COHOMOLOGY OF SEMIDIRECT PRODUCTS

A THESIS

SUBMITTED TO THE DEPARTMENT OF MATHEMATICS

AND THE GRADUATE SCHOOL OF ENGINEERING AND SCIENCE

OF BILKENT UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

MASTER OF SCIENCE

By

Ayşegül Tekin

May, 2012

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Ergün Yalçın(Advisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Özgün Ünlü

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Mustafa Korkmaz

Approved for the Graduate School of Engineering and Science:

Prof. Dr. Levent Onural
Director of the Graduate School

ABSTRACT

COHOMOLOGY OF SEMIDIRECT PRODUCTS

Ayşegül Tekin
M.S. in Mathematics
Supervisor: Assoc. Prof. Dr. Ergün Yalçın
May, 2012

Let $\Gamma = L \rtimes G$ be a semidirect product where G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice. Adem-Ge-Pan-Petrosyan [8] stated a conjecture which says that the cohomology of Γ is given by the following isomorphism, $H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z}))$. However, Langer-Lück [10] and later Petrosyan-Putrycz [9] showed that there are some groups which do not satisfy this isomorphism. In this thesis we do some explicit calculations related to this conjecture. Through the thesis we consider the case where $\dim L = 4$. In fact, dimension 4 is the lowest dimension for L where the semidirect group $\Gamma = L \rtimes G$ does not satisfy the conjecture. According to [9], in dimension 4 there are 44 non-isomorphic representations and only 2 of them do not satisfy the conjecture. We consider two of these 44 representations where one of them is counterexample for the conjecture of Adem-Ge-Pan-Petrosyan and the other one satisfies it. These results were already stated in [9]. In this thesis, we make detailed calculations for the cohomology of Γ for these representations by using the method of N. Petrosyan and B. Putrycz.

Keywords: Cohomology of semidirect products, Compatible action, Wall's Theorem.

ÖZET

YARI DİREKT ÇARPIMLARIN KOHOMOLOJİSİ

Ayşegül Tekin
Matematik, Yüksek Lisans
Tez Yöneticisi: Doç. Dr. Ergün Yalçın
Mayıs, 2012

G sonlu devirli bir group, L sonlu boyutlu bir $\mathbb{Z}G$ latisi, ve $\Gamma = L \rtimes G$ yarı direkt bir çarpım grubu olsun. Adem-Ge-Pan-Petrosyan [8] Γ'nın kohomoloji grubunun L'in kohomoloji grubunun katsayılarıyla hesaplanan G'nin kohomoloji grubu tarafından verildigini söyleyen bir sanı ortaya attı. Yani $H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z}))$ olduğunu iddia etti. Fakat, bu izomorfizmayı sağlamayan bazı grupların var olduğu daha sonra Langer-Lück [10] ve Petrosyan-Putrycz [9] tarafından gösterildi. Bu tezde biz L'nin 4 boyutlu olduğu durumu inceliyoruz. Aslında bu, yarı direkt çarpımlar için sanının doğru olmadığı en küçük boyut. Dört boyutlu durumda, 44 tane izomorfik olmayan grup temsili var [9] ve bunlardan sadece 2 tanesi sanıyı sağlamıyor. Biz bir tanesi Adem-Ge-Pan-Petrosyan'in sanısına karşıt örnek ve diğer bir tanesi onu sağlayan bir örnek olmak üzere G'nin iki farklı temsilini inceledik. Bu sonuçlar [9]'da verilmiştir. Bu tezde N. Petrosyan ve B. Putrycz'nin metodlarını kullanarak verilen temsiller için kohomoloji gruplarının detaylı hesaplamalarını yaptık.

Anahtar sözcükler: Yarı direkt çarpımların kohomolojisi, Uyumlu etki, Wall Teoremi.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Assoc. Prof. Dr. Ergün Yalçın for his excellence guidance, valuable suggestions, encouragement and patience. I am glad to have the chance to study with him.

I would also like to thank Asst. Prof. Dr. Özgün Ünlü and Prof. Mustafa Korkmaz for a careful reading of this thesis.

I would like to thank my husband Ali Osman Tekin for his encouragement, support, endless love and trust. My thanks also goes to my daughter, Zülal Tekin, for accepting that writing and mothering can go together. This thesis would never be possible without their countenance.

I would like to thank my sister, Betül Küçüköz, who never gave up trying to motivate me when I felt down. She is a great sister.

I am so grateful to have the chance to thank my parents, Sıtkı and Mukadder Küçüköz, who always unconditionally support me. I always knew that they would help to solve all kinds of problem that I had.

I would like to thank my friend Berrin Şentürk who helped me about Latex with all her patience.

Finally, I would like to thank all my friends in the department who increased my motivation, whenever I needed.

Contents

Chapter 1

Introduction

In this thesis we study cohomology of semidirect products. We consider the groups $\Gamma = L \rtimes G$ where G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice. Note that a group Γ of this type is called an n-dimensional crystallographic group of split type. Crystallographic groups are defined as the discrete subgroups of the group of isometries of $\mathbb{R}^n$ which act on $\mathbb{R}^n$ properly discontinuously and cocompactly [9]. The action of a group G on a topological space X is said to be cocompact if the quotient space X/G is a compact space or there is a compact subset K of X where under the action of G the image of K covers X. We say the action is proper if the map $G \times X \to X \times X$ sending (g, x) to (gx, x) is continuous and the inverse image of a compact subset is compact. To say that the action is properly discontinuous, we need G to be discrete.

A projective resolution for Γ can be described as a double complex. Let B_* and C_* be projective resolutions of $\mathbb{Z}$ over $\mathbb{Z}L$ and $\mathbb{Z}G$, respectively. If B_* admits a compatible G-action, then we can easily write a projective resolution for Γ. Otherwise, we first take B_* as follows

$$B_* : \cdots \longrightarrow B_n \xrightarrow{\partial'_n} B_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \longrightarrow B_1 \xrightarrow{\partial'_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Note that we have $\mathrm{Ind}_L^\Gamma B_* = B_* \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma$ is free over $\mathbb{Z}\Gamma$ and $\mathbb{Z} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma = \mathbb{Z}G$. Since taking tensor product of a complex with a projective module is an exact functor, $\mathrm{Ind}_L^\Gamma B_*$ with the induced differentials is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}G$. The induced complex $\mathrm{Ind}_L^\Gamma B_*$ is given by

$$\mathrm{Ind}_L^\Gamma B_* : \cdots \longrightarrow B_n \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \xrightarrow{\partial'_n \otimes \mathrm{id}} B_{n-1} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \longrightarrow \cdots \longrightarrow \mathbb{Z}G \longrightarrow 0.$$

Now we consider each module of $\mathrm{Ind}_L^\Gamma B_*$ with the trivial left G-action and define

$$A_{**} = \bigoplus\nolimits_{r,s} A_{r,s}$$

where

$$A_{r,s} := \mathrm{Ind}_L^\Gamma B_r \otimes_{\mathbb{Z}G} C_s.$$

To complete the construction of the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$, we need to make the double complex A_{**} a chain complex. In this step, we use Wall's Theorem [5] which makes A_{**} a chain complex by adding differentials d_n's. For the calculation of d_n's we define a contracting homotopy for a particular resolution of $\mathbb{Z}$ over $\mathbb{Z}L$. Then we let $d_n = -h(\Sigma_{i=1}^n d_i d_{n-i})$ for $n \geq 1$. If all the d_2 differentials become trivial on the cohomology groups calculated by using d_1's, then the cohomology of Γ is given by a cohomology of G with the coefficients in the cohomology of L. In this situation, the Lyndon-Hochschild-Serre spectral sequence associated to given $L \rtimes G$ collapses at E_2. For this case Adem-Ge-Pan-Petrosyan [8] made the following conjecture:

Conjecture 1.0.1. Suppose that G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice. Then for any $k \geq 0$ we have

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).$$

There are papers in the literature discussing the cases where the conjecture is satisfied. For $G = \mathbb{Z}_4$, the complete list of indecomposable representations given in [8]. In this paper, Adem-Ge-Pan-Petrosyan prove that for indecomposable modular representations $\rho_1, \rho_2, \rho_3, \rho_4$, there is a compatible action and as a consequence of this, the conjecture is satisfied. In addition, for ρ_5 and ρ_7 the conjecture is also satisfied. Although it is not known whether there is a compatible action for ρ_6, according to [8] Conjecture 1.0.1 also holds for ρ_6.

M. Langer and W. Lück [10] disprove the conjecture by giving a counterexample for $L = \mathbb{Z}^6$. Another article [9], written by N. Petrosyan and B. Putrycz, disprove Conjecture 1.0.1 by providing a complete list of counterexamples up to dimension 5. In this thesis, we consider the 4-dimensional case which is the lowest dimension of that type of semidirect groups for which the conjecture is not true. In [9], it is stated that in dimension 4, there are 44 non-isomorphic groups of the given type and the only 2 of them do not satisfy the conjecture. In [9], N. Petrosyan and B. Putrycz make the detailed calculations for one of these counterexamples and they stated that ρ_8 satisfies the conjecture where ρ_9 does not.

In this thesis we first give the basics of homological algebra. Then we describe group cohomology and discuss Wall's Theorem which is about constructing acyclic doubly graded complex. Then we discuss how we can construct a projective resolution for semidirect products. We give a practical way for the calculation of cohomology of the groups whose resolution admits a compatible action. After that we describe the method of calculation for Conjecture 1.0.1 due to Petrosyan and Putrycz [9]. Last we consider the group $\Gamma = L \rtimes (\mathbb{Z}/4)$ with two different actions given by the representations γ_2 and ρ_8 which are mentioned in [9]. As a result, we see that first representation is the counterexample for Conjecture 1.0.1 and second one satisfies it.

The thesis is organized as follows:

Chapter 2 is a preliminary section on the basics of homological algebra and group cohomology. In Chapter 3, we define the compatible action and consider two groups whose resolutions admits a compatible G-action. In Chapter 4, we make two calculations for the representations γ_2 and ρ_8.

Chapter 2

Homological Algebra and Group Cohomology

2.1 Preliminaries on Homological Algebra

This section contains some basic definitions and properties of homological algebra which are used in the following sections. For the definitions and results of this section, we follow [2], [5], and [6]. First, note that the sequence

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is said to be *exact* at B if Im $\alpha = \ker \beta$. Similarly, a sequence

$$\cdots \longrightarrow A_{n-1} \xrightarrow{\alpha_{n-1}} A_n \xrightarrow{\alpha_n} A_{n+1} \xrightarrow{\alpha_{n+1}} \cdots$$

is said to be an *exact sequence* if it is exact at A_n for every n. This definition gives the following.

Proposition 2.1.1. *Let A, B, and C be R-modules. Then,*

1. *The sequence* $0 \longrightarrow A \xrightarrow{\alpha} B$ *is exact at A if and only if α is injective.*
2. *The sequence* $B \xrightarrow{\beta} C \longrightarrow 0$ *is exact at C if and only if β is surjective.*

Example 2.1.2. Let $\theta : A \to B$ be any homomorphism. Then we can write an exact sequence

$$0 \longrightarrow \ker\theta \xrightarrow{i} A \xrightarrow{\theta} \mathrm{Im}\ \theta \longrightarrow 0$$

where i is the inclusion map.

Definition 2.1.3. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

and

$$0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow 0$$

be two short exact sequences of modules. We say a triple (f_0, f_1, f_2) is *a homomorphism of short exact sequences* if the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0. \end{array}$$

Definition 2.1.4. The short exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ of R-modules is said to be *split* if there exists an R-module homomorphism $s : C \to B$ where $\beta \circ s = \mathrm{id}_C$. In this case the extension is called by *a split extension of C by A*.

Proposition 2.1.5.

1. *Let A, B, and C be R-modules for some ring R. Then the short exact sequence of R-modules $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is split if and only if there is an R-module complement to $\alpha(A)$ in B. In this case, we can write $B = A \oplus C$.*

2. *Let A, B, and C be groups. Then the short exact sequence of groups $1 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 1$ is split if and only if there is a subgroup complement to $\alpha(A)$ in B. In this case, we can write $B = A \rtimes C$.*

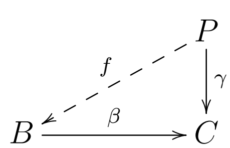
Proof of Proposition 2.1.5 is straightforward. We consider the first case. If we have $B \cong A \oplus C$, then we define s such that $s(c)$ is equal to preimage of c under the map β for all $c \in C$. Since β is surjective, s is well-defined and $\beta \circ s = \mathrm{id}_C$. Hence, the sequence is split. If the sequence is split, then there exists an R-module homomorphism s such that $\beta \circ s = \mathrm{id}_C$. Then we choose $C' = s(C) \subseteq B$, where C' is mapped isomorphically onto C by $\beta : \beta(C') \to C$. So $B = A \oplus C$. Note that a homomorphism s given in the Definition 2.1.4 is called a splitting homomorphism for the sequence.

One of the important concepts in homological algebra is the concept of the projective modules. Before giving the definition of a projective module, recall that $\mathrm{Hom}_R(D, -)$ is a left exact functor. This means that if $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence, then $0 \longrightarrow \mathrm{Hom}_R(D, A) \xrightarrow{\alpha'} \mathrm{Hom}_R(D, B) \xrightarrow{\beta'} \mathrm{Hom}_R(D, C)$ is exact. Now we define projective module.

Proposition 2.1.6. *Let P be an R-module. Then the following conditions are equivalent:*

1. *If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence of R-modules, then $0 \longrightarrow Hom_R(P, A) \xrightarrow{\alpha'} Hom_R(P, B) \xrightarrow{\beta'} Hom_R(P, C) \longrightarrow 0$ is also a short exact sequence.*

2. *Given a diagram*



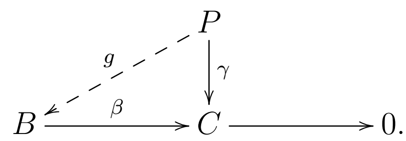
   with β is surjective, there exists a lifting $f : P \to B$ such that $\beta f = \gamma$.

3. *Every short exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0$ splits.*

4. *P is a direct summand of a free R-module.*

Proof. First we show that conditions (1) and (2) are equivalent. Now, assume that $\mathrm{Hom}_R(P, -)$ is an exact functor which means β' is surjective. That is for any

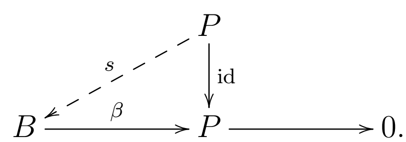
$\gamma \in \mathrm{Hom}_R(P, C)$ there exists $g \in \mathrm{Hom}_R(P, B)$ such that $\gamma = \beta'(g) = \beta g$. So for given maps β and γ, there is a lifting g where the following diagram commutes



Hence, (2) is true.

Conversely, if (2) is given, then for a given map $\gamma : P \to C$, there is a map $g : P \to B$ with $\beta g = \gamma$. This implies that for any $\gamma \in \mathrm{Hom}_R(P, C)$ there exists $g \in \mathrm{Hom}_R(P, B)$ with $\gamma = \beta g = \beta'(g)$. Then $\gamma \in \mathrm{Im}\,\beta'$ gives that β' is surjective. Therefore, $\mathrm{Hom}_R(P, -)$ is an exact functor.

Suppose that condition (2) is satisfied and $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} P \longrightarrow 0$ is a short exact sequence. Consider the identity map from P to P. By (2), this map lifts to a homomorphism s making the following diagram commute



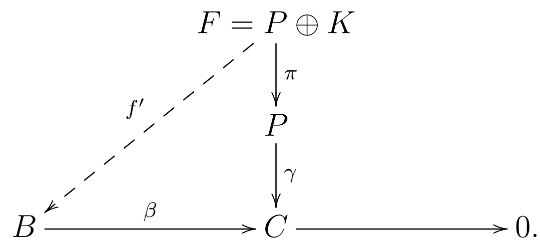
So $\beta \circ s = 1$ and s is the splitting homomorphism for the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} P \longrightarrow 0 .$$

This proves (3).

Note that every module P can be written as a quotient of a free module. For example, P is quotient of the free module on the set of elements in P. We denote this free module by F. Then there is always an exact sequence $0 \longrightarrow \ker\beta \xrightarrow{i} F \xrightarrow{\beta} P \longrightarrow 0$. Condition (3) implies that this sequence splits. From previous arguments, we can say F is isomorphic to the direct sum of $\ker\beta$ and P. Hence $F = \ker\beta \oplus P$. So, P is a direct summand of a free module. This proves (4).

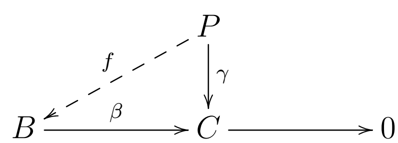
Finally, we need to show that (4) implies (2). Let us assume P is a direct summand of a free R-module F and we write $F = P \oplus K$. Also we have a homomorphism γ from P to C which is given in (2). Consider the homomorphism $\gamma \circ \pi : F \to C$ where π is the natural projection from F to P. Now define an element c_x such that for some element $x \in F$, we have $\gamma \circ \pi(x) = c_x \in C$, and $b_x \in B$ by $\beta(b_x) = c_x$. From the universal property of free modules, there exists a unique homomorphism f' from F to B with $f'(x) = b_x$. So the following diagram commutes



Now we define a map f from P to B by $f(p) = f'((p, 0))$ for $p \in P$. More precisely, $f = f' \circ i$ where i is the injection map from P to F. This shows that f is an R-module homomorphism. Then

$$\beta \circ f(p) = \beta \circ f'((p, 0)) = \gamma \circ \pi((p, 0)) = \gamma(p).$$

That is $\beta \circ f = \gamma$. Hence, we prove that the diagram



commutes and that (4) implies (2). This completes the proof. □

Definition 2.1.7. An R-module P is called *projective* if it satisfies one of the equivalent conditions given in Proposition 2.1.6.

Now, we continue with the definition of a chain complex. This is a sequence of abelian group homomorphisms where successive maps compose to zero.

Definition 2.1.8. Let C_* be a sequence of R-modules

$$C_* : \cdots \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow C_1 \xrightarrow{d_1} C_0 \longrightarrow 0\ .$$

1. If the composition of any two successive maps is zero, i.e., $d_{n-1} \circ d_n = 0$ for all $n \geq 0$, then the sequence C_* is called *a chain complex*.

2. If C_* is chain complex, then its *n-th homology group* is defined by $H_n(C) = \ker d_n/\mathrm{Im}\ d_{n+1}$.

Similarly, we define a cochain complex and a cohomology group of a cochain complex as follows.

Definition 2.1.9. Let C^* be a sequence of R-modules

$$C^* : 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} \cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} \cdots .$$

1. If the composition of any two successive maps is zero, i.e., $d^n \circ d^{n-1} = 0$ for all n, then the sequence C^* is called *a cochain complex*.
2. If C^* is cochain complex, then its *n-th cohomology group* is defined by $H^n(C) = \ker d^n / \text{Im } d^{n-1}$.

Note that the cochain complex C^* is exact means that all its cohomology groups are zero. Thus, the n-th cohomology group is a measure of the failure of exactness of a complex at the n-th stage (see [2]).

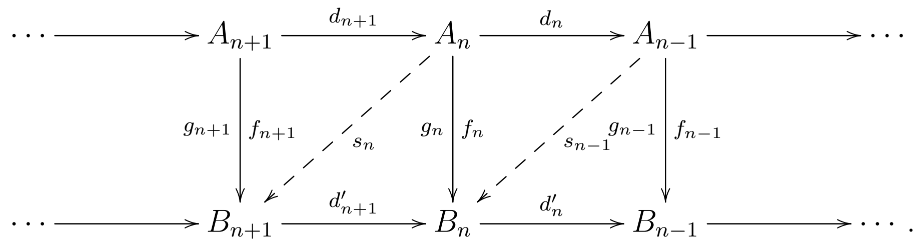
In the previous part, we defined the homomorphism of short exact sequence. Now we generalize this and define a homomorphism of complexes.

Definition 2.1.10. Let $A_* = \{A_n\}$ and $B_* = \{B_n\}$ be chain complexes. Then the map $f : A_* \to B_*$ which is a family of homomorphisms $f_n : A_n \to B_n$ is called *a homomorphism of complexes* or *a chain map* where the following diagram commutes for every $n \in \mathbb{Z}$

$$\begin{array}{ccccccc} \cdots \longrightarrow & A_n & \xrightarrow{d_n} & A_{n-1} & \longrightarrow \cdots \\ & \downarrow f_n & & \downarrow f_{n-1} & \\ \cdots \longrightarrow & B_n & \xrightarrow{d'_n} & B_{n-1} & \longrightarrow \cdots . \end{array}$$

One can show that a homomorphism f of chain complexes A_* and B_* induces a group homomorphism from $H_n(A_*)$ to $H_n(B_*)$ for $n \geq 0$. Observe the relation between two chain maps.

Definition 2.1.11. Let $A_* = \{A_n\}$, $B_* = \{B_n\}$ be two chain complexes and f, g be two chain maps from A_* to B_*. Then the homomorphism $s : A_* \to B_*$ is called *homotopy* from f to g if it satisfies $d'_{n+1} \circ s_n + s_{n-1} \circ d_n = f_n - g_n$. So, the following diagram commutes



In this case, we say f *is homotopic to* g and write $f \simeq g$. A chain map f is called a *homotopy equivalence* if there is another chain map f' such that $f \circ f'$ and $f' \circ f$ are homotopic to identity map, that is $f \circ f' \simeq \mathrm{id}_A$ and $f' \circ f \simeq \mathrm{id}_B$ and the chain complexes A_* and B_* are called *homotopy equivalent.*

Definition 2.1.12. Let $A_* = \{A_n\}$ be a chain complex. Then A_* is *contractible* if it is homotopy equivalent to the zero complex. In particular, the identity map on A_* is homotopic to zero map, i.e., $\mathrm{id}_A \simeq 0$. A homotopy from id_A to 0 is called *a contracting homotopy.*

One can show that any contractible chain complex is acyclic, so its all homology groups are zero, i.e., $H_n(C) = 0$ for all n. Also it is easy to see that homotopy relation $\simeq$ is an equivalence relation.

Now we define a projective resolution of an R-module A. Since for projective modules it is possible to lift various maps, they are used for the construction of resolution of A.

Definition 2.1.13. For any R-module A, define *a projective resolution of* A as an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0 \qquad (2.1.1)$$

where P_n is a projective R-module for all n.

For every R-module A, there is a projective resolution. To see this, first we choose P_0 such that it is a free R-module on a set of generators of A. Clearly, all free modules are also projective. Then we define an R-module homomorphism $\epsilon : P_0 \to A$ by $\epsilon(\sum_{i=1}^{n} r_i a_i) = \sum_{i=1}^{n} r_i \theta(a_i)$ where $\theta : A \to A$ is any map of sets. From the universal property of free modules, this map is uniquely defined. Now, the resolution begins with $P_0 \xrightarrow{\epsilon} A \longrightarrow 0$. Since ϵ is surjective then the sequence is exact. Second we choose a free module P_1 which is mapping onto the submodule $\ker \epsilon$ of P_0. After this stage the sequence becomes $P_1 \longrightarrow P_0 \longrightarrow A$. From construction the sequence is exact. Inductively, at the n-th stage we define a free R-module P_{n+1} that maps surjectively onto the submodule $\ker d_n$ of P_n. As a result, we obtain a projective resolution of A.

