DILATION THEOREMS FOR VH-SPACES

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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In the Appendix of the book *Leçons d'analyse fonctionnelle* by F. Riesz and B. Sz.-Nagy, B. Sz.-Nagy [15] proved an important theorem on operator valued positive definite maps on $*$-semigroups, which today can be considered as one of the pioneering results of dilation theory. In the same year W.F. Stinespring [11] proved another celebrated theorem about dilation of operator valued completely positive linear maps on C^*-algebras. Then F.H. Szafraniec [14] showed that these theorems are actually equivalent.

Due to reasons coming from multivariate stochastic processes R.M. Loynes [7], considered a generalization of B. Sz.-Nagy's Theorem for vector Hilbert spaces (that he called VH-spaces). These VH-spaces have "inner products" that are vector valued, into the so-called "admissible spaces".

This work is aimed at providing a detailed proof of R.M. Loynes Theorem that generalizes B. Sz.-Nagy, a detailed proof of the equivalence of Stinespring's Theorem in the Arveson formulation [2] for B^*-algebras with B. Sz.-Nagy's Theorem following the lines in [14] together with some ideas from [2], and to get VH-variants of Stinespring's Theorem for C^*-algebras and B^*-algebras. Relations between these theorems are also considered.

Keywords: C^*-Algebras , VH-Spaces, Completely positive maps, Dilation .

ÖZET

VH-UZAYLARINDA GENLEŞME TEOREMLERİ

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F. Riesz ve Sz.-Nagy tarafından yazılmış olan *Leçons d'analyse fonctionnelle* adlı kitabın ek bölümünde, Sz.Nagy [15] bugün genleşme teorisinin en önemli sonuçlarından biri sayılan $*$-semigruplar üzerinde pozitif tanımlı operatör değerli fonksiyonlarla ilgili bir teorem ispatladı. Aynı yıl W.F. Stinespring [11] de C^*-cebirleri üzerinde tamamen pozitif fonksiyonlar için başka bir teorem ispatladı. Daha sonra F.H. Szafraniec [14] bu iki teoremin aslında eşdeğer olduğunu gösterdi.

R.M. Loynes, motivasyonunu çok değiskenli stokastik modellerden aldığı uzerinde, değerini uygun seçilmiş bir topolojik uzayda alan, vektör değerli bir iç çarpım tanımlı olan VH-uzaylarını tanımlayarak B. Sz.-Nagy nin teoreminin bir versiyonunu bu uzaylar için ispatladı.

Bu tezin amacı ; R.M. Loynes'in yukarıda bahsedilen teoreminin ayıntılı bir ispatını verip, bu teoremin ve Steinspring teoreminin Arveson tarafından B^*-cebirleri için ispatlanan [2] versiyonunun [14] ü takip ederek ve [2] den fikirler kullanarak eşdeğer olduklarını gösterip, Steinspring teoreminin C^* ve B^*-cebirleri için VH-uzaylarında benzerlerini elde ederek bu teoremlerin R.M. Loynes'in teoremiyle olan ilişkilerini incelemektir.

Anahtar sözcükler: C^*-Cebirleri, VH-Uzay, Tamamen pozitif operatörler, Stinespring temsili .

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Contents

Dilation Theorems for VH-Spaces

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June 22, 2009

Chapter 1

Introduction

In the Appendix of the book *Leçons d'analyse fonctionnelle* by F. Riesz and B. Sz.-Nagy, B. Sz.-Nagy [15] proved an important theorem on operator valued positive definite maps on $*$-semigroups, which today can be considered as one of the pioneering results of dilation theory. In the same year W.F. Stinespring [11] proved another celebrated theorem on dilations of operator valued completely positive linear maps on C^*-algebras. Then F.H. Szafraniec [14] showed that these theorems are actually equivalent.

Due to reasons coming from multivariate stochastic processes, R.M. Loynes [7], considered a generalization of B. Sz.-Nagy's Theorem for vector Hilbert spaces (that he called VH-spaces). These VH-spaces have "inner products" that are vector valued, into the so-called "admissible spaces". There are of course reasons why studying such objects turns out to be important. Let A be a commutative C^*-algebra. By the important theorem of Gelfand-Naimark we know that A can be identified with the continuous functions $C(X)$ on a locally compact Hausdorff space X. When X is a Euclidean manifold it is natural to consider the tangent spaces at each point to study the manifold. However, this is more a geometric point of view. The important shift of approach might be considering a Hilbert space at each point of the manifold. If we are to express this in a technical way we can take a Hilbert space H_t at each $t \in X$. In any of these Hilbert spaces there is an inner product. In fact, all of these Hilbert spaces are glued together so as to form a vector bundle E. In this vector bundle we can define the inner product of two sections, say ξ and η, $\langle \xi, \eta \rangle$ as following function

$$t \longmapsto \langle \xi(t), \eta(t) \rangle.$$

As seen, with this definition the vector bundle E is now equipped with a $C(X)$-valued inner product. This is an important example from [6] which shows why the spaces having inner product in a more general space might be important.

One of the most important such objects are Hilbert C^*-modules in which case the inner product takes its values in a C^*-algebra. However, when one examines the proofs of several dilation theorems it might be seen that the techniques can even generalize to more general spaces than the Hilbert C^*-modules. The spaces we will examine in this thesis are VH-spaces. In the case of VH-spaces the inner product takes its values in a suitable topological vector space. The most important point is that VH-spaces lack the multiplicative structure, after all it is just a vector space. As we will see, yet this weak-structured spaces enjoy many useful properties of the usual Hilbert spaces. Some of the difficulties here are the lack of Riesz Representation Theorem [7] and the Schwarz inequality. In fact, it is not possible to expect a kind of Schwarz inequality since, as we mentioned, the inner product takes its values in a topological space lacking a multiplicative structure. However, many of the theorems and techniques can be adapted to this case, too.

This work is aimed at providing a detailed proof of R.M. Loynes Theorem that generalizes B. Sz.-Nagy, a detailed proof of the equivalence of Stinespring's Theorem in the Arveson formulation [2] for B^*-algebras, with B. Sz.-Nagy's Theorem following the lines in [14] together with some ideas from [2], and to get VH-variants of Stinespring's Theorem for C^*-algebras and B^*-algebras. Relations between these theorems are also considered.

Chapter 2

Preliminaries on C^* and B^*-Algebras

In this chapter we recall a few definitions and facts from the theory of operator algebras that we will use. We assume known all basic notions in Hilbert spaces and operators on Hilbert spaces, e.g. see [4].

Definition 2.1. By an *algebra* over $\mathbb{C}$ we mean a complex vector space A together with a binary operation representing multiplication $A \ni x, y \mapsto xy \in A$ satisfying

1. Bilinearity: For $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in A$ we have

$$(\alpha x + \beta y)z = \alpha \cdot xz + \beta \cdot yz,$$
$$x(\alpha \cdot y + \beta \cdot z) = \alpha \cdot xy + \beta \cdot xz.$$

2. Associativity: $x(yz) = (xy)z$.

Definition 2.2. A *normed algebra* is a pair $(A, \|\cdot\|)$ consisting of an algebra together with a norm $\|\cdot\| : A \mapsto [0, \infty)$ which is related to the multiplication as follows:

$$\|xy\| \leq \|x\|\|y\|, \qquad x, y \in A.$$

A *Banach algebra* is a normed algebra that is a (complete) Banach space relative to its given norm.

Definition 2.3. If A is a Banach algebra, an *involution* is a map $a \mapsto a^*$ of A into itself such that for all a and b in A all scalars α the following hold:

1. $(a^*)^* = a$
2. $(ab)^* = b^*a^*$
3. $(\alpha a + b)^* = \bar{\alpha}a^* + b^*$

Additionally, an algebra which has an identity is called *unital*.

Definition 2.4. A C^**-algebra* is a Banach algebra with involution such that

$$\|a^*a\| = \|a\|^2$$

for every a in A.

Definition 2.5. For every element x in a unital C^*-algebra A, the *spectrum* of x is defined as the set

$$\sigma(x) = \{\lambda \in \mathbb{C} : x - \lambda \notin A^{-1}\}$$

where A^{-1} denotes the set of all invertible elements in A.

Definition 2.6. If A is a C^*-algebra and $a \in A$, then:

- a is *hermitian* if $a = a^*$
- a is *normal* if $a^*a = aa^*$.
- when A is unital, a is *unitary* if $a^*a = aa^* = 1$

For any C^*-algebra A, A^h will denote the collection of hermitian elements of A.

Definition 2.7. If A is a C^*-algebra, an element a of A is *positive* if $a \in A^h$ and $\sigma(a) \subseteq \mathbb{R}_+$, the set of non-negative real numbers. This property is denoted by $a \geq 0$ and A_+ denotes the collection of all positive elements in A. We say that an element is *negative* if $-a \in A_+$. We can write this as $a \leq 0$ and A_- the collection of all negative elements in A.

Theorem 2.8. *If A is a C^*-algebra the following statements are equivalent*

1. $a \geq 0$
2. $a = b^2$ *for some* b *in* A_+
3. $a = x^*x$ *for some* x *in* A.

The set of all bounded operators on a Hilbert space is denoted by $\mathcal{B}(H)$. In fact, the following proposition gives an important property of positive operators on the Hilbert space $\mathcal{H}$.

Proposition 2.9. *If $\mathcal{H}$ is a Hilbert space and $A \in \mathcal{B}(H)$, then A is positive if and only if $\langle Ah, h\rangle \geq 0$ for every vector h.*

Definition 2.10. A map $\varphi : A \to \mathcal{B}(H)$, where A is a $*$-algebra, is said to be *positive definite* (shortly PD) if

$$\sum_{i,j} (\varphi(s_j^* s_i) f_i, f_j) \geq 0$$

for any finite number of $s_1, s_2, \ldots, s_n$ in A and $f_1, f_2, \ldots, f_n$ in H. A linear map $\mu : A \to \mathcal{B}(H)$, where A is a C^*-algebra, is said to be *completely positive* (shortly CP) if for each n, $\mu^{(n)}$ is a positive map of A^n into $B(H^n)$ where A^n is the C^*-algebra of all matrices (a_{ij}) with entries a_{ij} in A and $\mu^n((a_{ij})) = (\mu(a_{ij}))$. Since for any positive square matrix (a_{ij}) in A^n can be written as linear combination (with positive coefficients) of matrices of type $(b_j^* b_i)$, for a linear map on C^*-algebra positive definiteness and complete positivity coincide.

