using (5.4), changing measure to $\mathbb{P}_0$ by noting $1_{\{\tau_i\le\tau\}}\in\mathcal{F}^\delta_{\tau_i\wedge\tau}$, conditioning on θ and using independence of θ and $1_{\{\tau_i\le\tau\}}$ under $\mathbb{P}_0$ measure we get,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^\infty a1_{\{\tau_i\le\tau\}}\right] &= \sum_{i=1}^\infty a\mathbb{E}_0\left[Z^\delta_{\tau_i\wedge\tau}1_{\{\tau_i\le\tau\}}\right]\\
&= \sum_{i=1}^\infty a\mathbb{E}_0\left[\left(1_{\{\theta>\tau_i\}}+1_{\{\theta\le\tau_i\}}\frac{L^\delta_{\tau_i}}{L^\delta_\theta}\right)1_{\{\tau_i<\tau\}}\right]\\
&= \sum_{i=1}^\infty a\mathbb{E}_0\Bigg[\underbrace{\left((1-\pi)e^{-\lambda\tau_i}+\pi L^\delta_{\tau_i}+(1-\pi)\int_0^{\tau_i}\lambda e^{-\lambda s}\frac{L^\delta_{\tau_i}}{L^\delta_s}\mathrm{d}s\right)}_{(1-\pi)(1+\Phi^\delta_{\tau_i})e^{-\lambda\tau_i}}\cdot
\end{aligned}$$

$$
\begin{aligned}
&1_{\{\tau_i<\tau\}}\Bigg] \\
&= \sum_{i=1}^{\infty} a\mathbb{E}_0\left[(1-\pi)(1+\Phi^\delta_{\tau_i})e^{-\lambda\tau_i}1_{\{\tau_i\le\tau\}}\right] \\
&= a(1-\pi)\mathbb{E}_0\left[\sum_{i=1}^{\infty}(1+\Phi^\delta_{\tau_i})e^{-\lambda\tau_i}1_{\{\tau_i\le\tau\}}\right]
\end{aligned}
$$

We can further rewrite the summation inside the expectation as an integral by using (5.9).

$$
\begin{aligned}
&a(1-\pi)\mathbb{E}_0\left[\sum_{i=1}^{\infty}(1+\Phi^\delta_{\tau_i})e^{-\lambda\tau_i}1_{\{\tau_i\le\tau\}}\right] \\
&= a(1-\pi)\mathbb{E}_0\left[\sum_{0<s<\tau}\left(1+\Phi^\delta_s\right)e^{-\lambda s}\Delta\alpha^\delta_{on}(s)\right] \\
&= a(1-\pi)\mathbb{E}_0\left[\int_0^\tau e^{-\lambda s}\left(1+\Phi^\delta_s\right)\mathrm{d}\alpha^\delta_{on}(s)\right] \qquad \text{(A.3)}
\end{aligned}
$$

Changing measure to $\mathbb{P}_0$ by noting $\alpha^\delta(s)\in\mathcal{F}^\delta_s$, conditioning on θ and using independence of θ and $1_{\{s\le\tau\}}$, $\alpha^\delta(s)$ under $\mathbb{P}_0$ measure we get,

$$
\begin{aligned}
&\mathbb{E}\left[b\sum_{i=1}^{\infty}(\sigma_i\wedge\tau-\tau_i\wedge\tau)\right] \\
&= b\cdot\mathbb{E}\left[\int_0^\tau\alpha^\delta(s)\mathrm{d}s\right] \;=\; b\cdot\int_0^\infty\mathbb{E}_0\left[1_{\{s\le\tau\}}Z^\delta_s\alpha^\delta(s)\right]\mathrm{d}s \\
&= b\cdot\int_0^\infty\mathbb{E}_0\left[1_{\{s\le\tau\}}\left(1_{\{\theta>s\}}+1_{\{\theta\le s\}}\frac{L^\delta_s}{L^\delta_\theta}\right)\cdot\alpha^\delta(s)\right]\mathrm{d}s \\
&= b\cdot\int_0^\infty\mathbb{E}_0\left[\mathbb{E}_0\left[\left(1_{\{\theta>s\}}+1_{\{\theta\le s\}}\frac{L^\delta_s}{L^\delta_\theta}\right)\cdot\alpha^\delta(s)1_{\{s\le\tau\}}\middle|\mathcal{F}^\delta_s\right]\right]\mathrm{d}s \\
&= b\cdot\int_0^\infty\mathbb{E}_0\left[1_{\{s\le\tau\}}\alpha^\delta(s)\cdot\mathbb{E}_0\left[\left(1_{\{\theta>s\}}+1_{\{\theta\le s\}}\frac{L^\delta_s}{L^\delta_\theta}\right)\middle|\mathcal{F}^\delta_s\right]\right]\mathrm{d}s \\
&= b\cdot\int_0^\infty\mathbb{E}_0\left[1_{\{s\le\tau\}}\alpha^\delta(s)\cdot\left((1-\pi)e^{-\lambda s}+\pi L^\delta_s+(1-\pi)\int_0^s\lambda e^{-\lambda u}\frac{L^\delta_s}{L^\delta_u}\mathrm{d}u\right)\right]\mathrm{d}s \\
&= b\cdot\int_0^\infty\mathbb{E}_0\left[1_{\{s\le\tau\}}\alpha^\delta(s)\cdot(1-\pi)e^{-\lambda s}\cdot(1+\Phi_s)\right]\mathrm{d}s \\
&= b(1-\pi)\mathbb{E}_0\left[\int_0^\tau\alpha^\delta(s)\cdot e^{-\lambda s}\cdot(1+\Phi_s)\mathrm{d}s\right] \qquad \text{(A.4)}
\end{aligned}
$$

Using (A.1), (A.2), (A.3) and (A.4), we can reformulate (5.2) as (5.7).

Note. By defining the following process

$$\alpha^{\delta}_{off}(t) = \sum_{i=1}^{\infty} 1_{[\sigma_i,\infty)}(t)$$

the observation process can be written as,

$$\alpha^{\delta}(t) = \alpha^{\delta}_{on}(t) - \alpha^{\delta}_{off}(t)$$

where $\alpha^{\delta}_{on}(t)$ is defined as in (5.9).

A.2 Dynamics of L^{δ}_t

We compute the dynamics of L^{δ}_t as defined in (5.5). We begin with first principles,

$$L^{\delta}_t = L^{\delta}_0 + (L^{\delta}_{T_1} - L^{\delta}_0) + \dots + (L^{\delta}_t - L^{\delta}_{T_n}) = L^{\delta}_0 + (L^{\delta}_{T_1} - L^{\delta}_{T_1-}) + (L^{\delta}_{T_1-} - L^{\delta}_0) + \dots$$

$$+(L^{\delta}_t - L^{\delta}_{T_n}) = L^{\delta}_0 + \sum_{i=1}^{n}(L^{\delta}_{T_i} - L^{\delta}_{T_i-}) + \sum_{i=1}^{n}(L^{\delta}_{T_i-} - L^{\delta}_{T_{i-1}}) + (L^{\delta}_t - L^{\delta}_{T_n}) = L^{\delta}_0$$

$$+ \sum_{i=1}^{n}(L^{\delta}_{T_i} - L^{\delta}_{T_i-}) + \int_0^t \frac{\mathrm{d}\widetilde{L}^{\delta}_s}{\mathrm{d}s} \cdot \mathrm{d}s = L^{\delta}_0 + \int_0^t (L^{\delta}_s - L^{\delta}_{s-}) \cdot \mathrm{d}X_s + \int_0^t \frac{\mathrm{d}\widetilde{L}^{\delta}_s}{\mathrm{d}s} \cdot \mathrm{d}s,$$

where T_i's are arrival times and $\widetilde{L}^{\delta}_s$ is the continuous part of L^{δ}_s. We look at jump and continous parts of L^{δ}_s separately as follows,

$$(L^{\delta}_s - L^{\delta}_{s-}) = L^{\delta}_{s-} \cdot \left[\exp\left\{\ln\left(\frac{\lambda_1}{\lambda_0}\right) \cdot \sum_{k=1}^{\infty} 1_{(\tau_k,\,\sigma_k]}(s)\right\} - 1\right],$$

$$\frac{\mathrm{d}\widetilde{L}^{\delta}_s}{\mathrm{d}s} = -(\lambda_1 - \lambda_0) \cdot \widetilde{L}^{\delta}_s \cdot \left(\sum_{k=1}^{\infty} 1_{(\tau_k,\,\sigma_k]}(s)\right).$$

Thus the dynamics of the likelihood ratio process L_t^δ in (5.5) is given by the following differential equation

$$dL_t^\delta = L_{t^-}^\delta \left[\exp\left\{\ln\left(\frac{\lambda_1}{\lambda_0}\right)\alpha^\delta(t^-)\right\} - 1\right] \mathrm{d}X_t^\delta - (\lambda_1 - \lambda_0)L_t^\delta \alpha^\delta(t)\mathrm{d}t, \quad L_0^\delta = 1. \tag{A.5}$$

A.3 Dynamics of Φ_t^δ

The observation process X_t^δ as defined in (5.1) can be rewritten as,

$$X_t^\delta = X_0^\delta + \int_0^t \alpha^\delta(u)\mathrm{d}X_u \tag{A.6}$$

and the differential form of (A.6) can be substituted for $\mathrm{d}X_t^\delta$ term in (A.5). Since θ is independent of the process X and has an exponential distribution under $\mathbb{P}_0$, using the generalized Bayes result and (5.8) gives us,

$$\Phi_t^\delta = \frac{\mathbb{P}\left\{\theta \le t|\mathcal{F}_t^\delta\right\}}{\mathbb{P}\left\{\theta > t|\mathcal{F}_t^\delta\right\}} = \frac{\mathbb{E}[1_{\{\theta\le t\}}|\mathcal{F}_t^\delta]}{\mathbb{E}[1_{\{\theta> t\}}|\mathcal{F}_t^\delta]} = \frac{\mathbb{E}_0[Z_t^\delta 1_{\{\theta\le t\}}|\mathcal{F}_t^\delta]}{\mathbb{E}_0[Z_t^\delta 1_{\{\theta> t\}}|\mathcal{F}_t^\delta]} = \frac{e^{\lambda t}}{(1-\pi)}\cdot$$

$$\mathbb{E}_0\left[\left(1_{\{\theta>t\}} + 1_{\{\theta\le t\}}\cdot\frac{L_t^\delta}{L_\theta^\delta}\right)1_{\{\theta\le t\}}\middle|\mathcal{F}_t^\delta\right] = \frac{e^{\lambda t}}{(1-\pi)}\mathbb{E}_0\left[1_{\{\theta\le t\}}\frac{L_t^\delta}{L_\theta^\delta}\middle|\mathcal{F}_t^\delta\right]$$
$$= \frac{e^{\lambda t}L_t^\delta}{(1-\pi)}\left[\pi + (1-\pi)\int_0^t \frac{1}{L_s^\delta}\lambda e^{-\lambda s}\mathrm{d}s\right]$$

Let T_i be the i^{th} jump time of process X_t^δ. Then Φ_t^δ jumps whenever L_t^δ jumps and L_t^δ jumps whenever X_t^δ jumps.

$$\Phi_t^\delta = \Phi_0^\delta + \sum_{i=1}^{X_t^\delta}(\Phi_{T_i}^\delta - \Phi_{T_i-}^\delta) + \int_0^t \frac{\mathrm{d}\widehat{\Phi}_s^\delta}{\mathrm{d}s}\cdot\mathrm{d}s \;=\; \Phi_0^\delta + \int_0^t \overbrace{(\Phi_s^\delta - \Phi_{s-}^\delta)}^{J_s\text{: jump part}}\mathrm{d}X_s^\delta$$

$$+ \int_0^t \underbrace{\frac{\mathrm{d}\widehat{\Phi}_s^\delta}{\mathrm{d}s}}_{C_s\text{: continuous part}}\cdot\mathrm{d}s$$