In general, if A is not projective then a projective resolution of A has infinite length. In the case where A is projective, we have a trivial resolution such that $0 \longrightarrow A \xrightarrow{\text{id}} A \longrightarrow 0$. Now consider the projective resolution 2.1.1 and omit the term A then take homomorphisms of each of the terms into an R-module D. Note that by taking $\text{Hom}_R(-, D)$ of $A \longrightarrow B$, we get $\text{Hom}_R(B, D) \longrightarrow \text{Hom}_R(A, D)$. That is Hom functor reverses the direction of the homomorphisms. Then we obtain the following

$$0 \longrightarrow \text{Hom}_R(P_0, D) \xrightarrow{d_1'} \text{Hom}_R(P_1, D) \longrightarrow \cdots$$
$$\cdots \longrightarrow \text{Hom}_R(P_{n-1}, D) \xrightarrow{d_n'} \text{Hom}_R(P_n, D) \xrightarrow{d_{n+1}'} \cdots . \tag{2.1.2}$$

The corresponding cohomology group has a special name.

Definition 2.1.14. Let A and D be two R-modules. For a projective resolution of A as in 2.1.1 define $d_n' : \text{Hom}_R(P_{n-1}, D) \to \text{Hom}_R(P_n, D)$ as in 2.1.2. Then

$$\text{Ext}_R^n(A, D) = \ker d_{n+1}' / \text{Im } d_n'$$

for $n \geq 1$ and $\text{Ext}_R^0(A, D) = \ker d_1'$. This group is called *the n-th Ext-group.* Note that this is the *n-th cohomology group derived from the functor $Hom_R(-, D)$.*

It is easy to see that $\text{Ext}_R^0(A, D) = \text{Hom}_R(A, D)$.

Definition 2.1.15. Let A and D be two R-modules and

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} A \longrightarrow 0$$

be a projective resolution of A over R. By deleting the term A, take tensor product of the resolution with D then we get the following

$$\cdots \longrightarrow D \otimes P_n \xrightarrow{1 \otimes d_n} D \otimes P_{n-1} \xrightarrow{1 \otimes d_{n-1}} \cdots \xrightarrow{1 \otimes d_1} D \otimes P_0 \longrightarrow 0.$$

Then define

$$\mathrm{Tor}_n^R(D, A) = \ker(1 \otimes d_n)/\mathrm{Im}(1 \otimes d_{n+1}).$$

for $n \geq 1$ and $\mathrm{Tor}_0^R(D, A) = (D \otimes P_0)/\mathrm{Im}(1 \otimes d_1)$. The group $\mathrm{Tor}_n^R(D, A)$ is called *the n-th Tor-group*. It is *the n-th homology group derived from the functor* $(D \otimes -)$.

Now we consider the following examples which can also be found in [1] and [2].

Example 2.1.16.

1. As we mentioned above if F is a free R-module, then there is a free resolution of F as in the following

$$0 \longrightarrow F \xrightarrow{\mathrm{id}} F \longrightarrow 0\,.$$

2. Let $R = \mathbb{Z}[t]/(t^2 - 1)$ and $A = \mathbb{Z}$. Since $t^2 - 1 = (t - 1)(t + 1)$, it is clear that an element of R is canceled by $(t+1)$ (respectively, $(t-1)$) if and only if it is divisible by $(t - 1)$ (respectively, $(t + 1)$). Hence we can write the following projective resolution which is of infinite length

$$\cdots \xrightarrow{t-1} R \xrightarrow{t+1} R \xrightarrow{t-1} R \xrightarrow{\epsilon} A \longrightarrow 0\,.$$

3. As a generalization of the above example, take $R = \mathbb{Z}[t]/(t^n - 1)$ and $A = \mathbb{Z}$. Since $(t^{n-1} + t^{n-2} + ... + t + 1)(t - 1) = t^n - 1$, then we get a similar projective resolution of A

$$\cdots \xrightarrow{t-1} R \xrightarrow{\Sigma t} R \xrightarrow{t-1} R \xrightarrow{\epsilon} A \longrightarrow 0$$

where $\Sigma t = (t^{n-1} + t^{n-2} + ... + t + 1)$.

4. Now we give an example for the calculation of Ext-groups. Let $R = \mathbb{Z}$ and $A = \mathbb{Z}_n$. First we write the projective resolution of A. We have $\mathbb{Z}$ is projective module as an R-module. Then the exactness of the following sequence makes it a projective resolution of A

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times n} \mathbb{Z} \longrightarrow \mathbb{Z}_n \longrightarrow 0.$$

Note that $(\times n)$ denotes the map of multiplication by n. From the definition, we can say that $\text{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_n, D) \cong \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, D) \cong {}_nD$, where ${}_nD$ is a set of elements d of D which satisfy $nd = 0$. Now take $\text{Hom}_{\mathbb{Z}}(-, D)$ of the above sequence. Since $\text{Hom}_{\mathbb{Z}}(\mathbb{Z}, D) \cong D$, then we obtain the following

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, D) \longrightarrow D \xrightarrow{\times n} D \longrightarrow 0.$$

So, Ext-groups can be calculated easily as follows;

- $\text{Ext}^0_{\mathbb{Z}}(\mathbb{Z}_n, D) \cong {}_nD$,
- $\text{Ext}^1_{\mathbb{Z}}(\mathbb{Z}_n, D) \cong D/nD$,
- $\text{Ext}^k_{\mathbb{Z}}(\mathbb{Z}_n, D) \cong 0$ for all $k \geq 2$.

Let A and A' be R-modules and C_* and C'_* be a projective resolutions of A and A', respectively. Assume f is a homomorphism from A to A'. Then we have the following proposition.

Proposition 2.1.17. *Let C_* and C'_* be as above and $f : A \to A'$ is a homomorphism of R-modules. Then for any $n \geq 0$, there is a lifting f_n of f where the following diagram commutes*

$$\begin{array}{ccccccccccc} C_* : \cdots & \xrightarrow{d_3} & P_2 & \xrightarrow{d_3} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A & \longrightarrow & 0 \\ & & \downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} & & \downarrow{f} & & \\ C'_* : \cdots & \xrightarrow{d'_3} & P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\epsilon'} & A' & \longrightarrow & 0. \end{array}$$

Proof. For a given diagram since P_0 is projective then there exists a lifting $f_0 : P_0 \to P'_0$ such that $\epsilon' f_0 = f\epsilon$. Proceeding inductively, assume f_n has been defined and it makes above diagram commute. Also note that, $d'_m(f_m d_{m+1}) = \underbrace{d'_m(d'_{m+1}}_{0} f_{m+1}) = 0$. So, $\text{Im}(f_m d_{m+1}) \subseteq \ker d'_m$ for all $0 \leq m \leq n$. Now the projectivity of P_{n+1} implies that there is a lifting $f_{n+1} : P_{n+1} \to P'_{n+1}$ such that $f_n d_{n+1} = d'_{n+1} f_{n+1}$. This completes the proof.

□

We continue with the one of the important theorems of homological algebra.

Theorem 2.1.18 (Fundamental Theorem of Homological Algebra)**.** *Let P_* be a chain complex of projective R-modules and C_* be an acyclic chain complex. Then for a given homomorphism $\varphi : H_0(P_*) \to H_0(C_*)$, there is a chain map $f_* : P_* \to C_*$ inducing φ on H_0. Furthermore, any two such chain maps are chain homotopic.*

This fundamental theorem implies that any two projective resolutions of an R-module M are homotopy equivalent.

Corollary 2.1.19. *Let M be an R-module. Then any two projective resolutions of M are chain homotopy equivalent.*

Proof. Let P_* and Q_* be two projective resolutions of M where $H_0(P_*) \cong M \cong H_0(Q_*)$. Consider the following diagram:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

From 2.1.17, the identity map $\mathrm{id} : M \to M$ can be lifted. Then we obtain the chain maps f_* and g_*. Note that the chain map $g_* \circ f_* : P_* \to P_*$ induces the identity map on M. But the identity map $\mathrm{id}_* : P_* \to P_*$ also induces identity map on M. So by the second statement of the fundamental theorem of homological algebra, we get $g_* \circ f_* \cong \mathrm{id}_{P*}$. Similarly, $f_* \circ g_*$ induces identity on homology, so $f_* \circ g_* \cong \mathrm{id}_{Q*}$. Thus P_* homotopy equivalent to Q_*. □

Hence this corollary implies the independence of Ext-groups from the choice of projective resolution.

Corollary 2.1.20. *The Ext-group $Ext^n_R(N, M)$, which maps (N, M) into $H^n(Hom_R(P_*(N), M))$, does not depend on the projective resolution $P_*(N)$ of N.*

As a last part of that section, we give the Universal Coefficient Theorem and the Künneth Theorem.

Theorem 2.1.21 (Künneth Theorem)**.** *Let C_* and C'_* be chain complexes over R where R is principle ideal domain and C_* is dimension-wise free. Then the following sequences are exact and naturally split*

$$0 \longrightarrow \bigoplus_{p\in\mathbb{Z}} H_p(C)\otimes_R H_{n-p}(C') \longrightarrow H_n(C\otimes_R C')$$

$$\longrightarrow \bigoplus_{p\in\mathbb{Z}} Tor_1^R(H_p(C), H_{n-p-1}(C')) \longrightarrow 0$$

and

$$0 \longrightarrow \prod_{p\in\mathbb{Z}} Ext_R^1(H_p(C), H_{n+p+1}(C')) \longrightarrow H_n(Hom_R(C, C'))$$

$$\longrightarrow \prod_{p\in\mathbb{Z}} Hom_R(H_p(C), H_{p+n}(C')) \longrightarrow 0.$$

Now assume that C_* as in the above theorem and C'_* consists of a single module M. So C'_* is

$$C' : 0 \longrightarrow M \longrightarrow 0.$$

In this special case, we have a Universal Coefficient Theorem.

Theorem 2.1.22 (Universal Coefficient Theorem)**.** *Let C_* and C'_* be as above. Then we have the following exact sequences*

$$0 \longrightarrow H_n(C)\otimes_R M \longrightarrow H_n(C\otimes_R M) \longrightarrow Tor_1^R(H_{n-p}(C), M) \longrightarrow 0$$

and

$$0 \longrightarrow Ext_R^1(H_{n-1}(C), M) \longrightarrow H^n(Hom_R(C, M)) \longrightarrow Hom_R(H_n(C), M) \longrightarrow 0.$$

2.2 Group Cohomology

In this section, we define a group cohomology and consider some applications. For more details you can see [2]. Let A be an abelian group and G be a group.

If G act on A as an automorphism, then we say A is a G-module. We denote the set of elements of A which are fixed by all elements of G by A^G. That is $A^G = \{a \in A \mid ga = a \text{ for all } g \in G\}$. For A is a G-module, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$ is a group of all $\mathbb{Z}G$-module homomorphisms from $\mathbb{Z}$ to A with trivial G-action. Every G-module homomorphism from $\mathbb{Z}$ to A is uniquely determined by the image of 1. We say $f_x(1) = x$. Then $x = f_x(1) = f_x(g \cdot 1) = g \cdot f_x(1) = g \cdot x$ for all elements of G. So one can show that for any element $x \in A^G$, the map $f_x \to x$ is an isomorphism. Hence, $A^G \cong \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$. For a short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0,$$

we obtain again an exact sequence such that

$$0 \longrightarrow A^G \longrightarrow B^G \longrightarrow C^G \ .$$

This shows that by considering as a $\mathbb{Z}G$-module, any projective resolution of $\mathbb{Z}$ gives a long exact sequence extending above sequence. One of this type of resolutions is standard resolution of $\mathbb{Z}$. Consider the following sequence

$$\cdots \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} F_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \ . \tag{2.2.1}$$

Here, for $n \geq 0$, F_n is a free $\mathbb{Z}$-module generated by the $(n+1)$-tuples $(g_0, \ldots, g_n)$ where $g_i \in G$ for all i. The action of G is given by $g \cdot (g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$ and the boundary operator is defined by

$$d_n(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \widehat{g_i}, \ldots, g_n).$$

The augmentation map ϵ maps g_0 to 1. One can show the exactness of the above sequence. Then this sequence is called by *the standard resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}G$. Now we change the basis of a free $\mathbb{Z}G$-module F_n. The $(n+1)$-tuples whose first element is 1 and in the form $(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$ forms a basis, since these represent the G-orbits of $(n+1)$-tuples. Then we introduce the following *bar notation*

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n).$$

In that case, the action of G on F_n is given by $g \cdot [g_1|g_2|...|g_n] = [gg_1|...|g_n]$. Note that the *augmentation map* ϵ from F_0 to $\mathbb{Z}$ and d_1 are defined as in the following

$$\epsilon(\sum_{g\in G} r_g g) = \sum_{g\in G} r_g$$

and

$$d_1([g]) = g - 1.$$

Other maps are more complicated. For $n \geq 2$, d_n is given by

$$d_n[g_1|...|g_n] = g_1 \cdot [g_2|...|g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1|...|g_{i-1}|g_i g_{i+1}|g_{i+2}|...|g_n] + (-1)^n [g_1|...|g_{n-1}].$$

One can show that the above resolution is projective, in fact a free resolution. In the above notation, resulting resolution is called *bar resolution.*

Now we take $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ of resolution 2.2.1 in Chapter 2 and replace the first term by 0 to obtain

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(F_0, A) \xrightarrow{d^1} \mathrm{Hom}_{\mathbb{Z}G}(F_1, A) \xrightarrow{d^2} \mathrm{Hom}_{\mathbb{Z}G}(F_2, A) \xrightarrow{d^3} \cdots .$$

This sequence gives the cohomology groups which are $\mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$. To make the notion clear, we consider the elements of $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ that are uniquely determined by their images on the $\mathbb{Z}G$-basis elements of F_n. Lets identify this image by n-tuple $(g_1, g_2, ..., g_n)$ where $g_i \in G$. So we can identify $\mathrm{Hom}_{\mathbb{Z}G}(F_n, A)$ with the set of functions from $G^n = G \times \cdots \times G$ (n-times) to A. Note that for $n = 0$, $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A)$ is given by A.

Now we define a new concept.

Definition 2.2.1. Let G be a finite group and A be a G-module. Define $C^n(G, A)$ as a collection of all maps from G^n to A for $n \geq 1$ and $C^0(G, A) = A$. Then the elements of $C^n(G, A)$ are called *n-cochains.*

One can show that each $C^n(G, A)$ is an additive abelian group.

Definition 2.2.2. We define the *n-th coboundary* homomorphism for $n \geq 0$ from $C^n(G, A)$ to $C^{n+1}(G, A)$ by

$$\begin{aligned} d^n(f)(g_1, ..., g_{n+1}) = &\; g_1 \cdot f(g_2, ..., g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, ..., g_{i-1}, g_i g_{i+1}, g_{i+2}, ..., g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, ..., g_n). \end{aligned}$$

Definition 2.2.3.

1. The elements of $Z^n(G, A) = \ker d^n$ are called *n-cocycles.*
2. The elements of $B^n(G, A) = \text{Im } d^{n-1}$ are called *n-coboundaries.*

By definition it is easy to show that $d^n \circ d^{n-1} = 0$ for $n \geq 1$. This implies $\text{Im } d^{n-1} \subseteq \ker d^n$ so $B^n(G, A) \subseteq Z^n(G, A)$. Now we define the group cohomology.

Definition 2.2.4. Let A be a G-module, we say $H^n(G, A) = Z^n(G, A)/B^n(G, A)$ is the *n-th cohomology group of G with coefficients in A.*

Remark that $H^n(G, A) \cong \text{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, A)$ and these groups can be calculated by using any projective resolution of $\mathbb{Z}$. Now we observe the following examples which can be found in [2].

Example 2.2.5.

1. As a first example, consider the trivial group G. Then G^n is also a trivial group. Any element of f of $C^n(G, A)$ is determined by the image of n-tuple $(1, ..., 1)$. So $C^n(G, A) \cong A$. If $f = a \in A$, then the map
$$d^n(f)(1, ..., 1) = a + \sum (-1)^i a + (-1)^{n+1} a$$
returns 0 if n is even and returns a if n is odd. Hence, $d_n = 0$ if n is even and $d_n = \text{id}$ if n is odd. Then the cohomology groups are given as follows
   - $H^0(G, A) = A^G = A$, and
   - $H^n(G, A) = 0$ for all n.
2. In this example, we discuss the cohomology group of finite cyclic group which is used in the following chapters. Assume $G = \langle x \rangle$ is a cyclic group of order n. We use the following projective resolution
$$\cdots \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \cdots \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

where ϵ is the augmentation map and $\Sigma x = x^{n-1} + ... + x + 1$. Take $\mathrm{Hom}_{\mathbb{Z}G}(-, A)$ of the resolution and replace first term by 0. By using the fact of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, A) \cong A$, we get the cochain complex

$$0 \longrightarrow A \xrightarrow{x-1} A \xrightarrow{\Sigma x} A \xrightarrow{x-1} \cdots .$$

Then cohomology groups are as in the following

- $H^0(G, A) = A^G$,
- $H^n(G, A) = A^G/(\Sigma x)A$ if n is even, and
- $H^n(G, A) = \{a \in A \mid (\Sigma x)a = 0\}/(x-1)A$ if n is odd.

3. As a special case, do the above example for $n = 2$ and $A = \mathbb{Z}$. Let $G = Z_2 = \langle t \mid t^2 = 1\rangle$. Then the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ is given by

$$\cdots \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{t+1} \mathbb{Z}G \xrightarrow{t-1} \cdots \xrightarrow{t+1} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Take $\mathrm{Hom}(-, \mathbb{Z})$ of the resolution and replace the first term by 0. Note that $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G, \mathbb{Z}) \cong \mathbb{Z}$. Then we get

$$0 \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{0} \cdots .$$

So the cohomology groups are $H^i(G, \mathbb{Z}) = (\mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}_2, \ldots)$ for $n \geq 0$.

Note that we say A is cohomologically trivial if $H^n(G, A) = 0$ for all $n \geq 1$. Let A be a G-module and A' be a G'-module. We take two group homomorphisms α and β such that $\alpha : G' \to G$ and $\beta : A \to A'$. We want to define a homomorphism between the cohomology groups $H^n(G, A)$ and $H^n(G', A')$. To do this we need α and β to be compatible.

Definition 2.2.6. Suppose that α and β be as defined above. Then α and β are said to be *compatible* if $\beta(\alpha(g')a) = g'\beta(a)$ for all $a \in A$ and $g' \in G'$.

Now we define a homomorphism h_n such that

$$\begin{aligned} h_n : C^n(G, A) &\to C^n(G', A') \\ f &\to \beta \circ f \circ \alpha^n. \end{aligned}$$

If α and β are compatible, one can show the commutativity of h_n with coboundary maps, that is $h_{n+1} \circ d_n = d_n \circ h_n$. Then h_n maps cocycles to cocycles and coboundaries to coboundaries. So, it induces the homomorphism on cohomologies

$$h_n : H^n(G, A) \to H^n(G', A').$$

We discuss the following examples that can be found in [2].

Example 2.2.7.

1. Let A be a G-module. Then for any subgroup H of G, A is also an H-module. We consider the maps $\text{inc} : H \to G$ and $\text{id} : A \to A$ which are the inclusion map and the identity map, respectively. Since $\text{id}(\text{inc}(h)a) = \text{id}(ha) = ha = h(\text{id}(a))$, then inclusion map and identity are compatible. The corresponding induced group homomorphism on cohomology groups is called the *restriction homomorphism*

$$\text{Res} : H^n(G, A) \to H^n(H, A).$$

2. Now suppose H is a normal subgroup of G and A is a G-module. Note that A^H is a G/H-module under the action of $(gH) \cdot a = g \cdot a$. Observe the maps projection and inclusion, that are $\pi : G \to G/H$ and $\text{inc} : A^H \to A$. Since $\text{inc}(\pi(g)a) = \text{inc}((gH)a) = ga = g(\text{inc}(a))$, then these maps are compatible. The corresponding induced group homomorphism on cohomology of groups is called the *inflation homomorphism*

$$\text{Inf} : H^n(G/H, A^H) \to H^n(G, A)$$

One of the notion that will be used in following chapters is the tensor product of projective resolutions which gives a double complex. Assume R is a ring. Let A be a right R-module and B be a left R-module. Tensor product of A and B over R, i.e., $A \otimes_R B$, is an abelian group generated by $a \otimes b$ with the following properties;

$$(a + a') \otimes b = a \otimes b + a' \otimes b,$$
$$a \otimes (b + b') = a \otimes b + a \otimes b',$$
$$ar \otimes b = a \otimes rb,$$

where $a \in A$, $b \in B$, and $r \in R$.

Definition 2.2.8. A *double complex* is a commutative diagram with each row and column is a complex:

$$\begin{array}{ccccccccc}
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
C_{30} & \xleftarrow{d'_{31}} & C_{31} & \xleftarrow{d'_{32}} & C_{32} & \xleftarrow{d'_{33}} & C_{33} & \leftarrow & \cdots \\
{\scriptstyle d_{30}}\downarrow & & {\scriptstyle d_{31}}\downarrow & & {\scriptstyle d_{32}}\downarrow & & {\scriptstyle d_{33}}\downarrow & & \\
C_{20} & \xleftarrow{d'_{21}} & C_{21} & \xleftarrow{d'_{22}} & C_{22} & \xleftarrow{d'_{23}} & C_{23} & \leftarrow & \cdots \\
{\scriptstyle d_{20}}\downarrow & & {\scriptstyle d_{21}}\downarrow & & {\scriptstyle d_{22}}\downarrow & & {\scriptstyle d_{23}}\downarrow & & \\
C_{10} & \xleftarrow{d'_{11}} & C_{11} & \xleftarrow{d'_{12}} & C_{12} & \xleftarrow{d'_{13}} & C_{13} & \leftarrow & \cdots \\
{\scriptstyle d_{10}}\downarrow & & {\scriptstyle d_{11}}\downarrow & & {\scriptstyle d_{12}}\downarrow & & {\scriptstyle d_{13}}\downarrow & & \\
C_{00} & \xleftarrow{d'_{01}} & C_{01} & \xleftarrow{d'_{02}} & C^{02} & \xleftarrow{d'_{03}} & C_{03} & \leftarrow & \cdots
\end{array}$$

We define a module C_n such that $C_n = \bigoplus_{i+j=n} C_{ij}$ with the differential ∂_n where $\partial_n = d_{ij} + (-1)^i d'_{ij}$ for $i + j = n$. That is $\partial_n : C_n \to C_{n-1}$ satisfies $\partial_n(c_{ij}) = d_{ij}(c_{ij}) + (-1)^i d'_{ij}(c_{ij})$ where $c_{ij} \in C_{ij}$.

Note that $\partial_{n-1}\partial_n(c) = d_{i-1,j}d_{ij}(c) + d'_{i,j-1}d'_{ij}(c) \pm (d'_{i-1,j}d_{ij}(c) - d_{i,j-1}d'_{ij}(c))$. Since each column and row is a complex, then $d^2 = d'^2 = 0$. Also commutativity of diagram gives $dd'(c) = d'd(c)$. Hence $\partial^2 = 0$.