Definition 2.11. A *Banach $*$-algebra* (or B^*-algebra) is a Banach algebra A that is endowed with an involution $x \mapsto x^*$ satisfying $\|x^*\| = \|x\|,\ x \in A$.

Definition 2.12. A *representation* of a Banach $*$-algebra is a homomorphism $\pi : A \to \mathcal{B}(H)$ of A into the $*$-algebra of bounded operators on some Hilbert space satisfying $\pi(x^*) = \pi(x)^*$ for all $x \in A$.

Proposition 2.13. *Let A be a B^*-algebra with unit. Let R be the set of representations of A. For each $x \in A$, we define*

$$\|x\|' = \sup_{\pi \in R} \|\pi(x)\|.$$

We have that $\|x\|' \leq \|x\|$. Also, the map $x \mapsto \|x\|'$ is a semi-norm on A which satisfies

- $\|xy\|' \leq \|x\|'\|y\|'$
- $\|x^*\|' = \|x\|'$
- $\|x^*x\|' = \|x\|'^2$

With the notation as in the previous proposition, let I be the set of $x \in A$ such that $\|x\|' = 0$. Observe that I is a closed self-adjoint two-sided ideal of A. The map $x \mapsto \|x\|'$ defines a norm on the quotient A/I. Equipped with this norm A/I satisfies the axioms of a C^*-algebra except that A/I is not complete in general. The completion B of A/I is a C^*-algebra which is called the *enveloping C^*-algebra of A*.

Chapter 3

VH-spaces

In this chapter we review most of the definitions and theorems on VH-spaces, an acronym for *vector Hilbert spaces*, introduced and studied first by R.M. Loynes, cf. [7], [8], and [9].

3.1 Definitions and Basic Theorems

In this part, we give the definition of a VH-space and prove some theorems in order to establish the basic properties of a VH-space. In fact, the proof of the theorem which shows the continuity of addition could have been omitted. But we intentionally tried to provide the essential steps in order to demonstrate what kind of techniques are used to prove things in a VH-space.

Definition 3.1. A linear topological vector space Z is called *admissible* if:

1. Z has an involution, that is, a mapping shown by $x \longmapsto x^*$ of Z onto itself which satisfies:
    - $(z^*)^* = z$
    - $(az_1 + bz_2)^* = \bar{a}z_1^* + \bar{b}z_2^*$.

If Z is taken to be a real vector space, involution might be just identity map.

2. Z contains a closed convex cone P with $P \cap -P = \{0\}$, which may be used to define a partial order in Z. The partial order is defined by $z_1 \geq z_2$ iff $z_1 - z_2 \in P$.

3. The topology is compatible with the ordering. By this, we mean that there exist a basic set of neighborhoods, say $\{N_0\}$ of the origin such that $x \in N_0$ and $0 \leq y \leq x$ implies $y \in N_0$. In particular, Z is locally convex. Throughout the text whenever we talk about neighborhoods we mean the neighborhoods $\{N_0\}$.

4. The elements of P satisfies: if $x \in P$ then $x^* = x$. Observe that this is trivial if Z is real vector space.

5. Z is a complete topological space.

In order to substantiate this definition, we give a few relavant examples.

Examples 3.2. *C^*-Algebras.* If $\mathcal{A}$ is a C^*-algebra then it is an admissible space with the cone of positive elements and normed topology. In particular, this is the case for the C^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on a complex Hilbert space H, as well as for the C^*-algebra $C(X)$ of all complex valued continuous functions on a compact Hausdorff space X.

Locally C^*-Algebras. A complex $*$- algebra $\mathcal{A}$ is a *locally C^*-algebra* if it is endowed with a family of seminorms $\{p_\alpha\}$ that are submultiplicative, that is, $p_\alpha(xy) \leq p_\alpha(x)p_\alpha(y)$ for all $x, y \in \mathcal{A}$ and all α, satisfy the C^*-algebra condition $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in \mathcal{A}$ and all α, and is complete with respect to the topology induced by this family of seminorms. The notion of positive element is the same as in the case of a C^*-algebra.

$\mathcal{B}(X, X^*)$. Let X be a complex Banach space and X^* its topological dual. On the vector space $\mathcal{B}(X, X^*)$ of all bounded linear operators $T\colon X \to X^*$ a natural notion of positive operator can be defined: T is *positive* if $(Tx)x \geq 0$ for all $x \in X$. Then $\mathcal{B}_+(X, X^*)$, the collection of all positive operators is a strict cone that is closed with respect to the weak operator topology. The involution in $\mathcal{B}(X, X^*)$ is defined in the following way: for any $T \in \mathcal{B}(X, X^*)$, the adjoint of T is the restriction to X of the dual operator $T^* \colon X^{**} \to X^*$. With respect to these, $\mathcal{B}(X, X^*)$ becomes an admissible space.

Definition 3.3. A linear space E is called a *VE-space* if there is given a map $(x, y) \mapsto [x, y]$ from $E \times E$ into an admissible space (cf. Definition 3.1) Z, subject to the following properties:

1. $[x, x] \geq 0$ for all $x \in E$, and $[x, x] = 0$ if and only if $x = 0$.
2. $[x, y] = [y, x]^*$ for all $x, y \in E$.
3. $[ax_1 + bx_2, y] = a[x_1, y] + b[x_2, y]$ for all $a, b \in \mathbb{C}$ and all $x_1, x_2 \in E$.

This map is called the *(vector) inner product* on E, or the *gramian*.

We will show that infact any VE-space can be made in a natural way into a locally convex space, cf. [7].

Theorem 3.4. *Given a VE-space E, we define the following topology on E by taking the sub-base of all neighborhoods of origin as the sets*

$$U_0 = \{x : [x, x] \in N_0\}, \tag{3.1}$$

where N_0 are the sets as in Definition 3.1 of the admissible space Z. Then, E becomes a locally convex (Hausdorff) linear topological space. Moreover, $[x, y]$ is a continuous function on $E \times E$ and E satisfies the first axiom of countability if Z does.

Proof. We first show that addition is continuous. Applying twice the Proposition I.3.3 from [10] to N_0 in order to find N_φ in Z such that,

$$N_\varphi - N_\varphi + N_\varphi - N_\varphi \subseteq N_0.$$

Then, we have that for any given neighbourhood U_0 as in (3.1), there exist a set U_φ in E such that

$$U_\varphi - U_\varphi + U_\varphi - U_\varphi \subseteq U_0.$$

For $x, y \in U_\varphi$ we have the following

$$\begin{aligned}
&[x+y, x+y] = [x,x] + [x,y] + [y,x] + [y,y] \geq 0 \\
&[x-y, x-y] = [x,x] - [x,y] - [y,x] + [y,y] \geq 0 \qquad \text{which implies that} \\
&[x-y, x-y] \leq [x-y, x-y] + [x+y, x+y] \leq 2[x,x] + 2[y,y].
\end{aligned} \tag{3.2}$$

So, by the definition of the topology we have $2[x,x] + 2[y,y] \in N_0$. Now we can use the admissibility condition together with the above inequality to conclude that $x - y \in U_0$. Throughout, sometimes we will need to do more modifications in order to compensate the lack of Schwarz inequality.

We also need to show that αx is jointly continuous in α and x. However, the proof in this case is no different from that of in topological vector spaces. For a given α and x, it is enough to show that $\alpha y + \delta x + \delta y$ is contained in an arbitrary neighborhood of origin if δ and y are small enough. But this is just a consequence of topological property we have just shown.

For the convexity, by using (3.2) we have the following expression

$$\begin{aligned}
[px + (1-p)y, px + (1-p)y] &= p^2[x,x] + (1-p)^2[y,y] \ + p(1-p)([x,y] + [y,x]) \\
&\leq p^2[x,x] + (1-p)^2[y,y] + p(1-p)([x,x] + [y,y])
\end{aligned}$$

Obviously, the right hand side belongs to N_0 since N_0 is convex. Hence U_0 is convex. The countability condition and Hausdorff condition easily follows from the definition.

Now we show the continuity of $[x,y]$ on $E \times E$. We have

$$[x+h, y+k] - [x,y] = [h,y] + [x,k] + [h,k].$$

All we need to show is that the right hand side tends to zero as h, k go to zero. Since the polar decomposition formula

$$[x,y] = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{1}{4}i(\|x+iy\|^2 - \|x-iy\|^2) \tag{3.3}$$

is a purely algebraic property of an inner product, it also holds for the vector-valued inner product we defined. In this case, $[h, k]$ is just a linear combination of $[h \pm k, h \pm k]$ and $[h \pm ik, h \pm ik]$ and these tend to zero as h, k tend to zero. In a similar fashion, $[h, y]$ is a linear combination of $p^{-1}[h \pm py, h \pm py]$ and $p^{-1}[h \pm ipy, h \pm ipy]$ which can indeed be made as small as we want by choosing small p then h for a fixed y. The term $[x, k]$ also goes to zero by the same argument. $\square$

Observe that the topology on a VE-space is taken to make the map $x \mapsto [x, x]$ continuous. In fact, obviously, this is the most natural topology one can think of. So, it should not be a big surprise that the inner product turns out to be continuous by the polarization identity.

First we give the definition for a VH-space.

Definition 3.5. A linear space is a VH-space if it is a VE-space which is complete as a topological space.

In order to substantiate this definition we present some relevant examples.