Letting $\widehat{\Phi}^{\delta}_{s}$, $\widehat{L}^{\delta}_{s}$ denote the continuous parts of processes Φ^{δ}_{s} and L^{δ}_{s} respectively, we compute expressions for the J_s and C_s as follows,

$$C_s = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{e^{\lambda s}\widehat{L}^{\delta}_{s}}{(1-\pi)}\left[\pi + (1-\pi)\int_0^s \frac{1}{\widehat{L}^{\delta}_{u}}\lambda e^{-\lambda u}\mathrm{d}u\right]\right) = \lambda + \widehat{\Phi}^{\delta}_{s}(\lambda - (\lambda_1 - \lambda_0)\alpha^{\delta}(s))$$

$$J_s = \frac{e^{\lambda t}L^{\delta}_{s}}{(1-\pi)}\left[\pi + (1-\pi)\int_0^s \frac{1}{L^{\delta}_{u}}\lambda e^{-\lambda u}\mathrm{d}u\right] - \frac{e^{\lambda t}L^{\delta}_{s^-}}{(1-\pi)}\left[\pi + (1-\pi)\int_0^{s^-} \frac{1}{L^{\delta}_{u}}\cdot\right.$$
$$\left.\lambda e^{-\lambda u}\mathrm{d}u\right] = \Phi^{\delta}_{s^-}\left(\exp\left\{\ln\left(\frac{\lambda_1}{\lambda_0}\right)\alpha^{\delta}(s^-)\right\} - 1\right)$$

Putting together the continuous and jump parts we have in differential form,

$$\begin{aligned} &\mathrm{d}\Phi^{\delta}_{t} \\ &= \left(\left(\frac{\lambda_1}{\lambda_0}\right)^{\alpha^{\delta}(t^-)} - 1\right)\Phi^{\delta}_{t^-}\mathrm{d}X^{\delta}_{t} + \left[\lambda + \Phi^{\delta}_{t}\left\{\lambda - (\lambda_1-\lambda_0)\alpha^{\delta}(t)\right\}\right]\mathrm{d}t, \\ &\Phi^{\delta}_{0} = \phi = \frac{\pi}{1-\pi}. \end{aligned} \tag{A.7}$$

Appendix B

Long proofs

B.1 Proof of Theorem 6.1.9

We adapt (Davis, 1993, Lemma A2.3, p. 261) to our problem and it forms a basic building block of the proof along with the strong Markov property.

Lemma B.1.1. *For every $\mathbb{F}-$ stopping time τ and every $n \in \mathbb{N}_0$, there are $\mathcal{F}^{\delta}_{\tau_i}-$measurable random variables $x,\, y : \Omega \mapsto [0,\, \infty]$ such that $\tau \wedge \sigma_i \wedge T_{(n-k)-(2i-3)} = (\tau_i + x) \wedge (\tau_i + y) \wedge T_{(n-k)-(2i-3)}$ $\mathbb{P}_0-$ a.s. on $\{\tau \geq \tau_i\}$.*

Proof. To prove $V_n(\alpha,\, \phi) \leq v_n(\alpha,\, \phi)$, $n \in \mathbb{N}$ we first establish (6.5). The inequality we are looking for follows from there since $U^{\epsilon}_n \leq \rho_n$ $\mathbb{P}_0-$a.s.. We first prove for the case when $\alpha = 1$. The case when $\alpha = 0$ should follow similar arguments. When $n = 1$,

$$\begin{aligned}
&\mathbb{E}^{(1,\,\phi)}_0\left[\int_0^{U^{\epsilon}_1} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s + \underbrace{\int_0^{U^{\epsilon}_1} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s)}_{=0 \text{ since } \alpha^{\delta}_s \text{ jumps at } \tau_i\text{'s}}\right] \\
&= \mathbb{E}^{(1,\,\phi)}_0\left[\int_0^{t^{\epsilon}_0(1,\,\Phi^{\delta}_0)\wedge s^{\epsilon}_0(1,\,\Phi^{\delta}_0)\wedge T_1} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s\right] \\
&= (Jv_0)(t^{\epsilon}_0(1,\, \Phi^{\delta}_0),\, s^{\epsilon}_0(1,\, \Phi^{\delta}_0),\, 1,\, \phi)
\end{aligned}$$

$$\leq (J_0 v_0)(1,\, \phi) + \epsilon$$
$$= v_1(1,\, \phi) + \epsilon$$

Assuming it holds for $n = k$, we need to prove for $n = k+1$. We have $U^{\epsilon}_{k+1} \wedge s^{\epsilon/3}_k(1,\, \Phi^{\delta}_0) \wedge T_1 = t^{\epsilon/3}_k(1,\, \Phi^{\delta}_0) \wedge s^{\epsilon/3}_k(1,\, \Phi^{\delta}_0) \wedge T_1$ a.s. .

$$\begin{aligned}
&\mathbb{E}^{(1,\,\phi)}_0 \left[\int_0^{U^{\epsilon}_{k+1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s + \int_0^{U^{\epsilon}_{k+1}} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s) \right] \\
&= \mathbb{E}^{(1,\,\phi)}_0 \Bigg[\int_0^{U^{\epsilon}_{k+1} \wedge s^{\epsilon/3}_k \wedge T_1} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s + 1_{\{s^{\epsilon/3}_k < U^{\epsilon}_{k+1} \wedge T_1\}} \int_{s^{\epsilon/3}_k}^{U^{\epsilon}_{k+1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s \\
&+ 1_{\{T_1 < U^{\epsilon}_{k+1} \wedge s^{\epsilon/3}_k\}} \int_{T_1}^{U^{\epsilon}_{k+1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s + \int_0^{U^{\epsilon}_{k+1} \wedge s^{\epsilon/3}_k \wedge T_1} e^{-\lambda s} h(\, \Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s) \\
&+ 1_{\{s^{\epsilon/3}_k < U^{\epsilon}_{k+1} \wedge T_1\}} \int_{s^{\epsilon/3}_k}^{U^{\epsilon}_{k+1}} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s) \\
&+ 1_{\{T_1 < U^{\epsilon}_{k+1} \wedge s^{\epsilon/3}_k\}} \int_{T_1}^{U^{\epsilon}_{k+1}} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s) \Bigg] \\
&= \mathbb{E}^{(1,\,\phi)}_0 \Bigg[\int_0^{t^{\epsilon/3}_k \wedge s^{\epsilon/3}_k \wedge T_1} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s \\
&+ \underbrace{1_{\{s^{\epsilon/3}_k < t^{\epsilon/3}_k \wedge T_1\}} \int_{s^{\epsilon/3}_k}^{s^{\epsilon/3}_k + U^{\epsilon/3}_k \circ \theta_{s^{\epsilon/3}_k}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s}_{(a)} \\
&+ \underbrace{1_{\{T_1 < t^{\epsilon/3}_k \wedge s^{\epsilon/3}_k\}} \int_{T_1}^{T_1 + U^{\epsilon/3}_k \circ \theta_{T_1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\mathrm{d}s}_{(b)} + \int_0^{t^{\epsilon/3}_k \wedge s^{\epsilon/3}_k \wedge T_1} e^{-\lambda s} h(\, \Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s) \\
&+ \underbrace{1_{\{s^{\epsilon/3}_k < t^{\epsilon/3}_k \wedge T_1\}} \int_{s^{\epsilon/3}_k}^{s^{\epsilon/3}_k + U^{\epsilon/3}_k \circ \theta_{s^{\epsilon/3}_k}} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s)}_{(c)} \\
&+ \underbrace{1_{\{T_1 < t^{\epsilon/3}_k \wedge s^{\epsilon/3}_k\}} \int_{T_1}^{T_1 + U^{\epsilon/3}_k \circ \theta_{T_1}} e^{-\lambda s} h(\Phi^{\delta}_s)\mathrm{d}\alpha^{\delta}_{on}(s)}_{(d)} \Bigg] \qquad \text{(B.1)}
\end{aligned}$$

Using the strong Markov property of X_s^δ we can simplify (a) as follows,

$$\mathbb{E}_0^{(1,\phi)}\left[1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}\int_{s_k^{\epsilon/3}}^{s_k^{\epsilon/3}+U_k^{\epsilon/3}\circ\theta_{s_k^{\epsilon/3}}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s\right]$$

$$=\mathbb{E}_0^{(1,\phi)}\left[\mathbb{E}_0^{(1,\phi)}\left\{1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}\int_{s_k^{\epsilon/3}}^{s_k^{\epsilon/3}+U_k^{\epsilon/3}\circ\theta_{s_k^{\epsilon/3}}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s\middle|\mathcal{F}^\delta_{s_k^{\epsilon/3}}\right\}\right]$$

$$=\mathbb{E}_0^{(1,\phi)}\left[1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}\mathbb{E}_0^{(1,\phi)}\left\{\int_0^{U_k^{\epsilon/3}\circ\theta_{s_k^{\epsilon/3}}} e^{-\lambda(s+s_k^{\epsilon/3})}g(\alpha^\delta_{s+s_k^{\epsilon/3}},\,\Phi^\delta_{s+s_k^{\epsilon/3}})\cdot\right.\right.$$

$$\left.\left.\mathrm{d}s\middle|\mathcal{F}^\delta_{s_k^{\epsilon/3}}\right\}\right]$$

$$=\mathbb{E}_0^{(1,\phi)}\left[1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}e^{-\lambda s_k^{\epsilon/3}}\mathbb{E}_0^{(1,\phi)}\left\{\left(\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\right)\circ\theta_{s_k^{\epsilon/3}}\mathrm{d}s\middle|\mathcal{F}^\delta_{s_k^{\epsilon/3}}\right\}\right]$$

$$=\mathbb{E}_0^{(1,\phi)}\left[1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}e^{-\lambda s_k^{\epsilon/3}}\mathbb{E}_0^{(0,\,\Phi^\delta_{s_k^{\epsilon/3}})}\left\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s\right\}\right]$$

Using the above approach we could also simplify (b), (c), (d). (B.1) becomes,

$$=\mathbb{E}_0^{(1,\phi)}\left[\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s+1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}e^{-\lambda s_k^{\epsilon/3}}\cdot\right.$$

$$\mathbb{E}_0^{(0,\,\Phi^\delta_{s_k^{\epsilon/3}})}\left\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s\right\}$$

$$+1_{\{T_1<t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\}}e^{-\lambda T_1}\mathbb{E}_0^{(1,\,\Phi^\delta_{T_1})}\left\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s\right\}$$

$$+\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s}h(\,\Phi_s^\delta)\mathrm{d}\alpha^\delta_{on}(s)$$

$$+1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}}e^{-\lambda s_k^{\epsilon/3}}\mathbb{E}_0^{(0,\,\Phi^\delta_{s_k^{\epsilon/3}})}\left\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}h(\Phi_s^\delta)\mathrm{d}\alpha^\delta_{on}(s)\right\}$$

$$\left.+1_{\{T_1<t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\}}e^{-\lambda T_1}\mathbb{E}_0^{(1,\,\Phi^\delta_{T_1})}\left\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s}h(\Phi_s^\delta)\mathrm{d}\alpha^\delta_{on}(s)\right\}\right]$$