By using the notion of double complex, we can calculate the resolution $A_* \otimes B_*$ where $A_* = \{A_n\}$ and $B_* = \{B_n\}$ are projective resolutions of R. Assume A_* and B_* be as in the following:

$$A_* : \ldots \longrightarrow A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \longrightarrow R \longrightarrow 0$$

$$B_* : \ldots \longrightarrow B_3 \xrightarrow{d'_3} B_2 \xrightarrow{d'_2} B_1 \xrightarrow{d'_1} B_0 \longrightarrow R \longrightarrow 0$$

Then $A_* \otimes B_*$ gives the following double complex:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_3 \otimes B_0 & \xleftarrow{d'_{13}} & A_3 \otimes B_1 & \xleftarrow{d'_{23}} & A_3 \otimes B_2 & \xleftarrow{d'_{33}} & A_3 \otimes B_3 & \leftarrow & \cdots \\
\downarrow {\scriptstyle d_{03}} & & \downarrow {\scriptstyle d_{13}} & & \downarrow {\scriptstyle d_{23}} & & \downarrow {\scriptstyle d_{33}} & & \\
A_2 \otimes B_0 & \xleftarrow{d'_{12}} & A_2 \otimes B_1 & \xleftarrow{d'_{22}} & A_2 \otimes B_2 & \xleftarrow{d'_{32}} & A_2 \otimes B_3 & \leftarrow & \cdots \\
\downarrow {\scriptstyle d_{02}} & & \downarrow {\scriptstyle d_{12}} & & \downarrow {\scriptstyle d_{22}} & & \downarrow {\scriptstyle d_{32}} & & \\
A_1 \otimes B_0 & \xleftarrow{d'_{11}} & A_1 \otimes B_1 & \xleftarrow{d'_{21}} & A_1 \otimes B_2 & \xleftarrow{d'_{31}} & A_1 \otimes B_3 & \leftarrow & \cdots \\
\downarrow {\scriptstyle d_{01}} & & \downarrow {\scriptstyle d_{11}} & & \downarrow {\scriptstyle d_{21}} & & \downarrow {\scriptstyle d_{31}} & & \\
A_0 \otimes B_0 & \xleftarrow{d'_{10}} & A_0 \otimes B_1 & \xleftarrow{d'_{20}} & A_0 \otimes B_2 & \xleftarrow{d'_{30}} & A_0 \otimes B_3 & \leftarrow & \cdots
\end{array}
$$

In the above diagram we get $C_n = \bigoplus_{p+q=n}(A_p \otimes B_q)$.

2.3 Wall's Theorem

In this section, we discuss the construction of doubly graded complex and Wall's Theorem which defines the boundary maps to make the complex exact. To do this, we follow the methods of [5]. Before this process, we observe the following result. Let B_* be the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}L$ where L is normal subgroup of Γ and B_* is given by

$$B_* : \cdots \longrightarrow B_n \longrightarrow \cdots \longrightarrow B_1 \longrightarrow B_0 \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Note that $\mathbb{Z} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \cong \mathbb{Z}(\Gamma/L)$. Indeed, assume $L \cup g_1 L \cup \cdots \cup g_n L \cup \cdots$ be a coset decomposition of Γ. Then the free $\mathbb{Z}$-module basis of $\mathbb{Z} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma$ is $\{[1], [g_1], [g_2], \cdots\}$ where the action of Γ is given by $g[g_i] = g_r l[g_i] = [g_r l g_i] = [g_r g_i]$, with $g = g_r l$. This result is given in [5]. By using this property, taking tensor product of B_* with $\mathbb{Z}\Gamma$ over $\mathbb{Z}L$ results the resolution of $\mathbb{Z}(\Gamma/L)$ over $\mathbb{Z}\Gamma$ which is given in the following

$$B'_* : \cdots \longrightarrow B_n \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \longrightarrow \cdots \longrightarrow B_0 \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \longrightarrow \mathbb{Z}(\Gamma/L) \longrightarrow 0\,. \quad (2.3.1)$$

After this observation, now we write the resolution of $\mathbb{Z}$ over $\mathbb{Z}(\Gamma/L)$ which is given by

$$C_* : \cdots \longrightarrow C_n \xrightarrow{\partial} \cdots \longrightarrow C_1 \xrightarrow{\partial} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Then we can resolve each C_n over $\mathbb{Z}\Gamma$ by using above argument and B'_*. In particular, for all n, we take the tensor product of B'_* and C_n over $\mathbb{Z}(\Gamma/L)$ where $\mathbb{Z}(\Gamma/L) \otimes_{\mathbb{Z}(\Gamma/L)} C_n = C_n$. Now we denote the resulting resolution as follows

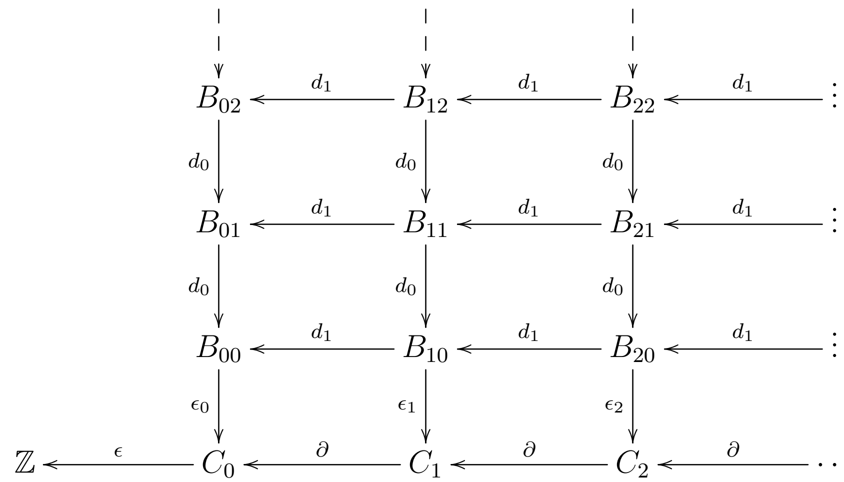
$$\cdots \longrightarrow B_{n2} \xrightarrow{d_0} B_{n1} \xrightarrow{d_0} B_{n0} \xrightarrow{\epsilon_n} C_n \longrightarrow 0.$$

Here, we get the following

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & B_{02} & & B_{12} & & B_{22} & & \vdots \\
 & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \\
 & & B_{01} & & B_{11} & & B_{21} & & \vdots \\
 & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \\
 & & B_{00} & & B_{10} & & B_{20} & & \vdots \\
 & & \downarrow{\scriptstyle \epsilon_0} & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & \\
\mathbb{Z} & \xleftarrow{\epsilon} & C_0 & \xleftarrow{\partial} & C_1 & \xleftarrow{\partial} & C_2 & \xleftarrow{\partial} & \cdots
\end{array}$$

Then we lift the boundary maps ∂ in the resolution of $\mathbb{Z}$. We make a guess that the differential map of doubly graded complex is of the form $d(b_{ij}) = d_0(b_{ij}) + (-1)^i d_1(b_{ij})$ where d_1 is the lifted boundary and d_0 is a vertical differential and

$b_{ij} \in B_{ij}$. So, we obtain a new diagram



Now the problem is d may not be a differential. This is because

$$\begin{aligned} d^2 &= (d_0 + (-1)^i d_1)^2 \\ &= d_0^2 + (-1)^i d_0 d_1 + (-1)^{i-1} d_1 d_0 + d_1 d_1 \\ &= d_1 d_1 \end{aligned}$$

may not be zero where $d_1 : B_{ij} \to B_{i-1,j}$. To make $d^2 = 0$, we add correction terms. We have that $\epsilon_i d_1 d_1 = \partial\partial\epsilon_{i+2} = 0$. So the chain map $d_1 d_1$ is homotopic to zero and there are maps d_2 from $B_{i,j}$ to $B_{i+1,j-2}$ where $d_0 d_2 + d_2 d_0 = d_1 d_1$. The map d_2 is the first correction term and the total differential becomes $d = d_0 + (-1)^i d_1 + d_2$. Then we continue the same procedure until to find $d^2 = 0$. Following theorem, gives the idea of these maps.

Theorem 2.3.1 (Wall [5]). *In the above situation, there is a series of* $\mathbb{Z}\Gamma$*-homomorphisms* $d_n : B_{i,j} \to B_{i+n-1,j-n}$ *such that if we set* $D_m = \bigoplus_{t=0}^{m} B_{t,m-t}$ *and* $d = d_0 + (-1)^i d_1 + \sum_{l=2}^{m} d_l$*, then* $d^2 = 0$ *and the resulting complex is a resolution of* $\mathbb{Z}$ *over* $\mathbb{Z}\Gamma$*. In particular, each* d_n *can be chosen so that*

1. d_0 *is the vertical differential,*
2. $\partial\epsilon_i = \epsilon_{i-1} d_1$,
3. $\sum_{i=0}^{k} d_i d_{k-i} = 0$ *for all* $k \geq 2$.

Moreover, any map which satisfies the properties (1)*,* (2)*, and* (3) *is a differential that makes the above complex acyclic.*

Proof. In this proof we follow the method of [5]. Now, we first show that any map with (1), (2), and (3) makes the total complex acyclic. To see this result, we use a spectral sequence which converges to a homology group of the given complex. Note that property (1) gives that the differential in E^0 is exactly the map d_0. Hence in E^1 we get just C_* which is a free $\mathbb{Z}(\Gamma/L)$-resolution of $\mathbb{Z}$. By (2), d^1 is precisely the differential of C_* that is ∂. Then exactness of C_* gives the result. That is $E^2 \cong E^\infty \cong \mathbb{Z}$. So, the above complex is acyclic.

Conversely, we need to show that there exist maps d_k for $k \geq 3$ which satisfies given properties. Assume that we define d_i for all $i < k$ and d_k for $B_{r-1,s}$ that satisfy (3). Claim of theorem says that there is a map d_k such that $d_0 d_k = f$ where $f = -\Sigma_{i=1}^k d_i d_{k-i}$. It is equivalent to say that f is in the kernel of d_0. By direct calculation, we can see the result.

$$\begin{aligned} d_0 f &= -\sum_{i=1}^{k} d_0 d_i d_{k-i} \\ &= -(d_0 d_1 d_{k-1} + d_0 d_2 d_{k-2} + \ldots) \\ &= \sum_{i=1}^{k} \sum_{j=1}^{i} d_j d_{i-j} d_{k-i} \\ &= \sum_{j=1}^{k} d_j \sum_{i=j}^{k-j} d_{i-j} d_{k-i} \\ &= 0 \end{aligned}$$

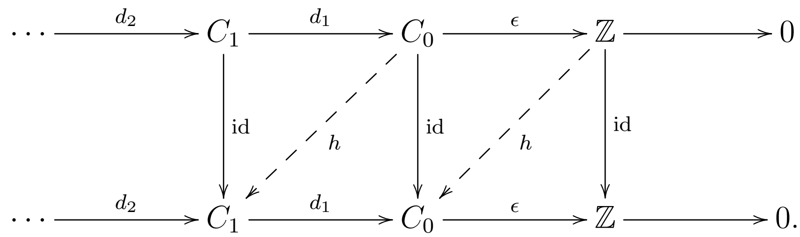
Hence, this completes the proof. □

For the calculation of d_n which is defined in Wall's theorem, we need to take the preimage of f under d_0 where f is in $\ker d_0 = \text{Im } d_0$. For this process, an efficient computational method is to use a contracting homotopy.

Assume that we are given a free and acyclic chain complex over $\mathbb{Z}G$ as in the following

$$\cdots \longrightarrow C_i \xrightarrow{d} C_{i-1} \longrightarrow \cdots \longrightarrow C_1 \xrightarrow{d} C_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

We consider the following contracting homotopy h.



Recall that a *contracting homotopy* is a chain map $h : C_i \to C_{i+1}$ such that $hd_n + d_{n+1}h = \mathrm{id}$ where we denote $\mathbb{Z} = C_{-1}$ and $\epsilon = d_0$. Here we consider some examples of construction of contracting homotopy given in [5].

Example 2.3.2. 1. Let $\Gamma = \mathbb{Z}$ with generater t. We consider the following resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$

$$0 \longrightarrow \mathbb{Z}(\mathbb{Z})[e_1] \xrightarrow{t-1} \mathbb{Z}(\mathbb{Z})[e_0] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0\ .$$

Then the contracting homotopy h is given by

$$h(1) = e_0 \text{ and } h(t^i e_0) = \begin{cases} \sum_{j=0}^{i-1} t^j e_1, & i > 0; \\ -\sum_{j=1}^{-i} t^{-j} e_1, & i < 0; \\ 0, & i = 0. \end{cases}$$

One can show that h satisfies the contracting homotopy condition;

$$\begin{aligned} hd(t^i e_0) + dh(t^i e_0) &= h(1) + d((t^{i-1} + \ldots + 1)e_1) \\ &= e_0 + (t^i - 1)e_0 \\ &= t^i e_0 \end{aligned}$$

for all i.

2. Now we choose $\Gamma = \mathbb{Z}/2$ and take the standard $\mathbb{Z}(\mathbb{Z}/2)$-resolution of $\mathbb{Z}$

$$\cdots \xrightarrow{t-1} \mathbb{Z}\Gamma[e_2] \xrightarrow{t+1} \mathbb{Z}\Gamma[e_1] \xrightarrow{t-1} \mathbb{Z}\Gamma[e_0] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

The contracting homotopy is the following

$$h(1) = e_0,\ h(e_i) = 0, \text{ and } h(te_i) = e_{i+1}.$$

3. In general, we consider the case of $\Gamma = \mathbb{Z}_p$ where p is an odd prime. Then the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}\Gamma$ is that

$$\cdots \xrightarrow{t-1} \mathbb{Z}\Gamma[e_2] \xrightarrow{\Sigma t} \mathbb{Z}\Gamma[e_1] \xrightarrow{t-1} \mathbb{Z}\Gamma[e_0] \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0.$$

Then the following formula is a contracting homotopy for the above resolution

$$h(1) = e_0$$

and

$$h(t^i e_{2n+1}) = \begin{cases} 0, & i \neq p-1 \\ e_{2n+2}, & i = p-1 \end{cases}$$

$$h(t^i e_{2n}) = \begin{cases} 0, & i = 0 \\ \sum_{j=0}^{i-1} t^j e_{2n+1}, & i \neq 0 \end{cases}$$

for $n \geq 0$.

Chapter 3

Cohomology of Semidirect Products

In general, cohomology groups of semidirect products $\Gamma = L \rtimes G$ are not easy to calculate. However, if the projective resolution for L admits a compatible G-action, then the calculation of cohomology groups becomes easier. Hence for a given semidirect group, the first thing to check is whether there is a compatible action on the projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}L$-module. In this chapter, first we give a definition of a compatible action. When there is a compatible action, if there is minimal L-resolution on which G acts then the cohomology group of Γ is given by the cohomology of G with coefficients in the cohomology group of L. Otherwise, group cohomology calculation is much more difficult and the details of that case is explained in Chapter 4.

3.1 Compatible Action

In this section, we consider a compatible action which gives a practical way for calculating cohomology of semidirect products. For details you can see [7]. Let $\Gamma = L \rtimes G$. We define the action of G on L by ${}^{g}l = \varphi(g)l$ for all $l \in L$ and $g \in G$ where φ is a homomorphism $\varphi : G \to \mathrm{Aut}(L)$. Now we take the free resolution

(P_*, d_*) of $\mathbb{Z}$ over $\mathbb{Z}L$ with the augmentation map $\epsilon : P_0 \to \mathbb{Z}$. Then the definition of compatible action is given as follows:

Definition 3.1.1. Given a free $\mathbb{Z}L$-resolution (P_*, d_*) of $\mathbb{Z}$ with augmentation ϵ, we say P_* admits *a compatible G action with respect to φ* if for all $g \in G$ there is an augmentation preserving chain map $f(g) : P_* \to P_*$ where the following properties are satisfied

1. $f(g)(l \cdot x) = ({}^g l) \cdot (f(g)x)$ for all $l \in L$ and $x \in P_n$,
2. $f(g)f(g') = f(gg')$ for all $g, g' \in G$,
3. $f(1) = \mathrm{id}_{P_*}$.

If there is a compatible action, then we can define a $\mathbb{Z}\Gamma$-module structure on P. Note that all the elements of Γ can be written uniquely as $\gamma = lg$ where $l \in L$ and $g \in G$. In that case, the following equation gives the $\mathbb{Z}\Gamma$-module structure

$$\gamma \cdot x = (lg) \cdot x = l \cdot f(g)x.$$

Now we give some basic properties of compatible actions which are given in [7].

1. Let P'_* and P''_* be projective resolutions of $\mathbb{Z}$ over L_1 and L_2 with compatible actions f_1 and f_2, respectively. Note that $P'_* \otimes P''_*$ is a projective resolution of $\mathbb{Z}$ over $L = L_1 \times L_2$. Then there is a compatible action of G on $P'_* \otimes P''_*$ with

$$f(g)(x \otimes y) = f_1(g)(x) \otimes f_2(g)(y)$$

   where $x \in P'_*$ and $y \in P''_*$.

2. Let $G = G_1 \times G_2$ and M is a G_i-module for $i = 1, 2$. Assume a compatible action of G_i on a resolution of $\mathbb{Z}$ over M, P_*, is given by f_i where $f_1(g_1)f_2(g_2) = f_2(g_2)f_1(g_1)$. Then a compatible action of G on P_* is given by $f(g)(x) = f_1(g)f_2(g)x$.

3. Assume $\varphi : G_2 \to G_1$ is a group homomorphism and P_* is a resolution of $\mathbb{Z}$ over L where L is G_1-module. If G_1 acts compatibly on P_* by f', then G_2 also acts compatibly on P_* by

$$f(g)x = f'(\varphi(g))x$$

for any $g \in G_2$ and $x \in P_n$.

3.2 Projective Resolutions for Semidirect Products

In this section, we define projective resolutions for semidirect product $\Gamma = L \rtimes G$ in the case where there is a compatible G-action. The details on the material in this section can be found in [3]. Let R be a ring and X_* and W_* be projective resolutions of R over RL and RG, respectively. An action of G on X_* is defined as a map $g : X_* \to X_*$ which satisfies

$$g(lx) = ({}^{g}l)g(x)$$

for $g \in G$, $l \in L$, and $x \in X$. We write X_* and W_* as in the following

$$X_* : \cdots \longrightarrow X_3 \xrightarrow{d_3} X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow R \longrightarrow 0,$$

$$W_* : \cdots \longrightarrow W_3 \xrightarrow{d_3} W_2 \xrightarrow{d_2} W_1 \xrightarrow{d_1} W_0 \longrightarrow R \longrightarrow 0.$$

Assume W is projective RG-module and X is projective RL-module on which G acts as above.

Now we want to construct a projective $R(L \rtimes G)$-resolution of R which depends on X_* and W_*. Note that we can make $X_* \otimes W_*$ into an $(L \rtimes G)$-complex by defining

$$(l, g)(x \otimes w) = l(g(x)) \otimes gw$$

with $l \in L$, $g \in G$, $x \in X$, and $w \in W$. Then we get the following result.

Theorem 3.2.1. *Let X, W be as above with the given G-action. Then $X \otimes W$ is a projective $R(L \rtimes G)$-module.*

Proof. First we show that there is a natural equivalence of functors $\mathrm{Hom}_{L \rtimes G}(X \otimes W, M)$ and $\mathrm{Hom}_G(W, \mathrm{Hom}_L(X, M))$. To see this we define a map θ such that

$$\theta : \mathrm{Hom}_{L \rtimes G}(X \otimes W, M) \to \mathrm{Hom}_G(W, \mathrm{Hom}_L(X, M))$$
$$f \to \theta(f)$$

where $f(x \otimes w) = \theta(f)(w)(x)$. We need to verify that $\theta(f)$ is a G-homomorphism and $\theta(f)(w)$ is a L-homomorphism. Note that f is a $L \rtimes G$-homomorphism. So it satisfies $f((l, 1)(x \otimes w)) = lf(x \otimes w)$ and $f((1, g)(x \otimes w)) = gf(x \otimes w)$. Then

$$\theta(f)(w)(lx) = f(lx \otimes w) = f((l, 1)(x \otimes w)) = lf(x \otimes w) = l(\theta(f)(w)(x))$$

gives that $\theta(f)(w)$ is a L-homomorphism. To show that $\theta(f)$ is a G-homomorphism, $\theta(f)(gw)(x) = f(x \otimes gw)$ should be equal to $g(\theta(f)(w)(x))$. By the following argument we can prove this

$$\begin{aligned} g(\theta(f)(w)(x)) &= g\theta(f)(w)({}^{g^{-1}}x) \\ &= gf({}^{g^{-1}}x \otimes w) \\ &= f((1, g)({}^{g^{-1}}x \otimes w)) \\ &= f(x \otimes gw). \end{aligned}$$

Now we define a map φ such that

$$\varphi : \mathrm{Hom}_G(W, \mathrm{Hom}_L(X, M)) \to \mathrm{Hom}_{L \rtimes G}(X \otimes W, M)$$

with $\varphi(t)(x \otimes w) = t(w)(x)$. Similarly, we show that $\varphi(t)$ is a $L \rtimes G$-homomorphism. Since t is a G-homomorphism and $t(w)$ is a L-homomorphism, then we get

$$\begin{aligned} \varphi(t)((l, g)(x \otimes w)) &= \varphi(t)(l({}^{g}x) \otimes gw) \\ &= t(gw)(l({}^{g}x)) \\ &= l(t(gw)({}^{g}x)) \\ &= (l, g)(t(w)(x)) \\ &= (l, g)(\theta(t)(x \otimes w)). \end{aligned}$$

Obviously, $\theta\varphi(t)(x \otimes w) = \varphi\theta(f)(w)(x) = \text{id}$. Hence, θ is an isomorphism.

We know that X and W are projective R-modules. By Proposition2.1.6, $\text{Hom}_L(X, -)$ and $\text{Hom}_G(W, -)$ are exact functors. Now take composition of two functors

$$\text{Hom}_G(W, \text{Hom}_L(X, M)) \cong \text{Hom}_{L \rtimes G}(X \otimes W, M)$$

which is again an exact functor as a composition of two exact functors. Again from 2.1.6, $X \otimes W$ is a projective $R(L \rtimes G)$-module. This completes the proof.

□

Now, we are ready to prove our main theorem.

Theorem 3.2.2. *Let X_* be a projective RL-resolution of R and W_* be a projective RG-resolution of R.Suppose that the map $g : X_* \to X_*$ satisfies $g(lx) = ({}^g l)g(x)$. Then $X_* \otimes W_*$ is a projective $R(L \rtimes G)$-resolution of R.*

Proof. Let X_* and W_* be as in the theorem. We consider $X_* \otimes W_*$ as an $R(L \rtimes G)$-complex where $(X \otimes W)_n = \bigoplus_{p+q=n}(X_p \otimes W_q)$. By Theorem 3.2.1, we have $(X \otimes W)_n$ is projective $R(L \rtimes G)$-module for all n.

To see the exactness of the complex $X_* \otimes W_*$, we use the Künneth Theorem which says that the following sequence is exact

$$0 \longrightarrow \bigoplus_{p \in \mathbb{Z}}(H_p(X_*) \otimes H_{n-p}(W_*)) \longrightarrow H_n(X_* \otimes W_*)$$

$$\longrightarrow \text{Tor}_1^R(H_p(X_*), H_{n-p-1}(W_*)) \longrightarrow 0.$$

From the exactness of X_* and W_*, we get

$$H_p(X) \cong H_{n-p}(W) \cong \text{Tor}_1^R(H_p(X), H_{n-p-1}(W)) \cong 0.$$

Hence $H_n(X_* \otimes W_*) \cong 0$ for all $n > 0$. So we prove that $(X_* \otimes W_*)$ is an $R(L \rtimes G)$-projective resolution of R.

□

As a result, we see that the above construction gives a projective resolution for semidirect products when X_* admits a compatible G-action. This result can be used to calculate the cohomology of semidirect products. In particular, if we take X_* as a minimal L-resolution on which G acts, then we have the following isomorphisms

$$H^*(L \rtimes G, M) \cong H^*(G, \mathrm{Hom}_L(X_*, M)) \cong H^*(G, H^*(L, M)).$$

Now we give some applications of above results. First we consider the Dihedral group of order 8, D_8 and take $R = \mathbb{F}_2$. This group has some special properties which make this application possible. This is a simple illustration of Theorem 3.2.2. After that we work on more general cases.