Examples 3.6. *Hilbert C^*-Modules.* Let A be a C^*-algebra. An *inner-product A-module* is a linear space E which is a right A-module together with a map $E \times E \ni (x, y) \mapsto \langle x, y\rangle \in A$ such that: (i) $\langle x, ya + zb\rangle = \langle x, y\rangle a + \langle x, z\rangle b$, (ii) $\langle x, ya\rangle = \langle x, y\rangle a$, (iii) $\langle y, x\rangle = \langle x, y\rangle^*$, (iv) $\langle x, x\rangle \geq 0$ and if $\langle x, x\rangle = 0$ then $x = 0$. A norm on E can be given by $\|x\| = \|\langle x, x\rangle\|$ and, if E is complete with respect to this norm then E is called a *Hilbert C^*-module.* Clearly, this is an example of VH-space. These objects are intensively studied, e.g. see [6].

Hilbert Modules over Locally C^*-Algebras. In the above definition, one can replace the C^*-algebra A by a locally C^*-algebra and get the notion of *Hilbert modules over locally C^*-algebras.* Again, this is an example of a VH-space.

In the sequel, we fix a VH-space H and a VE-space E. The notation we use for the inner product will be either $[\cdot,\cdot]$ or $(\cdot,\cdot)$ which will be clear from the context.

Theorem 3.7. *Any VE-space can be embedded as a dense subspace of a VH-space which is uniquely determined up to isomorphism.*

Proof. Since E is equipped with a locally convex space topology, we just take the completion of E as a topological vector space in which case we get H. The only non-standard argument here is how to extend the inner product to the completion, which can be done as in the case of Hilbert spaces using nets instead of sequences. That is to say, we can show that if (x_α) and (y_α) are Cauchy nets in E it follows that $[x_\alpha, y_\alpha]$ is a Cauchy net in Z. Now, the conditions of the inner product are shown to be satisfied easily but the second condition. But this condition also holds in the completion by the polarization identity given by (3.3). □

3.2 Linear Operators on VH-Spaces

In this section we show that most of the definitions for operators in a Hilbert space can be translated to the case of VH-spaces, with remarkable exceptions.

In our case, the continuity of an operator corresponds to the existence of a neighborhood of origin N_ϕ for any neighborhood of the origin N_θ, such that we have

$$[x,x] \in N_\phi \Rightarrow [Ax,Ax] \in N_\theta.$$

Unlike the case of Hilbert spaces we will consider a special class of continuous operators namely the bounded operators $\mathcal{B}(H)$ which are defined in a similar way: a linear operator $A\colon H \to H$ is *bounded*, equivalently $A \in \mathcal{B}(H)$, if there exist a constant k such that

$$[Ax,Ax] \le k[x,x], \quad x \in H. \tag{3.1}$$

For a bounded operator A we define the *operator norm* of A, $\|A\|$ to be the square root of the least k satisfying (3.1), in which case we have

$$[Ax, Ax] \leq \|A\|^2[x, x], \quad x \in H.$$

Theorem 3.8. *The class $\mathcal{B}(H)$ of all bounded operators on H forms a Banach algebra under the operator norm.*

Proof. We want to show that $\|A\|$ is a norm. The other properties of the norm trivially hold but the triangle inequality. For triangle inequality we have for any $p > 0$,

$$[pAx - Bx, pAx - Bx] = p^2[Ax, Ax] - p[Ax, Bx] - p[Bx, Ax] + [Bx, Bx] \geq 0$$

which implies that

$$p^2[Ax, Ax] + [Bx, Bx] \geq p[Ax, Bx] + p[Bx, Ax]. \tag{3.2}$$

We also have

$$[(A + B)x, (A + B)x] = [Ax, Ax] + [Ax, Bx] + [Bx, Ax] + [Bx, Bx].$$

By multiplying and dividing the above inequality by p and using (3.2) we obtain

$$[(A + B)x, (A + B)x] \leq (\|A\|^2 + \|B\|^2)[x, x] + p[Ax, Ax] + p^{-1}[Bx, Bx].$$

Since the case for $\|A\| = 0$ is trivially true we can take $\|A\| \neq 0$. Putting $p = \|B\|/\|A\|$ yields

$$[(A + B)x, (A + B)x] \leq (\|A\|^2 + \|B\|^2)[x, x].$$

Hence, it follows that

$$\|A + B\| \leq \|A\| + \|B\|.$$

We also have to show that $\mathcal{B}(H)$ is complete. Suppose we have a Cauchy sequence (A_n) in $\mathcal{B}(H)$. Then,

$$[(A_n - A_m)x, (A_n - A_m)x] \leq \|A_n - A_m\|^2[x, x]$$

where the left side approaches to 0 as n, m tend to infinity. This implies that $(A_n x)$ is a Cauchy sequence in H which has a limit Ax. The linearity of A is clear and the rest of the proof is the same as in the Banach space case. □

Suppose A is a bounded linear operator in H. If there exists a bounded operator A^* such that for all $x, y \in H$

$$[Ax, y] = [x, A^*y]$$

we call this operator A^* the *adjoint* of A. We denote by $\mathcal{B}^*(H)$ the collection of all adjointable elements in $\mathcal{B}(H)$. We emphasize the fact that, in a general VH-space setting, not all bounded operators are adjointable. This is mostly due to lack of an analog of the Riesz Representation Theorem. The definitions of *self-adjoint*, *unitary* and *normal* operators are same as in the Hilbert space case.

We define a *contraction* to be a linear operator T such that $[Tx, Tx] \leq [x, x]$. We prove the following important result which we will refer quite frequently in the sequel.

Lemma 3.9. *If T is a contraction which has an adjoint on a dense linear manifold, say M, of a VH-space H, then the adjoint T^* is a contraction, too. Hence, for any bounded operator $T \in \mathcal{B}^*(H)$ we have $\|T\| = \|T^*\|$.*

Proof. For the first part, we use the fact that

$$\begin{aligned} (u - v, u - v) &\geq 0 \\ (u, u) - (u, v) - (v, u) + (v, v) &\geq 0 \\ (u, u) + (v, v) &\geq (u, v) + (v, u). \end{aligned}$$

Then,

$$\begin{aligned}(T^*x, T^*x) &= \frac{1}{2}((T^*x, T^*x) + (T^*x, T^*x)) \\ &= \frac{1}{2}((x, TT^*x) + (TT^*x, x)) \\ &= \frac{1}{2}((TT^*x, TT^*x) + (x, x)) \leq \frac{1}{2}((T^*x, T^*x) + (x, x)).\end{aligned}$$

Observe that the above calculation gives us

$$(T^*x, T^*x) \leq (x, x) \text{ for } x \in M. \tag{3.3}$$

This gives us the continuity of T^* on M. That is, for any neighborhood U of the origin we have

$$(x, x) \in U \Rightarrow (T^*x, T^*x) \in U$$

by the condition 3 of Definition 3.1.

Now, we want to extend T^* to the completion. For, we know that for any element w in the completion we can find a net $z_\lambda \to w$ [4]. For any neighborhood U of the origin we can find μ_u such that if $\lambda, \eta \geq \mu_u$ then $z_\lambda - z_\eta \in U$. In order to find μ_u, we take an open set W such that $W - W \subseteq U$. Since, the net $z_\lambda - w \to 0$ we can find μ_w which satisfies,

$$\lambda \geq \mu_w \Rightarrow (z_\lambda - w) \in W.$$

Then if $\lambda, \eta \geq \mu_w$ we have $(z_\lambda - w) - (z_\eta - w) = z_\lambda - z_\eta \in U$. We can set $\mu_u = \mu_w$.

We define the set $N_\mu = \{T^*(z_\lambda) \mid \lambda \geq \mu\}$. We denote by $\mathfrak{F}$ the elementary filter generated by N_μ [10]. By (3.3) it follows that $\mathfrak{F}$ is a Cauchy filter, hence converges to a point in the completion [10]. So, we define

$$T^*(w) := \lim_{\lambda} T^*(z_\lambda).$$

By the continuity of the inner product and closedness of the cone we have

$$(T^*w, T^*w) \leq (w, w).$$

Now for the second part we simply apply the first part to the operator $T/\|T\|$ which is a contraction and whose adjoint is $T^*/\|T\|$. By symmetry we obtain $\|T\| = \|T^*\|$. $\square$

We will also need the following lemma in the sequel.

Lemma 3.10. *If we have* $[f, f] = 0, f \in H$, *for a vector valued sesqui-linear function* $[\cdot,\cdot]\colon H \times H \to Z$ *on a VE-space* H, *then* $[f, f'] = [f', f] = 0$ *for all* $f' \in H$.

Proof. For any $\lambda \in \mathbb{C}$ and $f' \in H$, we have

$$\begin{aligned}[f+\lambda f', f+\lambda f'] &= \overbrace{[f,f]}^{0} + \lambda[f', f] + \bar{\lambda}[f, f'] + |\lambda|^2[f', f'] \\ &= \lambda[f', f] + \bar{\lambda}[f, f'] + |\lambda|^2[f', f'] \geq 0.\end{aligned}$$

If we put $\lambda = |\lambda|e^{i\theta}$, divide both sides by $|\lambda|$ and take $|\lambda| = 0$ we get,

$$e^{i\theta}[f', f] + e^{-i\theta}[f, f'] \geq 0.$$

Taking $\theta = 0\ , \pi, \pi/2$ and $-\pi/2$ yields,

$$\begin{aligned}[][f', f] + [f, f'] &\geq 0 \\ -([f', f] + [f, f']) &\geq 0 \qquad\qquad \text{and} \\ i([f', f] - [f, f']) &\geq 0 \\ i([f, f'] - [f', f]) &\geq 0.\end{aligned}$$

By symmetry, we obtain $[f', f] = [f, f'] = 0$. $\square$

3.3 Self-Adjoint Operators in $\mathcal{B}^*(H)$

It is obvious that A is self-adjoint if and only if $[Ax, y] = [x, Ay]$ for all $x, y \in H$. It is clear that an operator A is self-adjoint if and only if

$$[Ax, x] = [Ax, x]^*, \quad x \in H. \tag{3.1}$$

The following is an important result about self-adjoint operators which we will refer frequently. The importance of this inequality is that it may replace the Schwarz inequality which in general does not hold for a VH-space.