Combining (a), (c) and (b), (d) we get,

$$= \mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s} h(\,\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)$$

$$+ 1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}} e^{-\lambda s_k^{\epsilon/3}} \mathbb{E}_0^{(0,\,\Phi^\delta_{s_k^{\epsilon/3}})}\Bigg\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_0^{U_k^{\epsilon/3}} e^{-\lambda s} h(\Phi_s^\delta)\cdot$$

$$\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} + 1_{\{T_1<t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\}} e^{-\lambda T_1} \mathbb{E}_0^{(1,\,\Phi_{T_1}^\delta)}\Bigg\{\int_0^{U_k^{\epsilon/3}} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s$$

$$+ \int_0^{U_k^{\epsilon/3}} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\}\Bigg]$$

$$\leq \mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \overbrace{\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s} h(\,\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)}^{=0}$$

$$+1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}} e^{-\lambda s_k^{\epsilon/3}} v_k(0,\,\Phi^\delta_{s_k^{\epsilon/3}}) + 1_{\{T_1<t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\}} e^{-\lambda T_1} v_k(1,\,\Phi_{T_1}^\delta)\Bigg] + \frac{2\epsilon}{3}$$

$$= \mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s$$

$$+1_{\{s_k^{\epsilon/3}<t_k^{\epsilon/3}\wedge T_1\}} e^{-\lambda s_k^{\epsilon/3}} v_k(0,\,\Phi^\delta_{s_k^{\epsilon/3}}) + 1_{\{T_1<t_k^{\epsilon/3}\wedge s_k^{\epsilon/3}\}} e^{-\lambda T_1} v_k(1,\,\Phi_{T_1}^\delta)\Bigg] + \frac{2\epsilon}{3}$$

$$= (Jv_k)(t_k^{\epsilon/3},\,s_k^{\epsilon/3},\,1,\,\phi) + \frac{2\epsilon}{3}$$

$$\leq (J_0 v_k)(1,\,\phi) + \frac{\epsilon}{3} + \frac{2\epsilon}{3}$$

$$= v_{k+1}(1,\,\phi) + \epsilon$$

We now prove for the case $\alpha = 0$. When $n = 1$,

$$\mathbb{E}_0^{(0,\,\phi)}\left[\int_0^{U_1^\epsilon} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_0^{U_1^\epsilon} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\right]$$

$$= \mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{t_0^\epsilon(0,\,\Phi_0^\delta)\wedge q_0^\epsilon(0,\,\Phi_0^\delta)} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + 1_{\{q_0^\epsilon(0,\,\Phi_0^\delta)<t_0^\epsilon(0,\,\Phi_0^\delta)\}} e^{-\lambda q_0^\epsilon(0,\,\Phi_0^\delta)}\cdot$$

$$h(\Phi^\delta_{q_0^\epsilon(0,\,\Phi_0^\delta)})\Bigg]$$

$$= (Jv_0)(t_0^\epsilon(0,\,\Phi_0^\delta),\,q_0^\epsilon(0,\,\Phi_0^\delta),\,0,\,\phi)$$

$$\leq (J_0 v_0)(0,\, \phi) + \epsilon$$
$$= v_1(0,\, \phi) + \epsilon$$

Assuming it holds for $n = k$, we need to prove for $n = k+1$. We have $U^{\epsilon}_{k+1} \wedge q^{\epsilon/2}_k(0,\, \Phi^\delta_0) = t^{\epsilon/2}_k(0,\, \Phi^\delta_0) \wedge q^{\epsilon/2}_k(0,\, \Phi^\delta_0)$ a.s. .

$$\begin{aligned}
&\mathbb{E}^{(0,\,\phi)}_0\left[\int_0^{U^\epsilon_{k+1}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s + \int_0^{U^\epsilon_{k+1}} e^{-\lambda s} h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)\right] \\
&= \mathbb{E}^{(0,\,\phi)}_0\Bigg[\int_0^{U^\epsilon_{k+1}\wedge q^{\epsilon/2}_k} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s + 1_{\{q^{\epsilon/2}_k < U_{k+1}\}}\int_{q^{\epsilon/2}_k}^{U^\epsilon_{k+1}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s \\
&+ \int_0^{U^\epsilon_{k+1}\wedge q^{\epsilon/2}_k} e^{-\lambda s} h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s) + 1_{\{q^{\epsilon/2}_k < U^\epsilon_{k+1}\}}\int_{q^{\epsilon/2}_k}^{U^\epsilon_{k+1}} e^{-\lambda s} h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)\Bigg] \\
&= \mathbb{E}^{(0,\,\phi)}_0\Bigg[\int_0^{t^{\epsilon/2}_k\wedge q^{\epsilon/2}_k} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s + 1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s \\
&+ \int_0^{t^{\epsilon/2}_k\wedge q^{\epsilon/2}_k} e^{-\lambda s} h(\,\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s) + 1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)\Bigg] \\
&= \mathbb{E}^{(0,\,\phi)}_0\Bigg[\int_0^{t^{\epsilon/2}_k\wedge q^{\epsilon/2}_k} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s + \underbrace{1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s}_{(a)} \\
&+ 1_{\{t^{\epsilon/2}_k > q^{\epsilon/2}_k\}} e^{-\lambda q^{\epsilon/2}_k} h(\Phi^\delta_{q^{\epsilon/2}_k}) + \underbrace{1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)}_{(b)}\Bigg]
\end{aligned} \tag{B.2}$$

Owing to the strong Markov property of X^δ_s we can simplify (a) as follows,

$$\begin{aligned}
&\mathbb{E}^{(0,\,\phi)}_0\left[1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s\right] \\
&= \mathbb{E}^{(0,\,\phi)}_0\left[\mathbb{E}^{(0,\,\phi)}_0\left\{1_{\{q^{\epsilon/2}_k < t^{\epsilon/2}_k\}}\int_{q^{\epsilon/2}_k}^{q^{\epsilon/2}_k + U^{\epsilon/2}_k\circ\theta_{q^{\epsilon/2}_k}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s)\mathrm{d}s\,\middle|\,\mathcal{F}^\delta_{q^{\epsilon/2}_k}\right\}\right]
\end{aligned}$$

$$= \mathbb{E}_0^{(0,\,\phi)}\left[1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}\mathbb{E}_0^{(0,\,\phi)}\left\{\int_0^{U_k^{\epsilon/2}\circ\theta_{q_k^{\epsilon/2}}} e^{-\lambda(s+q_k^{\epsilon/2})}g(\alpha^\delta_{s+q_k^{\epsilon/2}},\,\Phi^\delta_{s+q_k^{\epsilon/2}})\mathrm{d}s\middle|\mathcal{F}^\delta_{q_k^{\epsilon/2}}\right\}\right]$$

$$= \mathbb{E}_0^{(0,\,\phi)}\left[1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\mathbb{E}_0^{(0,\,\phi)}\left\{\left(\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}g(\alpha^\delta_s,\,\Phi^\delta_s)\right)\circ\theta_{q_k^{\epsilon/2}}\mathrm{d}s\middle|\mathcal{F}^\delta_{q_k^{\epsilon/2}}\right\}\right]$$

$$= \mathbb{E}_0^{(0,\,\phi)}\left[1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\mathbb{E}_0^{(1,\,\Phi^\delta_{q_k^{\epsilon/2}})}\left\{\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s\right\}\right]$$

Using the above approach we could also simplify (b). (B.2) becomes,

$$= \mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/2}\wedge q_k^{\epsilon/2}} e^{-\lambda s}g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s + 1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\mathbb{E}_0^{(1,\,\Phi^\delta_{q_k^{\epsilon/2}})}\Bigg\{\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}\cdot$$
$$g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s\Bigg\}$$
$$+ 1_{\{t_k^{\epsilon/2}>q_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}h(\Phi^\delta_{q_k^{\epsilon/2}}) + 1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\mathbb{E}_0^{(1,\,\Phi^\delta_{q_k^{\epsilon/2}})}\Bigg\{\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}\cdot$$
$$h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)\Bigg\}\Bigg]$$
$$= \mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/2}\wedge q_k^{\epsilon/2}} e^{-\lambda s}g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s + 1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\Bigg\{\mathbb{E}_0^{(1,\,\Phi^\delta_{q_k^{\epsilon/2}})}\Bigg\{\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}\cdot$$
$$g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s\Bigg\} + h(\Phi^\delta_{q_k^{\epsilon/2}}) + \mathbb{E}_0^{(1,\,\Phi^\delta_{q_k^{\epsilon/2}})}\Bigg\{\int_0^{U_k^{\epsilon/2}} e^{-\lambda s}h(\Phi^\delta_s)\mathrm{d}\alpha^\delta_{on}(s)\Bigg\}\Bigg\}\Bigg]$$
$$\leq \mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{t_k^{\epsilon/2}\wedge q_k^{\epsilon/2}} e^{-\lambda s}g(\alpha^\delta_s,\,\Phi^\delta_s)\mathrm{d}s + 1_{\{q_k^{\epsilon/2}<t_k^{\epsilon/2}\}}e^{-\lambda q_k^{\epsilon/2}}\Bigg\{h(\Phi^\delta_{q_k^{\epsilon/2}})$$
$$+ v_k(1,\,\Phi^\delta_{q_k^{\epsilon/2}})\Bigg\}\Bigg] + \epsilon/2$$
$$= (Jv_k)(t_k^{\epsilon/2},\,q_k^{\epsilon/2},\,0,\,\phi) + \epsilon/2$$
$$\leq (J_0v_k)(0,\,\phi) + \epsilon/2 + \epsilon/2$$
$$= v_{k+1}(0,\,\phi) + \epsilon$$

In order to prove the opposite inequality i.e., $V_n(\alpha,\,\phi) \geq v_n(\alpha,\,\phi),\ n \in \mathbb{N}$ we proceed by defining some notations and use induction arguments to establish the inequality. $Y_n(\tau,\,\delta)$ = total risk of $(\tau,\,\delta)$ if the problem is automatically terminated at the n^{th} non terminating event.

$$Y_n(\tau,\,\delta) = \int_0^{\tau\wedge\rho_n} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_0^{\tau\wedge\rho_n} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)$$

We would like to show,

$$\mathbb{E}_0^{(1,\,\phi)}\left[Y_n(\tau,\,\delta)\right] \geq v_n(1,\,\phi), \quad (\tau,\,\delta)\in\mathcal{M}. \tag{B.3}$$

Alternatively we could prove the following inequality for $k = 1,\,\cdots,\,n+1$,

$$\mathbb{E}_0^{(1,\,\phi)}\left[Y_n(\tau,\,\delta)\right] \geq \mathbb{E}_0^{(1,\,\phi)}\left[Y_{n-k+1}(\tau,\,\delta) + NT_{k-1}\right] =: RHS_{k-1} \tag{B.4}$$

where,

$$\begin{aligned} NT_{k-1} &:= 1_{\{\tau\geq\rho_{n-k+1}\}} e^{-\lambda\rho_{n-k+1}} v_{k-1}(\alpha_{\rho_{n-k+1}}^\delta,\,\Phi_{\rho_{n-k+1}}^\delta) \\ \beta^{n-k+1=2m} &:= \left\{\sigma_1,\,\tau_2,\,\sigma_2,\,\cdots,\,\tau_m,\,\sigma_m,\,T_2,\,T_4,\,\cdots,\,T_{2m}\right\}. \end{aligned} \tag{B.5}$$

As defined earlier ρ_{n-k+1} is the last non-terminating event that occurs in a particular realization of the problem when we are allowed to wait until atmost the $(n-k+1)^{\text{st}}$ non-terminating event. Let β^{n-k+1} be the set of all possible non-terminating events that could occur at the $(n-k+1)^{\text{st}}$ stage if we are to wait until the $(n-k+1)^{\text{st}}$ non-terminating event. We consider the cases when $(n-k+1)$ is odd and even separately. Let us also define the following function,

$$\Lambda^{n-k}(\rho_{n-k}) = \left\{\begin{array}{ll} \tau_{i+1}, & \text{if } \rho_{n-k} = \sigma_i,\ i = 1,\cdots,\,m \\ \sigma_i \wedge T_{(n-k)-(2i-3)}, & \text{if } \rho_{n-k} = \tau_i,\ i = 2,\,\cdots,\,m \\ T_{i+1}\wedge\sigma_{\frac{n-k-i}{2}}, & \text{if } \rho_{n-k} = T_i,\ i = 1,\,3,\,\cdots,\,2m-1 \end{array}\right\} \tag{B.6}$$

