MEASURING SELF-SELECTIVITY VIA GENERALIZED CONDORCET RULES

A Master's Thesis

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July 2011

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray
Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

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July 2011

In this thesis, we introduce a method to measure self-selectivity of social choice functions. Due to Koray [2000], a neutral and unanimous social choice function is known to be universally self-selective if and only if it is dictatorial. Therefore, in this study, we confine our set of test social choice functions to particular singleton-valued refinements of *generalized Condorcet rules*. We show that there are some non-dictatorial self-selective social choice functions. Moreover, we define the notion of *self-selectivity degree* which enables us to compare social choice functions according to the strength of their self-selectivities. We conclude that the family of generalized Condorcet functions is an appropriate set of test social choice functions when we localize the notion of self-selectivity.

Keywords: Social choice, Self-selectivity, Self-selectivity degree, Generalized Condorcet rules

ÖZET

GENELLEŞTİRİLMİŞ CONDORCET KURALLARI İLE KENDİNİ-SEÇERLİĞİN ÖLÇÜLMESİ

ALTUNTAŞ, Açelya

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Bu tez çalışmamızda, sosyal seçim fonksiyonlarının kendini-seçerliğini ölçmeye yarayan bir yöntem sunulmaktadır. Koray [2000]'dan dolayı, nötr ve oybirlikçi bir sosyal seçim fonksiyonu ancak ve sadece diktatörlük olduğunda evrensel kendini-seçerdir. Bu yüzden, bu çalışmada, sosyal seçim fonksiyonlarının test kümesi, tek-değerli genelleştirilmiş Condorcet kuralları inceltmelerine sınırlandırılmaktadır. Bu kısıtlama altında, diktatörlük olmayan kendini-seçer sosyal seçim fonksiyonları olduğu gösterilmektedir. Ayrıca, sosyal seçim fonksiyonlarının kendini-seçerlik kuvvetlerine göre karşılaştırılmasını sağlayan kendini-seçerlik derecesi kavramı tanıtılmaktadır. Kendini-seçerlik kavramı yerel hale getirildiği zaman, elde edilen genelleştirilmiş Condorcet fonksiyonlarının sosyal seçim fonksiyonlarının test kümesi için uygun olduğu gösterilmektedir.

Anahtar Kelimeler: Sosyal seçim, Kendini-seçerlik, Kendini-seçerlik derecesi, genelleştirilimiş Condorcet kuralları

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CHAPTER 1

INTRODUCTION

Self-selectivity of a social choice function (SCF) is concerned with "choosing how to choose". We imagine a society, which is going to make a choice from a given set A of alternatives, is also to choose the choice function to be employed in its choice from A. Here a natural question arises that concerns consistency between the choice from the set A of alternatives and the set $\mathcal{A}$ of available SCFs. More specifically, the society's preference profile on A induces a preference profile on $\mathcal{A}$ where the SCFs are ranked according to the alternatives they choose over the initial preference profile on A. So, the question now is whether an SCF F chooses itself, if it is used to make the choice of the choice function from among any finite set of SCFs including F. If it does so, then F will be called as self-selective. If it does not, then this failure can be regarded as a lack of consistency on the part of this SCF F.

By Koray [2000], it is well known that a unanimous and neutral SCF is universally self-selective if and only if it is dictatorial. The universality of self-selectivity of an SCF F is that it selects itself among *any* finite set of SCFs including F itself. There are two frequently used methods in social choice theory when one wishes to escape impossibility results. One is the restriction of the domain of preference profiles. The other one allows the social choice rules (SCR) considered to be set-valued rather than singleton-valued. In addition to these two approaches, there is a third way which is peculiar to self-selectivity. It consists of restricting the set of SCFs against which self-selectivity is to be tested. In this study, we focus on restricting our test SCFs to a particular family which is different than all families that have been employed in previous studies.

Either of these three methods may or may not end up with escaping dictatoriality depending upon the particular way the method in question is employed. In order to escape impossibility, Ünel [1999] restricts the domain of preference profiles to single-peaked ones and thereby provides a whole class of non-dictatorial self-selective SCFs. Another result that allows the existence of non-dictatorial self-selective SCRs is achieved by Koray [1998]. By allowing the SCRs considered to be set-valued, he proves that any neutral top-majoritarian SCR which is self-selective at preference profiles where Condorcet winner exists is a refinement of Condorcet rule. That is, he concludes that the Condorcet rule is the maximal neutral and self-selective SCR at such preference profiles. More recently, Koray and Slinko [2008] also find some self-selective non-dictatorial SCFs by relaxing universal self-selectivity. They start with a social choice correspondence (SCC) which can be thought of as a constitutional rule reflecting the norms that a society wishes to adhere, and restrict their test functions to singleton-valued refinements thereof. In particular, they prove that if an SCF is a refinement of Pareto correspondence and self-selective relative to any set of test SCFs which are refinements of Pareto correspondence, then it is either dictatorial or Pareto anti-dictatorial. Although Koray and Ünel [2003] utilize a similar method to Koray and Slinko [2008], they end up with only dictatorial SCFs. The difference is that they restrict the set of available SCFs to tops-only ones. However, it turns out that dictatoriality cannot be escaped by this particular restriction of test SCFs.