Theorem 3.11. *If $A \in \mathcal{B}^*(H)$ is self-adjoint, then we have*

$$-\|A\|[x, x] \leq [Ax, x] \leq \|A\|[x, x]$$

Proof. By putting $p = 1/\|A\|$ and $B = I$ in (3.2) we obtain

$$\begin{aligned} 2[Ax, x] =& [Ax, x] + [x, Ax] \\ \leq& \|A\|^{-1}[Ax, Ax] + \|A\|[x, x] \\ \leq& 2\|A\|[x, x]. \end{aligned}$$

which gives one part of the inequality. Second part easily follows if we put $-A$ to this result. □

3.4 Accessible Subspaces and Projections

Definition 3.12. A subspace M of a VH-space H is *accessible* if every element $x \in H$ can be written as $x = y + z$ where y is in M and z is such that $[z, m] = 0$ for all $m \in M$, that is orthogonal to M.

Observe that if such a decomposition exists it is unique and we write $y = Px$ where P is the *orthogonal projection* onto M.

Theorem 3.13. *Any orthogonal projection P is self-adjoint and idempotent. Conversely, any self-adjoint idempotent operator is an orthogonal projection onto its range subspace. Also, P is a positive contraction with $[Px, x] = [Px, Px]$ and any accessible subspace is closed.*

Proof. By using the notation above we have

$$[Px, y] = [x, y]$$

for all $x \in$ VH and $y \in M$. So, for some z in VH we can write $z = Pz + (z - Pz)$. Observe that we have, for any $m \in M$, $[z - Pz, m] = [z, m] - [Pz, m] = 0$ by the above equality. Putting everything together we obtain

$$[x, Pz] = [Px, Pz] = [Px, z].$$

Hence, P is a self-adjoint operator. P is idempotent by definition.

Conversely, if P is idempotent and self-adjoint we have

$$[x, y] = [x, Py] = [Px, y]$$

in which case for any element $z \in H$ we have the decomposition $z = Pz + (z - Pz)$ just as above.

For the second part, we have

$$[x, x] = [x - Px, x - Px] + [Px, Px]$$

so that, $[Px, Px] \leq [x, x]$. Hence a projection P is a continuous operator and it follows from the first part that any accessible space is closed. $\square$

Chapter 4

Dilations of $\mathcal{B}^*(H)$ Valued Maps

The main theorem of this paper is the following:

Theorem 4.1 (R.M. Loynes [7]). *Let Γ be a unital $*$-semigroup with unit ε and $T_\xi(\xi \in \Gamma)$ a family of continuous linear operators in $\mathcal{B}^*(H)$ for some VH-space H, satisfying the following conditions:*

(a) $T_\varepsilon = I$, $(T_\xi)^* = T_{\xi^*}$ *for all* $\xi \in \Gamma$.

(b) T_ξ *is positive definite as a function of* ξ*, in the sense that if* g_ξ $(\xi \in \Gamma)$ *is a function from* Γ *to* H *which vanishes except for a finite number of indices, then*

$$\sum_{\xi,\eta\in\Gamma} [T_{\xi^*\eta} g_\eta, g_\xi] \ \geq 0.$$

(c) For any given α *in* Γ *and any given neighborhood* N_0 *of the origin in* Z *there exists a neighborhood* N_0^α *of the origin in* Z *such that if* g_ξ *is a function from* Γ *to* H *which vanishes except for a finite number of indices, then*

$$\sum_{\xi,\eta\in\Gamma} [T_{\xi^*\eta} g_\eta, g_\xi] \in N_0^\alpha$$

implies that

$$\sum_{\xi,\eta\in\Gamma} [T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi] \in N_0$$

Then there exists a VH-space $\widehat{H}$, in which H can be isomorphically embedded as an accessible subspace, and a representation D_ξ of Γ in $\widehat{H}$, such that if P is the orthogonal projection onto H then

$$T_\xi = PD_\xi|_H, \quad \xi \in \Gamma. \tag{4.1}$$

Moreover, there exists such an $\widehat{H}$ which is minimal in the sense that it is generated by elements of the form $D_\xi f$, where $f \in H$ and $\xi \in \Gamma$, and this minimal $\widehat{H}$ is uniquely determined up to isomorphism.

The proof to this theorem follows closely the lines of the the proof of B. Sz.-Nagy for the Hilbert space case, but with important differences caused by the anomalies of VH-spaces, when compared to Hilbert spaces.

Proof. We divide the proof into five steps:

Step 1. *Construction of the space $\widehat{H}$*:

We define $\mathbf{G}$ to be the space of functions from Γ into H which vanishes on all but finitely many elements of Γ. Let $\mathbf{F}$ denote the linear space of functions from Γ to H which has a representation

$$f_\xi = \sum_\eta T_{\xi^*\eta} g_\eta, \quad \text{where } g \in \mathbf{G}. \tag{4.2}$$

Let us denote this relation simply as $f = \hat{g}$. We define the following vector inner product on $\mathbf{F}$, namely, for $f, f' \in \mathbf{F}$

$$[f, f'] := \sum_{\xi \in \Gamma} [f_\xi, g'_\xi]_H, \quad f = \hat{g}, f' = \hat{g}', \; g, g' \in \mathbf{G}. \tag{4.3}$$

We need to check that this definition is independent of the particular representation of f and f'. We check whether this is well defined by plugging (4.2) in (4.3):

$$
\begin{aligned}
[f, f'] &= \sum_{\xi\in\Gamma}[f_\xi, g'_\xi] \\
&= \sum_{\xi,\eta}[T_{\xi^*\eta}g_\eta, g'_\xi] \\
&= \sum_{\xi,\eta}[g_\eta, T_{\eta^*\xi}g'_\xi] \\
&= \sum_{\eta\in\Gamma}[g_\eta, f'_\eta] && \text{by the fact that } f' = \hat{g}'.
\end{aligned}
$$

So we obtain

$$[f, f'] = \sum_{\xi\in\Gamma}[f_\xi, g'_\xi] = \sum_{\eta\in\Gamma}[g_\eta, f'_\eta].$$

Observe that in the above equality the rightmost term is independent of g' and the middle term is independent of g which establishes the fact that the inner product is well-defined.

The linearity is clear and positivity is a direct consequence of the condition (b) in the theorem. For positive definiteness, we have to show that $[f, f] = 0$ implies $f = 0$. In the Hilbert space case this is a trivial consequence of Schwarz inequality which we do not have for a VH-space.

We take $f \in \mathbf{F}$ such that $[f, f] = 0$. By Lemma 3.10, we get $[f, f'] = 0$ for any $f' \in \mathbf{F}$. For any $\eta \in \Gamma$ and $h \in H$ we define $\delta_\eta h$ as the following function

$$(\delta_\eta h)_\xi = \begin{cases} h, & \xi = \eta \\ 0, & \text{otherwise} \end{cases} \tag{4.4}$$

We take a function $g = \widehat{(\delta_\eta f_\eta)}$. By the definition of the inner-product we get $[f, g] = [f, \delta_\eta f_\eta] = \sum_\xi [f_\xi, (\delta_\eta f_\eta)_\xi] = [f_\eta, f_\eta] = 0$. This implies that $f_\eta = 0$ for any $\eta \in \Gamma$, so $f = 0$.

So far, we showed that $\mathbf{F}$ is a VE-space equipped with the vector inner product $[\cdot, \cdot]$. By taking the abstract completion of $(\mathbf{F}, [\cdot, \cdot])$ we obtain the $\mathrm{VH} - space$ $\widehat{H}$

as desired.

H can be naturally identified with a subspace of $\mathbf{F}$ in the following way: $f \in H, f \longmapsto (T_{\xi^*}f)_{\xi\in\Gamma} \in \mathbf{F}$. We observe that

$$(T_{\xi^*}f)_\xi = \widehat{\delta_\epsilon f} \tag{4.5}$$

where the δ-function is as defined in (4.4).

If we denote the natural inclusion from H into $\mathbf{F}$ by J, we take the projection as $P_H = J^*$. By definition it is clear that P_H is a self-adjoint and idempotent operator. Thus, its range, namely H, is an accessible subspace by Theorem 3.13. By the following calculations we can find P_H concretely:

$$\begin{aligned}
[P_H f, h] &= [f, {P_H}^* h] \\
&= [f, Jh] = [f, (T_{\xi^*}h)_{\xi\in\Gamma}] \\
&= \sum_{\xi\in\Gamma} [f_\xi, (\delta_\epsilon h)_\xi] = [f_\epsilon, h]_H
\end{aligned}$$

So, we showed that $P_H f = f_\epsilon$. Since we calculate the adjoint here explicitly and the operator J is an isometry by (4.5), it follows by Lemma 3.9 that the adjoint is also bounded and even has norm equals 1.

Step 2. *The representation D.*

For arbitrary $\xi \in \Gamma$, D_ξ is defined first on the vector space $\mathbf{F}$: for any $f \in \mathbf{F}$,

$$D_\xi f := (f_{\xi^*\eta})_{\eta\in\Gamma} \quad \text{this means, for } \xi \in \Gamma, \xi \longmapsto (f_{\xi^*\eta})_\eta. \tag{4.6}$$

gives a representation of Γ in $\mathcal{B}(\widehat{H})$. However we need to check that the right hand side of (4.6) really belongs to $\mathbf{F}$. More precisely, we need to find a g such that $(D_\xi f)_\eta = \hat{g}$. If we plug the right side of (4.6) into (4.2) we get

$$f_{\xi^*\eta} = \sum_{\gamma\in\Gamma} T_{\eta^*\xi\gamma} g_\gamma. \tag{4.7}$$

By introducing a new variable $\zeta = \eta\gamma$ and defining the function,

$$h_\zeta^\xi = \sum_{\substack{\xi\gamma=\zeta \\ \gamma\in\Gamma}} g_\gamma,$$

the equation (4.7) now becomes,

$$f_{\xi^*\eta} = \sum_{\zeta\in\Gamma} T_{\eta^*\zeta} h_\zeta^\xi.$$

This shows that the image of D lies in $\mathbf{F}$ as claimed.

