PRICING AND OPTIMAL EXERCISE OF PERPETUAL AMERICAN OPTIONS WITH LINEAR PROGRAMMING

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By
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January, 2010

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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January, 2010

An American option is the right but not the obligation to purchase or sell an underlying equity at any time up to a predetermined expiration date for a predetermined amount. A perpetual American option differs from a plain American option in that it does not expire. In this study, we solve the optimal stopping problem of a perpetual American option with methods from the linear programming literature. Under the assumption that the underlying's price follows a discrete time and discrete state Markov process, we formulate the problem with an infinite dimensional linear program using the excessive and majorant properties of the value function. This formulation allows us to solve complementary slackness conditions efficiently, revealing an optimal stopping strategy which highlights the set of stock-prices for which the option should be exercised. Under two different stock-price movement scenarios (simple and geometric random walks), we show that the optimal strategy is to exercise the option when the stock-price hits a special critical value. The analysis also reveals that such a critical value exists only for some special cases under the geometric random walk, dependent on a combination of state-transition probabilities and the economic discount factor. We further demonstrate that the method is useful for determining the optimal stopping time for combinations of plain vanilla options, by solving the same problem for spread and strangle positions under simple random walks.

Keywords: Difference equations, Markov processes, Infinite dimensional linear programming, Perpetual American options.

ÖZET

VADESİZ AMERİKAN TİPİ OPSİYONLARIN DOĞRUSAL PROGRAMLAMA İLE FİYATLANDIRILMASI VE EN İYİ KULLANIM DEĞERLERİNİN BELİRLENMESİ

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Vadesiz Amerikan opsiyonları, klasik Amerikan opsiyonlarından farklı olarak sahiplerine bir hisse senedini, herhangi bir bitiş zamanı olmaksızın, ileri bir tarihte önceden belirlenmiş bir fiyat üzerinden alma veya satma hakkı verirler. Bu çalışmada, söz konusu opsiyonların yazıldığı tarihten itibaren en iyi kazancı verecek şekilde ne zaman kullanılması gerektiği problemi ele alınmıştır. Farklı ayrık durumlu ve ayrık zamanlı Markov rassal süreçleri altında incelenen problem, değer fonksiyonunun "excessive" ve "majorant" özellikleri kullanılarak sonsuz değişkenli doğrusal programlama ile modellenmiştir. En iyi opsiyon kullanım zamanını veren strateji, kuvvetli çifteşlik (strong duality) özelliği de gösteren problemde tümler gevşeklik (complementary slackness) koşulları kullanılarak karakterize edilmiş, opsiyonun hangi hisse değerlerinde kullanılması gerektiği belirlenmiştir. İkili, üçlü basit rassal yürüyüş ile geometrik rassal yürüyüş senaryolarında elde edilen sonuçlar paralellik göstermekte, en iyi opsiyon kazancının hesaplanan en iyi kullanım noktasından itibaren elde edileceğini belirtmektedir. Elde edilen diğer bir sonuç, geometrik rassal yürüyüş modelinde opsiyonun en iyi kullanım noktalarının, durum geçiş olasılıkları ve ekonomik iskonto çarpanının belirlediği bir faktör doğrultusunda, sadece belirli özel durumlarda var olabileceğini göstermektedir. Çalışma, farklı alım ve satım opsiyonlarının birleştirilmesinden meydana gelen "spread" ve "strangle" tipi kazanç fonksiyonlarının en iyi kullanım aralıklarının basit rassal yürüyüş altında belirlenmesiyle tamamlanmıştır.

Anahtar sözcükler: Fark denklemleri, Markov rassal süreçleri, Sonsuz değişkenli doğrusal programlama, Vadesiz amerikan opsiyonu.

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Chapter 1

Introduction

Mathematical finance is undoubtedly one of the contemporary fields of applied mathematics that has enjoyed a continuous interest and a vast amount of research from mathematicians, physicists, economists and engineers. Although originated as an area with the aim of describing the complex behaviour of financial markets, investor actions and the optimal allocation of financial resources; the topic, in its own right, frequently had a major impact on the research trends of modern applied mathematics, especially in the last century. Numerous subjects of stochastic analysis and optimization theory were studied with the motivation resulting from the constant interest of researchers into the field of financial economics. Many results, as later shown, had even far reaching connections to other disciplines that are not usually thought to be related to financial economics.

A researcher interested in financial mathematics will see two main research directions that have played a significant role in the development of financial markets in the modern sense, especially in the second half of the 20^{th} century. The first of these, known as the modern portfolio theory, has its roots in the seminal 1952 paper of Harry Markowitz. His work, which was later popularized as the mean-variance analysis, has constructed the fundamental connection between asset risks and returns. The latter direction, which also constitutes the basis for this work, is the study of financial derivatives; in particular options and futures. Major financial markets in the world have seen a rapid explosion in the trading volumes of financial derivatives in the last 25 years. The public interest to these instruments have been so noticeable that derivative markets founded chronologically after traditional stock markets grew beyond major stock markets in trading volume (7). This huge interest eventually created the need for derivative pricing models. These models aim to determine whether a derivative is under-priced or over-priced in a market by calculating a theoretical price under certain assumptions.

For a typical investor in a derivatives market, the fundamental question is to determine the market price of an instrument. Derivatives as trading contracts have more complexity when compared with traditional equities and it is not always straightforward to decide on the correct amount that should be given to purchase the claim. The matter can further be complicated when the holder of the claim is allowed to use her contract at a time of her own choice in the future, as it is the case in so-called American options. In this work, our main objective will be to construct an optimal trading strategy for the holder of an American option. We will show that it is possible, when the stock price follows a discrete time and discrete state random walk, to associate certain states of the world with a trading decision which will enable the trader to use his/her option contract to capture the best expected future pay-off.

A large emphasis will be given, in this introductory chapter, to the description of the derivatives markets in general to familiarize the reader with the workings of the financial contracts traded within. We will first look at the classic definitions of options and futures in derivatives markets.

1.1 Options and Futures in Derivatives Markets

A *derivative*, as defined by Hull, is a financial instrument whose value depends on (or derives from) the values of other, more basic, underlying variables (7). As this definition implies, the conditions defining an investor's gain or loss are determined, in the case of derivatives, with other assets' or contracts' defining characteristics. In mathematical terms, the *payoff* of a derivative, that is the amount of loss or gain in a certain state of the world, is a function of the underlying asset's payoff.

Futures and options are two major types of derivatives. A *future contract* is an agreement between two parties to buy or sell an asset at a certain time in the future for a certain price. An *option contract*, on the other hand, is an agreement between two parties to have the right but not the obligation to buy or sell an asset at a certain time in the future for a certain price. This means with options an additional condition on trade is imposed, where the owner of a future contract has the obligation to honor the contract at the date of expiration while for the owner of an option contract this is not the case. For this reason, it costs the participants nothing to enter into future contracts while an investor willing to enter into an option contract will have to pay a certain amount, known as the *premium* for the option.

In principle, an option contract can be written on any form of security whose price exhibits randomness into the future, including common stocks, exchanges and commodities. The most frequent trading of option contracts are encountered in stock option markets, although the use of commodities as underlyings are historically older than the use of common stocks. This historic relation implies that the idea of option contracts is actually older than the emergence of modern stock markets. In this work, it is assumed that the option under study is a common stock option unless otherwise stated.

There is little (yet fundamental) difference, in terms of business contracts, between future and option contracts. For this reason, we will not go into much detail regarding the mechanics of futures markets. Hull [7] is an excellent source to get familiarized with the basic definitions and types of future contracts. We will instead turn our attention to options. Mathematical models of options introduce various characteristics for different contracts. It is, therefore, important to understand these specifications to have a working knowledge of the option contracts.

1.1.1 Basic Terminology & Specifications

A *call* option is the right but not the obligation to buy an asset at a certain time in the future for a previously agreed price, whereas, a *put* option is the right but not the obligation to sell an asset in a similar fashion. The agreed price is called the *strike price*. The issuer of an option contract is said to *write* the option and the date at which the right of exercising an option expires is known as the *maturity date*. When the holder of an option uses the contract, which means (s)he buys or sells the asset at the strike price, we say that the holder has *exercised* the option. In a given state of the world, the amount that the holder of the option gains or loses is captured as a function of the underlying's payoff. This is defined as the *option payoff*. In the remainder of this work, we will use S for the strike price, T for the maturity date and the real-valued function $f : E \to \mathbb{R}$ for the option payoff where E is the set of all possible states of the world.

An option whose exercise is only possible at the maturity date T is said to be a *European* type option while for the *American* type options, early exercise is allowed in the period $[0, T]$. These two types of options are known as *plain vanilla options* and they form the basis of option pricing literature. In this volume, we study the American type options. This type of option, having a time period rather than a single point in time for exercising, involves a dynamic valuation process. The holder of the option must observe the price of the underlying throughout the life of the option and must decide on a time which maximizes his earnings. This type of analysis is not present in European type options since the only consideration there is the probability distribution of the underlying in the maturity date.

Options that do not fall under the category of plain vanilla options are known as *Exotic Options*. These options are non-standard and the trading volume is relatively smaller than the plain vanilla options, but they are more complicated trading agreements. Since they are outside the scope of this thesis, the interested reader is referred to (7).

Assume that for some state of the world $x \in E$, the payoff of the underlying is captured with the variable $X(x)$. For the owner of a call option, if $X(x)$ is greater than the strike price S at the maturity date, it is meaningful for the holder to exercise the option for an immediate gain of $X(x) - S$, since the contract gives her the right to buy a unit of the underlying at the price S. Then, by selling this unit in the original market for its real market value $X(x)$, the owner can have the specified gain. If, the price of the underlying, however, is lower than S, it will not be profitable to exercise the option because the same asset is already available cheaper in the exchange market. For a call option, the payoff function corresponds to:

$$f(x) = max\{X(x) - S, 0\} = (X(x) - S)^+.$$

In the case of a put option, the condition on trade is reversed, and the owner has the right to sell the option at the maturity date. Note that this strategy is only profitable when $X(x) < S$, hence, the payoff of a put option is:

$$f(x) = max\{S - X(x), 0\} = (S - X(x))^+.$$

We say that the option is *in-the-money* if the payoff function yields a positive value, *at-the-money* if the price of the underlying is equal to the strike price and *out-of-the-money* if it is not profitable to exercise the option. The conditions for being in one of these states depends on the type of trade agreement. For a call option, the option is in the money if $X(x) > S$, at-the-money if $X(x) = S$ and out-of-the-money if $X(x) < S$. For a put option, the conditions are reversed: it is in-the-money if $X(x) < S$ and out-of-the-money if $X(x) > S$.

In the remainder of this thesis, an option is assumed to be a call option unless otherwise stated. Thus, whenever $f(x) > 0$ the option is in-the-money. If $f(x) = 0$, the option is either at-the-money or out-of-the-money. Note that these definitions are based on the option payoff at maturity, however, in the case of American options, the trader is allowed to exercise the option prior to the maturity date. The payoff of the option, in this case, can be modelled with the real valued function $f : E \times T \to \mathbb{R}$, where T is an index set representing time. We reserve the symbol $X_t(x)$ for the payoff of the underlying asset at time t and state x, and define $f_t(x)$ to be the image of $(x, t) \in E \times T$ for some $x \in E$ and $t \in T$.

1.1.2 Option Positions

The payoff of an option contract is directly related to the position of the investor into the contract. Classic finance terms *long* and *short* also apply in the case of options: An investor is said to be in a long position if (s)he has bought one option contract and in a short position if (s)he has sold one. In a portfolio setting, a long position corresponds to a positive weight in the portfolio, whereas, a short position corresponds to some negative weight.

For any position, the payoff is derived from the payoff of the unit option with an appropriate real coefficient. For a single option, the payoff of a short position will simply be $-f_t(x)$ regardless of the put/call attribute. Let C be a set defined as $C := \{c_1, c_2, \ldots, c_n\}$ where $c_i, i = 1 \ldots n$ is the number of options held or sold for the option type i defined over a total of n different options. Similarly, define $f_t^i(x)$ be the payoff of the i^{th} option contract. The payoff of the option portfolio, $f_t^P(x)$, will be the linear combination:

$$f_t^P(x) = \sum_{i=1}^{n} c_i f_t^i(x).$$

One can also introduce the profit function in a similar way. Recall that the writer of an option collects a certain amount called the premium. If we denote this amount by P^C (P^P) for call (put) options, the profit functions for call(put) options, $p_t^C(x)$ $(p_t^P(x))$, will respectively become;

$$p_t^C(x) = (X_t(x) - S)^+ - P^C \quad \text{and} \quad p_t^P(x) = (X_t(x) - S)^+ - P^P.$$

The profit function of an option portfolio is then defined similarily as a linear combination of option coefficients (or weights). Cases for $n = 1$ and $c_1 \in \{1, -1\}$ are simple cases for both put and call options and are shown in Figure 1.1.

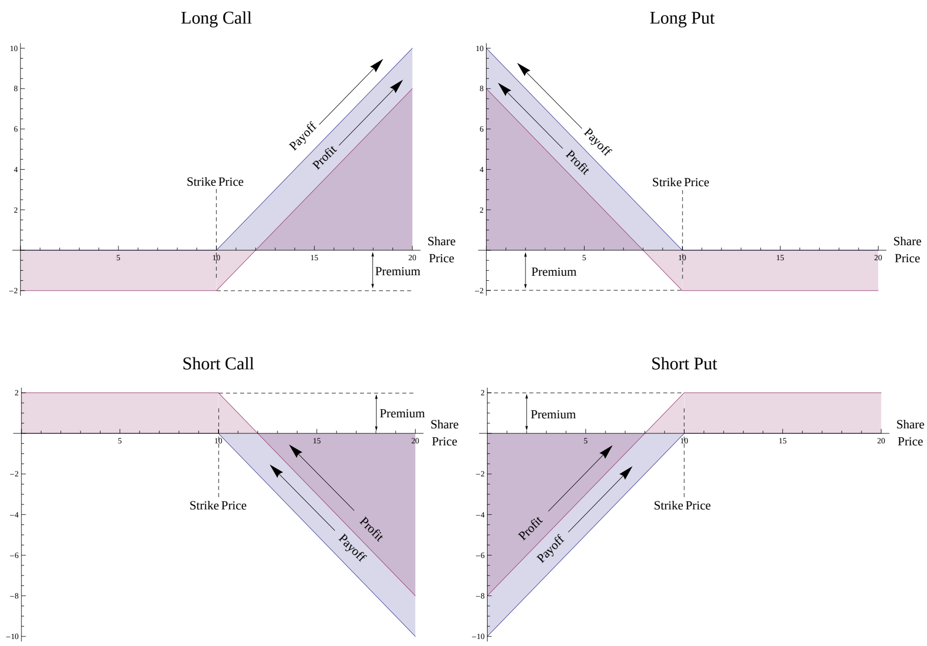

Figure 1.1: Option payoffs and profits for different types and positions.

Option portfolios involving two or more different options are generally used as trading strategies that portray the trader's beliefs on the behaviour of the underlying. Financial engineers use options of different types with different strike prices to create combinations of future payoffs that favor a particular type of price movement while ignoring other directions. Spreads, strips, straps, straddles, and strangles are widely known trading strategies that involve different option combinations. For a detailed analysis on these trading strategies, the reader is referred to [7].

1.2 Stochastic Nature of Option Pricing Models

Option pricing models historically benefited heavily from the general theory of probability. Due to the need to study a series of future payoffs in financial settings, probabilistic methods are indispensible tools of the option pricing literature.

The very concept of fundamental asset pricing equation, for this reason, involves expectations of payoffs into the future. In Cochrane's notation [4], the price p_t of an asset at a given time t is the expected value of the discounted future payoff for times $s > t$:

$$p_t = \mathbb{E}_t\big[\, m_s x_s \,\big].$$

The operator $\mathbb{E}_t\big[\,\cdot\,\big]$, here, represents the conditional expectation and is equivalent to:

$$\mathbb{E}_t\big[\,\cdot\,\big] = \mathbb{E}\big[\,\cdot \mid x_t\,\big].$$

The discount factor m_t applied to future payoffs in this equation can either be deterministic or stochastic. A very classic discounting method, which is derived from continous compounding formula, is to discount future payoffs with an exponential function of the risk-free interest rate and time. If we set:

$$m_s = e^{-r(s-t)}$$

where r is the risk-free interest rate for times $s > t$, our pricing equation becomes:

$$p_t = \mathbb{E}_t\big[\, e^{-r(s-t)} x_s \,\big].$$

Stochastic discounting factors form the basis of consumption-based pricing equations. In these models, the discount factor m_s is a function of the ratio of marginal utility functions based on current and future consumptions. For a detailed treatment of the subject, the reader is referred to [4].

In the case of options, basic pricing equations also apply. The investor is willingly faced with a situation where (s)he is asked to pay some amount for the option to obtain the right of exercise. Then, at the time of exercise, the pay-off of the option which is also a stochastic process dependent on the price process of the stock will yield a random positive pay-off which will be the input of a pricing equation. In this work, we will discount future pay-offs with a fixed discounting factor and construct a strategy based on the discounted future expectations of possible pay-offs.

1.3 Motivation

The holder of an American type option will be interested in determining the correct moment to exercise the contract. The characterization of optimal exercising rules where the decision is made with respect to the expected pay-off in the future is the main objective of this work. Our problem, therefore, is to obtain such states of the world where it is no longer meaningful for the trader to retain the rights to the underlying.

In order to make a decision, the trader must possess the knowledge of the best possible pay-off in the future, at each state of the world. Having such an information will allow the trader to compare what he can get at a particular point in time to the best he can do in the future. Delaying the decision to exercise when the best future pay-off cannot beat the immediate pay-off will clearly be suboptimal, due to the time value of money.

We will call the best future pay-off at each state of the world the value of the option. Suppose the underlying stock follows a stochastic process X_t on the state space E. For any initial state $x \in E$ and at any future time $t > 0$, we can denote the expected pay-off with $E[f(X_t)|X_0 = x]$. The maximum of such functions over the time-index set will be our value function, which will be denoted with v. In mathematical terms, v corresponds to

$$v(x) = \max_{t \in T} \mathbb{E}_x \left[\alpha^t f(X_t) \right] .$$

Our problem is to find a subset OPT of the state space E where for all $x \in OPT$ we have $v(x) = f(x)$. Note that it is not possible to have $v(x) < f(x)$ since $v(x) = \max_{t \in T} \mathbb{E}_x [\alpha^t f(X_t)] \geq \mathbb{E}_x [\alpha^0 f(X_0)] = f(x)$. Thus, for any $x \notin OPT$ we must have $v(x) > f(x)$ which means that the best expected future pay-off is larger than what is readily available. Then, the correct decision must be to wait further to exercise the contract.

In this work, we are mainly interested in determining the correct value function and the set OPT to understand when to make an optimal stopping decision. Throughout the thesis, we will have the following assumptions:

1. The stock price process follows a discrete time and discrete state random walk.
2. There is a fixed discount rate $\alpha \in (0,1)$ per period due to the time value of money.
3. The option contract under study may be written without an expiration date.

Note that there are two directions of study in this topic: the first one being the underlying probability space and stochastic process, and the second being the payoff structure that the option yields. It is possible to adjust the analysis studying different cases of random walks and different options. What is common is the optimal stopping framework and the characterization of optimal stopping criteria. In this volume, we will derive exercising regions under settings which follow both directions of study.

The remainder of the thesis is organized as follows:

In chapter 2, we will provide a brief review of the relevant literature for the valuation of American type options.

In chapter 3, the theoretical grounds of our study will be presented. Key results from the theory of Markov processes, specifically the connection between excessive functions and optimal stopping, will be given. A key result in this section forms the basis for the linear programming constructions of the problem at hand.

In chapter 4, we will develop an optimization framework towards the solution of optimal exercising under simple random walks. The solution technique based on duality and complementary slackness, which enables us to give an exact solution will be discussed. The study of the problem under this simple random walk will further be extended to a more general stock movement scenario.

In chapter 5, we will study the same problem under a geometric random walk scenario. It will be shown that the same analysis can be applied to this second case only if the discounting factor for future pay-offs restricts the movement of the price-process .

In chapter 6, we will analyze certain special cases. Two particular option strategies of interest, the spread and the strangle positions will be studied. It will be shown that these latter cases differ from regular options with altered exercise regions and the critical points identifying these regions will be derived. These examples serve as a useful tool in understanding the behaviour of the value function under varying pay-offs.

Finally, in chapter 7, we will conclude this work with a discussion of our contributions and point to some possible future research directions.

Chapter 2

Literature Review

The subject of determining correct market prices for contingent claims, in its own right, is a well documented and widely studied branch of mathematical finance. Valuation of options has consistently been in the center of the derivative pricing literature. One of the most significant distinctions within the published works in the field is the study of European versus American type options. As introduced in the previous section, we will study the valuation of American options without an expiration date written on stocks that follow discrete time and state random walks.

It is possible to find many collective texts and surveys on this subject classification. Comprehensive treatments of option pricing can be found in Hobson [6] and specifically of American options in Myneni [10]. Many now-standard topics such as the Black-Scholes option pricing model can be accessed from Hull's text on derivatives (see [7]).

Studies on option pricing started with the analysis of European type options. Bachelier [1], having provided the first analytic treatment of the problem in 1900, is considered the founder of mathematical finance. Later, Samuelson [11] provided a comprehensive treatment on the theory of warrant pricing. The subject has further been collectively developed by the contributions of Black and Scholes [2], in their famous 1973 paper, and Merton [9]. In their work, Black and Scholes show that in a frictionless and arbitrage-free market, the price of an option solves a special differential equation, a variant of the heat equation arising in physical problems. Their assumption that the stock-price follows a geometric Brownian motion has been a very key and much cited assumption.

The problem of determining correct prices for American type contingent claims was first handled by McKean upon a question posed by Samuelson (see appendix of [12]). In his response, McKean transformed the problem of pricing American options into a free boundary problem. The formal treatment of the problem from an optimal stopping perspective was later done by Moerbeke [14] and Karatzas [8], who used hedging arguments for financial justification. Wong, in a recent study, has collected the optimal stopping problems arising in the financial markets (16).

In this thesis, we attempt to provide an alternative approach to solving the pricing problem of perpetual American options when the underlying stock follows discrete time and discrete state Markov processes. Our objective will be to determine the optimal stopping region(s) for exercising the option contract. This is a relatively simpler problem compared to its continuous counterpart and allows a linear programming formulation. It is well known that the value function of an optimal stopping problem for a Markov process is the minimal excessive function majorizing the pay-off of the reward process (see [5] and [3]). The value function, then, can be obtained by solving an infinite dimensional linear programming model using duality. This approach is taken, for instance, in [13] to treat singular stochastic control problems. In a recent paper, Vanderbei and Pınar [15] use this approach to propose an alternative method for the pricing of American perpetual warrants. Under mild assumptions, they find that the optimal stopping region can be characterized with a critical threshold which leads to the decision to exercise when exceeded. In this thesis, we will mainly extend their analysis on simple random walks by providing a more general optimal stopping criterion and give a solution to the geometric random walk case.