A natural question concerning a non-dictatorial, thus a non-universally self-selective SCF F is "how self-selective it is". F may not be choosing itself from a particular set of test SCFs rendering it non-self-selective. However, it is only natural to consider an SCF F to be more self-selective in case it beats more rivals by choosing itself from among them. If self-selectivity is regarded as a particular measure of consistency on the part of an SCF, then it becomes important to introduce a proper measure of self-selectivity. One obvious candidate is associating with each SCF the maximal sets of SCFs that it beats in terms of self-selectivity. In this study, we employ a special family of test SCFs, namely singleton-valued refinements of generalized Condorcet rules, to that end.

Roughly speaking, for each $q \in [0, 1]$, an alternative is a *q-Condorcet winner* if it defeats any other alternative in pairwise q-majority. The usual definition of a Condorcet winner corresponds to $q = \frac{1}{2}$. There are three main reasons why we take particular singleton-valued refinements of generalized Condorcet rules as our test functions for self-selectivity. Firstly, we can hardly disclaim the central position that the Condrocet rule occupies in social choice theory, which is only confirmed by its closeness to self-selectivity established by Koray [1998]. Secondly, different q-Condorcet functions exhibit a well-behaved pattern concerning self-selectivity in the sense that the degree of self-selectivity increases as q increases. Finally, in this framework, testing a given SCF for self-selectivity against each test function separately turns out to be equivalent to testing it against collections of arbitrary sets of SCFs of finite sizes. In addition to the simplicity it brings to the analysis, one can also expect the measure of self-selectivity introduced via q-Condorcet rules to reflect a genuine yardstick for self-selectivity.

After formally defining the notion of *self-selectivity degree* relative to q-Condorcet rules, we apply this notion to *q-Condorcet* functions, *p-qualified majority* functions, some special scoring functions and majoritarian compromise. We modify the notion of self-selectivity degree when we deal with k-plurality rules as strictly speaking the degree notion does not apply to them directly as it stands. We thereby obtain examples of non-dictatorial SCFs which are not universally self-selective, but self-selective to a large extent.

In the next chapter, we introduce some basic definitions. Chapter 3 starts with an illustrative example and shows some useful properties of the family of generalized Condorcet rules. Chapter 4 reports a sequence of results about some families of SCFs. Finally, Chapter 5 closes the thesis with some concluding remarks.

CHAPTER 2

PRELIMINARIES

Let N be a finite nonempty set of individuals with $\mid N \mid= n$. Let $\mathbb{N}$ denote the set of natural numbers, set $I_m = \{1, \ldots, m\}$ and denote the set of all linear orders on I_m by $\mathcal{L}(I_m)$ for each $m \in \mathbb{N}$.

Definition 1. A function $F : \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n \rightarrow \mathbb{N}$ is called a *social choice function* (SCF) if, for each $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, one has $F(R) \in I_m$. We denote the set of all SCFs by $\mathcal{F}$.

Take any finite set A with $\mid A \mid= m \in \mathbb{N}$. Let $\mu : I_m \rightarrow A$ be a bijection, i.e., a one-to-one and onto function. Now, any linear order profile L on A induces a linear order profile L_μ on I_m as follows: For all $i \in N$ and $k, l \in I_m$, one has $kL^i_\mu l$ if and only if $\mu(k)L^i\mu(l)$. We define $F(L) = \mu(F(L_\mu))$, where μ is a bijection from I_m to A.

For each $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ and permutation σ_m on I_m, we define the permuted linear order profile R_{σ_m} on I_m as follows: For all $v \in N$, $a_i, a_j \in I_m$ one has $a_i R^v_{\sigma_m} a_j \iff \sigma_m(a_i)R^v\sigma_m(a_j)$.

Definition 2. $F \in \mathcal{F}$ is called *neutral* if, for each $m \in \mathbb{N}$, σ_m on I_m, one has $\sigma_m(F(R_{\sigma_m})) = F(R)$. We denote the set of all neutral SCFs by $\mathcal{N}$

Note that, neutrality of an SCF F implies that the labelling of the alternatives does not matter and, also, it allows us to extend the domain of F to linear order profiles on any finite nonempty set. It is clear that $\mu(F(L_\mu)) = v(F(L_v))$ for any two bijections $\mu, v : I_m \to A$ if F is neutral. However, as we also consider SCFs which are not neutral in this thesis, the bijection μ that is used will matter.