We now show that

$$[D_\alpha f, f'] = [f, D_{\alpha^*} f'], \quad f, f' \in \mathbf{F},\ \alpha \in \Gamma. \tag{4.8}$$

First we show that D is a representation on $\mathbf{F}$, that is,

$$D_{\alpha\beta} = D_\alpha D_\beta, \quad \alpha, \beta\ \in \Gamma. \tag{4.9}$$

Let $f \in \mathbf{F}$ and $g_\eta = (D_\beta f)_\eta = (f_{\beta^*\eta})_\eta$, and then $D_\alpha g_\eta = g_{\alpha^*\eta}$, so $g_{\alpha^*\eta} = f_{\beta^*\alpha^*\eta} = D_{\alpha\beta} f$, hence (4.9) is proven.

Now, letting $f = \hat{g}$ and $f' = \hat{g}'$ for some $g, g' \in \mathbf{G}$ we have,

$$\begin{aligned}
[D_\alpha f, f'] &= \sum_{\xi\in\Gamma} [f_{\alpha^*\xi}, g'_\xi] \\
&= \sum_{\xi\in\Gamma} \sum_{\eta\in\Gamma} [T_{\xi^*\alpha\eta} g_\eta, g'_\xi] \\
&= \sum_{\xi\in\Gamma} \sum_{\eta\in\Gamma} [g_\eta, T_{\eta^*\alpha^*\xi} g'_\xi] \\
&= \sum_{\eta\in\Gamma} [g_\eta, f'_{\alpha\eta}] = [f, D_{\alpha^*} f']
\end{aligned}$$

and hence the formula (4.8) is proven.

Observe that so far D_ξ is defined only in $\mathbf{F}$. In order to show that D_ξ extends from $\mathbf{F}$ to $\widehat{H}$, we have to show that D exhibits the boundedness property. This is a result of the following observation together with the condition (c) in the theorem as explained before,

$$\begin{aligned} [D_\alpha f, D_\alpha f] &= [D_{\alpha^*} D_\alpha f, f] = [D_{\alpha^*\alpha} f, f] \\ &= \sum_{\xi,\eta} [T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi]. \end{aligned} \tag{4.10}$$

Condition (c) says that for each given α, and a given neighborhood of the origin N_0 in Z there exists a neighborhood N_0^α of origin such that $[f, f] \in N_0^\alpha$ implies $[D_\alpha f, D_\alpha f] \in N_0$. Thus, D_ξ extends by continuity as a continuous linear operator $\widehat{H} \to \widehat{H}$. Finally, since D_{ξ^*} extends also by continuity and taking into account of (4.8), it follows that $D_{\xi^*} = D_\xi^*$, in particular for any $\xi \in \Gamma$ the operator D_ξ is adjointable.

Step 3. $T_\xi = P_H D_\xi|_H$.

Recall that $P_H f = f_\epsilon$ for $f \in \mathbf{F}$. We know that H is identified with the subspace $\{(T_{\xi^*} f)_{\xi\in\Gamma} | f \in H\}$. If we consider $g_\eta = T_{\eta^*} f$ then,

$$D_\xi g_\eta = g_{\xi^*\eta}$$

and then, letting $\eta = \epsilon$ we get

$$g_{\xi^*\epsilon} = g_{\xi^*} = T_\xi f,$$

which shows that $T_\xi = P_H D_\xi|_H$.

Step 4. *The closure of the span of* $\{D_\alpha H \mid \alpha \in \Gamma\} = \widehat{H}$.

We have to show that

$$\overline{\operatorname{lin}\{D_\alpha H \mid \alpha \in \Gamma\}} = \widehat{H}. \tag{4.11}$$

To this end, we recall the fact that $\mathbf{F}$ contains a copy of H which are exactly the elements of the form $(T_{\xi^*} f)_{\xi\in\Gamma}$, where $f \in H$. Hence (4.11) is a consequence of the fact that $D_\alpha(T_{\xi^*} f) = T_{\alpha^*\xi^*} f$ and the Definition 4.2.

Step 5. *The uniqueness of $\widehat{H}$.*

By (4.10) if we have two different extensions, say $\widehat{H}$ and $\widehat{H}'$, with corresponding D and D', we have

$$[D_\alpha f, D_\alpha f] = [D^{'}_\alpha f, D^{'}_\alpha f] \qquad \text{for all } f \in \mathbf{F}. \tag{4.12}$$

It follows that there is an isometry $\mathbf{U}$ with

$$\sum_{\alpha\in\Gamma} D_\alpha f_\alpha \overset{\mathbf{U}}{\longmapsto} \sum_{\alpha\in\Gamma} D^{'}_\alpha f_\alpha. \tag{4.13}$$

Again since $\mathbf{F}$ is dense in $\widehat{H}$, $\mathbf{U}$ extends to an isometry,

$$\mathbf{U} : \widehat{H} \to \widehat{H}'.$$

We also observe that $\mathbf{U}$ satisfies:

$$\mathbf{U}|_H = \mathbf{I}_H \tag{4.14}$$

$$\mathbf{U}D_\xi = D^{'}_\xi \mathbf{U} \quad \text{for all } \xi \in \Gamma. \tag{4.15}$$

This establishes the fact that different extensions are isomorphic. □

The next corollary shows that the construction provided by the previous theorem carries over to the case when some linearity properties occur.

Corollary 4.2. *If $T_{\xi\alpha\eta} = T_{\xi\beta\eta} + T_{\xi\gamma\eta}$ for some fixed α, β, γ and all ξ, η in Γ then $D_\alpha = D_\beta + D_\gamma$.*

Proof. We know that the elements of $f \in \mathbf{F}$ are of the form

$$f = \sum_\eta T_{\xi^*\eta} g_\eta = \sum_\eta D_\xi(T_\eta g_\eta).$$

So, it follows that $D_{\alpha^*\xi^*} = D_{\beta^*\xi^*} + D_{\gamma^*\xi^*}$. Since, by the above theorem, T is the restriction of D we have $T_{\alpha^*\xi^*} = T_{\beta^*\xi^*} + T_{\gamma^*\xi^*}$. It is evident that $(T_{\xi^*}g_{\xi^*})_\xi$ also spans $\mathbf{F}$. Hence, we obtain $D_\alpha = D_\beta + D_\gamma$. □

From now on, we consider only complex VH-spaces. We observe that in fact it is possible to derive the condition $(T_\xi)^* = T_{\xi^*}$ from the positive definiteness of T in the complex case. We prove this as a lemma.

Lemma 4.3. *Let φ be a map from a $*$-semigroup to $\mathcal{B}^*(H)$ for some (complex) VH-Space H. Suppose that φ satisfies positive definiteness, namely,*

$$\sum_{i,j}(\varphi(s_i^*s_j)f_j, f_i) \geq 0 \tag{4.16}$$

for finitely supported $\{f_i\} \subseteq H$ and $\{s_i\} \subseteq S$. Then, it follows that $\varphi(a^*) = \varphi^*(a)$.

Proof. If we write positive definiteness for $s_1 = 1, s_2 = a, f_1 = x, f_2 = y$ we obtain,

$$(\varphi(a)y, x) + (\varphi(a^*)x, y) + (\varphi(1)x, x) + (\varphi(a^*a)y, y) \geq 0.$$

Since by positivity we have,

$$(\varphi(1)x, x) + (\varphi(a^*a)y, y) \geq 0$$

this means that the expression $(\varphi(a)y, x) + (\varphi(a^*)x, y)$ is in the real span of the cone. Hence, the expression is equal to its adjoint by Definition 3.1, namely,

$$(\varphi(a)y, x) + (\varphi(a^*)x, y) = (x, \varphi(a)y) + (y, \varphi(a^*)x).$$

If we rearrange the terms we obtain

$$((\varphi(a^*) - \varphi^*(a))x, y) + (y, (\varphi^*(a) - \varphi(a^*))x,) = 0.$$

Letting $y = -i(\varphi(a^*) - \varphi^*(a))x$ yields,

$$2i((\varphi(a^*) - \varphi^*(a))x, (\varphi(a^*) - \varphi^*(a))x) = 0. \quad \text{For all } x \text{ in } H.$$

So, we conclude that $\varphi(a^*) = \varphi^*(a)$.

□

In this paper we also want to prove the equivalence of the B. Sz-Nagy's Theorem with other dilation theorems. However, in order to achieve this for the case of VH-spaces we need a stronger version. What we need actually is the following:

Corollary 4.4. *Let S be a $*$-semigroup with a unit ϵ and H be a (complex) VH-space. Let $\varphi\colon S \to \mathcal{B}^*(H)$. The following assertions are equivalent:*

(1) φ has the form

$$\varphi(s) = V^*\Phi(s)V \qquad s \in S \tag{4.17}$$

where V is an adjointable bounded linear operator from H to a VH-space K and Φ is an involution preserving semigroup homomorphism of S into $\mathcal{B}^*(K)$.

(2) φ satisfies the positive definiteness

$$\sum_{i,j} (\varphi(s_i^* s_j) f_j, f_i) \geq 0 \tag{4.18}$$

and the boundedness condition

$$\sum_{i,j} (\varphi(s_i^* u^* u s_j) f_j, f_i) \leq c(u)^2 \quad \sum_{i,j} (\varphi(s_i^* s_j) f_j, f_i) \tag{4.19}$$

for all $u \in S$, and finitely supported $\{f_i\} \subseteq H$, $\{s_i\} \subseteq S$, and the nonnegative constant $c(u)$ is independent of s_i and f_i.

Moreover, φ is unital if and only if K contains H isometrically and $\phi(s) = P_H \Phi(s)|_H$ for all $s \in S$.

Proof. We use theorem 4.1. It is straightforward that all the conditions are fulfilled, including the condition $\varphi(a^*) = \varphi^*(a)$ by Lemma 4.3, but condition (c). Consider a neighborhood N_0 of 0. We take N_0^u to be $\frac{N_0}{c(u)^2}$. Now the condition (c) follows from the third admissibility condition given in definition 3.1 in a VH-space using inequality (4.19). $\square$

In the sequel, we will call the inequality (4.19) $c(u)$-boundedness.