Chapter 3

Preliminaries

The aim of this chapter is to present a set of mathematical definitions and tools to lay the groundwork for deriving rules for exercising perpetual American type options. The majority of results in this chapter are from the optimal stopping literature on stochastic processes, especially on the ramifications of the Markov hypothesis. The reader is encouraged to see [5] for a foundational yet readable treatment of Markov processes. The notation throughout the chapter is inherited from Çınlar's introductory text on stochastic processes (3).

3.1 Markov Processes on $\mathbb{R}$

Let the triplet $(\Omega, \mathcal{F}, P)$ be a probability space. We will start with the classic definitions of stochastic processes and Markov processes.

Definition 3.1.1. A stochastic process $\boldsymbol{X} = \{X_t, t \in T\}$ with the state space E is a collection of E-valued random variables indexed by a set T, often interpreted as the *time*. We say that $\boldsymbol{X}$ is a discrete-time stochastic process if T is countable and a continuous-time stochastic process if T is uncountable. Likewise, $\boldsymbol{X}$ is called a discrete-state stochastic process if E is countable and called a continous-state stochastic process if E is an uncountable set.

For any $i \in E$, we define $\mathbb{P}_i(X_t)$ to be the conditional probability distribution of the stochastic process at time t conditional on the initial state i. Similarily $\mathbb{E}_i(X_t)$ is defined to be the conditional expectation of the value of the stochastic process at time t conditional on the initial state i. Thus, we have;

$$\mathbb{P}_i(X_t) = \mathbb{P}\big[\, X(t) = x \mid X(0) = i \,\big]$$

and

$$\mathbb{E}_i(X_t) = \mathbb{E}\big[\, X(t) \mid X(0) = i \,\big].$$

In the vast literature of stochastic processes, one assumption known as the *Markov Hypothesis* is of central importance. This hypothesis gives an arbitrary stochastic process a memoryless property, where the future of the process only depends on the current state of the process. In this work we will assume, in all cases, that the stock prices follow a stochastic process with the Markov property.

Definition 3.1.2. A stochastic process $\boldsymbol{X}$ having the property,

$$\mathbb{P}\{\boldsymbol{X}(t+h) = y \mid \boldsymbol{X}(s), \forall s \leq t\} = \mathbb{P}\{\boldsymbol{X}(t+h) = y \mid \boldsymbol{X}(t)\}$$

for all $h > 0$ is said to have the Markov property. A stochastic process with the Markov property is called a Markov process.

The general definition of a Markov process permits any set S to qualify as a state space. In this study, we use discrete state Markov processes defined on $\mathbb{R}^+$ to model stock prices. Note that the stock prices having non-negative values is a valid assumption: a company with a zero stock value practically has no value in the market at all and the prices can never drop below zero since this would mean that the company pays you an additional amount for purchasing its stocks. The states that the process attains and the transition probabilities are, however, completely dependent on the scenario under study.

3.2 Potentials and Excessive Functions

The study of potentials and excessive functions is of significant importance in the optimal stopping literature. In a sense, these functions are tools to connect the underlying stochastic process to the outcomes associated with the movement of the process in time. Immediate consequences of a stochastic process as it continues its path on the sample space in time are modelled through the use of the so-called reward functions.

Definition 3.2.1. A real-valued function $g : E \to \mathbb{R}$ is called the *reward* function of a stochastic process $\boldsymbol{X}$.

A reward function defined on the states of the process represents a quantity acquired once the process enters a particular state in time. In practice, reward functions are very useful in modeling random phenomena that give some form of a payoff. Our motivation in considering reward functions, of course, comes from the need to model the payoff of an option contract depending on the payoff of the underlying stock.

The potential of a particular state of the stochastic process is defined as the expectation of all future rewards of the process $\boldsymbol{X}$ when the process initiates from this particular state. The notion of a potential is useful when the future of the stochastic process given a state needs to be quantified in terms of the reward function. The following definition captures the notion of a potential:

Definition 3.2.2. Let $g : E \to \mathbb{R}$ be a reward function defined on E. The function $Rg : E \times T \to \mathbb{R}$ defined as

$$Rg_t(i) = \mathbb{E}_i \left[\sum_{h=0}^{\infty} g(X_{t+h}) \right]$$

is called the *potential* of g.

In some applications, future rewards of a stochastic process need to be discounted by a factor $\alpha \in [0, 1]$. This is a very common case in financial applications, where possible future payoffs are discounted to today's dollars to compensate for the time value of money. With each transition in time, the value of a constant future cash flow in today's dollars decreases, thus, the amount of discount must be greater for further points in time. To capture this effect, we define α-potentials:

Definition 3.2.3. Let $g : E \to \mathbb{R}$ be a reward function defined on E and $\alpha \in (0, 1]$. The function $R^\alpha g : E \times T \to \mathbb{R}$ defined as

$$R^\alpha g_t(i) = \mathbb{E}_i \left[\sum_{h=0}^{\infty} \alpha^h g(X_{t+h}) \right]$$

is called the α-*potential* of g.

Next, we introduce the family of α-*excessive* functions which plays a key role in characterizing the value function of a stochastic process.

Definition 3.2.4. Let f be a finite-valued function defined on E and P be a transition matrix. The function $f \geq 0$ is said to be α-*excessive* if $f \geq \alpha P f$. If f is 1-*excessive*, it is simply called *excessive*.

The notion of an α-excessive function is useful when defining a particular reward function with the following property: At any given state, the associated reward at time 0 is always greater than or equal to the discounted expected value of any future reward. To see this, consider a Markov process X and the transition matrix P associated with it. Note that, by construction, we have:

$$\alpha P f(i) = E_i \left[\alpha f(X_1) \right].$$

When f is α-excessive, $f \geq \alpha P f$. By multiplying both side of this inequality with αP we get:

$$\alpha P f \geq \alpha^2 P^2 f$$

which implies $f \geq \alpha^2 P^2 f$. By repeating this procedure, we can get the following inequality for any $k \in \mathbb{N}$:

$$f(i) \geq \alpha^k P^k f(i) = E_i \left[\alpha^k f(X_k) \right]$$

which essentially implies that for a reward function with the α-excessive property, the initial reward at time 0 is greater than the discounted future expectation of reward at time t for any $t > 0$.

Note that a reward function on E need not necessarily be α-excessive though as we will later see, the value function of a stochastic process must be α-excessive. To define this value function, we will look at the optimal stopping problem of a Markov process.

3.3 Optimal Stopping on Markov Processes

Suppose we have the Markov process X, the transition matrix P and the reward function f defined on the state space E. Let $t = 0$ denote time zero, the initial period at the beginning of analysis and suppose $X_0 = i$. A valid measure of assessing a particular state's value can be defined as:

$$\sup_{\tau} E_i \left[\alpha^{\tau} f(X_{\tau})\right]$$

which gives the supremum of the discounted expected future rewards over all stopping times τ when the initial state is i. In other words, it gives the highest expected value of future rewards when the current state of the process is i. Throughout this work, we will use this measure to make a stopping decision for exercising an option. We can, thus, define the value of the game under study as a function from E to the reals, which gives the highest possible expected pay-off per state.

Definition 3.3.1. The real valued function v on E given by

$$v(i) = \sup_{\tau} E_i \left[\alpha^{\tau} f(X_{\tau})\right]$$

is said to be the *value function* of a game associated with the Markov process X and the reward function f.

Note that we need not restrict ourselves on the discounted rewards, however, due to the time value of money, this will always be the case in the upcoming applications.

In order to make a stopping decision, we need to determine the set of states, say $OPT \subset E$, such that $v(j) = f(j)$, $\forall j \in OPT$. Note that for any state with this property, it is meaningless, in terms of the expected future reward, to continue pursuing the game. Since the participant will never get a better value in the future, of course on average, the correct decision is to stop playing the game and collecting the reward. In our setting, this will correspond to exercising the option. The optimal strategy can, therefore, be characterized as to exercise the option as soon as the underlying stock process attains a value in OPT.

When working with finite state spaces, it can be guaranteed that $OPT \neq \emptyset$. However, if the state space is infinite, we need another notion of an optimal stopping set. Let $\epsilon > 0$ be an arbitrary positive real value. We define the set:

$$OPT_\epsilon = \{t \in T : v(i) - \epsilon \leq E_i \left[\alpha^t f(X_t)\right]\}$$

to be the set of all stopping times such that the supremum pay-off of state i is arbitrarily close to the expected future pay-off of state i when the process is stopped. In the majority of cases we will study, though, we will be able to characterize OPT properly whereas in some special cases, the value function can at most be ϵ-close to the pay-off available.

With these definitions from the optimal stopping literature in mind, we note that the problem of pricing a perpetual American option and determining the optimal stopping strategy is equivalent to computing a value function for the underlying stock-price process and determining the set of states where the value function is equal to the pay-off of the option. The essence of our study will be the application of the following key theorem within the option pricing framework.

3.4 The Fundamental Theorem

We close this chapter with a fundamental result from the optimal stopping literature. This result is especially useful in a discrete time and discrete space Markov setting since it allows the problem to be formulated with Linear Programming methods.

Theorem 3.4.1. *Let f be a bounded function on E. The value function v is the minimal α-excessive function greater than or equal to the pay-off function f.*

Proof. Given both in [5], p.105 and [3], p.221. □

Now, suppose that both E and T are countable. We can safely model the assertion of Theorem 3.4.1 as the following LP using the definition of an α-excessive function:

$$
\begin{array}{lll}
\min & \sum_{i \in E} v(i) & \\
\text{s.t.} & v(i) \geq f(i) & \forall i \in E \\
& v(i) \geq \alpha P v & \forall i \in E \\
& v(i) \geq 0 & \forall i \in E.
\end{array}
$$

Note that Theorem 3.4.1 requires f to be a bounded function. In our problem, this requirement will be violated most of the time since we will allow the pay-off function f grow unboundedly as time passes. In a non-discounted setting this would imply that it is never reasonable to exercise the option. This will not be the case, however, for discounted pay-offs since the discount factor will, most of the time, force the pay-off for arbitrarily large states to tend to zero. The required conditions will be given as different cases are studied. We will start with the pricing of perpetual American options under simple random walks.

Chapter 4

Pricing and Optimal Exercise Under Simple Random Walks

The aim of this chapter is to introduce a linear programming framework for modelling the optimal stopping problem for discrete-time, perpetual American type options. Relying on the fundamental concepts of the optimal stopping literature discussed in the previous chapter, we will model the problem in hand by defining decision variables corresponding to each state in a countable state space. The minimal excessive-majorant property of the value function will, then, correspond to the optimal solution of the proposed model under certain constraints. It will be shown that this model, assuming a certain stock price movement, can be solved to optimality, revealing a single point in the state space to be used as a decision point for the exercise of the option.

4.1 An Optimization Framework For Pricing Perpetual American Options

As briefly introduced in Chapter 1, the aim of the holder of a perpetual American option is to decide whether to exercise the option or wait, given a certain state of the world. In particular, (s)he would like to know if on any date, it is possible to be better-off by delaying the action (exercising the option), with the sole knowledge of the price of the stock. One way to capture this decision is to consider the expected pay-off of the option on any period starting from the initial period.

Following the notation in the previous section, let $t_0 = 0$ be the initial period and assume that the value of the stock-price process, X_t, is known and equal to $X_0 = x$ at this initial period. For any stopping time $\tau > 0$, the pay-off for the holder of the option will be $f(X_\tau)$, which corresponds to the reward function defined in Chapter 3. Let $\alpha < 1$ be an appropriate positive discount factor for any monetary amount in a single transition between two consecutive periods. The discounted expected value of the future pay-off, for the stopping time τ will be:

$$\dot{v}(x,\tau) = \mathbb{E}_x\left[\alpha^\tau f(X_\tau)\right] = \mathbb{E}\big[\,\alpha^\tau f(X_\tau) \mid X_0 = x\,\big].$$

Note that a state-dependent function $v : E \to \mathbb{R}$ can be obtained by defining the maximum of such expectations over all possible stopping times $\tau \geq 0$:

$$v(x) = \max_{\tau\geq 0}\{\ \dot{v}(x,\tau)\ \} = \max_{\tau\geq 0}\mathbb{E}_x\left[\alpha^\tau f(X_\tau)\right]. \tag{4.1}$$

This will precisely give the value function of our option trading problem, outputting the best discounted expected pay-off that the trader may get in all possible stopping times starting with the current period. Note that when $\tau = 0$, the pay-off of exercising the option will be $f(x)$, which is the initial pay-off. This implies, by definition of $v(x)$, that $v(x) \geq f(x)$, $\forall x \in E$.

If this inequality is strictly satisfied, then, there must be at least one stopping time $\dot{\tau}$ such that $\mathbb{E}_x\left[\alpha^{\dot{\tau}} f(X_{\dot{\tau}})\right] > f(x)$. Knowing this, the trader will wait rather than exercise, due to the fact that for some stopping time $\dot{\tau}$, the expected pay-off will be larger than what (s)he can get at that initial moment.

It is, therefore, important to characterize the *optimal stopping set*, $OPT = \{x \in E,\ v(x) = f(x)\}$, since for any $x \in OPT$, the stock-price process, having started in state x, never attains a certain state $\dot{x} \neq x$ with its discounted expected pay-off strictly larger than current pay-off, when stopped. In other words, it is never possible to do better than the current pay-off, in expectation. Therefore, the decision in such a state x must be to exercise the option immediately.

In what follows, we will describe a method based on linear programming and duality, for characterizing the set OPT, as well as determining the function v, which will prove to be an efficient method for solving the optimal exercise problem for perpetual American options, both under different stock-price processes and various trading positions.

With the assumption that the stock-price process under consideration is a Markov process, the fundemental result of Chapter 3 tells us that the value function v is the minimal α-excessive function greater than or equal to the pay-off function f. Let P be the state-transition matrix of a certain Markov Process X. By Definition 3.1.2, since v is α-excessive, for any $n > 0$, $n \in \mathbb{N}$, we have $v(x) \geq \alpha^n P^n v(x)$. The quantity $P^n v(x)$, here, is the regular matrix multiplication when the discrete function $v(x)$ is thought as a column vector. Note that for any stopping time $\tau > 0$, we have $\alpha^\tau P^\tau v(x) = \alpha^\tau \mathbb{E}_x \left[v(X_\tau)\right]$; thus, for $\tau = 1$, we have $v(x) \geq \alpha P v(x) = \alpha \mathbb{E}_x \left[v(X_1)\right]$.

Our aim is, then, to find the minimal discrete function v subject to the constraint system:

$$v(x) \geq f(x) \tag{4.2}$$

$$v(x) \geq \alpha P v(x) = \alpha \mathbb{E}_x \left[v(X_1)\right]. \tag{4.3}$$

Another characterization of (4.2) and (4.3) can be made using dynamic programming. At each period $t > 0$, the value $v(x)$ must be equal to $\max\{f(x), \alpha P v(x)\}$, that is, the value of the option in state x is equal to the maximum of the current pay-off and the expectation of the value function in the next period. The value function, thus, satisfies (4.2) and (4.3). Note that for $t = 1$, the value function cannot be deterministic due to the unknown behaviour of X_1 and is obtained by conditioning $v(X_1)$ on possible states reached from the initial state x at $t = 0$.

We can now define a linear programming model that, when solved to optimality, will characterize the value function v. Let v_j be a decision variable defined on the set E denoting the value of state $j\Delta x$, v be the solution vector, and the parameter f_j to be the pay-off $f(j)$ for some $j \in E$. We can, then, obtain the function $v(x)$ by solving the problem:

$$\begin{aligned} \text{(P) min} \quad & \sum_{j \in E} v_j \\ \text{s.t.} \quad & v_j \geq f_j \qquad \forall j \in E \\ & v_j \geq \alpha P v \qquad \forall j \in E. \end{aligned}$$

Note that for a finite state space E, we have a finite number of variables and a finite number of relatively simple constraints. On the other hand, it does not seem likely to model the problem using linear programming when the state space is uncountable. Nevertheless, if we restrict ourselves to countable subsets of $\mathbb{R}$, the exercising region can usually be determined under certain conditions.

We shall denote the main optimization model defined above with (P), and number the variations of this model as different cases are studied in the consequent chapters. First, we will consider a simple setting where the underlying stock follows a binomial random walk.

4.2 A Simple Discrete Random Walk on $\mathbb{R}$

Let Δx be a fixed positive integer in the interval $(0, 1]$. We will use Δx to represent any required precision level for the difference between two consecutive stock prices in two consecutive periods. We define the set $E_1 = \{j \cdot \Delta x,\ j \in \mathbb{N}\}$ to be the state space for the stock-price process. Note that E_1 constitutes a mesh on the real line with equal spacing of Δx. Let t_0 denote the beginning period of analysis and define the collection $\{X_t,\ t \in \mathbb{N}\}$ of random variables for each $X_t \in E_1$ to be the stock-price process. One very basic yet reasonable random walk on E_1 can be defined by letting:

$$X_{t+1} = \begin{cases} X_t + \Delta x & \text{w.p.} \qquad p \\ X_t - \Delta x & \text{w.p.} \qquad q = 1 - p \end{cases}$$

for each period $t \in \mathbb{N}$ and $X_t \in E_1 - \{0\}$. It is also convenient to define state 0 as an absorbing state, since once the stock price hits value 0, it is unlikely to gain some value again: the company has probably gone bankrupt! Thus, if for some $t' \in \mathbb{N}$, $X_{t'} = 0$, then for all $t > t'$, $X_t = 0$.

Let $X_0 \in E_1$ to be the initial state at the beginning of analysis and define j_0 to be $\{j : X_0 = j\Delta x\}$. Note that this set is a singleton on $\mathbb{R}$ and that j_0 has a unique value. For any fixed $t > 0$, the conditional probability mass function $\mathbb{P}\{X_t = x \mid X_0\}$ can be used to calculate the expected value of the stock price t periods into the future. If $t \leq j_0$, then, the value of the random variable $\{X_t \,|\, X_0\}$ will be in the sub-space $E_1^t = \{X_0 + j \cdot \Delta x, j \in \{-t, -(t-2), -(t-4), \ldots, (t-4), (t-2), t\}\}$ with $t+1$ possible distinct values and with a probability mass function:

$$\Phi(t, j, p) = \mathbb{P}\big[\, X_t = X_0 + j \cdot \Delta x \mid X_0 \,\big] = \binom{t}{\phi(j)} q^{t-\phi(j)} p^{\phi(j)}$$

where $\phi(j)$ is a labeling of E_1^t with the set of natural numbers. More precisely, $\phi(j) = k - 1$ if j is the k^{th} element in E_1^t. The expected value of the stock price at the t^{th} period in the future will, then, be:

$$\sum_{x_j \in E_1^t} x_j \cdot \Phi(t, j, p).$$

When $t > j_0$, however, there is a positive probability that the process enters state 0. When this happens, the probability of being in state $x_j \in E_1^t$ in exactly t transitions is reduced by the probability of visiting x_j in exactly t transitions following paths that include state 0. Moreover, the set of possible states in t transitions further reduces to the non-negative elements of E_1^t. Although it may be possible to give a closed form p.m.f for these states, this task is not undertaken here since j_0^{th} row of the t-step transition matrix A^t where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ q & 0 & p & 0 & \ldots \\ 0 & q & 0 & p & \ldots \\ 0 & 0 & q & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

necessarily gives the desired distribution function.

In general, the pay-off f is a function on the state space E. Since we will be working with a state space which is a subset of $\mathbb{R}$, we need to update the definition of f accordingly. Limited to this chapter, we define the pay-off function $f : E_1 \to \mathbb{R}$ as:

$$f(x) = \max\{0, (x - S)\}$$

where $x = j\Delta x$ for some $j \in \mathbb{N}$. Note that there is a one-to-one correspondence between $\mathbb{N}$ and E_1. Thus, we can use the substitute $f : \mathbb{N} \to \mathbb{R}$:

$$f_j = \max\{0, (j\Delta x - S)\}. \tag{4.4}$$

4.3 Pricing and Optimal Exercise Under a Simple Random Walk

Under this simple random walk on mesh E_1, problem (P) will be:

$$\begin{array}{llllll}
\text{(P1)} & min & \sum_{j\in\mathbb{N}} v_j & & & \\
 & s.t. & v_j \geq & f_j & \forall j \in \mathbb{N} & (4.5) \\
 & & v_j \geq & \alpha\,(qv_{j-1} + pv_{j+1}) & \forall j \in \mathbb{N}\setminus\{0\} & (4.6)
\end{array}$$

where the decision variable v_j corresponds to the numerical value of the value function at $j\Delta x$, that is, $v_j = v(j\Delta x)$ and the right hand side parameter f_j corresponds to the numerical value of the pay-off function at $j\Delta x$, that is $f_j = f(j\Delta x)$. Note that for $j = 0$, we have the constraint $v_0 \geq \alpha \cdot v_0$ which will be a redundant constraint due to the choice of α.

Problem (P1) will stand for the optimal stopping problem of a perpetual American option contingent on a stock obeying the simple random walk on E_1. The optimal solution to (P1), if it exists, will reveal crucial information for the trader since by observing the gap between v and f, the trader will be able to decide on which states of the world to exercise the option and on which states to delay the action. Our aim here, therefore, will be to solve (P1) to optimality and characterize the states to take action, that is to obtain a set OPT such that OPT$= \{j \in \mathbb{N} : v_j = f_j\}$.

Our solution method relies heavily on LP-duality and complementary slackness conditions. For this reason, we provide the dual problem (D1) to (P1), and the associated CS conditions here. Consider the dual problem:

$$
\begin{array}{llrcll}
\text{(D1)} & max & \sum_{j=0}^{\infty} f_j \cdot y_j & & & \\
& s.t. & y_0 - \alpha q z_1 & = & 1 & (4.7) \\
& & y_1 + z_1 - \alpha q z_2 & = & 1 & (4.8) \\
& & y_j - \alpha p z_{j-1} + z_j - \alpha q z_{j+1} & = & 1, \quad j \geq 2 & (4.9) \\
& & y_j & \geq & 0, \quad j \geq 0 & (4.10) \\
& & z_j & \geq & 0, \quad j \geq 1. & (4.11)
\end{array}
$$

The dual problem (D1) has two sets of decision variables y_j and z_j that correspond to constraints (4.5) and (4.6) in (P1), respectively. Constraints 4.9 in (D2) results from the specific structure of (4.6) in (P1). The reader can easily verify that (4.7) and (4.8) cannot be included in (4.9) due to the existence of the absorbing state 0. Consequently, the complementary slackness conditions for the pair (P1)-(D1) are:

$$(f_j - v_j) \cdot y_j = 0, \quad j \geq 0 \tag{4.12}$$

$$\big(\alpha(pv_{j+1} + qv_{j-1}) - v_j\big) \cdot z_j = 0, \quad j \geq 1 \tag{4.13}$$

$$v_0 \cdot (1 - y_0 + \alpha q z_1) = 0 \tag{4.14}$$

$$v_1 \cdot (1 - y_1 - z_1 + \alpha q z_2) = 0 \tag{4.15}$$

$$v_j \cdot \big(1 - y_j - z_j + \alpha(p z_{j-1} + q z_{j+1})\big) = 0, \quad j \geq 2. \tag{4.16}$$

Let us call the optimal solution to (P1) v^*, which is an infinite dimensional real-valued vector indexed with the set of natural numbers. Finite approximations of (P1) suggest that there exists a certain $j^* \in \mathbb{N}$ such that:

$$
\begin{array}{lclcll}
v_0^* & = & f_0, & & & \\
v_j^* & = & \alpha(pv_{j+1}^* + qv_{j-1}^*) & > & f_j & \forall 0 < j < j^*, \\
v_j^* & = & f_j & > & \alpha(pv_{j+1}^* + qv_{j-1}^*) & \forall j^* \leq j.
\end{array}
$$

Indeed, it can be shown that there is a threshold value $X^* \in E_1$ and a $j^* \in \mathbb{N}$ associated with it such that a trader, who has to decide whether to exercise the option or not on the realization of the stock price, always exercises the option for stock prices higher than X^*. To prove this result, we will first make an educated guess of a candidate value function and show that our guess is feasible to (P1) by proving a feasibility lemma, under a certain condition. Then, we will show that the candidate solution satisfies the complementary slackness conditions derived above. Finally, we will show that our guess and the corresponding dual variables do not produce a duality gap and that strong duality holds, which will imply that our candidate indeed solves (P1) to optimality. First we make a simplifying assumption:

Assumption. There exists a $j_S \in \mathbb{N}$ such that $S = j_S \Delta x$.

