PROJECTIVE RESOLUTIONS OVER EI-CATEGORIES

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Representations of EI-categories occur naturally in algebraic K-theory and algebraic topology (see [4], [10], [12]). In this thesis, we study EI-category representations with finite projective dimension. We apply this general theory to orbit categories of finite groups and prove Rim's theorem for the orbit category (Theorem B in [5]). It follows from this theorem that, for a fixed prime p, the constant functor over the orbit category of a finite group G with respect to the family of p-subgroups and with coefficients in $\mathbb{Z}_{(p)}$ has finite projective dimension, which we denote by $\mathrm{pd}(G, p)$. In this thesis, we calculate $\mathrm{pd}(S_4, 2)$ and $\mathrm{pd}(S_5, 2)$ explicitly, which are among the first nontrivial cases. We also prove that the constant functor over the orbit category of all subgroups with prime power order and with integral coefficients never has a finite projective resolution unless G itself has prime power order.

Keywords: EI-categories, projective dimension, constant functor, orbit categories, Rim's theorem.

ÖZET

EI-KATEGORİLERİ ÜZERİNDE PROJEKTİF ÇÖZÜCÜLER

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EI-kategori temsilleri cebirsel K-teorisinde ve cebirsel topolojide doğal bir şekilde ortaya çıkmaktadır (bkz. [4], [10], [12]). Bu tezde sonlu projektif boyuta sahip EI-kategori temsillerini inceledik. Bu genel teoriyi sonlu grupların orbit kategorilerine uyguladık ve orbit kategorisi için Rim'in teoreminin bir versiyonunu ([5]'teki Theorem B) ispatladık. Bu teoremin bir sonucu bize, sabit bir p asalı için, sonlu bir G grubunun p-altgruplarının verdiği orbit kategoride $\mathbb{Z}_{(p)}$ katsayılı sabit funktorun projektif boyutunun sonlu olduğunu söylüyor. Bu boyut $\mathrm{pd}(G, p)$ olarak gösteriliyor. Bu tezde ilk bariz olmayan durumlardan olan $\mathrm{pd}(S_4, 2)$ ve $\mathrm{pd}(S_5, 2)$ değerlerini tam olarak hesapladık. Ayrıca orbit kategorisinde altgrup ailesini herhangi bir asalın kuvveti kadar elemana sahip bütün altgruplar ve katsayı halkasını tamsayılar olarak aldığımız zaman sabit funktorun projektif boyutunun, $|G|$'nin birden fazla asal çarpanı olduğu sürece sonlu olamayacağını ispatladık.

Anahtar sözcükler: EI-kategoriler, projektif boyut, sabit funktor, orbit kategorileri, Rim'in teoremi.

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Contents

Chapter 1

Introduction

A common approach to study a mathematical object is to *linearize* it with a (nonzero) commutative ring R. Perhaps the most striking example of this is character theory where a group is linearized with $\mathbb{C}$, whose basics are enough to prove results which have no known proofs in pure group theory. Modular representation theory shows that rings other than $\mathbb{C}$ can also be used with great success to understand a (finite) group G.

Instead of a group G, representations of a *small category* Γ can be studied, which gives rise to a more general theory. The representation of small categories actually serves as a general framework for various representation theories: If Γ consists of a single object we get monoid representations; if Γ is a partially ordered set (poset) we get *incidence algebras*; if Γ is the category generated by a graph we obtain *quiver representations.*

In this thesis, we are especially interested in representations of (finite) *EI-categories*, which are by definition categories where every endomorphism is invertible. EI-categories and their representations were first studied by Lück [4] and tom Dieck [10] in the context of algebraic K-theory. The theory of finite G-spaces provides various EI-categories. And for example fusion systems and transporter systems studied by Broto, Levi and Oliver in [11] are EI-categories. The general theme of this thesis is an investigation of *finite* projective resolutions over (finite) EI-categories as it is done in [4]. In general, restricting a category representation to an individual object of a category yields a monoid representation; for EI-categories, we get instead a group representation which behaves much more nicely. This fact, together with the natural poset structure on the isomorphism classes of an EI-category allows one to obtain general results towards necessary or sufficient conditions for the existence of finite projective resolutions and bounds for projective dimensions.

As an application of the material we develop in the EI-category setting, we consider orbit categories over a finite group G. The theory of orbit categories was first introduced by Bredon [12]; the motivation being that orbit categories provide a useful setting to study G-actions on topological spaces when the family of isotropy subgroups is given. The main result we prove about orbit categories is a theorem of Hambleton, Pamuk and Yalçın [5]:

Theorem 1.1. *Let G be a finite group and p a prime. Let $R = \mathbb{Z}_{(p)}$. Let Γ_G be the orbit category given by an (isotropy) family consisting of p-subgroups and let P be a Sylow p-subgroup. Then an $R\Gamma_G$-module M has a finite projective resolution if and only if $\mathrm{Res}^G_P(M)$ has a finite projective resolution.*

This theorem roughly states that in a "p-local setting", the existence of a finite projective resolution over the orbit category can be detected by restricting to a Sylow p-subgroup. This result is similar to a theorem of Rim [8] which states that the projectivity of a $\mathbb{Z}G$-module can be detected by restricting to Sylow subgroups of G; hence can be referred as "Rim's theorem for the orbit category". Proving this result is our main application of the theory of finite projective resolutions over EI-categories.

As a consequence of Rim's theorem for the orbit category, we obtain the following:

Corollary 1.2. *Let G be a finite group and p a prime. Let $R = \mathbb{Z}_{(p)}$. Let Γ_G denote the orbit category of G with respect to the family of all p-subgroups of G. Then the constant functor $\underline{R} : \Gamma_G \to R\text{-}\mathsf{Mod}$ has a finite projective resolution.*

Therefore in this case $\underline{R}$ has finite projective dimension. Since this dimension only depends on the group G and the prime p, we denote it by $\mathrm{pd}(G,p)$. We prove some general results which relate $\mathrm{pd}(G,p)$ to the more intrinsic properties of G. We also make calculations for the specific cases $p = 2$ and $G = S_4, S_5$ and obtain the following result:

Proposition 1.3. $\mathrm{pd}(S_4, 2) = 1$ *and* $\mathrm{pd}(S_5, 2) = 2$.

Finally we also prove a result which states that the constant functor almost never has a finite projective resolution when $R = \mathbb{Z}$ and $\mathcal{F}$ is the family of all subgroups of G which have prime power order. This shows that the situation is drastically different in the "integral setting" than the "p-local setting".

The thesis is organized as follows:

In Chapter 2, we establish the basics of the general theory of category representations and category algebras.

In Chapter 3, we study the representations of EI-categories. We introduce the notion of length. With the help of the established results in group representations and group cohomology, we build the necessary theory for EI-category representations with finite projective resolutions.

In Chapter 4, we define and study the basic properties of orbit categories. We then define a restriction functor which restricts an orbit category of G to an orbit category of a given subgroup H. We prove that this functor preserves projectives and then by using Rim's theorem for group rings and the results in the previous chapters, we prove Rim's theorem for the orbit category.

In Chapter 5 we discuss how the constant functor behaves in the p-local and integral settings. Proposition 1.3 is also proved in this chapter.

Chapter 2

Representations of small categories

In this chapter, we introduce the theory of category representations and category algebras. Our main source for the material in this chapter is [1].

We begin by recalling group representations and group algebras. Given a group G and a commutative ring R there are two equivalent ways to bring them together:

- Form an R-algebra RG (called the group algebra) and consider RG-modules.
- Consider G-actions on R-modules.

By a G-action on an R-module M, we mean a group homomorphism $\rho : G \to \mathrm{Aut}_R(M)$; this means G acts on M as R-linear automorphisms. Such a ρ is often called a *representation* of G over R.

In what follows, we will replace the group G with a small category Γ (A category Γ is called *small* if the collection of morphisms in Γ, shortly $\mathrm{Mor}(\Gamma)$, forms a set) and generalize both viewpoints above.

2.1 Category algebra and representations of categories

Definition 2.1. The *category algebra* $R\Gamma$ is the free R-module generated by the set $\mathrm{Mor}(\Gamma)$ endowed with the following multiplication on basis elements (which is then extended to whole $R\Gamma$ bilinearly which makes $R\Gamma$ an R-algebra):
Given $\alpha, \beta \in \mathrm{Mor}(\Gamma)$

$$\beta\alpha := \begin{cases} \beta \circ \alpha & \text{if } \mathrm{cod}(\alpha) = \mathrm{dom}(\beta) \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.2. The *rng* (A rng is what we get when we drop the existence of a multiplicative identity out of the *ring* axioms) $R\Gamma$ has a multiplicative identity if and only if Γ has finitely many objects. Indeed, if $u = \sum_{\alpha \in \mathrm{Mor}(\Gamma)} r_\alpha \alpha$ is the identity, for every $x \in \mathrm{Obj}(\Gamma)$ we have

$$\mathrm{id}_x = u \, \mathrm{id}_x = \left(\sum_{\alpha \in \mathrm{Mor}(\Gamma)} r_\alpha \alpha \right) \mathrm{id}_x = \sum_{\mathrm{dom}(\alpha) = x} r_\alpha \alpha \, .$$

Hence $r_{\mathrm{id}_x} = 1$. Since only finitely many r_α are nonzero, it follows that $\mathrm{Obj}(\Gamma)$ is finite. Conversely if $\mathrm{Obj}(\Gamma)$ is finite, the element $\sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x \in R\Gamma$ is clearly the identity.

Definition 2.3. A *representation of* Γ *over* R is a covariant functor $M : \Gamma \to R\text{-}\mathsf{Mod}$.

Example 2.4. Given a group G, let Γ be the category with a single object x and $\mathrm{Hom}_\Gamma(x, x) = G$ where compositions are given by the group multiplication in G. Then clearly the category algebra $R\Gamma$ is the group algebra RG. Also the datum of a covariant functor $M : \Gamma \to R\text{-}\mathsf{Mod}$ (a representation of Γ over R) is just a group homomorphism $G \to \mathrm{Aut}_R(M)$ (via the functor axioms), which is a representation of G over R.

Hence the category algebra is a generalization of the group algebra and the representations of categories is a generalization of the representations of groups.

We also note that if Γ is actually a poset, $R\Gamma$ is precisely what is known as the *incidence algebra*. Representations of *quivers* are similarly a special case of representations of categories. Thus the theory of category algebras and category representations can be seen as a general framework for various representation theories.

2.2 $R\Gamma$-Mod versus R-Mod$^\Gamma$

In this section, we discuss the relation between left $R\Gamma$-modules and representations of Γ over R. Note that both collections form a category: First is the module category $R\Gamma\text{-Mod}$ and the second is the functor category $R\text{-Mod}^\Gamma$. We know that these categories are equivalent when Γ is given by a group G as in Example 2.4. A theorem of Mitchell (see [3], Theorem 7.1) explains their relation in general:

Proposition 2.5 ([1], Proposition 2.1). *There are functors $F : R\text{-Mod}^\Gamma \to R\Gamma\text{-Mod}$ and $G : R\Gamma\text{-Mod} \to R\text{-Mod}^\Gamma$ such that*

- $G \circ F \cong \mathrm{id}_{R\text{-Mod}^\Gamma}$
- *If Γ has finitely many objects, $F \circ G \cong \mathrm{id}_{R\Gamma\text{-Mod}}$.*

Proof. First we define F. Given $M \in \mathrm{Obj}(R\text{-Mod}^\Gamma)$, that is, given a covariant functor $M : \Gamma \to R\text{-Mod}$, define

$$F(M) = \bigoplus_{x \in \mathrm{Obj}(\Gamma)} M(x)\,.$$

Note that a generic element of the R-module $F(M)$ can be written uniquely of the form

$$m = \sum_{x \in \mathrm{Obj}(\Gamma)} m_x$$

where $m_x \in M(x)$ and only finitely many m_x are nonzero.

Let $\mathrm{supp}(m) = \{x : m_x \neq 0\}$. Again, $\mathrm{supp}(m)$ is a finite subset of $\mathrm{Obj}(\Gamma)$ for every $m \in F(M)$.

For a morphism $\alpha : y \to z$ in Γ, define

$$\Xi_\alpha : F(M) \to F(M)$$

by

$$(\Xi_\alpha(m))_x = \begin{cases} 0 & \text{if } x \neq z \\ M(\alpha)(m_y) & \text{if } x = z \end{cases}$$

Note that if $y \notin \operatorname{supp}(m)$, $m_y = 0$ and hence $(\Xi_\alpha(m))_y = 0$.

Observe that for $r \in R$, $m, n \in F(M)$, we have

$$\begin{aligned}
(\Xi_\alpha(rm+n))_x &= \begin{cases} 0 & \text{if } x \neq z \\ M(\alpha)((rm+n)_y) & \text{if } x = z \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq z \\ M(\alpha)(rm_y + n_y) & \text{if } x = z \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq z \\ rM(\alpha)(m_y) + M(\alpha)(n_y) & \text{if } x = z \end{cases} \\
&= r\,(\Xi_\alpha(m))_x + (\Xi_\alpha(n))_x \\
&= (r\Xi_\alpha(m) + \Xi_\alpha(n))_x
\end{aligned}$$

where the third equality holds because $M(\alpha) : M(y) \to M(z)$ is an R-module homomorphism. Since the above holds for every x, $\Xi_\alpha(rm+n) = r\Xi_\alpha(m) + \Xi_\alpha(n)$; so Ξ_α is an R-module homomorphism. We have obtained a function

$$\begin{aligned} \Xi : \operatorname{Mor}(\Gamma) &\to \operatorname{End}_R(F(M)) \\ \alpha &\mapsto \Xi_\alpha \,. \end{aligned}$$

As $\operatorname{End}_R(F(M))$ is an R-module (R is commutative), by the universal property of free modules Ξ extends to an R-module homomorphism

$$\Xi : R\Gamma \to \operatorname{End}_R(F(M)) \,.$$

Now we claim that Ξ is also a rng homomorphism. It suffices to check that for every $\alpha, \beta \in \mathrm{Mor}(\Gamma)$

$$\Xi_\beta \circ \Xi_\alpha = \begin{cases} \Xi_{\beta\circ\alpha} & \text{if } \mathrm{dom}(\beta) = \mathrm{cod}(\alpha) \\ 0 & \text{otherwise.} \end{cases}$$

Indeed, for every $m \in F(M)$ and $x \in \mathrm{Obj}(\Gamma)$

$$\begin{aligned}
\left[(\Xi_\beta \circ \Xi_\alpha)(m)\right]_x &= \left[\Xi_\beta(\Xi_\alpha(m))\right]_x \\
&= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\beta) \\ M(\beta)(\Xi_\alpha(m)_{\mathrm{dom}(\beta)}) & \text{if } x = \mathrm{cod}(\beta) \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\beta) \\ M(\beta)(0) & \text{if } x = \mathrm{cod}(\beta) \text{ and } \mathrm{dom}(\beta) \neq \mathrm{cod}(\alpha) \\ M(\beta)(M(\alpha)(m)) & \text{if } x = \mathrm{cod}(\beta) \text{ and } \mathrm{dom}(\beta) = \mathrm{cod}(\alpha) \end{cases} \\
&= \begin{cases} M(\beta\circ\alpha)(m) & \text{if } x = \mathrm{cod}(\beta) \text{ and } \mathrm{dom}(\beta) = \mathrm{cod}(\alpha) \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \left[\Xi_{\beta\circ\alpha}(m)\right]_x & \text{if } \mathrm{dom}(\beta) = \mathrm{cod}(\alpha) \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

as desired.

So for every $M \in \mathrm{Obj}(R\text{-}\mathsf{Mod}^\Gamma)$, $F(M)$ becomes an $R\Gamma$-module via an R-algebra homomorphism $\Xi^M : R\Gamma \to \mathrm{End}_R(F(M))$ as described above (We didn't write superscripts above Ξ before because M was fixed). This finishes the definition of the action of F on objects.

Now let $\nu : M \to N$ be a morphism in $R\text{-}\mathsf{Mod}^\Gamma$, that is, a natural transformation between covariant functors $M, N : \Gamma \to R\text{-}\mathsf{Mod}$. We want to define an $R\Gamma$-module homomorphism

$$F(\nu) : F(M) \to F(N)\,.$$

Since there is an R-module homomorphism

$$\nu_x : M(x) \to N(x)$$

for all $x \in \mathrm{Obj}(\Gamma)$, we get an R-module homomorphism

$$F(\nu) : F(M) = \bigoplus_{x \in \mathrm{Obj}(\Gamma)} M(x) \to \bigoplus_{x \in \mathrm{Obj}(\Gamma)} N(x) = F(N)\,.$$

We will show that $F(\nu)$ is actually an $R\Gamma$-module homomorphism. It suffices to check

$$F(\nu)(\alpha \cdot m) = \alpha \cdot F(\nu)(m)$$

for every $\alpha \in \mathrm{Mor}(\Gamma)$ and $m \in F(M)$. Now

$$F(\nu)(\alpha \cdot m) = F(\nu)(\Xi_\alpha^M(m))$$

and so

$$\begin{aligned}
[F(\nu)(\alpha \cdot m)]_x &= \nu_x(\Xi_\alpha^M(m)_x) \\
&= \begin{cases} \nu_x(0) & \text{if } x \neq \mathrm{cod}(\alpha) \\ \nu_x(M(\alpha)(m_{\mathrm{dom}(\alpha)})) & \text{if } x = \mathrm{cod}(\alpha) \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ \left(\nu_{\mathrm{cod}(\alpha)} \circ M(\alpha)\right)(m_{\mathrm{dom}(\alpha)}) & \text{if } x = \mathrm{cod}(\alpha) \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ \left(N(\alpha) \circ \nu_{\mathrm{dom}(\alpha)}\right)(m_{\mathrm{dom}(\alpha)}) & \text{if } x = \mathrm{cod}(\alpha) \end{cases} \\
&= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ N(\alpha)\left([F(\nu)(m)]_{\mathrm{dom}(\alpha)}\right) & \text{if } x = \mathrm{cod}(\alpha) \end{cases} \\
&= \left[\Xi_\alpha^N(F(\nu)(m))\right]_x \\
&= [\alpha \cdot F(\nu)(m)]_x
\end{aligned}$$

as desired. Note that we use the naturality of ν in the fourth equality. Having defined the action of F on the objects and morphisms, we now check that F satisfies the functor axioms:

Consider $F(\mathrm{id}_M) : F(M) \to F(M)$. Since

$$[F(\mathrm{id}_M)(m)]_x = (\mathrm{id}_M)_x(m_x) = \mathrm{id}_{M(x)}(m_x) = m_x$$

for every m, x we have $F(\mathrm{id}_M) = \mathrm{id}_{F(M)}$.
Say $\mu : M \to N$ and $\nu : N \to P$ are morphisms in $R\text{-}\mathsf{Mod}^\Gamma$. Then

$$\begin{aligned}[F(\nu \circ \mu)(m)]_x &= (\nu \circ \mu)_x(m_x) \\ &= (\nu_x \circ \mu_x)(m_x) \\ &= \nu_x([F(\mu)(m)]_x) \\ &= [F(\nu)(F(\mu)(m))]_x \\ &= [(F(\nu) \circ F(\mu))(m)]_x\end{aligned}$$

for all m, x; so $F(\nu \circ \mu) = F(\nu) \circ F(\mu)$. Thus F is a legitimate functor.

We also need to define a functor $G : R\Gamma\text{-}\mathsf{Mod} \to R\text{-}\mathsf{Mod}^\Gamma$. To define G on objects, given an $R\Gamma$-module U we should define a covariant functor

$$G(U) : \Gamma \to R\text{-}\mathsf{Mod}\,.$$

Let $G(U)(x) = \mathrm{id}_x U$ for every $x \in \mathrm{Obj}(\Gamma)$.

Note that $\mathrm{id}_x U = \{u \in U : \mathrm{supp}(u) \subseteq \mathrm{cod}^{-1}(x)\}$ where $\mathrm{cod}^{-1}(x)$ means all the morphisms in Γ with codomain x. For a morphism $\alpha : x \to y$ in Γ, define

$$\begin{aligned}G(U)(\alpha) : \mathrm{id}_x U &\to \mathrm{id}_y U \\ u &\mapsto \alpha u\,.\end{aligned}$$

$G(U)(\alpha)$ is a well-defined function because $\mathrm{supp}(u) \subseteq \mathrm{cod}^{-1}(x)$ implies $\mathrm{supp}(\alpha u) \subseteq \mathrm{cod}^{-1}(y)$. Moreover $G(U)(\alpha)$ is clearly additive and every $r \in R$ commutes with α by definition of the category algebra. Hence we have

$$G(U)(\alpha)(ru) = \alpha r u = r \alpha u = r\,(G(U)(\alpha)(u))$$

so $G(U)(\alpha)$ is an R-module homomorphism. So we have defined the action of $G(U)$ on the objects and morphisms of Γ. Now clearly $G(U)(\mathrm{id}_x) = \mathrm{id}_{\mathrm{id}_x U} = \mathrm{id}_{G(U)(x)}$ and

$$G(U)(\beta \circ \alpha)(u) = (\beta \circ \alpha)u = \beta\alpha u = \beta\,[G(U)(\alpha)(u)] = (G(U)(\beta) \circ G(U)(\alpha))\,(u)$$

hence $G(U)(\beta \circ \alpha) = G(U)(\beta) \circ G(U)(\alpha)$. Thus $G(U)$ is really a functor and we are done defining G on the objects of $R\Gamma\text{-}\mathsf{Mod}$.

Next, given an $R\Gamma$-module homomorphism

$$\varphi : U \to V$$

we define a natural transformation

$$G(\varphi) : G(U) \to G(V)\,.$$

So for each $x \in \mathrm{Obj}(\Gamma)$ we need an R-module homomorphism

$$G(\varphi)_x : \mathrm{id}_x U \to \mathrm{id}_x V\,.$$

Simply define $G(\varphi)_x$ to be the restriction $\varphi|_{\mathrm{id}_x U}$, that is,

$$G(\varphi)_x(\mathrm{id}_x u) := \varphi(\mathrm{id}_x u) = \mathrm{id}_x \varphi(u)$$

where the equality holds because φ is an $R\Gamma$-module homomorphism.
$G(\varphi)_x$ is an R-module homomorphism because

$$\begin{aligned} G(\varphi)_x(r \cdot \mathrm{id}_x u + \mathrm{id}_x u') &= G(\varphi)_x(\mathrm{id}_x \cdot (ru + u')) \\ &= \mathrm{id}_x \varphi(ru + u') \\ &= r\,\mathrm{id}_x \varphi(u) + \mathrm{id}_x \varphi(u') \\ &= rG(\varphi)_x(\mathrm{id}_x u) + G(\varphi)_x(\mathrm{id}_x u')\,. \end{aligned}$$

Now we check the naturality of the $G(\varphi)_x$'s. Given a morphism $\alpha : x \to y$ in Γ, the diagram

$$\begin{array}{ccc} \mathrm{id}_x U & \xrightarrow{G(U)(\alpha)} & \mathrm{id}_y U \\ {\scriptstyle G(\varphi)_x}\downarrow & & \downarrow{\scriptstyle G(\varphi)_y} \\ \mathrm{id}_x V & \xrightarrow[G(V)(\alpha)]{} & \mathrm{id}_y V \end{array}$$

should commute. Indeed

$$\begin{aligned}
(G(\varphi)_y \circ G(U)(\alpha))\,(\mathrm{id}_x u) &= G(\varphi_y)(\alpha\,\mathrm{id}_x u) \\
&= G(\varphi_y)(\alpha u) \\
&= G(\varphi_y)(\mathrm{id}_y \alpha u) \\
&= \varphi(\mathrm{id}_y \alpha u) \\
&= \varphi(\alpha u) \\
&= \alpha\varphi(u) \\
&= \alpha\,\mathrm{id}_x \varphi(u) \\
&= \alpha G(\varphi)_x(\mathrm{id}_x u) \\
&= (G(V)(\alpha) \circ G(\varphi)_x)\,(\mathrm{id}_x u)\,.
\end{aligned}$$

Thus the collection of $G(\varphi)_x$'s does define a natural transformation $G(\varphi) : G(U) \to G(V)$. We have

- $G(\mathrm{id}_U)_x = \mathrm{id}_U|_{\mathrm{id}_x U} = \mathrm{id}_{\mathrm{id}_x U} = \mathrm{id}_{G(U)(x)}$
- For $R\Gamma$-module homomorphisms $\varphi : U \to V$ and $\psi : V \to W$

$$\begin{aligned}
G(\psi \circ \varphi)_x &= (\psi \circ \varphi)|_{\mathrm{id}_x U} \\
&= \psi|_{\mathrm{id}_x V} \circ \varphi|_{\mathrm{id}_x U} \\
&= G(\psi)_x \circ G(\varphi)_x \\
&= (G(\psi) \circ G(\varphi))_x
\end{aligned}$$

for every x, therefore G is a functor.

To see how to establish a natural isomorphism $G \circ F \cong \mathrm{id}_{R\text{-}\mathsf{Mod}^\Gamma}$, we first investigate $(G \circ F)(M)$ for a fixed representation M. Both M and $G(F(M))$ are covariant functors from Γ to R-Mod. Given $y \in \mathrm{Obj}(\Gamma)$,

$$G(F(M))(y) = \mathrm{id}_y F(M) = \mathrm{id}_y F(M) = \Xi_{\mathrm{id}_y}(F(M))\ .$$

For $m \in F(M)$ and $x \in \mathrm{Obj}(\Gamma)$,

$$\begin{aligned} \left[\Xi_{\mathrm{id}_y}(m)\right]_x &= \begin{cases} 0 & \text{if } x \neq y \\ M(\mathrm{id}_y)(m_y) & \text{if } x = y \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq y \\ m_y & \text{if } x = y\,. \end{cases} \end{aligned}$$

Therefore there is an R-module isomorphism

$$\begin{aligned} j_y : M(y) &\to G(F(M))(y) \\ w &\mapsto m \end{aligned}$$

such that $m_x = 0$ if $x \neq y$ and $m_y = w$.