$\big\{$Note in the above equation: $n-k = 2m-1.\big\}$

The function $\Lambda^x(\cdot)$ maps the current non-terminating event to the immediate next one when our problem runs at most until the x^{th} non-terminating event. Thus we could also see that $\Lambda^{n-k}(\rho_{n-k}) = \rho_{n-k+1}$. In (B.4), the base case when $k = 1$ holds as an equality. We assume (B.4) holds for $k-1$ and would now like to show it also holds for k. We achieve our objective (B.3) when $k = n+1$. Using the above definitions we could rewrite RHS_{k-1} as follows,

$$
\begin{aligned}
& RHS_{k-1} \\
& = \mathbb{E}_0^{(1,\,\phi)}\left[Y_{n-k+1}(\tau,\,\delta) + NT_{k-1}\right] \\
& = \mathbb{E}_0^{(1,\,\phi)}\Bigg[Y_{n-k}(\tau,\,\delta) + \Bigg(\sum_{i=1}^{m} 1_{\{\rho_{n-k}=\tau_i\}} + \sum_{i=1}^{2m-1} 1_{\{\rho_{n-k}=T_i\}} + \sum_{i=1}^{m} 1_{\{\rho_{n-k}=\sigma_i\}}\Bigg)\cdot \\
& 1_{\{\tau\geq\rho_{n-k}\}}\Bigg\{\int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s}\cdot g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} \\
& + \Bigg(\sum_{i=1}^{m} 1_{\{\rho_{n-k}=\tau_i\}} \\
& + \sum_{i=1}^{2m-1} 1_{\{\rho_{n-k}=T_i\}} + \sum_{i=1}^{m} 1_{\{\rho_{n-k}=\sigma_i\}}\Bigg) NT_{k-1}\Bigg] \\
& = \mathbb{E}_0^{(1,\,\phi)}\Bigg[Y_{n-k}(\tau,\,\delta) \\
& + \sum_{i=1}^{m} 1_{\{\rho_{n-k}=\tau_i\}}\Bigg(1_{\{\tau\geq\rho_{n-k}\}}\Bigg\{\int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s}\cdot \\
& h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} + NT_{k-1}\Bigg) \\
& + \sum_{i=1}^{2m-1} 1_{\{\rho_{n-k}=T_i\}}\Bigg(1_{\{\tau\geq\rho_{n-k}\}}\Bigg\{\int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s}\cdot \\
& h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} + NT_{k-1}\Bigg) \\
& + \sum_{i=1}^{m} 1_{\{\rho_{n-k}=\sigma_i\}}\Bigg(1_{\{\tau\geq\rho_{n-k}\}}\Bigg\{\int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} g(\alpha_s^\delta,\,\Phi_s^\delta)\mathrm{d}s + \int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s}\cdot \\
& h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} + NT_{k-1}\Bigg)\Bigg]
\end{aligned}
$$

$$\left\{\begin{array}{l}\text{In the last equation we denote the first, second and the third} \\ \text{summation by r.v.'s } A,\, B \text{ and } C\text{, respectively.}\end{array}\right\}$$

We would show that $\mathbb{E}_0^{(1,\,\phi)}[A+B+C] \geq \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{\tau\geq\rho_{n-k}\}}e^{-\lambda\rho_{n-k}}v_k(\alpha_{\rho_{n-k}}^\delta,\,\Phi_{\rho_{n-k}}^\delta)\right]$.

Case (i). On the event $\{\rho_{n-k} = \tau_i\}$.

$$
\begin{aligned}
&\mathbb{E}_0^{(1,\phi)}[A] \\
&= \mathbb{E}_0^{(1,\phi)}\Bigg[\sum_{i=1}^{m} 1_{\{\rho_{n-k}=\tau_i\}}\Bigg(1_{\{\tau\geq\rho_{n-k}\}}\Bigg\{\int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s \\
&+ \int_{\rho_{n-k}}^{\tau\wedge\rho_{n-k+1}} e^{-\lambda s} h(\Phi_s^\delta)\mathrm{d}\alpha_{on}^\delta(s)\Bigg\} + NT_{k-1}\Bigg)\Bigg] \\
&= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\phi)}\Bigg[1_{\{\tau_i<\sigma_i\wedge T_{n-k-(2i-3)}\}}\Bigg(1_{\{\tau\geq\tau_i\}}\Bigg\{\int_{\tau_i}^{\tau\wedge\Lambda^{n-k}(\tau_i)} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s\Bigg\} \\
&+ NT_{k-1}\Bigg)\Bigg] \\
&= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\phi)}\Bigg[1_{\{\tau_i<\sigma_i\wedge T_{n-k-(2i-3)}\}}\Bigg(1_{\{\tau\geq\tau_i\}}\Bigg\{\int_{\tau_i}^{\tau\wedge\sigma_i\wedge T_{(n-k)-(2i-3)}} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s\Bigg\} \\
&+ NT_{k-1}\Bigg)\Bigg] \qquad \text{(B.7)}
\end{aligned}
$$

The reason we do not have a $1_{\{\tau_i<\tau\}}e^{-\lambda\tau_i}h(\Phi_{\tau_i}^\delta)$ term is because the definition of $Y_{n-k}(\tau,\, \delta)$ includes all but the cost to go. Using (B.5) and $\rho_{n-k+1} = \Lambda^{n-k}(\rho_{n-k}) = \Lambda^{n-k}(\tau_i) = \sigma_i \wedge T_{(n-k)-(2i-3)}$ we can write out NT_{k-1} as,

$$
\begin{aligned}
&NT_{k-1} \\
&= 1_{\{\tau\geq\rho_{n-k+1}\}} e^{-\lambda\rho_{n-k+1}} v_{k-1}(\alpha_{\rho_{n-k+1}}^\delta,\, \Phi_{\rho_{n-k+1}}^\delta) \\
&= 1_{\{\sigma_i\leq\tau\wedge T_{(n-k)-(2i-3)}\}} e^{-\lambda\sigma_i} v_{k-1}(0,\, \Phi_{\sigma_i}^\delta) + 1_{\{T_{(n-k)-(2i-3)}\leq\tau\wedge\sigma_i\}} e^{-\lambda T_{(n-k)-(2i-3)}}. \\
&v_{k-1}(1,\, \Phi_{T_{(n-k)-(2i-3)}}^\delta)
\end{aligned}
$$

Putting it back in (B.7) we get,

$$
\begin{aligned}
&= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\phi)}\Bigg[1_{\{\tau_i<\sigma_i\wedge T_{n-k-(2i-3)}\}}\Bigg(1_{\{\tau\geq\tau_i\}}\Bigg\{\int_{\tau_i}^{\tau\wedge\sigma_i\wedge T_{(n-k)-(2i-3)}} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\mathrm{d}s\Bigg\} \\
&+ 1_{\{\sigma_i\leq\tau\wedge T_{(n-k)-(2i-3)}\}} e^{-\lambda\sigma_i} v_{k-1}(0,\, \Phi_{\sigma_i}^\delta) + 1_{\{T_{(n-k)-(2i-3)}\leq\tau\wedge\sigma_i\}} e^{-\lambda T_{(n-k)-(2i-3)}}. \\
&v_{k-1}\left(1,\, \Phi_{T_{(n-k)-(2i-3)}}^\delta\right)\Bigg)\Bigg]
\end{aligned}
$$

Also, $1_{\{\sigma_i \le \tau \wedge T_{(n-k)-(2i-3)}\}} = 1_{\{\tau \ge \tau_i\}} \cdot 1_{\{\sigma_i \le \tau \wedge T_{(n-k)-(2i-3)}\}}$ and $1_{\{T_{(n-k)-(2i-3)} \le \tau \wedge \sigma_i\}} = 1_{\{\tau \ge \tau_i\}} \cdot 1_{\{T_{(n-k)-(2i-3)} \le \tau \wedge \sigma_i\}}$ since $\tau_i < T_{(n-k)-(2i-3)} \wedge \sigma_i$ which follows from (B.6). Therefore we can factor out $1_{\{\tau_i \le \tau\}}$ from last three terms as follows,

$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} \Bigg\{ \int_{\tau_i}^{\tau \wedge \sigma_i \wedge T_{(n-k)-(2i-3)}} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta) \mathrm{d}s$$
$$+ 1_{\{\sigma_i \le \tau \wedge T_{(n-k)-(2i-3)}\}} e^{-\lambda \sigma_i} v_{k-1}(0,\, \Phi_{\sigma_i}^\delta) + 1_{\{T_{(n-k)-(2i-3)} \le \tau \wedge \sigma_i\}} e^{-\lambda T_{(n-k)-(2i-3)}}$$
$$v_{k-1}(1,\, \Phi_{T_{(n-k)-(2i-3)}}^\delta) \Bigg\} \Bigg] \tag{B.8}$$

By Lemma B.1.1, there are $\mathcal{F}_{\tau_i}^\delta$ -measurable random variables x, y such that $\tau \wedge \sigma_i \wedge T_{(n-k)-(2i-3)} = (\tau_i + x) \wedge (\tau_i + y) \wedge T_{(n-k)-(2i-3)}$ $\mathbb{P}_0-$ a.s. on the event $\{\tau \ge \tau_i\}$. Also note that $T_{(n-k)-(2i-3)} - \tau_i \stackrel{d}{=} T_1$. Making these substitutions into (B.8) gives us,

$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[\mathbb{E}_0^{(1,\,\phi)} \Bigg\{ 1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} \Bigg(\int_{\tau_i}^{(\tau_i + x) \wedge (\tau_i + y) \wedge T_{(n-k)-(2i-3)}} e^{-\lambda s} \cdot$$
$$g(\alpha_s^\delta,\, \Phi_s^\delta) \mathrm{d}s + 1_{\{(\tau_i + y) \le (\tau_i + x) \wedge T_{(n-k)-(2i-3)}\}} e^{-\lambda(\tau_i + y)} v_{k-1}(0,\, \Phi_{\sigma_i}^\delta)$$
$$+ 1_{\{T_{(n-k)-(2i-3)} \le (\tau_i + x) \wedge (\tau_i + y)\}} e^{-\lambda T_{(n-k)-(2i-3)}} v_{k-1}(1,\, \Phi_{T_{(n-k)-(2i-3)}}^\delta) \Bigg) \Bigg| \mathcal{F}_{\tau_i}^\delta \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} \mathbb{E}_0^{(1,\,\phi)} \Bigg\{ \int_{\tau_i}^{(\tau_i + x) \wedge (\tau_i + y) \wedge T_{(n-k)-(2i-3)}} e^{-\lambda s} \cdot$$
$$g(\alpha_s^\delta,\, \Phi_s^\delta) \mathrm{d}s + 1_{\{y \le x \wedge (T_{(n-k)-(2i-3)} - \tau_i)\}} e^{-\lambda(\tau_i + y)} v_{k-1}(0,\, \Phi_{\sigma_i}^\delta)$$
$$+ 1_{\{(T_{(n-k)-(2i-3)} - \tau_i) \le x \wedge y\}} e^{-\lambda T_{(n-k)-(2i-3)}} v_{k-1}(1,\, \Phi_{T_{(n-k)-(2i-3)}}^\delta) \Bigg| \mathcal{F}_{\tau_i}^\delta \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} e^{-\lambda \tau_i} \mathbb{E}_0^{(1,\,\Phi_{\tau_i}^\delta)} \Bigg\{ \int_{0}^{x \wedge y \wedge (T_{(n-k)-(2i-3)} - \tau_i)} e^{-\lambda s} \cdot$$
$$g(\alpha_s^\delta,\, \Phi_s^\delta) \mathrm{d}s + 1_{\{y \le x \wedge (T_{(n-k)-(2i-3)} - \tau_i)\}} e^{-\lambda y} v_{k-1}(0,\, \Phi_y^\delta)$$
$$+ 1_{\{(T_{(n-k)-(2i-3)} - \tau_i) \le x \wedge y\}} e^{-\lambda (T_{(n-k)-(2i-3)} - \tau_i)} v_{k-1}(1,\, \Phi_{(T_{(n-k)-(2i-3)} - \tau_i)}^\delta) \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Big[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} e^{-\lambda \tau_i} \mathbb{E}_0^{(1,\,\Phi_{\tau_i}^\delta)} \Big\{ \int_{0}^{x \wedge y \wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta) \mathrm{d}s$$

$$+ 1_{\{y \le x \wedge T_1\}} e^{-\lambda y} v_{k-1}(0,\, \Phi^{\delta}_{y}) + 1_{\{T_1 \le x \wedge y\}} e^{-\lambda T_1} v_{k-1}(1,\, \Phi^{\delta}_{T_1}) \Big\} \Big]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Big[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} e^{-\lambda \tau_i} (J v_{k-1})(x,\, y,\, 1,\, \Phi^{\delta}_{\tau_i}) \Big]$$
$$\ge \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Big[1_{\{\tau_i < \sigma_i \wedge T_{n-k-(2i-3)}\}} \cdot 1_{\{\tau \ge \tau_i\}} e^{-\lambda \tau_i} v_k(1,\, \Phi^{\delta}_{\tau_i}) \Big]$$
$$= \mathbb{E}_0^{(1,\,\phi)} \left[\sum_{i=1}^{m} 1_{\{\rho_{n-k} = \tau_i\}} \cdot 1_{\{\tau \ge \rho_{n-k}\}} e^{-\lambda \rho_{n-k}} v_k(\alpha^{\delta}_{\rho_{n-k}},\, \Phi^{\delta}_{\rho_{n-k}}) \right].$$

$$\left\{ \begin{array}{l} \text{since we are working on the event} \\ \{\rho_{n-k} = \tau_i\} \text{ in this part.} \end{array} \right\}$$