Note that $j_S = \max\{k \in \mathbb{N} : \; f_k = 0\}$. We know that, since $S > 0$, j_S must also be strictly positive. We will need the quantities

$$\xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p} \qquad \xi_+ = \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p}$$

$$\zeta_+ = 1/\xi_- \;=\; \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q} \qquad \zeta_- = 1/\xi_+ \;=\; \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q}$$

and the characterization of a critical point given in terms of ξ_- and ξ_+:

$$j^* \;=\; \begin{cases} j_S + 1 & \text{if} \quad \left[\frac{f_{j_S+2}}{f_{j_S+1}} < \frac{\xi_+^{j_S+2} - \xi_-^{j_S+2}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}}\right] \\ \max\left\{k : f_k \, \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^{k} - \xi_-^{k}} > f_{k-1}\right\} & \text{otherwise} \end{cases} \tag{4.17}$$

in our formulation of the theorem. It will turn out that j^*, taking one of the two possible values, will turn out to be the defining integer of the critical state we are seeking. The following lemma proves the existence of j^*.

Lemma 4.3.1. *There exists a finite number j^* defined as in 4.17.*

Proof. Since j_S is finite, we either have

$$\frac{f_{j_S+2}}{f_{j_S+1}} < \frac{\xi_+^{j_S+2} - \xi_-^{j_S+2}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}}$$

or its negation. In the first case there is nothing to prove since $j^* = j_S + 1$. Now suppose the LHS above is greater than or equal to the RHS. Note that the ratio

$\frac{f_k}{f_{k-1}}$ is only defined for $k > j_S + 1$. Whenever this ratio is defined we have:

$$\frac{f_k}{f_{k-1}} = \frac{k\Delta x - S}{(k-1)\Delta x - S}$$

Clearly, this is a decreasing sequence in $\{(j_S+2), (j_S+3,)\ldots\}$ whose limit is 1. On the other hand the sequence:

$$\left\{\frac{\xi_+^k - \xi_-^k}{\xi_+^{k-1} - \xi_-^{k-1}}\right\},\ k > j_S + 1$$

decreases to ξ_- as $k \to \infty$ since $\xi_- > 1$ and $\xi_+ < 1$. But we have:

$$\frac{f_{j_S+2}}{f_{j_S+1}} \geq \frac{\xi_+^{j_S+2} - \xi_-^{j_S+2}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}}$$

which implies that there exists only finitely many elements of $\{(j_S+2), (j_S+3), \ldots\}$ such that

$$\frac{f_k}{f_{k-1}} \geq \frac{\xi_+^k - \xi_-^k}{\xi_+^{k-1} - \xi_-^{k-1}}$$

since the LHS converges to 1 while the RHS converges to $\xi_- > 1$. The result follows then, since the maximum of a finite set always exists. $\square$

Next, we will define a particular function from the set $\mathbb{N}$ to $\mathbb{R}$ using this newly defined j^*. Let v^* be a function on $\mathbb{N}$ given by:

$$v_j^* = \begin{cases} 0 & j = 0 \\ f_{j^*}\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}, & 0 < j < j^* \\ f_j & j^* \leq j. \end{cases} \tag{4.18}$$

We will first prove a feasibility result which shows that this function can be feasible to (P1) provided that j^* satisfies a certain condition.

Lemma 4.3.2. *v^* is feasible for (P1) if and only if p and α are chosen such that*

$$j^* + 1 \geq \frac{S}{\Delta x} + \frac{\alpha(p-q)}{(1-\alpha)}.$$

Proof. First we show that this condition is necessary for v^* to be feasible. Suppose v^* is feasible to (P1). Then, v^* satisfies (4.6). Let $k = j^* + i$ where $i = 1, 2, 3, \ldots$. By definition of v^*, we have $v_k = f_k$. Thus, the system of inequalities

$$f_{j^*+i} \geq \alpha(pf_{j^*+i+1} + qf_{j^*+i-1}), \quad i = 1, 2, 3, \ldots$$

must hold since v^* is feasible. Note that by definition of j^*, $f_{j^*+i} > 0$ for all $i = 1, 2, 3, \ldots$. Then $f_{j^*+i} = (j^* + i)\Delta x - S$. By substituting the value of f_{j^*+i} into the above system and rearranging the terms we obtain:

$$j^* \geq \frac{S}{\Delta x} + \frac{\alpha(p-q)}{1-\alpha} - i, \quad i = 1, 2, 3, \ldots$$

Clearly, the right hand side is maximized when $i = 1$ which also implies that the given condition holds.

Now, we will show that the given condition is also sufficient for the feasibility of v^*. First suppose that $j^* = j_S + 1$ and consider (4.5). For $j \geq j_S + 1$ these are satisfied trivially by definition of v^*. So, suppose $0 < j < j_S + 1$. Since $f_{j_S+1} > 0$ and we have

$$\frac{\xi_+^j - \xi_-^j}{\xi_+^{j_S+1} - \xi_-^{j_S+1}} \in (0, 1)$$

for $0 < j < j_S + 1$, the constraints are also satisfied in this region because $f_j = 0$ for $j < j_S + 1$. The constraint $v_0 \geq f_0$ is also satisfied trivially by definition of v^* since both sides are equal to zero in this case. Thus, v^* satisfies (4.5). Now, consider (4.6) and suppose $0 < j < j_S + 1$. We claim that, regardless of the value of j^*, v_j^* for $0 < j < j^*$ is a solution to the second order difference equation:

$$v_j - \alpha(pv_{j+1} + qv_{j-1}) = 0, \quad 0 < j < j^* \tag{4.19}$$

with the boundary conditions:

$$v_0 = 0$$
$$v_{j^*} = f_{j^*}$$

and therefore satisfies (4.6) with equality . To see this, take an arbitrary j such that $0 < j < j^*$ and substitute into (4.19). After some algebra, it is easy to show that v^* indeed solves the difference equation for $0 < j < j^*$ (see A.1). Suppose, now, that $j \geq j_S + 1$. We have:

$$j^* \geq \frac{S}{\Delta x} + \frac{\alpha(p-q)}{1-\alpha} - 1 \geq \frac{S}{\Delta x} + \frac{\alpha(p-q)}{1-\alpha} - i, \quad i = 1, 2, 3, \ldots$$

which is equivalent to (4.6) with $v_j^* = f_j$ for $j > j^*$ as we have deduced in the first part of the proof. The only remaining point to be checked, then, is $j = j_S + 1$.

Consider the solution of (4.19) extended to all natural numbers and let ω_j denote this function. Then, we have:

$$\omega_{j_S+1} = \alpha p \omega_{j_S+2} + \alpha q \omega_{j_S}.$$

Note that $\omega_{j_S+1} = f_{j_S+1} = v^*_{j_S+1}$ and ω_{j_S} coincides with $v^*_{j_S}$. Then, (4.6) at $j = j_S + 1$ is satisfied if and only if $f_{j_S+2} \leq \omega_{j_S+2}$. But this is already satisfied since, by the choice of j^*, we have:

$$\frac{f_{j_S+2}}{f_{j_S+1}} < \frac{\xi_+^{j_S+2} - \xi_-^{j_S+2}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}}$$

and

$$\omega_{j_S+2} = f_{j_S+1} \left(\frac{\xi_+^{j_S+2} - \xi_-^{j_S+2}}{\xi_+^{j_S+1} - \xi_-^{j_S+1}} \right).$$

Thus, v^* satisfies (4.6) $\forall j \in \mathbb{N} \setminus \{0\}$. This concludes that v^* is feasible to (P1) when $j^* = j_S + 1$. Now, we look at the case where $j^* \neq j_S + 1$. Then, by definition of j^* we have:

$$j^* = \max \left\{ k : f_k \frac{\xi_+^{k-1} - \xi_-^{k-1}}{\xi_+^k - \xi_-^k} > f_{k-1} \right\}.$$

Note that by the above discussion, which holds regardless of the value of j^*, v^* satisfies (4.6) except for the point $j = j^*$. But we know that (4.6) at $j = j^*$ is satisfied if and only if

$$f_{j^*+1} \leq \omega_{j^*+1} = f_{j^*} \left(\frac{\xi_+^{j^*+1} - \xi_-^{j^*+1}}{\xi_+^{j^*} - \xi_-^{j^*}} \right).$$

Suppose that this does not hold. Then, we could not have picked j^* because $j^* + 1$ also satisfies the selection criterion, which clearly is a contradiction. The above inequality must hold then, which also implies that the (4.6) hold for all values of $j \in \mathbb{N} \setminus \{0\}$. To complete the proof, we need to show (4.5) are also satisfied with this second value of j^*. For $j \geq j^*$, they are satisfied with $v_j^* = f_j$ by definition. Then, suppose $j < j^*$. The definition of j^* implies that $j^* > j_S$. When $j \leq j_S$, the constraints are satisfied with

$$f_{j^*} \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}} > 0$$

since both terms on the left are strictly positive. In addition, when $j = 0$ we have $v_0^* = 0 = f_0$. Thus, the only region to check is $j_S < j < j^*$. To show $v_j^* \geq f_j$ in this region, we look at the difference:

$$\begin{aligned} D = v_j^* - f_j &= (j^*\Delta x - S)\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}} - (j\Delta x - S) \\ &= j^*\Delta x\left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right) - j\Delta x + S\left(1 - \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right) \\ &= \left[j^*\left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right) - j\right]\Delta x + S\left(1 - \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right). \end{aligned}$$

If D is non-negative, we are done. Note that the second term above is always strictly positive. Then, it suffices to show that if the first term is negative, it's absolute value is less than the second term. In mathematical terms, we need:

$$\left[j - j^*\left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right)\right]\Delta x < S\left(1 - \left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right)\right).$$

By the choice of j^*, for any j in the region $j_S < j < j^*$, we have:

$$\frac{f_j^*}{f_j} = \frac{j^*\Delta x - S}{j\Delta x - S} > \frac{\xi_+^{j^*} - \xi_-^{j^*}}{\xi_+^j - \xi_-^j}.$$

Rearranging the terms in this inequality, we can obtain the desired requirement above which implies that $D > 0$. Then, v^* satisfies constraints (4.5) for any $j \in \mathbb{N}$. We can, thus, conclude that v^* is feasible to (P1) if the given condition holds, which completes the proof. □

We are now ready to present the main result of this chapter. The following theorem establishes that our guess solves (P1) to optimality, which in turn will imply that j^* is the critical value we seek for optimal exercise.

Theorem 4.3.1. *v^* solves (P1) to optimality if and only if*

$$j^* + 1 \geq \frac{S}{\Delta x} + \frac{\alpha(p - q)}{(1 - \alpha)}. \tag{4.20}$$

Proof. Suppose v^* solves (P1) to optimality. Then, v^* is feasible, which implies that the given condition holds by Lemma 4.3.2. The condition, therefore, is necessary for v^* to be optimal. We will show that it is also sufficient for optimality. Again, by Lemma 4.3.2, if the given condition holds, than v^* is feasible. For optimality, v^* must satisfy the CS conditions (4.12)-(4.16). To show this, we need to know the corresponding solution vector (y^*, z^*) to (D1). Consider two functions $y^* : \mathbb{N} \to \mathbb{R}$ and $z^* : \mathbb{N} \setminus \{0\} \to \mathbb{R}$ defined as:

$$y_j^* = \begin{cases} 1 + \alpha q z_1^* & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1}^* & j = j^* \\ 1 & j^* < j \end{cases} \tag{4.21}$$

$$z_j^* = \begin{cases} \left(1 - \frac{\zeta_-^{j^*-1}}{\zeta_-^{j^*} - \zeta_+^{j^*}} \zeta_+^j - \frac{\zeta_+^{j^*-1}}{\zeta_+^{j^*} - \zeta_-^{j^*}} \zeta_-^j \right) \Big/ (1-\alpha) & 0 < j < j^* \\ 0 & j^* \leq j \end{cases} \tag{4.22}$$

First we need to check whether they are feasible. Constraint (4.7) is trivially satisfied with equality for the values of y^* at 0 and z^* at 1. To show (4.8), we will make the following remark:

Remark. For $0 < j < j^*$, z_j^* is a solution to the second order difference equation:

$$z_j - \alpha(p z_{j-1} + q z_{j+1}) = 1, \quad 0 < j < j^* \tag{4.23}$$

with the boundary conditions:

$$z_0 = 0$$
$$z_j^* = 0.$$

Having this in mind, it is easy to show that (4.8) and (4.9) are satisfied. We skip (4.10) for the moment and consider (4.11). For $j \geq j^*$, $z_j^* = 0$, thus there is nothing to prove. Now, suppose that $0 < j < j^*$. Assume, to the contrary, that there exists a $0 < j' < j^*$ such that $z_{j'} < 0$. Then, there must be a $k \in \{1, \ldots, j^* - 1\}$ such that $z_k < 0$ and that z_k is a local minimum. That is, $z_k \leq z_{k+1}$ and $z_k \leq z_{k-1}$. If j' is the index of the local minimum, that is $j' = k$, we have:

$$\begin{aligned} z_{j'} &= \alpha p z_{j'-1} + \alpha q z_{j'+1} + 1 \\ &\geq \alpha p z_{j'} + \alpha q z_{j'} + 1 \\ &= \alpha z_{j'} + 1 \end{aligned}$$

which implies that $z_{j'} \geq 1/(1-\alpha)$. Clearly this contradicts with $z_{j'}$ being negative. If j' is not the index of the local minimum then we either have $z_{j'-1} > z_{j'} > z_{j'+1}$ or $z_{j'-1} < z_{j'} < z_{j'+1}$. Note that due to the boundary conditions, we must encounter at least one local minimum as we change j' towards 0 or j^* which will again lead to a contradiction. Hence, $z_j \geq 0$ for all $0 < j < j^*$, which means (4.11) is satisfied with z_j^*. Since we have established that $z_j^* \geq 0$, it is also easy to verify the non-negativity of y_j by using values of z_1 and z_{j^*-1} on (4.10). We can, thus, conclude that the vector pair (y^*, z^*) is a feasible solution to (D1).

We are now ready to check whether the primal-dual solution pair $(v^*)-(y^*, z^*)$ satisfies the complementary slackness conditions. We will handle each one of the five conditions one by one. First consider (4.12). For $j = 0$ we have $f_0 = v_0^* = 0$, for $0 < j < j^*$ we have $y_j^* = 0$ and for $j \geq j^*$ we have $v_j^* - f_j = 0$. Thus (4.12) is satisfied with the pair $(v^*) - (y^*, z^*)$. Now consider (4.13). For $j \geq j^*$ we have $z_j^* = 0$ and for $0 < j < j^*$ we have $v_j^* - \alpha(p v_{j+1}^* + q v_{j-1}^*) = 0$. Since $v_0^* = 0$, (4.14) is immediately satisfied. Also, by (4.7) and the value of y_1^*, (4.15) is satisfied. Finally, consider (4.16). For $j < j^*$ we have $y_j^* = 0$ and $z_j^* - \alpha(p z_{j-1}^* + q z_{j+1}^*) = 1$ which, when used together, yields $1 - y_j^* - z_j^* + \alpha(p z_{j-1}^* + q z_{j+1}^*) = 0$. For $j > j^*$ we have $y_j^* = 1$ and $z_j^* = 0$ setting $1 - y_j^* - z_j^* + \alpha(p z_{j-1}^* + q z_{j+1}^*)$ again to 0. When $j = j^*$, the left hand side of (4.16) will again be 0 since $y_{j^*}^* = 1 + \alpha p z_{j^*-1}^*$ and $z_{j^*}^*$, $z_{j^*+1}^*$ are both 0. Thus, it follows that the solution pair $(v^*) - (y^*, z^*)$ satisfies the CS conditions (4.12)-(4.16).

It remains to show that the given solutions do not produce a duality gap, that is:

$$\sum_{j \in \mathbb{N}} v_j^* = \sum_{j \in \mathbb{N}} f_j \cdot y_j^* \,.$$

Consider the tails of the both sides of the equality. For $j > j^*$ we have $v_j^* = f_j$ and $y_j^* = 1$. Thus, $\sum_{j>j^*} v_j^* = \sum_{j>j^*} 1 \cdot f_j = \sum_{j>j^*} y_j^* \cdot f_j$. For this reason, it suffices to show that:

$$\sum_{j=0}^{j^*} v_j^* = \sum_{j=0}^{j^*} f_j \cdot y_j^*.$$

The algebra leading to the desired result is cumbersome, thus, we will break it down to a few steps and provide details in the appendix. For some useful properties relating the roots ξ_+, ξ_-, ζ_+, ζ_-, please refer to A.2.

It is possible,for instance, to write $z_{j^*-1}^*$ using these relations (see A.3) as:

$$z_{j^*-1}^* = \Big(\frac{1}{1-\alpha}\Big) \cdot \left(1 - \frac{\xi_+^{j^*} - 1}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_- + \frac{\xi_-^{j^*} - 1}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_+\right).$$

By using the finite summation formula for the power series, we get:

$$\begin{aligned}
\sum_{j=0}^{j^*} v_j^* &= \sum_{j=0}^{j^*-1} f_{j^*} \cdot \frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}} + f_{j^*} \\
&= f_{j^*}\left(\frac{1}{\xi_+^{j^*} - \xi_-^{j^*}} \sum_{j=0}^{j^*-1} \big(\xi_+^j - \xi_-^j\big) + 1\right) \\
&= f_{j^*}\left(\frac{1}{\xi_+^{j^*} - \xi_-^{j^*}} \left(\frac{1-\xi_+^{j^*}}{1-\xi_+} - \frac{1-\xi_-^{j^*}}{1-\xi_-}\right) + 1\right) \\
&= f_{j^*}\left(\frac{1}{\xi_+^{j^*} - \xi_-^{j^*}} \left(\frac{(1-\xi_-)(1-\xi_+^{j^*}) - (1-\xi_+)(1-\xi_-^{j^*})}{(1-\xi_+)(1-\xi_-)}\right) + 1\right) \\
&= f_{j^*}\left(\frac{1}{\xi_+^{j^*} - \xi_-^{j^*}} \left(\frac{-\xi_+^{j^*} + \xi_-^{j^*} + \xi_-\xi_+^{j^*} - \xi_- - \xi_+\xi_-^{j^*} + \xi_+}{1 - (\xi_+ + \xi_-) + \xi_+\xi_-}\right) + 1\right) \\
&= f_{j^*}\left(\frac{-1}{1 - \frac{1}{\alpha p} + \frac{q}{p}} \left(\frac{\xi_+^{j^*} - \xi_-^{j^*} - (\xi_-\xi_+^{j^*} - \xi_-) + (\xi_+\xi_-^{j^*} - \xi_+)}{\xi_+^{j^*} - \xi_-^{j^*}}\right) + 1\right) \\
&= f_{j^*}\left(\frac{-\alpha p}{\alpha - 1}\left(1 - \frac{\xi_+^{j^*} - 1}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_- + \frac{\xi_-^{j^*} - 1}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_+\right) + 1\right) \\
&= f_{j^*}\big(\alpha p z_{j^*-1}^* + 1\big) = f_{j^*} y_{j^*}^* \\
&= \sum_{j=0}^{j^*} f_j \cdot y_j^*
\end{aligned}$$

which establishes that strong duality holds between (P1) and (D1). Note that the last equality holds since $y_j^* = 0$ for $0 < j < j^*$ and $f_0 = 0$. Hence, the solution pair (v^*)-(y^*, z^*) is an optimal pair to (P1)-(D1). □

We can now use Theorem 3.4.1 to conclude that the holder of the option should exercise when the realized stock price at any period is larger than or equal to $j^*\Delta x$. In a discrete-time and discrete-state setting the solution to (P1) will yield a minimal function from the set of natural numbers to the set of real numbers with the two additional properties as discussed in Section 3. Then, by Theorem 4.3.1 and 3.4.1, v^* must be the valuation function we are interested in. Note that the only requirement of Theorem 3.4.1, that is the pay-off function f must be a bounded function, seems not to be satisfied in our problem setting. The pay-off function of the simple random walk discussed here allows arbitrarily large pay-offs and thus is not bounded on the state space E_1. However, we are interested in the existence of a maximum taken with respect to the time index. In other words, we are trying to obtain a certain measure for a given state which will dictate the maximum expected pay-off that can be available in time. The constant discounting factor α ensures that the sequence,

$$\left\{\mathbb{E}_x\left[\alpha^t f(X_t)\right], \forall t \in \mathbb{N}\right\}$$

for a given state $x \in E_1$ converges to 0 as $t \to \infty$ since the growth of the pay-off function is only linear compared to the exponential decay resulting from the discount factor. It is therefore not possible to expect arbitrarily large pay-offs in today's dollars since a very distant pay-off in the future will have an equivalent close to zero. The initial state may still be arbitrarily large but we can say that the gap between $v(x)$ and $f(x)$ is bounded. Note that due to this additional property of the simple random walk model the existence of $v(x)$ is ensured. This does not mean, however, that an exercise region always exists with the bounded gap property. Theorem 4.3.1 shows that the gap is actually 0 for certain states in E_1 which is enough for characterizing the states to exercise the claim.

4.4 A Visual Representation of Theorem 4.3.1

The results introduced in the previous section are best appreciated when displayed visually. Consider the simplest case and set $\Delta x = 0.1$, $p = 0.50$, $\alpha = 0.999$. Figure 4.1 clearly shows that the value function calculated with the given parameters coincides with the payoff function for $j \geq j^* = 112$. Note that for $0 < j < 112$ it is not optimal since the value of the option exceeds the pay-off available for the share price $j\Delta x$. The dual variables y^* and z^* are shown in Figure 4.2.