Chapter 5

Stinespring and Sz.-Nagy Theorems

In this section we prove the equivalence of two important theorems for the case of complex Hilbert spaces following the ideas of Szafraniec [14], namely the classical non-linear dilation theorem of B. Sz.-Nagy [15] and the theorem of Stinespring for the case of B^*-algebras which is infact the reformulated version of the Steinspring Theorem for C^*-algebras. This reformulated version was proved in [2]. Our main purpose is to investigate a corresponding equivalence for the case of VH-spaces, that will be done in the next section. We will make use of the notions of complete positivity (CP) and positive definiteness (PD) which we explained in Chapter 2.

Theorem 5.1 (B. Sz-Nagy, [15])**.** *Let S be a $*$-semigroup with a unit. Then a necessary and sufficient condition that $\varphi\colon S \to B(H)$ have the form*

$$\varphi = V^* \Phi(s) V \qquad s \in S \tag{5.1}$$

where V is a bounded linear map of H to a Hilbert space K containing H and Φ is an involution preserving semigroup homomorphism of S into $B(K)$, is that φ be a positive definite map satisfying the boundedness condition

$$\sum_{i,j} (\varphi(s_i^* u^* u s_j) f_j, f_i) \leq c(u)^2 \ \sum_{i,j} (\varphi(s_i^* s_j) f_j, f_i), \tag{5.2}$$

for all $u \in S$ and all finitely supported $\{f_i\} \subseteq H$, $\{s_i\} \subseteq S$ where the nonnegative constant $c(u)$ is independent of s_i and f_i.

In the theorem above, we do not assume that $\varphi(1)$ is the identity operator. In this case, the only difference is that the copy of H in K is not isometric to H. Also, the adjointness condition $\varphi(a^*) = \varphi^*(a)$ follows from the fact that the Hilbert space is complex as in Lemma 4.3.

Theorem 5.2 (Stinespring, [11],[2])**.** *Let A be a unital B^*-algebra with normalized unit, H a Hilbert space, and $\mu\colon A \to B(H)$ a linear map. Then a necessary and sufficient condition that μ have the form*

$$\mu(a) = V^*\Omega(a)V \qquad (a \in A), \tag{5.3}$$

where V is a bounded linear operator from H to a Hilbert space K and $\Omega\colon A \to B(K)$ is a $*$-representation, is that μ be positive definite.

We show the equivalence of these two theorems.

Theorem 5.3. *Theorem 5.1 is equivalent with Theorem 5.2.*

The proof of this theorem, which will use ideas from [2], has a real difficulty for the implication Stinespring's Theorem implies Sz.-Nagy Theorem, because in this case we are somehow in a position to construct a B^*-algebra by using the $*$-semigroup. Before we prove this implication we quote the following lemma due to Szafraniec [12].

Lemma 5.4. *Suppose $\varphi : S \to B(H)$ is positive definite. Then the following conditions are equivalent:*

- *φ satisfies the boundedness condition* (5.2)*.*
- *There exists a function $\alpha\colon S \to [0, +\infty)$ such that $\|\varphi(s)\| \leq C\alpha(s)$, where $\alpha(st) \leq \alpha(s)\alpha(t), t, s \in S$ and $\alpha(1) = 1$.*

We also need the following lemma which is an exercise from [1]. This lemma is very useful which we will also use when we prove the VH-variant of Steinspring theorem for B^*-algebras.

Lemma 5.5. *Suppose A is a B^*-algebra. Then, for every self-adjoint element x in the open unit ball of A, $1-x$ has a self adjoint square root in A.*

Proof. For, $0 < \alpha < 1$ we have that

$$(1-z)^{\alpha} = 1 - \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n \geq 0$ and $\sum_{n=1}^{\infty} c_n = 1$.

This implies that for elements $\|x\| < 1$ in a Banach algebra we get, for $\alpha = 1/2$

$$(1-x)^{1/2} = 1 - \sum_{n=1}^{\infty} c_n x^n.$$

That is to say $1 - x = y^2$ for some y. Moreover if we are in a B^*-algebra observe that we have $\|x^*\| = \|x\| < 1$ which implies that

$$(1-x^*)^{1/2} = 1 - \sum_{n=1}^{\infty} c_n (x^*)^n,$$

from which we get $(1-x^*)^{1/2} = y^* = (1-x)^{1/2} = y$ since x is a self-adjoint element. So, we obtain $1 - x = y^*y = y^2$. Hence the result. □

Proof of Theorem 5.3. Sz.-Nagy's Theorem ⇒ Stinespring's Theorem. A B^*-algebra becomes a multiplicative $*$-semigroup. By positive definiteness we have

$$\sum_{i,j} (\mu(s_i^* u^* u s_j) f_j, f_i) \geq 0 \tag{5.4}$$

We want to obtain the condition (5.2) of Theorem 5.1. In Lemma 5.5 we take $x = u^*u/2\|u^*u\|$ which is in the open unit ball of A. By the lemma it follows that

$1-x=y^2$ for some self-adjoint y ,that is $y^*y=1-x$. Replacing u^*u in (5.4) by $1-x$ yields

$$\sum_{i,j}(\mu(s_j^*u^*us_i)f_i,f_j) \;\leq\; 2\|uu^*\| \;\sum_{i,j}(\mu(s_j^*s_i)f_i,f_j).$$

Thus, we apply Sz.-Nagy's Theorem with $\varphi=\mu$. The linearity of the map $\Omega=\Phi$ follows from Corollary 4.2.

Stinespring's Theorem $\Rightarrow$ Sz.-Nagy's Theorem.

Suppose that we have a PD map satisfying the boundedness condition (5.2). So we have

$$(\varphi(s^*u^*us)f,f)\leq c(u)^2(\varphi(s^*s)f,f) \tag{5.5}$$

By using an idea of Arveson in [2], we take $c(u)$ to be the maximum of 1 and the best $c(u)$ which satisfies (5.5). Observe that $c(u)$ is submultiplicative even without taking maximum with 1. This can be seen by replacing s with vs in (5.5), which gives us $c(uv)\leq c(u)c(v)$.

If we put $s=1$ in (5.5) we obtain

$$(\varphi(u^*u)f,f)\leq c(u)^2(\varphi(1)f,f)\leq c(u)^2\|\varphi(1)\|(f,f). \tag{5.6}$$

Besides, by using the Schwarz inequality for PD maps on $*$-semigroups [13] we obtain

$$\|\varphi(s)f\|^2\leq\|\varphi(1)\|(\varphi(s^*s)f,f). \tag{5.7}$$

Now if we use (5.6) in order to estimate the right side of (5.7) and take square root of both sides of the inequality we get

$$\|\varphi(s)\|\leq\|\varphi(1)\|c(s). \tag{5.8}$$

This condition is also a result of the Lemma 5.4. In the proof of Lemma 5.4 in

[12] for $c(u)$, which is chosen to be minimal for (5.5), we have $c(s) = \alpha(s)^{\frac{1}{2}}$. It follows that $c(s^*) = c(s)$, since for $\alpha(s)$ we have $\|\varphi(s^*)\| = \|\varphi^*(s)\| = \|\varphi(s)\| \leq \|\varphi(1)\|\alpha(s)$. The equality $\varphi(s^*) = \varphi^*(s)$ is a result of Lemma 4.3.

Now we define by $\ell^1(S, c)$ as the set of complex functions ξ on S which satisfies

$$\sum_s |\xi(s)|c(s) < +\infty. \tag{5.9}$$

This space is a subspace of $\ell^1(S)$ and becomes a B^*-algebra the norm of ξ given by (5.9). The multiplication is given by the convolution

$$(\xi * \eta)(u) = \begin{cases} \sum\limits_{st=u} \xi(s)\eta(t) & \text{if the sum has at least one term,} \\ 0 & \text{otherwise.} \end{cases}.$$

If we denote by $\delta(s)$ the function taking value 1 at s and zero elsewhere, it is clear that $\delta(1)$ is the normalized unit of the B^*-algebra with these definitions of norm and multiplication. We define the involution as $\xi^*(s) = \overline{\xi(s^*)}$. We want to extend the map $\varphi(s)$ to the $\ell^1(S, c)$. The inequality (5.8) enables us to define a map $\hat{\varphi} : l^1(S, c) \to \mathcal{B}(H)$ as $\hat{\varphi}(\xi) = \sum_s \xi(s)\varphi(s)$. By using (5.8) we obtain

$$\|\hat{\varphi}(\xi)\| \leq \|\varphi(1)\|\|\xi\|. \tag{5.10}$$

In the definition of $\hat{\varphi}$ we take ξ to be a function which has finite support on S. But observe that $\hat{\varphi}$ can be extended to the whole $l^1(S, c)$ since any function in $l^1(S, c)$ can be norm approximated by finitely supported functions. It is obvious that $\hat{\varphi}$ is linear. By using the fact that φ is PD we can now check that $\hat{\varphi}$ is PD.

We have

$$\begin{aligned} \sum_{i,j} (\hat{\varphi}(\xi_j^*\xi_i)f_i, f_j) &= \sum_{i,j} (\sum_{s^*,t} (\xi_j^*(s^*)\xi_i(t)\varphi(s^*t)f_i, f_j) \\ &= \sum_{s^*,t} (\varphi(s^*t)(\sum_i \xi_i(t)f_i), (\sum_j \xi_j(s)fj)) \geq 0 \end{aligned}$$

where the last inequality follows from the positive definiteness of φ. Observe that we can interchange the sums since ξ has finite support which implies that all the sums are finite.

Observe that we can in fact go back to φ by putting $\widehat{\varphi}(\delta(s)) = \varphi(s)$ where $\delta(s)$ is the point mass at s. Now we can use Stinespring's Theorem to get (5.1) in Sz.Nagy's Theorem. $\square$

Chapter 6

Dilation Theorems for VH-Spaces

6.1 Stinespring's Theorem for VH-Spaces

In this section we prove an analogue of Stinespring theorem for the case of VH-spaces. In fact, we prove two theorems respectively for the representation of C^* and B^*-algebras in VH-spaces.