Now we claim that for every morphism $\alpha : y \to z$ in Γ, the diagram

$$\begin{array}{ccc} M(y) & \xrightarrow{M(\alpha)} & M(z) \\ \Big\downarrow{\scriptstyle j_y} & & \Big\downarrow{\scriptstyle j_z} \\ G(F(M))(y) & \xrightarrow[G(F(M))(\alpha)]{} & G(F(M))(z) \end{array}$$

commutes. Indeed for $w \in M(y)$, we have

$$\begin{aligned} (G(F(M))(\alpha) \circ j_y)(w) &= G(F(M))(\alpha)(j_y(w)) \\ &= \alpha \cdot j_y(w) \\ &= \Xi_\alpha(j_y(w)) \end{aligned}$$

and for every $x \in \mathrm{Obj}(\Gamma)$, we have

$$\begin{aligned} [\Xi_\alpha(j_y(w))]_x &= \begin{cases} 0 & \text{if } x \neq z \\ M(\alpha)(j_y(w)_y) & \text{if } x = z \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq z \\ M(\alpha)(w) & \text{if } x = z \end{cases} \\ &= [j_z(M(\alpha)(w))]_x\,. \end{aligned}$$

Hence $(G(F(M))(\alpha) \circ j_y)(w) = \Xi_\alpha(j_y(w)) = (j_z \circ M(\alpha))(w)$ as desired. It follows that j_y's define a natural *isomorphism*

$$\tilde{j} : M \to (G \circ F)(M)\,.$$

In what follows M will not be fixed, so we will write j^M for this isomorphism. That is, for every $M \in \mathrm{Obj}(R\text{-}\mathsf{Mod}^\Gamma)$ there is an isomorphism

$$j^M : M \to (G \circ F)(M)\,.$$

We claim that j^M's are natural in M. So we show that given a natural transformation $\nu : M \to N$, the diagram

$$\begin{array}{ccc} M & \xrightarrow{\nu} & N \\ \downarrow{\scriptstyle j^M} & & \downarrow{\scriptstyle j^N} \\ G(F(M)) & \xrightarrow[G(F(\nu))]{} & G(F(N)) \end{array}$$

commutes. It suffices to check that

$$\begin{array}{ccc} M(y) & \xrightarrow{\nu_y} & N(y) \\ \downarrow{\scriptstyle j^M_y} & & \downarrow{\scriptstyle j^N_y} \\ \mathrm{id}_y F(M) = G(F(M))(y) & \xrightarrow[G(F(\nu))_y]{} & G(F(N))(y) = \mathrm{id}_y F(N) \end{array}$$

commutes for each $y \in \mathrm{Obj}(\Gamma)$. Indeed for $w \in M(y)$

$$\begin{aligned} (G(F(\nu))_y \circ j^M_y)(w) &= G(F(\nu))_y(j^M_y(w)) \\ &= F(\nu)(j^M_y(w)) \end{aligned}$$

and for each $x \in \mathrm{Obj}(\Gamma)$

$$\begin{aligned} \left[F(\nu)(j^M_y(w))\right]_x &= \nu_x(\left[j^M_y(w)\right]_x) \\ &= \begin{cases} \nu_x(0) & \text{if } x \neq y \\ \nu_x(w) & \text{if } x = y \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq y \\ \nu_y(w) & \text{if } x = y \end{cases} \\ &= \left[j^N_y(\nu_y(w))\right]_x\,. \end{aligned}$$

Therefore $(G(F(\nu))_y \circ j_y^M)(w) = (j_y^N \circ \nu_y)(w)$ as desired. So the collection of j^M's defines an isomorphism

$$j : \mathrm{id}_{R\text{-}\mathsf{Mod}^\Gamma} \to G \circ F$$

and this finishes the first part of the proposition.

For the second part of the proposition, assume Γ has finitely many objects. We first find an $R\Gamma$-module isomorphism

$$\varepsilon_U : U \to F(G(U))$$

for every left $R\Gamma$-module[1] U. Since

$$F(G(U)) = \bigoplus_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x U$$

we define

$$\varepsilon_U(u) = \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x u\,.$$

Note that the sum in this definition makes sense because $\mathrm{Obj}(\Gamma)$ is a finite set. Now for every $u, u' \in U$ and $r \in R$, we have

$$[\varepsilon_U(ru + u')]_x = \mathrm{id}_x(ru + u') = r\,\mathrm{id}_x u + \mathrm{id}_x u' = [r\varepsilon_U(u) + \varepsilon_U(u')]_x$$

hence ε_U is an R-module homomorphism. Moreover, given a morphism $\alpha : y \to z$ in Γ,

$$[\varepsilon_U(\alpha u)]_x = \mathrm{id}_x \alpha u = \begin{cases} 0 & \text{if } \mathrm{cod}(\alpha) \neq x \\ \alpha u & \text{if } \mathrm{cod}(\alpha) = x. \end{cases}$$

whereas

$$\begin{aligned} [\alpha \cdot \varepsilon_U(u)]_x &= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ G(U)(\alpha)([\varepsilon_U(u)]_{\mathrm{dom}(\alpha)}) & \text{if } x = \ \mathrm{cod}(\alpha) \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ G(U)(\alpha)(\mathrm{id}_{\mathrm{dom}(\alpha)} u) & \text{if } x = \ \mathrm{cod}(\alpha) \end{cases} \\ &= \begin{cases} 0 & \text{if } x \neq \mathrm{cod}(\alpha) \\ \alpha\,\mathrm{id}_{\mathrm{dom}(\alpha)} u & \text{if } x = \ \mathrm{cod}(\alpha). \end{cases} \end{aligned}$$

[1]Note that since $R\Gamma$ has an identity $1_{R\Gamma}$ in this case, we assume that U satisfies $1_{R\Gamma} \cdot u = u$ for every $u \in U$. We will use this assumption soon.

Thus $\varepsilon_U(\alpha u) = \alpha \cdot \varepsilon_U(u)$. So ε_U is actually an $R\Gamma$-module homomorphism. To show that it is an isomorphism we write an inverse

$$\delta_U : F(G(U)) \to U \,.$$

Firstly, for every $x \in \mathrm{Obj}(\Gamma)$, define

$$\delta_{U,x} : \mathrm{id}_x U \to U$$

as the inclusion map, which is definitely an R-module homomorphism. By the universal property of direct sums, $\delta_{U,x}$'s give the map we want:

$$\delta_U : F(G(U)) = \bigoplus_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x U \to U$$

Let $l \in F(G(U)) = \bigoplus_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x U$. Then writing $l = \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x u^x$ with $u^x \in U$,

$$\begin{aligned}
[\varepsilon_U(\delta_U(l))]_y &= \mathrm{id}_y \delta_U(l) \\
&= \mathrm{id}_y \delta_U \left(\sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x u^x \right) \\
&= \mathrm{id}_y \sum_{x \in \mathrm{Obj}(\Gamma)} \delta_{U,x}(\mathrm{id}_x u^x) \\
&= \mathrm{id}_y \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x u^x \\
&= \mathrm{id}_y u^y \\
&= l_y \,.
\end{aligned}$$

Thus $\varepsilon_U \circ \delta_U = \mathrm{id}_{F(G(U))}$. Now let $u \in U$. Then

$$\begin{aligned}
\delta_U(\varepsilon_U(u)) &= \sum_{x \in \mathrm{Obj}(\Gamma)} [\varepsilon_U(u)]_x \\
&= \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x u \\
&= \left(\sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x \right) u \\
&= 1_{R\Gamma} \cdot u \\
&= u \,.
\end{aligned}$$

Hence $\delta_U \circ \varepsilon_U = \mathrm{id}_U$.

Finally, we show that ε_U's are natural in U; so they define an isomorphism

$$\varepsilon : \mathrm{id}_{R\Gamma\text{-}\mathsf{Mod}} \to F \circ G$$

which finishes the proof. Let $\varphi : U \to V$ be an $R\Gamma$-module homomorphism. We need to show that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\quad\varphi\quad} & V \\ {\scriptstyle \varepsilon_U}\downarrow & & \downarrow{\scriptstyle \varepsilon_V} \\ F(G(U)) & \xrightarrow[F(G(\varphi))]{} & F(G(V)) \end{array}$$

commutes. Indeed,

$$\begin{aligned} \left[\left(F(G(\varphi)) \circ \varepsilon_U\right)(u)\right]_x &= \left[F(G(\varphi))(\varepsilon_U(u))\right]_x \\ &= G(\varphi)_x \left(\varepsilon_U(u)_x\right) \\ &= G(\varphi)_x(\mathrm{id}_x u) \\ &= \varphi(\mathrm{id}_x u) \\ &= \mathrm{id}_x \varphi(u) \\ &= \left[\varepsilon_V(\varphi(u))\right]_x \\ &= \left[\left(\varepsilon_V \circ \varphi\right)(u)\right]_x . \end{aligned}$$

This completes the proof. □

Remark 2.6. In the rest of this thesis, we will always assume that Γ has finitely many objects. In this case Proposition 2.5 establishes a category equivalence between $R\Gamma$-Mod and R-Mod^{Γ} so we can talk about modules of the category algebra $R\Gamma$ and representations of Γ over R interchangeably; this will be useful.

2.3 Right modules, bimodules, tensor product and adjunction

The category of *left* $R\Gamma$-modules is equivalent to the category of *covariant* functors from Γ to R-Mod by Proposition 2.5. For *right* $R\Gamma$-modules the following series of category equivalences give the expected result:

$$\mathsf{Mod}\text{-}R\Gamma \equiv (R\Gamma)^{\mathrm{op}}\text{-}\mathsf{Mod} \equiv R\Gamma^{\mathrm{op}}\text{-}\mathsf{Mod} \equiv R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}$$

Thus, the category of *right* $R\Gamma$-modules is equivalent to the category of *contravariant* functors from Γ to R-Mod. Note that the second equivalence above is because the opposite ring of $R\Gamma$ is isomorphic to $R\Gamma^{\mathrm{op}}$ (R is commutative!); the third equivalence is by Proposition 2.5.

If Λ is another small category with finitely many objects, we can talk about $R\Gamma$-$R\Lambda$-bimodules. We write the general Hom - tensor adjunctions for this case:

Theorem 2.7. *Let B be a $R\Gamma$-$R\Lambda$-bimodule. Then*

- $B \otimes_{R\Lambda} - : R\Lambda\text{-}\mathsf{Mod} \to R\Gamma\text{-}\mathsf{Mod}$ *is left adjoint to* $\mathrm{Hom}_{R\Gamma}(B, -) : R\Gamma\text{-}\mathsf{Mod} \to R\Lambda\text{-}\mathsf{Mod}$.
- $- \otimes_{R\Gamma} B : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R\Lambda$ *is left adjoint to* $\mathrm{Hom}_{R\Lambda}(B, -) : \mathsf{Mod}\text{-}R\Lambda \to \mathsf{Mod}\text{-}R\Gamma$.

We will often shortly write $F \dashv G$ instead of saying F is left adjoint to G. For the general theory of adjunctions, see [6], Chapter 9.

An important source of $R\Gamma$-$R\Lambda$-bimodules is left $R(\Gamma \times \Lambda^{\mathrm{op}})$-modules (so we are considering modules of the category algebra for the product category $\Gamma \times \Lambda^{\mathrm{op}}$). Indeed, a left $R(\Gamma \times \Lambda^{\mathrm{op}})$-module B is a left $R\Gamma$-module via

$$\alpha \cdot w = \left(\sum_{y \in \mathrm{Obj}(\Lambda)} (\alpha, \mathrm{id}_y) \right) w$$

for $\alpha \in \mathrm{Mor}(\Gamma)$ and a right $R\Lambda$-module via

$$w \cdot \beta = \left(\sum_{x \in \mathrm{Obj}(\Gamma)} (\mathrm{id}_x, \beta) \right) w$$

for $\beta \in \mathrm{Mor}(\Lambda) = \mathrm{Mor}(\Lambda^{\mathrm{op}})$. Clearly the left action of $R\Gamma$ commutes with the right action of $R\Lambda$, which makes B an $R\Gamma$-$R\Lambda$-bimodule[2].

On the other hand, we know that $R(\Gamma \times \Lambda^{\mathrm{op}})\text{-}\mathsf{Mod} \equiv R\text{-}\mathsf{Mod}^{\Gamma \times \Lambda^{\mathrm{op}}}$, so it is enough to write a covariant functor $B : \Gamma \times \Lambda^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$ to obtain an $R\Gamma$-$R\Lambda$-bimodule. We will do this to define induction and coinduction in the next section.

2.4 Restriction and induction

Let $F : \Gamma \to \Lambda$ be a covariant functor. (Γ, Λ are small categories with finitely many objects as always) It is natural to expect that F induces a restriction functor

$$\mathrm{Res}^F : R\Lambda\text{-}\mathsf{Mod} \to R\Gamma\text{-}\mathsf{Mod}\,.$$

Thinking in terms of the category algebras, it is tempting to think that F extends to a ring homomorphism $R\Gamma \to R\Lambda$; so then we can take Res^F as the restriction of scalars functor of this ring homomorphism. That would be just the generalization of the common approach in defining $\mathrm{Res}_\varphi : RG\text{-}\mathsf{Mod} \to RH\text{-}\mathsf{Mod}$ when $\varphi : H \to G$ is a group homomorphism. However F generally does not extend to a ring homomorphism:

Proposition 2.8 ([2], Proposition 3.2.5)**.** *Consider the R-module homomorphism $\tilde{F} : R\Gamma \to R\Lambda$ uniquely extending $F : \Gamma \to \Lambda$. Then*

- *$\tilde{F}$ is a rng homomorphism if and only if F is injective on objects.*
- *$\tilde{F}$ is a ring homomorphism if and only if F is bijective on objects.*

[2]The main difference between a left $R(\Gamma \times \Lambda^{\mathrm{op}})$-module and a general $R\Gamma$-$R\Lambda$-bimodule is that the left and right multiplications by elements of R can be distinct in the latter.

Proof. Assume $\tilde{F}$ is a rng homomorphism. Let $x, y \in \mathrm{Obj}(\Gamma)$ such that $F(x) = F(y)$. Then

$$\begin{aligned}
\tilde{F}(\mathrm{id}_x \mathrm{id}_y) &= \tilde{F}(\mathrm{id}_x)\tilde{F}(\mathrm{id}_y) \\
&= F(\mathrm{id}_x)F(\mathrm{id}_y) \\
&= \mathrm{id}_{F(x)} \mathrm{id}_{F(y)} \\
&= \mathrm{id}_{F(x)} \mathrm{id}_{F(x)} \\
&= \mathrm{id}_{F(x)} \circ \mathrm{id}_{F(x)} \\
&= \mathrm{id}_{F(x)} \\
&\neq 0
\end{aligned}$$

hence $\mathrm{id}_x \mathrm{id}_y \neq 0$. Therefore $\mathrm{cod}(\mathrm{id}_y) = \mathrm{dom}(\mathrm{id}_x)$, that is, $y = x$.

Conversely, assume F is injective on objects. It suffices to check that $\tilde{F}$ preserves the multiplications of the basis elements of $R\Gamma$ to deduce that $\tilde{F}$ is a rng homomorphism. Let $\alpha, \beta \in \mathrm{Mor}(\Gamma)$. Then

$$\begin{aligned}
\tilde{F}(\beta)\tilde{F}(\alpha) &= F(\beta)F(\alpha) \\
&= \begin{cases} 0 & \text{if } \mathrm{cod}(F(\alpha)) \neq \mathrm{dom}(F(\beta)) \\ F(\beta) \circ F(\alpha) & \text{if } \mathrm{cod}(F(\alpha)) = \mathrm{dom}(F(\beta)) \end{cases} \\
&= \begin{cases} 0 & \text{if } F(\mathrm{cod}(\alpha)) \neq F(\mathrm{dom}(\beta)) \\ F(\beta) \circ F(\alpha) & \text{if } F(\mathrm{cod}(\alpha)) = F(\mathrm{dom}(\beta)) \end{cases} \\
&= \begin{cases} 0 & \text{if } \mathrm{cod}(\alpha) \neq \mathrm{dom}(\beta) \\ F(\beta) \circ F(\alpha) & \text{if } \mathrm{cod}(\alpha) = \mathrm{dom}(\beta) \end{cases} \\
&= \begin{cases} 0 & \text{if } \mathrm{cod}(\alpha) \neq \mathrm{dom}(\beta) \\ F(\beta \circ \alpha) & \text{if } \mathrm{cod}(\alpha) = \mathrm{dom}(\beta) \end{cases} \\
&= \tilde{F}\left(\begin{cases} 0 & \text{if } \mathrm{cod}(\alpha) \neq \mathrm{dom}(\beta) \\ \beta \circ \alpha & \text{if } \mathrm{cod}(\alpha) = \mathrm{dom}(\beta) \end{cases} \right) \\
&= \tilde{F}(\beta\alpha)
\end{aligned}$$

and we are done. Here the 3rd and 5th equalities hold because F is a functor and the 4th equality holds because F is injective on objects.

For the second part of the proposition, first assume $\tilde{F}$ is a ring homomorphism. In particular $\tilde{F}$ is a rng homomorphism and so F is injective on objects. Moreover ring homomorphisms preserve units, so

$$\begin{aligned} \sum_{y \in \mathrm{Obj}(\Lambda)} \mathrm{id}_y &= 1_{R\Lambda} \\ &= \tilde{F}(1_{R\Gamma}) \\ &= \tilde{F}\left(\sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x \right) \\ &= \sum_{x \in \mathrm{Obj}(\Gamma)} F(\mathrm{id}_x) \\ &= \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_{F(x)} \,. \end{aligned}$$

It follows that $\displaystyle\sum_{y \in \mathrm{Obj}(\Lambda) - F(\mathrm{Obj}(\Gamma))} \mathrm{id}_y = 0$. Hence $\mathrm{Obj}(\Lambda) = F(\mathrm{Obj}(\Gamma))$, that is, F is surjective on objects.

Conversely, assume F is bijective on objects. Then by the first part $\tilde{F}$ is a rng homomorphism. Moreover

$$\begin{aligned} \tilde{F}(1_{R\Gamma}) &= \tilde{F}\left(\sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_x \right) \\ &= \sum_{x \in \mathrm{Obj}(\Gamma)} F(\mathrm{id}_x) \\ &= \sum_{x \in \mathrm{Obj}(\Gamma)} \mathrm{id}_{F(x)} \\ &= \sum_{y \in \mathrm{Obj}(\Lambda)} \mathrm{id}_y \\ &= 1_{R\Lambda} \end{aligned}$$

where the 4th equality is by bijectivity of F on objects. Hence $\tilde{F}$ is a ring homomorphism. □

So constructing $\mathrm{Res}^F : R\Lambda\text{-}\mathsf{Mod} \to R\Gamma\text{-}\mathsf{Mod}$ as a restriction of scalars functor of a ring (or even rng) homomorphism is not an option in general. We can bypass this issue by considering the categories $R\text{-}\mathsf{Mod}^\Gamma$ and $R\text{-}\mathsf{Mod}^\Lambda$ instead of their equivalent counterparts $R\Gamma\text{-}\mathsf{Mod}$ and $R\Lambda\text{-}\mathsf{Mod}$. Here there is an obvious way to define the restriction:

$$\begin{aligned}\mathrm{Res}^F : R\text{-}\mathsf{Mod}^\Lambda &\to R\text{-}\mathsf{Mod}^\Gamma \\ N &\mapsto N \circ F\end{aligned}$$

for every $N \in \mathrm{Obj}(R\text{-}\mathsf{Mod}^\Lambda)$, i.e for every covariant functor $N : \Lambda \to R\text{-}\mathsf{Mod}$. And for a morphism $\nu : N \to N'$ in $R\text{-}\mathsf{Mod}^\Lambda$, we define

$$\mathrm{Res}^F(\nu) : N \circ F = \mathrm{Res}^F(N) \to \mathrm{Res}^F(N') = N' \circ F$$

by $\mathrm{Res}^F(\nu)_x = \nu_{F(x)} : N(F(x)) \to N'(F(x))$ for every $x \in \mathrm{Obj}(\Gamma)$. Res^F is clearly a functor with these assignments.

Remark 2.9. At this point we change our convention and start working with *right* modules instead of left; in other words we work with categories of the form $\mathsf{Mod}\text{-}R\Gamma$ instead of $R\Gamma\text{-}\mathsf{Mod}$. Clearly the theory of left and right modules coincide by taking the opposite categories; however in the future we will specifically work with orbit categories and be interested in their right modules, rather than left modules.

For instance given a covariant functor $F : \Gamma \to \Lambda$, we will deal with the restriction functor

$$\mathrm{Res}_F : \mathsf{Mod}\text{-}R\Lambda \equiv R\text{-}\mathsf{Mod}^{\Lambda^{\mathrm{op}}} \to R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}} \equiv \mathsf{Mod}\text{-}R\Gamma$$

defined in the same way as above.

Res_F is clearly an exact functor, so we suspect that it might have both left and right adjoints. Indeed it does, and the strategy for constructing these adjoints is to use the general adjunction in Theorem 2.7. We will define an $R\Gamma$-$R\Lambda$-bimodule B such that the functor

$$\mathrm{Hom}_{R\Lambda}(B, -) : \mathsf{Mod}\text{-}R\Lambda \to \mathsf{Mod}\text{-}R\Gamma$$

will be isomorphic to Res_F. As we discussed at the end of the previous section we define B as a covariant functor $B : \Gamma \times \Lambda^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$. Let B be the composition

$$\Gamma \times \Lambda^{\mathrm{op}} \xrightarrow{F\times \mathrm{id}} \Lambda \times \Lambda^{\mathrm{op}} \cong \Lambda^{\mathrm{op}} \times \Lambda \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} \mathsf{Set} \xrightarrow{\mathfrak{F}} R\text{-}\mathsf{Mod}$$

where $\mathfrak{F}$ denotes the free functor, which sends every set to the free R-module that it generates. Note that the forgetful functor $\mathfrak{G} : R\text{-}\mathsf{Mod} \to \mathsf{Set}$ is right adjoint to $\mathfrak{F}$.

For a right $R\Lambda$-module N, equivalently a contravariant functor $N : \Lambda \to R\text{-}\mathsf{Mod}$, considering $\mathrm{Hom}_{R\Lambda}(B, N)$ as a contravariant functor from Γ to $R\text{-}\mathsf{Mod}$, for every $x \in \mathrm{Obj}(\Gamma)$ we have isomorphisms (as sets)

$$\begin{aligned}
\mathrm{Hom}_{R\Lambda}(B, N)(x) &= \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\Lambda^{\mathrm{op}}}}(B(F(x), -), N)\\
&= \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\Lambda^{\mathrm{op}}}}(\mathfrak{F} \circ \mathrm{Hom}_\Lambda(-, F(x)), N)\\
&\cong \mathrm{Hom}_{\mathsf{Set}^{\Lambda^{\mathrm{op}}}}(\mathrm{Hom}_\Lambda(-, F(x)), \mathfrak{G} \circ N)\\
&\cong \mathfrak{G}(N(F(x))
\end{aligned}$$

where the first isomorphism is by the adjunction $\mathfrak{F} \dashv \mathfrak{G}$ and the second is by the Yoneda lemma[3]. It is easy to see that the resulting bijection between the sets $\mathrm{Hom}_{R\Lambda}(B, N)(x)$ and $N(F(x))$ is actually R-linear, i.e we have an R-module isomorphism

$$\mathrm{Hom}_{R\Lambda}(B, N)(x) \cong N(F(x)) = \mathrm{Res}_F(N)(x)\,.$$

Moreover this isomorphism is natural in x and N by naturality of adjunction and the Yoneda lemma. Hence we get

$$\mathrm{Hom}_{R\Lambda}(B, -) \cong \mathrm{Res}_F$$

as desired. In this case we denote the functor $- \otimes_{R\Gamma} B$ as Ind_F so we have

$$\mathrm{Ind}_F \dashv \mathrm{Res}_F\ .$$

This adjunction, together with the exactness of Res_F immediately yields the following:

Corollary 2.10. $\mathrm{Ind}_F : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R\Lambda$ *sends projectives to projectives.*

[3]We will have a very similar situation in the next section where the Yoneda lemma is crucial.

Realizing Res_F as a *left* adjoint can be done with a similar approach. Here is a sketch: Note that given a $R\Lambda$-$R\Gamma$-bimodule C, we have functors

$-\otimes_{R\Lambda} C : \mathsf{Mod}\text{-}R\Lambda \to \mathsf{Mod}\text{-}R\Gamma$ and $\mathrm{Hom}_{R\Gamma}(C,-) : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R\Lambda$ such that

$$-\otimes_{R\Lambda} C \dashv \mathrm{Hom}_{R\Gamma}(C,-)$$

by Theorem 2.7 (just interchange Γ and Λ in the theorem). There is a specific $R\Lambda$-$R\Gamma$-bimodule C such that $\mathrm{Res}_F \cong -\otimes_{R\Lambda} C$. This C as a covariant functor $\Lambda \times \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$ is given by the following composition:

$$\Lambda \times \Gamma^{\mathrm{op}} \xrightarrow{\mathrm{id}\times F} \Lambda \times \Lambda^{\mathrm{op}} \cong \Lambda^{\mathrm{op}} \times \Lambda \xrightarrow{\mathrm{Hom}_\Lambda(-,-)} \mathsf{Set} \xrightarrow{\mathfrak{F}} R\text{-}\mathsf{Mod}$$

It remains to check for every $N \in \mathrm{Obj}(R\text{-}\mathsf{Mod}^{\Lambda^{\mathrm{op}}})$ and $x \in \mathrm{Obj}(\Gamma)$, there is an R-module isomorphism as below

$$(N\otimes_{R\Lambda}C)(x) = N\otimes_{R\Lambda}C(-,x) = N\otimes_{R\Lambda}(\mathfrak{F}\circ \mathrm{Hom}_\Lambda(F(x),-)) \cong N(F(x)) = \mathrm{Res}_F(N)(x)$$

which is natural in N and x. Then it follows that $-\otimes_{R\Lambda} C \cong \mathrm{Res}_F$ and calling the functor $\mathrm{Hom}_{R\Gamma}(C,-)$ as Coind_F we get

$$\mathrm{Res}_F \dashv \mathrm{Coind}_F$$

and hence

Corollary 2.11. $\mathrm{Coind}_F : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R\Lambda$ *sends injectives to injectives.*

We will be mostly interested in with specific inclusion functors and the restriction and induction functors they yield. Here is the setup: Fix $x \in \mathrm{Obj}(\Gamma)$. Let Γ_x be the subcategory of Γ with the single object x and with $\mathrm{End}_{\Gamma_x}(x) = \mathrm{Aut}_\Gamma(x)$. That is, Γ_x contains only x and its automorphisms. We have an inclusion functor

$$F : \Gamma_x \to \Gamma\,.$$

Hence we get the restriction and induction functors $\mathrm{Res}_F : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R\Gamma_x$ and $\mathrm{Ind}_F : \mathsf{Mod}\text{-}R\Gamma_x \to \mathsf{Mod}\text{-}R\Gamma$. In this case we write Res_x instead of Res_F and E_x instead of Ind_F.

Note that the category algebra $R\Gamma_x$ is precisely the group algebra $R\operatorname{Aut}_\Gamma(x)$. Hence $\mathsf{Mod}\text{-}R\Gamma_x = \mathsf{Mod}\text{-}R\operatorname{Aut}_\Gamma(x)$. We usually write shortly $R[x]$ instead of $R\operatorname{Aut}_\Gamma(x)$ when Γ is clear from context. With these considerations we see that for a right $R\Gamma$-module N,

$$\operatorname{Res}_x(N) = N(x)$$

where we consider $N(x)$ not only as an R-module but as an $R[x]$-module. And for a right $R[x]$-module M, for every $y \in \operatorname{Obj}(\Gamma)$, we have

$$E_x(M)(y) = M \otimes_{R[x]} R\operatorname{Hom}_\Gamma(y, x)$$

where the left $R\operatorname{Aut}_\Gamma(x)$-module structure on $R\operatorname{Hom}_\Gamma(y, x)$ is given by the R-linearization of the left $\operatorname{Aut}_\Gamma(x)$-action on $\operatorname{Hom}_\Gamma(y, x)$.