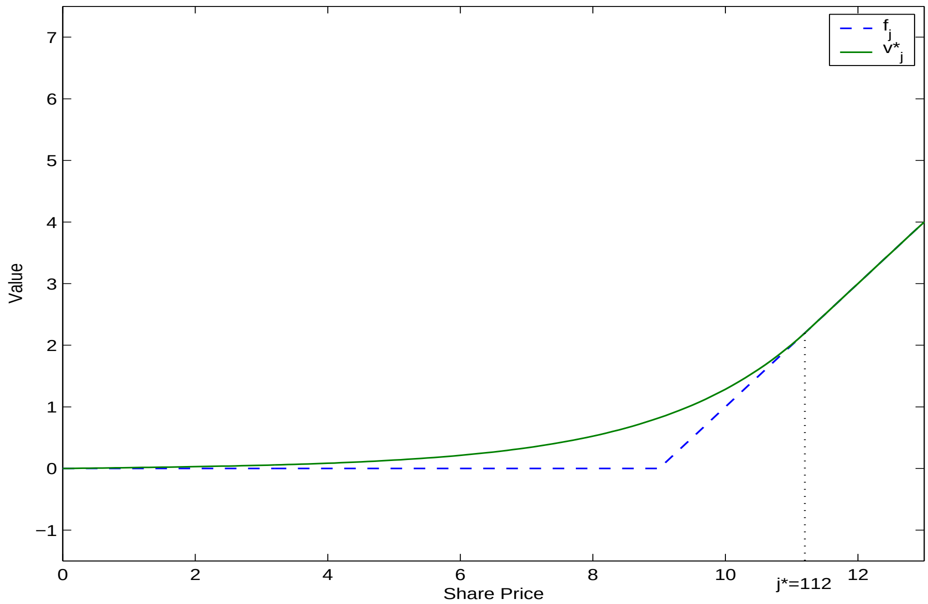

Figure 4.1: Plot of v_j^* and f_j versus stock price when $\Delta x = 0.1$, $p = 0.50$, $\alpha = 0.999$.

Next, we shall consider a hypothetical case in which the critical state is miscalculated. Suppose that we were unable to determine the critical state correctly and instead chose a $j' < j^*$. The resulting value function is shown in Figure 4.3. We will, then, exercise the option once the price of the stock hits the value $j'\Delta x$ which is less than $j^*\Delta x$. Note that $v'(j') < v^*(j')$ which implies that at that moment in time, there exists another strategy with a greater expected value of some future cash-flow. We will, therefore, miss the chance to choose a better strategy due to our miscalculation. It also should be noted that in technical terms, this case corresponds to $v'(j)$ being non-excessive. In other words, the second set of constraints in our optimization model are not satisfied in this case. This result is also consistent with the fact that $v^*(j)$ is the minimal excessive function greater than the payoff function f because $v'(j)$ cannot be the function describing the optimal stopping strategy since it is not excessive.

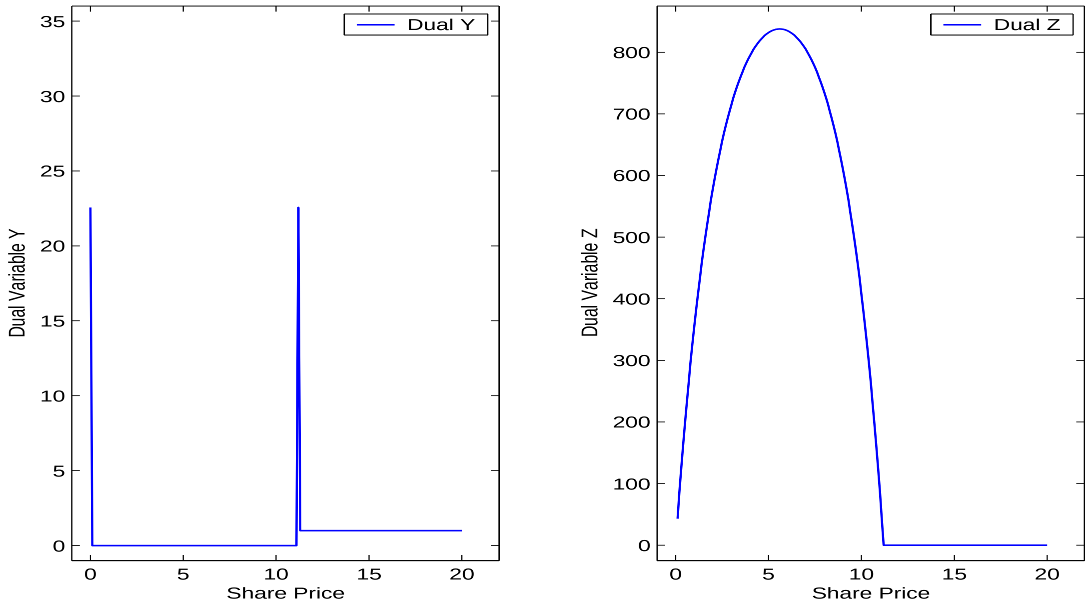

Figure 4.2: Plot of the dual variables y^* and z^* versus stock price when $\Delta x = 0.1$, $p = 0.50$, $\alpha = 0.999$.

The second case, where we believe that the optimal value to exercise is higher than the suggested value $j^*\Delta x$, is shown in Figure 4.4. This strategy will also be sub-optimal because it leads to an additional amount of waiting time after the stock hits $j^*\Delta x$. In fact, the value function given in this case is even meaningless because on a certain region, the option's reported value is less than the payoff of the option. This corresponds to $v'(j)$ not being a majorant function over $f(j)$ in our analysis. That is, with this false belief, we will have a value function which fails to be larger than the payoff function for a set of points in our range and it will violate the first set of constraint in our LP formulation. The fact that $v'(j)$ is not an optimal strategy, therefore, follows from the infeasibility of this solution.

It will also be useful to see how the optimal exercise region is affected with varying p and α. Figure 4.5 shows a series of plots calculated with different values of the discount factor α. For a fixed α, the reader will observe that the critical point of exercise increases significantly when $p > 0.5$. This is a direct consequence of the lower bound of j^* provided in (4.20) which, to recall, was

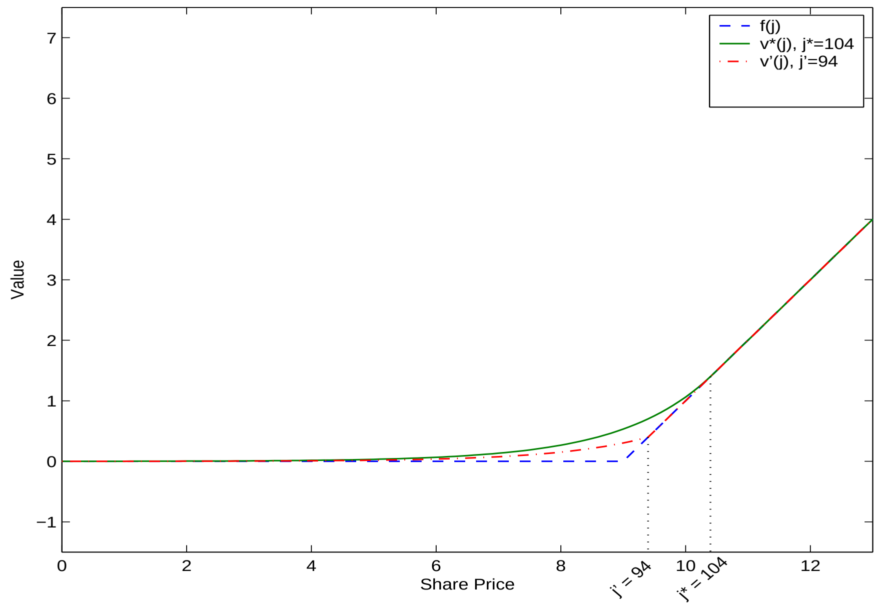

Figure 4.3: Plot of v_j^* and f_j when j^* is chosen too small.

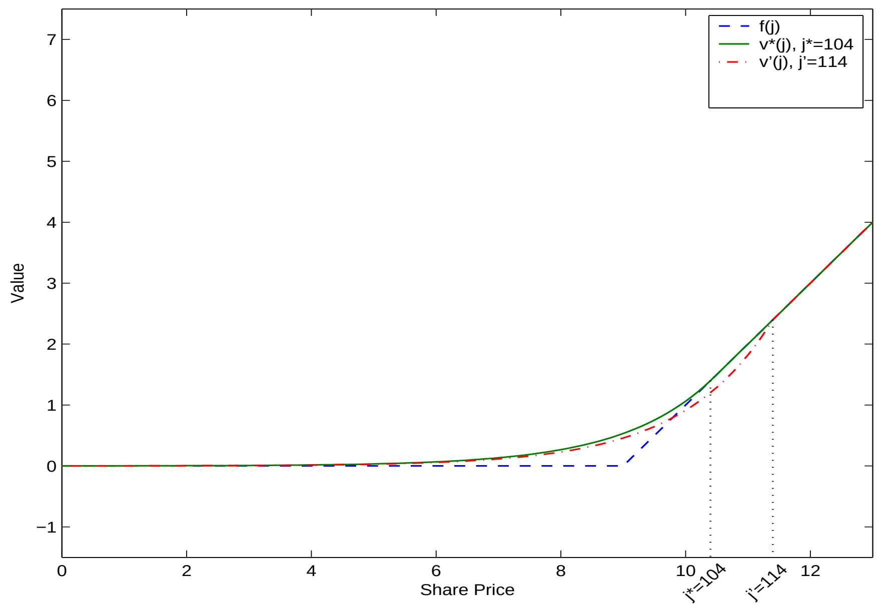

Figure 4.4: Plot of v_j^* and f_j when j^* is chosen too large.

especially binding for $p > 0.5$. On the other hand, as we decrease p, the critical point of exercise tends to the value $\min_{j\in\mathbb{N}}\{f_j : f_j > 0\}$, which is the least possible positive pay-off. The intuition is as follows: the backward probability is so high that the expected value of the discounted positive future pay-offs can never be better than the least possible positive pay-off. Thus, we will need to exercise as soon as we see a positive pay-off.

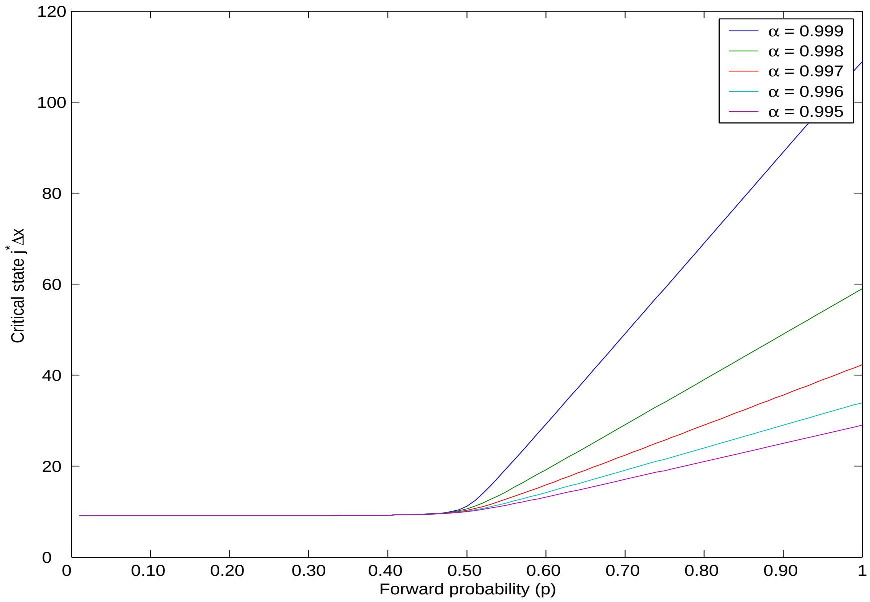

Figure 4.5: Critical point of exercise as a function of the forward probability p and the discount factor α.

4.5 Extending the Simple Random Walk Case

The aim of this section is to study the same optimal stopping problem under a more generalized random walk. Suppose that we no longer require $p + q = 1$, but instead we allow the stock-price process X_t on E_1 to stay the same at each transition with a positive probability. Assuming that $p + q < 1$, if the price of the stock at some future date t is X_t, then the price at the next period, X_{t+1}, will either be $X_t + \Delta x$, $X_t - \Delta x$ or X_t, staying unchanged, with probabilities p,q

and $1-(p+q)$, respectively. That is,

$$X_{t+1} = \begin{cases} X_t + \Delta x & \text{w.p.} \quad p \\ X_t & \text{w.p.} \quad 1-(p+q) \\ X_t - \Delta x & \text{w.p.} \quad q \end{cases}$$

We will repeat our analysis for this generalized version of the simple random walk. Let, similarly, v and f be the value and pay-off functions respectively as defined in (4.1) and (4.4). Our aim is to derive a similar closed formula for the value function which will identify the optimal stopping region. Note that the transition matrix is now:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ q & 1-p-q & p & 0 & \dots \\ 0 & q & 1-p-q & p & \dots \\ 0 & 0 & q & 1-p-q & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Thus, $\alpha P v_j$ will be equal to

$$\alpha(pv_{j+1} + (1-p-q)v_j + qv_{j-1}), \quad \forall j \in \mathbb{N} \setminus \{1\}$$

and (4.6) in our original formulation (P1) needs to be replaced with

$$v_j \geq \alpha(pv_{j+1} + (1-p-q)v_j + qv_{j-1}) \quad \forall j \in \mathbb{N} \setminus \{0\}. \tag{4.24}$$

The linear program to solve, then, will be :

$$\text{(P2)} \qquad \begin{array}{rlll} min & \sum_{j \in \mathbb{N}} v_j & & \\ s.t. & v_j \geq & f_j & \forall j \in \mathbb{N} \\ & v_j \geq & \alpha(pv_{j+1} + (1-p-q)v_j + qv_{j-1}) & \forall j \in \mathbb{N} \setminus \{0\} \end{array}$$

with its dual:

$$\text{(D2)} \qquad \begin{array}{rl} max & \sum_{j=0}^{\infty} f_j \cdot y_j \\ s.t. & y_0 - \alpha q z_1 \quad = 1 \end{array}$$

$$\begin{array}{rcll} y_1 + (1-\alpha+\alpha p+\alpha q)z_1 - \alpha q z_2 & = & 1 & \\ y_j - \alpha p z_{j-1} + (1-\alpha+\alpha p+\alpha q)z_j - \alpha q z_{j+1} & = & 1 & \forall j \geq 2 \\ y_j & \geq & 0 & \forall j \geq 0 \\ z_j & \geq & 0 & \forall j \geq 1 \end{array}$$

and the CS conditions:

$$(f_j - v_j)\cdot y_j = 0 \quad \forall j \geq 0 \tag{4.25}$$

$$\big(\alpha p v_{j+1} + (\alpha - \alpha p - \alpha q - 1)v_j + \alpha q v_{j-1})\big)\cdot z_j = 0 \quad \forall j \geq 1 \tag{4.26}$$

$$v_0 \cdot (1 - y_0 + \alpha q z_1) = 0 \tag{4.27}$$

$$v_1 \cdot (1 - y_1 - (1-\alpha+\alpha p+\alpha q)z_1 + \alpha q z_2) = 0 \tag{4.28}$$

$$v_j \cdot \big(1 - y_j + \alpha p z_{j-1} - (1-\alpha+\alpha p+\alpha q)z_j + \alpha q z_{j+1})\big) = 0 \quad \forall j \geq 2 \tag{4.29}$$

As in the simple random walk case, it will turn out that there also exists a critical j^* associated with this type of a random walk denoting the optimal set of states to exercise the option. This time, however, we will also discuss how we can obtain a solution once we have a guess on the behaviour of the value function. Defining once again the optimal solution to (P2) as the infinite dimensional real valued vector v^*, we assume that there exists a certain $j^* \in \mathbb{N}$ such that $v_j = f_j$ for $j \geq j^*$ and $v_j > f_j$ for $0 < j < j^*$. But since $v_j^* = \max\{f_j, \alpha(pv_{j+1}^* + (1-p-q)v_j^* + qv_{j-1}^*)\}$ we must have $v_j^* = \alpha(pv_{j+1}^* + (1-p-q)v_j^* + qv_{j-1}^*)$ for $0 < j < j^*$. The value function, then, must satisfy the following equations:

$$\begin{aligned} & v_0^* = f_0, \\ & v_j^* = \alpha(pv_{j+1}^* + (1-p-q)v_j^* + qv_{j-1}^*) \quad > \quad f_j, \quad 0 < j < j^*, \\ & v_j^* = f_j \quad > \quad \alpha(pv_{j+1}^* + (1-p-q)v_j^* + qv_{j-1}^*), \quad j^* \leq j. \end{aligned}$$

Due to our assumption on the behaviour of v^*, we already know the values of v^* for $j \geq j^*$. To determine v_j^* for $0 < j < j^*$, we need to solve the second order homogeneous difference equation:

$$v_j - \alpha(pv_{j+1} + (1-p-q)v_j + qv_{j-1}) = 0, \quad 0 < j < j^* \tag{4.30}$$

with the boundary conditions:

$$v_{j^*} = f_{j^*} \tag{4.31}$$

$$v_0 = 0 \tag{4.32}$$

Appendix A.4 gives a general treatment of solving such problems with given boundary conditions. The reader can verify that if the triplet $(\alpha p, -(1-\alpha+\alpha p+\alpha q), \alpha q)$ is substituted for the coefficients in A.2, the following roots will be obtained:

$$\bar{\xi}_- = \frac{(1-\alpha+\alpha p+\alpha q) + \sqrt{(1-\alpha+\alpha p+\alpha q)^2 - 4\alpha^2 pq}}{2\alpha p}$$

$$\bar{\xi}_+ = \frac{(1-\alpha+\alpha p+\alpha q) - \sqrt{(1-\alpha+\alpha p+\alpha q)^2 - 4\alpha^2 pq}}{2\alpha p}.$$

We know that the solution will be in the form:

$$v_j = C_+ \bar{\xi}_+^j + C_- \bar{\xi}_-^j.$$

To obtain C_+ and C_-, we can use the boundary conditions (4.31) and (4.32). When $j = 0$ we have $C_+ + C_- = 0$, thus $C_+ = -C_-$. Substituting the value of C_- into (4.31) we obtain:

$$C_+ = \frac{f_{j^*}}{\bar{\xi}_+^{j^*} - \bar{\xi}_-^{j^*}}.$$

Finally, substituting the values of C_+ and C_-, we find the solution for (4.30) as:

$$v_j = f_{j^*} \frac{\bar{\xi}_+^j - \bar{\xi}_-^j}{\bar{\xi}_+^{j^*} - \bar{\xi}_-^{j^*}} \quad \forall 0 < j < j^*,$$

which gives the candidate solution v^*:

$$v_j^* = \begin{cases} 0 & j = 0 \\ f_{j^*} \frac{\bar{\xi}_+^j - \bar{\xi}_-^j}{\bar{\xi}_+^{j^*} - \bar{\xi}_-^{j^*}}, & 0 < j < j^* \\ f_j & j^* \leq j \end{cases} \tag{4.33}$$

defined on the generalized roots $\bar{\xi}_-$ and $\bar{\xi}_+$. Note that when $p + q = 1$, we have $\bar{\xi}_- = \xi_-$ and $\bar{\xi}_+ = \xi_+$ and (4.33) coincides with the solution (4.18). We have now a solution for the primal variables, although the question of making a reasonable guess for the dual variables (y^*, z^*) remains. The derivation of z^* would be similar to that of v^* if we knew that the candidate v^* is indeed optimal to (P1). For the moment, let's assume that this is the case. Then, v^* must satisfy the CS conditions (4.25) - (4.29). We know that $v_j^* \neq f_j$ for $0 < j < j^*$, thus, $y_j^* = 0$ when $0 < j < j^*$ and (4.29) reduces to:

$$\big(1 + \alpha p z_{j-1} - (1 - \alpha + \alpha p + \alpha q) z_j + \alpha q z_{j+1})\big) = 0 \quad \forall 0 < j < j^*. \tag{4.34}$$

Note that this is a second-order non-homogeneous difference equation in j with the boundaries $z_{j^*} = 0$ and $z_0 = 0$. The first boundary follows from (4.26) and the second one is just an additional variable introduced to the model, which does not affect the problem formulation. To solve this system, suppose z can be written as $z_h + \bar{z}$ where z_h^* is the solution to the homogeneous case and $\bar{z}$ is any particular solution. If the particular solution is a constant denoted by c, it will be $\frac{1}{1-\alpha}$ which is obtained by substituting c into (4.34). The steps leading to the solution of the homogeneous part are identical to what we have done for v^*. The coefficient triplet $(\alpha q, -(1 - \alpha + \alpha p + \alpha q), \alpha p)$ will yield the generalized roots:

$$\bar{\zeta}_- = \frac{(1 - \alpha + \alpha p + \alpha q) + \sqrt{(1 - \alpha + \alpha p + \alpha q)^2 - 4\alpha^2 pq}}{2\alpha q}$$

$$\bar{\zeta}_+ = \frac{(1 - \alpha + \alpha p + \alpha q) - \sqrt{(1 - \alpha + \alpha p + \alpha q)^2 - 4\alpha^2 pq}}{2\alpha q}.$$

We know that the solution to the homogeneous part is of the form $C_+ \bar{\zeta}_+^j + C_- \bar{\zeta}_-^j$. Combining this with the particular solution we get:

$$z_j = C_+ \bar{\zeta}_+^j + C_- \bar{\zeta}_-^j + \frac{1}{1-\alpha}.$$

To obtain the corresponding coefficients C_+ and C_- we need to solve:

$$C_+ + C_- + \frac{1}{1-\alpha} = 0$$
$$C_+ \bar{\zeta}_+^{j^*} + C_- \bar{\zeta}_-^{j^*} + \frac{1}{1-\alpha} = 0.$$

The reader can verify that the solution to the above system is:

$$C_+ = \frac{1}{\bar{\zeta}_+^{j^*} - \bar{\zeta}_-^{j^*}} \cdot \left(\frac{\bar{\zeta}_-^{j^*} - 1}{1-\alpha} \right)$$

$$C_- = \frac{1}{\bar{\zeta}_+^{j^*} - \bar{\zeta}_-^{j^*}} \cdot \left(\frac{1 - \bar{\zeta}_+^{j^*}}{1 - \alpha} \right)$$

which will lead to the solution:

$$z_j = \left(1 - \frac{\bar{\zeta}_-^{j^*} - 1}{\bar{\zeta}_-^{j^*} - \bar{\zeta}_+^{j^*}} \bar{\zeta}_+^j - \frac{\bar{\zeta}_+^{j^*} - 1}{\bar{\zeta}_+^{j^*} - \bar{\zeta}_-^{j^*}} \bar{\zeta}_-^j \right) \cdot \frac{1}{1 - \alpha}, \quad 0 < j < j^*$$

when C_+ and C_- are substituted into z_j and the terms are organized. We also know that $z_j = 0$ for $j \geq j^*$ because in (4.26) the first factor will be strictly positive. Thus, combining these two results, we obtain the candidate dual solution:

$$z_j^* \quad = \quad \begin{cases} \left(1 - \frac{\bar{\zeta}_-^{j^*} - 1}{\bar{\zeta}_-^{j^*} - \bar{\zeta}_+^{j^*}} \bar{\zeta}_+^j - \frac{\bar{\zeta}_+^{j^*} - 1}{\bar{\zeta}_+^{j^*} - \bar{\zeta}_-^{j^*}} \bar{\zeta}_-^j \right) \cdot \frac{1}{1-\alpha} & 0 < j < j^* \\ 0 & j^* \leq j. \end{cases} \tag{4.35}$$

It remains to find the values of y^*. The dual constraint ensure that $y_j^* = 1$ for $j > j^*$ since $z{*}_j = 0$ in this region. Then, one can obtain the vector:

$$y_j^* \quad = \quad \begin{cases} 1 + \alpha q z_1^* & j = 0 \\ 0 & 0 < j < j^* \\ 1 + \alpha p z_{j^*-1}^* & j = j^* \\ 1 & j > j^* \end{cases} \tag{4.36}$$

which is given in terms of z^*.