Example 3.2.3. Let $\Gamma = D_8$ where $D_8 = \langle a, b \mid a^4 = b^2 = 1, bab = a^{-1}\rangle$ and $R = \mathbb{F}_2$. Note that in $\mathbb{F}_2$, -1 and $+1$ are same things. Let $C_4 = \langle a \mid a^4 = 1\rangle$ and $C_2 = \langle b \mid b^2 = 1\rangle$. Then $D_8 = C_4 \rtimes C_2$ where b acts on C_4 is by ${}^b a = a^{-1}$. Recall that $\mathbb{F}_2 C_4$ and $\mathbb{F}_2 C_2$ have projective resolutions in the following forms, respectively,

$$X_* : \cdots \xrightarrow{1+a} \mathbb{F}_2C_4 \xrightarrow{\Sigma a} \mathbb{F}_2C_4 \xrightarrow{1+a} \mathbb{F}_2C_4 \xrightarrow{\epsilon} \mathbb{F}_2 \longrightarrow 0$$

$$W_* : \cdots \xrightarrow{1+b} \mathbb{F}_2C_2 \xrightarrow{1+b} \mathbb{F}_2C_2 \xrightarrow{1+b} \mathbb{F}_2C_2 \xrightarrow{\epsilon'} \mathbb{F}_2 \longrightarrow 0$$

where $\Sigma a = 1 + a + a^2 + a^3$. Now we consider the action of b on C_4 and with this action all free $\mathbb{F}_2C_4$-modules can be considered as $\mathbb{F}_2C_2$-modules. We denote this new complex by X'_*. Note that there is a chain map between X_* and X'_* which is extended from $\mathrm{id} : \mathbb{F}_2 \to \mathbb{F}_2$, by Theorem 2.1.17. Let us denote this chain map by f_n where $f_n(hx) = ({}^b h) f_n(x)$. In particular, f_n satisfies $f_n(a^i x) = a^{-i} f_n(x)$. Since X'_* consists of $\mathbb{F}_2C_2$-modules, we define a map θ on X^1_* to return the same module structure with X_* where $\theta(a) = a^{-1}$. Now we get a following picture

$$\begin{array}{ccccccccccc}
X_* : \cdots & \xrightarrow{1+a} & \mathbb{F}_2C_4 & \xrightarrow{\Sigma a} & \mathbb{F}_2C_4 & \xrightarrow{1+a} & \mathbb{F}_2C_4 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{\epsilon} & 0 \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
X'_* : \cdots & \xrightarrow{1+a} & \mathbb{F}_2C_4 & \xrightarrow{\Sigma a} & \mathbb{F}_2C_4 & \xrightarrow{1+a} & \mathbb{F}_2C_4 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{\epsilon} & 0 \\
 & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
X''_* : \cdots & \xrightarrow{1+a^{-1}} & \mathbb{F}_2C_4 & \xrightarrow{\Sigma a} & \mathbb{F}_2C_4 & \xrightarrow{1+a^{-1}} & \mathbb{F}_2C_4 & \longrightarrow & \mathbb{F}_2 & \xrightarrow{\epsilon} & 0
\end{array}$$

We denote $\theta f_n = \alpha_n$. We can calculate each α_n by using commutativity of the above diagram. Recall that we have $\alpha_0 = \mathrm{id}$. Then

(i) $(1+a) = (1+a^{-1})\alpha_1 \Rightarrow \alpha_1 = a$ and

(ii) $a(1+a+a^2+a^3) = (1+a+a^2+a^3)\alpha_2 \Rightarrow \alpha_2 = 1$.

Note that α is two periodic. As a special case for D_8, we can assume θ is multiplication by a^2, that is $\theta(a) = a^3 = a^{-1}$. By using these information, it is possible to calculate each f_n where

- $\theta f_{2n}(a) = a \Rightarrow f_0(a) = a^3 \Rightarrow f_0$ is multiplication by a^2 and
- $\theta f_{2n+1}(a) = a^2 \Rightarrow f_0(a) = a^2 \Rightarrow f_1$ is multiplication by a

for $n \geq 0$. Remember the action of a and b on $x \otimes w$. We have that $a(x \otimes w) = ax \otimes w$ where $b(x \otimes w) = {}^{b}x \otimes bw$. For the calculations of differentials of double complex, it is easy to see that the vertical maps do not change which are trivially dependent on multiplication by a. However, for the horizontal maps, we should consider the action of b. In each coordinate the action of b is given by f_n's. In particular, the first horizontal differential map $(1+b)$ becomes $1+a^2b$ and the second one becomes $1+ab$. So we get the following diagram:

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \\
RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \\
RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \xleftarrow{1+ab} & RC_4 \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \downarrow{\scriptstyle 1+a} & & \\
RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \xleftarrow{1+a^2b} & RC_4 \otimes RC_2 & \leftarrow & \cdots
\end{array}
$$

Let us observe the commutativity of double complex. The relations

$$(1+a)(1+ab)+(1+a^2b)(1+a)=0$$

and

$$(1+a+a^2+a^3)(1+a^2b)+(1+ab)(1+a+a^2+a^3)=0$$

proves the commutativity. As a result, we obtain a projective resolution of D_8.

Now we check the minimality of $\mathbb{F}_2C_4$-resolution. Note that X_* is said to be a minimal resolution if $\mathrm{Hom}(X_*, \mathbb{F}_2)$ has zero differentials. Clearly, by applying $\mathrm{Hom}(-, \mathbb{F}_2)$ to the resolution X_*, differentials, $(1+a)$ and $(1+a+a^2+a^3)$, becomes multiplication by 2 and 4, respectively. Since we have $R = \mathbb{F}_2$, then all the differentials of $\mathrm{Hom}(X_*, \mathbb{F}_2)$ are zero. That is

$$\mathrm{Hom}(X_*, \mathbb{F}_2) : 0 \longrightarrow \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \mathbb{F}_2 \xrightarrow{0} \cdots .$$

So X_* is a minimal resolution. Now the cohomology of D_8 can be calculated easily. First we calculate cohomology in each row. Note that, $\mathrm{Hom}_{R(C_4 \rtimes C_2)}(RC_4 \otimes RC_2, R) \cong \mathrm{Hom}_{RC_2}(RC_2, \mathrm{Hom}_{RC_4}(RC_4, R)) \cong R$ and all differentials in rows and columns return to zero since we choose $R = \mathbb{F}_2$. As a result, we get $H^i(G, H^j(L, \mathbb{F}_2)) = \mathbb{F}_2$ for all $0 \leq i, j$. Hence, we conclude that the cohomology of D_8 is given by

$$H^n(D_8, \mathbb{F}_2) = \bigoplus_{n+1} \mathbb{F}_2.$$

Now we consider the more general case and make same calculations for D_{2n}.

Example 3.2.4. Let us observe $D_{2n} = C_n \rtimes C_2$ with $C_n = \langle a \mid a^n = 1 \rangle$, $C_2 = \langle b \mid b^2 = 1 \rangle$ and the action given by ${}^b a = a^{-1}$. In this example, we take $R = \mathbb{Z}$. Projective resolutions of C_n and C_2 are X_* and W_*, respectively, where

$$X_* : \cdots \xrightarrow{1-a} RC_n \xrightarrow{\Sigma a} RC_n \xrightarrow{1-a} RC_n \xrightarrow{\epsilon} R \longrightarrow 0$$

$$W_* : \cdots \xrightarrow{1-b} RC_2 \xrightarrow{1+b} RC_2 \xrightarrow{1-b} RC_2 \xrightarrow{\epsilon'} R \longrightarrow 0.$$

Let f_n and θ be as in the previous example. So we get a following commutative diagram

$$\begin{array}{ccccccccccc}
X_1 : \cdots & \xrightarrow{1-a} & RC_n & \xrightarrow{\Sigma a} & RC_n & \xrightarrow{1-a} & RC_n & \xrightarrow{\epsilon} & R & \longrightarrow & 0 \\
 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow \mathrm{id} & & \\
X_2 : \cdots & \xrightarrow{1-a} & RC_n & \xrightarrow{\Sigma a} & RC_n & \xrightarrow{1-a} & RC_n & \xrightarrow{\epsilon} & R & \longrightarrow & 0 \\
 & & \downarrow \theta & & \downarrow \theta & & \downarrow \theta & & \downarrow \mathrm{id} & & \\
X_3 : \cdots & \xrightarrow{1-a^{-1}} & RC_n & \xrightarrow{\Sigma a} & RC_n & \xrightarrow{1-a^{-1}} & RC_n & \xrightarrow{\epsilon} & R & \longrightarrow & 0
\end{array}$$

where $\Sigma a = 1 + a + a^2 + ... + a^{n-1}$. By using commutativity of above diagram we find each $\alpha_n = \theta f_n$.

- $\alpha_0 = 1$
- $(1-a) = (1-a^{-1})\alpha_1 \Rightarrow \alpha_1 = -a$
- $(-a)(\Sigma a) = (\Sigma a)\alpha_2 \Rightarrow \alpha_2 = -1$
- $-(1-a) = (1-a^{-1})\alpha_3 \Rightarrow \alpha_3 = a$
- $a(\Sigma a) = (\Sigma a)\alpha_4 \Rightarrow \alpha_4 = 1$

So α has period 4. Now we want to find the differentials of the complex $X_* \otimes W_*$. Note that $a(x \otimes w) = ax \otimes w$ and $b(x \otimes w) = {}^b x \otimes bw$. Since $\alpha_0 = \mathrm{id}$, the first differential map in the double complex does not change and we get $f_0 = \theta$. So the differential $d : X_0 \otimes W_1 \to X_0 \otimes W_0$ is identified by $(1-b)$ which fixes the b action. We get $b(x_0 \otimes w) = f_0 x_0 \otimes bw$ where $x_0 \in C_0(X_*)$. To define the second map, we observe $b(x_1 \otimes w) = {}^b x_1 \otimes bw = f_1 x_1 \otimes bw$. Affect of f_1 can be seen by the following:

$$f_1 x_1 = \theta(-a)x_1 = -a^{-1}\theta x_1 = -a^{-1} f_0 x_1.$$

Then the former equation becomes

$$b(x_1 \otimes w) = -a^{-1}(f_0 x_1 \otimes bw).$$

Hence for this differential $-a^{-1}$ comes in front of b. As a result, the map $d : X_1 \otimes W_1 \to X_1 \otimes W_0$ is $(1 + a^{-1}b)$. By using same method we can find other differentials.

- $b(x_2 \otimes w) = f_2 x_2 \otimes bw = -\theta x_2 \otimes w = -f_0 x_2 \otimes w = -(f_0 x_2 \otimes w)$
- $b(x_3 \otimes w) = f_3 x_3 \otimes bw = \theta(a x_3) \otimes w = a^{-1} f_0 x_3 \otimes w = a^{-1}(f_0 x_3 \otimes w)$
- $b(x_4 \otimes w) = f_4 x_4 \otimes bw = \theta x_4 \otimes w = (f_0 x_4 \otimes w)$

So we complete the maps of the double complex which is given below

$$
\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
RC_n \otimes RC_2 & \xleftarrow{1-b} & RC_n \otimes RC_2 & \xleftarrow{1+b} & RC_n \otimes RC_2 & \xleftarrow{1-b} & RC_n \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \\
RC_n \otimes RC_2 & \xleftarrow{-(1-a^{-1}b)} & RC_n \otimes RC_2 & \xleftarrow{-(1+a^{-1}b)} & RC_n \otimes RC_2 & \xleftarrow{-(1-a^{-1}b)} & RC_n \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \\
RC_n \otimes RC_2 & \xleftarrow{1+b} & RC_n \otimes RC_2 & \xleftarrow{1-b} & RC_n \otimes RC_2 & \xleftarrow{1+b} & RC_n \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \downarrow{\scriptstyle \Sigma a} & & \\
RC_n \otimes RC_2 & \xleftarrow{-(1+a^{-1}b)} & RC_n \otimes RC_2 & \xleftarrow{-(1-a^{-1}b)} & RC_n \otimes RC_2 & \xleftarrow{-(1+a^{-1}b)} & RC_n \otimes RC_2 & \leftarrow & \cdots \\
\downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \downarrow{\scriptstyle 1-a} & & \\
RC_n \otimes RC_2 & \xleftarrow{1-b} & RC_n \otimes RC_2 & \xleftarrow{1+b} & RC_n \otimes RC_2 & \xleftarrow{1-b} & RC_n \otimes RC_2 & \leftarrow & \cdots
\end{array}
$$

Clearly, commutativity of the above diagram is satisfied. As an example, we can consider the followings

$$
\begin{aligned}
-(1-a)(1+a^{-1}b) + (1-b)(1-a) &= 0 \\
(\Sigma a)(1+b) - (1+a^{-1}b)(\Sigma a) &= 0 \\
-(1-a)(1-a^{-1}b) + (1+b)(1-a) &= 0.
\end{aligned}
$$

So we obtain the projective resolution of D_{2n} which is $X_* \otimes W_*$. Let us denote $(X_* \otimes W_*) = (C_n)_{n \geq 0}$ where $C_n = \bigoplus_{n+1}(RC_n \otimes RC_2)$ and maps come from the above double complex. Note that X_* is not a minimal resolution since when we observe $\mathrm{Hom}(X_*, \mathbb{Z})$, its differentials $(1 - a)$ and Σa, becomes multiplication by 0 and n, respectively. Then we cannot use the same method with the previous example. Calculation of cohomology of D_{2n} requires the usage of spectral sequence.

Chapter 4

Calculations for γ_2 and ρ_8

In this chapter we make calculations for the conjecture of Adem-Ge-Pan-Petrosyan. We assume $\Gamma = L \rtimes G$, where G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice. Conjecture of Adem-Ge-Pan-Petrosyan says that the cohomology group of Γ is given by the cohomology group of G with the coefficient in the cohomology group of L. We consider the 4-dimensional case, that is $L = \mathbb{Z}^4$. Actually, it is the lowest dimension of that type of semidirect groups for which the conjecture is not true. According to [9], in dimension 4 there are 44 non-isomorphic groups of the given type and the only 2 of them do not satisfy the Conjecture 1.0.1. In both of these groups G is a cyclic group of order 4. The action of G on L is given by a left multiplication by matrices:

$$\gamma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\gamma_2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

Calculations for the representation γ_1 is given in [9]. In this chapter, I make detailed calculations for the representation γ_2 which is a counterexample for Conjecture 1.0.1. In [9], it is also stated that the example of ρ_8 satisfies the conjecture where ρ_8 is one of the indecomposables for $G = \mathbb{Z}_4$ which are defined in [8]. I also calculate this example in detail. We recall the conjecture of Adem-Ge-Pan-Petrosyan:

Conjecture 4.0.5. Suppose that G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice. Then for any $k \geq 0$ we have

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).$$

Actually the right hand side of the conjecture is easy to calculate. The main difficulty is the calculation of the left hand side. For that calculation we use a method of Petrosyan and Putrycz. In the following section we explain this method step by step.

4.1 The Method of Petrosyan and Putrycz

Let $\Gamma = L \rtimes G$ where G is a finite cyclic group and L is a finitely generated $\mathbb{Z}G$-lattice.

Step 1: Suppose (B_*, ∂') and (C_*, ∂) are free $\mathbb{Z}L$ and $\mathbb{Z}G$-resolutions of $\mathbb{Z}$, respectively. Note that the induced module $\mathrm{Ind}_L^\Gamma B_* = B_* \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma$ is free over $\mathbb{Z}\Gamma$ and $\mathbb{Z} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma = \mathbb{Z}G$. Since taking tensor product of a complex with a projective module is an exact functor, then $\mathrm{Ind}_L^\Gamma B_*$ with the induced differentials is a free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}G$. For given B_*

$$B_* : \cdots \longrightarrow B_n \xrightarrow{\partial'_n} B_{n-1} \xrightarrow{\partial'_{n-1}} \cdots \longrightarrow B_1 \xrightarrow{\partial'_1} B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0,$$

we get the following induced complex

$$\mathrm{Ind}_L^\Gamma B_* : \cdots \longrightarrow B_n \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \xrightarrow{\partial'_n \otimes \mathrm{id}} B_{n-1} \otimes_{\mathbb{Z}L} \mathbb{Z}\Gamma \longrightarrow \cdots \longrightarrow \mathbb{Z}G \longrightarrow 0.$$

Now we consider each module of $\mathrm{Ind}_L^\Gamma B_*$ with the trivial left G-action and define

$$A_{r,s} := \mathrm{Ind}_L^\Gamma B_r \otimes_{\mathbb{Z}G} C_s.$$

By denoting the graded complex $\bigoplus_r \mathrm{Ind}_L^\Gamma B_r$ by $\mathrm{Ind}_L^\Gamma B_*$ we set

$$D_s = \bigoplus_r A_{r,s} = \mathrm{Ind}_L^\Gamma B_* \otimes_{\mathbb{Z}G} C_s$$

and

$$A_{**} = \bigoplus_{r,s} A_{r,s}.$$

Also we named the differentials of each complex D_s by d_0. So, we get the following

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
A_{2,0} & \longleftarrow & A_{2,1} & \longleftarrow & A_{2,2} & \longleftarrow & A_{2,3} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \\
A_{1,0} & \longleftarrow & A_{1,1} & \longleftarrow & A_{1,2} & \longleftarrow & A_{1,3} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \downarrow{\scriptstyle d_0} & & \\
A_{0,0} & \longleftarrow & A_{0,1} & \longleftarrow & A_{0,2} & \longleftarrow & A_{0,3} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle \epsilon_0} & & \downarrow{\scriptstyle \epsilon_1} & & \downarrow{\scriptstyle \epsilon_2} & & \downarrow{\scriptstyle \epsilon_3} & & \\
C_0 & \xleftarrow{\partial} & C_1 & \xleftarrow{\partial} & C_2 & \xleftarrow{\partial} & C_3 & \longleftarrow & \cdots
\end{array}$$

In our case $L = \mathbb{Z}^n$. To obtain a free $\mathbb{Z}L$-resolution B_* of $\mathbb{Z}$, first consider the $n = 1$ case. Now we have a trivial resolution with generators e and e_1 as in the following

$$0 \longrightarrow \mathbb{Z}(\mathbb{Z})[e_1] \longrightarrow \mathbb{Z}(\mathbb{Z})[e] \longrightarrow \mathbb{Z} \longrightarrow 0. \qquad (4.1.1)$$

For $n = 2$, we just need to take the tensor product of above resolution by itself. So we get

$$0 \longrightarrow \mathbb{Z}(\mathbb{Z}^2)[e_{12}] \longrightarrow \bigoplus_{i=1}^{2} \mathbb{Z}(\mathbb{Z}^2)[e_i] \longrightarrow \mathbb{Z}(\mathbb{Z}^2)[e] \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Inductively, if we define a resolution for $n = k-1$, by taking tensor product of this resolution with 4.1.1, we get a resolution for $n = k$. By this construction, we have each module B_m in the following form

$$B_m = \langle e_{i_1 \dots i_m} \mid 1 \le i_1 \le \dots \le i_m \le n \rangle_{\mathbb{Z}L} \text{ for } 0 \le m \le n$$

and the differentials are given by

$$d_m^B(e_{i_1 \dots i_m}) = \sum_{j=1}^{m} (-1)^{j-1}(t_{i_j} - 1)e_{i_1 \dots \widehat{i_j} \dots i_m}.$$

Now we need to define a free $\mathbb{Z}G$-resolution of $\mathbb{Z}$ which is called C_*. In our computations, G is a finite cyclic group. That is $G = \langle x \mid x^m = 1 \rangle$. So we can take a standard 2-periodic resolution

$$\cdots \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0$$

where Σx denote $x^{m-1} + \ldots + x + 1$. Hence

$$\begin{aligned} A_{r,s} &= \operatorname{Ind}_L^\Gamma B_r \otimes_{\mathbb{Z}G} C_s \\ &= \operatorname{Ind}_L^\Gamma B_r \otimes_{\mathbb{Z}G} \mathbb{Z}G \\ &\cong \operatorname{Ind}_L^\Gamma B_r. \end{aligned}$$

Now the complex A_{**} is given by the following doubly graded complex:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\oplus_{i<j}\mathbb{Z}\Gamma e_{ij} & \longleftarrow & \oplus_{i<j}\mathbb{Z}\Gamma e_{ij} & \longleftarrow & \oplus_{i<j}\mathbb{Z}\Gamma e_{ij} & \longleftarrow & \oplus_{i<j}\mathbb{Z}\Gamma e_{ij} & \longleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\oplus_i\mathbb{Z}\Gamma e_i & \longleftarrow & \oplus_i\mathbb{Z}\Gamma e_i & \longleftarrow & \oplus_i\mathbb{Z}\Gamma e_i & \longleftarrow & \oplus_i\mathbb{Z}\Gamma e_i & \longleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\mathbb{Z}\Gamma e & \longleftarrow & \mathbb{Z}\Gamma e & \longleftarrow & \mathbb{Z}\Gamma e & \longleftarrow & \mathbb{Z}\Gamma e & \longleftarrow & \cdots \\
\downarrow \epsilon_0 & & \downarrow \epsilon_1 & & \downarrow \epsilon_2 & & \downarrow \epsilon_3 & & \\
\mathbb{Z}G & \xleftarrow{\partial} & \mathbb{Z}G & \xleftarrow{\partial} & \mathbb{Z}G & \xleftarrow{\partial} & \mathbb{Z}G & \longleftarrow & \cdots
\end{array}$$

Step 2: In the following steps, we need to find inverse of d_0 for elements in $\ker d_0 = \text{Im } d_0$. An efficient computational method for this is to use a contracting homotopy. Recall that a contracting homotopy of an acyclic complex C_* is a chain map $h : C_i \to C_{i+1}$ such that $hd_0 + d_0h = \text{id}$. Then for $c \in \ker d_0$, we have $d_0h(c) = c$. Hence h maps c to its preimage under d_0. This is the result that we want.

In general, we study on a chain complex of the form $C_* \otimes_{\mathbb{Z}} C'_*$, where C_* is free over $\mathbb{Z}H$ and C'_* is free over $\mathbb{Z}H'$. Note that the tensor product of two acyclic complexes, that are free over $\mathbb{Z}$, is again acyclic and free over $\mathbb{Z}$. Hence $C_* \otimes_{\mathbb{Z}} C'_*$ is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}H \otimes_{\mathbb{Z}} \mathbb{Z}H' = \mathbb{Z}(H \times H')$. Then the contracting homotopy of this resolution can be constructed by using the following lemma which appears in [5].

Lemma 4.1.1. *Let $C_* \otimes_{\mathbb{Z}} C'_*$ be as above. Assume h_H, a contracting homotopy for C_*, and $h_{H'}$, a contracting homotopy for C'_*, are given. Then a contracting homotopy for $C_* \otimes_{\mathbb{Z}} C'_*$ can be written by the formula*

$$h^{\otimes}(c \otimes c') = [h_H \otimes 1 + h_H\epsilon \otimes h_{H'}](c \otimes c') \tag{4.1.2}$$

where $c \in C_n$ and $c' \in C'_n$.