Observe that by Corollary 2.10, we can use the functor E_x to get projective modules in $\mathsf{Mod}\text{-}R\Gamma$ by using projective $R[x]$-modules.

We can say even more when Γ satisfy a freeness condition. Here is the freeness condition:

Definition 2.12. Γ is called a *free category* if for every $x, y \in \operatorname{Obj}(\Gamma)$, the set $\operatorname{Hom}_\Gamma(y, x)$ is a free left $\operatorname{Aut}_\Gamma(x)$-set, that is, for every $f \in \operatorname{Hom}_\Gamma(y, x)$ and $g \in \operatorname{Aut}_\Gamma(x)$, the equation $g \circ f = f$ implies $g = \mathrm{id}_x$.

And here is what we can say more when Γ is free:

Proposition 2.13. *If Γ is free, then $E_x : \mathsf{Mod}\text{-}R[x] \to \mathsf{Mod}\text{-}R\Gamma$ is an exact functor for every $x \in \operatorname{Obj}(\Gamma)$.*

Proof. Recall that given a right $R[x]$-module M, we have

$$E_x(M)(y) = M \otimes_{R[x]} R\operatorname{Hom}_\Gamma(y, x)$$

for every $y \in \operatorname{Obj}(\Gamma)$. If

$$0 \longrightarrow M'' \xrightarrow{\lambda} M \xrightarrow{\mu} M' \longrightarrow 0$$

is an exact sequence of right $R[x]$-modules, when we apply E_x to it, we get

$$0 \longrightarrow E_x(M'') \xrightarrow{E_x(\lambda)} E_x(M) \xrightarrow{E_x(\mu)} E_x(M') \longrightarrow 0$$

We claim that the above is an exact sequence. This amounts to checking the exactness of

$$0 \longrightarrow E_x(M'')(y) \xrightarrow{E_x(\lambda)_y} E_x(M)(y) \xrightarrow{E_x(\mu)_y} E_x(M')(y) \longrightarrow 0$$

for every $y \in \mathrm{Obj}(\Gamma)$. But this is precisely the sequence obtained after applying the functor $- \otimes_{R[x]} R\,\mathrm{Hom}(y, x)$ to the original exact sequence

$$0 \longrightarrow M'' \xrightarrow{\lambda} M \xrightarrow{\mu} M' \longrightarrow 0$$

Finally by freeness of Γ, $\mathrm{Hom}(y, x)$ is a free left $\mathrm{Aut}_\Gamma(x)$-set and hence $R\,\mathrm{Hom}(y, x)$ is a free left $R[x]$-module. Therefore the functor $- \otimes_{R[x]} R\,\mathrm{Hom}(y, x)$ is exact and we are done. □

It follows by Corollary 2.10 and Proposition 2.13 that when Γ is free, E_x sends *projective resolutions* to *projective resolutions*. This yields a useful method to calculate Ext-groups of $R\Gamma$-modules:

Proposition 2.14. *If* Γ *is free, then* $\mathrm{Ext}^*_{R\Gamma}(E_x(M), N) \cong \mathrm{Ext}^*_{R[x]}(M, \mathrm{Res}_x(N))$

Proof. Take a projective resolution $\mathbf{P}$ of M. Then as Γ is free, $E_x(\mathbf{P})$ is a projective resolution of $E_x(M)$. Thus

$$\begin{aligned} \mathrm{Ext}^n_{R\Gamma}(E_x(M), N) &= H^n(\mathrm{Hom}_{R\Gamma}(E_x(\mathbf{P}), N)) \\ &\cong H^n(\mathrm{Hom}_{R[x]}(\mathbf{P}, \mathrm{Res}_x(N))) \\ &= \mathrm{Ext}^n_{R[x]}(M, \mathrm{Res}_x(N)) \end{aligned}$$

where the isomorphism is due to the adjunction $E_x \dashv \mathrm{Res}_x$. □

2.5 Yoneda lemma and projectives in Mod-$R\Gamma$

In this section, we give an important class of projective objects by making use of the Yoneda lemma. Fix $x \in \mathrm{Obj}(\Gamma)$. Consider the composition of functors

$$R\Gamma(-, x) : \Gamma \xrightarrow{\mathrm{Hom}_\Gamma(-,x)} \mathsf{Set} \xrightarrow{\mathfrak{F}} R\text{-}\mathsf{Mod}$$

where $\mathfrak{F}$ denotes the free functor. Note that $R\Gamma(-,x)$ is contravariant hence lies in $R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}$.

Now for every $M \in \mathrm{Obj}(R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}})$,

$$\begin{aligned}\mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}}(R\Gamma(-,x),M) &= \mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}}(\mathfrak{F}\circ\mathrm{Hom}_\Gamma(-,x),M)\\ &\cong \mathrm{Hom}_{\mathsf{Set}^{\Gamma^{\mathrm{op}}}}(\mathrm{Hom}_\Gamma(-,x),\mathfrak{G}\circ M)\\ &\cong (\mathfrak{G}\circ M)(x)\end{aligned}$$

where $\mathfrak{G}: R\text{-}\mathsf{Mod} \to \mathsf{Set}$ denotes the forgetful functor. The first isomorphism is by the adjunction $\mathfrak{F} \dashv \mathfrak{G}$ and the second isomorphism is by the Yoneda lemma. Now as a set, $(\mathfrak{G}\circ M)(x) = M(x)$. It is easy to see that the bijection above preserves the R-module structures in $\mathrm{Hom}_{R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}}(R\Gamma(-,x),M)$ and $M(x)$. Moreover both isomorphisms given by adjunction and the Yoneda lemma are natural in M. Hence we conclude that the functor

$$\mathrm{Hom}_{R\Gamma}(R\Gamma(-,x),-): R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}} \to R\text{-}\mathsf{Mod}$$

is isomorphic to the evaluation functor

$$\begin{aligned}\mathrm{ev}_x : R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}} &\to R\text{-}\mathsf{Mod}\\ M &\mapsto M(x)\,.\end{aligned}$$

Since ev_x is clearly an exact functor, $\mathrm{Hom}_{R\Gamma}(R\Gamma(-,x),-)$ is an exact functor. Therefore $R\Gamma(-,x)$ is a projective object in $R\text{-}\mathsf{Mod}^{\Gamma^{\mathrm{op}}}$.

We constructed $R\Gamma(-,x)$ as a contravariant functor from Γ to $R\text{-}\mathsf{Mod}$. We know that $R\Gamma(-,x)$ corresponds to a right module of the category algebra $R\Gamma$. This module is precisely the right ideal $\mathrm{id}_x R\Gamma$ of $R\Gamma$. Actually the regular right $R\Gamma$-module has the decomposition

$$R\Gamma_{R\Gamma} = \bigoplus_{x\in\mathrm{Obj}(\Gamma)} \mathrm{id}_x R\Gamma$$

which is another way of seeing the projectivity of $R\Gamma(-,x)$'s.

We will use the following criteria to check whether a functor with domain $\mathsf{Mod}\text{-}R\Gamma$ preserves projectives:

Proposition 2.15. *Let* A *be an abelian category an* $F : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{A}$ *a functor which preserves direct sums (coproducts). Then the following are equivalent:*

1. F *sends projectives to projectives.*
2. $F(\mathrm{id}_x R\Gamma)$ *is projective for every* $x \in \mathrm{Obj}(\Gamma)$*.*
3. $F(R\Gamma_{R\Gamma})$ *is projective.*

Proof. (1) ⇒ (2) is obvious. (2) ⇒ (3) is by

$$F(R\Gamma_{R\Gamma}) = F\left(\bigoplus_{x\in\mathrm{Obj}(\Gamma)} \mathrm{id}_x R\Gamma\right) \cong \bigoplus_{x\in\mathrm{Obj}(\Gamma)} F(\mathrm{id}_x R\Gamma)\,.$$

For (3) ⇒ (1), note that (3) gives that F sends free right $R\Gamma$-modules to projectives in A. Now if P is a projective right $R\Gamma$-module, $P \oplus Q$ is free for some Q and hence $F(P) \oplus F(Q) \cong F(P \oplus Q)$ is projective. Since a direct summand of a projective is projective in any abelian category, $F(P)$ is projective in A. □

Here is a quick application:

Corollary 2.16. *Let* $x \in \mathrm{Obj}(\Gamma)$*. The evaluation functor* $ev_x : \mathsf{Mod}\text{-}R\Gamma \to R\text{-}\mathsf{Mod}$ *sends projectives to projectives.*

Proof. For every $y \in \mathrm{Obj}(\Gamma)$,

$$\mathrm{ev}_x(R\Gamma(-, y)) = R\,\mathrm{Hom}_\Gamma(x, y)$$

is a free, hence projective R-module. By Proposition 2.15, we are done. □

Chapter 3

EI-categories and their representations

A category Γ is called an *EI-category* if every endomorphism is an isomorphism. Representations of EI-categories enjoy several properties that general category representations do not have. We will focus on orbit categories in the next chapter, which turn out to be EI-categories. Many of the interesting properties of the orbit categories stem from their EI-property, so we study general EI-categories in this chapter and prove the relevant results that we will use in the future.

Throughout this section, Γ is an EI-category with finitely many objects and R is a nonzero commutative ring. Our main source of references for this section are [4] and [5].

3.1 The length of a representation

In this section, we define the notion of length in an EI-category and study the related properties.

We denote the isomorphism class of an object x in Γ by $\overline{x}$ and the set of isomorphism classes of Γ by $\mathrm{Iso}(\Gamma)$.

Proposition 3.1. *There is a partial order* $\leq$ *on* $\mathrm{Iso}(\Gamma)$ *defined by*

$$\overline{x} \leq \overline{y} \iff \mathrm{Hom}_\Gamma(x, y) \neq \emptyset\,.$$

Proof. The relation is well-defined since if $x \cong x'$ and $y \cong y'$ there is a bijection between $\mathrm{Hom}_\Gamma(x, y)$ and $\mathrm{Hom}_\Gamma(x', y')$; in particular

$$\mathrm{Hom}_\Gamma(x, y) \neq \emptyset \iff \mathrm{Hom}_\Gamma(x', y') \neq \emptyset\,.$$

Reflexivity of $\leq$ is by the existence of identity morphisms and transitivity of $\leq$ is by composition. We need the EI-property for antisymmetry; indeed if $\overline{x} \leq \overline{y} \leq \overline{x}$, there exist morphisms $\alpha : x \to y$ and $\beta : y \to x$ in Γ. Since Γ is EI, $\beta \circ \alpha$ and $\alpha \circ \beta$ are isomorphisms. Since $\beta \circ \alpha$ has a left inverse, α has a left inverse. And since $\alpha \circ \beta$ has a right inverse, α has a right inverse. Thus α is an isomorphism[1] and hence $\overline{x} = \overline{y}$. □

The fact that $\mathrm{Iso}(\Gamma)$ has a poset structure and that Γ has finitely many objects allows us to define various notions of lengths:

Definition 3.2. Given a chain $\overline{x_0} < \overline{x_1} < \cdots < \overline{x_n}$ in $\mathrm{Iso}(\Gamma)$, n is called the *length of the chain.*

Definition 3.3. Given $\overline{x} \in \mathrm{Iso}(\Gamma)$, the length of the longest chain in Γ that ends with $\overline{x}$ is denoted by $l(\overline{x})$.

So for instance if $\overline{x}$ is a minimal element, $l(\overline{x}) = 0$.

Definition 3.4. The number $\max\{l(\overline{x}) : x \in \mathrm{Obj}(\Gamma)\}$ is called the *length of* Γ and denoted by $l(\Gamma)$.

Definition 3.5. Given a nonzero right $R\Gamma$-module M, the largest number in the set $\{l(\overline{x}) : M(x) \neq 0\}$ is called the *length of* M and denoted by $l(M)$. If x is an object such that $M(x) \neq 0$ and $M(y) = 0$ whenever $\overline{y} > \overline{x}$, x is called a *maximal object of* M. For $M = 0$, we write $l(M) = -1$.

[1] Similarly β and actually *every* morphism between x and y is an isomorphism in this case.

In other words, $l(M)$ is the length of a longest chain whose last term does not vanish under M. The length of an $R\Gamma$-module provides an important handle to employ induction in proofs.

Example 3.6. Let $y \in \mathrm{Obj}(\Gamma)$. Then $l(R\Gamma(-, y)) = l(\overline{y})$. This is because

$$R\Gamma(-, y)(x) = R\,\mathrm{Hom}_\Gamma(x, y) \neq 0 \iff \mathrm{Hom}_\Gamma(x, y) \neq \emptyset \iff \overline{x} \leq \overline{y}$$

Example 3.7. Let $x \in \mathrm{Obj}(\Gamma)$ and let A be a nonzero right $R[x]$-module. Then the induced module $E_x(A)$ has length $l(\overline{x})$. Indeed

$$E_x(A)(y) = A \otimes_{R[x]} R\,\mathrm{Hom}_\Gamma(y, x) \neq 0$$

implies $\overline{y} \leq \overline{x}$ and

$$E_x(A)(x) = A \otimes_{R[x]} R\,\mathrm{Hom}_\Gamma(x, x) = A \otimes_{R[x]} R[x] \cong A \neq 0\,.$$

Note how we used the EI-property of Γ in the last example. This will be even more apparent when we investigate the adjunctions $E_x \dashv \mathrm{Res}_x$.

Proposition 3.8. *Let*

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$$

be an exact sequence of right $R\Gamma$-modules. Then $l(M) = \max\{l(L), l(N)\}$.

Proof. Clearly for any $x \in \mathrm{Obj}(\Gamma)$, if $M(x) = 0$ then $L(x) = N(x) = 0$. That is $L(x) \neq 0$ implies $M(x) \neq 0$ and $N(x) \neq 0$ implies $M(x) \neq 0$. The former implication gives $l(L) \leq l(M)$ and the latter gives $l(N) \leq l(M)$; hence $l(M) \geq \max\{l(L), l(M)\}$.

For the converse, suppose $l(M) > l(L)$ and $l(M) > l(N)$. So $M \neq 0$ and M has a maximal object x. Since $l(L) < l(M) = l(\overline{x})$, $L(x) = 0$ and similarly $N(x) = 0$. This is absurd because we have an exact sequence

$$0 \longrightarrow L(x) \longrightarrow M(x) \longrightarrow N(x) \longrightarrow 0$$

of R-modules. This finishes the proof. $\square$

Proposition 3.9. *Let M be a right $R\Gamma$-module. Then there exists an epimorphism*

$$\phi : P \to M$$

where P is a projective right $R\Gamma$-module such that $l(P) = l(M)$.

Proof. Note that for every $x \in \mathrm{Obj}(\Gamma)$ and every $m \in M(x)$, by the Yoneda lemma there is a morphism

$$\phi^{x,m} : R\Gamma(-, x) \to M$$

such that ${\phi^{x,m}}_x(\mathrm{id}_x) = m$. Let $\mathcal{S} = \{x \in \mathrm{Obj}(\Gamma) : M(x) \neq 0\}$ and let

$$P = \bigoplus_{x \in \mathcal{S}} \bigoplus_{m \in M(x)} R\Gamma(-, x)\,.$$

As a direct sum of projectives, P is projective. By the universal property of direct sums, $\phi^{x,m}$'s yield a morphism $\phi : P \to M$. Here ϕ is an epimorphism because for every $x \in \mathrm{Obj}(\Gamma)$, the R-module homomorphism $\phi_x : P(x) \to M(x)$ is surjective, as for any $m \in M(x)$ we have $\phi_x(\mathrm{id}_x) = m$.
Finally, by using Proposition 3.8 and Example 3.6 we observe that

$$\begin{aligned} l(P) &= l(\bigoplus_{x \in \mathcal{S}} \bigoplus_{m \in M(x)} R\Gamma(-, x)) \\ &= \max_{x \in \mathcal{S}} \{l(\bigoplus_{m \in M(x)} R\Gamma(-, x))\} \\ &= \max_{x \in \mathcal{S}} \{l(R\Gamma(-, x))\} \\ &= \max_{x \in \mathcal{S}} \{l(\overline{x})\} \\ &= l(M)\,. \end{aligned}$$

□

Definition 3.10. Let

$$\mathbf{C} : \dots C_n \to C_{n-1} \to \dots C_1 \to C_0 \to C_{-1} \to \dots$$

be a chain complex of right $R\Gamma$-modules. Then, we define the length of $\mathbf{C}$ by

$$l(\mathbf{C}) := \max\{l(C_i) : i \in \mathbb{Z}\}\,.$$

Corollary 3.11. *Let M be a right $R\Gamma$-module. Then M has a projective resolution $\mathbf{P} \to M$ such that $l(\mathbf{P}) \leq l(M)$.*

Proof. We construct $\mathbf{P}$ inductively. By Proposition 3.9 there is a short exact sequence

$$0 \longrightarrow K_0 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_0 is projective and $l(P_0) = l(M)$. Note that $l(K_0) \leq l(P_0) = l(M)$. Assume we have an exact sequence

$$0 \longrightarrow K_n \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \dots \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_i is projective and $l(P_i) \leq l(M)$ for all $i = 1, \dots, n$. Then $l(K_n) \leq l(P_n)$ and by Proposition 3.9 there is a short exact sequence

$$0 \longrightarrow K_{n+1} \longrightarrow P_{n+1} \longrightarrow K_n \longrightarrow 0$$

with $l(P_{n+1}) = l(K_n) \leq l(P_n) \leq l(M)$.
This finishes the definition of $\mathbf{P}$ which has the desired properties.

□

Proposition 3.8 is informative but not enough to admit inductive proofs on the length of modules. We may very well have $l(L) = l(M) = l(N)$ above, but we need something like $l(L) < l(N)$ to employ induction on the length. Here is a condition that ensures this:

Lemma 3.12. *Let*

$$0 \longrightarrow L \longrightarrow M \xrightarrow{\mu} N \longrightarrow 0$$

be a short exact sequence of right $R\Gamma$-modules with $M \neq 0$. If $\mu_x : M(x) \to N(x)$ is an isomorphism of R-modules for every maximal object x of M, then

$$l(L) < l(M) = l(N)\,.$$

Proof. The condition on μ ensures that the maximal objects of M and N are the same. Hence $l(M) = l(N)$. If $L = 0$, $l(L) = -1 < l(M)$ and we are done.

Otherwise let y be a maximal object of L. Since $L(y) \neq 0$, $M(y) \neq 0$; hence $l(M) \geq l(\overline{y})$. Suppose $l(M) = l(\overline{y})$, but then y is a maximal object of M. By the exact sequence

$$0 \longrightarrow L(y) \longrightarrow M(y) \xrightarrow{\mu_y} N(y) \longrightarrow 0$$

we get $L(y) = 0$ since μ_y is an isomorphism. This is a contradiction, therefore we have $l(L) = l(\overline{y}) < l(M)$. □

Remark 3.13. Recalling the restriction functors $\mathrm{Res}_x : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R[x]$, the condition on μ in Proposition 3.12 is equivalent to saying that $\mathrm{Res}_x(\mu)$ is an isomorphism for every maximal object x of M.

If $\mu : M \to N$ is a morphism which satisfies all the properties in Proposition 3.12 except being an epimorphism, the situation can be "fixed" in a harmless way:

Lemma 3.14. *Let $\mu : M \to N$ be a morphism of nonzero right $R\Gamma$-modules such that $l(M) = l(N)$ and $\mu_x : M(x) \to N(x)$ is an isomorphism for every maximal object x of M. Then there is a projective right $R\Gamma$-module Q such that $l(Q) < l(N)$ and a morphism $\theta : Q \to N$ such that the induced morphism $[\mu,\theta] : M \oplus Q \to N$ is an epimorphism.*

Proof. Let $\gamma : N \to C$ be a cokernel of μ. Clearly $l(C) \leq l(N)$. Suppose $l(C) = l(N)$, so $C \neq 0$. Let y be a maximal object of C. Then since $C(y) \neq 0$ and $\gamma_y : N(y) \to C(y)$ is surjective, $N(y) \neq 0$. Hence $l(N) \geq l(\overline{y}) = l(C) = l(N)$, that is, $l(N) = l(\overline{y})$ and y is a maximal object of N. But we have an exact sequence

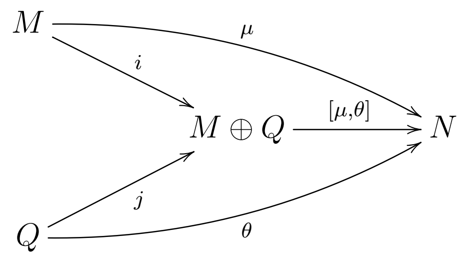
$$M(y) \xrightarrow{\mu_y} N(y) \xrightarrow{\gamma_y} C(y) \longrightarrow 0$$

which forces $C(y) = 0$ since μ_y is an isomorphism. This is a contradiction.

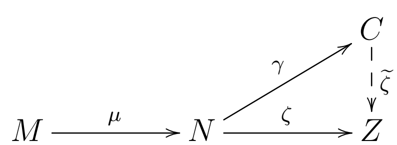
Thus $l(C) < l(N)$. Now by Proposition 3.9 there is an epimorphism $\pi : Q \to C$ such that Q is projective and $l(Q) = l(C)$. By the lifting property of projectives, there is a morphism $\theta : Q \to N$ such that $\pi = \gamma \circ \theta$. Finally we check that

$$[\mu,\theta] : M \oplus Q \to N$$

is an epimorphism. We show that if a morphism $\zeta : N \to Z$ satisfies $\zeta \circ [\mu,\theta] = 0$, then $\zeta = 0$. Since $[\mu,\theta]$ is given by the universal property of direct sums, we have a commutative diagram



Therefore $\zeta \circ \mu = \zeta \circ [\mu,\theta] \circ i = 0$. So by universal property of cokernels, ζ factors (uniquely) through γ, say via $\widetilde{\zeta}$:



Now

$$\begin{aligned}\widetilde{\zeta} \circ \pi &= \widetilde{\zeta} \circ \gamma \circ \theta \\ &= \zeta \circ \theta \\ &= \zeta \circ [\mu,\theta] \circ j \\ &= 0 \circ j = 0\,.\end{aligned}$$

But π is an epimorphism, so $\widetilde{\zeta} = 0$. Thus $\zeta = 0$. □

Remark 3.15. In the situation of Lemma 3.14, for every maximal object x of M, $[\mu,\theta]_x$ is an isomorphism. This is because $l(Q) < l(N) = l(M) = l(\overline{x})$, which yields $Q(x) = 0$. So adding θ to μ gives an epimorphism while preserving the crucial property of μ.

3.2 $E_x \dashv \mathrm{Res}_x$ adjunctions and related constructions

We can say more about the adjunctions $E_x \dashv \mathrm{Res}_x$ now as Γ is an EI-category:

Proposition 3.16. *Let $x \in \mathrm{Obj}(\Gamma)$. The unit $\eta^x : \mathrm{id}_{\mathsf{Mod}\text{-}R[x]} \to \mathrm{Res}_x E_x$ of the adjunction $E_x \dashv \mathrm{Res}_x$ is an isomorphism.*

Proof. It suffices to check that for every right $R[x]$-module A,

$$\eta^x_A : A \to \mathrm{Res}_x E_x(A)$$

is an isomorphism. Indeed,

$$\mathrm{Res}_x E_x(A) = E_x(A)(x) = A \otimes_{R[x]} R\,\mathrm{Hom}_\Gamma(x, x)$$

and η^x_A is given by

$$\begin{aligned} \eta^x_A : A &\to A \otimes_{R[x]} R\,\mathrm{Hom}_\Gamma(x, x) \\ a &\mapsto a \otimes \mathrm{id}_x \ . \end{aligned}$$

Since Γ is EI, $R\,\mathrm{Hom}_\Gamma(x, x)$ on the right hand side of the tensor product is nothing but $R[x]$ as the regular left $R[x]$-module and hence η^x_A is an isomorphism. □

Given x, let $\eta^x : \mathrm{id}_{\mathsf{Mod}\text{-}R[x]} \to \mathrm{Res}_x E_x$ be the unit (as above) and let $\epsilon^x : E_x \mathrm{Res}_x \to \mathrm{id}_{\mathsf{Mod}\text{-}R\Gamma}$ be the counit of the adjunction $E_x \dashv \mathrm{Res}_x$. These yield natural transformations

$$\eta^x \mathrm{Res}_x : \mathrm{Res}_x \to \mathrm{Res}_x E_x \mathrm{Res}_x$$

$$\mathrm{Res}_x \epsilon^x : \mathrm{Res}_x E_x \mathrm{Res}_x \to \mathrm{Res}_x$$

and by the general properties of adjoints ([6], Proposition 10.1), we have

$$\mathrm{Res}_x \epsilon^x \circ \eta^x \mathrm{Res}_x = \mathrm{id}_{\mathrm{Res}_x} .$$

We observed that η^x is an isomorphism, hence $\eta^x \mathrm{Res}_x$ is certainly an isomorphism. By the above identity we conclude that $\mathrm{Res}_x \epsilon^x$ is also an isomorphism. Shortly:

Proposition 3.17. *For every right $R\Gamma$-module M,*

$$\mathrm{Res}_x(\epsilon^x_M) : \mathrm{Res}_x E_x \mathrm{Res}_x(M) \to \mathrm{Res}_x(M)$$

is an isomorphism.

In other words, if we evaluate the natural transformation ϵ_M^x at x we get an isomorphism. Note that each $E_x \operatorname{Res}_x$ lives in the endofunctor category $\mathsf{Mod}\text{-}R\Gamma^{\mathsf{Mod}\text{-}R\Gamma}$ as an object. Since $\mathsf{Mod}\text{-}R\Gamma^{\mathsf{Mod}\text{-}R\Gamma}$ is an abelian category, we can form the direct sum

$$E = \bigoplus_{\overline{x} \in \operatorname{Iso}(\Gamma)} E_x \operatorname{Res}_x .$$

Note that E does not depend on the choice of representatives of isomorphism classes in $\operatorname{Iso}(\Gamma)$ because if $x \cong x'$ then $E_x \operatorname{Res}_x \cong E_{x'} \operatorname{Res}_{x'}$.

Since each $\epsilon^x : E_x \operatorname{Res}_x \to \operatorname{id}_{\mathsf{Mod}\text{-}R\Gamma}$ is a morphism in $\mathsf{Mod}\text{-}R\Gamma^{\mathsf{Mod}\text{-}R\Gamma}$, the universal property of direct sums yield a morphism $\epsilon : E \to \operatorname{id}_{\mathsf{Mod}\text{-}R\Gamma}$.