The pair of candidate solutions for the primal-dual pair (P2)-(D2) presented above were obtained with an a priori assumption on the behaviour of the value function. In what follows, we will show that this pair *is* the optimal pair as we did in the simple random walk case.

Before doing so, we need to define the critical point j^*. An intuitive way to define this critical value is to use the assumption made on the behaviour of the value function. Note that the candidate function is defined in such a way that the ratio of v_j^* to f_j reduces to 1 as j gets closer to j^*. For the values $j \geq j^*$, we have $\frac{v_j^*}{f_j} = 1$. For this reason, we will seek the maximum value of j for which the inequality below holds:

$$\frac{v_{j-1}^*}{f_{j-1}} > \frac{v_j^*}{f_j}.$$

In mathematical terms we have:

$$j^* = \max\left\{j : \frac{v^*_{j-1}}{f_{j-1}} > \frac{v^*_j}{f_j}\right\}.$$

Substituting the values of v^*, we will find j^* to be:

$$j^* = \max\left\{j : \frac{f_j}{f_{j-1}} > \frac{\bar{\xi}_+^j - \bar{\xi}_-^j}{\bar{\xi}_+^{j-1} - \bar{\xi}_-^{j-1}}\right\}.$$

Note that in our formulation of the problem, we have assumed that the strike price S is a multiple of Δx, that is, there exists a natural number j_S such that $S = j_S \Delta x$. Due to this assumption and the linear stock movement scenario of this chapter, the sequence $\frac{f_j}{f_{j-1}}$, which is defined for $j \geq j_S + 2$, will precisely be:

$$\left\{2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\right\}.$$

On the other hand, the term:

$$\frac{\bar{\xi}_+^j - \bar{\xi}_-^j}{\bar{\xi}_+^{j-1} - \bar{\xi}_-^{j-1}}$$

can get arbitrarily large since the limit of this sequence is $\bar{\xi}_-$, which grows unboundedly for the values of p close to 0. Therefore, we may encounter a situation, due to the choice of p and q, where we have:

$$\bar{\xi}_- > 2$$

which also implies that

$$\frac{f_j}{f_{j-1}} < \frac{\bar{\xi}_+^j - \bar{\xi}_-^j}{\bar{\xi}_+^{j-1} - \bar{\xi}_-^{j-1}}$$

for all values of $j \geq j_S + 2$. To overcome this gap in the definition of j^*, we introduce another case and finally arrive at:

$$j^* \quad = \quad \begin{cases} j_S + 1 & \text{if } \left[\frac{f_{j_S+2}}{f_{j_S+1}} < \frac{\bar{\xi}_+^{j_S+2} - \bar{\xi}_-^{j_S+2}}{\bar{\xi}_+^{j_S+1} - \bar{\xi}_-^{j_S+1}}\right] \\ \max\left\{j : f_j \, \frac{\bar{\xi}_+^{j-1} - \bar{\xi}_-^{j-1}}{\bar{\xi}_+^{j} - \bar{\xi}_-^{j}} > f_{j-1}\right\} & \text{otherwise.} \end{cases} \tag{4.37}$$

In addition to constructing a reasonable solution vector for our problem, we have also defined the critical value j^* which describes the trading strategy for the holder of the option. What remains is to show that these selections indeed solve our problem to optimality.

The proof for the existence of this critical value is identical to that of Lemma 4.3.1 except that the roots of the difference equation have now been generalized. The reader will realize that all arguments in this proof identically hold for the generalized case if we have $\bar{\xi}_- > 1$. The following lemma establishes this fact.

Lemma 4.5.1. *The generalized root $\bar{\xi}_-(\bar{\xi}_+)$ of the difference equation (4.30) is strictly greater(less) than 1.*

Proof. Since $\alpha \in (0,1)$ we have:

$$4\alpha p > 4\alpha^2 p.$$

By collecting all the terms on the RHS and adding $(1-\alpha r)^2 - 4\alpha^2 pq$ to both sides we obtain:

$$(1-\alpha r)^2 - 4\alpha^2 pq > (1-\alpha r)^2 - 4\alpha^2 pq - 4\alpha p + 4\alpha^2 p.$$

Now, we add and subtract $4\alpha^2 p^2$ from the RHS to obtain:

$$\begin{aligned}(1-\alpha r)^2 - 4\alpha^2 pq &> (1-\alpha r)^2 - 4\alpha^2 pq - 4\alpha p + 4\alpha^2 p + 4\alpha^2 p^2 - 4\alpha^2 p^2 \\ &= (1-\alpha r)^2 + 4\alpha^2 p^2 - 4\alpha p(1-\alpha+\alpha q+\alpha p) \\ &= (1-\alpha r)^2 + 4\alpha^2 p^2 - 4\alpha p(1-\alpha r) \\ &= \left[(1-\alpha r) - 2\alpha p\right]^2. \end{aligned} \tag{4.38}$$

We claim that the LHS of the above inequality is strictly positive. To see this, we substitute the value of r into the LHS and rearrange the terms:

$$\begin{aligned}(1-\alpha r)^2 - 4\alpha^2 pq &= \left[1-\alpha(1-(p+q))\right]^2 - 4\alpha^2 pq \\ &= 1 + \alpha^2(1-p-q)^2 - 2\alpha(1-p-q) - 4\alpha^2 pq \\ &= 1 + \alpha^2(1-2(p+q)+(p+q)^2) - 2\alpha + 2\alpha(p+q) - 4\alpha^2 pq \\ &= 1 + \alpha^2 - 2\alpha^2(p+q) + \alpha^2 p^2 + 2\alpha^2 pq \\ &\quad + \alpha^2 q^2 - 2\alpha + 2\alpha(p+q) - 4\alpha^2 pq \\ &= 1 - 2\alpha + \alpha^2 + 2(p+q)(\alpha-\alpha^2) \\ &\quad + \alpha^2(p^2 - 2\alpha^2 pq + q^2) \\ &= (1-\alpha)^2 + 2(p+q)(\alpha-\alpha^2) + \alpha^2(p-q)^2. \end{aligned}$$

From this equality, then, one can see that $(1-\alpha r)^2 - 4\alpha^2 pq$ is strictly positive since, due to the choice of α, all three terms above are non-negative and the first two are strictly positive. We can now take the square roots of both sides in 4.38. Since the absolute value of the LHS is greater than the absolute value of the RHS, taking the square roots leads to the following inequalities wherein both sides are real numbers:

$$\sqrt{(1-\alpha r)^2 - 4\alpha^2 pq} > (1-\alpha r) - 2\alpha p$$

and

$$\sqrt{(1-\alpha r)^2 - 4\alpha^2 pq} > 2\alpha p - (1-\alpha r).$$

Rearranging the terms in the above inequalities will give the desired result, which completes the proof. □

Our next task will be to determine whether v^* is feasible to (P2) and whether there are some additional requirements for this feasibility. Note that proving the feasibility of v^* for the entire constraint system in (P2) reduces to showing that:

1. $v_j^* \geq f_j$ when $0 < j < j^*$ and
2. $f_j \geq \alpha(pf_{j+1} + (1-p-q)f_j + qf_{j-1})$ when $j > j^*$.

The reason for this follows from the definition of v^* which has also been discussed with slight modifications in the proof of Lemma 4.3.1. Note that in the second set of equations above, the excluded region is satisfied with equality since v^* is already the solution to this second-order difference equation. On the other hand, a comparison to the previously established result will reveal that the remaining set of inequalities is a source for an additional requirement on the feasibility of v^*. The following lemma and its corollary establishes that this requirement is the same to the one we have derived previously.

Lemma 4.5.2. *The system of inequalities*

$$f_j \geq \alpha(pf_{j+1} + (1-p-q)f_j + qf_{j-1}) \qquad \forall j > j^*$$

is equivalent to

$$j^* + k \geq j_S + \frac{\alpha(p-q)}{(1-\alpha)} \qquad \forall k \in \{1, 2, 3, \ldots\}.$$

Proof. Recalling that $f_j = \max\{(j\Delta x - S), 0\}$ and $j^* \geq j_S$ where $S = j_S\Delta x$ for some $j_S \in \mathbb{N}$, we can write this system of inequalities as follows:

$$\begin{aligned}(j^* + k)\Delta x - j_S\Delta x \geq \alpha\big[&p((j^* + k + 1)\Delta x - j_S\Delta x)\\ &+ (1 - p - q)((j^* + k)\Delta x - j_S\Delta x)\\ &+ q((j^* + k - 1)\Delta x - j_S\Delta x)\big] \qquad \forall k \in \{1, 2, 3, \ldots\}.\end{aligned}$$

Cancelling Δx from both sides and rearranging the terms, we obtain:

$$\begin{aligned}j^* + k - j_S &\geq \alpha\big[(j^* + k)(p + 1 - p - q + q) + p - q + j_S(p + 1 - p - q + q)\big]\\ &\geq \alpha(j^* + k - j_S) + \alpha(p - q) \qquad \forall k \in \{1, 2, 3, \ldots\}.\end{aligned}$$

This is clearly equivalent to:

$$j^* + k \geq j_S + \frac{\alpha(p - q)}{(1 - \alpha)} \qquad \forall k \in \{1, 2, 3, \ldots\}.$$

which completes the proof. □

Corollary 4.5.1. *The inequalities*

$$f_j \geq \alpha(pf_{j+1} + (1 - p - q)f_j + qf_{j-1}) \qquad \forall j > j^*$$

are satisfied if and only if

$$j^* + 1 \geq j_S + \frac{\alpha(p - q)}{(1 - \alpha)}.$$

Proof. By lemma 4.5.2 the inequalities are equivalent to

$$j^* + k \geq j_S + \frac{\alpha(p - q)}{(1 - \alpha)} \qquad \forall k \in \{1, 2, 3, \ldots\}.$$

Suppose this system holds. Then, the conclusion will clearly hold since it is included in the system. Now, suppose the conclusion holds. For any $k \in \{1, 2, 3, \ldots\}$, we have:

$$j^* + k \geq j^* + 1 \geq j_S + \frac{\alpha(p - q)}{(1 - \alpha)}$$

which means that the inequalities hold for all $k \in \{1, 2, 3, \ldots\}$. Hence, the condition given in the conclusion is both necessary and sufficient for the inequalities to hold, which completes the proof. □

The lemma and its corollary above reveals that the value function with the generalized roots remains excessive as long as the same feasibility condition of the previous section is satisfied. With this in mind, we will conclude that our value function, $v_j^*(\bar{\xi}_-, \bar{\xi}_+)$, which is given as a function of $\bar{\xi}_-$ and $\bar{\xi}_+$ in order to avoid confusion, is feasible as long as the feasibility condition is satisfied. The result follows from identical arguments discussed in the proof of Lemma 4.3.2, where the roots ξ_- and ξ_+ are replaced by their generalized counterparts. For this reason, the proof of the following lemma is omitted.

Lemma 4.5.3. *The generalized value function $v_j^*(\bar{\xi}_-, \bar{\xi}_+)$ is feasible for (P2) if and only if p, q and α are chosen so that,*

$$j^* + 1 \geq j_S + \frac{\alpha(p-q)}{(1-\alpha)}$$

holds.

Proof. Omitted. □

As expected, the value function we have generalized for the present case also solves (P2) to optimality as long as it remains feasible. Roughly speaking, the result follows from the fact that when v_j^* is feasible, the primal-dual solution pair $(v^*, (y^*, z^*))$ satisfies the complementary slackness conditions of (P2)-(D2), which is enough for optimality. The formal proof will again be omitted since it is very similar to that of theorem 4.3.1 and can be done with changing the roots accordingly.

Theorem 4.5.1. *$v_j^*(\bar{\xi}_-, \bar{\xi}_+)$ solves (P2) to optimality if and only if*

$$j^* + 1 \geq j_S + \frac{\alpha(p-q)}{(1-\alpha)}. \tag{4.39}$$

Proof. Omitted. □

Theorem 4.5.1 implies that the function $v_j^*(\bar{\xi}_-, \bar{\xi}_+)$ is indeed the value function of the optimal stopping (exercising) problem of a perpetual American option. As its definition suggests, this value function characterizes both the set of states to exercise the option and the set of states to wait. We will conclude this section with some further remarks on the implication of Theorem 4.5.1.

4.6 Closing Remarks

Theorem 4.3.1 establishes a certain value function v^* which can be used, at any time period, to calculate the value of an option, given the price of the underlying stock. The trader can calculate the value of the threshold value $j^*\Delta x$ with the given formula, and compare the current value of the stock to this threshold value. The function v_j^* suggests that if $X_0 < j^*\Delta x$, the exercise action should be delayed for at least one period and if $X_0 \geq j^*\Delta x$, the option should be exercised immediately.

Note that this linear programming approach allows us a certain level of flexibility, from two perspectives. First, it might be possible to study variations of the simple random walk introduced within this chapter. Note that (4.3) varies directly with the changes in the behaviour of the random walk. As we have discussed previously, if it is possible to make a rough guess of the set OPT, then the CS conditions can be solved in a certain way to reveal the value function v.

It might also be useful to determine the regions of exercise under different pay-off structures. The LP formulation allows us to alter the right hand side of (4.2), which corresponds to the pay-off of the position under study. Later, we will see how valuation is affected under changes in this pay-off function.

Within the current framework, our initial guess characterizing OPT to be $\{j \in \mathbb{N} : \ j \geq j^*\}$ allowed us to reduce the complementary slackness conditions to a pair of second order difference equations with certain boundary conditions, which were then solved with a well-known solution technique (see A.4). It is worth mentioning that this approach, when the boundaries are generalized to be arbitrary values, will be a useful tool to study other positions.

One practical and intuitive outcome of this setting, due to the requirement

(4.39), is that as the forward probability p and the discount factor α approach 1, it becomes harder to capture the optimal stopping state at a lower stock value. The intuition is as follows: given an initial state, as we increase the forward probability, the probabilistic weights of being in states with a higher value at some future date increase, which, in turn, increases the future expectation of the pay-off. To compensate this increased pay-off, the option must be exercised at instances with relatively higher stock prices. This effect, however, is dampened with decreasing α since a lower α value reduces the increased weight on states with higher value drastically.

Chapter 5

Pricing and Optimal Exercise Under Geometric Random Walks

It is possible to use the machinery of infinite dimensional linear programming studied in the previous chapters on different types of random walks. In this chapter, we will derive similar conditions for the optimal exercise of perpetual American options under geometric random walks. It will be shown, by using the boundary conditions for arbitrarily large or small values of the state space, that the optimal exercise threshold can be obtained by a much simpler formula. The analysis reveals that the knowledge of the condition OPT $\neq \emptyset$ suffices to solve a large set of problem instances due to the sole dependence to the pay-off function. We will begin with a description of the underlying's price movement scenario.

5.1 A Geometric Random Walk Model on $\mathbb{R}$

Let φ be a real number slightly greater than 1 and $X_0 > 0$ be the current price of the underlying stock. Without loss of generality, we take $t = 0$ as the current time. Consider the state space $E_2 = \{X_0 \cdot \varphi^j : j \in \mathbb{Z}\}$. Note that $\lim_{j \to -\infty} \varphi^j = 0$, thus, the set E_2 spans $\mathbb{R}^{++}$. Similar to the simple random walk case, we will define the collection of E_2 valued random variables $\{X_t : t \in \mathbb{N}\}$ to be the stock-price process. Let t be a non-negative integer denoting future periods. Suppose at each period $t \geq 0$, the stock-price in $t+1$ obeys the random progression:

$$X_{t+1} = \begin{cases} X_t \cdot \varphi & \text{w.p.} \quad p \\ X_t \cdot \varphi^{-1} & \text{w.p.} \quad q = 1 - p. \end{cases}$$

The process described above leads to a binomial conditional p.m.f. similar to the one we discussed in Chapter 4. Let $t > 0$ be an arbitrary future period and define the set $E_2^t = \{X_0\varphi^{-t}, X_0\varphi^{-(t-2)}, \ldots, X_0\varphi^{t-2}, X_0\varphi^t\}$ to be the set of possible values of the stock t periods into the future. Clearly $E_2^t \subset E_2$. By construction, E_2 does not have any absorbing states and the conditional p.m.f of X_t is given by:

$$\Phi^g(t, j, p) = \mathbb{P}\big[\, X_t = X_0 \cdot \varphi^j \mid X_0 \,\big] = \binom{t}{\phi(j)} q^{t-\phi(j)} p^{\phi(j)}$$

where $j \in \{-t, -(t-2), \ldots, (t-2), t\}$ and $\phi(j)$ is the number of times the random process has moved forward in t units of time. Unlike the simple random walk case, though, this process will always have an undisturbed binomial distribution due to the absence of an absorbing state, yielding a much simpler distribution function. Note that this case is very similar to the simple random walk probabilistically, however, the value that the process attains is entirely different.

5.2 Pricing and Optimal Exercise Under the Geometric Random Walk

Under the geometric random walk scenario, we wish to solve the problem:

$$\begin{array}{llrcll}
\text{(P3)} & min & \sum_{j \in \mathbb{Z}} v_j & & & \\
& s.t. & v_j & \geq & f_j & \forall j \in \mathbb{Z} \qquad (5.1) \\
& & v_j & \geq & \alpha\{qv_{j-1} + pv_{j+1}\} & \forall j \in \mathbb{Z}. \qquad (5.2)
\end{array}$$

Note that (P3) is a modified version of (P1), extended to the set of integers with a modified right hand side. Here, f_j is given by $\max\{(X_0 \cdot \varphi^j - S), 0\}$. Constraints

(5.1) and (5.2), defined over the set of integers, again correspond to the majorant and excessive properties of the value function.

As it turns out, a similar solution strategy based on LP duality is applicable to the geometric random walk case. Let us consider the dual problem (D3) to the original problem:

$$\text{(D3)} \qquad max \quad \sum_{j\in\mathbb{Z}} f_j \cdot y_j$$

$$s.t.$$

$$\begin{aligned} y_j - \alpha p z_{j-1} + z_j - \alpha q z_{j+1} &= 1 \quad \forall j \in \mathbb{Z} &(5.3)\\ y_j &\geq 0 \quad \forall j \in \mathbb{Z} &(5.4)\\ z_j &\geq 0 \quad \forall j \in \mathbb{Z} &(5.5)\end{aligned}$$

which will yield the CS conditions:

$$(f_j - v_j)\cdot y_j = 0, \quad j \in \mathbb{Z} \tag{5.6}$$

$$\big(\alpha(pv_{j+1} + qv_{j-1}) - v_j\big)\cdot z_j = 0 \quad \forall j \in \mathbb{Z} \tag{5.7}$$

$$v_j \cdot \big(1 - y_j - z_j + \alpha(pz_{j-1} + qz_{j+1})\big) = 0 \quad \forall j \in \mathbb{Z}. \tag{5.8}$$

As in the previous chapter, we are interested in finding the optimal solution to (P3) in order to characterize the states to take action. Let v^* denote the optimal solution to (P3). Following a similar methodology, we will first try to construct the value function under the assumption that a similar critical threshold denoted by j^* exists with the properties

$$v_j = f_j \quad \forall\, j \geq j^* \tag{5.9}$$

$$v_j > f_j \quad \forall\, j < j^*. \tag{5.10}$$

We will now show that the existence of this critical state is guaranteed only when the parameters chosen for the model satisfy certain conditions. First let us make the following assumption:

Assumption. There exists a $j_S \in \mathbb{Z}$ such that $S = X_0 \cdot \varphi^{j_S}$.

The following lemma restricts the existence of j^* to a certain condition.

Lemma 5.2.1. *A critical value $j^* \in \mathbb{Z}^{++}$ defined as:*

$$j^* = \begin{cases} j_S + 1 & \text{if } \frac{f_{j_S+2}}{f_{j_S+1}} \leq \xi_- \\ \max\left\{k : \frac{f_k}{f_{k-1}} > \xi_-\right\} & \text{otherwise} \end{cases} \tag{5.11}$$

exists if and only if $\varphi < \xi_-$, where ξ_- is given by

$$\xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p}.$$

Proof. To begin, let us observe that $f_j > 0$ when $j \geq j_S + 1$ and thus the ratio $\frac{f_j}{f_{j-1}}$ is only defined when $j \geq j_S + 2$. The ratio is also monotonically decreasing. Furthermore, it converges to φ as j tends to infinity and $\frac{f_j}{f_{j-1}} > \varphi$ for all $j \geq j_S + 2$. Now, suppose that j^* as defined above exists. Then, we either have $j^* = j_S + 1$ or $j^* = \max\{k : \frac{f_k}{f_{k-1}} > \xi_-\}$. If $j^* = j_S + 1$, $\frac{f_{j_S+2}}{f_{j_S+1}} > \varphi$ and the definition of j^* in 5.11 together imply:

$$\xi_- \geq \frac{f_{j_S+2}}{f_{j_S+1}} > \varphi.$$

If, on the other hand, $j^* = \max\{k : \frac{f_k}{f_{k-1}} > \xi_-\}$, we have $\frac{f_{j^*+1}}{f_{j^*}} \leq \xi_-$. Suppose, for a contradiction, that $\xi_- \leq \varphi$. Since $\frac{f_j}{f_{j-1}} > \varphi$ for all $j \geq j_S + 2$, we also have $\frac{f_{j^*+1}}{f_{j^*}} > \varphi$. This contradicts with $\frac{f_{j^*+1}}{f_{j^*}} \leq \xi_- \leq \varphi$. Thus, the condition $\varphi < \xi_-$ is necessary for the existence of j^*.