Case (ii) & (iii). Case (ii) on the event $\{\rho_{n-k} = T_i\}$, calculations are very identical to case (i) since next possible non-terminating event is $T_{i+1} \wedge \sigma_{\frac{n-k-i}{2}}$ i.e., a new arrival or we stop the observation control. The case (iii) when $\rho_{n-k} = \sigma_i$ looks a little different and we present the proof for that here. Using the above notations we have,

$$\mathbb{E}_0^{(1,\,\phi)}[B]$$
$$= \mathbb{E}_0^{(1,\,\phi)} \left[\sum_{i=1}^{m} 1_{\{\rho_{n-k} = \sigma_i\}} \left(1_{\{\tau \ge \sigma_i\}} \left\{ \int_{\sigma_i}^{\tau \wedge \rho_{n-k+1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s) \mathrm{d}s + \int_{\sigma_i}^{\tau \wedge \rho_{n-k+1}} e^{-\lambda s} \cdot \right.\right.\right.$$
$$\left.\left.\left. h(\Phi^{\delta}_s) \mathrm{d}\alpha^{\delta}_{on}(s) \right\} + NT_{k-1} \right) \right]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \left[1_{\{\rho_{n-k} = \sigma_i\}} \left(1_{\{\tau \ge \sigma_i\}} \left\{ \int_{\sigma_i}^{\tau \wedge \tau_{i+1}} e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s) \mathrm{d}s + 1_{\{\tau_{i+1} < \tau\}} e^{-\lambda \tau_{i+1}} \cdot \right.\right.\right.$$
$$\left.\left.\left. h(\Phi^{\delta}_{\tau_{i+1}}) \right\} + NT_{k-1} \right) \right] \tag{B.9}$$

Using (B.5) and $\rho_{n-k+1} = \Lambda^{n-k}(\rho_{n-k}) = \Lambda^{n-k}(\sigma_i) = \tau_{i+1}$ we can write out NT_{k-1} as,

$$NT_{k-1} = 1_{\{\tau \ge \rho_{n-k+1}\}} e^{-\lambda \rho_{n-k+1}} v_{k-1}(\alpha^{\delta}_{\rho_{n-k+1}},\, \Phi^{\delta}_{\rho_{n-k+1}}) = 1_{\{\tau \ge \tau_{i+1}\}} e^{-\lambda \tau_{i+1}} \cdot$$

$$v_{k-1}(\alpha^\delta_{\tau_{i+1}},\, \Phi^\delta_{\tau_{i+1}})$$

Putting it back in (B.9) we get,

$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \Bigg(1_{\{\tau\geq\sigma_i\}} \Bigg\{ \int_{\sigma_i}^{\tau\wedge\tau_{i+1}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s) \mathrm{d}s$$
$$+ 1_{\{\tau_{i+1}<\tau\}} e^{-\lambda\tau_{i+1}} h(\Phi^\delta_{\tau_{i+1}}) \Bigg\} + 1_{\{\tau\geq\tau_{i+1}\}} e^{-\lambda\tau_{i+1}} v_{k-1}(\alpha^\delta_{\tau_{i+1}},\, \Phi^\delta_{\tau_{i+1}}) \Bigg) \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \cdot 1_{\{\tau\geq\sigma_i\}} \Bigg\{ \int_{\sigma_i}^{\tau\wedge\tau_{i+1}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s) \mathrm{d}s$$
$$+ 1_{\{\tau_{i+1}<\tau\}} e^{-\lambda\tau_{i+1}} h(\Phi^\delta_{\tau_{i+1}}) + 1_{\{\tau\geq\tau_{i+1}\}} e^{-\lambda\tau_{i+1}} v_{k-1}(\alpha^\delta_{\tau_{i+1}},\, \Phi^\delta_{\tau_{i+1}}) \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \cdot 1_{\{\tau\geq\sigma_i\}} \Bigg\{ \int_{\sigma_i}^{\tau\wedge\tau_{i+1}} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s) \mathrm{d}s$$
$$+ 1_{\{\tau_{i+1}<\tau\}} e^{-\lambda\tau_{i+1}} \left(h(\Phi^\delta_{\tau_{i+1}}) + v_{k-1}(\alpha^\delta_{\tau_{i+1}},\, \Phi^\delta_{\tau_{i+1}}) \right) \Bigg\} \Bigg] \tag{B.10}$$

By Lemma B.1.1, there are $\mathcal{F}^\delta_{\sigma_i}$ -measurable random variables x, y such that $\tau \wedge \tau_{i+1} = (\sigma_i + x) \wedge (\sigma_i + y)$ $\mathbb{P}_0-$ a.s. on the event $\{\tau \geq \sigma_i\}$. Making these substitutions into (B.10) gives us,

$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[\mathbb{E}_0^{(1,\,\phi)} \Bigg\{ 1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \cdot 1_{\{\tau\geq\sigma_i\}} \Bigg(\int_{\sigma_i}^{(\sigma_i+x)\wedge(\sigma_i+y)} e^{-\lambda s} \cdot$$
$$g(\alpha^\delta_s,\, \Phi^\delta_s) \mathrm{d}s + 1_{\{y<x\}} e^{-\lambda(\sigma_i+y)} \left(h(\Phi^\delta_{(\sigma_i+y)}) + v_{k-1}(1,\, \Phi^\delta_{(\sigma_i+y)}) \right) \Bigg) \Bigg| \mathcal{F}^\delta_{\sigma_i} \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \Bigg[1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \cdot 1_{\{\tau\geq\sigma_i\}} e^{-\lambda\sigma_i} \mathbb{E}_0^{(0,\,\Phi^\delta_{\sigma_i})} \Bigg\{ \int_0^{x\wedge y} e^{-\lambda s} g(\alpha^\delta_s,\, \Phi^\delta_s) \mathrm{d}s$$
$$+ 1_{\{y<x\}} e^{-\lambda y} \left(h(\Phi^\delta_y) + v_{k-1}(1,\, \Phi^\delta_y) \right) \Bigg| \mathcal{F}^\delta_{\sigma_i} \Bigg\} \Bigg]$$
$$= \sum_{i=1}^{m} \mathbb{E}_0^{(1,\,\phi)} \left[1_{\{\sigma_i<\tau_{i+1}\wedge T_{n-k-2(i-1)}\}} \cdot 1_{\{\tau\geq\sigma_i\}} e^{-\lambda\sigma_i} (Jv_{k-1})(x,\, y,\, 0,\, \Phi^\delta_{\sigma_i}) \right]$$
$$\geq \mathbb{E}_0^{(1,\,\phi)} \left[\sum_{i=1}^{m} 1_{\{\rho_{n-k}=\sigma_i\}} \cdot 1_{\{\tau\geq\rho_{n-k}\}} e^{-\lambda\rho_{n-k}} v_k(\alpha^\delta_{\rho_{n-k}},\, \Phi^\delta_{\rho_{n-k}}) \right]$$

Upon combining the inequalities obtained in cases (i), (ii) and (iii), we get $RHS_{k-1} \geq RHS_k$. This ends our proof for the case when $(n-k+1)$ is even and $\tau_1 = 0$. The proof for the case when $(n-k+1)$ is odd and $\tau_1 = 0$ follows similar arguments. However, β^{n-k+1} and $\Lambda^{n-k}(\rho_{n-k})$ have a different definition which we present here. $\beta^{n-k+1=2m-1} := \{\sigma_1, \tau_2, \sigma_2, \cdots, \tau_m, \sigma_m, T_1, T_3, \cdots, T_{2m-1}\}$.

$$\Lambda^{n-k}(\rho_{n-k}) = \left\{ \begin{array}{ll} \tau_{i+1}, & \text{if } \rho_{n-k} = \sigma_i,\ i = 1, \cdots, m-1 \\ \sigma_i \wedge T_{(n-k)-(2i-2)}, & \text{if } \rho_{n-k} = \tau_i,\ i = 2, \cdots, m \\ T_{i+1} \wedge \sigma_{\frac{n-k-i+2}{2}}, & \text{if } \rho_{n-k} = T_i,\ i = 2, 4, \cdots, 2(m-1) \end{array} \right\}$$

$\big\{$Note in the above equation: $n-k = 2(m-1)$.$\big\}$

□

B.2 Proof of Lemma 6.1.10

Proof. Let us first check the second part of the lemma as follows,

$$(J_0\, w_1)(\alpha, \phi)$$
$$= \begin{cases} \inf_{(\tau, \sigma_1) \in \mathcal{M}} \ \mathbb{E}_0 \Big[\int_0^{\tau \wedge \sigma_1 \wedge T_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \mathrm{d}s + 1_{\{\sigma_1 < \tau \wedge T_1\}} e^{-\lambda \sigma_1} w_1(0, \Phi_{\sigma_1}^\delta) \\ +1_{\{T_1 < \sigma_1 \wedge \tau\}} e^{-\lambda T_1} w_1(1, \Phi_{T_1}^\delta) \Big], \quad \text{when } \alpha = 1 \\ \\ \inf_{(\tau, \tau_1) \in \mathcal{M}} \ \mathbb{E}_0 \Bigg[\int_0^{\tau \wedge \tau_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \mathrm{d}s + 1_{\{\tau_1 < \tau\}} e^{-\lambda \tau_1} \Bigg(h(\Phi_{\tau_1}^\delta) \\ +w_1(1, \Phi_{\tau_1}^\delta) \Bigg) \Bigg], \ \text{when } \alpha = 0 \end{cases}$$

$$\leq \begin{cases} \inf_{(\tau,\,\sigma_1)\in\mathcal{M}} \; \mathbb{E}_0 \Big[\int_0^{\tau\wedge\sigma_1\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \mathrm{d}s + 1_{\{\sigma_1<\tau\wedge T_1\}} e^{-\lambda\sigma_1} w_2(0, \Phi_{\sigma_1}^\delta) \\ +1_{\{T_1<\sigma_1\wedge\tau\}} e^{-\lambda T_1} w_2(1, \Phi_{T_1}^\delta) \Big], \quad \text{when } \alpha = 1 \\ \\ \inf_{(\tau,\,\tau_1)\in\mathcal{M}} \; \mathbb{E}_0 \Bigg[\int_0^{\tau\wedge\tau_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \mathrm{d}s + 1_{\{\tau_1<\tau\}} e^{-\lambda\tau_1} \Big(h(\Phi_{\tau_1}^\delta) \\ +w_2(1, \Phi_{\tau_1}^\delta) \Big) \Bigg], \text{ when } \alpha = 0 \end{cases}$$

$$= (J_0\, w_2)(\alpha,\, \phi)$$

where we used (5.13), (5.14) and the domination principle of expectation. Moving to the first part of (i), we have from (5.13), (7.3), (7.5) and (7.6) the following,

$$\begin{aligned} &\mathbb{E}_0 \Bigg[\int_0^{t_0\wedge s_0\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta) \mathrm{d}s + 1_{\{s_0<\tau\wedge T_1\}} e^{-\lambda s_0} w(0, \Phi_{s_0}^\delta) + 1_{\{T_1<s_0\wedge t_0\}} e^{-\lambda T_1} \cdot \\ &w(1, \Phi_{T_1}^\delta) \Bigg] = \int_0^{t_0\wedge s_0} e^{-(\lambda+\lambda_0)s} g(1, x(s,\, \phi)) \mathrm{d}s \\ &+ 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} w(0, x(s_0,\, \phi)) + \int_0^{s_0\wedge t_0} e^{-(\lambda_0+\lambda)u} w\left(1, \frac{\lambda_1}{\lambda_0} x(u,\, \phi)\right) \lambda_0 \mathrm{d}u \\ &\geq \int_0^{t_0\wedge s_0} e^{-(\lambda+\lambda_0)s} \left(\frac{-\lambda}{c} - \lambda_0 \|w\| \right) \mathrm{d}s - 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} \|w\| \\ &= \left(\frac{-\lambda}{c} - \lambda_0 \|w\| \right) \left(\frac{1}{\lambda+\lambda_0} \right) \left(1 - e^{-(\lambda+\lambda_0)(t_0\wedge s_0)}\right) - 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} \|w\|, \end{aligned} \tag{B.11}$$

where the first inequality follows since $g(1,\phi) \geq \frac{b-\lambda}{c} \geq \frac{-\lambda}{c}$ from (5.11) and $\|w\| := \sup_{\substack{\alpha\in\{0,1\},\\ \phi\in\mathbb{R}_+}} |w(\alpha,\, \phi)| < \infty$ therefore, $-\|w\| \leq w(\cdot,\cdot) \leq \|w\|$. Taking the infimum over $t_0,\, s_0 \in \overline{\mathbb{R}}_+$ on both sides of the inequality (B.11), the inequality is still preserved and the L.H.S. is nothing but $(J_0\, w)\,(1,\phi)$ (from 7.1).