Theorem 6.1. *Let A be a unital C^*-algebra, H be a VH-space and $\mu\colon A \to \mathcal{B}^*(H)$ be a linear map. Then μ has the form*

$$\mu(a) = V^*\rho(a)V \qquad (a \in A)$$

where V is an adjointable bounded linear operator from H into a VH-space K and $\rho\colon A \to \mathcal{B}^*(K)$ is a $*$-representation, if and only if μ satisfies the following condition:

$$\sum_{i,j} (\mu(a_j^* a_i)x_i, x_j) \geq 0 \tag{6.1}$$

for all $a_i \in A$ and $x_i \in H$ finitely supported.

Proof. For necessity, we know that ρ is a $*$-representation. We have $\mu(a) =$

$V^*\rho(a)V$, so

$$\begin{aligned}
&\sum_{i,j}(\mu(a_j^*a_i)x_i, x_j) = \sum_{i,j}(V^*\rho(a_j^*a_i)Vx_i, x_j) \\
&= \sum_{i,j}(\rho(a_j^*a_i)Vx_i, Vx_j) = \sum_{i,j}(\rho(a_i)Vx_i, \rho(a_j)Vx_j) \\
&= (\sum_i \rho(a_i)Vx_i, \sum_i \rho(a_i)Vx_i) \geq 0.
\end{aligned}$$

For sufficiency, we consider the algebraic tensor product $A \otimes H$. The elements of this tensor product are of the form

$$\xi = \sum_i a_i \otimes x_i \tag{6.2}$$

$$\eta = \sum_j b_j \otimes y_j, \tag{6.3}$$

where $a_i, b_j \in A$ and $x_i, y_j \in H$ are finitely supported. On $A \otimes H$ we define the vector inner product by

$$(\xi, \eta) = \sum_{i,j}(\mu(b_j^*a_i)x_i, y_j), \quad \text{where } \xi, \eta \in A \otimes H. \tag{6.4}$$

Observe that this is positive by (6.1). Also by the linearity of μ, it follows that this is sesqui-linear. There is a natural mapping ρ' from A into the set of all linear transformations on $A \otimes H$ given by

$$\rho'(a)\left(\sum_i a_i \otimes x_i\right) = \sum_i aa_i \otimes x_i. \tag{6.5}$$

For all $a \in A$ and $\xi \in A \otimes H$ we want to find an estimate for $(\rho'(a)\xi, \rho'(a)\xi)$. By replacing a_i in (6.1) by aa_i we obtain

$$\sum_{i,j}(\mu(a_j^* a^* a a_i)x_i, x_j) \geq 0. \tag{6.6}$$

We know that the following inequality holds in a C^*-algebra

$$-\|a^*a\| \leq a^*a \leq \|a^*a\|. \tag{6.7}$$

Then it follows that $\|a^*a\| - a^*a \geq 0$. In a C^*-algebra any positive element is of the form v^*v for some v. This allows us to replace a^*a in (6.6) by $\|a^*a\| - a^*a$. By the linearity of μ this yields

$$\sum_{i,j}(\mu(a_j^* a^* a a_i)x_i, x_j) \;\leq\; \|a^*a\| \sum_{i,j}(\mu(a_j^* a_i)x_i, x_j), \tag{6.8}$$

equivalently,

$$(\rho'(a)\xi, \rho'(a)\xi) \;\leq\; \|a\|^2(\xi, \xi) \tag{6.9}$$

which is the estimate we need.

We define $N = \{\ \xi \in A \otimes H \mid (\xi, \xi) = 0\ \}$. N is a linear manifold by Lemma 3.10. Also, N is invariant under $\rho'(a)$ by (6.9). Hence, the quotient space $(A \otimes H)/N$ is a VE-space. By taking the abstract completion of a VE-space, as we explained in the preliminaries, we obtain the VH-space K. By using (6.9) we can extend ρ' to ρ in the completion.

We define $Vx = 1 \otimes x + N$ for all $x \in K$. We have $(1 \otimes x, 1 \otimes x) = (\mu(1)x, x)$. Since $(\mu(1)x, x) \geq 0$ by (6.1), as in the proof of Theorem 3.11 we have

$$\begin{aligned} 2(\mu(1)x, x) = (\mu(1)x, x) + (x, \mu(1)x) &\leq (\mu(1)x, \mu(1)x) + (x, x) \\ &\leq (\|\mu(1)\|^2 + 1)(x, x) \end{aligned}$$

from which it follows that V is a bounded operator. Different from the standard case it is not clear here why V should be adjointable. But since μ is adjointable it turns out that we can find the adjoint of V too: $V^*(a \otimes y) = \mu^*(a^*)y$. We check that this is really the adjoint of V by writing

$$(Vx, a \otimes y) = (1 \otimes x, a \otimes y) = (\mu(a^*)x, y) = (x, \mu^*(a^*)y) = (x, V^*y). \quad (6.10)$$

By Lemma 4.3 we have that $\mu(a^*) = \mu^*(a)$ which implies $V^*(a \otimes y) = \mu(a)y$. We extend V^* linearly to the whole space. However, it is not clear why V^* is a well-defined operator. For, choose any $\xi = \sum_i a_i \otimes x_i \in N$, that is $(\xi, \xi) = 0$. Observe that we have, for any $x \in H$,

$$(1 \otimes x, \xi) = (1 \otimes x, \sum_i a_i \otimes x_i) = \sum_i (\mu(a_i^*)x, x_i) = (x, \sum_i \mu(a_i)x_i). \quad (6.11)$$

By Lemma 3.10, we have $(1 \otimes x, \xi) = 0$. We choose $x = \sum_i \mu(a_i)x_i$. By (6.11), we obtain $\sum_i \mu(a_i)x_i = V^*(\xi) = 0$. So that, V^* is well defined. Also, by Lemma (3.9) V^* is bounded.

Consequently, we have

$$\begin{aligned}
(V^*\rho(a)Vx, y) &= (\rho(a)Vx, Vy) \\
&= (\rho'(a)1 \otimes x, 1 \otimes y) \\
&= (a \otimes x, 1 \otimes y) \\
&= (\mu(a)x, y)
\end{aligned}$$

Letting $y = V^*\rho(a)Vx - \mu(a)x$, we obtain $(V^*\rho(a)Vx - \mu(a)x, V^*\rho(a)Vx - \mu(a)x) = 0$. Hence, $\mu(a) = V^*\rho(a)V$ which completes the proof of the theorem. □

We observe that different from the Hilbert space case we had to find the adjoint of V precisely. This is because in a VH-space H we do not know whether every bounded operator is adjointable. Observe that the only place where we use a property of a C^*-algebra is when we find an estimate for $(\rho'(a)\xi, \rho'(a)\xi)$. However, it turns out that by using Lemma 5.5, we are able to prove VH-space analogue of the Stinespring theorem for B^*-algebras as well.

Theorem 6.2. *Let A be a unital B^*-algebra, H be a VH-space, and $\mu\colon A \to \mathcal{B}^*(H)$ a linear map. Then μ has the form*

$$\mu(a) = V^*\rho(a)V \qquad (a \in A)$$

where V is an adjointable bounded linear operator from H into a VH-space K and $\rho : A \to \mathcal{B}^*(K)$ is a $*$-representation if and only if μ satisfies the following condition

$$\sum_{i,j} (\mu(a_j^* a_i)x_i, x_j) \geq 0, \tag{6.12}$$

for all $a_i \in A$ and $x_i \in H$ finitely supported.

Proof. The proof is the same as the proof of Theorem 6.1 but the derivation of the estimate for $(\rho'(a)\xi, \rho'(a)\xi)$. However, this is an easy consequence of Lemma 5.5. If $a^*a = 0$ then (6.8) is trivially true, if $a^*a \neq 0$ then in Lemma 5.5 we take $x = a^*a/2\|a^*a\|$ which is obviously an element in the unit ball of A. By the lemma it follows that $1 - x$ is of the form y^2 for some self-adjoint y which means we have $y^*y = 1 - x$. We now replace a^*a in (6.6) by $1 - x$ from which we get

$$\sum_{i,j} (\mu(a_j^* a^* a a_i)x_i, x_j) \ \leq \ 2\|a^*a\| \ \sum_{i,j} (\mu(a_j^* a_i)x_i, x_j). \tag{6.13}$$

The other parts of the proof transfers exactly to this case. □

Observe that in (6.13) the constant 2 on the right side can be taken 1. For, it is enough to consider a sequence $t_n \geq 1$ and $t_n \to 1$. In the proof of Theorem 6.2, we put $x = a^*a/t_n\|a^*a\|$ which is in the open unit ball. We can take the limit as $n \to \infty$ by the closedness of the cone. Hence, the bound for the case of B^*-algebras is not worse than that of C^*-algebras.

6.2 A Comparison of Dilation Theorems for VH-Spaces

In Chapter 4 we obtained Corrollary 4.4 which is a stronger version of Theorem 4.1. In the preceeding section we proved analogs of Stinespring Theorem for the case of B^* and/or C^*-algebras and VH-spaces. In this section we prove the equivalence of these theorems.