In order to show that it is also sufficient, let us suppose $\varphi < \xi_-$. The existence of j_S is guaranteed by our starting assumption. Therefore, when $\frac{f_{j_S+2}}{f_{j_S+1}} \leq \xi_-$, we can use $j^* = j_S + 1$ which readily shows the existence of j^*. On the other hand, suppose that $\frac{f_{j_S+2}}{f_{j_S+1}} > \xi_-$. Since the sequence

$$\left\{\frac{f_j}{f_{j-1}}, \; j \geq j_S + 2\right\}$$

is a monotonically decreasing sequence whose limit is φ, $\varphi < \xi_-$ implies that there exist finitely many $j \in \mathbb{Z}$ with $j \geq j_S + 2$ such that $\frac{f_j}{f_{j-1}} > \xi_-$. The existence of $j^* = \max\{k : \frac{f_k}{f_{k-1}} > \xi_-\}$, then, follows since the maximum of a finite set always exists. Hence, $\varphi < \xi_-$ is also a sufficient condition for the existence of j^*, which completes the proof. □

Having shown that there exists a j^* which satisfies conditions 5.10, we will now formally derive the corresponding value function. Since $v_j = \max\{f_j, \alpha(pv_{j+1} + qv_{j-1})\}$ and $v_j \neq f_j$ for $j < j^*$, due to our assumption, we have $v_j = \alpha(pv_{j+1} + qv_{j-1})$ for $j < j^*$. To determine the value function, then, we need to solve the second order homogeneous difference equation:

$$v_j - \alpha(pv_{j+1} + qv_{j-1}) = 0, \quad j < j^* \tag{5.12}$$

with the boundary conditions:

$$v_{j^*} = f_{j^*} \tag{5.13}$$

$$v_{-\infty} = 0. \tag{5.14}$$

Using the general solution technique given in Appendix A.4, we can obtain a solution for the system (5.12)-(5.14). Since we are working with a homogeneous equation, we know that v_j is of the form $C_+\xi_+^j + C_-\xi_-^j$ where ξ_- and ξ_+ are the same roots as in chapter 4. They will be repeated here for the sake of completeness:

$$\xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p} \qquad \xi_+ = \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p}.$$

From the boundary (5.14), we get:

$$C_+\left(\lim_{j\to-\infty} \xi_+^j\right) + C_-\left(\lim_{j\to-\infty} \xi_-^j\right) = 0.$$

Note that the roots ξ_- and ξ_+ are greater than and less than 1 respectively. Since $\xi_- > 1$, we have $\lim\limits_{j\to-\infty} \xi_-^j = 0$ which reduces the boundary condition to:

$$C_+\left(\lim_{j\to-\infty} \xi_+^j\right) = 0.$$

Clearly, this is only possible when $C_+ = 0$ since ξ_+^j grows without bound with decreasing j. Now, utilizing the boundary (5.13), we will find C_- to be:

$$\begin{aligned} C_+\xi_+^{j^*} + C_-\xi_-^{j^*} &= f_{j^*} \\ C_-\xi_-^{j^*} &= f_{j^*} \\ C_- &= \frac{f_{j^*}}{\xi_-^{j^*}}. \end{aligned}$$

The coefficients C_+ and C_-, thus, lead to the value function:

$$v_j^* = f_{j^*} \cdot \xi_-^{(j-j^*)}$$

for $j < j^*$. For other values of j we already know that it is equal to f_j. The arguments given above, therefore, suggest that we have a candidate v^* given by:

$$v_j^* \quad = \quad \begin{cases} f_{j^*} \cdot \xi_-^{j-j^*} & j < j^* \\ f_j & j^* \leq j \end{cases} \tag{5.15}$$

which solves (P3) to optimality, which is to be verified. Note that this function can only be constructed when j^* exists, which should be kept in mind in the remainder of the chapter since the results which will follow consequently all assume that the existence condition is satisfied.

To make a reasonable guess of the dual variables (y^*, z^*), we will use a similar method. To begin with, we assume that $f_{j^*} > 0$, which makes sense because it is unlikely to have a state where the decision is to exercise the option while the pay-off is zero. The pay-off f_{j^*} being positive implies, by definition of v^*, that $v_j^* > 0$ for all $j \in \mathbb{Z}$. From the CS condition (5.6), we have $y_j^* = 0$ for $j < j^*$. This, together with v_j^* being positive implies that z_j^* satisfies the non-homogeneous second order difference equation:

$$z_j - \alpha(qz_{j+1} + pz_{j-1}) = 1 \quad \forall j < j^*. \tag{5.16}$$

Since $v_j^* \neq \alpha(pv_{j+1}^* + qv_{j-1}^*)$ for $j \geq j^*$, CS condition (5.7) implies that $z_j^* = 0$ for $j \geq j^*$. Furthermore, as $j \to -\infty$, we have:

$$z_{j+1} = z_{j-1} = z_j = z_{-\infty}.$$

Thus, from (5.16) we have:

$$z_{-\infty} \to \frac{1}{1-\alpha}.$$

Using these two boundary conditions, we can obtain a solution to (5.16). As in the simple random walk case, the particular solution is $\frac{1}{1-\alpha}$. For the homogeneous solution, we will proceed as in Appendix A.4. Note that the roots ζ_+ and ζ_- are also applicable to (5.16). The two boundary conditions give the following system, which needs to be solved to determine z_j^*:

$$C_+\zeta_+^{j^*} + C_-\zeta_-^{j^*} + \frac{1}{1-\alpha} = 0 \tag{5.17}$$

$$C_+\left(\lim_{j\to-\infty}\zeta_+^j\right)+C_-\left(\lim_{j\to-\infty}\zeta_-^j\right)+\frac{1}{1-\alpha}=\frac{1}{1-\alpha}. \tag{5.18}$$

Since $\zeta_+ < 1$ and $\zeta_- > 1$, a similar line of reasoning to the v^* case applied to equation (5.18) reveals that $C_+ = 0$. Using this and the equation (5.17), we find C_- to be:

$$-\left(\frac{1}{1-\alpha}\right)\zeta_-^{-j^*}.$$

The dual variables z^* associated with the candidate optimal solution v^* will, therefore, be:

$$z_j^* = \begin{cases} \frac{1}{1-\alpha}\left(1-\zeta_-^{j-j^*}\right) & j < j^* \\ 0 & j^* \le j. \end{cases} \tag{5.19}$$

The second set of dual variables, y^*, are also determined using the CS conditions, yielding a function which is very similar to the one in the previous chapter. Since $v_j \neq f_j$ for $j < j^*$, from (5.6) we have $y_j = 0$ for $j < j^*$. For $j > j^*$ we have $z_j = 0$ which, in turn, will set $y_j = 1$ due to the equation (5.8). Finally, for $j = j^*$ the same equation suggests that $1 - y_{j^*} + \alpha p z_{j^*-1} = 0$. The resulting piecewise function will be:

$$y_j^* = \begin{cases} 0 & j < j^* \\ 1+\alpha p z_{j^*-1}^* & j = j^* \\ 1 & j > j^*. \end{cases} \tag{5.20}$$

We have constructed a pair of candidate solutions for (P3) and (D3) assuming that a candidate threshold value j^* exists. Now we need to show that v^* defined in terms of the critical threshold j^* is in fact the optimal solution to problem (P3). To show this, we need to show (a) v^* is primal feasible, (b) (y^*, z^*) is dual feasible, and (c) v^*, y^* and z^* together satisfy the CS conditions.

Lemma 5.2.2. *v^* is feasible for (P3) if and only if p and α are chosen such that*

$$\varphi^{j^*+1} \ge \left[\frac{1-\alpha}{1-\alpha p\varphi-\alpha q\varphi^{-1}}\right]\varphi^{js}.$$

Proof. We first show that the given condition is necessary for feasibility. Let us assume that v^* as given in (5.15) is feasible to (P3). Then, v^* must satisfy (5.2).

This implies that the system of inequalities

$$v^*_{j^*+k} \geq \alpha p v^*_{j^*+k+1} + \alpha q v^*_{j^*+k-1}, \quad k = 1, 2, 3, \ldots$$

hold with $v^*_j = f_j$. By substituting the values of f_j into the above inequalities and rearranging the terms in the sense of proof for Lemma 4.5.2, one obtains the following system of inequalities:

$$\varphi^{j^*+k} \geq \left[\frac{1-\alpha}{1-\alpha p\varphi - \alpha q \varphi^{-1}}\right] \varphi^{js}, \quad j = 1, 2, 3, \ldots$$

which also implies that the given condition holds. This shows that the given condition is necessary for the feasibility of v^*.

Next, we show that it is also sufficient for feasibility. Suppose that the condition above holds. We first concentrate on (5.2). Let $j > j^*$. Since $\varphi^{j^*+k} \geq \varphi^{j^*+1}$ for all $k \geq 1$, we have:

$$\varphi^{j^*+k} \geq \varphi^{j^*+1} \geq \left[\frac{1-\alpha}{1-\alpha p\varphi - \alpha q \varphi^{-1}}\right] \varphi^{js}, \quad j = 1, 2, 3, \ldots$$

which follows from the given condition. As noted in the first part of the proof, this system is equivalent to having:

$$f_{j^*+k} \geq \alpha p f_{j^*+k+1} + \alpha q f_{j^*+k-1}, \quad k = 1, 2, 3, \ldots$$

which shows that (5.2) are satisfied with $v^*_j = f_j$ for $j > j^*$. When $j < j^*$, on the other hand, we already have $v^*_j = \alpha p v^*_{j+1} + \alpha q v^*_{j-1}$ since v^* is necessarily the solution to this difference equation. Therefore, it remains to check for the case when $j = j^*$. Consider the solution of the difference equation:

$$\begin{aligned} \omega_j &= \alpha p \omega_{j+1} + \alpha q \omega_{j-1} \\ \omega_{j^*} &= f_{j^*} \\ \lim_{j \to -\infty} \omega_j &= 0 \end{aligned}$$

which extends over the set of integers, i.e. $\forall j \in \mathbb{Z}$. Then, $\omega_{j^*+1} = f_{j^*}\xi_-$. Since $\omega_{j^*} = \alpha p \omega_{j^*+1} + \alpha q \omega_{j^*-1}$ and $\omega_{j^*-1} = v^*_{j^*-1}$, $f_{j^*} = \omega_{j^*} \geq \alpha p f_{j^*+1} + \alpha q v^*_{j^*-1}$ if and only if $f_{j^*+1} \leq \omega_{j^*+1} = f_{j^*}\xi_-$. This means that it is sufficient to check $\frac{f_{j^*+1}}{f_{j^*}} \leq \xi_-$ in order to show feasibility of v^* for (5.2) at $j = j^*$. Recall that j^* may attain one of the two values defined in (5.11). First, suppose $j^* = j_S + 1$. Then, by definition of j^*, we have $\frac{f_{j^*+1}}{f_{j^*}} = \frac{f_{j_S+2}}{f_{j_S+1}} \leq \xi_-$, hence the desired result. Now, let $j^* = \max\left\{k : \frac{f_k}{f_{k-1}} > \xi_-\right\}$. Definition of j^* in this case implies that j^* is the maximum integer k with the property $\frac{f_k}{f_{k-1}} > \xi_-$ which further implies that for j^*+1, we have $\frac{f_{j^*+1}}{f_{j^*}} \leq \xi_-$. Thus, we can conclude (5.2) is satisfied with v_j^* at $j = j^*$ which completes the feasibility of v^* for (5.2) for all $j \in \mathbb{Z}$.

Now, we turn our attention to (5.1). Note that, by definition of v^*, these are satisfied trivially for $j \geq j^*$. So, let $j < j^*$. Then, $v_j^* = f_{j^*}\xi_-^{j-j^*}$. Under the assumption that j^* exists, we have $j^* \geq j_S + 1$ which implies that $f_{j^*} > 0$. Then, it follows that $v_j^* > 0$. Note that for any $j \leq j_S$, we have $f_j = 0$ and thus $v_j^* > f_j$ which is sufficient for the feasibility of (5.1). Therefore, it remains to check feasibility for the values of j where $j_S < j < j^*$. In order to do this, let us consider the difference

$$D = v_j^* - f_j$$

for $j_S < j < j^*$. We have:

$$\begin{aligned} D &= f_{j^*}\xi_-^{j-j^*} - f_j \\ &= \left(X_0\varphi^{j^*} - S\right)\xi_-^{j-j^*} - \left(X_0\varphi^j - S\right) \\ &= X_0\left(\varphi^{j^*}\xi_-^{j-j^*} - \varphi^j\right) + S\left(1 - \xi_-^{j-j^*}\right). \end{aligned}$$

Note that we need to show $D \geq 0$. By the choice of j and the fact that $\xi_- > 1$, the second term in D above is always strictly positive. If the first term is also greater than or equal to 0, we are done. So, suppose that we have $X_0\left(\varphi^{j^*}\xi_-^{j-j^*} - \varphi^j\right) < 0$. In this case, in order for $D \geq 0$, we need:

$$X_0\left(\varphi^j - \varphi^{j^*}\xi_-^{j-j^*}\right) \leq S\left(1 - \xi_-^{j-j^*}\right).$$

By rearranging the terms, we can obtain:

$$\frac{X_0\varphi^{j^*} - S}{X_0\varphi^j - S} \geq \xi_-^{j^*-j}.$$

Note that by the choice of j^*, for any $j_S < j < j^*$ we have $(j^* - j)$ inequalities satisfying:

$$\frac{f_{j+1}}{f_j} \geq \xi_-$$
$$\frac{f_{j+2}}{f_{j+1}} \geq \xi_-$$
$$\vdots$$
$$\frac{f_{j^*}}{f_{j^*-1}} \geq \xi_-.$$

Since all terms in these inequalities are positive, the inequality obtained by multiplying all terms in both sides of these inequalities does not change their direction. This operation will yield:

$$\frac{f_{j^*}}{f_j} \geq \xi_-^{j^*-j}$$

which is precisely the desired property for D to be non-negative. Since $D \geq 0$, then, we have $v_j^* \geq f_j$ for all $j_S < j < j^*$, which concludes that (5.1) are satisfied with v^* for all $j \in \mathbb{Z}$. Since we have also shown that (5.2) are also satisfied, we will conclude that the condition presented in this lemma is a sufficient condition for the feasibility of v^* to (P3), which completes the proof. □

The next thing to do in this sequel is to show that the candidate dual solution (y^*, z^*) is feasible to (D3), which is necessary for the optimality of v^*.

Lemma 5.2.3. *The pair of dual variables y^* and z^* given in (5.20) and (5.19) respectively is a feasible solution to (D3).*

Proof. Let us first consider (5.5). Note that for $j \geq j^*$, we have $z_j^* = 0$ which satisfies non-negativity of z^*. Now, let $j < j^*$. Since $j < j^*$ and $\varphi > 1$, we have $\varphi^{j-j^*} \in (0, 1)$. Then, since $\alpha < 1$, both $\frac{1}{1-\alpha}$ and $(1 - \varphi^{j-j^*})$ are strictly positive, which implies that $z_j^* > 0$ for $j < j^*$. Therefore, z_j^* satisfies (5.5) for all $j \in \mathbb{Z}$.

Now, we show y^* satisfies (5.4). By definition of y^*, for any $j \neq j^*$ we already have $y^* \geq 0$. Thus, the only case to verify is when $j = j^*$. Since $z_j^* \geq 0$ for any $j \in \mathbb{Z}$, we have $z_{j^*-1}^* \geq 0$. Then, $y_{j^*}^* = 1 + \alpha p z_{j^*-1}^* \geq 0$ which concludes that $y_j^* \geq 0$ for all $j \in \mathbb{Z}$ and that y^* satisfies (5.4).

Finally, we show that y^* and z^* taken together satisfy (5.3). First, let $j < j^*$. Then, by definition, $y_j^* = 0$. The LHS of (5.3), therefore, reduce to $z_j - \alpha p z_{j-1} - \alpha q z_{j+1}$. Since z^* solves the difference equation $z_j - \alpha(q z_{j+1} + p z_{j-1}) = 1$, we have $z_j^* - \alpha(q z_{j+1}^* + p z_{j-1}^*) = 1$ which shows that (5.3) are satisfied for $j < j^*$. Now, suppose that $j > j^*$. In this case, we have $y_j^* = 1$ and $z_j^* = 0$. Thus,

$$y_j^* - \alpha p z_{j-1}^* + z_j^* - \alpha q z_{j+1}^* = 1 - 0 + 0 - 0 = 1$$

which shows that (5.3) are also satisfied when $j > j^*$. Then, it remains to check they also hold when $j = j^*$. But since $y_{j^*}^* = 1 + \alpha p z_{j^*-1}^*$, $z_{j^*}^* = 0$ and $z_{j^*+1}^* = 0$, we have:

$$\begin{aligned} y_{j^*}^* - \alpha p z_{j^*-1}^* + z_{j^*}^* - \alpha q z_{j^*+1}^* &= 1 + \alpha p z_{j^*-1}^* - \alpha p z_{j^*-1}^* \\ &= 1. \end{aligned}$$

Therefore, y^* and z^* taken together satisfy (5.3) for all $j \in \mathbb{Z}$, which completes the proof. □

Finally, we need our candidate solutions to satisfy the CS conditions (5.6) - (5.8) for optimality. Note that proving this result will relatively be easy, since we have mainly used these conditions to obtain the candidate solutions.

Lemma 5.2.4. *v^*, y^* and z^* as given in (5.15), (5.20) and (5.19) respectively, satisfy the CS conditions (5.6) - (5.8).*

Proof. We first consider equations (5.6). For $j \geq j^*$, we have $v_j^* = f_j$. Then, $(f_j - v_j^*) \cdot y_j^* = 0$. When $j < j^*$, we have $y_j^* = 0$, which again implies that $(f_j - v_j^*) \cdot y_j^* = 0$. Thus, (5.6) are satisfied for all $j \in \mathbb{Z}$.

Next, consider condition (5.7). For $j \geq j^*$, we have $z_j^* = 0$ and the result follows similarly. For $j < j^*$, we know that v_j^* solves $v_j - \alpha(p v_{j+1} + q v_{j-1}) = 0, j < j^*$ with the corresponding boundary conditions. Thus, $v_j^* - \alpha(p v_{j+1}^* + q v_{j-1}^*) = 0$. It follows, then, (5.7) are also satisfied by the choice of v^* and z^*.

Finally, we check (5.8). From feasibility of y^* and z^*, we have

$$y_{j^*}^* - \alpha p z_{j^*-1}^* + z_{j^*}^* - \alpha q z_{j^*+1}^* = 1$$

for all $j \in \mathbb{Z}$. Then, $v_j^* \cdot \big(1 - y_j^* - z_j^* + \alpha(pz_{j-1}^* + qz_{j+1}^*)\big) = 0$. Therefore, (5.8) are also satisfied by the choice of v^*, y^* and z^*; and we conclude that all CS conditions hold. □

Using these properties for the candidate solution v^*, we can now show that it is indeed optimal to problem (P3). The following theorem establishes this fact, however, it should be noted that it is valid whenever an appropriate j^* exists. Recalling the result in Lemma 5.2.1, we now that such a j^* exists only when $\varphi < \xi_-$.

Theorem 5.2.1. *Provided that j^*, as defined in (5.11), exists; the solution v^* given by (5.15) is optimal to problem (P3) if and only if p and α are chosen such that*

$$\varphi^{j^*+1} \geq \left[\frac{1-\alpha}{1-\alpha p\varphi - \alpha q\varphi^{-1}}\right]\varphi^{js}.$$

Proof. Showing the given condition is necessary for optimality is straightforward. Suppose v^* is optimal to (P3). Then, v^* must be feasible. By Lemma 5.2.2, then, we can conclude that the given condition must hold, which is the same for saying it is a necessary condition to optimality.

Now, we will show that it is also sufficient under the assumption that j^* exists. Suppose this condition holds. This is enough, by Lemma 5.2.2, for feasibility to problem (P3). Furthermore, we know that the pair of dual variables (y^*, z^*) are also feasible to (P3), by Lemma 5.2.3. Therefore, it remains to show that they satisfy CS conditions and there is no duality gap between the objective functions $P(v^*)$ and $D(y^*, z^*)$. But we have already shown, through Lemma 5.2.4, that these solutions satisfy the CS conditions. The only remaining issue, then, is to show that there is no duality gap between $P(v^*)$ and $D(y^*)$, that is, $P(v^*) = D(y^*)$. Separating the infinite sums P and D, in the sense:

$$\begin{aligned} P(v^*) &= f_{j^*} + \sum_{j<j^*} f_{j^*}\xi_-^{j-j^*} + \sum_{j>j^*} f_j \\ &= f_{j^*} + f_{j^*}\sum_{j<j^*}\xi_-^{j-j^*} + \sum_{j>j^*} f_j \end{aligned}$$

and

$$D(y^*) = f_{j^*}\left(1 + \frac{\alpha p}{1-\alpha}\left(1 - \zeta_-^{j^*-1-j^*}\right)\right) + \sum_{j>j^*} f_j$$
$$= f_{j^*} + \frac{\alpha p}{1-\alpha} f_{j^*}\left(1 - \left(\frac{1}{\alpha p} - \xi_-\right)\right) + \sum_{j>j^*} f_j$$

we can see that it suffices to show:

$$\frac{\alpha p}{1-\alpha}\left(1 - \frac{1}{\alpha p} + \xi_-\right) = \sum_{j<j^*} \xi_-^{j-j^*}.$$

Note that we have used the first and fifth properties in Appendix A.2 to reach the second line of the expansion for $D(y^*)$. Now, the RHS above equals:

$$\sum_{j<j^*} \xi_-^{j-j^*} = \sum_{j>0}\left(\frac{1}{\xi_-}\right)^j = \frac{1}{1-\frac{1}{\xi_-}} - 1 = \frac{1}{\xi_- - 1}$$

which follows from the formula for the summation of the infinite power series and the fact that $\xi_- > 1$. It can be shown, in a couple of simple steps, that

$$\frac{\alpha p}{1-\alpha} = \frac{-1}{\left(1 - \frac{1}{\alpha p} + \frac{q}{p}\right)}.$$

Then, by property 1 and 3, we have:

$$\frac{\alpha p}{1-\alpha}\left(1 - \frac{1}{\alpha p} + \xi_-\right) = -\left(\frac{1 - \frac{1}{\alpha p} + \xi_-}{1 - \frac{1}{\alpha p} + \frac{q}{p}}\right)$$
$$= -\left(\frac{1 - (\xi_- + \xi_+) + \xi_-}{1 - (\xi_- + \xi_+) + \xi_+ \cdot \xi_-}\right)$$
$$= -\left(\frac{1 - \xi_+}{1 - \xi_+ + \xi_-(\xi_+ - 1)}\right) = \frac{1}{\xi_- - 1}$$

which is enough to show that $P(v^*) = D(y^*)$. Then, there is no duality gap between the primal and dual objective functions, which means that all requirements for optimality are satisfied as long as the given condition in the statement of the theorem holds. Therefore, the choice of v^* solves (P3) to optimality, as long as an appropriate j^* exists. □

This theorem is a parallel result to the Theorem 4.3.1, which is particularly useful in identifying the optimal stopping region when the underlying stock-price follows a geometric random walk on $\mathbb{R}^{++}$. The theorem, of course, is to be used in conjunction with the fundamental result in chapter 3, establishing the fact that the function v^* correctly gives the maximum of the discounted expected future pay-offs in the case of a geometric random walk scenario. With its structure, the function v^* suggests that there exists a critical point j^* identifying a corresponding state $X_0 \cdot \varphi^{j^*} \in E_2$, which separates the state space into subsets. Whenever the stock price is less than this critical threshold, the value per state is strictly greater than the current pay-off, implying that the decision to exercise should be delayed. On the other hand, when the price of the stock is greater than or equal to this critical value, the value function gives the same amount with the existing pay-off, which means that it is no longer meaningful to hold the option for one more period.