Proposition 3.18. *ϵ is an epimorphism.*

Proof. It suffices to check that

$$\epsilon_M : \bigoplus_{\overline{x} \in \operatorname{Iso}(\Gamma)} E_x \operatorname{Res}_x(M) = E(M) \to M$$

is an epimorphism for every right $R\Gamma$-module M. And for that it suffices to check that

$$\{\epsilon_M\}_y : \left(\bigoplus_{\overline{x} \in \operatorname{Iso}(\Gamma)} E_x \operatorname{Res}_x(M) \right)(y) \to M(y)$$

is a surjective R-module homomorphism for every $y \in \operatorname{Obj}(\Gamma)$. But by Proposition 3.17

$$\{\epsilon_M^y\}_y : \left(E_y \operatorname{Res}_y(M)\right)(y) \to M(y)$$

is already an isomorphism, so $\{\epsilon_M\}_y$ is an epimorphism. □

Letting $\iota : K \to E$ to be a kernel of ϵ, we obtain an exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} E \xrightarrow{\epsilon} \operatorname{id}_{\mathsf{Mod}\text{-}R\Gamma} \longrightarrow 0$$

in $\mathsf{Mod}\text{-}R\Gamma^{\mathsf{Mod}\text{-}R\Gamma}$. Fix a right $R\Gamma$-module M. Evaluating the above at M, we get a short exact sequence

$$0 \longrightarrow K(M) \xrightarrow{\iota_M} E(M) \xrightarrow{\epsilon_M} M \longrightarrow 0$$

Now $K(M)$ is also a right $R\Gamma$-module, so we also have the following short exact sequence:

$$0 \longrightarrow K^2(M) \xrightarrow{\iota_{K(M)}} EK(M) \xrightarrow{\epsilon_{K(M)}} K(M) \longrightarrow 0$$

Splicing, we get an exact sequence

$$0 \longrightarrow K^2(M) \longrightarrow EK(M) \longrightarrow E(M) \longrightarrow M \longrightarrow 0$$

Continuing this procedure we get

$$\ldots \longrightarrow EK^3(M) \longrightarrow EK^2(M) \longrightarrow EK(M) \longrightarrow E(M) \longrightarrow M \longrightarrow 0$$

We call this long exact sequence the *EK-resolution* of M. The following result says that this resolution is finite:

Proposition 3.19. *For every right $R\Gamma$-module M, $EK^t(M) = 0$ whenever $t > l(M)$.*

Proof. We employ induction on $l(M)$. For $l(M) = -1$, $M = 0$ and there is nothing to show. Now assume the claim holds for every module of length smaller than l and let M be a module of length l. Consider the short exact sequence

$$0 \longrightarrow K(M) \xrightarrow{\iota_M} E(M) \xrightarrow{\epsilon_M} M \longrightarrow 0$$

and write $\mu = \epsilon_M$. We claim that this short exact sequence satisfies the hypothesis of Lemma 3.12: Let y be a maximal object of $E(M)$. Since $l(M) \leq l(E(M)) = l(\overline{y})$, $M(x) = 0$ whenever $\overline{x} > \overline{y}$. On the other hand $R\operatorname{Hom}_\Gamma(y, x) \neq 0$ if and only if $\overline{y} \leq \overline{x}$. Therefore

$$\begin{aligned}
E(M)(y) &= \left(\bigoplus_{\overline{x} \in \mathrm{Iso}(\Gamma)} E_x \operatorname{Res}_x(M) \right)(y) \\
&= \bigoplus_{\overline{x} \in \mathrm{Iso}(\Gamma)} E_x(\operatorname{Res}_x(M))(y) \\
&= \bigoplus_{\overline{x} \in \mathrm{Iso}(\Gamma)} M(x) \otimes_{R[x]} R \operatorname{Hom}_\Gamma(y, x) \\
&= \bigoplus_{\overline{x} \geq \overline{y}} M(x) \otimes_{R[x]} R \operatorname{Hom}_\Gamma(y, x) \\
&= \bigoplus_{\overline{x} = \overline{y}} M(x) \otimes_{R[x]} R \operatorname{Hom}_\Gamma(y, x) \\
&= M(y) \otimes_{R[y]} R \operatorname{Hom}_\Gamma(y, y) \\
&= M(y) \otimes_{R[y]} R[y]
\end{aligned}$$

and μ_y is given by

$$\begin{aligned}
\mu_y : E(M)(y) = M(y) \otimes_{R[y]} R[y] &\to M(y) \\
m \otimes \alpha &\mapsto M(\alpha)(m)
\end{aligned}$$

which is clearly an isomorphism. Thus by Lemma 3.12,

$$l(K(M)) < l(E(M)) = l(M) = l\,.$$

Finally, for every $t > l(M)$ we have $t - 1 > l(K(M))$ and by the inductive hypothesis we get $EK^t(M) = EK^{t-1}(K(M)) = 0$. □

Given a nonzero right $R\Gamma$-module M, let

$$\max(M) = \{\overline{x} \in \mathrm{Iso}(\Gamma) : x \text{ is a maximal object of } M\}$$

Now instead of going all the way to $E(M)$, let us be more economic and consider the module

$$D = \bigoplus_{\overline{x} \in \max(M)} E_x \operatorname{Res}_x(M)\,.$$

Again we have a morphism $\nu : D \to M$ induced by ϵ^x's. Note that

$$\begin{aligned} l(D) &= \max\{l(E_x \operatorname{Res}_x(M)) : \overline{x} \in \max(M)\} \\ &= \max\{l(\overline{x}) : \overline{x} \in \max(M)\} \\ &= l(M) \end{aligned}$$

where the first equality is by Proposition 3.8 and the second is by Example 3.7. Moreover for every maximal object y of D, ν_y is an isomorphism by the exact same reasoning in the proof of Proposition 3.19 which shows μ_y is an isomorphism. Therefore we can apply Lemma 3.14 to obtain a projective module Q such that $l(Q) < l(M)$ and an epimorphism

$$\rho : D \oplus Q \to M \,.$$

Now by Remark 3.15, for every maximal object x of $D \oplus Q$, ρ_x is an isomorphism. Therefore letting $\iota : L \to D \oplus Q$ to be a kernel of ρ, the short exact sequence

$$0 \longrightarrow L \xrightarrow{\ \iota\ } D \oplus Q \xrightarrow{\ \rho\ } M \longrightarrow 0$$

satisfies the hypothesis of Lemma 3.12; and hence $l(L) < l(M)$. In summary:

Proposition 3.20. *Let M be a nonzero right $R\Gamma$-module. There is a short exact sequence of right $R\Gamma$-modules*

$$0 \longrightarrow L \longrightarrow \bigoplus_{\overline{x} \in \max(M)} E_x \operatorname{Res}_x(M) \;\oplus\; Q \longrightarrow M \longrightarrow 0$$

such that Q is projective, $l(Q) < l(M)$, $l(L) < l(M)$.

We will use Proposition 3.20 and its consequences several times in this thesis. Here is an important corollary:

Corollary 3.21. *Let M be a nonzero right $R\Gamma$-module such that for every maximal object x of M, $\operatorname{Res}_x(M)$ is a projective right $R[x]$-module. Then there exists a short exact sequence*

$$0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$$

such that P is projective and $l(L) < l(M)$.

Proof. By Proposition 3.20 we have an exact sequence

$$0 \longrightarrow L \longrightarrow \bigoplus_{\overline{x} \in \max(M)} E_x \operatorname{Res}_x(M) \;\oplus\; Q \longrightarrow M \longrightarrow 0$$

such that Q is projective and $l(L) < l(M)$. Let

$$P = \bigoplus_{\overline{x} \in \max(M)} E_x \operatorname{Res}_x(M) \;\oplus\; Q\,.$$

By assumption, for each $\overline{x} \in \max(M)$, $\operatorname{Res}_x(M)$ is projective. Since for any x the functor E_x sends projectives to projectives and direct sum of projectives is projective, P is projective. □

Corollary 3.21 will be most useful when we consider modules with finite projective resolutions.

3.3 Inclusion and splitting functors

EI-property of Γ allows us to define new functors between $\mathsf{Mod}\text{-}R\Gamma$ and $\mathsf{Mod}\text{-}R[x]$ (other than Res_x and E_x) which are important tools to transfer information between these categories. We will first define a peculiar $R\Gamma$-$R\Gamma$-bimodule whose very existence will crucially depend on Γ being an EI-category. As before we will obtain this bimodule by a functor $T : \Gamma \times \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$. For this, we will use the following *bifunctor lemma* from general category theory:

Lemma 3.22 ([6], Lemma 7.14)**.** *Let Γ, Λ, Ψ be categories. A pair of maps for objects and morphisms*

$$T_0 : \mathrm{Obj}(\Gamma) \times \mathrm{Obj}(\Lambda) \to \mathrm{Obj}(\Psi)$$
$$T_1 : \mathrm{Mor}(\Gamma) \times \mathrm{Mor}(\Lambda) \to \mathrm{Mor}(\Psi)$$

defines a functor $T : \Gamma \times \Lambda \to \Psi$ if and only if

1. *T is functorial in each argument: $T(x, -) : \Lambda \to \Psi$ and $T(-, y) : \Gamma \to \Psi$ are functors for all $x \in \mathrm{Obj}(\Gamma)$ and $y \in \mathrm{Obj}(\Lambda)$.*

2. *T satisfies the following interchange law. Given $\alpha : x \to x' \in \mathrm{Mor}(\Gamma)$ and $\beta : y \to y' \in \mathrm{Mor}(\Lambda)$, the following commutes:*

$$\begin{array}{ccc} T(x,y) & \xrightarrow{T(\mathrm{id}_x,\beta)} & T(x,y') \\ \Big\downarrow{\scriptstyle T(\alpha,\mathrm{id}_y)} & & \Big\downarrow{\scriptstyle T(\alpha,\mathrm{id}_{y'})} \\ T(x',y) & \xrightarrow[T(\mathrm{id}_{x'},\beta)]{} & T(x',y') \end{array}$$

So to define a functor $T : \Gamma \times \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$, we define

$$T_0 : \mathrm{Obj}(\Gamma) \times \mathrm{Obj}(\Gamma^{\mathrm{op}}) \to \mathrm{Obj}(R\text{-}\mathsf{Mod})$$
$$(x,y) \mapsto \begin{cases} R\,\mathrm{Hom}_\Gamma(y,x) & \text{if } \overline{x} = \overline{y} \\ 0 & \text{otherwise} \end{cases}$$

and we define

$$T_1 : \mathrm{Mor}(\Gamma) \times \mathrm{Mor}(\Gamma^{\mathrm{op}}) \to \mathrm{Mor}(R\text{-}\mathsf{Mod})$$

as follows: Let $\alpha : x \to x'$ be a morphism in Γ and $\beta : y \to y'$ be a morphism in Γ^{op}. So $\beta : y' \to y$ is a morphism in Γ. Then we define the R-module homomorphism $T_1(\alpha,\beta) : T_0(x,y) \to T_0(x',y')$ as follows:

- If $\overline{x} \neq \overline{y}$ or $\overline{x'} \neq \overline{y'}$, $T_1(\alpha,\beta)$ is the zero morphism.
- Otherwise we have $\overline{x} = \overline{y}$ and $\overline{x'} = \overline{y'}$; hence $T_0(x,y) = R\,\mathrm{Hom}_\Gamma(y,x)$ and $T_0(x',y') = R\,\mathrm{Hom}_\Gamma(y',x')$. For $f \in \mathrm{Hom}_\Gamma(y,x)$, we define

  $$T_1(\alpha,\beta)(f) = \alpha \circ f \circ \beta$$

  and extend R-linearly.

Let us check that T is functorial in the second argument. For $y \in \mathrm{Obj}(\Gamma)$, we should check that

$$T(-,y) : \Gamma \to R\text{-}\mathsf{Mod}$$

is a functor. Clearly $T(\mathrm{id}_x, y) = \mathrm{id}_{T(x,y)}$ for every $x \in \mathrm{Obj}(\Gamma)$. Also if $\alpha : x \to x'$ and $\alpha' : x' \to x''$ are morphisms in Γ, the diagram

$$\overbrace{T(x,y) \xrightarrow{T(\alpha,\mathrm{id}_y)} T(x',y) \xrightarrow{T(\alpha',\mathrm{id}_y)} T(x'',y)}^{T(\alpha'\circ\alpha,\mathrm{id}_y)}$$

is commutative: If $\overline{x} \neq \overline{y}$, $T(x,y) = 0$ and there is nothing to check. Also if $\overline{x''} \neq \overline{y}$, $T(x'',y) = 0$ and again there is nothing to check. Otherwise we have $\overline{x} = \overline{y} = \overline{x''}$. Moreover the existence of the morphisms α and α' gives that $\overline{x} \leq \overline{x'} \leq \overline{x''}$. Thus by the antisymmetry of $\leq$, we get $\overline{x'} = \overline{y}$ and the diagram becomes

$$\overbrace{R\,\mathrm{Hom}_\Gamma(y,x) \xrightarrow{T(\alpha,\mathrm{id}_y)} R\,\mathrm{Hom}_\Gamma(y,x') \xrightarrow{T(\alpha',\mathrm{id}_y)} R\,\mathrm{Hom}_\Gamma(y,x'')}^{T(\alpha'\circ\alpha,\mathrm{id}_y)}$$

which is clearly commutative. Similarly T is functorial in the first argument.

Finally we verify the interchange law: Let $\alpha : x \to x' \in \mathrm{Mor}(\Gamma)$ and $\beta : y' \to y \in \mathrm{Mor}(\Gamma)$. Then

$$\begin{array}{ccc} T(x,y) & \xrightarrow{T(\mathrm{id}_x,\beta)} & T(x,y') \\ {\scriptstyle T(\alpha,\mathrm{id}_y)}\big\downarrow & & \big\downarrow{\scriptstyle T(\alpha,\mathrm{id}_{y'})} \\ T(x',y) & \xrightarrow[T(\mathrm{id}_{x'},\beta)]{} & T(x',y') \end{array}$$

commutes: Indeed if $\overline{x} \neq \overline{y}$ or $\overline{x'} \neq \overline{y'}$, there is nothing to check. Otherwise we have $\overline{x} = \overline{y}$ and $\overline{x'} = \overline{y'}$. Moreover the existence of α and β gives $\overline{x} \leq \overline{x'}$ and $\overline{y'} \leq \overline{y}$. Since

$$\overline{x} \leq \overline{x'} = \overline{y'} \leq \overline{y} = \overline{x}$$

we obtain

$$\overline{x} = \overline{x'} = \overline{y} = \overline{y'}$$

and hence the diagram becomes

$$\begin{array}{ccc} R\operatorname{Hom}_\Gamma(y,x) & \xrightarrow{T(\mathrm{id}_x,\beta)} & R\operatorname{Hom}_\Gamma(y',x) \\ \downarrow{\scriptstyle T(\alpha,\mathrm{id}_y)} & & \downarrow{\scriptstyle T(\alpha,\mathrm{id}_{y'})} \\ R\operatorname{Hom}_\Gamma(y,x') & \xrightarrow[T(\mathrm{id}_{x'},\beta)]{} & R\operatorname{Hom}_\Gamma(y',x') \end{array}$$

which is clearly commutative. Thus we have a legitimate functor

$$T : \Gamma \times \Gamma^{\mathrm{op}} \to R\text{-}\mathsf{Mod}$$

We also have the standard R-linearized Hom functor

$$\begin{aligned} H : \Gamma \times \Gamma^{\mathrm{op}} &\to R\text{-}\mathsf{Mod} \\ (x,y) &\mapsto R\operatorname{Hom}_\Gamma(y,x) \end{aligned}$$

We can construct an epimorphism $\theta : H \to T$ of bimodules as follows: For every $x, y \in \mathrm{Obj}(\Gamma)$, we know that $T(x,y) = H(x,y)$ if $\overline{x} = \overline{y}$ and $T(x,y) = 0$ otherwise. So we define

$$\theta_{(x,y)} : H(x,y) \to T(x,y)$$

to be the identity map if $\overline{x} = \overline{y}$ and zero otherwise. Clearly every $\theta_{(x,y)}$ is a surjective R-module homomorphism. To see that they define a natural transformation, let $\alpha : x \to x'$ and $\beta : y' \to y$ be morphisms in Γ and consider the square

$$\begin{array}{ccc} H(x,y) & \xrightarrow{H(\alpha,\beta)} & H(x',y') \\ \downarrow{\scriptstyle \theta_{(x,y)}} & & \downarrow{\scriptstyle \theta_{(x',y')}} \\ T(x,y) & \xrightarrow[T(\alpha,\beta)]{} & T(x',y') \end{array}$$

If $\overline{y} \nleq \overline{x}$, $H(x,y) = R\operatorname{Hom}_\Gamma(y,x) = 0$ and the square trivially commutes. And if $\overline{x'} \neq \overline{y'}$, $T(x',y') = 0$ and again the square trivially commutes. Otherwise $\overline{y} \leq \overline{x}$ and $\overline{x'} = \overline{y'}$. Moreover by α and β we have $\overline{x} \leq \overline{x'}$ and $\overline{y'} \leq \overline{y}$. So

$$\overline{y} \leq \overline{x} \leq \overline{x'} = \overline{y'} \leq \overline{y}$$

and hence $\overline{x} = \overline{x'} = \overline{y} = \overline{y'}$. Then $T(x, y) = H(x, y)$, $T(x', y') = H(x', y')$ and $\theta_{(x,y)}$, $\theta_{(x',y')}$ are identity maps. Also $T(\alpha, \beta) = H(\alpha, \beta)$ by definiton. Thus the square commutes.

Now fix $x \in \mathrm{Obj}(\Gamma)$. Let $F : \Gamma_x \to \Gamma$ be the inclusion functor. Consider the composition

$$B : \Gamma \times {\Gamma_x}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times F} \Gamma \times \Gamma^{\mathrm{op}} \xrightarrow{T} R\text{-}\mathsf{Mod}$$

B defines an $R\Gamma$-$R[x]$-bimodule and we know that the functor $- \otimes_{R\Gamma} B$ is left adjoint to $\mathrm{Hom}_{R[x]}(-, B)$. In this case we write S_x for $- \otimes_{R\Gamma} B$ and I_x for $\mathrm{Hom}_{R[x]}(-, B)$. Spelling out the adjunction again, the functor

$$S_x : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R[x]$$

is left adjoint to

$$I_x : \mathsf{Mod}\text{-}R[x] \to \mathsf{Mod}\text{-}R\Gamma.$$

S_x is called the *splitting functor* and I_x is called the *inclusion functor* along x.

Observe that if we use H instead of T to define a bimodule, that is, if we consider the composition

$$C : \Gamma \times {\Gamma_x}^{\mathrm{op}} \xrightarrow{\mathrm{id} \times F} \Gamma \times \Gamma^{\mathrm{op}} \xrightarrow{H} R\text{-}\mathsf{Mod}$$

the functor $- \otimes_{R\Gamma} C$ is isomorphic to Res_x. We previously constructed an epimorphism $\theta : H \to T$ of bimodules. Since C and B are just the restrictions of H and T respectively along $\mathrm{id} \times F$, θ gives an epimorphism $C \to B$. Finally since tensor products preserve epimorphisms, we get an epimorphism of functors

$$\xi : \mathrm{Res}_x = - \otimes_{R\Gamma} C \to - \otimes_{R\Gamma} B = S_x\,.$$

Proposition 3.23. *Let M be a right $R\Gamma$-module and $x \in \mathrm{Obj}(\Gamma)$. If $l(M) \leq l(\overline{x})$, then $\xi_M : \mathrm{Res}_x(M) \to S_x(M)$ is an isomorphism of right $R[x]$-modules.*

Proof. Let $\varphi : C \to B$ be the epimorphism of bimodules that we observed above.

Writing C and B really as bimodules of category algebras, we have

$$C = \bigoplus_{y \in \mathrm{Obj}(\Gamma)} R\,\mathrm{Hom}_\Gamma(x,y)$$
$$B = \bigoplus_{\overline{y}=\overline{x}} R\,\mathrm{Hom}_\Gamma(x,y)$$

and for any $h : x \to y$,

$$\varphi(h) = \begin{cases} h & \text{if } \overline{x} = \overline{y} \\ 0 & \text{otherwise.} \end{cases}$$

The claim is that the R-module homomorphism

$$\mathrm{id}_M \otimes \varphi : M \otimes_{R\Gamma} C \to M \otimes_{R\Gamma} B$$

is an isomorphism. This is equivalent to the claim that the $R\Gamma$-balanced map

$$\iota : M \times C \to M \otimes_{R\Gamma} B$$
$$(m,c) \mapsto m \otimes \varphi(c)$$

satisfies the universal property of the tensor product $M \otimes_{R\Gamma} C$. So let

$$\lambda : M \times C \to U$$

be an $R\Gamma$-balanced map, where U is an R-module. We first observe that if $h : x \to y$ is not an isomorphism, then $\lambda(m,h) = 0$ for any $m \in M$: Indeed,

$$\lambda(m,h) = \lambda(m, \mathrm{id}_y \cdot h) = \lambda(m \cdot \mathrm{id}_y, h) = \lambda(0,h) = 0$$

since $m \cdot \mathrm{id}_y \in M(y) = 0$ as $\overline{x} < \overline{y}$.

Now we will define an $R\Gamma$-balanced map $\kappa : M \times B \to U$. As an R-module, B is generated by the isomorphisms in Γ with domain x, so it suffices to define κ on these generators. Given $m \in M$ and an isomorphism $f : x \to y$, we simply define

$$\kappa(m,f) = \lambda(m,f)\,.$$

This makes sense because f is also an element of C. But we should be careful because the $R\Gamma$-actions on B and C are different. B is *not* an $R\Gamma$-submodule of

C, so we should verify that κ is $R\Gamma$-balanced. It suffices to check that for $m \in M$, $f : x \to y$ an isomorphism in Γ and $g : y \to z$ a morphism in Γ,

$$\kappa(m, g \cdot f) = \kappa(m \cdot g, f).$$

If g is not an isomorphism, on one hand we have

$$\kappa(m, g \cdot f) = \kappa(m, 0) = 0$$

and on the other hand

$$\kappa(m \cdot g, f) = \lambda(m \cdot g, f) = \lambda(m, g \cdot f) = \lambda(m, g \circ f) = 0$$

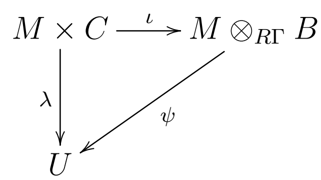
because $g \circ f$ is not an isomorphism. So we get the equality. And if g is an isomorphism, then

$$\kappa(m, g \cdot f) = \kappa(m, g \circ f) = \lambda(m, g \circ f) = \lambda(m, g \cdot f) = \lambda(m \cdot g, f) = \kappa(m \cdot g, f)\,.$$

Thus κ is $R\Gamma$-balanced. So by the universal property of tensor products, there is a unique R-module homomorphism

$$\psi : M \otimes_{R\Gamma} B \to U$$

such that $\psi(m \otimes f) = \kappa(m, f)$. We claim that ψ is the unique R-module homomorphism which makes the diagram



commute. Let $m \in M$ and $h : x \to y$. We have $(\psi \circ \iota)(m, h) = \psi(m \otimes \varphi(h))$. If h is not an isomorphism,

$$\psi(m \otimes \varphi(h)) = \psi(m \otimes 0) = 0 = \lambda(m, h)\,.$$

If h is an isomorphism,

$$\psi(m \otimes \varphi(h)) = \psi(m \otimes h) = \kappa(m, h) = \lambda(m, h)\,.$$

So ψ commutes the diagram. Let $\widetilde{\psi} : M \otimes_{R\Gamma} B \to U$ be another R-module homomorphism such that $\widetilde{\psi} \circ \iota = \lambda$. Then for $m \in M$ and $f : x \to y$ an isomorphism,

$$\begin{aligned}\widetilde{\psi}(m \otimes f) &= \widetilde{\psi}(m \otimes \varphi(f)) \\ &= \widetilde{\psi}(\iota(m, f)) \\ &= (\widetilde{\psi} \circ \iota)(m, f) \\ &= \lambda(m, f) \\ &= \kappa(m, f)\end{aligned}$$

hence $\widetilde{\psi} = \psi$. □

Corollary 3.24. $S_x E_x \cong \mathrm{id}_{\mathsf{Mod}\text{-}R[x]}$.

Proof. By Proposition 3.16, $\mathrm{Res}_x E_x \cong \mathrm{id}_{\mathsf{Mod}\text{-}R[x]}$. We have an epimorphism

$$\xi : \mathrm{Res}_x \to S_x$$

ξ induces an epimorphism

$$\xi E_x : \mathrm{Res}_x E_x \to S_x E_x \,.$$

We claim that ξE_x is actually an isomorphism. Indeed for every right $R[x]$-module N, the epimorphism

$$(\xi E_x)_N = \xi_{E_x(N)} : \mathrm{Res}_x E_x(N) \to S_x E_x(N)$$

is an isomorphism because $l(E_x(N)) = l(\overline{x})$ by Example 3.7. □

The following proposition complements Corollary 3.24.

Proposition 3.25. *If $\overline{x} \neq \overline{y}$, then $S_x E_y = 0$.*

Proof. Recall that S_x is the functor $- \otimes_{R\Gamma} B : \mathsf{Mod}\text{-}R\Gamma \to \mathsf{Mod}\text{-}R[x]$ where B is a $R\Gamma$-$R[x]$-bimodule given by

$$B = \bigoplus_{\overline{z}=\overline{x}} R\,\mathrm{Hom}_\Gamma(x, z)\,.$$

In particular $B(y) = 0$ since $\overline{y} \neq \overline{x}$. Thus for every right $R[y]$-module N,

$$\begin{aligned} S_x E_y(N) &= E_y(N) \otimes_{R\Gamma} B \\ &\cong N \otimes_{R[y]} B(y) \\ &= 0\,. \end{aligned}$$

□

Remark 3.26. The isomorphism in the proof of Proposition 3.25 comes from the following general fact: If R, S, T are rings and $\varphi : R \to S$ is a *rng* homomorphism, for every right R-module N and S-T-bimodule B, we have an isomorphism

$$\mathrm{Ind}_\varphi(N) \otimes_S B \cong N \otimes_R \mathrm{Res}_\varphi(B)$$

of right T-modules. The isomorphism in the proof follows from considering the rng homomorphism $R[y] \to R\Gamma$ induced by the inclusion functor $\Gamma_y \to \Gamma$ (see Proposition 2.8).

Our next aim is to show that the functor S_x preserves projectives (for any $x \in \mathrm{Obj}(\Gamma)$).