$$\begin{aligned} (J_0\, w)\,(1,\phi) \geq &\inf_{t_0,\, s_0\in\overline{\mathbb{R}}_+} -\left(\frac{\lambda}{c} + \lambda_0 \|w\| \right) \left(\frac{1}{\lambda+\lambda_0} \right) \left(1 - e^{-(\lambda+\lambda_0)(t_0\wedge s_0)}\right) \\ &- 1_{\{s_0<t_0\}} e^{-(\lambda+\lambda_0)s_0} \|w\| \\ &= \min\left\{\mathcal{H}_{S_1},\ \mathcal{H}_{S_2}\right\}. \end{aligned}$$

The infimum on the R.H.S. of the above inequality can be regarded as the minimum of the infimums of the same objective function over two disjoint sets $S_1 = \{(t_0,\, s_0) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ :\ t_0 \le s_0\}$ and $S_2 = \{(t_0,\, s_0) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ :\ t_0 > s_0\}$. $\mathcal{H}_{S_1}$ and $\mathcal{H}_{S_2}$ can be found as follows,

$$\begin{aligned}
&\mathcal{H}_{S_1} \\
&= \inf_{(t_0,\, s_0)\in S_1} \left\{ -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right)\left(1 - e^{-(\lambda+\lambda_0)t_0}\right)\right\} \\
&= -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) + \inf_{t_0\in\overline{\mathbb{R}}_+}\left\{\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) e^{-(\lambda+\lambda_0)t_0}\right\} \\
&= -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) \qquad \text{(B.12)}
\end{aligned}$$

$$\begin{aligned}
&\mathcal{H}_{S_2} \\
&= \inf_{(t_0,\, s_0)\in S_2} \left\{ -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right)\left(1 - e^{-(\lambda+\lambda_0)s_0}\right) - e^{-(\lambda+\lambda_0)s_0}\|w\|\right\} \\
&= -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) + \inf_{s_0\in\overline{\mathbb{R}}_+}\left\{ e^{-(\lambda+\lambda_0)s_0}\left[\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right)\right.\right. \\
&\left.\left. - \|w\|\right]\right\} \\
&= -\left(\frac{\lambda}{c} + \lambda_0 \|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) + \inf_{s_0\in\overline{\mathbb{R}}_+}\left\{ e^{-(\lambda+\lambda_0)s_0}\left[\left(\frac{1}{c} - \|w\|\right)\left(\frac{\lambda}{\lambda+\lambda_0}\right)\right]\right\} \\
&= \begin{cases} -\left(\frac{\lambda}{c} + \lambda_0\|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right), & \text{if } \frac{1}{c} \ge \|w\| \\ -\left(\frac{\lambda}{c} + \lambda_0\|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right) + \left(\frac{1}{c} - \|w\|\right)\left(\frac{\lambda}{\lambda+\lambda_0}\right), & \text{if } \frac{1}{c} < \|w\| \end{cases} \\
&= \begin{cases} -\left(\frac{\lambda}{c} + \lambda_0\|w\|\right)\left(\frac{1}{\lambda+\lambda_0}\right), & \text{if } \frac{1}{c} \ge \|w\| \\ -\|w\|, & \text{if } \frac{1}{c} < \|w\|. \end{cases} \qquad \text{(B.13)}
\end{aligned}$$

From (B.12) and (B.13), we can note that $\min\{\mathcal{H}_{S_1},\, \mathcal{H}_{S_2}\} = \mathcal{H}_{S_2}$. Hence,

$$(J_0\, w)(1,\, \phi) \ge \mathcal{H}_{S_2}, \qquad \text{(using (B.13))}$$

Similarly in order to compute the lower bound on $(J_0\, w)(0,\, \phi)$ we start with,

$$\int_0^{t_0\wedge r_0} e^{-\lambda s} g(\alpha_s^\delta, \Phi_s^\delta)\mathrm{d}s + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(h(\Phi_{r_0}^\delta) + w(1, \Phi_{r_0}^\delta)\right)$$

$$\geq \int_0^{t_0\wedge r_0} e^{-\lambda s}\left(\frac{-\lambda}{c}\right)\mathrm{d}s + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)$$

where the inequality follows since $g(0,\phi) \geq \frac{-\lambda}{c}$ from (5.11) and $h(\phi) \geq \frac{a}{c}$ from (5.12). Taking the infimum over $t_0,\, r_0 \in \overline{\mathbb{R}}_+$ on both sides of the inequality, the inequality is still preserved and the L.H.S. is nothing but $(J_0\, w)\,(0,\phi)$ (from 7.8).

$$\begin{aligned}
&(J_0\, w)\,(0,\phi)\\
&\geq \inf_{t_0,\, r_0 \in \overline{\mathbb{R}}_+} \left\{\int_0^{t_0\wedge r_0} e^{-\lambda s}\left(\frac{-\lambda}{c}\right)\mathrm{d}s + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)\right\}\\
&= \inf_{t_0,\, r_0 \in \overline{\mathbb{R}}_+} \left\{\left(\frac{-\lambda}{c}\right)\left(\frac{1}{\lambda}\right)\left(1 - e^{-\lambda(t_0\wedge r_0)}\right) + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)\right\}\\
&= \inf_{t_0,\, r_0 \in \overline{\mathbb{R}}_+} \left\{\left(\frac{-1}{c}\right)\left(1 - e^{-\lambda(t_0\wedge r_0)}\right) + 1_{\{r_0<t_0\}} e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)\right\}\\
&= \min\left\{\mathcal{G}_{R_1},\, \mathcal{G}_{R_2}\right\}.
\end{aligned}$$

where $R_1 = \left\{(t_0,\, r_0) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ :\ t_0 \leq\ r_0\right\}$ and $R_2 = \{(t_0,\, r_0) \in \overline{\mathbb{R}}_+ \times \overline{\mathbb{R}}_+ :\ t_0 >\ r_0\}$. $\mathcal{G}_{R_1}$ and $\mathcal{G}_{R_2}$ can be found as follows,

$$\mathcal{G}_{R_1} = \inf_{(t_0,\, r_0)\in R_1}\left\{\left(\frac{-1}{c}\right)\left(1 - e^{-\lambda t_0}\right)\right\} = \inf_{t_0 \in \overline{\mathbb{R}}_+}\left\{\left(\frac{-1}{c}\right)\left(1 - e^{-\lambda t_0}\right)\right\} = \frac{-1}{c}. \tag{B.14}$$

$$\begin{aligned}
\mathcal{G}_{R_2} &= \inf_{(t_0,\, r_0)\in R_2}\left\{\left(\frac{-1}{c}\right)\left(1 - e^{-\lambda r_0}\right) + e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)\right\}\\
&= \inf_{r_0 \in \overline{\mathbb{R}}_+}\left\{\left(\frac{-1}{c}\right)\left(1 - e^{-\lambda r_0}\right) + e^{-\lambda r_0}\left(\frac{a}{c} - \|w\|\right)\right\}\\
&= \frac{-1}{c} + \inf_{r_0 \in \overline{\mathbb{R}}_+}\left\{e^{-\lambda r_0}\left(\frac{1+a}{c} - \|w\|\right)\right\}\\
&= \begin{cases} \frac{-1}{c}, & \text{if } \frac{1+a}{c} \geq \|w\|\\ \frac{-1}{c} + \left(\frac{1+a}{c} - \|w\|\right), & \text{if } \frac{1+a}{c} < \|w\| \end{cases}
\end{aligned}$$

$$= \begin{cases} \frac{-1}{c}, & \text{if } \frac{1+a}{c} \geq \|w\| \\ \frac{a}{c} - \|w\|, & \text{if } \frac{1+a}{c} < \|w\|. \end{cases} \tag{B.15}$$

From (B.14) and (B.15), we can note that $\min\{\mathcal{G}_{R_1}, \mathcal{G}_{R_2}\} = \mathcal{G}_{R_2}$. Hence,

$$(J_0\, w)(0,\, \phi) \geq \mathcal{G}_{R_2}, \qquad (\text{using (B.15)}) .$$

We can finally conclude that (from (B.2), (B.2))

$$(J_0\, w)\,(\cdot,\, \cdot) \ \geq\ \min\{\mathcal{H}_{S_2},\, \mathcal{G}_{R_2}\} =\ K\ > -\infty$$

□

B.3 Proof of Proposition 8.2.1

Proof. We first consider the case when $\tau_1 = 0$. Let $u_1,\, u_2 \geq t > 0$. We have from earlier definitions that,

$$(Jw)(u_1,\, u_2,\, 1,\, \phi) = \mathbb{E}_0^{(1,\,\phi)}\Bigg[\int_0^{u_1\wedge u_2\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + 1_{\{u_2<u_1\wedge T_1\}} e^{-\lambda u_2}$$
$$w(0,\, \Phi_{u_2}^\delta) + 1_{\{T_1<u_1\wedge u_2\}} e^{-\lambda T_1} w(1,\, \Phi_{T_1}^\delta)\Bigg]$$

Let us consider two cases: one in which we have $u_1 \wedge u_2 = u_1$ and the other $u_1 \wedge u_2 = u_2$. It turns out that the result does not change with the two cases. So the display becomes,

$$\mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{u_1\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + 1_{\{T_1<u_1\}} e^{-\lambda T_1} w(1,\, \Phi_{T_1}^\delta)\right]$$
$$= \mathbb{E}_0^{(1,\,\phi)}\left[\left(\int_0^{t\wedge T_1} + 1_{\{T_1>t\}}\int_t^{u_1\wedge T_1}\right) e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + 1_{\{T_1<u_1\}} e^{-\lambda T_1} w(1,\, \Phi_{T_1}^\delta)\right]$$
$$= \mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{t\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + 1_{\{T_1<u_1\}} e^{-\lambda T_1} w(1,\, \Phi_{T_1}^\delta)\right]$$
$$+ \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}\int_t^{u_1\wedge T_1} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s\right]$$