A key observation in the scenario studied in this chapter is that, the existence of a critical state is not guaranteed. By Lemma 5.2.1, we know that the critical value to identify v^* does not always exist: the existence is guaranteed only when $\varphi < \xi_-$, which is a condition fully dependent on the parameters of the problem instance. It states that the upward movement in the stock price must be bounded with a quantity dependent on both the probability distribution of the random progression and the discount factor α. This result is, in fact, quite intuitive when compared to the simple random walk scenario studied in the previous chapter. In Chapter 4, we had arbitrarily large future pay-offs which grow at a linear rate. For this reason, the discount factor α which acts in a geometric progression suffices to reduce the very distant and large future pay-off to a present value close to zero. The expectation of a distant future pay-off, therefore, remained finite even if we had an unbounded pay-off function. In this chapter, the growth of the pay-off is geometric and of the same order with the discount factor. Therefore, in order to keep the future expectations bounded, the discount factor must be greater than the growth rate of the stock price. This relation, however, is not a straight comparison of φ and α, since the solution of a difference equation is involved in the calculation, but a comparison to the root of the difference equation ξ_-, which also involves the effect of the discount factor α.

Chapter 6

Applications

In this chapter we will mainly focus on different types of payoff structures. The LP formulation presented in the previous sections allows a certain level of flexibility in terms of the pay-off function, leading to a class of trading positions that can be studied under the same methodology. Here, we will apply the same formulation into two widely known trading strategies: the spread and the strangle positions. These positions, favouring only certain directions in the price movement of the stock are generally used in practice to prevent from excessive losses due to unexpected stock activity.

6.1 The Spread Position

Suppose we have a certain monetary amount to invest in an array of put/call options with varying strike prices. The spread position is an option portfolio with the same number of puts (or calls) with different strike prices and different long/short attributes. When there is a maturity date specified on the contract, the options involved in a spread position have the same maturity dates.

For example, suppose we purchase one call option with a strike price K_{C_1} and sell another with a strike K_{C_2} where $K_{C_1} < K_{C_2}$. The pay-off to the trader, then, will be:

$$f^S(x) = \max\left((x - K_{C_1}), 0\right) - \max\left((x - K_{C_2}), 0\right)$$

Note that since $K_{C_2} > K_{C_1}$, when $x > K_{C_2}$, we have the constant pay-off:

$$f^S(x) = x - K_{C_1} - x + K_{C_2} = K_{C_2} - K_{C_1}.$$

This is an example of a bull spread with two calls and the pay-off function is shown in Figure (6.1), with $K_{C_1} = 9$, $K_{C_2} = 12$ and the constant pay-off 3 for the values of stock price x greater than 12.

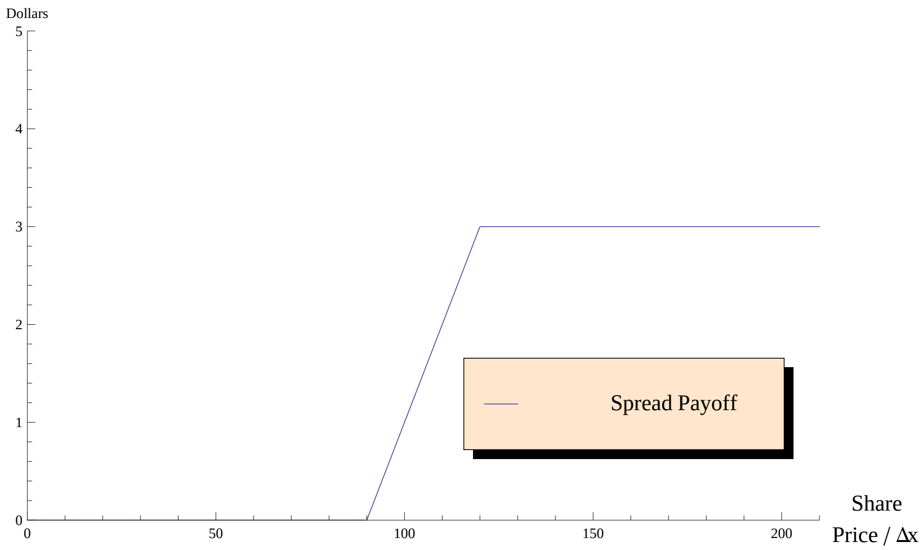

Figure 6.1: Pay-off function for a spread position ($K_{C_1} = 9$, $K_{C_2} = 12$).

We can use the solution framework discussed in the previous sections to derive the corresponding value function for such positions. Suppose that the price of the underlying stock follows a simple random walk as in Chapter 4. Then, we know that the value of the position solves the linear program:

$$\begin{aligned}
\text{(P4) min} \quad & \sum_{j\in\mathbb{N}} v_j & \\
\text{s.t.} \quad & v_j \geq f_j^S & \forall j \in \mathbb{N} \\
& v_j \geq \alpha(pv_{j+1} + qv_{j-1}) & \forall j \in \mathbb{N}
\end{aligned}$$

to optimality. Note that the only change in this formulation is the modified RHS in the first set of constraints.

If we were to guess the correct value function for this position, it would again be meaningful to assume that the value function will be equal to the pay-off function after a critical point, just as in the pay-off for the plain call option. Suppose, then, we have a point j^* such that for $j \geq j^*$ we have $v_j = f_j$ and for $j < j^*$, we have $v_j > f_j$. If we have $K_{C_1} < j^*\Delta x < K_{C_2}$, the solution is exactly the same as the plain call option, due to the similarity of the boundary conditions. If, however, we have $j^* \geq K_{C_2}$, it turns out that the value function must meet with the pay-off function at $j^* = \frac{K_{C_2}}{\Delta x}$, that is j^* cannot be greater than K_{C_2}. This is shown in Figures (6.2) and (6.5), respectively.

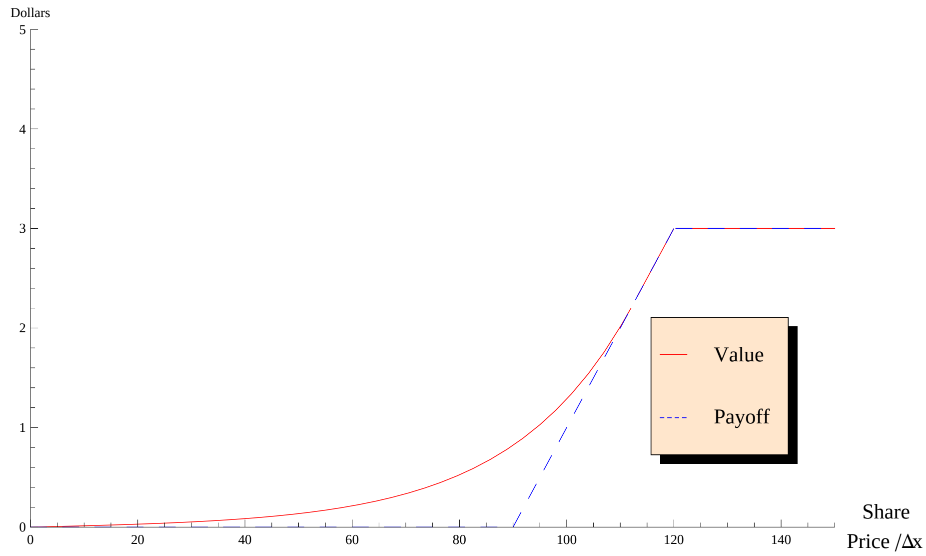

Figure 6.2: Value vs. pay-off for a bull spread when $j^*\Delta x < K_{C_2}$.($K_{C_1} = 9$, $K_{C_2} = 12$).

Two cases plotted in these figures depict either of the two possible scenarios. Recalling from Chapter 4, we know that the optimal point of exercise, characterized by j^*, occurs when the ratio $\frac{f_j}{f_{j-1}}$ drops below the power term $\frac{\xi_+^j - \xi_-^j}{\xi_+^{j-1} - \xi_-^{j-1}}$. Note that when we have a plain call option, the former ratio converges to 1 only with arbitrarily large j, whereas, in the case of a bull spread, the ratio is exactly 1 for $j > K_{C_2}$. Since we are dealing with a discrete state-space, we have $\frac{f_{K_{C_2}}}{f_{K_{C_2-1}}} > 1$ and $\frac{f_{K_{C_2+1}}}{f_{K_{C_2}}} = 1$. The intuition, then, is as follows: It is for sure that the pay-off ratio will reduce to 1 after $j = K_{C_2}$. The power term, on the other hand, is strictly larger than one, its magnitude being dependent on the forward and backward probabilities p and q. If the power term is greater than $\frac{f_{K_{C_2}}}{f_{K_{C_2-1}}}$ for all values of $j < K_{C_2}$, the pay-off ratio reduces below the power term for some $j < K_{C_2}$. This corresponds to the first scenario and was shown in Figure 6.2. For further clarification, these critical ratios are shown in Figure 6.3. Note that the power term is very close 1, resulting in a constant-looking plot, although, it has a positive (and decreasing) gap from 1.

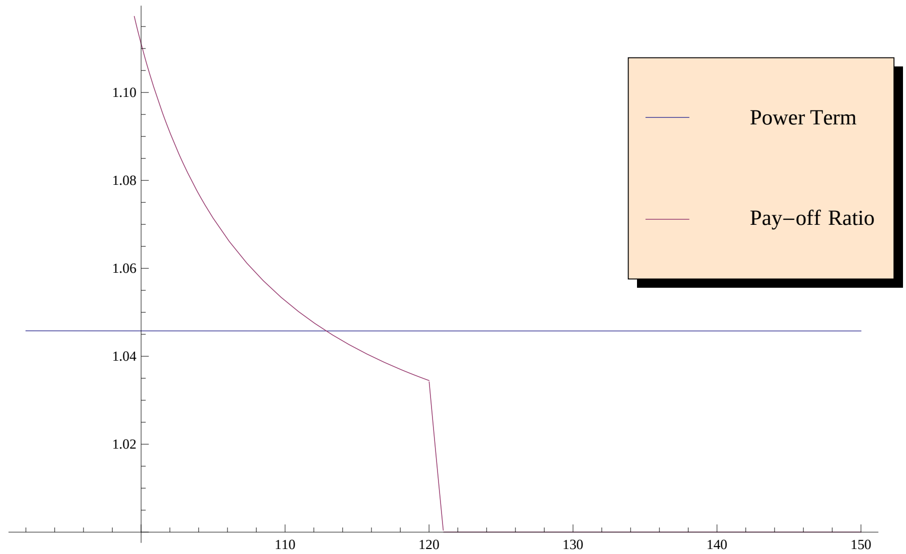

Figure 6.3: The incremental pay-off ratio and the power term for the bull spread when the optimal exercise point is less than K_{C_2}.

In the second scenario, the forward probability p is large enough to set the power term at K_{C_2} below the pay-off ratio. Since the pay-off ratio is strictly decreasing until K_{C_2} and jumps to 1 at K_{C_2+1}, it is guaranteed that the first point when the pay-off ratio is less than the power term is K_{C_2+1}, making K_{C_2} the optimal point to exercise. Then, the trader must exercise instantly when the stock price hits K_{C_2}, which is intuitive since the future pay-off for higher states is always the same, independent on the forward probability p. This scenario is given in Figure 6.4.

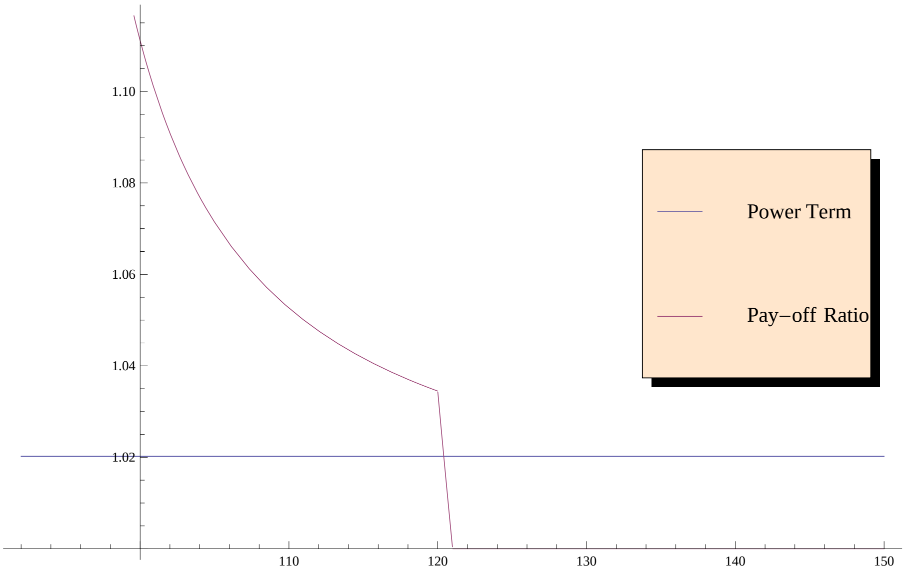

Figure 6.4: The incremental pay-off ratio and the power term for the bull spread when the optimal exercise point is K_{C_2}.

6.2 The Strangle Position

Given a certain monetary amount to invest, suppose we purchase one call and one put written on the same stock with different strike prices. Let K_c be the strike price of the call and K_p the strike price of the put. At any time $t > 0$, the pay-off of our portfolio will be:

$$f^S(X_t) = \max\left((X_t - K_c), 0\right) + \max\left((K_p - X_t), 0\right).$$

If, we have $K_c > K_p$, this function gives a V shaped function which takes the value 0 between K_p and K_c. This is shown in Figure 6.6.

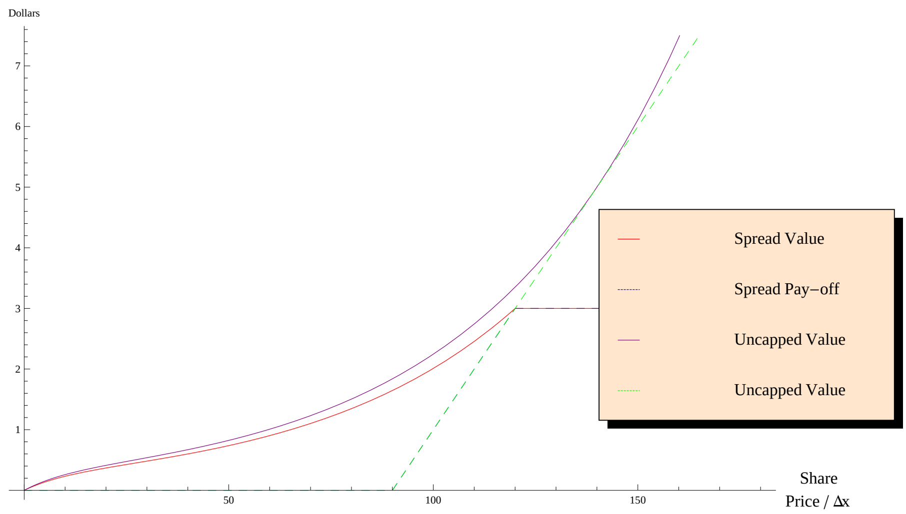

Figure 6.5: Value vs. pay-off for a bull spread when $j^*\Delta x = K_{C_2}$. The optimal exercise point for an uncapped pay-off and the corresponding value function are also shown for a comparison ($K_{C_1} = 9$, $K_{C_2} = 12$).

Note that this position is particularly useful when the stock price either gets relatively higher or lower if at the time of purchase, the stock price is between K_p and K_c. This is equivalent to saying that large movements in the stock price are expected, as opposed to small movements that do not affect the current price so much. We further note that when $K_c = K_p$, we have a perfectly V shaped function, which means that any change in the price is immediately appreciated. Of course, we expect from such positions that they are more valuable, since they have a larger profit region.

Now, suppose that the stock-price obeys a simple random walk as discussed in Chapter 4. We can find a value function $v^S(x)$ for any state $x \in E_1$ by adjusting the right hand side of the constraints in our LP formulation that correspond to the majorant property of the value function. The solution of the linear program,

$$\begin{aligned}
\text{(P5) min} \quad & \sum_{j \in \mathbb{N}} v_j \\
\text{s.t.} \quad & v_j \geq f_j^S && \forall j \in \mathbb{N} \\
& v_j \geq \alpha(pv_{j+1} + qv_{j-1}) && \forall j \in \mathbb{N}
\end{aligned}$$

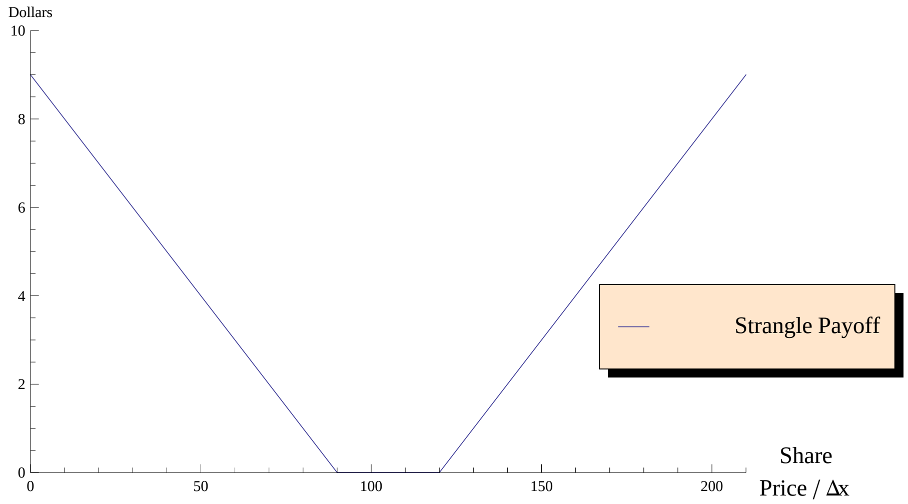

Figure 6.6: Pay-off function for a strangle position ($K_c = 12$, $K_p = 9$).

will, then, give the corresponding value function of a strangle position. In this formulation, v_j stands for $v(j\Delta x)$ and f_j^S stands for $\max((j\Delta x - K_c), 0) + \max((K_p - j\Delta x), 0)$.

The reader will recall that we have used the CS conditions to derive a candidate (probably optimal) solution to a much simpler problem in the previous chapters. But this approach involves making a guess on the behaviour of the value function. Here, let us first assume that the value function becomes equal to the pay-off function in at least two points, say j_1 and j_2, giving two boundary conditions for the difference equation derived from the CS condition. Furthermore, let us assume that the value function is strictly larger than the pay-off function between j_1 and j_2 while it is exactly equal to the pay-off function outside of this region. This means that the value function, in the given region, should satisfy the difference equation:

$$v_j - \alpha(pv_{j+1} + qv_{j-1}) = 0 \quad \forall j_1 < j < j_2 \tag{6.1}$$

with two boundary conditions:

$$v_{j_1} = f_{j_1} \tag{6.2}$$

$$v_{j_2} = f_{j_2}. \tag{6.3}$$

Appendix A.4 provides the general solution to such second-order difference equations. Thus, using the equality condition to the pay-off function as described above, we obtain the value function:

$$v_j^* = \begin{cases} \left(\frac{\xi_-^{j1} f_{j_2}^g - \xi_-^{j2} f_{j_1}^g}{\xi_-^{j1}\xi_+^{j2} - \xi_+^{j1}\xi_-^{j2}}\right)\xi_+^j + \left(\frac{\xi_+^{j2} f_{j_1}^g - \xi_+^{j1} f_{j_2}^g}{\xi_-^{j1}\xi_+^{j2} - \xi_+^{j1}\xi_-^{j2}}\right)\xi_-^j & j_1 < j < j_2 \\ f_j & \text{otherwise.} \end{cases} \tag{6.4}$$

The function described above is shown in Figure 6.7, where $j_1 = 6.6$ and $j_2 = 14.4$.

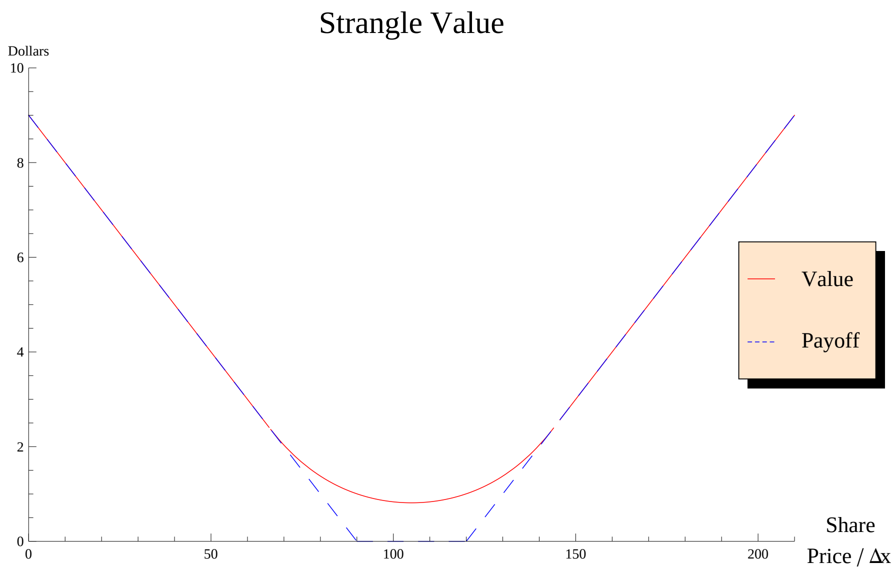

Figure 6.7: Pay-off vs. value for a strangle position ($K_c = 12$, $K_p = 9$, $p, q = 0.50$).

The critical points j_1 and j_2 given in this formulation can be obtained in a similar fashion to the one we have used in Chapter 4. Note that j_1 is the minimum integer j such that:

$$\frac{v_{j+1}}{f_{j+1}} > \frac{v_j}{f_j}$$

and j_2 is the greatest integer j such that

$$\frac{v_{j-1}}{f_{j-1}} > \frac{v_j}{f_j}.$$

Thus, we can see, according to the value function above, that the optimal decision to exercise the strangle position is when the stock price is not in the region (j_1, j_2) where j_1 and j_2 solves the system:

$$j_1 = \min\left\{ j : \frac{f_j^g}{f_{j+1}^g} > \frac{A\xi_+^j + B\xi_-^j}{A\xi_+^{j+1} + B\xi_-^{j+1}} \right\}$$
$$j_2 = \max\left\{ j : \frac{f_j^g}{f_{j-1}^g} > \frac{A\xi_+^j + B\xi_-^j}{A\xi_+^{j-1} + B\xi_-^{j-1}} \right\}$$

and

$$A = \frac{\xi_-^{j_1} f_{j_2}^g - \xi_-^{j_2} f_{j_1}^g}{\xi_-^{j_1}\xi_+^{j_2} - \xi_+^{j_1}\xi_-^{j_2}}$$
$$B = \frac{\xi_+^{j_2} f_{j_1}^g - \xi_+^{j_1} f_{j_2}^g}{\xi_-^{j_1}\xi_+^{j_2} - \xi_+^{j_1}\xi_-^{j_2}}.$$

It is algebraically challenging to give a closed form solution for j_1 and j_2 only in terms of the pay-off function f^g and the roots ξ_+, ξ_- as we did in the previous chapters. Nevertheless, it is much easier to verify, once we assume that they are known, that v^* is an optimal solution to P4. We will not go into further detail to state a formal theorem, but a major part of the proof lies in establishing the fact that v^* solves the corresponding CS conditions, which partially follows from the construction of v^* using the same CS conditions.