Proposition 3.27. *I_x is an exact functor.*

Proof. By definition for a right $R[x]$-module M we have

$$I_x(M) : \Gamma \to R\text{-}\mathsf{Mod}$$
$$y \mapsto \begin{cases} \mathrm{Hom}_{R[x]}(R\,\mathrm{Hom}_\Gamma(x,y), M) & \overline{x} = \overline{y} \\ 0 & \overline{x} \neq \overline{y} \end{cases}$$

Let

$$0 \longrightarrow M'' \xrightarrow{\lambda} M \xrightarrow{\mu} M' \longrightarrow 0$$

be an exact sequence of right $R[x]$-modules. We claim that

$$0 \longrightarrow I_x(M'') \xrightarrow{I_x(\lambda)} I_x(M) \xrightarrow{I_x(\mu)} I_x(M') \longrightarrow 0$$

is an exact sequence of right $R\Gamma$-modules. So we should check that for every $y \in \mathrm{Obj}(\Gamma)$,

$$0 \longrightarrow I_x(M'')(y) \xrightarrow{I_x(\lambda)_y} I_x(M)(y) \xrightarrow{I_x(\mu)_y} I_x(M')(y) \longrightarrow 0$$

is exact. Indeed if $\overline{y} \neq \overline{x}$, the above is just a sequence of zero modules, hence trivially exact. If $\overline{y} = \overline{x}$, the sequence is exactly the image of the original exact sequence

$$0 \longrightarrow M'' \xrightarrow{\lambda} M \xrightarrow{\mu} M' \longrightarrow 0$$

under the covariant functor $\mathrm{Hom}_{R[x]}(R\,\mathrm{Hom}_\Gamma(x,y), -)$. Now since $\overline{x} = \overline{y}$, every morphism in $\mathrm{Hom}_\Gamma(x,y)$ is an isomorphism, as Γ is EI. Therefore as a right $\mathrm{Aut}_\Gamma(x)$-set, $\mathrm{Hom}_\Gamma(x,y)$ is free (if $f \circ g = f$, $g = \mathrm{id}_x$). Thus $R\,\mathrm{Hom}_\Gamma(x,y)$ is a free right $R[x]$-module. Hence $\mathrm{Hom}_{R[x]}(R\,\mathrm{Hom}_\Gamma(x,y), -)$ is an exact functor. □

Corollary 3.28. *S_x sends projectives to projectives.*

Proof. $S_x \dashv I_x$ and I_x is exact. □

We now have enough machinery to transfer information between $\mathsf{Mod}\text{-}R\Gamma$ and $\mathsf{Mod}\text{-}R[x]$'s.

Proposition 3.29. *Let P be a projective right $R\Gamma$-module. If $l(P) \leq l(\overline{x})$, then $\mathrm{Res}_x(P)$ is a projective right $R[x]$-module.*

Proof. By Proposition 3.23 and Corollary 3.28 $\mathrm{Res}_x(P) \cong S_x(P)$ is projective. □

The final result we prove in this chapter is a decomposition theorem for projective right $R\Gamma$-modules. First we define the *support* of a module:

Definition 3.30. Let M be a right $R\Gamma$-module. We denote the set

$$\{\overline{x} \in \mathrm{Iso}(\Gamma) : M(x) \neq 0\}$$

by $\mathrm{supp}(M)$ and call it the *support* of M.

We prove a general diagram chasing lemma:

Lemma 3.31. *Assume A, B are abelian categories, $F, G : \mathsf{A} \to \mathsf{B}$ are covariant functors and $\nu : F \to G$ a natural transformation. Then*

1. *Assume F and G preserve coproducts (direct sums). Let $(A_\lambda)_{\lambda \in I}$ be a collection of objects in A such that the object A serves as their coproduct when equipped with morphisms $i_\lambda : A_\lambda \to A$. If every*

$$\nu_{A_\lambda} : F(A_\lambda) \to G(A_\lambda)$$

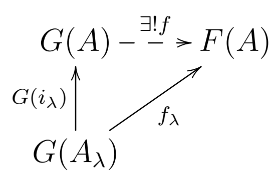
has a left inverse, then

$$\nu_A : F(A) \to G(A)$$

has a left inverse.

2. *Let C, D be objects in A and $C \oplus D$ their direct sum. If $\nu_{C \oplus D}$ has a left inverse, ν_C (and ν_D) has a left inverse.*

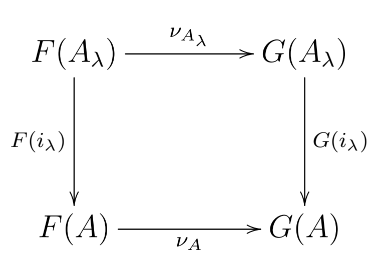
Proof. For the first part, choose a left inverse $s_\lambda : G(A_\lambda) \to F(A_\lambda)$ for each ν_{A_λ}. Let $f_\lambda = F(i_\lambda) \circ s_\lambda : G(A_\lambda) \to F(A)$. Since G preserves coproduct diagrams, there exists a unique $f : G(A) \to F(A)$ making the diagram



commute for each λ. We claim that f is a left inverse of ν_A. Now since F preserves coproduct diagrams, it suffices to verify $f \circ \nu_A \circ F(i_\lambda) = F(i_\lambda)$ for every λ. Indeed

$$\begin{aligned} f \circ \nu_A \circ F(i_\lambda) &= f \circ G(i_\lambda) \circ \nu_{A_\lambda} \\ &= f_\lambda \circ \nu_{A_\lambda} \\ &= F(i_\lambda) \circ s_\lambda \circ \nu_{A_\lambda} \\ &= F(i_\lambda) \end{aligned}$$

where in the first equality, we use the commutative diagram



that the naturality of ν yields.

For the second part, say the morphisms $i_C : C \to C \oplus D$ and $i_D : D \to C \oplus D$ make $C \oplus D$ a coproduct. We are in an abelian category, so i_C has a left inverse π_C. Also by assumption, $\nu_{C\oplus D}$ has a left inverse s. Considering the morphism

$$t = F(\pi_C) \circ s \circ G(i_C)$$

and the commutative diagram

$$\begin{array}{ccc} F(C) & \xrightarrow{\nu_C} & G(C) \\ {\scriptstyle F(i_C)}\downarrow & & \downarrow{\scriptstyle G(i_C)} \\ F(C\oplus D) & \xrightarrow[\nu_{C\oplus D}]{} & G(C\oplus D) \end{array}$$

we have

$$\begin{aligned} t \circ \nu_C &= F(\pi_C) \circ s \circ G(i_C) \circ \nu_C \\ &= F(\pi_C) \circ s \circ \nu_{C\oplus D} \circ F(i_C) \\ &= F(\pi_C) \circ F(i_C) \\ &= \mathrm{id}_{F(C)} \ . \end{aligned}$$

□

Now we prove the key lemma towards the decomposition theorem:

Lemma 3.32. *Let P be a projective right $R\Gamma$-module. Let $\mathcal{T}$ be a subset of* $\mathrm{Iso}(\Gamma)$ *such that $\overline{x} \in \mathcal{T}$ implies $l(\overline{x}) \geq l(P)$. Consider the natural transformation*

$$\nu : \bigoplus_{\overline{x}\in\mathcal{T}} E_x \operatorname{Res}_x \to \mathrm{id}_{\textsf{Mod-}R\Gamma}$$

Then ν_P has a left inverse.

Proof. Assume $(P_\lambda)_{\lambda\in I}$ is a collection of projective right $R\Gamma$-modules which satisfy the assertion of the lemma. We claim that if $P = \bigoplus_{\lambda\in I} P_\lambda$, P also satisfies the assertion. Let $\mathcal{T}$ be a subset as in the statement. Then in particular for every λ,

$\overline{x} \in \mathcal{T}$ implies $l(\overline{x}) \geq l(P_\lambda)$. So by our initial assumption, ν_{P_λ} has a left inverse for every λ.

Noting that both the domain and the codomain of ν preserves direct sums, we conclude by the first part of Lemma 3.31 that ν_P has a left inverse. Thus, the class of projectives satisfying the lemma is closed under direct sums.

Now we show that projectives of the form $P = R\Gamma(-, y)$ satisfy the assertion. Let $\mathcal{T}$ be as in the statement. Note that

$$\bigoplus_{\overline{x} \in \mathcal{T}} E_x \operatorname{Res}_x(P) = \bigoplus_{\overline{x} \in \mathcal{T} \cap \operatorname{supp}(P)} E_x \operatorname{Res}_x(P)\,.$$

Let $\overline{x} \in \mathcal{T} \cap \operatorname{supp}(P)$. Since $\overline{x} \in \mathcal{T}$, $l(\overline{x}) \geq l(P) = l(\overline{y})$ and since $\overline{x} \in \operatorname{supp}(P)$, $\overline{x} \leq \overline{y}$. These force $\overline{x} = \overline{y}$, so $\mathcal{T} \cap \operatorname{supp}(P) \subseteq \{\overline{y}\}$; hence either $\mathcal{T} \cap \operatorname{supp}(P) = \emptyset$ or $\mathcal{T} \cap \operatorname{supp}(P) = \{\overline{y}\}$. In the former case,

$$\bigoplus_{\overline{x} \in \mathcal{T}} E_x \operatorname{Res}_x(P) = 0$$

hence the zero morphism is the left inverse of ν_P. In the latter case,

$$\bigoplus_{\overline{x} \in \mathcal{T}} E_x \operatorname{Res}_x(P) = E_y \operatorname{Res}_y(P)$$

and since $P = E_y(R[y])$,

$$\nu_P : E_y \operatorname{Res}_y E_y(R[y]) \to E_y(R[y])$$

is actually an isomorphism: Indeed if we let ϵ to be the counit and η to be the unit of the adjunction $E_y \dashv \operatorname{Res}_y$, we have $\nu_P = \epsilon_P = \epsilon_{E_y(R[y])}$ and by general properties of adjoint functors ([6], Proposition 10.1) we have

$$\epsilon_{E_y(R[y])} \circ E_y(\eta_{R[y]}) = \operatorname{id}_{E_y(R[y])}$$

Now η is an isomorphism by Proposition 3.16, hence ν_P is an isomorphism.

Finally we prove the assertion for an arbitrary projective P. By Proposition 3.9 (and its proof), there is a projective right $R\Gamma$-module F which is a direct sum of projectives of the form $R\Gamma(-, y)$ such that $l(F) = l(P)$ and an epimorphism

$\phi : F \to P$. As P is projective, ϕ splits so we can identify $F = P \oplus Q$ for some Q. By our previous arguments, F satisfies the assertion of the lemma. Let $\mathcal{T}$ be as in the statement. Note that if $\overline{x} \in \mathcal{T}$, $l(\overline{x}) \geq l(P) = l(F)$. Therefore $\nu_F = \nu_{P \oplus Q}$ has a left inverse. By the second part of Lemma 3.31, ν_P has a left inverse. □

Theorem 3.33. *Let P be a projective right $R\Gamma$-module. Then*

$$P \cong \bigoplus_{\overline{x} \in \operatorname{supp}(P)} E_x S_x(P)$$

Proof. Employ induction on $l(P)$. If $l(P) = -1$, then $P = 0$ and there is nothing to show. For $l(P) \geq 0$, letting $\mathcal{T} = \max(P)$ in Lemma 3.32, we obtain a *split* short exact sequence

$$0 \longrightarrow \bigoplus_{\overline{x} \in \max(P)} E_x \operatorname{Res}_x(P) \xrightarrow{\mu} P \longrightarrow C \longrightarrow 0$$

We know that μ_x is an isomorphism whenever $\overline{x} \in \max(P)$, so by the dual of Lemma 3.12, $l(C) < l(P)$. Since the sequence above is split,

$$P \cong \bigoplus_{\overline{x} \in \max(P)} E_x \operatorname{Res}_x(P) \oplus C \tag{*}$$

and C is a projective right $R\Gamma$-module. So by the induction hypothesis,

$$C \cong \bigoplus_{\overline{x} \in \operatorname{supp}(C)} E_x S_x(C)\,.$$

For every $\overline{x} \in \operatorname{supp}(P) - \max(P)$, applying S_x to (*) and using Proposition 3.25, we get

$$S_x(P) \cong S_x(C)\,.$$

Noting that $\operatorname{supp}(C) \subseteq \operatorname{supp}(P) - \max(P)$, we get

$$\begin{aligned}\bigoplus_{\overline{x} \in \operatorname{supp}(P) - \max(P)} E_x S_x(P) &\cong \bigoplus_{\overline{x} \in \operatorname{supp}(P) - \max(P)} E_x S_x(C) \\ &= \bigoplus_{\overline{x} \in \operatorname{supp}(C)} E_x S_x(C) \\ &\cong C\,.\end{aligned}$$

Hence finally

$$
\begin{aligned}
P &\cong \bigoplus_{\overline{x}\in\max(P)} E_x \operatorname{Res}_x(P) \oplus C \\
&\cong \bigoplus_{\overline{x}\in\max(P)} E_x \operatorname{Res}_x(P) \quad \oplus \bigoplus_{\overline{x}\in\operatorname{supp}(P)-\max(P)} E_x S_x(P) \\
&\cong \bigoplus_{\overline{x}\in\max(P)} E_x S_x(P) \quad \oplus \bigoplus_{\overline{x}\in\operatorname{supp}(P)-\max(P)} E_x S_x(P) \\
&\cong \bigoplus_{\overline{x}\in\operatorname{supp}(P)} E_x S_x(P)\,.
\end{aligned}
$$

The third isomorphism is by Proposition 3.23. □

3.4 Finite projective resolutions

In this section we investigate right $R\Gamma$-modules with a finite projective resolution. The length of the shortest projective resolution of such M is called the *projective dimension* of M and denoted by $\operatorname{pd}(M)$. More precisely $\operatorname{pd}(M) = n$ means that there is a projective resolution of M of the form

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and M does not have a shorter projective resolution (P_i's need not be finitely generated).

In addition to our already existing assumptions in this chapter, we further assume in this section that not only $\operatorname{Obj}(\Gamma)$ is finite, but also $\operatorname{Mor}(\Gamma)$ is finite. This assumption ensures that $\operatorname{Aut}_\Gamma(x)$ is a finite group for any x, hence useful tools like cohomology of finite groups become accessible while we go back and forth between $\mathsf{Mod}\text{-}R\Gamma$ and $\mathsf{Mod}\text{-}R[x]$.

Lemma 3.34. *Let*

$$0 \longrightarrow K \longrightarrow P_{n-1} \longrightarrow P_{n-2} \longrightarrow \ldots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence of right $R\Gamma$-modules where P_j's are projective. Then for every right $R\Gamma$-module V and every $i \geq 1$, we have $\operatorname{Ext}^i_{R\Gamma}(K, V) \cong \operatorname{Ext}^{i+n}_{R\Gamma}(M, V)$.

Proof. Employ induction on n. If $n = 1$, the given sequence is of the form

$$0 \longrightarrow K \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is projective. Since we have a short exact sequence, applying the contravariant functor $\mathrm{Hom}_{R\Gamma}(-, V)$ yields a long exact sequence of Ext-groups. In particular for every i there is an exact sequence

$$\mathrm{Ext}^i_{R\Gamma}(P, V) \longrightarrow \mathrm{Ext}^i_{R\Gamma}(K, V) \longrightarrow \mathrm{Ext}^{i+1}_{R\Gamma}(M, V) \longrightarrow \mathrm{Ext}^{i+1}_{R\Gamma}(P, V)\,.$$

Since P is projective, for $i \geq 1$ we have $\mathrm{Ext}^i_{R\Gamma}(P, V) = \mathrm{Ext}^{i+1}_{R\Gamma}(P, V) = 0$. Thus

$$\mathrm{Ext}^i_{R\Gamma}(K, V) \cong \mathrm{Ext}^{i+1}_{R\Gamma}(M, V)\,.$$

Now we show the result for $n + 1$ while assuming it for n. So let

$$0 \longrightarrow K \longrightarrow P_n \xrightarrow{\varphi} P_{n-1} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an exact sequence where P_j's are projective. Letting $C = \mathrm{coker}(\varphi)$, we can break the sequence into two exact sequences:

$$0 \longrightarrow K \longrightarrow P_n \longrightarrow C \longrightarrow 0$$

$$0 \longrightarrow C \longrightarrow P_{n-1} \longrightarrow P_n \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Now for every $i \geq 1$ and every V,

$$\mathrm{Ext}^i_{R\Gamma}(K, V) \cong \mathrm{Ext}^{i+1}_{R\Gamma}(C, V) \cong \mathrm{Ext}^{i+1+n}_{R\Gamma}(M, V)$$

where the first isomorphism is by the first part and the second is by the inductive hypothesis. □

Proposition 3.35. *Let M be a right $R\Gamma$-module with a finite projective resolution. Then* $\mathrm{pd}(M) = \max\{r : \mathrm{Ext}^r_{R\Gamma}(M, -) \neq 0\}$.

Proof. Let $n = \mathrm{pd}(M)$. So there is an exact sequence

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow P_{n-2} \xrightarrow{\varphi} P_{n-3} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where P_j's are projective.

Clearly $\mathrm{Ext}^r_{R\Gamma}(M,-) = 0$ when $r > n$ because we can calculate the Ext-groups by the above resolution. Now suppose $\mathrm{Ext}^n_{R\Gamma}(M,-) = 0$. Then letting $K = \ker(\varphi)$ we get an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-2} \xrightarrow{\varphi} P_{n-3} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

and for every V, we have $\mathrm{Ext}^1_{R\Gamma}(K,V) \cong \mathrm{Ext}^n_{R\Gamma}(M,V) = 0$ by Lemma 3.34; thus K is projective. But this is a contradiction because we obtained a projective resolution with length less than $n = pd(M)$. Hence $\mathrm{Ext}^n_{R\Gamma}(M,-) \neq 0$ and we are done. □

From here it follows that *every* projective resolution of M can be trimmed to get a projective resolution of minimum length:

Corollary 3.36. *Let M be a right $R\Gamma$-module with $\mathrm{pd}(M) = n$. Let*

$$\dots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \xrightarrow{\varphi} P_{n-2} \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

be an arbitrary projective resolution of M. Then $K = \ker(\varphi)$ is projective.

Proof. There is an exact sequence

$$0 \longrightarrow K \longrightarrow P_{n-1} \xrightarrow{\varphi} P_{n-2} \longrightarrow \dots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Now by Lemma 3.34 and Proposition 3.35,

$$\mathrm{Ext}^1_{R\Gamma}(K,V) \cong \mathrm{Ext}^{n+1}_{R\Gamma}(M,V) = 0$$

for every V; hence K is projective. □

Remark 3.37. Note that the proof of Corollary 3.36 shows that if the r-th Ext-functor $\mathrm{Ext}^r_{R\Gamma}(M,-)$ is zero for some r, then M has a finite projective resolution. Because taking any projective resolution of M and trimming it at a suitable place yields a finite projective resolution of M.

So we get the following important characterization:

Proposition 3.38. *A right $R\Gamma$-module M has a finite projective resolution if and only if there exists an integer n such that $\mathrm{Ext}^r_{R\Gamma}(M,-) = 0$ for all $r \geq n$.*

Corollary 3.39. *Let*

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0$$

be a short exact sequence of right $R\Gamma$-modules. If two of the modules in the sequence have a finite projective resolution, then so does the third.

Proof. Given a right $R\Gamma$-module V, apply $\mathrm{Hom}_{R\Gamma}(-, V)$ to the given short exact sequence to get a long exact sequence of Ext-groups. Then apply Proposition 3.38. □

All the results we obtained about finite projective resolutions up to now directly generalizes to any abelian category with enough projectives. We begin to make use of our assumptions with the next proposition:

Proposition 3.40. *Let M be a right $R\Gamma$-module with a finite projective resolution. Let x be a maximal object of M. Then the right $R[x]$-module $\mathrm{Res}_x(M)$ has a finite projective resolution.*

Proof. By Corollary 3.11, there is a projective resolution $\mathbf{P} \to M$ such that

$$l(\mathbf{P}) \leq l(M) = l(\overline{x})\,.$$

Since M has finite projective dimension, we can assume $\mathbf{P}$ is a chain complex with finitely many nonzero terms. As $l(\mathbf{P}) \leq l(\overline{x})$, by Proposition 3.29 the finite resolution $\mathrm{Res}_x(\mathbf{P}) \to \mathrm{Res}_x(M)$ of $\mathrm{Res}_x(M)$ is a projective resolution. □

Proposition 3.40 enables us to use the following result from the representation theory and cohomology of finite groups:

Theorem 3.41. *Let G be a finite group. If N is an R-projective RG-module with a finite projective resolution, then N is RG-projective.*

Proof. See [7], Corollary 5.5. □

This yields the following result:

Theorem 3.42. *Let M be a right $R\Gamma$-module which admits a finite projective resolution. Let x be a maximal object of M such that* $\mathrm{Res}_x(M) = M(x)$ *is R-projective. Then* $\mathrm{Res}_x(M)$ *is a projective right $R[x]$-module.*

Proof. By Proposition 3.40, $\mathrm{Res}_x(M)$ has a finite projective resolution. Since $\mathrm{Res}_x(M)$ is R-projective and $\mathrm{Aut}_\Gamma(x)$ is a finite group, by Theorem 3.41 $\mathrm{Res}_x(M)$ is a projective right $R[x]$-module. □

Theorem 3.43. *Let M be a nonzero right $R\Gamma$-module such that $M(x)$ is a projective R-module for all $x \in$* $\mathrm{Obj}(\Gamma)$*. If M has a finite projective resolution, then* $\mathrm{pd}(M) \leq l(M)$*.*

Proof. We employ induction on $l(M)$. For every maximal object x of M, by Theorem 3.42, $\mathrm{Res}_x(M)$ is a projective right $R[x]$-module. Thus Corollary 3.21 is applicable and we get a short exact sequence

$$0 \longrightarrow L \longrightarrow P \longrightarrow M \longrightarrow 0$$

where P is projective and $l(L) < l(M)$.

If $l(M) = 0$, we get $L = 0$ so $M \cong P$ is projective, hence $\mathrm{pd}(M) = 0 = l(M)$; this finishes the basis step of the induction.

For $l(M) > 0$, if $L = 0$ we again get $M \cong P$; hence $\mathrm{pd}(M) = 0 \leq l(M)$. So we can assume $L \neq 0$. Note that L has a finite projective resolution (by Corollary 3.39 for instance). Moreover for every $y \in \mathrm{Obj}(\Gamma)$, the short exact sequence

$$0 \longrightarrow L(y) \longrightarrow P(y) \longrightarrow M(y) \longrightarrow 0$$

of R-modules splits since $M(y)$ is projective. But $P(y)$ is also projective (Corollary 2.16), so $L(y)$ is projective. Hence by the inductive hypothesis we get $\mathrm{pd}(L) \leq l(L)$. Say $\mathrm{pd}(L) = n$, so there is a projective resolution of L of the form

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow L \longrightarrow 0\,.$$

Splicing,

$$0 \longrightarrow Q_n \longrightarrow Q_{n-1} \longrightarrow \ldots \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow P \longrightarrow M \longrightarrow 0$$

is a projective resolution of M. Hence

$$\mathrm{pd}(M) \leq n + 1 = \mathrm{pd}(L) + 1 \leq l(L) + 1 \leq l(M)\,.$$

□

Chapter 4

Orbit categories and Rim's theorem

In this chapter, we first introduce the orbit category of a finite group with respect to a family of subgroups. We show that the orbit category is a free EI-category with finitely many morphisms, hence is subject to every result we have developed in the previous chapters.

We also state Rim's theorem in this chapter, which says that the projectivity of a $\mathbb{Z}G$-module (where G is a finite group) can be detected by restriction to Sylow subgroups of G. There is also a p-local version of Rim's theorem, which is about detecting the projectivity of a $\mathbb{Z}_{(p)}G$-module by restriction to a Sylow p-subgroup.

The main result of this chapter is a generalization of the p-local version of Rim's theorem to the orbit category, as it is done in [5].

Throughout this chapter R is a nonzero commutative ring, G is a finite group and $\mathcal{F}$ is a family of subgroups of G which is closed under conjugation and taking subgroups.

4.1 Orbit categories

Definition 4.1. The *orbit category* $\mathsf{Or}_{\mathcal{F}}G$ of G with respect to the family $\mathcal{F}$ is the category given by:

- $\mathrm{Obj}(\mathsf{Or}_{\mathcal{F}}G) = \mathcal{F}$
- For $K, H \in \mathcal{F}$, $\mathrm{Hom}_{\mathsf{Or}_{\mathcal{F}}G}(K, H)$ is the set of G-maps from the left G-set G/K to the left G-set G/H, shortly $\mathrm{Hom}_{\mathsf{Or}_{\mathcal{F}}G}(K, H) = \mathrm{Hom}_G(G/K, G/H)$.

We will shortly write Γ_G for $\mathsf{Or}_{\mathcal{F}}G$. Here is the first useful property of Γ_G:

Proposition 4.2. *Γ_G is a free category.*

Proof. Given $H, K \in \mathrm{Obj}(\Gamma_G) = \mathcal{F}$ we check that $\mathrm{Hom}_{\Gamma_G}(K, H) = \mathrm{Hom}_G(G/K, G/H)$ is a free left $\mathrm{Aut}_{\Gamma_G}(H) = \mathrm{Aut}_G(G/H)$-set. Indeed, assume we have $f \in \mathrm{Aut}_G(G/H)$ and $u \in \mathrm{Hom}_G(G/K, G/H)$ such that $f \circ u = u$. Say $f(H) = aH$ and $u(K) = bH$. Then

$$bH = u(K) = f(u(K)) = f(bH) = bf(H) = baH$$

and hence $a = b^{-1}ba \in H$. Thus $f(H) = H$, so $f = \mathrm{id}_{G/H}$. □

Now we give a useful interpretation of morphisms in Γ_G:

Proposition 4.3. *For $K, H \in \mathcal{F}$, there is a bijection between $\mathrm{Hom}_{\Gamma_G}(K, H) = \mathrm{Hom}_G(G/K, G/H)$ and $(G/H)^K$, the set of K-fixed points of the left G-set G/H.*

Proof. Define a function

$$\begin{aligned} \Phi : \mathrm{Hom}_G(G/K, G/H) &\to (G/H)^K \\ f &\mapsto f(K)\,. \end{aligned}$$

Φ is well-defined because for every $k \in K$ we have $kf(K) = f(kK) = f(K)$. This gives $f(K) \in (G/H)^K$. To go backwards, define

$$\Psi : (G/H)^K \to \mathrm{Hom}_G(G/K, G/H)$$

by

$$\Psi(aH) : G/K \to G/H$$
$$gK \mapsto gaH$$

for every $aH \in (G/H)^K$. To see $\Psi(aH)$ is well-defined for a given $aH \in (G/H)^K$, we observe that if $gK = tK$, we have $(ga)^{-1}(ta) = a^{-1}g^{-1}ta = (g^{-1}t)^a \in H$ because $KaH = aH$, that is, $K^a \subseteq H$ and $g^{-1}t \in K$; thus $gaH = taH$. It is clear that $\Psi(aH)$ is a G-map. Finally Ψ is well-defined because $aH = bH$ implies $gaH = gbH$. Φ and Ψ are mutually inverse:

- For $f \in \mathrm{Hom}_G(G/K, G/H)$ we have $\Psi(\Phi(f)) = \Psi(f(K))$ and $\Psi(f(K))(gK) = gf(K) = f(gK)$; hence $\Psi(\Phi(f)) = f$.
- For $aH \in (G/H)^K$ we have $\Phi(\Psi(aH)) = \Psi(aH)(K) = aH$.