Proof. One can see the result by direct calculation

$$\begin{aligned} d^{\otimes}h^{\otimes} + h^{\otimes}d^{\otimes} =& (d \otimes 1 + (-1)^{sgn} \otimes d)(h_H \otimes 1 + h_H\epsilon \otimes h_{H'}) \\ &+ (h_H \otimes 1 + h_H\epsilon \otimes h_{H'})(d \otimes 1 + (-1)^{sgn} \otimes d) \\ =& (dh_H \otimes 1) + (h_H\epsilon \otimes dh_{H'}) + ((-1)^{sgn} \otimes d)(h_H \otimes 1) \\ &+ (h_Hd \otimes 1) + (h_H\epsilon \otimes h_{H'}d) + (h_H \otimes 1)((-1)^{sgn} \otimes d). \end{aligned}$$

Note that the terms $((-1)^{sgn} \otimes d)(h_H \otimes 1)$ and $(h_H \otimes 1)((-1)^{sgn} \otimes d)$ cancel out. After simplification, we first consider the case where c is not in C_0. In this case

$h_H\epsilon(c) = 0$, hence we obtain the following

$$\begin{aligned}[d^{\otimes}h^{\otimes} + h^{\otimes}d^{\otimes}](c \otimes c') &= [(dh_H + h_H d) \otimes 1 + h_H\epsilon \otimes (dh_{H'} + h_{H'}d)](c \otimes c') \\ &= [1 \otimes 1](c \otimes c') \\ &= c \otimes c'.\end{aligned}$$

Otherwise, if $c \in C_0$, then $h_H d(c)$ becomes 0. Hence we obtain

$$\begin{aligned}[d^{\otimes}h^{\otimes} + h^{\otimes}d^{\otimes}](c \otimes c') &= [(dh_H + h_H d) \otimes 1 + h_H\epsilon \otimes (dh_{H'} + h_{H'}d)](c \otimes c') \\ &= [dh_H \otimes 1 + h_H\epsilon \otimes (dh_{H'} + h_{H'}d)](c \otimes c') \\ &= [dh_H \otimes 1 + h_H\epsilon \otimes 1](c \otimes c') \\ &= [(dh_H + h_H\epsilon) \otimes 1](c \otimes c') \\ &= c \otimes c'.\end{aligned}$$

This finishes the proof. □

Recall that in Section 2.3, for $n = 1$ we define a contracting homotopy denoted by ${}^1h$ as follows

$${}^1h(1) = e \text{ and}$$

$${}^1h(t^i e) = \begin{cases} \sum_{j=0}^{i-1} t^j e_1, & i > 0; \\ -\sum_{j=1}^{-i} t^{-j} e_1, & i < 0; \\ 0, & i = 0. \end{cases}$$

Alternatively, ${}^1h$ can be written in a more simple way

$${}^1h(t^i e) = \tfrac{t^i - 1}{t-1} e_1.$$

Lets $(B_*, d_*^B, n = 1)$ be a free $\mathbb{Z}(\mathbb{Z})$-resolution of $\mathbb{Z}$. Since for each $k \geq 1$, the resolution $(B_*, d_*^B, n = k+1)$ is isomorphic to the tensor product of $(B_*, d_*^B, n = k)$ and $(B_*, d_*^B, n = 1)$, then from Lemma 4.1.1 we can define a contracting homotopy by

$$^{k+1}h = {}^{k}h \otimes 1 + ({}^{k}h\epsilon) \otimes {}^{1}h$$

where ${}^{k}h$ is the contracting homotopy for $(B_*, d_*^B, n = k)$. Before the generalization of h, we make clear the notation. While writing the resolution $(B_*, d_*^B, n = k+1)$ which is isomorphic to the tensor product of $(B_*, d_*^B, n = k)$ and $(B_*, d_*^B, n = 1)$, we call the generators of $(B_*, d_*^B, n = 1)$ by e and e_k. By taking the tensor product, we obtain the generators at the dimension $n = k+1$. As an illustration, we take tensor product of $(B_*, d_*^B, n = 2)$ with $(B_*, d_*^B, n = 1)$ to obtain a basis for $(B_*, d_*^B, n = 3)$.

$$\begin{aligned}
e \otimes e &= e \\
e_1 \otimes e &= e_1 \\
e_2 \otimes e &= e_2 \\
e_{12} \otimes e &= e_{12} \\
e \otimes e_3 &= e_3 \\
e_1 \otimes e_3 &= e_{13} \\
e_2 \otimes e_3 &= e_{23} \\
e_{12} \otimes e_3 &= e_{123}
\end{aligned}$$

Now we construct the contracting homotopy for $n = 2$.

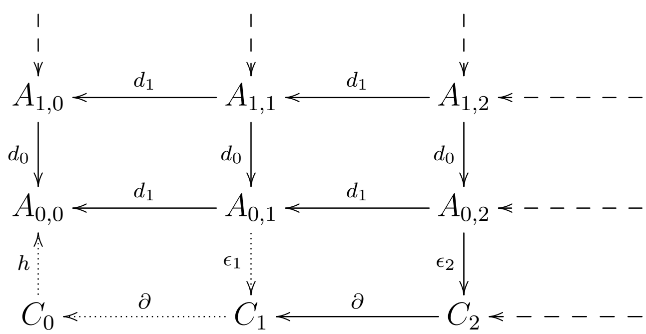
$$\begin{aligned}
{}^{2}h(t_1^n t_2^m e) &= ({}^{1}h \otimes 1)(t_1^n e \otimes t_2^m e) + ({}^{1}h\epsilon \otimes {}^{1}h)(t_1^n e \otimes t_2^m e) \\
&= \frac{t_1^n - 1}{t_1 - 1} t_2^m e_1 \otimes e + \frac{t_2^m - 1}{t_2 - 1} e \otimes e_2 \\
&= \frac{t_1^n - 1}{t_1 - 1} t_2^m e_1 + \frac{t_2^m - 1}{t_2 - 1} e_2 \\
{}^{2}h(t_1^n t_2^m e_1) &= ({}^{1}h \otimes 1)(t_1^n e_1 \otimes t_2^m e) + ({}^{1}h\epsilon \otimes {}^{1}h)(t_1^n e_1 \otimes t_2^m e) \\
&= 0 \\
{}^{2}h(t_1^n t_2^m e_2) &= ({}^{1}h \otimes 1)(t_1^n e \otimes t_2^m e_2) + ({}^{1}h\epsilon \otimes {}^{1}h)(t_1^n e \otimes t_2^m e_2) \\
&= \frac{t_1^n - 1}{t_1 - 1} t_2^m e_1 \otimes e_2 \\
&= \frac{t_1^n - 1}{t_1 - 1} t_2^m e_{12}
\end{aligned}$$

By induction, one can show that the general formula of h for $n = k$ is as in the following

$$
{}^k h(t_1^{n_1} t_2^{n_2} \ldots t_k^{n_k} e_{i_1 \ldots i_m}) = \begin{cases} \sum_{j=0}^{i_1-1} A_j^k e_{j i_1 \ldots i_m}, & \text{if } m > 0; \\ \\ \sum_{j=0}^{k} A_j^k e_j, & \text{if } m = 0; \end{cases}
$$

where $A_j^k = \frac{t_j^{n_j}-1}{t_j-1} t_{j+1}^{n_{j+1}} \ldots t_k^{n_k}$ and $A_0^k = 0$.

Step 3: Now we want to complete the construction of free $\mathbb{Z}\Gamma$- resolution of $\mathbb{Z}$. To do this we need to add some differentials to the complex A_{**} as discussed in Section 2.3. Let us $r = 0$ and α be a generator of $A_{0,s}$. We define the first differential d_1 by $d_1(\alpha) = h(\partial(\epsilon_s(\alpha))) \in A_{0,s-1}$.



For $r = 1$, we get $\epsilon_{s-1} d_1 d_0 = \epsilon_{s-1} h \partial \epsilon_s d_0 = \partial \epsilon_s d_0 = 0$. Hence $d_1 d_0$ from $A_{1,s}$ to $A_{0,s-1}$ maps into $\ker \epsilon_s = \text{Im } d_0$.

Step 4: Similarly, we can define $d_1(\alpha) = -h(d_1(d_0(\alpha)))$ for $\alpha \in A_{r,s}$ and $r \geq 1$. According to Theorem 2.3.1, there are maps d_k for $k \geq 2$ such that $\Sigma_{i=0}^k d_i d_{k-i} = 0$. In Section 2.3, it is proved that d_k satisfies the equation $d_0 d_k = f$ where $f = -\Sigma_{i=1}^k d_i d_{k-i}$ and f is in the kernel of d_0. To define a map d_k, we need to find the preimage of f under d_0. Note that the image of f is in the kernel of d_0. This is the case which we discuss in Step 2. By using a contracting homotopy, we get the result. Then $d_k = -h(\Sigma_{i=1}^k d_i d_{k-i})(\alpha)$ for $\alpha \in A_{r,s}$ and this completes the construction of free $\mathbb{Z}\Gamma$-resolution of $\mathbb{Z}$.

Step 5: Now we can calculate the cohomology of Γ. First we apply the functor $\text{Hom}_{\mathbb{Z}\Gamma}(-, \mathbb{Z})$ to the resulting resolution (A_{**}, d). Then we obtain a cochain complex of finitely generated $\mathbb{Z}$-free modules denoted by (F_*, δ_*).

Step 6: Now we consider the following extension

$$
0 \longrightarrow \ker \delta_{i+1}/\text{Im } \delta_i \longrightarrow F_{i+1}/\text{Im } \delta_i \longrightarrow \text{Im } \delta_{i+1} \longrightarrow 0.
$$

Since Im δ_{i+1} is a free $\mathbb{Z}$-module, then the above extension splits. So we have the isomorphism

$$F_{i+1}/\text{Im } \delta_i \cong H^{i+1}(\Gamma) \oplus \text{Im } \delta_{i+1}. \tag{4.1.3}$$

By using the above formula, we can calculate $H^{i+1}(\Gamma)$ for all $i \geq 0$. Note that each boundary map $\delta_i : F_i \to F_{i+1}$, for $0 \leq i \leq n+1$, is represented by a matrix given by the differentials d_k's. To make our work easy we use the Smith Normal Form (SNF) of these matrices which is a transformation of a matrix to the diagonal form.

4.2 Calculations for the Representation γ_2

As we mentioned earlier, the representation γ_2 gives a 4-dimensional counterexample for Conjecture 1.0.1. In this section, we show the detailed calculations for this example by using the steps given in Section 4.1. We consider $\Gamma = L \rtimes G$ where $L = \mathbb{Z}^4$, G is a cyclic group of order 4 with generator x. G is acting on L by a left multiplication given by the matrix

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}.$$

Proposition 4.2.1. *The cohomology group of* Γ *is given as in the following*

$$H^i(\Gamma) = \begin{cases} \mathbb{Z}, & i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2, & i = 2, 3; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2^3, & i \geq 4. \end{cases}$$

Proof. Note that we describe (B_*, d'), a free $\mathbb{Z}L$-resolution of $\mathbb{Z}$, as in Section 4.1. We consider the case of $n = 4$. Also we know the contracting homotopy ${}^4h$ given by

$$
{}^4h(t_1^{n_1}...t_4^{n_4}e_{i_1...i_m}) = \begin{cases} \sum_{j=0}^{i_1-1} A_j^4 e_{ji_1...i_m}, & \text{if } m > 0; \\ \\ \sum_{j=0}^{k} A_j^4 e_j, & \text{if } m = 0; \end{cases} \tag{4.2.1}
$$

where $A_j^4 = \frac{t_j^{n_j}-1}{t_j-1} t_{j+1}^{n_{j+1}}...t_4^{n_4}$ and $A_0^4 = 0$. By denoting $x = (1, M) \in L \rtimes G$, we write the standard $\mathbb{Z}G$-resolution (C_*, ∂) as it follows

$$
\cdots \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\Sigma x} \mathbb{Z}G \xrightarrow{x-1} \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0. \tag{4.2.2}
$$

In Section 4.1, we define

$$
D_s = \bigoplus_r A_{r,s} \cong \mathrm{Ind}_L^{\Gamma} B_* \otimes_{\mathbb{Z}G} \mathbb{Z}G \cong \mathrm{Ind}_L^{\Gamma} B_*
$$

and

$$
A_{**} = \bigoplus_{r,s} A_{r,s} \cong \bigoplus_s \mathrm{Ind}_L^{\Gamma} B_*.
$$

Now we use the notation of [9] and we add superscript to the generators of B_* to make clear which $A_{r,s}$ they belong. We say

$$
1 \otimes_L e^s_{i_1 i_2 \dots i_r} \in A_{r,s} \text{ for } r, s \geq 0 \text{ and } 1 \leq i_1 < \dots < i_r \leq 4,
$$

and

$$
ge^s_{i_1 i_2 \dots i_r} := g \otimes e^s_{i_1 i_2 \dots i_r}.
$$

So the resulting resolution is given by:

$$
\begin{array}{ccccccccc}
\mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \dashleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \dashleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \dashleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \dashleftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \dashleftarrow & \cdots \\
\downarrow \epsilon_0 & & \downarrow \epsilon_1 & & \downarrow \epsilon_2 & & \downarrow \epsilon_3 & & \\
\mathbb{Z}G & \xleftarrow{x-1} & \mathbb{Z}G & \xleftarrow{\Sigma x} & \mathbb{Z}G & \xleftarrow{x-1} & \mathbb{Z}G & \dashleftarrow & \cdots
\end{array}
$$

We have $d_k = -h(\Sigma_{i=1}^{k} d_i d_{k-i})$. By using this result, all the maps $d_k : A_{r,s} \to A_{r+k-1,s-k}$ can be calculated. For $s \geq 1$ and $k = 1$, we get the followings:

$$
\begin{aligned}
d_1(e^{2s-1}) =& (x-1)e \\
d_1(e^{2s}) =& (x^3 + x^2 + x + 1)e
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_1^{2s-1}) =& x t_1^{-1} e_1 + e_1 \\
d_1(e_2^{2s-1}) =& x t_4^{-1} e_4 + e_2 \\
d_1(e_3^{2s-1}) =& - x t_4 e_2 - x e_4 + e_3 \\
d_1(e_4^{2s-1}) =& x t_3^{-1} t_4 e_3 - x e_4 + e_4
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_1^{2s}) =& x^3 t_1^{-1} e_1 - x^2 e_1 + x t_1^{-1} e_1 - e_1 \\
d_1(e_2^{2s}) =& - x^3 t_3 e_2 - x^3 e_3 - x^2 t_4^{-1} e_3 + x^2 t_4^{-1} e_4 + x t_4^{-1} e_4 - e_2 \\
d_1(e_3^{2s}) =& x^3 t_2^{-1} t_4^{-1} e_2 + x^3 t_4^{-1} e_4 + x^2 t_3^{-1} e_3 - x t_4 e_2 - x e_4 - e_3 \\
d_1(e_4^{2s}) =& x^3 t_2^{-1} e_2 + x^2 t_2^{-1} t_3^{-1} e_2 + x^2 t_3^{-1} e_3 + x t_3^{-1} t_4 e_3 - x e_4 - e_4
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{12}^{2s-1}) =& xt_1^{-1}t_4^{-1}e_{14} - e_{12}\\
d_1(e_{13}^{2s-1}) =& -xt_1^{-1}t_4e_{12} - xt_1^{-1}e_{14} - e_{13}\\
d_1(e_{14}^{2s-1}) =& xt_1^{-1}t_3^{-1}t_4e_{13} - xt_1^{-1}e_{14} - e_{14}\\
d_1(e_{23}^{2s-1}) =& -e_{23} + xe_{24}\\
d_1(e_{24}^{2s-1}) =& -e_{24} - xt_3^{-1}e_{34}\\
d_1(e_{34}^{2s-1}) =& -xt_3^{-1}t_4^2e_{23} + xt_4e_{24} - e_{34} + xt_3^{-1}t_4e_{34}
\end{aligned}
$$

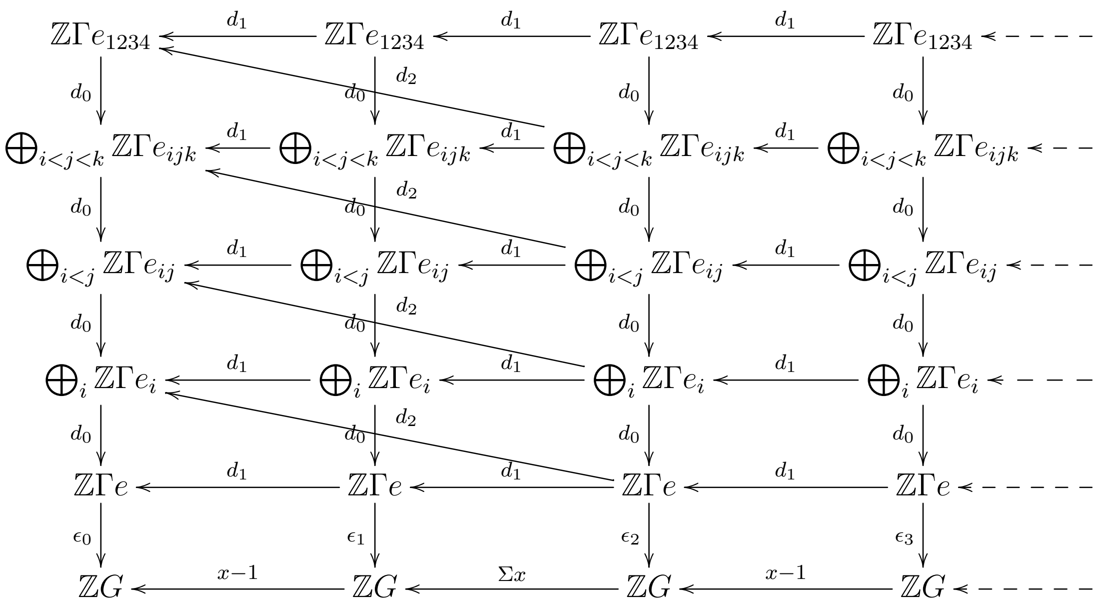
$$
\begin{aligned}
d_1(e_{12}^{2s}) =& -x^3t_1^{-1}t_3e_{12} - x^3t_1^{-1}e_{13} + x^2t_4^{-1}e_{13} - x^2t_4^{-1}e_{14} + xt_1^{-1}t_4^{-1}e_{14} + e_{12}\\
d_1(e_{13}^{2s}) =& x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{12} + x^3t_1^{-1}t_4^{-1}e_{14} - x^2t_3^{-1}e_{13} - xt_1^{-1}t_4e_{12} - xt_1^{-1}e_{14} + e_{13}\\
d_1(e_{14}^{2s}) =& x^3t_1^{-1}t_2^{-1}e_{12} - x^2t_2^{-1}t_3^{-1}e_{12} - x^2t_3^{-1}e_{13} + xt_1^{-1}t_3^{-1}t_4e_{13} - xt_1^{-1}e_{14} + e_{14}\\
d_1(e_{23}^{2s}) =& -x^3t_3t_4^{-1}e_{24} - x^3t_4^{-1}e_{34} + e_{23} + x^3t_2^{-1}t_4^{-1}e_{23} - x^2t_3^{-1}t_4^{-1}e_{34} + xe_{24}\\
d_1(e_{24}^{2s}) =& e_{24} + x^3t_2^{-1}e_{23} + x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{23} - x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{24} - x^2t_3^{-1}t_4^{-1}e_{34} - xt_3^{-1}e_{34}\\
d_1(e_{34}^{2s}) =& -xt_3^{-1}t_4^2e_{23} + xt_4e_{24} + e_{34} - x^3t_2^{-1}t_4^{-1}e_{24} - x^2t_2^{-1}t_3^{-2}e_{23} + xt_3^{-1}t_4e_{34}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{123}^{2s-1}) =& e_{123} + xt_1^{-1}e_{124}\\
d_1(e_{124}^{2s-1}) =& e_{124} - xt_1^{-1}t_3^{-1}e_{134}\\
d_1(e_{134}^{2s-1}) =& -xt_1^{-1}t_3^{-1}t_4^2e_{123} + xt_1^{-1}t_4e_{124} + e_{134} + xt_1^{-1}t_3^{-1}t_4e_{134}\\
d_1(e_{234}^{2s-1}) =& e_{234} - xt_3^{-1}t_4e_{234}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{123}^{2s}) =& -x^3t_1^{-1}t_3t_4^{-1}e_{124} - x^3t_1^{-1}t_4^{-1}e_{134} - e_{123} + x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{123}\\
&+ x^2t_3^{-1}t_4^{-1}e_{134} + xt_1^{-1}e_{124}\\
d_1(e_{124}^{2s}) =& -e_{124} + x^3t_1^{-1}t_2^{-1}e_{123} - x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{123} + x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{124}\\
&+ x^2t_3^{-1}t_4^{-1}e_{134} - xt_1^{-1}t_3^{-1}e_{134}\\
d_1(e_{134}^{2s}) =& -xt_1^{-1}t_3^{-1}t_4^2e_{123} + xt_1^{-1}t_4e_{124} - e_{134} - x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{124}\\
&+ x^2t_2^{-1}t_3^{-2}e_{123} + xt_1^{-1}t_3^{-1}t_4e_{134}\\
d_1(e_{234}^{2s}) =& -e_{234} - xt_3^{-1}t_4e_{234} - x^3t_2^{-1}t_4^{-1}e_{234} - x^2t_2^{-1}t_3^{-2}t_4^{-1}e_{234}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{1234}^{2s-1}) =& -e_{1234} - xt_1^{-1}t_3^{-1}t_4e_{1234}\\
d_1(e_{1234}^{2s}) =& e_{1234} - xt_1^{-1}t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{1234} + x^2t_2^{-1}t_3^{-2}t_4^{-1}e_{1234}
\end{aligned}
$$

For d_2 we have:



$$d_2(e^s) = 0$$

$$\begin{aligned}
d_2(e_1^{2s+1}) =& 0 \\
d_2(e_2^{2s+1}) =& -x^3 e_{23} + t_4^{-1} x^2 e_{34} \\
d_2(e_3^{2s+1}) =& t_2^{-1} e_{23} + x^3 t_2^{-1} t_3^{-1} t_4^{-1} e_{23} - x^3 t_2^{-1} t_3^{-1} t_4^{-1} e_{24} - x^3 t_3^{-1} t_4^{-1} e_{34} - x^2 t_3^{-1} e_{34} \\
d_2(e_4^{2s+1}) =& x^3 t_2^{-1} t_3^{-1} e23 - x^2 t_2^{-1} t_3^{-1} e_{24} - x^2 t_3^{-1} e_{34}
\end{aligned}$$

$$\begin{aligned}
d_2(e_1^{2s}) =& 0 \\
d_2(e_2^{2s}) =& e_{24} + x^2 t_4^{-1} e_{34} \\
d_2(e_3^{2s}) =& -x^3 t_2^{-1} t_4^{-1} e_{24} - x^2 t_3^{-1} e_{34} \\
d_2(e_4^{2s}) =& -x^2 t_2^{-1} t_3^{-1} e_{24} - x^2 t_3^{-1} e_{34}
\end{aligned}$$

$$\begin{aligned}
d_2(e_{12}^{2s+1}) =& -x^3t_1^{-1}e_{123} - x^2t_4^{-1}e_{134}\\
d_2(e_{13}^{2s+1}) =& -t_2^{-1}e_{123} + x^3t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}e_{123} - x^3t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}e_{124}\\
&- x^3t_1^{-1}t_3^{-1}t_4^{-1}e_{134} + t_3^{-1}x^2e_{134}\\
d_2(e_{14}^{2s+1}) =& x^3t_1^{-1}t_2^{-1}t_3^{-1}e_{123} + x^2t_2^{-1}t_3^{-1}e_{124} + x^2t_3^{-1}e_{134}\\
d_2(e_{23}^{2s+1}) =& x^3t_4^{-1}e_{234}\\
d_2(e_{24}^{2s+1}) =& -x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{234} - x^2t_2^{-1}t_3^{-1}e_{234}\\
d_2(e_{34}^{2s+1}) =& t_2^{-1}e_{234} + x^3t_2^{-2}t_3^{-2}t_4^{-1}e_{234} + x^2t_2^{-1}t_3^{-2}t_4e_{234}\\
&+ x^3t_2^{-1}t_3^{-1}t_4^{-1}e_{234} + x^2t_2^{-1}t_3^{-2}e_{234}
\end{aligned}$$