On the event $\{T_1 > t\}$ we have $u_1 \wedge T_1 = (t + (u_1 - t)) \wedge (t + T_1 \circ \theta_t) = t + ((u_1 - t) \wedge T_1 \circ \theta_t)$. Therefore we have

$$
\begin{aligned}
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}} \int_t^{t+((u_1-t)\wedge T_1\circ\theta_t)} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s\right] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[\mathbb{E}_0^{(1,\,\phi)}\left\{1_{\{T_1>t\}} \int_t^{t+((u_1-t)\wedge T_1\circ\theta_t)} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s \middle| \mathcal{F}_t^\delta\right\}\right] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\phi)}\left\{\left(\int_t^{t+((u_1-t)\wedge T_1)} e^{-\lambda s} g(\alpha_{s-t}^\delta,\, \Phi_{s-t}^\delta)\,\mathrm{d}s\right)\circ\theta_t \middle| \mathcal{F}_t^\delta\right\}\right] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{\int_t^{t+((u_1-t)\wedge T_1)} e^{-\lambda s} g(\alpha_{s-t}^\delta,\, \Phi_{s-t}^\delta)\,\mathrm{d}s\right\}\right] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}e^{-\lambda t}\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{\int_0^{((u_1-t)\wedge T_1)} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s\right\}\right] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\Bigg[1_{\{T_1>t\}}e^{-\lambda t}\Bigg(\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\Bigg\{\int_0^{((u_1-t)\wedge T_1)} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s \\
&+ 1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\Bigg\} - \mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\right\}\Bigg)\Bigg] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\Bigg[1_{\{T_1>t\}}e^{-\lambda t}\Bigg((Jw)(u_1-t,\, u_2-t,\, 1,\, \Phi_t^\delta) \\
&- \mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\right\}\Bigg)\Bigg] \\
&= \cdots + \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}\underbrace{e^{-\lambda t}(Jw)(u_1-t,\, u_2-t,\, 1,\, x(t,\,\phi))}_{\text{is deterministic}}\right] \\
&- \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}e^{-\lambda t}\mathbb{E}_0^{(1,\,x(t,\,\phi))}\left\{1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\right\}\right] \\
&= \cdots + e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\, u_2-t,\, 1,\, x(t,\,\phi)) \\
&- \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}e^{-\lambda t}\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\right\}\right] \qquad \text{(B.16)}
\end{aligned}
$$

We could simplify the expectation inside the last term as follows

$$
1_{\{T_1>t\}}e^{-\lambda t}\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{1_{\{T_1<u_1-t\}}e^{-\lambda T_1}w(1,\, \Phi_{T_1}^\delta)\right\}
$$

$$
\begin{aligned}
&= 1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\Phi_t^\delta)}\left\{1_{\{T_1+t<u_1\}}e^{-\lambda(T_1+t)}w(1,\,\Phi_{T_1}^\delta)\right\}\\
&= 1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\phi)}\left\{\left(1_{\{T_1+t<u_1\}}e^{-\lambda(T_1+t)}w(1,\,\Phi_{T_1}^\delta)\right)\circ\theta_t\middle|\mathcal{F}_t^\delta\right\}\\
&= 1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\phi)}\left\{1_{\{T_1\circ\theta_t+t<u_1\}}e^{-\lambda(T_1\circ\theta_t+t)}w(1,\,\Phi_{T_1}^\delta\circ\theta_t)\middle|\mathcal{F}_t^\delta\right\}\\
&= 1_{\{T_1>t\}}\mathbb{E}_0^{(1,\,\phi)}\left\{1_{\{T_1<u_1\}}e^{-\lambda T_1}w(1,\,\Phi_{T_1}^\delta)\middle|\mathcal{F}_t^\delta\right\}
\end{aligned}
$$

because we have $T_1 = t + T_1\circ\theta_t$ and $\Phi_{T_1}^\delta\circ\theta_t = \Phi_{T_1}^\delta$ on $\{T_1 > t\}$. Putting it back into (B.16) we have,

$$
\begin{aligned}
&= \cdots + e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\,u_2-t,\,1,\,x(t,\,\phi))\\
&- \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}e^{-\lambda t}\mathbb{E}_0^{(1,\,\phi)}\left\{1_{\{T_1<u_1\}}e^{-\lambda T_1}w(1,\,\Phi_{T_1}^\delta)\middle|\mathcal{F}_t^\delta\right\}\right]\\
&= \cdots + e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\,u_2-t,\,1,\,x(t,\,\phi))\\
&- \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}1_{\{T_1<u_1\}}e^{-\lambda T_1}w(1,\,\Phi_{T_1}^\delta)\right]\\
&= \mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{t\wedge T_1} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\,\mathrm{d}s + 1_{\{T_1<u_1\}}e^{-\lambda T_1}w(1,\,\Phi_{T_1}^\delta)\right]\\
&+ e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\,u_2-t,\,1,\,x(t,\,\phi)) - \mathbb{E}_0^{(1,\,\phi)}\left[1_{\{T_1>t\}}1_{\{T_1<u_1\}}e^{-\lambda T_1}\cdot\right.\\
&\left.w(1,\,\Phi_{T_1}^\delta)\right]\\
&= \mathbb{E}_0^{(1,\,\phi)}\left[\int_0^{t\wedge T_1} e^{-\lambda s}g(\alpha_s^\delta,\,\Phi_s^\delta)\,\mathrm{d}s + 1_{\{T_1\le t\}}e^{-\lambda T_1}w(1,\,\Phi_{T_1}^\delta)\right]\\
&+ e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\,u_2-t,\,1,\,x(t,\,\phi))\\
&= (Jw)(t,\,t+s,\,1,\,\phi) + e^{-(\lambda+\lambda_0)t}(Jw)(u_1-t,\,u_2-t,\,1,\,x(t,\,\phi)).
\end{aligned}
$$

Taking infimum on both sides over u_1, u_2 such that $u_1\wedge u_2 > t$ gives us the required result and s is any positive constant. □

B.4 Proof of Proposition 8.2.6

Proof. Let $u_1,\, u_2 \geq t > 0$. We have,

$$(Jw)(u_1,\, u_2,\, 0,\, \phi) = \mathbb{E}_0^{(0,\,\phi)}\Bigg[\int_0^{u_1\wedge u_2} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + 1_{\{u_2<u_1\}} e^{-\lambda u_2}\Big(h(\Phi_{u_2}^\delta)$$

$$+w(1,\, \Phi_{u_2}^\delta)\Big)\Bigg]$$

Let us consider the case $u_1 \wedge u_2 = u_2 \geq t > 0$. The proof for the other case is identical.

$$= \mathbb{E}_0^{(0,\,\phi)}\left[\int_0^{u_2} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda u_2}\Big(h(\Phi_{u_2}^\delta) + w(1,\, \Phi_{u_2}^\delta)\Big)\right]$$

$$= \mathbb{E}_0^{(0,\,\phi)}\left[\left(\int_0^t + \int_t^{u_2}\right) e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda u_2}\Big(h(\Phi_{u_2}^\delta) + w(1,\, \Phi_{u_2}^\delta)\Big)\right]$$

$$= \mathbb{E}_0^{(0,\,\phi)}\left[\int_0^t e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s\right]$$

$$+ \mathbb{E}_0^{(0,\,\phi)}\left[\int_t^{u_2} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda u_2}\Big(h(\Phi_{u_2}^\delta) + w(1,\, \Phi_{u_2}^\delta)\Big)\right]$$

$$= \cdots + \mathbb{E}_0^{(0,\,\phi)}\Bigg[\mathbb{E}_0^{(0,\,\phi)}\Bigg\{\int_0^{u_2-t} e^{-\lambda(s+t)} g(\alpha_{s+t}^\delta,\, \Phi_{s+t}^\delta)\,\mathrm{d}s + e^{-\lambda u_2}\Big(h(\Phi_{u_2-t}^\delta\circ\theta_t)$$

$$+ w(1,\, \Phi_{u_2-t}^\delta\circ\theta_t)\Big)\Bigg|\mathcal{F}_t^\delta\Bigg\}\Bigg]$$

$$= \cdots + \mathbb{E}_0^{(0,\,\phi)}\Bigg[\mathbb{E}_0^{(0,\,\phi)}\Bigg\{\Bigg(\int_0^{u_2-t} e^{-\lambda(s+t)} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda u_2}\Big(h(\Phi_{u_2-t}^\delta)$$

$$+ w(1,\, \Phi_{u_2-t}^\delta)\Big)\Bigg)\circ\theta_t\Bigg|\mathcal{F}_t^\delta\Bigg\}\Bigg]$$

$$= \cdots + \mathbb{E}_0^{(0,\,\phi)}\Bigg[e^{-\lambda t}\mathbb{E}_0^{(0,\,\phi)}\Bigg\{\Bigg(\int_0^{u_2-t} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda(u_2-t)}\Big(h(\Phi_{u_2-t}^\delta)$$

$$+ w(1,\, \Phi_{u_2-t}^\delta)\Big)\Bigg)\circ\theta_t\Bigg|\mathcal{F}_t^\delta\Bigg\}\Bigg]$$

$$= \cdots + \mathbb{E}_0^{(0,\,\phi)}\Bigg[e^{-\lambda t}\mathbb{E}_0^{(0,\,\Phi_t^\delta)}\Bigg\{\int_0^{u_2-t} e^{-\lambda s} g(\alpha_s^\delta,\, \Phi_s^\delta)\,\mathrm{d}s + e^{-\lambda(u_2-t)}\Big(h(\Phi_{u_2-t}^\delta)$$

$$
\begin{aligned}
&+ w(1,\, \Phi^{\delta}_{u_2 - t})\Big)\Big\}\Big] \\
&= \mathbb{E}_0^{(0,\,\phi)}\left[\int_0^t e^{-\lambda s} g(\alpha^{\delta}_s,\, \Phi^{\delta}_s)\,\mathrm{d}s\right] + \mathbb{E}_0^{(0,\,\phi)}\left[e^{-\lambda t}(Jw)(u_1 - t,\, u_2 - t,\, 0,\, y(t,\, \phi))\right] \\
&= (Jw)(t,\, t + q,\, 0,\, \phi) + e^{-\lambda t}(Jw)(u_1 - t,\, u_2 - t,\, 0,\, y(t,\, \phi))
\end{aligned}
$$

and (8.1) follows by taking infimum on both sides of the last equation over $u_1 \wedge u_2 \geq t$ and q is any positive constant. □

Appendix C

Code

We present here the codes (written in MATLAB) for the case when $\widehat{a} < 0$. The case when a equal to or less than zero have similar routines. Following is a brief description of the routines used:

`constant_1.m`: File to which we supply the constant values λ, λ_0, λ_1, a, b, c.

`of().m`: Contains the operator $J_0(\cdot,\, \cdot)$(5.13 - 5.14), and calls the routines `J11.m` and `J12.m` in order to solve the deterministic optimization problem in (7.6); and calls the routine `J01.m` to solve the deterministic optimization problem in (7.13).

`J11.m`: This routine computes the function $w(0, x(s_0,\, \phi))$, needed for $J_0(1,\, \cdot)$, using linear interpolation.

`J12.m`: Computes $\int_0^{s_0 \wedge t_0} e^{-(\lambda_0+\lambda)u} \cdot w\left(1, \frac{\lambda_1}{\lambda_0} x(u,\, \phi)\right) \lambda_0 \mathrm{d}u$, needed for $J_0(1,\, \cdot)$, using linear interpolation and computes the integral using quadrature techniques (provided as `quad.m` in MATLAB).

`J01.m`: Computes $w\left(1, (1+\phi)e^{\lambda r_0} - 1\right)$ needed for $J_0(0,\, \cdot)$, using linear interpolation.