□

As we observed above,

$$(G/H)^K = \{aH : KaH = aH\} = \{aH : K^a \subseteq H\}.$$

In particular for $H = K$, we get

$$(G/H)^H = \{aH : H^a \subseteq H\} = \{aH : H^a = H\} = \mathbf{N}_G(H)/H$$

where the second equality holds since G is finite. Note that both $\mathbf{N}_G(H)/H$ and $\mathrm{End}_{\Gamma_G}(H)$ naturally admit binary operations; the cosets in the former can be multiplied since $H \trianglelefteq \mathbf{N}_G(H)$ and we have composition for the latter. Proposition 4.3 gives the following bijection:

$$\Phi : \mathrm{End}_{\Gamma_G}(H) = \mathrm{End}_G(G/H) \to \mathbf{N}_G(H)/H$$
$$f \mapsto f(H).$$

Observe that

- $\Phi(\mathrm{id}_{G/H}) = \mathrm{id}_{G/H}(H) = H$

- For $f, f' \in \mathrm{End}_G(G/H)$, with $f(H) = aH$ and $f'(H) = a'H$, we have

$$\begin{aligned}\Phi(f' \circ f) &= (f' \circ f)(H)\\ &= f'(f(H))\\ &= f'(aH)\\ &= af'(H)\\ &= aa'H\\ &= aH \cdot a'H\\ &= f(H) \cdot f'(H)\\ &= \Phi(f) \cdot \Phi(f')\,.\end{aligned}$$

So Φ does not preserve, but *reverses* the binary operations. In other words Φ establishes a *monoid* isomorphism between $\mathrm{End}_{\Gamma_G}(H)$ and $(\mathbf{N}_G(H)/H)^{\mathrm{op}}$. But $(\mathbf{N}_G(H)/H)^{\mathrm{op}} \cong \mathbf{N}_G(H)/H$ [1] is not only a monoid, but a group. Hence $\mathrm{End}_{\Gamma_G}(H)$ must be a group under composition, which means $\mathrm{End}_{\Gamma_G}(H) = \mathrm{Aut}_{\Gamma_G}(H)$. Thus we have established that Γ_G is a free EI-category and clearly $\mathrm{Mor}(\Gamma_G)$ is finite.

4.2 Restricting the orbit category to a subgroup

In this section, given a subgroup H of G, we define an orbit category Γ_H for H (which depends on $\mathcal{F}$) and then construct a restriction functor from $\mathsf{Mod}\text{-}R\Gamma_G$ to $\mathsf{Mod}\text{-}R\Gamma_H$. Finally we prove that this functor preserves projectives, which is crucial for Rim's theorem for the orbit category.

Consider the projective $R\Gamma_G$-modules coming from the Yoneda Lemma, they are of the form $R\Gamma_G(-, K)$, where $K \in \mathcal{F}$. By Proposition 4.3, for every $L \in \mathcal{F}$ we have an isomorphism between the free R-modules $R\Gamma_G(-, K)(L) = R\,\mathrm{Hom}_{\Gamma_G}(L, K)$ and $R[(G/K)^L]$. Because of this reason we denote the projective right $R\Gamma_G$-module $R\Gamma_G(-, K)$ by $R[G/K^?]$.

[1] Every group is isomorphic to its opposite group via $x \mapsto x^{-1}$.

Now observe that a right $R\Gamma_G$-module $R[G/K^?]$ can be defined in the same way even when K is not in $\mathcal{F}$. The difference is that if $K \notin \mathcal{F}$, $R[G/K^?]$ may not be projective. Actually a right $R\Gamma_G$-module $R[S^?]$ can be defined for *any* left G-set S. Simply

$$R[S^?](L) = R\,\mathrm{Hom}_G(G/L, S)$$

for every $L \in \mathrm{Obj}(\Gamma_G) = \mathcal{F}$ and the action on morphisms is naturally defined.

Proposition 4.4. *Let S, T be left G-sets. Then there is an isomorphism*

$$R[(S \sqcup T)^?] \cong R[S^?] \oplus R[T^?]$$

of right $R\Gamma_G$-modules.

Proof. Here $S \sqcup T$ denotes the G-set formed by the disjoint union of S and T with the evident G-action. In categorical terms $S \sqcup T$ is the coproduct of S and T in the category of G-sets. We first establish an R-module isomorphism

$$\theta_K : R[(S \sqcup T)^?](K) \to (R[S^?] \oplus R[T^?])(K)$$

for every $K \in \mathcal{F}$. Note that

$$R[(S \sqcup T)^?](K) = R\,\mathrm{Hom}_G(G/K, S \sqcup T)$$

and

$$(R[S^?] \oplus R[T^?])(K) = R\,\mathrm{Hom}_G(G/K, S) \oplus R\,\mathrm{Hom}_G(G/K, T)\,.$$

Let $f \in \mathrm{Hom}_G(G/K, S \sqcup T)$. Since G/K is a transitive G-set, $\mathrm{im}\, f$ is a transitive G-subset of $S \sqcup T$; hence $\mathrm{im}\, f \subseteq S$ or $\mathrm{im}\, f \subseteq T$. In the former case, f restricts to a G-map

$$f_S : G/K \to S$$

and in the latter case to a G-map

$$f_T : G/K \to T\,.$$

So we define

$$\theta_K(f) = \begin{cases} (f_S, 0) & \text{if } \mathrm{im}\, f \subseteq S \\ (0, f_T) & \text{if } \mathrm{im}\, f \subseteq T \end{cases}$$

and extend R-linearly. To show that θ_K is bijective, we define an inverse β_K: Note that there are canonical injective G-maps

$$i_S : S \to S \sqcup T$$
$$i_T : T \to S \sqcup T$$

So we can define β_K by

$$\beta_K(u, 0) = i_S \circ u$$

for $u \in \mathrm{Hom}_G(G/K, S)$ and

$$\beta_K(0, v) = i_T \circ v$$

for $v \in \mathrm{Hom}_G(G/K, T)$, extended R-linearly. Then for $f \in \mathrm{Hom}_G(G/K, S \sqcup T)$,

$$\begin{aligned} \beta_K(\theta_K(f)) &= \begin{cases} \beta_K(f_S, 0) & \text{if } \mathrm{im}\, f \subseteq S \\ \beta_K(0, f_T) & \text{if } \mathrm{im}\, f \subseteq T \end{cases} \\ &= \begin{cases} i_S \circ f_S & \text{if } \mathrm{im}\, f \subseteq S \\ i_T \circ f_T & \text{if } \mathrm{im}\, f \subseteq T \end{cases} \\ &= f \end{aligned}$$

so $\beta_K \circ \theta_K = \mathrm{id}$ and similarly $\theta_K \circ \beta_K = \mathrm{id}$.

Finally we show that θ_K's are natural in K. So let $\alpha : K \to L$ be a morphism in Γ_G. So $K, L \in \mathcal{F}$ and

$$\alpha : G/K \to G/L$$

is a G-map. We must show that

$$\begin{array}{ccc} R\,\mathrm{Hom}_G(G/L, S \sqcup T) & \xrightarrow{\alpha^*} & R\,\mathrm{Hom}_G(G/K, S \sqcup T) \\ \downarrow{\scriptstyle \theta_L} & & \downarrow{\scriptstyle \theta_K} \\ R\,\mathrm{Hom}_G(G/L, S) \oplus R\,\mathrm{Hom}_G(G/L, T) & \xrightarrow[\alpha^*]{} & R\,\mathrm{Hom}_G(G/K, S) \oplus R\,\mathrm{Hom}_G(G/K, T) \end{array}$$

commutes. Let $f \in \mathrm{Hom}_G(G/L, S \sqcup T)$. Without loss of generality, we may assume $\operatorname{im} f \subseteq S$. Then

$$\begin{aligned}\alpha^*(\theta_L(f)) &= \alpha^*(f_S, 0)\\ &= (f_S \circ \alpha, 0)\\ &= ((f \circ \alpha)_S, 0)\\ &= ((\alpha^*(f))_S, 0)\\ &= \theta_K(\alpha^*(f))\end{aligned}$$

and we are done. $\square$

It is clear that the assignment $S \mapsto R[S^?]$ defines a covariant functor

$$\iota_G : G\text{-}\mathsf{Set} \to \mathsf{Mod}\text{-}R\Gamma_G\,.$$

Proposition 4.4 says that ι_G preserves finite coproducts.

Let H be a subgroup of G (we do not require that $H \in \mathcal{F}$). Every G-set can be seen as an H-set and there is a restriction functor

$$\operatorname{Res}^G_H : G\text{-}\mathsf{Set} \to H\text{-}\mathsf{Set}\,.$$

So we have the following diagram of categories and functors:

$$\begin{array}{ccc} G\text{-}\mathsf{Set} & \xrightarrow{\iota_G} & \mathsf{Mod}\text{-}R\Gamma_G \\ {\scriptstyle \operatorname{Res}^G_H}\big\downarrow & & \\ H\text{-}\mathsf{Set} & & \end{array}$$

We will define an orbit category Γ_H of H such that $\mathsf{Mod}\text{-}R\Gamma_H$ will complete the above diagram to a commutative square (up to a natural isomorphism). Let

$$\mathcal{F}_H = \{K \leq H : K \in F\}\,.$$

We observe the following:

- Let $K \in \mathcal{F}_H$ and $L \leq K$. Then $L \in \mathcal{F}$ since $K \in \mathcal{F}$ and $\mathcal{F}$ is closed under subgroups. Moreover $L \leq H$, so $L \in \mathcal{F}$.

- Let $K \in \mathcal{F}_H$ and $h \in H$. Then since $\mathcal{F}$ is closed under conjugation by any element of G, $K^h \in \mathcal{F}$. Also $K^h \leq H$, so $K^h \in \mathcal{F}_H$.

So $\mathcal{F}_H$ is a family of subgroups of H which is closed under taking subgroups and conjugation. Therefore we can form the orbit category $\mathsf{Or}_{\mathcal{F}_H}(H)$, which we denote shortly by Γ_H. We define a functor

$$F : \Gamma_H \to \Gamma_G$$

as follows: Note that $\mathrm{Obj}(\Gamma_H) = \mathcal{F}_H$ and $\mathrm{Obj}(\Gamma_G) = \mathcal{F}$. So we can define F on objects by inclusion: $F(K) = K$. For morphisms, let

$$f : K \to L$$

be a morphism in Γ_H. By Proposition 4.3, f is uniquely determined by a left coset $aL \in (H/L)^K \subseteq (G/L)^K$. Then again by Proposition 4.3, aL uniquely determines a morphism

$$F(f) : K \to L$$

in Γ_G. More transparently, by definition f is an H-map of the form

$$f : H/K \to H/L\,.$$

Say $f(K) = aL$. Then $F(f)$ is the G-map given by

$$\begin{aligned} F(f) : G/K &\to G/L \\ gK &\mapsto gaL\,. \end{aligned}$$

It is clear that F preserves identity morphisms. To see that F preserves compositions, let

$$f' : J \to K$$

$$f : K \to L$$

be morphisms in Γ_H. So we have H-maps

$$f' : H/J \to H/K\,,$$

$$f : H/K \to H/L\,,$$

$$f \circ f' : H/J \to H/L\,.$$

Say $f'(J) = a'K$ and $f(K) = aL$. Then

$$(f \circ f')(J) = f(f'(J)) = f(a'K) = a'f(K) = a'aL\,.$$

From these we get G-maps

$$\begin{aligned} F(f') : G/J &\to G/K \\ gJ &\mapsto ga'K\,, \end{aligned}$$

$$\begin{aligned} F(f) : G/K &\to G/L \\ gK &\mapsto gaL\,, \end{aligned}$$

$$\begin{aligned} F(f \circ f') : G/J &\to G/L \\ gJ &\mapsto ga'aL\,. \end{aligned}$$

Thus

$$(F(f) \circ F(f'))(gJ) = F(f)(F(f')(gJ)) = F(f)(ga'K) = ga'aL = F(f \circ f')(gJ)\,.$$

So the morphisms

$$\begin{aligned} F(f') &: J \to K \\ F(f) &: K \to L \\ F(f \circ f') &: J \to L \end{aligned}$$

in Γ_G satisfy $F(f \circ f') = F(f) \circ F(f')$, as desired. The functor $F : \Gamma_H \to \Gamma_G$ yields a restriction functor

$$\mathrm{Res}_F : \mathsf{Mod}\text{-}R\Gamma_G \to \mathsf{Mod}\text{-}R\Gamma_H\,.$$

We denote Res_F by Res^G_H in this case. We can now complete the square:

Theorem 4.5. *The diagram of functors*

$$\begin{array}{ccc} G\text{-}\mathsf{Set} & \xrightarrow{\iota_G} & \mathsf{Mod}\text{-}R\Gamma_G \\ \downarrow{\scriptstyle \mathrm{Res}^G_H} & & \downarrow{\scriptstyle \mathrm{Res}^G_H} \\ H\text{-}\mathsf{Set} & \xrightarrow{\iota_H} & \mathsf{Mod}\text{-}R\Gamma_H \end{array}$$

commutes up to a natural isomorphism.

Proof. Let S be a G-set. We should establish an isomorphism between the right $R\Gamma_H$-modules

$$(\mathrm{Res}^G_H \circ \iota_G)(S) = \mathrm{Res}^G_H(R[S^?]) = R[S^?] \circ F$$

and

$$(\iota_H \circ \mathrm{Res}^G_H)(S) = R[\mathrm{Res}^G_H(S)^?]\,.$$

So let $K \in \mathcal{F}_H$. We have

$$(R[S^?] \circ F)(K) = R[S^?](F(K)) = R\,\mathrm{Hom}_G(G/K, S)$$

and

$$R[\mathrm{Res}^G_H(S)^?](K) = R\,\mathrm{Hom}_H(H/K, S)\,.$$

Define

$$\epsilon_K : \mathrm{Hom}_G(G/K, S) \to \mathrm{Hom}_H(H/K, S)$$

by restriction, as every G-map from G/K to S restricts to an H-map from H/K to S. If $\epsilon_K(f) = \epsilon_K(f')$, then $f(K) = f'(K)$ and hence $f(gK) = gf(K) = gf'(K) = f'(gK)$ for every $g \in G$; thus ϵ_K is injective.

To see that ϵ_K is surjective, let $p : H/K \to S$ be an H-map. Define

$$\begin{aligned} f : G/K &\to S \\ gK &\mapsto g \cdot p(K)\,. \end{aligned}$$

f is well-defined because if $gK = tK$, $g^{-1}t \in K \subseteq H$; so as p is an H-map,

$$g \cdot p(K) = g \cdot p(g^{-1}tK) = g \cdot (g^{-1}t \cdot p(K)) = t \cdot p(K)\,.$$

It is clear that f is a G-map and

$$\epsilon_K(f)(hK) = f(hK) = h \cdot p(K) = p(hK)$$

for every $h \in H$; hence $\epsilon_K(f) = p$.

So ϵ_K is a bijection and so it extends to an R-module isomorphism

$$\epsilon_{S,K} : R\,\mathrm{Hom}_G(G/K, S) \to R\,\mathrm{Hom}_H(H/K, S)$$

We will show that $\epsilon_{S,K}$'s are natural in K. So let $\alpha : K \to L$ be a morphism in Γ_H; in other words $K, L \in \mathcal{F}_H$ and $\alpha : H/K \to H/L$ is an H-map. The diagram

$$\begin{array}{ccc} R\operatorname{Hom}_G(G/L,S) & \xrightarrow{(R[S^?]\circ F)(\alpha)} & R\operatorname{Hom}_G(G/K,S) \\ \downarrow{\scriptstyle \epsilon_{S,L}} & & \downarrow{\scriptstyle \epsilon_{S,K}} \\ R\operatorname{Hom}_H(H/L,S) & \xrightarrow[R[\operatorname{Res}^G_H(S)^?](\alpha)]{} & R\operatorname{Hom}_H(H/K,S) \end{array}$$

commutes: Let $f \in \operatorname{Hom}_G(G/L,S)$ and $h \in H$. On one hand

$$\left[\epsilon_{S,K} \circ (R[S^?] \circ F)(\alpha)\right](f) = \epsilon_{S,K}(f \circ F(\alpha))$$

and $\epsilon_{S,K}(f \circ F(\alpha))(hK) = (f \circ F(\alpha))(hK) = f(\alpha(hK))$.
On the other hand

$$\left[R[\operatorname{Res}^G_H(S)^?](\alpha) \circ \epsilon_{S,L}\right](f) = \epsilon_{S,L}(f) \circ \alpha$$

and $(\epsilon_{S,L}(f) \circ \alpha)(hK) = f(\alpha(hK))$.
Thus we get an isomorphism

$$\epsilon_S : R[S^?] \circ F \to R[\operatorname{Res}^G_H(S)^?]\,.$$

Finally we claim that ϵ_S's are natural in S. So let $\lambda : S \to T$ be a G-map. To verify that the diagram

$$\begin{array}{ccc} R[S^?] \circ F & \xrightarrow{(\operatorname{Res}^G_H \circ \iota_G)(\lambda)} & R[T^?] \circ F \\ \downarrow{\scriptstyle \epsilon_S} & & \downarrow{\scriptstyle \epsilon_T} \\ R[Res^G_H(S)^?] & \xrightarrow[(\iota_H \circ \operatorname{Res}^G_H)(\lambda)]{} & R[Res^G_H(T)^?] \end{array}$$

commutes, it suffices to check that

$$\begin{array}{ccc} R\operatorname{Hom}_G(G/K,S) & \xrightarrow{(\operatorname{Res}^G_H \circ \iota_G)(\lambda)_K} & R\operatorname{Hom}_G(G/K,T) \\ \downarrow{\scriptstyle \epsilon_{S,K}} & & \downarrow{\scriptstyle \epsilon_{T,K}} \\ R\operatorname{Hom}_H(H/K,S) & \xrightarrow[(\iota_H \circ \operatorname{Res}^G_H)(\lambda)_K]{} & R\operatorname{Hom}_H(H/K,T) \end{array}$$

commutes for every $K \in \mathcal{F}_H$. Indeed for $f \in \mathrm{Hom}_G(G/K, S)$, we have

$$\left[\epsilon_{T,K} \circ (\mathrm{Res}^G_H \circ \iota_G)(\lambda)_K\right](f) = \epsilon_{T,K}(\lambda \circ f)$$

and

$$\left[(\iota_H \circ \mathrm{Res}^G_H)(\lambda)_K \circ \epsilon_{S,K}\right](f) = \mathrm{Res}^G_H(\lambda) \circ \epsilon_{S,K}(f)$$

which are easily checked to be equal. Therefore we obtain a natural isomorphism

$$\epsilon : \mathrm{Res}^G_H \circ \iota_G \to \iota_H \circ \mathrm{Res}^G_H .$$

□

The functor $\mathrm{Res}^G_H : G\text{-}\mathsf{Set} \to H\text{-}\mathsf{Set}$ satisfies the well-known Mackey double coset formula:

Proposition 4.6. *Let K be any subgroup of G and consider the left G-set G/K. Let $E = \{HgK : g \in G\}$ be the set of H-K-double cosets. Then there is a left H-set isomorphism*

$$\mathrm{Res}^G_H(G/K) \cong \bigsqcup_{HgK \in E} H/(H \cap {}^gK)\,.$$

With the help of Theorem 4.5 we can transfer the Mackey formula to modules over the orbit category:

Proposition 4.7. *Let K be any subgroup of G and consider the right $R\Gamma_G$-module $R[G/K^?]$. Let $E = \{HgK : g \in G\}$ be the set of H-K-double cosets. Then there is a right $R\Gamma_H$-module isomorphism*

$$\mathrm{Res}^G_H(R[G/K^?]) \cong \bigoplus_{HgK \in E} R[H/(H \cap {}^gK)^?]\,.$$

Proof. We have

$$\begin{aligned}\mathrm{Res}^G_H(R[G/K^?]) &= (\mathrm{Res}^G_H \circ \iota_G)(G/K)\\ &\cong (\iota_H \circ \mathrm{Res}^G_H)(G/K)\\ &\cong \iota_H\left(\bigsqcup_{HgK\in E} H/(H\cap {}^gK)\right)\\ &\cong \bigoplus_{HgK\in E} \iota_H(H/(H\cap {}^gK))\\ &= \bigoplus_{HgK\in E} R[H/(H\cap {}^gK)^?]\end{aligned}$$

where the first isomorphism is by Theorem 4.5 and the second is by the Mackey formula. The third isomorphism holds since ι_H preserves finite coproducts by Proposition 4.4. □

Corollary 4.8. *The functor* Res^G_H *sends projectives to projectives.*

Proof. By Proposition 2.15, it is enough to check that Res^G_H sends projectives of the form $R[G/K^?]$ where $K \in \mathcal{F}$ to projectives. Indeed,

$$\mathrm{Res}^G_H(R[G/K^?]) \cong \bigoplus_{HgK\in E} R[H/(H\cap {}^gK)^?]$$

is a projective right $R\Gamma_H$-module because as $\mathcal{F}$ is closed under taking subgroups and conjugation, $H \cap {}^gK \in \mathcal{F}_H$ for every g; hence each $R[H/(H\cap {}^gK)^?]$ is projective.

□

4.3 Rim's theorem for the orbit category

Rim's original theorem is for group rings. Of course for every $H \leq G$ there is a restriction functor

$$\mathrm{Res}^G_H : RG\text{-}\mathsf{Mod} \to RH\text{-}\mathsf{Mod}\,.$$

Here is Rim's original theorem:

Theorem 4.9 ([8], Proposition 4.9)**.** *A $\mathbb{Z}G$-module N is projective if and only if the $\mathbb{Z}P$-module $\mathrm{Res}^G_P(N)$ is projective for every Sylow subgroup P of G.*

Let p be a prime. There is also a p-local version:

Theorem 4.10. *Let P be a Sylow p-subgroup of G. A $\mathbb{Z}_{(p)}G$-module N is projective if and only if the $\mathbb{Z}_{(p)}P$-module $\mathrm{Res}^G_P(N)$ is projective.*

The p-local version of Rim's theorem can be generalized to modules over the orbit category (see [5], Theorem B). To prove this result, we need an elementary lemma from Sylow theory:

Lemma 4.11. *Let Q, T be subgroups of G such that Q is a p-group and $Q \subseteq T$. Then there exists $P \in \mathrm{Syl}_p(G)$ such that $Q \subseteq P$ and $P \cap T \in \mathrm{Syl}_p(T)$.*

Proof. As Q is a p-subgroup of T, there exists $S \in \mathrm{Syl}_p(T)$ such that $Q \subseteq S$. Similarly S is a p-subgroup of G, so there exists $P \in \mathrm{Syl}_p(G)$ such that $S \subseteq P$. Now observe that $P \cap T$ is a p-subgroup of T which contains S, hence $P \cap T = S$. We are done. □

Corollary 4.12. *Let $Q \in \mathcal{F}$ be a p-subgroup of G. Then there exists $P \in \mathrm{Syl}_p(G)$ that contains Q such that the functor*

$$F : \Gamma_P \to \Gamma_G$$

that is used to define the restriction functor Res^G_P embeds $\mathrm{Aut}_{\Gamma_P}(Q)$ in $\mathrm{Aut}_{\Gamma_G}(Q)$ as a Sylow p-subgroup.

Proof. Apply Lemma 4.11 for $T = \mathbf{N}_G(Q)$ to get $P \in \mathrm{Syl}_p(G)$ which contains Q such that $P \cap \mathbf{N}_G(Q) = \mathbf{N}_P(Q)$ is a Sylow p-subgroup of $\mathbf{N}_G(Q)$. In particular

$$|\mathbf{N}_G(Q)|_p = |\mathbf{N}_P(Q)|$$

where the subscript p denotes the p-part of a number.
The functor F is clearly faithful so it gives an injective group homomorphism

$$F : \mathrm{Aut}_{\Gamma_P}(Q) \to \mathrm{Aut}_{\Gamma_G}(F(Q)) = \mathrm{Aut}_{\Gamma_G}(Q)$$

Now observe that

$$\begin{aligned} |\operatorname{Aut}_{\Gamma_P}(Q)| &= |\mathbf{N}_P(Q)/Q| \\ &= \frac{|\mathbf{N}_P(Q)|}{|Q|} \\ &= \frac{|\mathbf{N}_G(Q)|_p}{|Q|} \\ &= \left(\frac{|\mathbf{N}_G(Q)|}{|Q|}\right)_p \\ &= |\mathbf{N}_G(Q)/Q|_p \\ &= |\operatorname{Aut}_{\Gamma_G}(Q)|_p \end{aligned}$$

Thus the image of F is a Sylow p-subgroup of $\operatorname{Aut}_{\Gamma_G}(Q)$. □

We can now state and prove Rim's theorem for the orbit category:

Theorem 4.13 ([5], Theorem B). *Let $R = \mathbb{Z}_{(p)}$ and assume that every subgroup in $\mathcal{F}$ is a p-subgroup. Then a right $R\Gamma_G$-module M has a finite projective resolution if and only if the right $R\Gamma_P$-module $\operatorname{Res}^G_P(M)$ has a finite projective resolution for every Sylow p-subgroup P of G.*

Proof. The 'only if' direction is clear because Res^G_P is an exact functor which preserves projectives by Proposition 4.8.

Now assume $\operatorname{Res}^G_P(M)$ has a finite projective resolution for every $P \in \operatorname{Syl}_p(G)$. We will first prove the result assuming $M(K)$ is R-projective for every $K \in \mathcal{F}$: Employ induction on the length $l(M)$.

If $l(M) = -1$, $M = 0$ and there is nothing to show. For $l(M) > 0$, let Q be a maximal object of M. Apply Corollary 4.12 for Q to get $P \in \operatorname{Syl}_p(G)$. So $Q \in \mathcal{F}_P$ and moreover Q is a maximal object of $\operatorname{Res}^G_P(M)$. Hence by Theorem 3.42, $\operatorname{Res}^G_P(M)(Q) = M(Q)$ is projective as a right $R\operatorname{Aut}_{\Gamma_P}(Q)$-module.