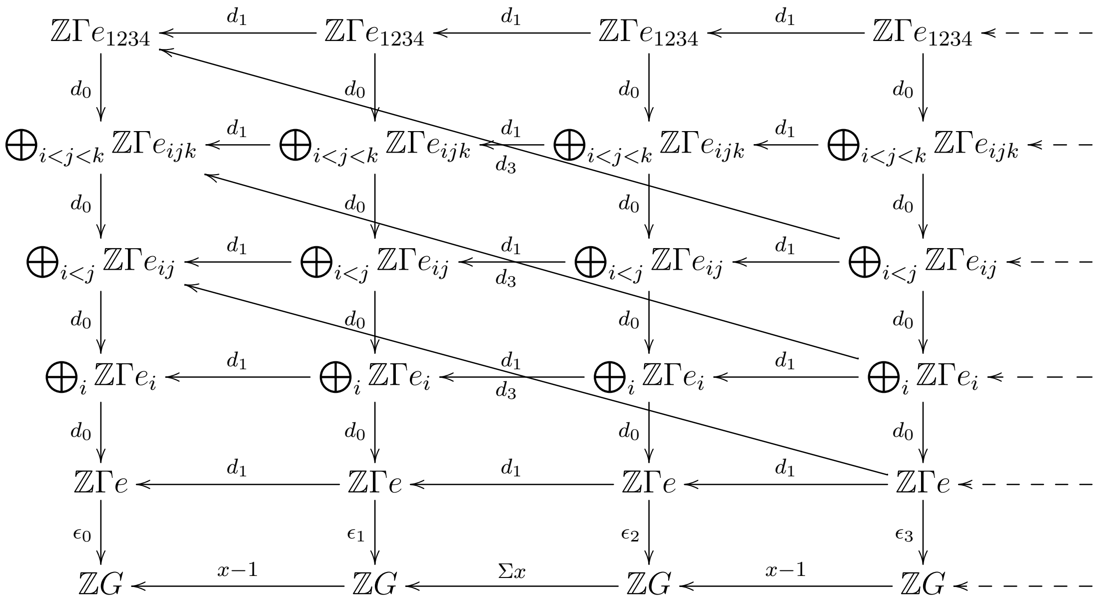
$$\begin{aligned}
d_2(e_{12}^{2s}) =& -e_{124} - x^2t_4^{-1}e_{134}\\
d_2(e_{13}^{2s}) =& -x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{124} + x^2t_3^{-1}e_{134}\\
d_2(e_{14}^{2s}) =& x^2t_2^{-1}t_3^{-1}e_{124} + x^2t_3^{-1}e_{134}\\
d_2(e_{23}^{2s}) =& -x^3t_2^{-1}t_4^{-1}e_{234} - e_{234}\\
d_2(e_{24}^{2s}) =& -x^2t_2^{-1}t_3^{-1}e_{234} - x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{234}\\
d_2(e_{34}^{2s}) =& x^2t_2^{-1}t_3^{-2}t_4e_{234} + x^2t_2^{-1}t_3^{-2}e_{234}
\end{aligned}$$

$$\begin{aligned}
d_2(e_{123}^{2s+1}) =& x^3t_1^{-1}t_4^{-1}e_{1234}\\
d_2(e_{124}^{2s+1}) =& x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{1234} + x^2t_2^{-1}t_3^{-1}e_{1234}\\
d_2(e_{134}^{2s+1}) =& -t_2^{-1}e_{1234} + x^3t_1^{-1}t_2^{-2}t_3^{-2}t_4^{-1}e_{1234} - x^2t_2^{-1}t_3^{-2}t_4e_{1234} + x^3t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}e_{1234}\\
&- x^2t_2^{-1}t_3^{-2}e_{1234}\\
d_2(e_{234}^{2s+1}) =& 0
\end{aligned}$$

$$\begin{aligned}
d_2(e_{123}^{2s}) =& -x^3t_1^{-1}t_2^{-1}t_4^{-1}e_{1234} + e_{1234}\\
d_2(e_{124}^{2s}) =& x^2t_2^{-1}t_3^{-1}e_{1234} + x^2t_2^{-1}t_3^{-1}t_4^{-1}e_{1234}\\
d_2(e_{134}^{2s}) =& -x^2t_2^{-1}t_3^{-2}t_4e_{1234} - x^2t_2^{-1}t_3^{-2}e_{1234}\\
d_2(e_{234}^{2s}) =& 0
\end{aligned}$$

$$d_2(e_{1234}^{s}) = 0$$

For d_3 we get:



$$d_3(e^s) = 0$$

$$\begin{aligned}
d_3(e_1^{2s+1}) =& 0 \\
d_3(e_2^{2s+1}) =& x^3 e_{234} \\
d_3(e_3^{2s+1}) =& -x^3 t_2^{-1} t_3^{-1} t_4^{-1} e_{234} - x^3 t_2^{-1} t_3^{-1} e_{234} \\
d_3(e_4^{2s+1}) =& -x^3 t_2^{-1} t_3^{-1} e_{234}
\end{aligned}$$

$$\begin{aligned}
d_3(e_1^{2s}) =& 0 \\
d_3(e_2^{2s}) =& x^3 e_{234} - x t_3^{-1} t_4^{-1} e234 \\
d_3(e_3^{2s}) =& -x^3 t_2^{-1} t_3^{-1} t_4^{-1} e_{234} - x^3 t_2^{-1} t_3^{-1} e_{234} \\
d_3(e_4^{2s}) =& x t_3^{-1} e_{234} - x^3 t_2^{-1} t_3^{-1} e_{234}
\end{aligned}$$

$$\begin{aligned}
d_3(e_{12}^{2s+1}) =& x^3 t_1^{-1} e_{1234} \\
d_3(e_{13}^{2s+1}) =& -x^3 t_1^{-1} t_2^{-1} t_3^{-1} t_4^{-1} e_{1234} - x^3 t_1^{-1} t_2^{-1} t_3^{-1} e_{1234} \\
d_3(e_{14}^{2s+1}) =& -x^3 t_1^{-1} t_2^{-1} t_3^{-1} e_{1234} \\
d_3(e_{23}^{2s+1}) =& d_3(e_{24}^{2s+1}) = d_3(e_{34}^{2s+1}) = 0
\end{aligned}$$

$$
\begin{aligned}
d_3(e_{12}^{2s}) =& x^3t_1^{-1}e_{1234} - xt_1^{-1}t_3^{-1}t_4^{-1}e_{1234} \\
d_3(e_{13}^{2s}) =& -x^3t_1^{-1}t_2^{-1}t_3^{-1}t_4^{-1}e_{1234} - x^3t_1^{-1}t_2^{-1}t_3^{-1}e_{1234} \\
d_3(e_{14}^{2s}) =& xt_1^{-1}t_3^{-1}e_{1234} - x^3t_1^{-1}t_2^{-1}t_3^{-1}e_{1234} \\
d_3(e_{23}^{2s}) =& d_3(e_{24}^{2s}) = d_3(e_{34}^{2s}) = 0
\end{aligned}
$$

Finally, we obtain $d_4 \equiv 0$. Now apply the functor $\mathrm{Hom}_{\mathbb{Z}\Gamma}(-, \mathbb{Z})$ to the resolution (A, d) and we named the resulting cochain complex by (F_*, δ_*) with the dimensions given below

$$
\dim(F_i) = \begin{cases} 1, & i = 0; \\ 5, & i = 2; \\ 11, & i = 3; \\ 15, & i = 4; \\ 16, & i \geq 4. \end{cases}
$$

By using the differentials d_1, d_2, and d_3 we can write matrices which represent the boundary maps δ_i for $0 \leq i \leq n+1$. During this process, we consider the lexicographical order of the generators. We find the matrices δ_1, δ_2, δ_3, δ_{2k}, and δ_{2k+1}. These matrices are given in Appendix A. Finally, we find the Smith Normal Form (SNF) of matrices as in the following:

	Diagonal of SNF
δ_0	$[0]$
δ_1	$[1, 1, 2, 4, 0]$
δ_2	$[1, 1, 1, 1, 2, 4, 0, 0, 0, 0, 0]$
δ_3	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0]$
δ_{2k}, $k \geq 2$	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0]$
δ_{2k+1}, $k \geq 2$	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0]$

Now we use the Formula 4.1.3 to calculate cohomology groups.

$$\begin{aligned}
\mathbb{Z}^5 =& H^1 \oplus \mathbb{Z}^4 \Rightarrow H^1 = \mathbb{Z} \\
\mathbb{Z}^7 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 =& H^2 \oplus \mathbb{Z}^6 \Rightarrow H^2 = \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
\mathbb{Z}^9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 =& H^3 \oplus \mathbb{Z}^8 \Rightarrow H^3 = \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^4 \oplus \mathbb{Z}^8 \Rightarrow H^4 = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^{2k+1} \oplus \mathbb{Z}^8 \Rightarrow H^{2k+1} = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^{2k+2} \oplus \mathbb{Z}^8 \Rightarrow H^{2k+2} = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3
\end{aligned}$$

This completes the proof. □

We computed the left hand side of the conjecture and now we consider the right hand side.

Proposition 4.2.2. *Right hand side of the conjecture is given by*

$$H^i(G, H^j(L,\Gamma)) = \begin{cases} \mathbb{Z}, & 0 \leq j \leq 3,\ i = 0; \\ \mathbb{Z}_2, & j = 1,\ i \geq 1; \\ \mathbb{Z}_2, & j = 2,\ i \geq 1,\ 2 | i; \\ \mathbb{Z}_4, & j = 2,\ i \geq 1,\ 2 \nmid i; \\ \mathbb{Z}_4, & j = 3,\ i \geq 1; \\ \mathbb{Z}_2, & j = 4,\ i \geq 1,\ 2 \nmid i; \\ \mathbb{Z}_4, & j = 0,\ i \geq 1,\ 2 | i; \\ 0, & \text{otherwise}. \end{cases}$$

Proof. In this proof, we use the similar way given in [9]. We observe that $H^1(L,\mathbb{Z}) \cong \mathrm{Hom}(L,\mathbb{Z}) \cong \mathbb{Z}^4$ generated by t_i, $1 \leq i \leq 4$. We can define $H^i(L,\mathbb{Z})$ as the j-th exterior power $\Lambda^j(H^1(L,\mathbb{Z}))$. Assume its generators are defined by

$$t_{i_1 \dots i_j} := t_{i_1} \wedge \dots \wedge t_{i_j} \text{ for } 1 \leq i_1 < \cdots < i_j \leq 4$$

and the action of G on $H^j(L,\mathbb{Z})$ is given by

$$M \cdot t_{i_1 \dots i_j} = M^T t_{i_1} \wedge M^T t_{i_2} \wedge \dots \wedge M^T t_{i_j}.$$

We have $t_i \wedge t_i = 0$ and $t_i \wedge t_j = -t_j \wedge t_i$. By taking the generators in lexicographical order, we get the following matrices for the action of M on $H^j(L,\mathbb{Z})$:

$$j=1: \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 \end{pmatrix}$$

$$j=2: \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

$$j=3: \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$j=4: (-1)$$

After the application of the functor $\mathrm{Hom}_{\mathbb{Z}G}(-, H^j(L,\mathbb{Z}))$ to the resolution 4.2.2, we get a cochain complex given by

$$0 \longrightarrow H^j(L,\mathbb{Z}) \xrightarrow{M-I} H^j(L,\mathbb{Z}) \xrightarrow{\Sigma M} H^j(L,\mathbb{Z}) \xrightarrow{M-I} \cdots .$$

Again by using Formula 4.1.3 and SNF of matrices we get the result. Now we consider the cases:

$$j = 0 : 0 \longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{\times 0} \cdots$$
$$H^i = (\mathbb{Z}, 0, \mathbb{Z}_4, 0, \mathbb{Z}_4, ...)$$

$$j = 1 : 0 \longrightarrow \mathbb{Z}^4 \xrightarrow{M-I} \mathbb{Z}^4 \xrightarrow{\Sigma M} \mathbb{Z}^4 \xrightarrow{M-I} \cdots$$
$$\mathrm{SNF}(M-I) = \mathrm{diag}([1,1,2,0])_{4\times 4}$$
$$\mathrm{SNF}(\Sigma M) = \mathrm{diag}([2,0,0,0])_{4\times 4}$$
$$H^i = (\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, ...)$$

$$j = 2 : 0 \longrightarrow \mathbb{Z}^6 \xrightarrow{M-I} \mathbb{Z}^6 \xrightarrow{\Sigma M} \mathbb{Z}^6 \xrightarrow{M-I} \cdots$$
$$\mathrm{SNF}(M-I) = \mathrm{diag}([1,1,1,1,4,0])_{6\times 6}$$
$$\mathrm{SNF}(\Sigma M) = \mathrm{diag}([2,0,0,0,0,0])_{6\times 6}$$
$$H^i = (\mathbb{Z}, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2, ...)$$

$$j = 3 : 0 \longrightarrow \mathbb{Z}^4 \xrightarrow{M-I} \mathbb{Z}^4 \xrightarrow{\Sigma M} \mathbb{Z}^4 \xrightarrow{M-I} \cdots$$
$$\mathrm{SNF}(M-I) = \mathrm{diag}([1,1,4,0])_{4\times 4}$$
$$\mathrm{SNF}(\Sigma M) = \mathrm{diag}([4,0,0,0])_{4\times 4}$$
$$H^i = (\mathbb{Z}, \mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_4, ...)$$

$$j = 4 : 0 \longrightarrow \mathbb{Z} \xrightarrow{M-I} \mathbb{Z} \xrightarrow{\Sigma M} \mathbb{Z} \xrightarrow{M-I} \cdots$$
$$\mathrm{SNF}(M-I) = \mathrm{diag}[2]$$
$$\mathrm{SNF}(\Sigma M) = \mathrm{diag}[0]$$
$$H^i = (0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, ...)$$

This finishes the proof. $\square$

Proposition 4.2.2 gives the following cohomology groups:

$$\begin{aligned} H^1 &= \mathbb{Z} \\ H^2 &= \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ H^3 &= \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\ H^4 &= \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2 \\ H^{2k+1} &= \mathbb{Z}_4^2 \oplus \mathbb{Z}_2 \\ H^{2k+2} &= \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2 \end{aligned}$$

for $k \geq 2$. As a result, we see there is a contradiction between Proposition 4.2.1 and Proposition 4.2.2. The first conflict is that

$$\bigoplus_{i+j=4} H^i(G, H^j(L, \mathbb{Z})) = \mathbb{Z}_4^2 \oplus \mathbb{Z}_2^2$$

$$H^4(L \rtimes G, \mathbb{Z}) = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3.$$

This is a counterexample for Conjecture 1.0.1.

4.3 Calculations for the Representation ρ_8

In this section, we make calculation for the semidirect product $\Gamma = L \rtimes G$ where G is a cyclic group of order 4 and G-action is given by the representation ρ_8. As we mentioned, it satisfies the conjecture of Adem-Ge-Pan-Petrosyan. The action of G on $L = \mathbb{Z}^4$ is given by a left multiplication by the matrix

$$M = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proposition 4.3.1. *The cohomology group of Γ is given as in the following*

$$H^i(\Gamma) = \begin{cases} \mathbb{Z}, & i = 1; \\ \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2, & i = 2, 3; \\ \mathbb{Z}_4 \oplus \mathbb{Z}_2^3, & i \geq 4. \end{cases}$$

Proof. By using the same notation with the Proposition 4.2.1, we have the free resolution of Γ as it follows:

$$\begin{array}{ccccccccc}
\mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \xleftarrow{d_1} & \mathbb{Z}\Gamma e_{1234} & \leftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \xleftarrow{d_1} & \bigoplus_{i<j<k} \mathbb{Z}\Gamma e_{ijk} & \leftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \xleftarrow{d_1} & \bigoplus_{i<j} \mathbb{Z}\Gamma e_{ij} & \leftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \xleftarrow{d_1} & \bigoplus_{i} \mathbb{Z}\Gamma e_{i} & \leftarrow & \cdots \\
\downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \downarrow d_0 & & \\
\mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \xleftarrow{d_1} & \mathbb{Z}\Gamma e & \leftarrow & \cdots \\
\downarrow \epsilon_0 & & \downarrow \epsilon_1 & & \downarrow \epsilon_2 & & \downarrow \epsilon_3 & & \\
\mathbb{Z}G & \xleftarrow{x-1} & \mathbb{Z}G & \xleftarrow{\Sigma x} & \mathbb{Z}G & \xleftarrow{x-1} & \mathbb{Z}G & \leftarrow & \cdots
\end{array}$$

We have the contracting homotopy 4.2.1. Again by proceeding the same method, we find the differentials. However, in this case we get nonzero results for d_4. For $s \geq 1$ and $k = 1$, we get the followings:

$$\begin{aligned}
d_1(e^{2s-1}) =&(x-1)e \\
d_1(e^{2s}) =&(x^3+x^2+x+1)e
\end{aligned}$$

$$\begin{aligned}
d_1(e_1^{2s-1}) =&xt_3^{-1}t_4e_3 - xe_4 + e_1 \\
d_1(e_2^{2s-1}) =&-xt_3^{-1}t_4e_1 + xt_3^{-1}t_4e_3 - xe_4 + e_2 \\
d_1(e_3^{2s-1}) =&-xt_3^{-1}e_2 + xt_3^{-1}e_3 + e_3 \\
d_1(e_4^{2s-1}) =&-xe_4 + e_4
\end{aligned}$$

$$\begin{aligned}
d_1(e_1^{2s}) =&x^3t_1^{-1}t_2e_1 - x^3e_2 + x^2t_2^{-1}t_3t_4e_2 - x^2t_4e_3 - x^2e_4 + xt_3^{-1}t_4e_3 - xe_4 - e_1 \\
d_1(e_2^{2s}) =&x^3t_1^{-1}t_3t_4e_1 - x^3t_4e_3 - x^3e_4 + x^2t_2^{-1}t_4^2e_2 - x^2t_4e_4 - x^2e_4 - xt_3^{-1}t_4e_1 \\
&+ xt_3^{-1}t_4e_3 - xe_4 - e_2 \\
d_1(e_3^{2s}) =&x^3t_1^{-1}t_4e_1 - x^3e_4 - x^2t_2^{-1}t_4e_1 + x^2t_2^{-1}t_4e_2 - x^2e_4 - xt_3^{-1}e_2 + xt_3^{-1}e_3 - e_3 \\
d_1(e_4^{2s}) =&-(x^3+x^2+x+1)e_4
\end{aligned}$$

$$
\begin{aligned}
d_1(e_{12}^{2s-1}) =& - e_{12} + xt_3^{-2}t_4^2e_{13} - xt_3^{-1}t_4e_{14} \\
d_1(e_{13}^{2s-1}) =& - e_{13} + xt_3^{-2}t_4e_{23} - xt_3^{-1}e_{24} + xt_3^{-1}e_{34} \\
d_1(e_{14}^{2s-1}) =& - xt_3^{-1}t_4e_{34} - e_{14} \\
d_1(e_{23}^{2s-1}) =& xt_3^{-2}t_4e_{12} - xt_3^{-2}t_4e_{13} - e_{23} + xt_3^{-2}t_4e_{23} - xt_3^{-1}e_{24} + xt_3^{-1}e_{34} \\
d_1(e_{24}^{2s-1}) =& xt_3^{-1}t_4e_{14} - xt_3^{-1}t_4e_{34} - e_{24} \\
d_1(e_{34}^{2s-1}) =& xt_3^{-1}e_{24} - xt_3^{-1}e_{34} - e_{34}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{12}^{2s}) =& - x^3t_1^{-1}t_2t_4e_{13} + x^3t_4e_{23} - x^3t_1^{-1}t_2e_{14} + x^3e_{24} - x^2t_2^{-1}t_3t_4^2e_{24} + x^2t_4^2e_{34} \\
& - x^2t_2^{-1}t_3t_4e_{24} + x^2t_4e_{34} + e_{12} + x^3t_1^{-1}t_3t_4e_{12} + x^2t_2^{-1}t_4^3e_{23} \\
& + x^2t_2^{-1}t_4^2e_{24} + xt_3^{-2}t_4^2e_{13} - xt_3^{-1}t_4e_{14} \\
d_1(e_{13}^{2s}) =& - x^3t_1^{-1}t_2e_{14} + x^3e_{24} - x^2t_2^{-1}t_3t_4e_{24} + x^2t_4e_{34} + e_{13} + x^3t_1^{-1}t_4e_{12} \\
& + x^2t_2^{-2}t_3t_4^2e_{12} - x^2t_2^{-1}t_4^2e_{13} + x^2t_2^{-1}t_4^2e_{23} - x^2t_2^{-1}t_4e_{14} \\
& + x^2t_2^{-1}t_4e_{24} + xt_3^{-2}t_4e_{23} - xt_3^{-1}e_{24} + xt_3^{-1}e_{34} \\
d_1(e_{14}^{2s}) =& - x^3t_1^{-1}t_2e_{14} + x^3e_{24} - x^2t_2^{-1}t_3t_4e_{24} + x^2t_4e_{34} - xt_3^{-1}t_4e_{34} + e_{14} \\
d_1(e_{23}^{2s}) =& - x^3t_1^{-1}t_3t_4e_{14} + x^3t_4e_{34} + xt_3^{-2}t_4e_{12} - xt_3^{-2}t_4e_{13} + e_{23} \\
& + x^3t_1^{-1}t_4^2e_{13} + x^3t_1^{-1}t_4e_{14} + x^2t_2^{-2}t_4^3e_{12} - x^2t_2^{-1}t_4^2e_{14} \\
& - x^2t_2^{-1}t_4e_{14} + x^2t_2^{-1}t_4e_{24} + xt_3^{-2}t_4e_{23} - xt_3^{-1}e_{24} + xt_3^{-1}e_{34} \\
d_1(e_{24}^{2s}) =& - x^3t_1^{-1}t_3t_4e_{14} + x^3t_4e_{34} - x^2t_2^{-1}t_4^2e_{24} + xt_3^{-1}t_4e_{14} - xt_3^{-1}t_4e_{34} + e_{24} \\
d_1(e_{34}^{2s}) =& - x^3t_1^{-1}t_4e_{14} + x^2t_2^{-1}t_4e_{14} - x^2t_2^{-1}t_4e_{24} + xt_3^{-1}e_{24} - xt_3^{-1}e_{34} + e_{34}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{123}^{2s+1}) =& e_{123} + xt_3^{-3}t_4^2e_{123} - xt_3^{-2}t_4e_{124} + xt_3^{-2}t_4e_{134} \\
d_1(e_{124}^{2s+1}) =& e_{124} - xt_3^{-2}t_4^2e_{134} \\
d_1(e_{134}^{2s+1}) =& e_{134} - xt_3^{-2}t_4e_{234} \\
d_1(e_{234}^{2s+1}) =& - xt_3^{-2}t_4e_{124} + xt_3^{-2}t_4e_{134} + e_{234} - xt_3^{-2}t_4e_{234}
\end{aligned}
$$

$$
\begin{aligned}
d_1(e_{123}^{2s}) =& x^3t_1^{-1}t_2t_4e_{134} - x^3t_4e_{234} - e_{123} - x^3t_1^{-1}t_3t_4e_{124} + xt_3^{-3}t_4^2e_{123} - xt_3^{-2}t_4e_{124} \\
& + xt_3^{-2}t_4e_{134} + x^3t_1^{-1}t_4^2e_{123} + x^3t_1^{-1}t_4e_{124} + x^2t_2^{-2}t_3t_4^3e_{124} - x^2t_2^{-1}t_4^3e_{134} \\
& + x^2t_2^{-2}t_3t_4^2e_{124} - x^2t_2^{-1}t_4^2e_{134} - x^2t_2^{-2}t_4^4e_{123} - x^2t_2^{-2}t_4^3e_{124} + x^2t_2^{-1}t_4^2e_{234} \\
d_1(e_{124}^{2s}) =& x^3t_1^{-1}t_2t_4e_{134} - e_{124} - x^3t_1^{-1}t_3t_4e_{124} - x^2t_2^{-1}t_4^3e_{234} - xt_3^{-2}t_4^2e_{134} - x^3t_4e_{234} \\
d_1(e_{134}^{2s}) =& - e_{134} - x^3t_1^{-1}t_4e_{124} - x^2t_2^{-2}t_3t_4^2e_{124} + x^2t_2^{-1}t_4^2e_{134} - x^2t_2^{-1}t_4^2e_{234} - xt_3^{-2}t_4e_{234} \\
d_1(e_{234}^{2s}) =& - xt_3^{-2}t_4e_{124} + xt_3^{-2}t_4e_{134} - e_{234} - x^3t_1^{-1}t_4^2e_{134} - x^2t_2^{-2}t_4^3e_{124} - xt_3^{-2}t_4e_{234}
\end{aligned}
$$

$$d_1(e_{1234}^{2s+1}) = - e_{1234} - xt_3^{-3}t_4^2e_{1234}$$
$$d_1(e_{1234}^{2s}) = e_{1234} - xt_3^{-3}t_4^2e_{1234} + x^2t_2^{-2}t_4^4e_{1234} - x^3t_1^{-1}t_4^2e_{1234}$$