Theorem 6.3. *Corollary 4.4 implies Theorem 6.1.*

Proof. A C^*-algebra A is also a $*$-semigroup. The boundedness condition (4.19) is obtained in exactly the same way as in the proof of Theorem 6.1. So, we can use Loynes's Theorem for $\varphi = \mu$ and $\Phi = \rho$. The only point which is not clear is that why would the map ρ be linear. μ is linear and we put $\varphi = \mu$ in the Loynes Theorem. By Corollary 4.2 we obtain, for $t, u, t+u \in A$, $\varphi(x(t+u)y) = \varphi(xty) + \varphi(xuy)$ which implies that $\Phi(t+u) = \Phi(t) + \Phi(u)$. Hence ρ is also linear. □

Theorem 6.4. *Corollary 4.4 implies Theorem 6.2.*

Proof. The boundedness condition (4.19) is obtained as in the proof of Theorem 6.2. The other parts of the proof is same as the previous theorem. □

An important point here is that whether the converse of Theorem 6.4 holds. The converse of this theorem holds for the Hilbert space case as we demonstrated in Chapter 5. We will show that the converse of Theorem 6.4 also holds for the VH-space case. However, we need the following lemma:

Lemma 6.5. *Let φ be a map from a $*$-semigroup to $\mathcal{B}^*(H)$ for some VH-Space H. Suppose that φ satisfies*

$$\sum_{i,j=1}^{2} (\varphi(s_i^* s_j) f_j, f_i) \geq 0 \tag{6.1}$$

namely, φ is 2-positive. Then, it follows that

$$(\varphi(s)f, \varphi(s)f) \leq \|\varphi(1)\|(\varphi(s^*s)f, f) \quad s \in S, f \in H. \tag{6.2}$$

Proof. In (6.1), by letting $s_1 = 1, s_2 = a, f_1 = -\varphi(a)f, f_2 = \|\varphi(1)\|f$ we obtain

$$\begin{aligned}(\varphi(1)\varphi(a)f, \varphi(a)f) - \|\varphi(1)\|(\varphi(a)f, \varphi(a)f) - \\ \|\varphi(1)\|(\varphi(a^*)\varphi(a)f, f) + \|\varphi(1)\|^2(\varphi(a^*a)f, f) \geq 0.\end{aligned} \tag{6.3}$$

By Lemma 4.3 we have $\varphi(1^*) = \varphi(1) = \varphi(1)^*$, so that $\varphi(1)$ is self-adjoint. By applying Theorem 3.11 we get

$$(\varphi(1)\varphi(a)f, \varphi(a)f) \leq \|\varphi(1)\|(\varphi(a)f, \varphi(a)f). \tag{6.4}$$

Replacing the first term of (6.3) by the right side of (6.4), after the cancellations, gives us

$$(\varphi(a^*)\varphi(a)f, f) \leq \|\varphi(1)\|(\varphi(a^*a)f, f).$$

Since we have $\varphi(a^*) = \varphi(a)^*$ by Lemma 4.3 we obtain,

$$(\varphi(a)f, \varphi(a)f) \leq \|\varphi(1)\|(\varphi(a^*a)f, f).$$

Hence, the result.

□

Observe that we can apply Lemma 6.5 if φ is positive definite. Since any positive definite map is 2-positive.

Theorem 6.6. *Theorem 6.2 implies Corollary 4.4.*

Proof. By the $c(u)$-boundedness in Corollary 4.4 we have

$$(\varphi(s^*u^*us)f, f) \leq c(u)^2(\varphi(s^*s)f, f). \tag{6.5}$$

Letting $s = 1$ yields

$$(\varphi(u^*u)f, f) \leq c(u)^2(\varphi(1)f, f). \tag{6.6}$$

As in the proof for the Hilbert space case, we take $c(u)$ to be the maximum of the best constant satisfying the $c(u)$-inequality (6.5) and 1. Twice application of the same inequality gives us $c : S \mapsto [1, \infty)$ to be submultiplicative.

By using (6.2) we obtain

$$\|\varphi(s)\| \leq \|\varphi(1)\| c(s). \tag{6.7}$$

Here in defining the B^*-algebra $\ell^1(S, c)$ we proceed exactly the same as in the proof Stinespring's Theorem $\Rightarrow$ Sz.-Nagy's Theorem in Chapter 5. We define a map $\hat{\varphi} : l^1(S, c) \to \mathcal{B}^*(H)$ as $\hat{\varphi}(\xi) = \sum_s \xi(s)\varphi(s)$. In Chapter 5 it was checked that $\hat{\varphi}$ satisfies positive definiteness which also applies to here. Also, similar to the Hilbert space case by using (6.7) we obtain

$$\|\hat{\varphi}(\xi)\| \leq \|\varphi(1)\| \|\xi\|. \tag{6.8}$$

However, the positive definiteness was checked only for functions ξ which vanishes all but only finitely many points. Because any function can be norm approximated by such functions, in order to check the positive definiteness of $\hat{\varphi}$ for any function we consider finitely supported sequences such that $\xi(n)_i \to \xi_i$ as n goes to infinity. We have that

$$\sum_{i,j} (\hat{\varphi}(\xi(n)_j^* \xi(n)_i) x_i, x_j) \geq 0.$$

Since $\ell^1(S, c)$ is a Banach space we have, if $\xi(n)_j^* \to \xi_j^*$ and $\xi(n)_i \to \xi_i$ it follows that $\xi(n)_j^* \xi(n)_i \to \xi_j^* \xi_i$. This is clear by the fact that

$$\begin{aligned} &\|\xi(n)_j^* \xi(n)_i - \xi_j^* \xi(n)_i + \xi_j^* \xi(n)_i - \xi_j^* \xi_i\| \\ &\leq \|\xi(n)_i\| \|\xi(n)_j^* - \xi_j^*\| + \|\xi_j^*\| \|\xi(n)_i - \xi_i\| \end{aligned}$$

By (6.8) we have that $\hat{\varphi}(\xi(n)_j^* \xi(n)_i) \to \hat{\varphi}(\xi_j^* \xi_i)$. The continuity of inner product gives

$$\sum_{i,j}(\hat{\varphi}(\xi(n)_j^*\xi(n)_i)x_i, x_j) \longrightarrow \sum_{i,j}(\hat{\varphi}(\xi_j^*\xi_i)x_i, x_j).$$

as $n \to \infty$. Since each term $\sum_{i,j}(\hat{\varphi}(\xi(n)_j^*\xi(n)_i)x_i, x_j) \geq 0$, by the closedness of the cone we obtain $\sum_{i,j}(\hat{\varphi}(\xi_j^*\xi_i)x_i, x_j) \geq 0$.

Observe that we have a way back to φ by putting $\varphi(s) = \hat{\varphi}(\delta_s)$ where δ_s is the point mass at s. We can apply Theorem 6.2 to $\hat{\varphi}$ in order to get the representation (4.17) in Corollary 4.4. □

Proposition 6.7. *Using the notation in Theorem 6.6 and its proof, we have that*

$$\sum_{i,j}(\hat{\varphi}(\xi_j^*\xi_i)f_i, f_j) \leq \|\varphi(1)\| \left(\sum_i \|\xi_i\|^2(f_i, f_i)\right).$$

Proof. By the definition of $\hat{\varphi}$ as in the proof of Theorem 6.6 we have

$$\sum_{i,j}(\hat{\varphi}(\xi_j^*\xi_i)f_i, f_j) = \sum_{i,j}(\sum_{s^*,t}(\varphi(s^*t)\xi_i(t)f_i, \xi_j(s)f_j). \tag{6.9}$$

Throughout the proof we will mainly refer to the right side of (6.9), which we denote by Σ. Since, $\hat{\varphi}$ is positive definite it follows that $\Sigma \geq 0$ hence $\Sigma = \Sigma^*$. Now we consider, $\Sigma + \Sigma^*$ and apply (3.2), for $p \geq 0$,

$$(u, v) + (v, u) \leq p(u, u) + p^{-1}(v, v)$$

to the adjoint terms in Σ and Σ^*. So that we have,

$$\begin{aligned}
&2\sum_{i,j}\sum_{s^*,t}(\varphi(s^*t)\xi_i(t)f_i, \xi_j(s)f_j) \\
&\leq \sum_{i,j}\sum_{s^*,t} p(\varphi(s^*t)\xi_i(t)f_i, \varphi(s^*t)\xi_i(t)f_i) + p^{-1}(\xi_j(s)f_j, \xi_j(s)f_j) \\
&\leq \sum_{i,j}\sum_{s^*,t} p\|\varphi(s^*t)\|^2\|\xi_i(t)\|^2(f_i, f_i) + p^{-1}\|\overline{\xi_j(s^*)}\|^2(f_j, f_j).
\end{aligned} \tag{6.10}$$

Since by (6.7) we have,

$$\|\varphi(s^*t)\|^2 \leq \|\varphi(1)\|^2 c(s^*)^2 c(t)^2 \tag{6.11}$$

by using the submultiplicativity of $c(s)$. By plugging (6.11) in (6.10) and putting $p = \frac{1}{\|\varphi(1)\|c(s^*)^2}$ we obtain,

$$\begin{aligned}
&\sum_{i,j}\sum_{s^*,t)} p\|\varphi(s^*t)\|^2\|\xi_i(t)\|^2(f_i,f_i) + p^{-1}\|\overline{\xi_j(s^*)}\|^2(f_j,f_j) \\
&\leq \sum_{i,j}\sum_{s^*,t} \|\varphi(1)\|c(t)^2\|\xi_i(t)\|^2(f_i,f_i) + \|\varphi(1)\|c(s^*)^2\|\overline{\xi_j(s^*)}\|^2(f_j,f_j) \\
&\leq \sum_{i,j} \|\varphi(1)\| \left(\sum_t c(t)|\xi_i(t)|\right)\left(\sum_t c(t)|\xi_i(t)|\right)(f_i,f_i) \\
&+ \|\varphi(1)\| \left(\sum_{s^*} c(s^*)|\overline{\xi_j(s^*)}|\right)\left(\sum_{s^*} c(s^*)|\overline{\xi_j(s^*)}|\right)(f_j,f_j)
\end{aligned}$$

So that we have,

$$\begin{aligned}
&\leq \|\varphi(1)\|\sum_i \|\xi_i\|^2(f_i,f_i) + \|\varphi(1)\|\sum_j \|\xi_j\|^2(f_j,f_j) \\
&= 2\|\varphi(1)\|\left(\sum_i \|\xi_i\|^2(f_i,f_i))\right).
\end{aligned}$$

where $\|\xi\|$ is the norm of ξ in $\ell^1(S,c)$ as mentioned in the proof of Theorem 5.3.

□

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