Chapter 7

Conclusion

In this thesis, we have studied the problem of finding optimal exercising point for perpetual American call options. These are the type of options that can be exercised by the holder at any point in time without a maturity date, and the key problem therefore is to decide if it is the right moment or not to exercise the option. The price movement of the underlying stock considered here is assumed to follow the principles of discrete-time discrete-state Markov processes. This means the price change is governed by a random process where different prices of the underlying correspond to different states of a Markov process, and the stock price moves among these different states according to a discrete probability distribution. Further assumptions have been made that the stock price exhibits the behavior of random walks in general, meaning the stock price can go up or down with a single incremental amount, or it can remain the same. In the end, we apply a linear-programming optimization framework where we show that the candidate solutions we analytically derive are also optimal solutions of the linear programs we formulate.

Looking closer into the process of random walks studied in this thesis, two possible alternatives are explored in detail and analytical results have been derived for each case. The first one is the case of a simple random walk where the stock price increases or decreases by a fixed amount, or it remains the same. We show that an optimal critical threshold value exists under all conditions, which characterizes the set of states at which the holder of the option should exercise. If the current stock price is above the level that corresponds to this threshold value, then the option must be exercised; otherwise it must be retained until this condition is satisfied.

The second case we study in this thesis is the case of geometric random walks where stock prices geometrically increase or decrease (the case of stock price remaining unchanged is not treated). Contrary to the simple walk scenario, this is the multiplicative case where the stock price is multiplied or divided by a predefined factor. We again show the existence and optimality of a critical threshold value that implies the level of stock price above which the option must be exercised. In this case, however, the threshold value exists (and is also optimal) only under certain conditions that are linked to the settings of the problem, including the rate of geometric change, the forward and backward probabilities associated with the geometric change and the discount factor. When these conditions are absent at the time the option holder is deciding, the analysis is indecisive, that is the holder of the option may choose not to exercise. However, depending on how close the expected value function comes near the current value of the option, the holder may still choose to exercise.

Figure 7.1 is a good graphical overview of the results obtained in this work. For a stock-price obeying a simple random walk an exercise region is guaranteed, regardless of the parameters of the model (Figure 7.1 (a) and (b)). The point of exercise, however, depends on the choice of the parameters: if the pay-off ratio is always less than the power term, as in (b), the trader exercises as soon as (s)he observes a positive pay-off from the option contract. For the geometric random walks, on the other hand, (Figure 7.1 (c),(d),(e), and (f)), an exercise region is not always guaranteed. In (c) and (d), we see similar instances to those of simple random walks, ensuring the existence of a critical point. In (e) and (f), the pay-off ratio fails to drop below the power term, thus failing to provide an optimal exercise region. Since the pay-off ratio converges to the geometric factor φ; if, by the choice of parameters, we have $\varphi = \xi_{-}$; for sufficiently large j (as in (e)), the value function can be arbitrarily close to the pay-off function, enabling the trader to define a satisfactory level depending on his/her own preference. Even this fails when $\varphi > \xi_-$ (as in (f)) and the trader always has the expectation of an infinitely growing pay-off.

The key contributions of this work in general are two-fold: a) the set of analytical results and conditions derived to show when a critical point in time is reached for the option to be exercised; and b) the linear programming based optimization approach in deriving the main results presented in the thesis. The former is clearly based on many settings of the problem we have studied, namely the fact that the problem is modeled as a discrete time and discrete state Markov process and that it follows the behaviors of simple and geometric random walks. The latter, however, is a technique that has not been fully exploited in the literature within the context of option pricing. While several studies mention the possibility of using this approach, no study, to the best of our knowledge, has gone as far as this work to actually implement it. The possibility of applying linear optimization in this fashion to other similar problems in this context and their variants is another added benefit resulting from this thesis.

The analytical approach we have taken in this thesis can also be applied to extend this work to address several extensions of this work. One important contribution would be to generalize the results derived for the case of geometric random walk, with the added treatment of the case where the stock price remains unchanged. Another extension would be consideration of perpetual American put options within the same framework. We contend that much of our work is equally applicable to put options; yet a formal study is nevertheless needed. Finally, it would be interesting to investigate if similar analytical results can be derived for option contracts written on multiple securities and the same linear programming optimization approach can be applied.

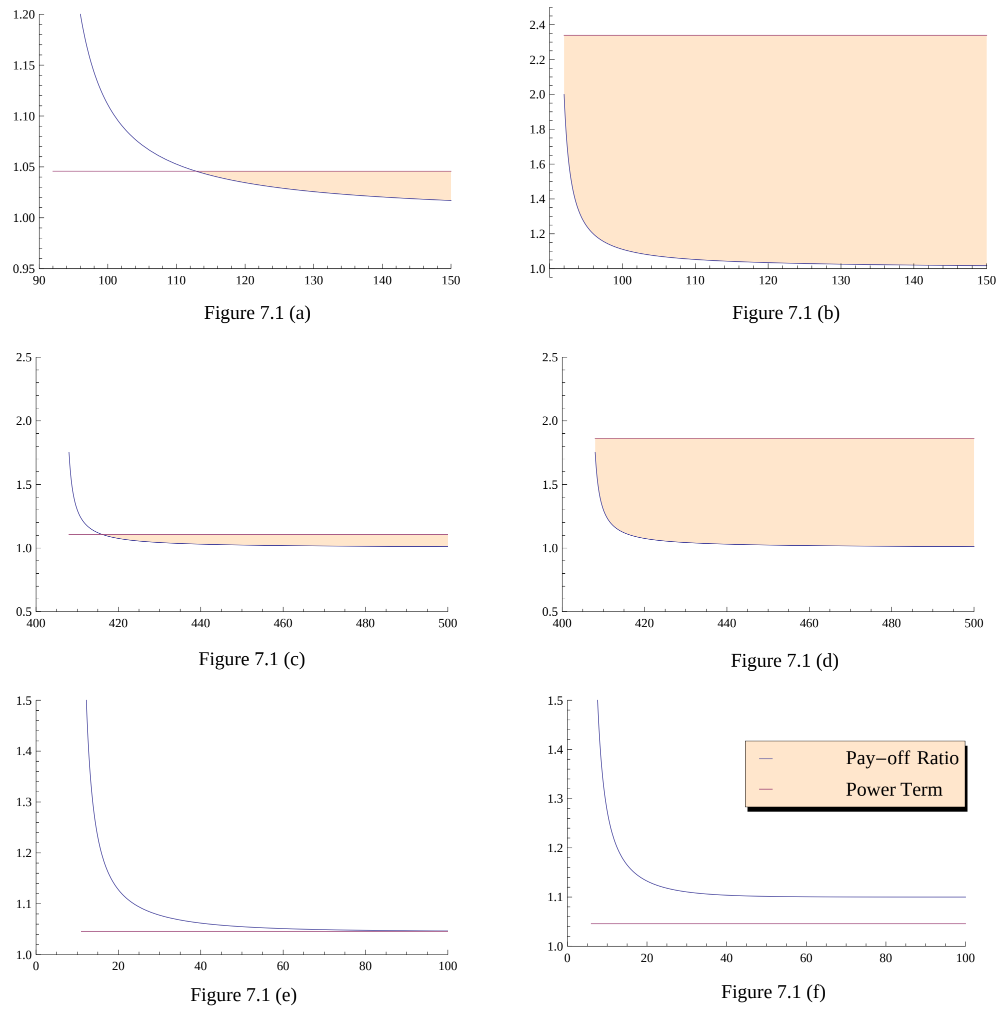

Figure 7.1: Various possibilities for exercise regions under different stock-price movement scenarios. (a) and (b) are for simple; (c),(d),(e) and (f) are for geometric random walks.

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Appendix A

Proofs and Supplementary Derivations

A.1 Showing that v^* is a solution to the difference equation (4.19) for $0 < j < j^*$

First we will show that the property $\xi_+ = \alpha(p\xi_+^2 + q)$ holds, which will be useful later. We have;

$$\begin{aligned}
\alpha(p\xi_+^2 + q) &= \alpha\left[p\left(\frac{-1+\sqrt{1-4\alpha^2pq}}{-2\alpha p}\right)^2 + q\right] \\
&= \alpha\left[p\left(\frac{1+1-4\alpha^2pq-2\sqrt{1-4\alpha^2pq}}{4\alpha^2p^2}\right) + q\right] \\
&= \alpha\left[\frac{1+1-4\alpha^2pq-2\sqrt{1-4\alpha^2pq}+4\alpha^2pq}{4\alpha^2p}\right] \\
&= \frac{2-2\sqrt{1-4\alpha^2pq}}{4\alpha p} = \frac{-1+\sqrt{1-4\alpha^2pq}}{-2\alpha p} = \xi_+.
\end{aligned}$$

Note that the property above also holds for ξ_-. Using this, we can write:

$$\begin{aligned}
\alpha p\Big(\xi_+^{j+1} - \xi_-^{j+1}\Big) + \alpha q\Big(\xi_+^{j-1} - \xi_-^{j-1}\Big) &= \alpha\Big(p\xi_+^{j+1} + q\xi_+^{j-1}\Big) - \alpha\Big(p\xi_-^{j+1} + q\xi_-^{j-1}\Big) \\
&= \alpha\xi_+^{j-1}\Big(p\xi_+^2 + q\Big) - \alpha\xi_-^{j-1}\Big(p\xi_-^2 + q\Big) \\
&= \xi_+^j - \xi_-^j
\end{aligned}$$

Then, for $0 < j < j^*$ we have:

$$\begin{aligned}
v_j &= \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \Big(\xi_+^j - \xi_-^j\Big) \\
&= \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \left[\alpha p\Big(\xi_+^{j+1} - \xi_-^{j+1}\Big) + \alpha q\Big(\xi_+^{j-1} - \xi_-^{j-1}\Big)\right] \\
&= \alpha p\left(f_{j^*} \cdot \frac{\xi_+^{j+1} - \xi_-^{j+1}}{\xi_+^{j^*} - \xi_-^{j^*}}\right) + \alpha q\left(f_{j^*} \cdot \frac{\xi_+^{j-1} - \xi_-^{j-1}}{\xi_+^{j^*} - \xi_-^{j^*}}\right) \\
&= \alpha p v_{j+1} + \alpha q v_{j-1}.
\end{aligned}$$

Thus, v^* is a solution to the difference equation 4.19 for the specified j values, as claimed.

A.2 Useful properties relating ξ_+, ξ_-,ζ_+, ζ_-

For the values α, p and q that make the roots

$$\xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p} \qquad \xi_+ = \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha p}$$

$$\zeta_+ = \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q} \qquad \zeta_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q}$$

real-valued, the following properties hold:

1. $\xi_+ \cdot \zeta_- = 1, \quad \xi_- \cdot \zeta_+ = 1$
2. $\xi_+^j = \zeta_-^{-j}, \quad \xi_-^j = \zeta_+^{-j}$
3. $\xi_+ \cdot \xi_- = \dfrac{q}{p}, \quad \zeta_+ \cdot \zeta_- = \dfrac{p}{q}$

4. $\xi_+^j = \left(\frac{q}{p}\right)^j \xi_-^{-j}$
   $\zeta_+^j = \left(\frac{p}{q}\right)^j \zeta_-^{-j}$

5. $\xi_+ + \xi_- = \frac{1}{\alpha p}$
   $\zeta_+ + \zeta_- = \frac{1}{\alpha q}$.

A.3 Transforming the function z_j^*

It is possible, by using relations in A.2, to transform $z_{j^*-1}^*$ to the given expression. We start with the definition of z_j^* at $j = j^*$:

$$z_{j^*-1}^* = \left(1 - \frac{\zeta_-^{j^*} - 1}{\zeta_-^{j^*} - \zeta_+^{j^*}}\zeta_+^{j^*-1} - \frac{\zeta_+^{j^*} - 1}{\zeta_+^{j^*} - \zeta_-^{j^*}}\zeta_-^{j^*-1}\right) \cdot \frac{1}{1-\alpha}.$$

Using property 2, we transform the corresponding roots:

$$z_{j^*-1}^* = \left(1 - \frac{\xi_+^{-j^*} - 1}{\xi_+^{-j^*} - \xi_-^{-j^*}}\xi_-^{1-j^*} - \frac{\xi_-^{-j^*} - 1}{\xi_-^{-j^*} - \xi_+^{-j^*}}\xi_+^{1-j^*}\right) \cdot \frac{1}{1-\alpha}.$$

Note that we have:

$$\begin{aligned}
\left(\frac{\xi_+^{-j^*} - 1}{\xi_+^{-j^*} - \xi_-^{-j^*}}\right) \cdot \xi_-^{1-j^*} &= \left(\frac{q}{p}\right)^{-j^*} \left(\frac{\xi_-^{j^*} - \left(\frac{q}{p}\right)^{j^*}}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_- \xi_+^{j^*}\right) \\
&= \left(\frac{q}{p}\right)^{-j^*} \left(\frac{\left(\frac{q}{p}\right)^{j^*} - \left(\frac{q}{p}\right)^{+j^*} \cdot \xi_+^{j^*}}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_-\right) \\
&= \frac{\xi_+^{j^*} - 1}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_-
\end{aligned}$$

and by similar operations:

$$\left(\frac{\xi_-^{-j^*} - 1}{\xi_-^{-j^*} - \xi_+^{-j^*}}\right) \cdot \xi_+^{1-j^*} = \frac{\xi_-^{j^*} - 1}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_+$$

leading to the desired result:

$$z^*_{j^*-1} = \left(\frac{1}{1-\alpha}\right) \cdot \left(1 - \frac{\xi_+^{j^*} - 1}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_- + \frac{\xi_-^{j^*} - 1}{\xi_-^{j^*} - \xi_+^{j^*}} \cdot \xi_+\right).$$

In fact it is even possible to express z^*_j in terms of ξ_+ and ξ_- for all $j \in \mathbb{N}$. The relation is given here without proof, which can be derived in a similar fashion to the derivation above.

$$z^*_j = \begin{cases} \frac{1}{1-\alpha} \cdot \left[1 - \frac{\xi_+^{j^*-j} - \xi_-^{j^*-j}}{\xi_+^{j^*} - \xi_-^{j^*}} - \left(\frac{q}{p}\right)^{j^*-j} \cdot \left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}}\right)\right] & \text{if} \quad 0 < j < j^* \\ 0 & \text{if} \quad j^* \leq j. \end{cases}$$

Note that this relation also implies:

$$z^*_j = \begin{cases} \frac{1}{1-\alpha} \cdot \left[1 - \frac{v^*_{j^*-j}}{f_{j^*}} - \left(\frac{q}{p}\right)^{j^*-j} \cdot \left(\frac{v^*_j}{f_{j^*}}\right)\right] & \text{if} \quad 0 < j < j^* \\ 0 & \text{if} \quad j^* \leq j. \end{cases} \tag{A.1}$$

A.4 Solution of second order difference equations with given boundary conditions

The system of linear equations,

$$a_0 v_j + a_1 v_{j-1} + a_2 v_{j-2} = b_0 \quad j \in \mathcal{I} \tag{A.2}$$

is known as a second order linear difference equation on the index set $\mathcal{I}$ where it is convenient to consider $\mathcal{I}$ to be a subset of the set of integers. It is said to be homogeneous if $b_0 = 0$ and non-homogeneous if otherwise. We shall give, here, the solution method adopted in this work to obtain a closed form formula for the unknown v.

It is assumed, a priori, that the solution is of the form ξ^j. If the difference equation encountered is a homogeneous equation, we have:

$$a_0 \xi^j + a_1 \xi^{(j-1)} + a_2 \xi^{(j-2)} = 0.$$

Dividing both sides of the equation by $\xi^{(j-2)}$, we get:

$$a_0\xi^2 + a_1\xi + a_2 = 0,$$

which will have the obvious roots:

$$\xi_+ = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_0} \quad \xi_- = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_0}.$$

The closed form formula for v, then, will be:

$$v_j = A \cdot \xi_+^j + B \cdot \xi_-^j,$$

where the coefficients A and B can be obtained by solving the system:

$$A \cdot \xi_+^{j_1} + B \cdot \xi_-^{j_1} = v_{j_1}$$
$$A \cdot \xi_+^{j_2} + B \cdot \xi_-^{j_2} = v_{j_2}$$

with two known values of v. If, we have a non-homogeneous equation, we can write:

$$v_j = v_j^h + \bar{v}_j$$

where v_j^h is the solution to the homogeneous case and $\bar{v}_j$ is any particular solution to the non-homogeneous case.

Using this method, we can find explicit solutions to both (4.19) and (4.23). First consider (4.19). We have:

$$v_j - \alpha(pv_{j+1} + qv_{j-1}) = 0$$
$$v_{j^*} = 0$$
$$v_0 = 0$$

with the two boundary conditions at $j = 0$ and $j = j^*$. Under the assumption that v_j is of the form ξ^j, dividing both sides of the equation with ξ^{j-1} yields:

$$\alpha p\xi^2 - \xi + \alpha q = 0$$

whose roots are:

$$\xi_+ = \frac{-1 + \sqrt{1 - 4\alpha^2pq}}{-2\alpha p} \quad \xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2pq}}{-2\alpha p}.$$

v_j must, then, be equal to:

$$v_j = C_+ \cdot \xi_+^j + C_- \cdot \xi_-^j.$$

To find coefficients C_+ and C_-, we use the boundary conditions. At $j = 0$ we have:

$$C_+ + C_- = 0 \quad \Rightarrow \quad C_+ = -C_-.$$

Using the first boundary condition, we get:

$$v_{j^*} = C_+ \xi_+^{j^*} - C_+ \xi_-^{j^*} \quad \Rightarrow \quad C_+ = \frac{v_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} = \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}}.$$

Thus, we obtain the function

$$v_j = \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_+^j - \frac{f_{j^*}}{\xi_+^{j^*} - \xi_-^{j^*}} \cdot \xi_-^j = f_{j^*} \left(\frac{\xi_+^j - \xi_-^j}{\xi_+^{j^*} - \xi_-^{j^*}} \right)$$

as a closed form solution of (4.19). Solution of (4.23) is obtained in a similar way. Note that we have a non-homogeneous difference equation in (4.23). We first assume that the non-homogeneous case has a particular solution:

$$\bar{z}_j = c.$$

Then, c must satisfy:

$$c - \alpha(pc + qc) = 1 \quad \Rightarrow \quad c = \frac{1}{1-\alpha}.$$

The homogeneous part is solved similarly outputting the following parameters:

$$\zeta_+ = \frac{-1 + \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q} \qquad \xi_- = \frac{-1 - \sqrt{1 - 4\alpha^2 pq}}{-2\alpha q}$$

and

$$C_+ = \frac{\zeta_-^{j^*} - 1}{(1-\alpha)(\zeta_+^{j^*} - \zeta_-^{j^*})} \qquad C_- = \frac{1 - \zeta_+^{j^*}}{(1-\alpha)(\zeta_+^{j^*} - \zeta_-^{j^*})}.$$

Since $z_j = z_j^h + \bar{z}_j$, we have:

$$z_j = \left(\frac{\zeta_-^{j^*} - 1}{(1-\alpha)(\zeta_+^{j^*} - \zeta_-^{j^*})} \right) \zeta_+^j + \left(\frac{1 - \zeta_+^{j^*}}{(1-\alpha)(\zeta_+^{j^*} - \zeta_-^{j^*})} \right) \zeta_-^j + \frac{1}{1-\alpha},$$

which leads to the given formula after rearranging the terms.

The technique given here can also be generalized to solve difference equations with arbitrary boundary values. Assume we are given a difference equation of the form (A.2) and any two points $j_1,\ j_2 \in \mathcal{I}$ such that $v_{j_1} = f_1$ and $v_{j_2} = f_2$. Roots of the homogeneous case can clearly be obtained with ease. Let's say these are λ_+ and λ_-. The boundaries of the difference equation yield the following system:

$$C_+\lambda_+^{j_1} + C_-\lambda_-^{j_1} = f_{j_1}$$
$$C_+\lambda_+^{j_2} + C_-\lambda_-^{j_2} = f_{j_2}$$

which has the solution:

$$C_+ = \frac{\lambda_-^{j_1} f_{j_2} - \lambda_-^{j_2} f_{j_1}}{\lambda_-^{j_1}\lambda_+^{j_2} - \lambda_+^{j_1}\lambda_-^{j_2}} \qquad C_- = \frac{\lambda_+^{j_2} f_{j_1} - \lambda_+^{j_1} f_{j_2}}{\lambda_-^{j_1}\lambda_+^{j_2} - \lambda_+^{j_1}\lambda_-^{j_2}}. \tag{A.3}$$

In the general sense, then, the homogeneous case has the solution

$$v_j = \left(\frac{\lambda_-^{j_1} f_{j_2} - \lambda_-^{j_2} f_{j_1}}{\lambda_-^{j_1}\lambda_+^{j_2} - \lambda_+^{j_1}\lambda_-^{j_2}}\right)\lambda_+^j + \left(\frac{\lambda_+^{j_2} f_{j_1} - \lambda_+^{j_1} f_{j_2}}{\lambda_-^{j_1}\lambda_+^{j_2} - \lambda_+^{j_1}\lambda_-^{j_2}}\right)\lambda_-^j \quad \forall j_1 < j < j_2.$$

The reader can easily verify that this general solution reduces to the solution we have just obtained for the $0 - j^*$ instance.

We may also encounter, on some instances of the problem, a boundary value of the form:

$$v_{\pm\infty} = 0$$

implying that the value function reduces asymptotically to 0 as the index value is increased without bound. An example to this is the geometric random walk case where the stock price never reduces to zero, but the value function must tend to zero accompanying the price of the stock. Suppose that this leads to the boundary condition:

$$C_+\left(\lim_{j\to-\infty}\lambda_+^j\right) + C_-\left(\lim_{j\to-\infty}\lambda_-^j\right) = 0.$$

If it is known that one of the roots is strictly greater than 1, as it is the case for models studied in this volume, the boundary condition reduces to an equation which is in terms of the second root. Say $\lambda_- > 1$. Clearly we have $\lim_{j\to-\infty}\lambda_-^j = 0$. Then, the reduced equation will be:

$$C_+\left(\lim_{j\to-\infty}\lambda_+^j\right) = 0.$$

Note that for $\lambda_+ < 1$, λ_+^j grows unboundedly as $j \to -\infty$. The only possibility for C_+ to satisfy the boundary condition is, thus, to have the value 0. The coefficient C_- can, then, be solved with the knowledge of another boundary condition. Say, for some integer j', the value of v is known, that is $v_{j'} = K$ where K represents this known value. C_- will clearly be equal to $K \cdot \lambda_-^{-j'}$ and the solution to the homogeneous case will be:

$$v_j = K \cdot \lambda_-^{j-j'}.$$

The opposite direction, that is a value function converging to zero as j tends to ∞ is left to the reader. Note that the arguments presented here are valid provided that a finite j' for which the value of v is known exists. Clearly, zero boundaries as j tends to ∞ and $-\infty$ is not applicable since it implies $C_- = C_+ = 0$. The various settings studied in this volume, however, will all have well-defined finite boundary points with finite values.