For d_2 we have:

$$d_2(e^s) = 0$$

$$d_2(e_1^{2s+1}) = e_{14} + x^3t_1^{-1}t_4e_{12} - x^3t_1^{-1}t_2e_{14} + x^3e_{24} + x^2t_2^{-1}t_4e_{23}$$
$$d_2(e_2^{2s+1}) = - e_{12} - x^3t_1^{-1}t_3t_4e_{12} + x^3t_1^{-1}t_2t_4e_{13} - x^3t_4e_{23} - x^2t_2^{-1}t_4^2e_{23} + x^2t_2^{-1}t_3t_4e_{24}$$
$$- x^2t_4e_{34} + e_{14} + x^3t_1^{-1}t_4e_{12} + x^2t_2^{-1}t_4e_{23}$$
$$d_2(e_3^{2s+1}) = - e_{13} - x^3t_1^{-1}t_4e_{12} + x^3t_1^{-1}t_2e_{14} - x^3e_{24} - x^2t_2^{-1}t_4e_{23} + x^2t_2^{-1}t_3e_{24}$$
$$- x^2e_{34} + x^3t_1^{-1}e_{12} + x^2t_2^{-1}e_{23}$$
$$d_2(e_4^{2s+1}) = 0$$

$$d_2(e_1^{2s}) = - t_3^{-1}t_4e_{13} + e_{14} + x^3t_1^{-1}t_2t_3^{-1}t_4e_{13} - x^3t_3^{-1}t_4e_{23} - x^3t_1^{-1}t_2e_{14} + x^3e_{24} + x^2t_2^{-1}t_4e_{23}$$
$$d_2(e_2^{2s}) = - t_3^{-1}t_4e_{23} + e_{24} + x^3t_1^{-1}t_4^2e_{13} - x^3t_1^{-1}t_3t_4e_{14} + x^3t_4e_{34} - x^2t_2^{-1}t_4^2e_{23}$$
$$+ x^2t_2^{-1}t_3t_4e_{24} - x^2t_4e_{34} + x^2t_2^{-1}t_4e_{23}$$
$$d_2(e_3^{2s}) = e_{34} - x^2t_2^{-1}t_4e_{23} + x^2t_2^{-1}t_3e_{24} - x^2e_{34} + x^2t_2^{-1}e_{23}$$
$$d_2(e_4^{2s}) = 0$$

$$d_2(e_{12}^{2s+1}) = - t_1e_{124} - x^3t_1^{-2}t_2t_3t_4e_{124} - x^3t_1^{-1}t_2t_3t_4e_{124} + x^3t_1^{-2}t_2^2t_4e_{134}$$
$$+ x^3t_1^{-1}t_2^2t_4e_{134} - x^3t_2t_4e_{234} + x^3t_1^{-2}t_2t_4^2e_{123} + x^3t_1^{-1}t_2t_4^2e_{123}$$
$$+ x^3t_1^{-2}t_2t_4e_{124} + x^3t_1^{-1}t_2t_4e_{124} + x^2t_2^{-2}t_3t_4^2e_{234} - x^3t_1^{-1}t_2t_4e_{134}$$
$$+ x^2t_2^{-1}t_4^2e_{234} - x^3t_1^{-1}t_2t_4e_{123}$$
$$d_2(e_{13}^{2s+1}) = - t_1e_{134} + x^3t_1^{-2}t_2e_{124} + x^2t_2^{-2}t_3t_4e_{234} + x^2t_2^{-1}t_4e_{234} + x^2t_2^{-2}t_4^2e_{123}$$
$$- x^3t_1^{-1}e_{124} - x^2t_2^{-1}e_{234}$$
$$d_2(e_{14}^{2s+1}) = x^3t_1^{-1}t_4e_{124} + x^2t_2^{-1}t_4e_{234}$$
$$d_2(e_{23}^{2s+1}) = - t_1e_{123} - e_{123} - x^3t_1^{-2}t_2t_3t_4e_{124} - x^3t_1^{-1}t_2t_3t_4e_{124} + x^3t_1^{-2}t_2^2t_4e_{134}$$
$$+ x^3t_1^{-1}t_2^2t_4e_{134} - x^3t_2t_4e_{234} - x^3t_4e_{234} + x^3t_1^{-2}t_2t_4^2e_{123} + x^3t_1^{-1}t_2t_4^2e_{123}$$
$$+ x^3t_1^{-2}t_2t_4e_{124} + x^3t_1^{-1}t_2t_4e_{124} - x^3t_1^{-2}t_2t_4e_{123} - x^3t_1^{-1}t_2t_4e_{123}$$
$$- t_1e_{134} + x^3t_1^{-1}t_4e_{124} + x^2t_2^{-1}t_4e_{234} - x^3t_1^{-1}e_{124} - x^2t_2^{-1}e_{234} - x^3t_1^{-1}t_3t_4e_{124}$$
$$+ x^3t_1^{-1}t_4^2e_{123} - x^2t_2^{-2}t_4^3e_{123} + x^2t_2^{-2}t_3t_4^2e_{124} + x^2t_2^{-2}t_4^2e_{123} - x^2t_2^{-1}t_4^2e_{134}$$

$$
\begin{aligned}
d_2(e_{24}^{2s+1}) =& -e_{124} - x^3t_1^{-1}t_3t_4e_{124} + x^3t_1^{-1}t_2t_4e_{134} - x^2t_2^{-1}t_4^2e_{234} + x^3t_1^{-1}t_4e_{124} \\
&+ x^2t_2^{-1}t_4e_{234} - x^3t_4e_{234} \\
d_2(e_{34}^{2s+1}) =& -e_{134} - x^3t_1^{-1}t_4e_{124} - x^2t_2^{-1}t_4e_{234} + x^3t_1^{-1}e_{124} + x^2t_2^{-1}e_{234}
\end{aligned}
$$

$$
\begin{aligned}
d_2(e_{12}^{2s}) =& t_3^{-2}t_4^2e_{123} - t_3^{-1}t_4e_{124} + x^3t_1^{-1}t_2t_3^{-1}t_4^2e_{134} - x^3t_3^{-1}t_4^2e_{234} \\
&+ x^3t_1^{-1}t_2t_3^{-1}t_4e_{134} - x^3t_3^{-1}t_4e_{234} + x^3t_1^{-1}t_3^{-1}t_4^3e_{123} - x^3t_1^{-1}t_4^2e_{124} \\
&+ x^2t_2^{-1}t_4^2e_{234} + t_3^{-1}t_4e_{123} - e_{124} + x^3t_1^{-1}t_2t_4e_{134} - x^3t_4e_{234} + x^3t_1^{-1}t_4^2e_{123} \\
&- x^3t_1^{-1}t_3t_4e_{124} + x^2t_2^{-2}t_3t_4^2e_{234} \\
d_2(e_{13}^{2s}) =& -t_3^{-1}t_4e_{134} + x^3t_1^{-1}t_2t_3^{-1}t_4e_{134} - x^3t_3^{-1}t_4e_{234} + x^2t_2^{-2}t_3t_4e_{234} \\
&+ x^2t_2^{-1}t_4e_{234} - x^2t_2^{-1}e_{234} - e_{134} + x^3t_1^{-1}t_3^{-1}t_4^2e_{123} - x^3t_1^{-1}t_4e_{124} + x^2t_2^{-2}t_4^2e_{123} \\
d_2(e_{14}^{2s}) =& -t_3^{-1}t_4e_{134} + x^3t_1^{-1}t_2t_3^{-1}t_4e_{134} - x^3t_3^{-1}t_4e_{234} + x^2t_2^{-1}t_4e_{234} \\
d_2(e_{23}^{2s}) =& -t_3^{-1}t_4e_{234} + x^2t_2^{-1}t_4e_{234} - x^2t_2^{-1}e_{234} - e_{234} - x^2t_2^{-2}t_4^3e_{123} \\
&+ x^2t_2^{-2}t_3t_4^2e_{124} - x^2t_2^{-1}t_4^2e_{134} + x^2t_2^{-2}t_4^2e_{123} \\
d_2(e_{24}^{2s}) =& -t_3^{-1}t_4e_{234} + x^3t_1^{-1}t_4^2e_{134} - x^2t_2^{-1}t_4^2e_{234} + x^2t_2^{-1}t_4e_{234} \\
d_2(e_{34}^{2s}) =& -x^2t_2^{-1}t_4e_{234} + x^2t_2^{-1}e_{234}
\end{aligned}
$$

$$
\begin{aligned}
d_2(e_{123}^{2s+1}) =& t_1^2e_{1234} + x^3t_1^{-3}t_2^2t_4e_{1234} + x^3t_1^{-2}t_2^2t_4e_{1234} + x^3t_1^{-2}t_2t_4^2e_{1234} - x^3t_1^{-2}t_2t_4e_{1234} \\
&- x^3t_1^{-1}t_2t_4e_{1234} - x^3t_1^{-1}t_2t_4e_{1234} - x^2t_2^{-3}t_3t_4^3e_{1234} - x^2t_2^{-2}t_4^3e_{1234} \\
d_2(e_{124}^{2s+1}) =& x^3t_1^{-2}t_2t_4^2e_{1234} + x^3t_1^{-1}t_2t_4^2e_{1234} - x^3t_1^{-1}t_2t_4e_{1234} \\
d_2(e_{134}^{2s+1}) =& x^2t_2^{-2}t_4^2e_{1234} \\
d_2(e_{234}^{2s+1}) =& x^3t_1^{-2}t_2t_4^2e_{1234} + x^3t_1^{-1}t_2t_4^2e_{1234} - t_1e_{1234} - e_{1234} - x^3t_1^{-2}t_2t_4e_{1234} \\
&- x^3t_1^{-1}t_2t_4e_{1234} - x^2t_2^{-2}t_4^3e_{1234} + x^2t_2^{-2}t_4^2e_{1234} + x^3t_1^{-1}t_4^2e_{1234}
\end{aligned}
$$

$$
\begin{aligned}
d_2(e_{123}^{2s}) =& e_{1234} + t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} - x^2t_2^{-2}t_4^3e_{1234} - x^2t_2^{-3}t_3t_4^3e_{1234} \\
&+ t_3^{-2}t_4^2e_{1234} \\
d_2(e_{124}^{2s}) =& t_3^{-2}t_4^2e_{1234} + x^3t_1^{-1}t_3^{-1}t_4^3e_{1234} + t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_4^2e_{1234} \\
d_2(e_{134}^{2s}) =& x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} + x^2t_2^{-2}t_4^2e_{1234} \\
d_2(e_{234}^{2s}) =& -x^2t_2^{-2}t_4^3e_{1234} + x^2t_2^{-2}t_4^2e_{1234}
\end{aligned}
$$

$$
d_2(e_{1234}^{s}) = 0
$$

For d_3 we get:

$$d_3(e^s) = 0$$

$$\begin{aligned}
d_3(e_1^{2s+1}) =& -x^3t_1^{-1}t_4e_{123} + x^3t_1^{-1}t_3e_{124} + x^3e_{234} - x^3t_1^{-1}t_2e_{134} + x^3t_1^{-1}e_{123} - t_3^{-1}t_4e_{134} \\
& + x^3t_1^{-1}t_3^{-1}t_4e_{123} - x^3t_1^{-1}e_{124} + x^3t_1^{-1}e_{134} \\
d_3(e_2^{2s+1}) =& -t_3^{-1}t_4e_{123} + e_{124} - x^3t_1^{-1}t_4e_{123} - x^3t_1^{-1}t_4e_{123} + x^3t_1^{-1}t_3e_{124} - x^3t_1^{-1}t_2e_{134} \\
& + x^3t_1^{-1}e_{123} - t_3^{-2}t_4^2e_{123} + t_3^{-1}t_4e_{124} - x^3t_1^{-1}t_3^{-1}t_4^2e_{123} + x^3t_1^{-1}t_4e_{124} - x^3t_1^{-1}t_4e_{134} \\
& - x^3t_1^{-1}t_2t_3^{-1}t_4e_{134} + x^3t_3^{-1}t_4e_{234} + x^3t_1^{-1}t_3^{-1}t_4e_{123} - x^3t_1^{-1}e_{124} \\
& + x^3t_1^{-1}e_{134} + x^3e_{234} - t_3^{-1}t_4e_{134} \\
d_3(e_3^{2s+1}) =& e_{134} + x^3t_1^{-1}t_4e_{123} - x^3t_1^{-1}t_3e_{124} + x^3t_1^{-1}t_2e_{134} - x^3e_{234} - x^3t_1^{-1}e_{123} - x^3t_1^{-1}e_{123} \\
& + x^3t_1^{-1}t_3t_4^{-1}e_{124} - x^3t_1^{-1}t_2t_4^{-1}e_{134} + x^3t_4^{-1}e_{234} + x^3t_1^{-1}t_4^{-1}e_{123} + t_3^{-1}t_4e_{134} \\
& - x^3t_1^{-1}t_3^{-1}t_4e_{123} + x^3t_1^{-1}e_{124} - x^3t_1^{-1}e_{134} - x^3t_1^{-1}t_2t_3^{-1}e_{134} + x^3t_3^{-1}e_{234} \\
& + x^3t_1^{-1}t_3^{-1}e_{123} - x^3t_1^{-1}t_4^{-1}e_{124} + x^3t_1^{-1}t_4^{-1}e_{134} \\
d_3(e_4^{2s+1}) =& 0
\end{aligned}$$

$$\begin{aligned}
d_3(e_1^{2s}) =& t_2^{-1}t_4^2e_{123} - t_2^{-1}t_3t_4e_{124} + t_4e_{134} - t_2^{-1}t_4e_{123} + e_{124} - x^3t_1^{-1}t_4e_{123} + x^3t_1^{-1}t_3e_{124} \\
& - x^3t_1^{-1}t_2e_{134} + x^3e_{234} + x^3t_1^{-1}e_{123} + x^3e_{124} + x^2t_1^{-1}t_3t_4e_{124} - x^2t_1^{-1}t_2t_4e_{134} \\
& - x^2t_1^{-1}t_4^2e_{123} - x^2t_1^{-1}t_4e_{124} - xt_2^{-1}t_4e_{234} - x^2t_2^{-1}t_3t_4e_{124} - xt_1^{-1}t_4^2e_{124} \\
& + xt_1^{-1}t_2t_3^{-1}t_4^2e_{134} - xt_3^{-1}t_4^2e_{234} + xt_1^{-1}t_3^{-1}t_4^3e_{123} + xt_1^{-1}t_3^{-1}t_4^2e_{124} - t_2^{-1}t_3^{-1}t_4^2e_{123} \\
& + t_2^{-1}t_4e_{124} - t_2^{-1}t_4e_{134} + t_2^{-1}t_4e_{234} + x^2t_4e_{134} - xt_1^{-1}t_4e_{124} - t_2^{-1}t_4e_{234} \\
& - t_3^{-1}t_4e_{134} - t_3^{-1}t_4e_{134} + x^2t_4e_{234} + x^3t_1^{-1}t_3^{-1}t_4e_{123} - x^3t_1^{-1}e_{124} + x^3t_1^{-1}e_{134} \\
d_3(e_2^{2s}) =& xt_3^{-1}t_4^2e_{123} + xt_3^{-1}t_4e_{124} + t_4e_{234} - xt_3^{-1}t_4e_{124} + e_{124} - x^3t_1^{-1}t_4e_{123} + e_{124} \\
& + x^3t_1^{-1}t_3e_{124} - x^3t_1^{-1}t_2e_{134} + x^3e_{234} + x^3t_1^{-1}e_{123} + x^3t_4e_{134} - x^2t_1^{-1}t_4e_{124} \\
& - x^2t_2^{-1}t_4^2e_{124} - xt_3^{-1}t_4^2e234 - xt_2^{-1}t_4^2e_{124} + xt_3^{-1}t_4^2e_{134} - t_3^{-1}t_4e_{234} + xt_3^{-1}t_4^2e_{134} \\
& - x^3t_1^{-1}t_3^{-1}t_4^2e_{123} + x^3t_1^{-1}t_4e_{124} - x^3t_1^{-1}t_4e_{134} + t_2^{-1}t_4^2e_{124} - x^3t_1^{-1}t_2t_3^{-1}t_4e_{134} \\
& + x^3t_3^{-1}t_4e_{234} - xt_3^{-1}t_4e_{134} + x^3t_1^{-1}t_3^{-1}t_4e_{123} - x^3t_1^{-1}e_{124} + x^3t_1^{-1}e_{134} \\
& - x^3t_1^{-1}t_4e_{123} - xt_2^{-1}t_4e_{234} - xt_2^{-1}t_3^{-1}t_4^2e_{123}
\end{aligned}$$

$$
\begin{aligned}
d_3(e_3^{2s}) =& - x^2t_2^{-1}t_4e_{124} + xt_3^{-1}t_4e_{123} + xt_3^{-1}e_{124} + e_{234} - e_{123} - t_4^{-1}e_{124} + x^3t_1^{-1}t_4e_{123} \\
& - x^3t_1^{-1}t_3e_{124} + x^3t_1^{-1}t_2e_{134} - x^3e_{234} + t_4^{-1}e_{124} - x^3t_1^{-1}e_{123} + t_4^{-1}e_{124} \\
& + x^3t_1^{-1}t_3t_4^{-1}e_{124} - x^3t_1^{-1}t_2t_4^{-1}e_{134} + x^3t_1^{-1}t_4^{-1}e_{123} + xt_3^{-1}t_4e_{134} \\
& + x^3t_1^{-1}e_{124} - x^3t_1^{-1}e_{134} + t_2^{-1}t_4e_{124} - x^3t_1^{-1}t_2t_3^{-1}e_{134} + x^3t_3^{-1}e_{234} \\
& + x^3t_1^{-1}t_3^{-1}e_{123} - x^3t_1^{-1}t_4^{-1}e_{124} + x^3t_1^{-1}t_4^{-1}e_{134} + x^3t_4^{-1}e_{234} - x^3t_1^{-1}e_{123} \\
& - x^3t_1^{-1}t_3^{-1}t_4e_{123} - xt_3^{-1}e_{134} \\
d_3(e_4^{2s}) =& 0 \\
d_3(e_{12}^{2s+1}) =& x^3t_1^{-1}t_2t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_2e_{1234} - x^3t_1^{-1}t_3t_4e_{1234} + t_1t_3^{-2}t_4^2e_{1234} \\
& + t_3^{-2}t_4^2e_{1234} + t_1t_3^{-3}t_4^3e_{1234} - x^3t_1^{-2}t_2t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_2t_3^{-1}t_4e_{1234} \\
& - x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} - x^3t_1^{-1}t_4e_{1234} + t_1t_3^{-1}t_4e_{1234} + x^3t_1^{-2}t_3t_4e_{1234} \\
& + x^3t_1^{-2}t_2t_4e_{1234} + x^3t_1^{-1}t_2t_4e_{1234} - x^3t_1^{-2}t_2e_{1234} - x^3t_1^{-1}t_2e_{1234} \\
& - t_3^{-2}t_4^2e_{1234} + x^3t_1^{-1}t_3t_4e_{1234} \\
d_3(e_{13}^{2s+1}) =& x^3t_1^{-1}t_2t_3^{-1}e_{1234} - x^3t_1^{-1}t_4e_{1234} + x^3t_1^{-2}t_2e_{1234} - x^3t_1^{-2}t_2t_4^{-1}e_{1234} \\
& + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} - x^3t_1^{-1}e_{1234} + x^3t_1^{-1}t_4^{-1}e_{1234} - x^3t_1^{-2}t_2t_3^{-1}e_{1234} \\
& - x^3t_1^{-1}t_2t_3^{-1}e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_3^{-1}e_{1234} \\
d_3(e_{14}^{2s+1}) =& x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234}
\end{aligned}
$$

$$
\begin{aligned}
d_3(e_{23}^{2s+1}) =& - x^3t_1^{-1}t_3t_4e_{1234} + x^3t_1^{-1}t_3e_{1234} + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} \\
& - x^3t_1^{-1}t_4e_{1234} + t_1t_3^{-1}t_4e_{1234} + t_3^{-1}t_4e_{1234} + t_1t_3^{-1}e_{1234} + t_3^{-1}e_{1234} \\
& + x^3t_1^{-1}t_2t_4e_{1234} + t_1e_{1234} + e_{1234} + x^3t_1^{-2}t_3t_4e_{1234} + x^3t_1^{-1}t_3t_4e_{1234} \\
& - t_1t_3^{-1}e_{1234} - t_3^{-1}e_{1234} - x^3t_1^{-2}t_3e_{1234} - x^3t_1^{-1}t_3e_{1234} - x^3t_1^{-2}t_2e_{1234} \\
& - x^3t_1^{-1}t_2e_{1234} + x^3t_1^{-2}t_2t_4^{-1}e_{1234} + x^3t_1^{-1}t_2t_4^{-1}e_{1234} + x^3t_1^{-2}t_2e_{1234} \\
& + x^3t_1^{-1}t_2e_{1234} - x^3t_1^{-2}t_2t_4^{-1}e_{1234} - x^3t_1^{-1}t_2t_4^{-1}e_{1234} + x^3t_1^{-2}t_2t_4e_{1234} \\
& + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} - x^3t_1^{-1}e_{1234} + x^3t_1^{-1}t_4^{-1}e_{1234} \\
& + t_3^{-2}t_4^2e_{1234} + t_1t_3^{-3}t_4^3e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_3^{-1}e_{1234} \\
& + t_1t_3^{-2}t_4^2e_{1234} + x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} \\
d_3(e_{24}^{2s+1}) =& t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_4e_{1234} + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} + t_3^{-2}t_4^2e_{1234} \\
& + x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234}
\end{aligned}
$$