Since $\operatorname{Aut}_{\Gamma_P}(Q)$ embeds in $\operatorname{Aut}_{\Gamma_G}(Q)$ as a Sylow p-subgroup, by p-local Rim's theorem $M(Q)$ is projective as an $R\operatorname{Aut}_{\Gamma_G}(Q)$-module. The maximal object Q

above was arbitrary, so by Corollary 3.21 there is a short exact sequence

$$0 \longrightarrow L \longrightarrow F \longrightarrow M \longrightarrow 0$$

of right $R\Gamma_G$-modules such that F is projective and $l(L) < l(M)$. For every $K \in \mathcal{F}$, the short exact sequence

$$0 \longrightarrow L(K) \longrightarrow F(K) \longrightarrow M(K) \longrightarrow 0$$

of R-modules splits as $M(K)$ is R-projective. $F(K)$ is also R-projective (see Corollary 2.16), hence $L(K)$ is R-projective. Therefore by the induction hypothesis, L has a finite projective resolution. Thus M has a finite projective resolution.

Now we prove the general case: Since $\mathsf{Mod}\text{-}R\Gamma_G$ has enough projectives, there exists an exact sequence

$$0 \longrightarrow U \longrightarrow V \longrightarrow M \longrightarrow 0$$

of right $R\Gamma_G$-modules where V is projective. So for every $K \in \mathcal{F}$

$$0 \longrightarrow U(K) \longrightarrow V(K) \longrightarrow M(K) \longrightarrow 0$$

is an exact sequence of R-modules where $V(K)$ is projective. Since $R = \mathbb{Z}_{(p)}$ is a PID, $U(K)$ is also R-projective (actually R-free) for every K.

Let $P \in \mathrm{Syl}_p(G)$. Then we have a short exact sequence of right $R\Gamma_P$-modules

$$0 \longrightarrow \mathrm{Res}^G_P(U) \longrightarrow \mathrm{Res}^G_P(V) \longrightarrow \mathrm{Res}^G_P(M) \longrightarrow 0$$

where $\mathrm{Res}^G_P(V)$ is projective by Corollary 4.8 and $\mathrm{Res}^G_P(M)$ has a finite projective resolution by assumption. Hence $\mathrm{Res}^G_P(U)$ has a finite projective resolution (Corollary 3.39). Therefore by the first part, U has a finite projective resolution; thus, so does M. $\square$

Remark 4.14. The phrase *for every Sylow p-subgroup* can be replaced by *for some Sylow p-subgroup* in Theorem 4.13. The reason is that if P, Q are Sylow

p-subgroups, $Q = P^g$ for some $g \in G$ and since $\mathcal{F}$ is closed under conjugation, there is a functor

$$\begin{aligned} c^g : \Gamma_P &\to \Gamma_Q \\ K &\mapsto K^g \end{aligned}$$

which is a category equivalence. Moreover we have the faithful functors

$$\begin{aligned} F : \Gamma_P &\to \Gamma_G \\ F' : \Gamma_Q &\to \Gamma_G \end{aligned}$$

which induce Res^G_P and Res^G_Q. It is straightforward to check that the diagram of functors

$$\begin{array}{ccccc} & & \Gamma_G & & \\ & \overset{F}{\nearrow} & & \overset{F'}{\nwarrow} & \\ \Gamma_P & & \xrightarrow{\quad c^g \quad} & & \Gamma_Q \end{array}$$

commutes up to a natural isomorphism. So this diagram induces a diagram

$$\begin{array}{ccccc} & & \mathsf{Mod}\text{-}R\Gamma_G & & \\ & \overset{\mathrm{Res}^G_P}{\swarrow} & & \overset{\mathrm{Res}^G_Q}{\searrow} & \\ \mathsf{Mod}\text{-}R\Gamma_P & & \xleftarrow[\quad (c^g)^* \quad]{} & & \mathsf{Mod}\text{-}R\Gamma_Q \end{array}$$

which is also commutative up to a natural isomorphism and $(c^g)^*$ is a category equivalence. In particular $(c^g)^*$ is an exact functor which preserves projectives. Thus for a right $R\Gamma_G$-module M, $\mathrm{Res}^G_Q(M)$ has a finite projective resolution if and only if $\mathrm{Res}^G_P(M)$ has a finite projective resolution.

Chapter 5

Resolving the constant functor

Given a small category Γ and a commutative ring R, we can always form a contravariant functor

$$\underline{R} : \Gamma \to R\text{-}\mathsf{Mod}$$

by letting $\underline{R}(x) = R$ for every $x \in \mathrm{Obj}(\Gamma)$ and $\underline{R}(\alpha) = \mathrm{id}_R$ for every $\alpha \in \mathrm{Mor}(\Gamma)$. $\underline{R}$ is called the constant functor and as a right $R\Gamma$-module, it plays the role of the trivial module of a group algebra. For example, using a projective resolution of $\underline{R}$, the *cohomology of* Γ (with coefficients in R) can be defined (see [1]) as a generalization of group cohomology.

In this chapter, as a consequence of Theorem 4.13, we show that for $R = \mathbb{Z}_{(p)}$, if Γ is the orbit category of a finite group G with respect to the family of *all* p-subgroups of G, then $\underline{R}$ has a finite projective resolution. After mentioning some general results about the projective dimension $\mathrm{pd}(\underline{R})$, we calculate $\mathrm{pd}(\underline{R})$ for some of the first nontrivial cases.

At the end of the chapter, we prove a negative result which states that for $R = \mathbb{Z}$ and

$$\mathcal{F} = \{\text{subgroups of } G \text{ with prime power order}\},$$

the constant functor $\underline{R}$ almost never has a finite projective resolution.

5.1 $\underline{R}$ in the p-local setting

Throughout this section, G is a finite group and p is a prime. Also $R = \mathbb{Z}_{(p)}$ and $\mathcal{F}$ is the family of *all* p-subgroups of G. As $\mathcal{F}$ is closed under subgroups and conjugation, we can form the orbit category Γ_G. So $\underline{R}$ is an object in $\mathsf{Mod}\text{-}R\Gamma_G$.

Proposition 5.1. *$\underline{R}$ has a finite projective resolution.*

Proof. Observe that the left G-set G/G is a singleton and $(G/G)^Q = G/G$ for any $Q \in \mathcal{F}$. It follows that there is a right $R\Gamma_G$-module isomorphism

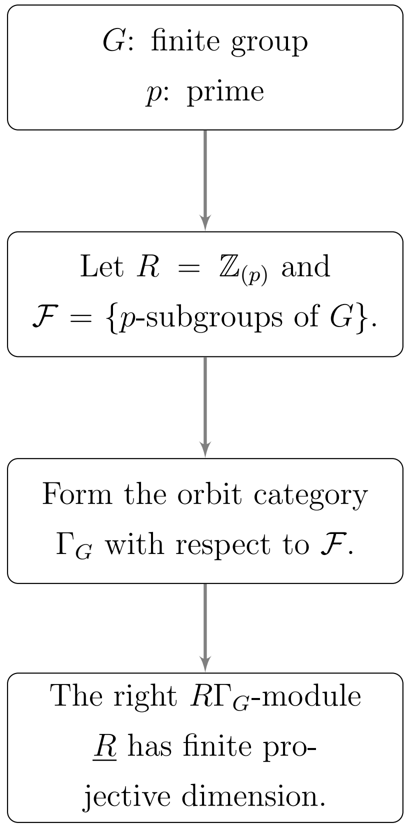
$$\underline{R} \cong R[G/G^?] \,.$$

Let $P \in \mathrm{Syl}_p(G)$. Then either by the simple fact that the restriction of a constant functor is another constant functor or using Corollary 4.7, we obtain a right $R\Gamma_P$-module isomorphism

$$\mathrm{Res}_P^G(\underline{R}) \cong R[P/P^?] \,.$$

But since $\mathcal{F}$ contains every p-subgroup, $P \in \mathcal{F}$. In particular $P \in \mathcal{F}_P$ and hence $R[P/P^?]$ is a projective right $R\Gamma_P$-module. Thus by Theorem 4.13, the right $R\Gamma_G$-module $\underline{R}$ has a finite projective resolution. $\square$

Observe how $\mathrm{pd}(\underline{R})$ only depends on G and p:



Hence we denote $\mathrm{pd}(\underline{R})$ by $\mathrm{pd}(G, p)$. Our major aim in this section is to relate the number $\mathrm{pd}(G, p)$ with the more intrinsic properties of the finite group G.

Remark 5.2. We have used Proposition 4.3 several times in this thesis and it gives a concrete picture of the morphisms in the orbit category. The bijection in the statement is adequate for most purposes but in fact it can be developed into a full-fledged category equivalence. We develop this equivalence here because it will serve very well for explicit calculations. Define the category Λ_G as follows:

- $\mathrm{Obj}(\Lambda_G) = \mathcal{F}$.
- For $H, K \in \mathcal{F}$, $\mathrm{Hom}_{\Lambda_G}(H, K) = (G/H)^K$. [1]
- For $H \in \mathcal{F}$, the identity morphism id_H is the trivial coset $H \in (G/H)^H$.

[1] The problem with this definition is that the Hom-sets may not be disjoint and their disjointness is necessary to define a category. This issue can be solved by introducing extra indices to distinguish the morphisms. We will not use such indices in our notation but assume they are present implicitly.

- Given $bH \in (G/H)^K$ and $aK \in (G/K)^L$, their composition is defined by
$$aK \circ bH = abH\,.$$

Let's check that the composition is well-defined: First, note that since $bH \in (G/H)^K$, $K^b \subseteq H$. Similarly $L^a \subseteq K$. Hence $L^{ab} \subseteq H$ and therefore $abH \in (G/H)^L = \mathrm{Hom}_{\Lambda_G}(H, L)$. Second, if $aK = cK$ and $bH = dH$, as $c^{-1}a \in K$ and bH is fixed by K, we have $c^{-1}abH = bH = dH$; thus $abH = cdH$. It is clear now that the composition is associative and identities act as expected.

Now we define a *contravariant* functor
$$\Phi : \Gamma_G \to \Lambda_G$$
by $\Phi(H) = H$ for every $H \in \mathcal{F}$ and for $f \in \mathrm{Hom}_{\Gamma_G}(K, H)$,
$$\Phi(f) = f(K) \in (G/H)^K = \mathrm{Hom}_{\Lambda_G}(H, K)\,.$$
We have checked in Proposition 4.3 that Φ is well-defined. Clearly Φ preserves identities. To see that Φ *reverses* compositions, let $f \in \mathrm{Hom}_{\Gamma_G}(L, K)$ and $u \in \mathrm{Hom}_{\Gamma_G}(K, H)$. So
$$f : G/L \to G/K$$
$$u : G/K \to G/H$$
are G-maps. Say $f(L) = aK$ and $u(K) = bH$. Then
$$\Phi(u \circ f) = (u \circ f)(L) = u(aK) = au(K) = abH = aK \circ bH = \Phi(f) \circ \Phi(u)\,.$$
Now Φ is full and faithful by Proposition 4.3. Also Φ is surjective on objects, thus Φ, interpreted as a covariant functor
$$\Phi : \Gamma_G^{\mathrm{op}} \to \Lambda_G$$
is a category equivalence. Consequently it yields a category equivalence
$$\mathsf{Mod}\text{-}R\Gamma_G \equiv R\Lambda_G\text{-}\mathsf{Mod}\,.$$
So for instance we can view the right $R\Gamma_G$-module $R[G/H^?]$ as a left $R\Lambda_G$-module such that $R[G/H^?](K) = R[(G/H)^K]$ has a left $R[(G/K)^K] = R[\mathbf{N}_G(K)/K]$-module structure by left multiplication. We will do this type of interpretations frequently in what follows without mentioning them explicitly.

Remark 5.3. There is a rather efficient way to start resolving $\underline{R}$: Let $P \in \mathrm{Syl}_p(G)$. The trivial G-map

$$\tau : G/P \to G/G$$

induces a morphism of right $R\Gamma_G$-modules

$$\epsilon : R[G/P^?] \to R[G/G^?]\,.$$

Note that $R[G/P^?]$ is projective because $P \in \mathcal{F}$. Also $R[G/G^?] \cong \underline{R}$. We claim that ϵ is an epimorphism. Considering ϵ as a morphism of left $R\Lambda_G$-modules, it suffices to check that

$$\epsilon_Q : R[(G/P)^Q] \to R[(G/G)^Q] = R[G/G]$$

is a surjective R-module homomorphism for every $Q \in \mathcal{F}$. By definition ϵ_Q is given by the set map

$$\begin{aligned}(G/P)^Q &\to G/G\\ aP &\mapsto G\end{aligned}$$

extended R-linearly; hence it suffices to check that this set map is surjective. Moreover as G/G is a singleton, it is enough to verify that $(G/P)^Q$ is nonempty. This is indeed the case because

$$(G/P)^Q = \{gP : Q^g \subseteq P\}$$

is nonempty by Sylow theory, since Q is a p-group and P is a Sylow p-subgroup.

Here is the first result about how $\mathrm{pd}(G,p)$ relates with G:

Proposition 5.4. $\mathrm{pd}(G,p) = 0$ *if and only if* G *has a normal Sylow* p*-subgroup.*

Proof. $\mathrm{pd}(G,p) = 0$ means that the constant functor $\underline{R}$ is projective. Consider the epimorphism

$$\epsilon : R[G/P^?] \to R[G/G^?] \cong \underline{R}$$

we obtained in Remark 5.3. Since $R[G/P^?]$ is projective, $\underline{R}$ is projective if and only if ϵ splits.

Assume ϵ splits, say via

$$s : \underline{R} \to R[G/P^?]\,.$$

Then evaluating at P, we get an $R[\mathbf{N}_G(P)/P]$-module homomorphism

$$s_P : R \to R[(G/P)^P] = R[\mathbf{N}_G(P)/P]$$

such that $\epsilon_P \circ s_P = \mathrm{id}$. Since R above has trivial $\mathbf{N}_G(P)/P$-action on it, it follows that

$$s_P(1) = \frac{1}{|\mathbf{N}_G(P)/P|}\left(\sum_{gP\in\mathbf{N}_G(P)/P} gP\right)\,.$$

This is because in general, for any group H and commutative ring k, if the augmentation map $kH \to k$ has a kH-splitting s, we have $s(1) = \frac{1}{|H|}\left(\sum_{h\in H} h\right)$.

Similarly, if we evaluate s at the trivial subgroup 1, we get an $R[\mathbf{N}_G(1)/1] = RG$-module homomorphism

$$s_1 : R \to R[(G/P)^1] = R[G/P]$$

such that $\epsilon_1 \circ s_1 = \mathrm{id}$. Hence

$$s_1(1) = \frac{1}{|G/P|}\left(\sum_{gP\in G/P} gP\right)\,.$$

Now note that $P \in (G/P)^1 = \mathrm{Hom}_{\Lambda_G}(P, 1)$. To keep this P in mind as a morphism in Λ_G, we denote it by ι. Applying the functor $R[G/P^?]$, we get an R-module homomorphism

$$\begin{aligned} R[G/P^?](\iota) : R[(G/P)^P] &\to R[(G/P)^1] \\ aP &\mapsto \iota \circ aP = P \circ aP = aP\,. \end{aligned}$$

Since s is a natural transformation, we have a commutative diagram

$$\begin{array}{ccc} R[(G/P)^P] & \xrightarrow{R[G/P^?](\iota)} & R[(G/P)^1] \\ {\scriptstyle s_P}\uparrow & & \uparrow{\scriptstyle s_1} \\ R & \xrightarrow[\underline{R}(\iota)=\mathrm{id}_R]{} & R \end{array}$$

Chasing the element $1 \in R$ and using our previously computed expressions for s_1 and s_P we obtain

$$\frac{1}{|\mathbf{N}_G(P)/P|}\left(\sum_{gP \in \mathbf{N}_G(P)/P} gP\right) = \frac{1}{|G/P|}\left(\sum_{gP \in G/P} gP\right).$$

Note that this is an equality that occurs in the (permutation) RG-module $R[(G/P)^1] = R[G/P]$. Thus it forces $\mathbf{N}_G(P) = G$, that is, $P \trianglelefteq G$.

Conversely, assume $P \trianglelefteq G$. Then P is the unique Sylow p-subgroup of G. Hence for every $Q \in \mathcal{F}$ and $g \in G$ we have $Q^g \subseteq P$; so $(G/P)^Q = G/P$. Therefore for each Q we can define an R-module homomorphism

$$\begin{aligned} s_Q : R &\to R[(G/P)^Q] = R[G/P] \\ 1 &\mapsto \frac{1}{|G/P|}\left(\sum_{gP \in G/P} gP\right) \end{aligned}$$

where we also use the fact that $|G/P|$ is an invertible element in $R = \mathbb{Z}_{(p)}$. The naturality of s_Q's is easily checked to yield a morphism

$$s : \underline{R} \to R[G/P^?].$$

Since $\epsilon_Q \circ s_Q = \mathrm{id}$ for every Q, s is a splitting for ϵ. □

Corollary 5.5. *G is nilpotent if and only if $\mathrm{pd}(G,p) = 0$ for every prime p.*

We will use the following lemma to obtain a bound for $\mathrm{pd}(G,p)$ by using the order of $|G|$.

Lemma 5.6. *Let p^n be the largest power of p that divides $|G|$. Then $l(\underline{R}) = n$.*

Proof. The reason is that the partial order on $\mathrm{Iso}(\Gamma_G)$ is just inclusion up to conjugation: That is, for $Q, T \in \mathcal{F}$

$$\begin{aligned} \overline{Q} \leq \overline{T} &\Leftrightarrow \mathrm{Hom}_{\Gamma_G}(Q,T) \neq \emptyset \\ &\Leftrightarrow Q^g \subseteq T \text{ for some } g \in G. \end{aligned}$$

Since p-groups have subgroups of every possible order, G has a chain of subgroups

$$Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_n$$

such that $|Q_i| = p^i$. This gives a chain

$$\overline{Q_0} < \overline{Q_1} < \ldots \overline{Q_n}$$

in $\mathrm{Iso}(\Gamma_G)$. Hence $l(\Gamma_G) \geq n$. Suppose $l(\Gamma_G) = m > n$. Then $\mathrm{Iso}(\Gamma_G)$ has a chain of the form

$$\overline{T_0} < \overline{T_1} < \ldots \overline{T_m}\,.$$

Now each T_j is a p-subgroup of G, so $|T_j|$ is one of $p^0, \ldots, p^n$. As $m > n$, by the pigeonhole principle there exist distinct $j, k \in \{0, \ldots, m\}$ such that

$$|T_j| = |T_k|\,.$$

Say $j < k$. Then $\overline{T_j} < \overline{T_k}$, so $(T_j)^g \subseteq T_k$ for some $g \in G$. Comparing the orders, we get $(T_j)^g = T_k$. Thus $\overline{T_j} = \overline{T_k}$, a contradiction.
Therefore $l(\Gamma_G) = n$. Since $\underline{R}$ does not vanish anywhere, $l(\underline{R}) = l(\Gamma_G) = n$. □

Proposition 5.7. *Let p^n be the largest power of p that divides $|G|$. Then $\mathrm{pd}(G, p) \leq n$.*

Proof. $\underline{R}(Q) = R$ is obviously R-projective for every $Q \in \mathcal{F}$. Hence by Theorem 3.43 and Lemma 5.6, $\mathrm{pd}(G, p) = \mathrm{pd}(\underline{R}) \leq l(\underline{R}) = n$. □

Propositions 5.4 and 5.7 render the calculation of $\mathrm{pd}(G, p)$ trivial for several cases. In particular, if $p \mid |G|$ and $p^2 \nmid |G|$, then we have

$$\mathrm{pd}(G,p) = \begin{cases} 0 & \text{if } G \text{ has a normal Sylow } p\text{-subgroup} \\ 1 & \text{otherwise} \end{cases}$$

which is easy to determine if $|G|$ is small.

Proposition 5.8. *Let H be a subgroup of G. Then $\mathrm{pd}(H, p) \leq \mathrm{pd}(G, p)$.*

Proof. Since $\mathcal{F}$ is the family of all p-subgroups of G, clearly

$$\mathcal{F}_H = \{K \leq H : K \in \mathcal{F}\}$$

is the family of all p-subgroups of H. Let Γ_H be the orbit category of H with respect to this family. So $\mathrm{pd}(H, p)$ equals the projective dimension of the right $R\Gamma_H$-module $R[H/H^?]$.

Now let $\mathrm{pd}(G, p) = n$, so there exists a projective resolution

$$\mathbf{P} \to R[G/G^?]$$

with length n. Applying the functor

$$\mathrm{Res}^G_H : \mathsf{Mod}\text{-}R\Gamma_G \to \mathsf{Mod}\text{-}R\Gamma_H$$

which we know to be exact and projective-preserving, we obtain a projective resolution

$$\mathrm{Res}^G_H(\mathbf{P}) \to \mathrm{Res}^G_H(R[G/G^?]) \cong R[H/H^?]$$

with length n. Thus $\mathrm{pd}(H, p) \leq n$. □

5.2 Calculation of $\mathrm{pd}(S_4, 2)$

We calculate $\mathrm{pd}(S_4, 2)$ in this section. The general results in the previous section give lower and upper bounds, but do not determine it. We compute the projective dimension by resolving the constant functor, starting with the epimorphism in Remark 5.3.

As our prime is 2, we take $R = \mathbb{Z}_{(2)}$ throughout this section. Also let Γ_{S_4} denote the orbit category of S_4 with respect to the family of all its 2-subgroups. We let Λ_{S_4} be defined accordingly, as in Remark 5.2.

As 2^3 is the largest power of 2 that divides $|S_4| = 24$, we have $\mathrm{pd}(S_4, 2) \leq 3$ by Proposition 5.7.

Let $P = \langle (13), (1234) \rangle$. Clearly P is isomorphic to D_8 and is a Sylow 2-subgroup of S_4. As P is not normal (S_4 has 3 Sylow 2-subgroups), we get

$\mathrm{pd}(S_4, 2) \geq 1$ by Proposition 5.4. So $\mathrm{pd}(S_4, 2)$ is either 1,2 or 3.
Consider the epimorphism

$$\epsilon : R[S_4/P^?] \to \underline{R}$$

as in the previous section. We have an exact sequence

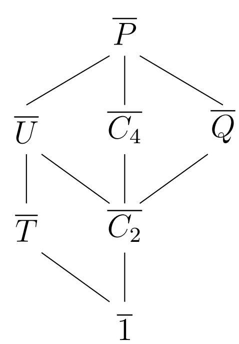
$$0 \longrightarrow \ker \epsilon \longrightarrow R[S_4/P^?] \xrightarrow{\epsilon} \underline{R} \longrightarrow 0$$

We will show that $\ker \epsilon$ is a projective right $R\Gamma_{S_4}$-module; it then follows that $\mathrm{pd}(S_4, 2) = 1$. Note that we can directly deduce just by the exact sequence above that $\ker \epsilon$ has finite projective dimension, by Proposition 5.1 and Corollary 3.39.

For a thorough analysis of the situation, we need the structure of the poset $\mathrm{Iso}(\Gamma_{S_4})$. $\mathrm{Iso}(\Gamma_{S_4})$ is the subgroup lattice of S_4 formed by its 2-subgroups, with conjugate subgroups identified. By Sylow theory every 2-subgroup of S_4 is conjugate to a subgroup of P. We pick and name representatives from the subgroups of P as follows:

$$\begin{aligned}
U &= \langle (13), (24) \rangle \\
C_4 &= \langle (1234) \rangle \\
Q &= \{1, (12)(34), (13)(24), (14)(23)\} \\
T &= \langle (13) \rangle \\
C_2 &= \langle (13)(24) \rangle
\end{aligned}$$

It is straightforward to check that every subgroup of P (hence every 2-subgroup of S_4) is S_4-conjugate to one of $P, U, C_4, Q, T, C_2, 1$. Hence $\mathrm{Iso}(\Gamma_{S_4})$ is given by the following lattice:



We will compute $(\ker\epsilon)(H) = \ker\epsilon_H$ for every 2-subgroup H of S_4. Note that

$$\epsilon_H : R[(S_4/P)^H] \to R$$

is given by augmentation, that is, for $aP \in (S_4/P)^H$, $\epsilon_H(aP) = 1$.
So we need to compute $(G/P)^H$ for each H. Observe that

$$S_4/P = \{P, (12)P, (14)P\}\,.$$

We also know that $aP \in (G/P)^H$ if and only if $H^a \subseteq P$. Now we can go case by case:

- $H = P$. Then $(S_4/P)^H = (S_4/P)^P = \mathbf{N}_{S_4}(P)/P = P/P = \{P\}$.
- $H = U$. Note that $(13)^{(12)} = (23) \in U^{(12)}$ but $(23) \notin P$. Similarly $(13)^{(14)} = (34) \in U^{(14)} - P$. Thus $(S_4/P)^U = \{P\}$.
- $H = C_4$. $(1234)^{(12)} = (1342) \notin P$ and $(1234)^{(14)} = (1423) \notin P$, hence $(S_4/P)^{C_4} = \{P\}$.
- $H = Q$. Note that $Q \trianglelefteq S_4$, so for any $a \in G$, $Q^a = Q \subseteq P$. So $(S_4/P)^Q = S_4/P$.
- $H = T$. We observed before that (13) fixes only P in S_4/P, so $(S_4/P)^T = \{P\}$.
- $H = C_2$. Since $C_2 \subseteq Q$, we have $(S_4/P)^Q \subseteq (S_4/P)^{C_2}$. Therefore $(S_4/P)^{C_2} = S_4/P$.
- $H = 1$. Clearly $(S_4/P)^1 = S_4/P$.

So for $H = P, U, C_4, T$ we have $(G/P)^H = \{P\}$. In that case ϵ_H is an isomorphism, so $\ker\epsilon_H = 0$. For $H = Q, C_2, 1$ we get nonzero kernel. In other words,

$$\mathrm{supp}(\ker\epsilon) = \{\overline{Q}, \overline{C_2}, \overline{1}\}\,.$$

Thus Q is a maximal object of $\mathrm{supp}(\ker\epsilon)$. We have a short exact sequence

$$0 \longrightarrow \ker\epsilon_Q \longrightarrow R[(S_4/P)^Q] \xrightarrow{\epsilon_Q} R \longrightarrow 0$$

of $R[\mathbf{N}_{S_4}(Q)/Q] = R[S_4/Q]$-modules. So $\ker \epsilon_Q$ is R-free; as the other two terms in the above sequence are R-free. We observed that $\ker \epsilon$ has a finite projective resolution, so by Theorem 3.42 $\ker \epsilon_Q = \operatorname{Res}_Q(\ker \epsilon)$ is a projective $R[S_4/Q]$-module. Therefore the induced right $R\Gamma_G$-module $E_Q \operatorname{Res}_Q(\ker \epsilon)$ is projective.