Note: the symbol Δ in the following code can be replaced with any variable name.

```
function [Δ lambda lambda_0 c lambda_1 a b ahat phi_d ...
a_1 xi UL] = constant_1()
Δ = 0.5;

lambda = 1;
lambda_0 = 3;
c = 0.1;
lambda_1 = 2*lambda_0;
a = 0;
b = 0.01;

ahat = lambda - (lambda_1-lambda_0);
phi_d = -lambda/ahat;
a_1 = 1/(lambda+lambda_0)*(phi_d + b/c*(1+phi_d) - lambda/c);
if (lambda +lambda_0)/c > ((lambda +lambda_0)/c ...
    -phi_d)*(lambda_1/(lambda+lambda_0)) +phi_d
    xi = (lambda +lambda_0)/c;
else
    xi = ((lambda +lambda_0)/c ...
        -phi_d)*(lambda_1/(lambda+lambda_0)) +phi_d;
end
UL = round(xi)+1;
end
```

```
function [fval fval_0 k] = of()
[Δ lambda lambda_0 c lambda_1 a b ahat phi_d a_1 xi UL] = ...
    constant_1();
k = 1;
error =100;

options = optimset('LargeScale','off');

% ,'Algorithm','sqp'// ,'Algorithm','active-set' //
```

```
%,'Algorithm','trust-region-reflective'(default)// ... ,'Algorithm','interior-point'

% % computing v1_0 and v1_1.

problem1 = ... createOptimProblem('fmincon','objective',@objfun,'x0',0,...
'lb',0,'ub',3,'options',options);
ms1 = MultiStart;

problem2 = ... createOptimProblem('fmincon','objective',@objfun_0,'x0',0,...
'lb',0,'ub',3,'options',options);
ms2 = MultiStart;

problem3 = ... createOptimProblem('fmincon','objective',@objfun_1,'x0',0,...
'lb',0,'ub',3,'options',options);
ms3 = MultiStart;


for jj =0:0.5:UL
    [x,fval(fix(jj/Δ +1),k)] = run(ms1,problem1,3);
    [x,fval_0(fix(jj/Δ +1),k)] = run(ms2,problem2,3);
end

% % computing v2_0, v2_1 and the others.
k  = 2;
while error >0.01
    fun_phi = fval(:,k-1);
    fun_phi_0 = fval_0(:,k-1);
    for jj =0:0.5:UL
        [x,fval(fix(jj/Δ +1),k)] = run(ms1,problem1,3);
        [x,uu] = run(ms2,problem2,3);
        if a > 1/(1+jj)
            fval_0(fix(jj/Δ +1),k) = uu;
        else
            [x,uu_0] = run(ms3,problem3,3);
            val_0 = [uu uu_0];
            fval_0(fix(jj/Δ +1),k) = min(val_0);
```

```
        end
    end
    for pp = 0:0.5:UL
        Diff(fix(pp/∆ +1)) = abs(fval(fix(pp/∆ +1),k) - ...
            fval(fix(pp/∆ +1),k-1));
        Diff_0(fix(pp/∆ +1)) = abs(fval_0(fix(pp/∆ +1),k) - ...
            fval_0(fix(pp/∆ +1),k-1));
    end
    error_1 = max(Diff);
    error_0 = max(Diff_0);
    error = max([error_0 error_1])
    k = k+1;
end
save v1.dat fval -ascii;
save v2.dat fval_0 -ascii;
%-------------------------------------
    function f = objfun(x)
        phi = jj;
        if k>1
            f = a_1 *(1-exp(-(lambda +lambda_0)*x)) ...
                +(phi-phi_d)/lambda_1*(1+b/c)*(1- ...
                exp(-lambda_1*x))+ ...
                exp(-(lambda_0+lambda).*x)*J11(x,phi,...
            fun_phi_0) + lambda_0*J12(x,phi,fun_phi);
         return;
        else
            f = a_1 *(1-exp(-(lambda +lambda_0)*x)) + ...
                (phi-phi_d)/lambda_1 *(1+b/c) * (1- ...
                exp(-lambda_1*x));
            return;
        end
    end
%---------------------------------------------
    function f = objfun_0(x)
        phi0 = jj;
        f = (1+phi0)*x - (1/lambda+1/c)*(1-exp(-lambda*x));
    end
    function f = objfun_1(x)
        phi0 = jj;
```

```
        f = (1+phi0)*x - (1/lambda+1/c)*(1-exp(-lambda*x)) + ...
            a/c*(1+ phi0)+ exp(-lambda*x)*J01(x,phi0,fun_phi);
    end
end
```

```
function [f] = J12(x,y,fun_phi)
    % note in our current scheme, we have that ahat <0. Note: ...
        if it
    % becomes greater or equal to 0, we need to change the ...
        whole thing.
    [∆ lambda lambda_0 c lambda_1 a b ahat phi_d a_1 xi UL] = ...
        constant_1();
    phi_0 = y;
    f = 0;
    if x == 0
        return;
    end
    % case where phi_0 > phi_d
    if phi_0>phi_d % then the integral goes from 0 ->
        % first determine where lower boundary point falls.
        f =0;
        low_bound = UL;
        upp_bound = UL;
        for ii = 0:∆:UL
            if lambda_1/lambda_0*phi_0 < ii
                low_bound = ii-∆;
                break;
            end
        end
        % then determine where upper boundary point falls.
        for ii = 0:∆:UL
            if lambda_1/lambda_0*(phi_d + (phi_0 - ...
                phi_d)*exp(ahat*x)) < ii
                upp_bound = ii;
                break;
            end
        end
        if lambda_1/lambda_0*(phi_d + (phi_0 - ...
            phi_d)*exp(ahat*x)) ≥ UL
```

```
            return
        end
        if low_bound < upp_bound
            f = f + quad(@fun2, 0, x);
            return;
        end

        net = 0;

        for i_1 = low_bound:-Δ:upp_bound+Δ
            if (lambda_0/lambda_1*(i_1-Δ) -phi_d) ≠ 0
                net = net + quad(@fun1, ...
                    1/ahat*log((lambda_0/lambda_1*i_1 -phi_d ...
                    )/(phi_0-phi_d)),1/ahat*log((lambda_0/...
                lambda_1*(i_1-Δ) -phi_d )/(phi_0-phi_d)));
            else
%                       disp('BAD CASE1');
               net = net +  quad(@fun1, ...
                   1/ahat*log((lambda_0/lambda_1*i_1 -phi_d ...
                   )/(phi_0-phi_d)),x);
            end
        end

        if low_bound == upp_bound && ...
           (lambda_0/lambda_1*(upp_bound-Δ) -phi_d) == 0
            f = quad(@fun2, 0, x);
            return;
        else
%                   disp('BAD CASE');
            if (lambda_0/lambda_1*low_bound -phi_d ...
               )/(phi_0-phi_d) == 1
                vam_2 = 0;
            else
                if low_bound == UL
                    low_bound = low_bound - Δ;
                    vam_2 = quad(@fun2, 0, ...
                        1/ahat*log((lambda_0/lambda_1*...
                    low_bound -phi_d )/(phi_0-phi_d));
                else
```

```
                    vam_2 = quad(@fun2, 0, ...
                       1/ahat*log((lambda_0/lambda_1*...
                    low_bound -phi_d )/(phi_0-phi_d)));
                end
            end
        end

        if lambda_0/lambda_1*upp_bound -phi_d == 0
            vam_1 = 0;
        else
            vam_1 = quad(@fun3,1/ahat*log((lambda_0/lambda_1...
            *upp_bound -phi_d )/(phi_0-phi_d)), x);
        end

        f = net +  vam_2 + vam_1;
        return;
    end

    if phi_0 < phi_d
        upp_bound_1 = UL;
        low_bound_1 = UL;
        f =0;
        % first determine where lower boundary point falls.
        for ii = 0:Δ:UL
            if lambda_1/lambda_0*phi_0 < ii
                low_bound_1 = ii;
                break;
            end
        end
        if lambda_1/lambda_0*phi_0 ≥ UL
            return;
        end
        % then determine where upper boundary point falls.
        for ii = 0:Δ:UL
            if lambda_1/lambda_0*(phi_d + (phi_0 - ...
               phi_d)*exp(ahat*x)) < ii
                upp_bound_1 = ii-Δ;
                break;
            end
        end
```

```
        net = 0;
        if low_bound_1 > upp_bound_1
            f = net + quad(@fun5, 0, x);
            return;
        end

        for i_2 = low_bound_1:Δ:upp_bound_1-Δ
            if lambda_0/lambda_1*(i_2+Δ) -phi_d ≠ 0
                net = net + quad(@fun4, ...
                    1/ahat*log((lambda_0/lambda_1*i_2 -phi_d ...
                    )/(phi_0-phi_d)),1/ahat*log((lambda_0/...
                lambda_1*(i_2+Δ) -phi_d )/(phi_0-phi_d)));
            else
                net = net + quad(@fun4, ...
                    1/ahat*log((lambda_0/lambda_1*i_2 -phi_d ...
                    )/(phi_0-phi_d)),x);
            end
        end

        if low_bound_1 == upp_bound_1 && ...
            (lambda_0/lambda_1*low_bound_1 -phi_d ) == 0
            f = quad(@fun5, 0, x);
            return;
        else
            vam_2 = quad(@fun5, 0, ...
                1/ahat*log((lambda_0/lambda_1*low_bound_1 ...
                -phi_d )/(phi_0-phi_d)));
        end

        if upp_bound_1 == UL || ...
            (lambda_0/lambda_1*upp_bound_1 -phi_d) == 0
            vam_1 = 0;
        else
            vam_1 =  quad(@fun6,1/ahat*log((lambda_0/lambda_1*...
            upp_bound_1 -phi_d )/(phi_0-phi_d)), x);
        end
        f = net + vam_2 + vam_1;
        return;
    end
    % ------------------------->>>>>>>>>>>
```

```
    if phi_0 == phi_d %case when phi = phi_d, Careful.
        f = 0;
        for ii = 0:Δ:UL
            if lambda_1/lambda_0*phi_d < ii
                constant = fun_phi(1/Δ*ii + 1)  + ...
                    (lambda_1/lambda_0*phi_d - ...
                    ii)*(fun_phi(1/Δ*ii+1)-fun_phi(1/Δ*ii))*1/Δ;
                f = constant*1/(lambda + ...
                    lambda_0)*(1-exp(-(lambda +lambda_0)*x));
                break;
            end
        end
        return;
    end


function g = fun1(t)
    g =  exp(-(lambda_0+ lambda).*t).*(fun_phi(1/Δ.*i_1 +1) + ...
        (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- ...
        i_1).*(fun_phi(1/Δ.*i_1+1)-fun_phi(1/Δ.*i_1))/Δ);
end

function g = fun2(t)
    g =  exp(-(lambda_0+ lambda).*t).*(fun_phi(1/Δ.*low_bound ...
        +1) + (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- ...
        low_bound).*(fun_phi(1/Δ.*low_bound+2)-fun_phi(1/Δ....
    *low_bound+1))*1/Δ);
end
function g = fun3(t)
    g =  exp(-(lambda_0+ lambda).*t).*(fun_phi(1/Δ.*upp_bound ...
        +1) + (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- ...
        upp_bound).*(fun_phi(1/Δ.*upp_bound+1)-fun_phi(1/Δ....
    *upp_bound))*1/Δ);
end
function g = fun4(t)
```

```
    g =  exp(-(lambda_0+ lambda).*t).*(fun_phi(1/Δ.*i_2 +1) + ...
        (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- ...
        i_2).*(fun_phi(1/Δ.*i_2+2)-fun_phi(1/Δ.*i_2+1))*1/Δ);
end
function g = fun5(t)
    var2 = fun_phi(1/Δ*low_bound_1);
    var3 = fun_phi(1/Δ*low_bound_1+1);
    g =  exp(-(lambda_0+ ...
        lambda).*t).*(fun_phi(1/Δ.*low_bound_1 +1) + ...
        (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- low_bound_1).*(var3-var2)/Δ);
end
function g = fun6(t)
    g =  exp(-(lambda_0+ ...
        lambda).*t).*(fun_phi(1/Δ*upp_bound_1 +1) + ...
        (lambda_1/lambda_0.*(phi_d + (phi_0 - ...
        phi_d).*exp(ahat.*t))- ...
        upp_bound_1).*(fun_phi(1/Δ.*upp_bound_1+2)-fun_phi...
    (1/Δ.*upp_bound_1+1))*1/Δ);
end
end
```

```
function f = J11(x,y,fun_phi_0)
    [Δ lambda lambda_0 c lambda_1 a b ahat phi_d a_1 xi UL] = ...
        constant_1();
    phi_0 = y;
    f = 0;
    for ii = 0:Δ:UL
        if (phi_d + (phi_0-phi_d )*exp(ahat*x)) < ii
            f = f + fun_phi_0(1/Δ*ii) + ((phi_d + ...
                (phi_0-phi_d )*exp(ahat*x)) - ...
                ii+Δ)*(fun_phi_0(1/Δ*ii+1) - fun_phi_0(1/Δ*ii))/Δ;
            break;
        end
    end
end
```

```
function f = J01(x,y,fun_phi_00)
    [△ lambda lambda_0 c lambda_1 a b ahat phi_d a_1 xi UL] = ...
        constant_1();

    phi_00 = y;
    f = 0;
    for ii = 0:△:UL
        if (1+phi_00)*exp(lambda*x)-1 < ii
            if ii == 0
                f = f + fun_phi_00(1);
            return;
            else
                f = f + fun_phi_00(1/△*ii) + ...
                    ((1+phi_00)*exp(lambda*x)-1- ...
                    ii+△)*(fun_phi_00(1/△*ii+1) - ...
                    fun_phi_00(1/△*ii))/△;
                return;
            end
        end
    end
end
```

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