$$\begin{aligned}
d_3(e_{34}^{2s+1}) =& -x^3t_1^{-1}t_4e_{1234} + x^3t_1^{-1}e_{1234} + x^3t_1^{-1}e_{1234} - x^3t_1^{-1}t_4^{-1}e_{1234} \\
&+ x^3t_1^{-1}t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_3^{-1}e_{1234} \\
d_3(e_{12}^{2s}) =& -t_4e_{1234} + x^3t_1^{-1}t_2e_{1234} + t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_2t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_3t_4e_{1234} \\
&- xt_2^{-1}t_3^{-1}t_4^2e_{1234} - xt_3^{-2}t_4^3e_{1234} - t_2^{-1}t_4^3e_{1234} + t_2^{-1}t_4e_{1234} - xt_2^{-2}t_4^3e_{1234} \\
&- xt_2^{-1}t_3^{-1}t_4^3e_{1234} + xt_2t_3^{-1}t_4^3e_{1234} + xt_3^{-1}t_4^3e_{1234} + xt_3^{-1}t_4^2e_{1234} + x^3t_1^{-2}t_3t_4e_{1234} \\
&+ x^3t_1^{-1}t_3t_4e_{1234} + x^3t_1^{-2}t_2t_4e_{1234} + x^3t_1^{-1}t_2t_4e_{1234} - x^3t_1^{-2}t_2e_{1234} - x^3t_1^{-1}t_2e_{1234} \\
&- x^3t_1^{-1}t_4e_{1234} + x^3t_1t_4e_{1234} + x^2t_1^{-2}t_2t_4^2e_{1234} + x^2t_1^{-1}t_2t_4^2e_{1234} + x^2t_1^{-1}t_4^3e_{1234} \\
&- xt_1^{-1}t_3^{-1}t_4^4e_{1234} + t_2^{-1}t_3^{-1}t_4^3e_{1234} - xt_1^{-1}t_3^{-1}t_4^3e_{1234} + t_2^{-1}t_3^{-1}t_4^2e_{1234} \\
&+ x^2t_1t_2^{-1}t_4^3e_{1234} + x^2t_2^{-1}t_4^3e_{1234} + xt_1^{-1}t_2t_3^{-1}t_4^3e_{1234} - xt_2^{-1}t_3^{-1}t_4^4e_{1234} \\
&- x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} + x^3t_4e_{1234} + x^2t_1^{-1}t_4^2e_{1234} + t_3^{-2}t_4^2e_{1234} \\
&- x^3t_1^{-2}t_2t_3^{-1}t_4e_{1234} - x^3t_1^{-1}t_2t_3^{-1}t_4e_{1234} \\
d_3(e_{13}^{2s}) =& -e_{1234} - x^3t_1^{-1}t_2e_{1234} + x^3t_1^{-1}t_2t_3^{-1}e_{1234} - x^3t_1^{-1}t_4e_{1234} + x^2t_2^{-1}t_4^2e_{1234} \\
&- t_2^{-1}t_4^2e_{1234} + t_2^{-1}t_4e_{1234} + x^2t_4^2e_{1234} + x^3t_1^{-1}t_2t_4^{-1}e_{1234} + xt_3^{-1}t_4e_{1234} \\
&+ xt_2t_3^{-2}t_4^2e_{1234} - t_2^{-1}t_3t_4e_{1234} - t_4e_{1234} + xt_2t_3^{-1}t_4^2e_{1234} + xt_3^{-1}t_4^2e_{1234} \\
&+ t_1t_2e_{1234} + t_2e_{1234} + t_1e_{1234} + e_{1234} + x^3t_1^{-2}t_2e_{1234} + x^3t_1^{-1}t_2e_{1234} \\
&- x^3t_1^{-1}t_2t_4^{-1}e_{1234} + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} - e_{1234} - x^3t_1^{-1}e_{1234} \\
&+ x^2t_1^{-1}t_4^2e_{1234} - xt_1^{-1}t_3^{-1}t_4^3e_{1234} + t_2^{-1}t_3^{-1}t_4^2e_{1234} - x^3t_1^{-2}t_2t_3^{-1}e_{1234} \\
&- x^3t_1^{-1}t_2t_3^{-1}e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_3^{-1}e_{1234} + x^2t_1t_2^{-1}t_4^2e_{1234} \\
&+ x^2t_2^{-1}t_4^2e_{1234} - xt_1^{-1}t_2t_3^{-1}t_4^3e_{1234} + xt_1^{-1}t_2t_3^{-1}t_4^3e_{1234} + xt_1^{-1}t_2t_3^{-1}t_4^2e_{1234} \\
&- x^2t_2^{-1}t_4^2e_{1234} + xt_1t_3^{-2}t_4e_{1234} + xt_3^{-2}t_4e_{1234} - x^3t_1^{-2}t_2t_4^{-1}e_{1234} + x^3t_1^{-1}t_4^{-1}e_{1234} \\
d_3(e_{14}^{2s}) =& -t_2^{-1}t_4^2e_{1234} + t_2^{-1}t_4e_{1234} + x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} + x^2t_1^{-1}t_4^2e_{1234} \\
&- xt_1^{-1}t_3^{-1}t_4^3e_{1234} + t_2^{-1}t_3^{-1}t_4^2e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234} \\
d_3(e_{23}^{2s}) =& -x^3t_1^{-1}t_3t_4e_{1234} - x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} + xt_2^{-1}t_3^{-1}t_4^2e_{1234} + t_1t_2e_{1234} \\
&+ t_2e_{1234} + x^3t_1^{-1}t_3e_{1234} + x^3t_1^{-2}t_3t_4e_{1234} + x^3t_1^{-1}t_3t_4e_{1234} + x^3t_1^{-2}t_2t_4e_{1234} \\
&+ x^3t_1^{-1}t_2t_4e_{1234} - x^3t_1^{-2}t_3e_{1234} - x^3t_1^{-1}t_3e_{1234} - x^3t_1^{-2}t_2e_{1234} - x^3t_1^{-1}t_2e_{1234} \\
&+ xt_1t_2t_3^{-2}t_4^2e_{1234} + xt_2t_3^{-2}t_4^2e_{1234} + xt_1t_3^{-2}t_4e_{1234} + xt_3^{-2}t_4e_{1234} \\
&+ xt_3^{-1}t_4e_{1234} + t_1t_2e_{1234} + t_2e_{1234} + t_1e_{1234} + x^3t_1^{-2}t_2e_{1234} + x^3t_1^{-1}t_2e_{1234} \\
&+ x^3t_1^{-1}t_4e_{1234} - x^3t_1^{-1}e_{1234} - xt_1t_3^{-2}t_4^2e_{1234} + t_4^2e_{1234} + x^3t_1^{-1}t_3^{-1}t_4^2e_{1234} \\
&- x^3t_1^{-1}e_{1234} + x^3t_1^{-1}t_4^{-1}e_{1234} - x^3t_1^{-1}t_3^{-1}t_4e_{1234} + x^3t_1^{-1}t_3^{-1}e_{1234}
\end{aligned}$$

$$
\begin{aligned}
d_3(e_{24}^{2s}) =& x t_2^{-1} t_3^{-1} t_4^2 e_{1234} - x t_3^{-1} t_4^2 e_{1234} + x^3 t_1^{-1} t_4 e_{1234} + x^3 t_1^{-1} t_4 e_{1234} \\
& - x^3 t_1^{-1} e_{1234} + x^3 t_1^{-1} t_3^{-1} t_4^2 e_{1234} - x^3 t_1^{-1} t_3^{-1} t_4 e_{1234} \\
d_3(e_{34}^{2s}) =& - x t_3^{-1} t_4 e_{1234} - x^3 t_1^{-1} t_4 e_{1234} + x^3 t_1^{-1} e_{1234} + e_{1234} + x^3 t_1^{-1} e_{1234} \\
& + x^3 t_1^{-1} t_3^{-1} t_4 e_{1234} - x^3 t_1^{-1} t_3^{-1} e_{1234} - x^3 t_1^{-1} t_4^{-1} e_{1234}
\end{aligned}
$$

For d_4 we obtain:

$$
\begin{aligned}
d_4(e_1^{2s+1}) =& e_{1234} - t_3^{-1} e_{1234} + x^2 t_4^2 e_{1234} + x t_2 t_3^{-1} t_4 e_{1234} + x t_3^{-1} t_4 e_{1234} - x^2 t_2^{-1} t_4^2 e_{1234} \\
& + x t_2^{-1} t_3^{-1} t_4^3 e_{1234} - t_3^{-2} t_4 e_{1234} + x^2 t_2^{-1} t_4 e_{1234} + x^2 t_1^{-1} t_4^2 e_{1234} + x^2 t_1 t_2^{-1} t_4 e_{1234} \\
& + x^2 t_2^{-1} t_4 e_{1234} + x t_1^{-1} t_2 t_3^{-1} t_4 e_{1234} - x^2 t_2^{-1} t_4 e_{1234} - t_4^{-1} e_{1234} \\
d_4(e_2^{2s+1}) =& e_{1234} - x^3 t_1^{-1} t_2 t_4 e_{1234} + e_{1234} - t_3^{-1} e_{1234} + x t_2^{-2} t_4^3 e_{1234} + x t_2^{-1} t_3^{-1} t_4^3 e_{1234} \\
& - x t_2 t_3^{-1} t_4^2 e_{1234} - x t_3^{-1} t_4^2 e_{1234} + t_2^{-1} t_4^2 e_{1234} - t_2^{-1} t_4 e_{1234} + x^2 t_4^2 e_{1234} \\
& + x t_2 t_3^{-1} t_4 e_{1234} + x t_3^{-1} t_4 e_{1234} - x^3 t_4 e_{1234} - x^2 t_1^{-1} t_4^2 e_{1234} - x^2 t_2^{-1} t_4^2 e_{1234} \\
& - x^2 t_2^{-1} t_4^2 e_{1234} + x t_2^{-1} t_3^{-1} t_4^3 e_{1234} - t_3^{-2} t_4 e_{1234} + x^2 t_2^{-1} t_4 e_{1234} - x^3 t_1 t_4 e_{1234} \\
& - x^3 t_4 e_{1234} - x^2 t_1^{-2} t_2 t_4^2 e_{1234} - x^2 t_1^{-1} t_2 t_4^2 e_{1234} - x^2 t_1^{-1} t_4^3 e_{1234} + x^3 t_4 e_{1234} \\
& \\
& - x^2 t_1 t_2^{-1} t_4^2 e_{1234} - x^2 t_2^{-1} t_4 e_{1234} - x t_1^{-1} t_2 t_3^{-1} t_4^2 e_{1234} + x^2 t_2^{-1} t_4^2 e_{1234} \\
& + x t_1^{-1} t_3^{-1} t_4^3 e_{1234} + x^2 t_1 t_2^{-1} t_4 e_{1234} + x^2 t_2^{-1} t_4 e_{1234} + x t_1^{-1} t_2 t_3^{-1} t_4 e_{1234} \\
& - x^2 t_2^{-1} t_4^2 e_{1234} + x^2 t_1^{-1} t_4^2 e_{1234} - t_4^{-1} e_{1234} - t_2^{-1} t_3^{-1} t_4^2 e_{1234} \\
d_4(e_3^{2s+1}) =& t_4^{-1} e_{1234} + t_3^{-1} e_{1234} + t_4^{-1} e_{1234} - t_3^{-1} t_4^{-1} e_{1234} - x t_2 t_3^{-1} t_4 e_{1234} - x t_3^{-1} t_4 e_{1234} \\
& + t_2^{-1} t_4 e_{1234} - t_2^{-1} e_{1234} + x^2 t_4 e_{1234} + x t_2 t_3^{-1} e_{1234} + x t_3^{-1} e_{1234} + x^2 t_2^{-1} t_4^2 e_{1234} \\
& - x t_2^{-1} t_3^{-1} t_4^3 e_{1234} + t_3^{-2} t_4 e_{1234} - x^2 t_2^{-1} t_4 e_{1234} - x^2 t_2^{-1} t_4 e_{1234} + x t_2^{-1} t_3^{-1} t_4^2 e_{1234} \\
& - t_3^{-2} e_{1234} - x^2 t_1^{-1} t_4^2 e_{1234} - x^2 t_1 t_2^{-1} t_4 e_{1234} - x^2 t_2^{-1} t_4 e_{1234} - x t_1^{-1} t_2 t_3^{-1} t_4 e_{1234} \\
& + x^2 t_2^{-1} t_4 e_{1234} + x t_1^{-1} t_3^{-1} t_4^2 e_{1234} - t_2^{-1} t_3^{-1} t_4 e_{1234} + x^2 t_1 t_2^{-1} e_{1234} \\
& + x^2 t_2^{-1} e_{1234} + x t_1^{-1} t_2 t_3^{-1} e_{1234} - t_4^{-2} e_{1234} - x^2 t_4^2 e_{1234} \\
d_4(e_4^{2s+1}) =& 0
\end{aligned}
$$

$$
\begin{aligned}
d_4(e_1^{2s}) =& e_{1234} - t_4^{-1}e_{1234} - t_3^{-1}e_{1234} - x^3t_3^{-1}t_4e_{1234} - x^2t_1^{-1}t_2^{-1}t_3t_4e_{1234} - x^2t_1^{-1}t_4e_{1234} \\
&+ x^2t_1^{-1}e_{1234} + x^2t_2^{-1}t_4^2e_{1234} + xt_1^{-1}t_2^{-1}t_4^2e_{1234} + xt_1^{-1}t_3^{-1}t_4^2e_{1234} \\
&- xt_1^{-1}t_3^{-1}t_4e_{1234} - xt_1^{-1}t_4e_{1234} + xt_1^{-1}e_{1234} - t_3^{-2}t_4e_{1234} - x^3t_3^{-2}t_4^2e_{1234} \\
&+ x^2t_1^{-1}t_3^{-1}t_4e_{1234} + x^2t_2^{-1}t_3^{-1}t_4^3e_{1234} + xt_1^{-1}t_3^{-1}t_4e_{1234} - xt_1^{-1}t_3^{-1}t_4e_{1234} \\
&+ xt_1^{-1}t_3^{-1}t_4e_{1234} \\
d_4(e_2^{2s}) =& - xt_3^{-2}t_4^2e_{1234} - xt_3^{-1}t_4e_{1234} + e_{1234} + e_{1234} - t_4^{-1}e_{1234} - x^2t_1^{-1}t_4e_{1234} \\
&+ x^2t_1^{-1}e_{1234} + x^2t_2^{-1}t_3^{-1}t_4^3e_{1234} + xt_2^{-1}t_3^{-1}t_4^2e_{1234} + t_2^{-1}t_4^2e_{1234} - t_3^{-1}e_{1234} \\
&- t_2^{-1}t_4e_{1234} - t_3^{-2}t_4e_{1234} + x^2t_1^{-1}t_3^{-1}t_4e_{1234} - t_2^{-1}t_3^{-1}t_4^2e_{1234} \\
d_4(e_3^{2s}) =& - xt_3^{-2}t_4e_{1234} + t_4^{-1}e_{1234} + t_3^{-1}e_{1234} + t_4^{-1}e_{1234} - t_4^{-2}e_{1234} - t_3^{-1}t_4^{-1}e_{1234} \\
&- x^2t_2^{-1}t_4e_{1234} + t_2^{-1}t_4e_{1234} - t_2^{-1}e_{1234} - t_2^{-1}t_3^{-1}t_4e_{1234} \\
d_4(e_4^{2s}) =& 0
\end{aligned}
$$

Now we apply the functor $\mathrm{Hom}_{\mathbb{Z}\Gamma}(-, \mathbb{Z})$ to the resolution (A, d). From previous example, we know the dimensions of the resulting cochain complex (F_*, δ_*). By using the differentials d_1, d_2, d_3, and d_4 we can represent each δ_i by a matrix. The list of matrices can be seen in Appendix B. Then we find the Smith Normal Form of the matrices as in the following:

	Diagonal of SNF
δ_0	$[0]$
δ_1	$[1, 1, 2, 4, 0]$
δ_2	$[1, 1, 1, 1, 2, 4, 0, 0, 0, 0, 0]$
δ_3	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0]$
δ_{2k}, $k \geq 2$	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0]$
δ_{2k+1}, $k \geq 2$	$[1, 1, 1, 1, 2, 2, 2, 4, 0, 0, 0, 0, 0, 0, 0, 0]$

By using Formula 4.1.3, we get the following cohomology groups.

$$
\begin{aligned}
\mathbb{Z}^5 =& H^1 \oplus \mathbb{Z}^4 \Rightarrow H^1 = \mathbb{Z} \\
\mathbb{Z}^7 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 =& H^2 \oplus \mathbb{Z}^6 \Rightarrow H^2 = \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
\mathbb{Z}^9 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 =& H^3 \oplus \mathbb{Z}^8 \Rightarrow H^3 = \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^4 \oplus \mathbb{Z}^8 \Rightarrow H^4 = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^{2k+1} \oplus \mathbb{Z}^8 \Rightarrow H^{2k+1} = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 \\
\mathbb{Z}^8 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_2^3 =& H^{2k+2} \oplus \mathbb{Z}^8 \Rightarrow H^{2k+2} = \mathbb{Z}_4 \oplus \mathbb{Z}_2^3
\end{aligned}
$$

So, this gives the result. □

Proposition 4.3.2. *Right hand side of the Conjecture 1.0.1 is given by*

$$
H^i(G, H^j(L,\Gamma)) = \begin{cases}
\mathbb{Z}, & 0 \le j \le 3,\ i = 0; \\
\mathbb{Z}_2, & j = 1,3,\ i \ge 1; \\
\mathbb{Z}_2, & j = 2,\ i \ge 1,\ 2 | i; \\
\mathbb{Z}_4, & j = 2,\ i \ge 1,\ 2 \nmid i; \\
\mathbb{Z}_2, & j = 4,\ i \ge 1,\ 2 \nmid i; \\
\mathbb{Z}_4, & j = 0,\ i \ge 1,\ 2 | i; \\
0, & \textit{otherwise.}
\end{cases}
$$

Proof. For the calculation of $H^i(G, H^j(L,\Gamma))$, we use the same method with the proof of Proposition 4.2.2. We define $H^i(L,\mathbb{Z})$ as the j-the exterior power $\bigwedge^j(H^1(L,\mathbb{Z}))$ with generators $t_{i_1\ldots i_j}$, which are specified in the proof of 4.2.2.

Then the action of M on $H^i(L,\mathbb{Z})$ is given by the following matrices:

$$j=1: \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

$$j=2: \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 & -1 & -1 \end{pmatrix}$$

$$j=3: \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$$j=4: (-1)$$

Now apply $\mathrm{Hom}_{\mathbb{Z}G}(-,H^j(L,\mathbb{Z}))$ to the resolution 4.2.2 and we get

$$0 \longrightarrow H^j(L,\mathbb{Z}) \xrightarrow{M-I} H^j(L,\mathbb{Z}) \xrightarrow{\Sigma M} H^j(L,\mathbb{Z}) \xrightarrow{M-I} \cdots .$$

Then Formula 4.1.3 gives the result:

$$\begin{aligned} j = 0 : 0 &\longrightarrow \mathbb{Z} \xrightarrow{\times 0} \mathbb{Z} \xrightarrow{\times 4} \mathbb{Z} \xrightarrow{\times 0} \cdots \\ H^i &= (\mathbb{Z}, 0, \mathbb{Z}_4, 0, \mathbb{Z}_4, ...) \\ \\ j = 1 : 0 &\longrightarrow \mathbb{Z}^4 \xrightarrow{M-I} \mathbb{Z}^4 \xrightarrow{\Sigma M} \mathbb{Z}^4 \xrightarrow{M-I} \cdots \\ &\mathrm{SNF}(M-I) = \mathrm{diag}([1,1,2,0])_{4\times 4} \\ &\mathrm{SNF}(\Sigma M) = \mathrm{diag}([2,0,0,0])_{4\times 4} \\ H^i &= (\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, ...) \end{aligned} \tag{4.3.1}$$

$$j = 2 : 0 \longrightarrow \mathbb{Z}^6 \xrightarrow{M-I} \mathbb{Z}^6 \xrightarrow{\Sigma M} \mathbb{Z}^6 \xrightarrow{M-I} \cdots$$

$$\text{SNF}(M - I) = \text{diag}([1, 1, 1, 1, 4, 0])_{6\times 6}$$

$$\text{SNF}(\Sigma M) = \text{diag}([2, 0, 0, 0, 0, 0])_{6\times 6}$$

$$H^i = (\mathbb{Z}, \mathbb{Z}_4, \mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_2, ...)$$

$$j = 3 : 0 \longrightarrow \mathbb{Z}^4 \xrightarrow{M-I} \mathbb{Z}^4 \xrightarrow{\Sigma M} \mathbb{Z}^4 \xrightarrow{M-I} \cdots$$

$$\text{SNF}(M - I) = \text{diag}([1, 1, 2, 0])_{4\times 4}$$

$$\text{SNF}(\Sigma M) = \text{diag}([2, 0, 0, 0])_{4\times 4}$$

$$H^i = (\mathbb{Z}, \mathbb{Z}_2, \mathbb{Z}_2, \mathbb{Z}_2, ...)$$

$$j = 4 : 0 \longrightarrow \mathbb{Z} \xrightarrow{M-I} \mathbb{Z} \xrightarrow{\Sigma M} \mathbb{Z} \xrightarrow{M-I} \cdots$$

$$SNF(M - I) = \text{diag}[2]$$

$$SNF(\Sigma M) = \text{diag}[0]$$

$$H^i = (0, \mathbb{Z}_2, 0, \mathbb{Z}_2, 0, ...)$$

This completes the proof. □

Hence the second example satisfies the claim of the conjecture of Adem-Ge-Pan-Petrosyan. That is:

$$H^k(L \rtimes G, \mathbb{Z}) \cong \bigoplus_{i+j=k} H^i(G, H^j(L, \mathbb{Z})).$$

Appendix A

List 1

$$\delta_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\delta_3 = \begin{pmatrix}
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & 0 & -1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}$$

$$\delta_{2k} =$$
$$\begin{pmatrix}
-2 & 0 & 2 & -2 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 1 & -2 & 5 & 0 & 0 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

$$\delta_{2k+1} = \begin{pmatrix}
0 & 1 & 2 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & 0 & -1 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -2 & 2 & 0 & 1 & -2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}$$

Appendix B

List 2

$$\delta_1 = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\delta_2 = \begin{pmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & -1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & -4 & -2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\delta_3 = \begin{pmatrix}
2 & 0 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 \\
-1 & 1 & 0 & -1 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
1 & -1 & 1 & 1 & 3 & -1 & 0 & -1 & 1 & 0 & -1 & -3 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -2 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
$$

$$
\delta_{2k} =
\begin{pmatrix}
-2 & 0 & 4 & 2 & 0 & 2 & -2 & -1 & 9 & 3 & 0 & 0 & -1 & -2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -2 & -2 & -2 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & -2 & -2 & -2 & 1 & 1 & 1 & -2 & -1 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 1 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -4 & -2 & -4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

$$\delta_{2k+1} =$$

$$\begin{pmatrix}
0 & -1 & 1 & 1 & -1 & 7 & 9 & 0 & 13 & 1 & 0 & 5 & -4 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 4 & 2 & 0 & 0 & 0 & 0 & 1 & -3 & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & -1 & -4 & -1 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 1 & -1 & 1 & 1 & 3 & -1 & 0 & -1 & 1 & 0 & -1 & -3 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & -2 & -1 & 0 & 1 & 2 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & -2 & 0 & -2 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 2 & 0 & 2 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Bibliography

[1] K.S. Brown, *Cohomology of Groups*, Springer-Verlag, New York 1982.

[2] D.S. Dummit and R.M.Foote, *Abstract Algebra*, John Wiley and Sonsi Inc., United States of America 2004.

[3] L. Evens, *The Cohomology of Groups*, Oxford Science Publications, New York 1991.

[4] J.J. Rotman, *An Introduction to Homological Algebra*, Springer Science+Business Media, New York 2009.

[5] A. Adem and R.J. Milgram, *Cohomology of Finite Groups*, Springer-Verlag, Germany 2004.

[6] S. Mac Lane, *Homology*, Springer-Verlag, Germany 1994.

[7] A. Adem and J. Pan. Toroidal Orbifolds, Gerbes And Group Cohomology. *Trans. Amer. Math. Soc.*, 358(9):3969-3983, 2006.

[8] A. Adem, J. Ge, J. Pan, and N. Petrosyan. Compatible Actions and Cohomology od Crystallographic Groups. *J. Algebra*, 320(1):341-353, 2008.

[9] N. Petrosyan and B. Putrycz, On Cohomology of Crystallographic Groups With Cyclic Holonomy of Split Type, arXiv:1106.4216v2, 2011.

[10] M. Langer and W. Lück, On The Group Cohomology of The Semidirect Products $\mathbb{Z}^n \rtimes_p \mathbb{Z}/m$ and a Conjecture of Adem-Ge-Pan-Petrosyan, arXiv:1105.4772v1, 2011.