Now using the counit

$$E_Q \operatorname{Res}_Q \to \mathrm{id}_{\mathsf{Mod}\text{-}R\Gamma_{S_4}}$$

of the adjunction $E_Q \dashv \operatorname{Res}_Q$, we obtain a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_Q \operatorname{Res}_Q(\ker \epsilon) & \longrightarrow & E_Q \operatorname{Res}_Q(R[S_4/P^?]) & \xrightarrow{E_Q \operatorname{Res}_Q(\epsilon)} & E_Q \operatorname{Res}_Q(\underline{R}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \zeta} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & \ker \epsilon & \longrightarrow & R[S_4/P^?] & \xrightarrow{\epsilon} & \underline{R} & \longrightarrow & 0
\end{array}$$

where ζ, η, θ are the evaluations of the counit at $\ker \epsilon$, $R[S_4/P^?]$ and $\underline{R}$ respectively. The top row is the image of the bottom row under the functor $E_Q \operatorname{Res}_Q$. Since the bottom row is exact and the functors E_Q and Res_Q are exact functors (Propositions 2.13 and 4.2), the top row is also exact.

We will show that ζ is an isomorphism and that verifies our claim that $\ker \epsilon$ is projective. The snake lemma yields the following exact sequence:

$$0 \longrightarrow \ker \zeta \longrightarrow \ker \eta \longrightarrow \ker \theta \longrightarrow \operatorname{coker} \zeta \longrightarrow \operatorname{coker} \eta \xrightarrow{\rho} \operatorname{coker} \theta \longrightarrow 0$$

We claim that $\ker \eta = \ker \theta = 0$ and ρ is an isomorphism. These conditions force $\ker \zeta = \operatorname{coker} \zeta = 0$, as desired. To verify our claims, we focus on the commutative square

$$\begin{array}{ccc}
E_Q \operatorname{Res}_Q(R[S_4/P^?]) & \xrightarrow{E_Q \operatorname{Res}_Q(\epsilon)} & E_Q \operatorname{Res}_Q(\underline{R}) \\
\downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \theta} \\
R[S_4/P^?] & \xrightarrow[\epsilon]{} & \underline{R}
\end{array}$$

Then for every 2-subgroup H of S_4 we have a commutative square

$$\begin{array}{ccc} E_Q \operatorname{Res}_Q(R[S_4/P^?])(H) & \xrightarrow{E_Q \operatorname{Res}_Q(\epsilon)_H} & E_Q \operatorname{Res}_Q(\underline{R})(H) \\ \downarrow{\scriptstyle \eta_H} & & \downarrow{\scriptstyle \theta_H} \\ R[(S_4/P)^H] & \xrightarrow[\epsilon_H]{} & R \end{array}$$

of R-modules. Note that

$$\begin{aligned} E_Q \operatorname{Res}_Q(R[S_4/P^?])(H) &= R[(S_4/Q)^H] \otimes_{R[\mathbf{N}_{S_4}(Q)/Q]} R[(S_4/P)^Q] \\ &= R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P]\,. \end{aligned}$$

Similarly

$$E_Q \operatorname{Res}_Q(\underline{R})(H) = R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R$$

and we have

$$\begin{aligned} E_Q \operatorname{Res}_Q(\epsilon)_H : R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P] &\to R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R \\ aQ \otimes bP &\mapsto aQ \otimes 1 \end{aligned}$$

$$\begin{aligned} \eta_H : R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P] &\to R[(S_4/P)^H] \\ aQ \otimes bP &\mapsto abP \end{aligned}$$

$$\begin{aligned} \theta_H : R[(S_4/Q)^H] \otimes_{R[S_4/Q]} R &\to R \\ aQ \otimes 1 &\mapsto 1\,. \end{aligned}$$

Now we can proceed case by case for H (up to isomorphism in Γ_G, that is, up to conjugacy in S_4):

- If H is one of P, U, C_4, T we have $(S_4/Q)^H = \emptyset$. Then the domain of η_H and θ_H are both zero, hence trivially $\ker \eta_H = \ker \theta_H = 0$. And since $(S_4/P)^H = \{P\}$, the commutative square becomes

$$\begin{array}{ccc} 0 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \eta_H} & & \downarrow{\scriptstyle \theta_H} \\ R\{P\} & \xrightarrow[\epsilon_H]{} & R \end{array}$$

  ϵ_H is an isomorphism. Hence ρ_H, which is the induced morphism between the cokernels of η_H and θ_H, is an isomorphism.

- If H is one of $Q, C_2, 1$ we have $(S_4/Q)^H = S_4/Q$ and $(S_4/P)^H = S_4/P$. Then

$$\begin{aligned} \eta_H : R[S_4/Q] \otimes_{R[S_4/Q]} R[S_4/P] &\to R[S_4/P] \\ aQ \otimes bP &\mapsto abP \end{aligned}$$

is an isomorphism. Similarly

$$\begin{aligned} \theta_H : R[S_4/Q] \otimes_{R[S_4/Q]} R &\to R \\ aQ \otimes 1 &\mapsto 1 \end{aligned}$$

is an isomorphism. Thus $\ker \eta_H = \ker \theta_H = 0$ and also $\operatorname{coker} \eta_H = \operatorname{coker} \theta = 0$, so ρ_H is an isomorphism.

This finishes our calculation, we proved:

Proposition 5.9. $\operatorname{pd}(S_4, 2) = 1$.

5.3 Calculation of $\operatorname{pd}(S_5, 2)$

In this section, we calculate $\operatorname{pd}(S_5, 2)$. Again $R = \mathbb{Z}_{(2)}$ and Γ_{S_5} and Λ_{S_5} are defined similar to the previous section.

Considering S_4 embedded in S_5 as the stabilizer of 5, we have a functor

$$F : \Gamma_{S_4} \to \Gamma_{S_5}$$

which can be used to define

$$\operatorname{Res}^{S_5}_{S_4} : \mathsf{Mod}\text{-}R\Gamma_{S_5} \to \mathsf{Mod}\text{-}R\Gamma_{S_4}$$

F induces a map

$$\beta : \operatorname{Iso}(\Gamma_{S_4}) \to \operatorname{Iso}(\Gamma_{S_5})$$

which preserves the ordering. We claim that β is a bijection, hence establishes a poset isomorphism:

Let K be a 2-subgroup of S_5 and denote its isomorphism class in $\mathrm{Iso}(\Gamma_{S_5})$ by $\widetilde{K}$. Noting that P is a Sylow 2-subgroup of S_5, $K^g \subseteq P$ for some $g \in S_5$. As K^g is a 2-subgroup of S_4, consider $\overline{K^g} \in \mathrm{Iso}(\Gamma_{S_4})$. Now

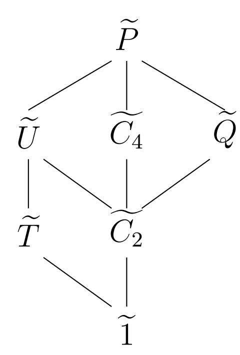
$$\beta(\overline{K^g}) = \widetilde{K^g} = \widetilde{K}$$

so β is surjective.

For injectivity, suppose $\beta(\overline{H}) = \beta(\overline{L})$ for some 2-subgroups H, L of S_4 but $\overline{H} \neq \overline{L}$. So H and L are conjugate in S_5 but not conjugate in S_4. First, $|H| = |L|$. By our knowledge of $\mathrm{Iso}(\Gamma_{S_4})$, there are two possibilities for a pair of non-conjugate 2-subgroups of the same order:

- $\overline{H}, \overline{L} \in \{\overline{U}, \overline{C_4}, \overline{Q}\}$. But the cycle structure of the elements in U,C_4 and Q can be distinguished. Only U has a transposition and only C_4 has a 4-cycle. So one of the three cannot be conjugate to another in S_5, a contradiction.
- $\overline{H}, \overline{L} \in \{\overline{T}, \overline{C_2}\}$. Again this is not possible because the nonidentity elements in T and C_2 have different cycle structures.

The contradiction yields $\overline{H} = \overline{L}$. So β is injective. Thus the lattice $\mathrm{Iso}(\Gamma_{S_5})$ is given by:



As before, we begin to resolve the constant functor $\underline{R}$ associated to Γ_{S_5} by the short exact sequence

$$0 \longrightarrow \ker \epsilon \longrightarrow R[S_5/P^?] \xrightarrow{\ \epsilon\ } \underline{R} \longrightarrow 0$$

The next step is to calculate $(S_5/P)^H$ for all H. To ease the calculations we make the following observation:

We can assume H is one of the representatives $P, U, C_4, Q, T, C_2, 1$ of $\mathrm{Iso}(\Gamma_{S_5})$. So H acts on the four letters $\{1, 2, 3, 4\}$. If this action has no fixed points, we claim that $(S_5/P)^H = (S_4/P)^H$. Indeed, assume $HgP = gP$ for some $g \in S_5$. So $H^g \subseteq P \subseteq S_4$. On the other hand observe that H^g acts on $\{g(1), g(2), g(3), g(4)\}$ without fixed points. Thus $5 \notin \{g(1), g(2), g(3), g(4)\}$, which means $g \in S_4$, so $gP \in S_4/P$. So we get $(S_5/P)^H \subseteq S_4/P$ and it follows that $(S_5/P)^H = (S_4/P)^H$. So we immediately obtain the following:

- $(S_5/P)^P = (S_4/P)^P = \{P\}$.
- $(S_5/P)^U = (S_4/P)^U = \{P\}$.
- $(S_5/P)^{C_4} = (S_4/P)^{C_4} = \{P\}$.
- $(S_5/P)^Q = (S_4/P)^Q = S_4/P$.
- $(S_5/P)^{C_2} = (S_4/P)^{C_2} = S_4/P$.

T and 1 does leave some points fixed. For them we have

- $(S_5/P)^T = \{P, (14)(235)P, (15432)P\}$ by inspection.
- $(S_5/P)^1 = S_5/P$ trivially.

Thus ϵ_H is an isomorphism if and only if $\widetilde{H} \in \{\widetilde{P}, \widetilde{U}, \widetilde{C_4}\}$ and hence

$$\mathrm{supp}(\ker \epsilon) = \{\widetilde{Q}, \widetilde{T}, \widetilde{C_2}, \widetilde{1}\}\,.$$

So Q and T are maximal objects of $\ker \epsilon$. Similar to the situation for the previous calculation, we obtain that $\mathrm{Res}_Q(\ker \epsilon)$ and $\mathrm{Res}_T(\ker \epsilon)$ are projective. And hence the induced right $R\Gamma_G$-modules $E_Q \,\mathrm{Res}_Q(\ker \epsilon)$ and $E_T \,\mathrm{Res}_T(\ker \epsilon)$ are projective. Again, by the counits of the adjunctions $E_Q \dashv \mathrm{Res}_Q$ and $E_T \dashv \mathrm{Res}_T$ we obtain two commutative diagrams with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_Q \operatorname{Res}_Q(\ker\epsilon) & \longrightarrow & E_Q \operatorname{Res}_Q(R[S_5/P^?]) & \xrightarrow{E_Q \operatorname{Res}_Q(\epsilon)} & E_Q \operatorname{Res}_Q(\underline{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\epsilon & \longrightarrow & R[S_5/P^?] & \xrightarrow{\epsilon} & \underline{R} & \longrightarrow & 0
\end{array}$$

and

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E_T \operatorname{Res}_T(\ker\epsilon) & \longrightarrow & E_T \operatorname{Res}_T(R[S_5/P^?]) & \xrightarrow{E_T \operatorname{Res}_T(\epsilon)} & E_T \operatorname{Res}_T(\underline{R}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker\epsilon & \longrightarrow & R[S_5/P^?] & \xrightarrow{\epsilon} & \underline{R} & \longrightarrow & 0
\end{array}$$

Summing the first rows, we obtain a single commutative diagram with exact columns (diagram is rotated for typesetting purposes):

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
E_Q \operatorname{Res}_Q(\ker\epsilon) \oplus E_T \operatorname{Res}_T(\ker\epsilon) & \xrightarrow{\zeta} & \ker\epsilon \\
\downarrow & & \downarrow \\
E_Q \operatorname{Res}_Q(R[S_5/P^?]) \oplus E_T \operatorname{Res}_T(R[S_5/P^?]) & \xrightarrow{\eta} & R[S_5/P^?] \\
\downarrow {\scriptstyle E_Q \operatorname{Res}_Q(\epsilon) \oplus E_T \operatorname{Res}_T(\epsilon)} & & \downarrow {\scriptstyle \epsilon} \\
E_Q \operatorname{Res}_Q(\underline{R}) \oplus E_T \operatorname{Res}_T(\underline{R}) & \xrightarrow{\theta} & \underline{R} \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

This time we will not be able to show that ζ is an isomorphism (because it isn't). We will recover most of its evaluations however, that is, for $\widetilde{H} > \widetilde{1}$, ζ_H *is* an isomorphism. To see this, we will use the exact sequence

$$0 \longrightarrow \ker \zeta_H \longrightarrow \ker \eta_H \longrightarrow \ker \theta_H \longrightarrow \operatorname{coker} \zeta_H \longrightarrow \operatorname{coker} \eta_H \xrightarrow{\rho_H} \operatorname{coker} \theta_H \longrightarrow 0$$

of R-modules, similar to what we did before. For every 2-subgroup H of S_5, we have the commutative square

$$\begin{array}{ccc} E_Q \operatorname{Res}_Q(R[S_5/P^?])(H) \oplus E_T \operatorname{Res}_T(R[S_5/P^?])(H) & \xrightarrow{\eta_H} & R[(S_5/P)^H] \\ \big\downarrow {\scriptstyle E_Q \operatorname{Res}_Q(\epsilon)_H \oplus E_T \operatorname{Res}_T(\epsilon)_H} & & \big\downarrow {\scriptstyle \epsilon_H} \\ E_Q \operatorname{Res}_Q(\underline{R})(H) \oplus E_T \operatorname{Res}_T(\underline{R})(H) & \xrightarrow{\theta_H} & R \end{array}$$

of R-modules. We have

$$\begin{aligned} E_Q \operatorname{Res}_Q(R[S_5/P^?])(H) &= R[(S_5/Q)^H] \otimes_{R[\mathbf{N}_{S_5}(Q)/Q]} R[(S_5/P)^Q] \\ &= R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P] \end{aligned}$$

$$E_T \operatorname{Res}_T(R[S_5/P^?])(H) = R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R[(S_5/P)^T]$$

$$\begin{aligned} \eta_H : \left(R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P]\right) \oplus \left(R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R[(S_5/P)^T]\right) &\to R[(S_5/P)^H] \\ (aQ \otimes bP, 0) &\mapsto abP \\ (0, cT \otimes dP) &\mapsto cdP \end{aligned}$$

and

$$E_Q \operatorname{Res}_Q(\underline{R})(H) = R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R$$

$$E_T \operatorname{Res}_T(\underline{R})(H) = R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R$$

$$\begin{aligned}\theta_H : \Big(R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R\Big) \oplus \Big(R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R\Big) &\to R \\ (aQ \otimes 1, 0) &\mapsto 1 \\ (0, cT \otimes 1) &\mapsto 1\end{aligned}$$

$$\begin{aligned}E_Q \operatorname{Res}_Q(\epsilon)_H : R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R[S_4/P] &\to R[(S_5/Q)^H] \otimes_{R[S_4/Q]} R \\ aQ \otimes bP &\mapsto aQ \otimes 1\end{aligned}$$

$$\begin{aligned}E_T \operatorname{Res}_T(\epsilon)_H : R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R[(S_5/P)^T] &\to R[(S_5/T)^H] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R \\ cT \otimes dP &\mapsto cT \otimes 1\end{aligned}$$

Now we analyze the cases:

- If H is one of P, U, C_4 we have $(S_5/Q)^H = (S_5/T)^H = \emptyset$ hence the domains of η_H and θ_H are zero. Also $(S_5/P)^H = \{P\}$, so the commutative square becomes

$$\begin{array}{ccc} 0 & \xrightarrow{\eta_H} & R\{P\} \\ \downarrow & & \downarrow{\scriptstyle \epsilon_H} \\ 0 & \xrightarrow{\theta_H} & R \end{array}$$

  hence $\ker \eta_H = \ker \theta_H = 0$ and ρ_H is an isomorphism.

- If H is Q or C_2 we have $(S_5/T)^H = \emptyset$. Also since both H acts on {1,2,3,4} without fixed points, $(S_5/Q)^H = (S_4/Q)^H = S_4/Q$ and $(S_5/P)^H = S_4/P$ So the commutative square becomes

$$\begin{array}{ccc} R[S_4/Q] \otimes_{R[S_4/Q]} R[S_4/P] & \xrightarrow{\eta_H} & R[S_4/P] \\ {\scriptstyle E_Q \operatorname{Res}_Q(\epsilon)_H}\downarrow & & \downarrow{\scriptstyle \epsilon_H} \\ R[S_4/Q] \otimes_{R[S_4/Q]} R & \xrightarrow{\theta_H} & R \end{array}$$

Hence η_H and θ_H are isomorphisms. Thus again we get that $\ker\eta_H = \ker\theta_H = 0$ and ρ_H is an isomorphism.

- If H is T, then $(S_5/Q)^H = \emptyset$ and $(S_5/T)^H = \mathbf{N}_{S_5}(T)/T$. So the commutative square becomes

$$\begin{array}{ccc} R[\mathbf{N}_{S_5}(T)/T] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R[(S_5/P)^T] & \xrightarrow{\eta_H} & R[(S_5/P)^T] \\ \downarrow{\scriptstyle E_T \operatorname{Res}_T(\epsilon)_H} & & \downarrow{\scriptstyle \epsilon_H} \\ R[\mathbf{N}_{S_5}(T)/T] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R & \xrightarrow{\theta_H} & R \end{array}$$

So in this case too, η_H and θ_H are isomorphisms.

Thus the above analysis proves that ζ_H is an isomorphism whenever $\widetilde{H} > \widetilde{1}$. Hence the only possible isomorphism class that $\ker\zeta$ and $\operatorname{coker}\zeta$ do not vanish is $\widetilde{1}$. Therefore

$$l(\ker\zeta) \leq 0$$
$$l(\operatorname{coker}\zeta) \leq 0\,.$$

By Proposition 3.9 there exists an epimorphism $\pi : F \to \operatorname{coker}\zeta$ such that F is projective and $l(F) = l(\operatorname{coker}\zeta)$. By the lifting property of projectives, we can lift π to $\sigma : F \to \ker\epsilon$. Letting

$$W = E_Q \operatorname{Res}_Q(\ker\epsilon) \oplus E_T \operatorname{Res}_T(\ker\epsilon)$$

we get a morphism

$$[\zeta, \sigma] : W \oplus F \to \ker\epsilon\,.$$

Since F surjects onto $\operatorname{coker}\zeta$ via σ, $[\zeta, \sigma]$ is an epimorphism. Let $L = \ker[\zeta, \sigma]$. As $l(F) \leq 0$ and $l(\ker\zeta) \leq 0$, $l(L) \leq 0$ (This argument overall is exactly the same as the one we used in proving Lemma 3.14). Hence we get a short exact sequence

$$0 \longrightarrow L \longrightarrow W \oplus F \xrightarrow{[\zeta,\sigma]} \ker\epsilon \longrightarrow 0$$

Note that $L(H)$ is R-projective for any H. Since $l(L) \leq 0$, either $L = 0$ or by Proposition 3.43 $\operatorname{pd}(L) \leq 0$. In any case L is projective. Therefore the exact sequence

$$0 \longrightarrow L \longrightarrow W \oplus F \longrightarrow R[S_5/P^?] \xrightarrow{\epsilon} \underline{R} \longrightarrow 0$$

that we get by splicing is a projective resolution of $\underline{R}$. Thus $\mathrm{pd}(S_5, 2) \leq 2$.

Finally, we show that $\ker \epsilon$ is not projective: Suppose $\ker \epsilon$ is projective. Then by Theorem 3.33

$$\ker \epsilon \cong \bigoplus_{\widetilde{H} \in \mathrm{Iso}(\Gamma_{S_5})} E_H S_H(\ker \epsilon)\,.$$

In particular there is an R-module isomorphism

$$\ker \epsilon_1 \cong \bigoplus_{\widetilde{H} \in \mathrm{Iso}(\Gamma_{S_5})} E_H S_H(\ker \epsilon)(1)\,.$$

These are free and finitely generated R-modules, so their ranks must match:

$$\mathrm{rank}_R(\ker \epsilon_1) = \mathrm{rank}_R \left(\bigoplus_{\widetilde{H} \in \mathrm{Iso}(\Gamma_{S_5})} E_H S_H(\ker \epsilon)(1) \right) \qquad (*)$$

Now

$$E_T \,\mathrm{Res}_T(R[S_5/P^?])(1) = R[S_5/T] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R[(S_5/P)^T]\,.$$

Note that S_5/T is a free right $\mathbf{N}_{S_5}(T)/T$-set (this follows from the freeness of orbit categories or can be checked by hand) with $|S_5 : \mathbf{N}_{S_5}(T)| = 10$ orbits. Therefore as an R-module

$$\begin{aligned} E_T \,\mathrm{Res}_T(R[S_5/P^?])(1) &\cong \left(R[(S_5/P)^T]\right)^{10} \\ &\cong R^{3\cdot 10} \\ &= R^{30} \end{aligned}$$

and similarly

$$E_T \,\mathrm{Res}_T(\underline{R})(1) = R[S_5/T] \otimes_{R[\mathbf{N}_{S_5}(T)/T]} R \cong R^{10}\,.$$

Therefore

$$E_T \,\mathrm{Res}_T(\ker \epsilon)(1) \cong R^{30-10} = R^{20}\,.$$

On the other hand, $R[(S_5/P)^1] \cong R^{15}$ as an R-module, so

$$\ker \epsilon_1 \cong R^{15-1} = R^{14}\,.$$

But now by (*),

$$\begin{aligned} 14 &= \operatorname{rank}_R(\ker \epsilon_1) \\ &\geq \operatorname{rank}_R(E_T S_T(\ker \epsilon)(1)) \\ &= \operatorname{rank}_R(E_T \operatorname{Res}_T(\ker \epsilon)(1)) \\ &= 20 \end{aligned}$$

which is a contradiction. Hence $\ker \epsilon$ is not projective. So we proved

Proposition 5.10. $\operatorname{pd}(S_5, 2) = 2$.

5.4 $\underline{R}$ in the integral setting

We stated Rim's theorem for group rings in its integral and p-local versions (Theorem 4.9 and Theorem 4.10, respectively) in the previous chapter. Theorem 4.13 gives a generalization of the p-local version to the orbit category setting. In this section we show that the natural analogue of this generalization is not valid for the integral setting by demonstrating that the constant functor fails to have finite projective dimension except in trivial cases.

Naturally, we take $R = \mathbb{Z}$ in this section (G is a finite group, as always). Now what should the family $\mathcal{F}$ be? The family which enabled the constant functor to have finite projective dimension in the p-local setting was the family of all p-subgroups. Since we are not localizing at a prime now, we include them all without discriminating and let

$$\mathcal{F} = \{H \leq G : H \text{ has prime power order}\}.$$

Clearly $\mathcal{F}$ is closed under taking subgroups and conjugation. So we can form the orbit category Γ_G of G with respect to $\mathcal{F}$.

The critical ingredient in our argument is the following deep result which at least partly depends on the classification of finite simple groups:

Theorem 5.11 ([9], Corollary 1.3)**.** *Let p, q be distinct primes dividing $|G|$. Let $P \in \operatorname{Syl}_p(G)$ and $Q \in \operatorname{Syl}_q(G)$. Then both P and Q cannot be self-normalizing, that is, either $\mathbf{N}_G(P) > P$ or $\mathbf{N}_G(Q) > Q$.*

The following immediate corollary is all we need:

Corollary 5.12. *If every Sylow subgroup of G is self-normalizing, G has prime power order.*

Proof. The claim trivially holds for $G = 1$. Assuming $G > 1$, there exists a prime p dividing $|G|$. Let q also be a prime dividing $|G|$. Let $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$. Since both P and Q are self-normalizing by assumption, Theorem 5.11 forces $p = q$. So p is the only prime dividing $|G|$, thus $|G| = p^n$ for some $n > 0$. □

Theorem 5.13. *Let $\underline{\mathbb{Z}}$ denote the constant functor in $\mathsf{Mod}\text{-}\mathbb{Z}\Gamma_G$. The following are equivalent:*

1. *G has prime power order.*
2. *$\underline{\mathbb{Z}}$ is projective.*
3. *$\underline{\mathbb{Z}}$ has a finite projective resolution.*

Proof. (1) ⇒ (2): We have $G \in \mathcal{F}$, so $\underline{\mathbb{Z}} \cong \mathbb{Z}[G/G^?]$ is projective.
(2) ⇒ (3): Trivial.
(3) ⇒ (1): Let P be a Sylow subgroup of G. So $P \in \mathcal{F}$ and $\underline{\mathbb{Z}}(P) = \mathbb{Z}$ is obviously $\mathbb{Z}$-projective. Hence by Theorem 3.42, $\underline{\mathbb{Z}}(P) = \mathbb{Z}$ is projective as a $\mathbb{Z}[\mathbf{N}_G(P)/P]$-module. This forces $\mathbf{N}_G(P)/P = 1$, that is, $P = \mathbf{N}_G(P)$. As every Sylow subgroup of G is self-normalizing, by Corollary 5.12, G has prime power order. □

Bibliography

[1] Peter Webb, *An introduction to the representations and cohomology of categories*, Group representation theory, EPFL Press, Lausanne, 2007, pp. 149-173.

[2] F. Xu, *Homological properties of category algebras*, Ph.D. thesis, University of Minnesota 2006.

[3] B. Mitchell, *Rings with several objects*, Advances in Math. 8 (1972), 1-161.

[4] W. Lück, *Transformation groups and algebraic K-theory*, Lecture Notes in Mathematics, vol. 1408, Springer-Verlag, Berlin, 1989, Mathematica Gottingensis.

[5] I. Hambleton, S. Pamuk, and E. Yalçın, *Equivariant CW-complexes and the orbit category*, (arXiv:0807.3357v3 [math.AT]), 2010.

[6] S. Awodey, *Category Theory*, Oxford Logic Guides, 2006.

[7] D. J. Benson, *Complexity and varieties for infinite groups I*, J. Algebra 193 (1997) 260-287.

[8] D. S. Rim, *Modules over finite groups*, Ann. of Math. vol. 69 (1959) pp. 700-712.

[9] Robert M. Guralnick, Gunter Malle, and Gabriel Navarro, *Self-normalizing Sylow subgroups*, Proc. Amer. Math. Soc. 132 (2004), no. 4, 973979 (electronic).

[10] T. t. Dieck, *Transformation groups*, Arch. Math. (Basel) 8 (1987), x+312.

[11] C. Broto, R. Levi, and B. Oliver, *The homotopy theory of fusion systems*, Journal Amer. Math. Soc. 16 (2003), 779-856.

[12] G. E. Bredon, *Equivariant cohomology theories*, Lecture Notes in Mathematics, No. 34, Springer - Verlag, Berlin, 1967.