Take any $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ and nonempty finite subset $\mathcal{A}$ of $\mathcal{F}$. Define for all $F, G \in \mathcal{A}$ and $i \in N$, $FR^i_{\mathcal{A}}G$ if and only if $F(R)R^iG(R)$. Note that $R^i_{\mathcal{A}}$ is a complete preorder on $\mathcal{A}$ as more than one SCF in $\mathcal{A}$ can choose the same alternative in I_m. Thus, any linear order profile $R \in \mathcal{L}(I_m)^n$ induces a preference profile $R_{\mathcal{A}}$ on any nonempty finite subset $\mathcal{A}$ of $\mathcal{F}$.

Definition 3. Let $R^i_{\mathcal{A}}$ be a complete preorder on $\mathcal{A}$. A linear order L^i is said to be *compatible* with $R^i_{\mathcal{A}}$ if, for all $F, G \in \mathcal{A}$, $FR^i_{\mathcal{A}}G$ is implied by FL^iG. The set of all linear order profiles on $\mathcal{A}$ is denoted by $L(\mathcal{A})^n$.

Definition 4. For all $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ and nonempty finite subset $\mathcal{A}$ of $\mathcal{N}$, define the set of all linear order profiles on $\mathcal{A}$ induced by R, $\mathcal{L}(\mathcal{A}, R)$, as follows: $\mathcal{L}(\mathcal{A}, R) = \{L \in L(\mathcal{A})^n \mid L^i$ is a linear order on $\mathcal{A}$ compatible with $R^i_{\mathcal{A}}$ for each $i \in N\}$.

For each nonempty finite subset $\mathcal{A}$ of $\mathcal{F}$, choose and fix a bijection $\mu_{\mathcal{A}} : I_m \to \mathcal{A}$, where $\mid \mathcal{A} \mid = m$. Given an SCF $F : \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n \to \mathbb{N}$, for each nonempty finite subset $\mathcal{A}$ of $\mathcal{F}$, we obtain an extension $F : \mathcal{L}(\mathcal{A})^n \to \mathcal{A}$ of F via $\mu_{\mathcal{A}}$. Note that here we use the same symbol F for both the given SCF and its extension to $\mathcal{L}(\mathcal{A})^n$, which we will continue to do in the sequel. This will lead to no ambiguity so long as the family of bijection $\{\mu_{\mathcal{A}}\}$ is kept fixed.

Definition 5. i. Given $F \in \mathcal{F}$, $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ and a finite subset $\mathcal{A}$ of $\mathcal{F}$ with $F \in \mathcal{A}$, we say that F is *self-selective at R relative to $\mathcal{A}$ with respect to* $\{\mu_{\mathcal{A}}\}$ if there exists some $L \in \mathcal{L}(\mathcal{A}, R)$ such that $F = F(L)$.

ii. F is said to be *self-selective at R with respect to* $\{\mu_{\mathcal{A}}\}$ if F is self-selective at R relative to any finite subset $\mathcal{A}$ of $\mathcal{F}$ with $F \in \mathcal{A}$ with respect to $\{\mu_{\mathcal{A}}\}$.

iii. F is said to be *universally self-selective with respect to* $\{\mu_{\mathcal{A}}\}$ if F is self-selective at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$ relative to any finite subset $\mathcal{A}$ of $\mathcal{F}$ with $F \in \mathcal{A}$ with respect to $\{\mu_{\mathcal{A}}\}$.

Definition 6. Let $\mid N \mid= n$, $\mathcal{A} \subseteq \mathcal{F}$ be given. An SCF $F \in \mathcal{F}$ is said to be *self-selective relative to* $\mathcal{A}$ if there is some $\{\mu_{\mathcal{A}}\}$ such that F is self-selective at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$ relative to $\mathcal{A}$ with respect to $\{\mu_{\mathcal{A}}\}$.

Definition 7. An SCF $F \in \mathcal{F}$ is said to be *unanimous* if, for all $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ and $a \in I_m$ we have $[\forall i \in N, \forall b \in I_m : aR^i b] \Rightarrow F(R) = a$.

Definition 8. An SCF $F \in \mathcal{F}$ is said to be *dictatorial* if and only if $\exists i \in N, \forall m \in \mathbb{N}, \forall R \in \mathcal{L}(I_m)^n$ such that $F(R) = \arg\max_{I_m} R^i$.

Koray [2000] shows that when $m \geq 3$ any neutral and unanimous SCF F is universally self-selective if and only if it is dictatorial.

Remark 1. Take any non-dictatorial SCF $F \in \mathcal{F}$. Let F be tested only against itself, i.e. $\mathcal{A} = \{F\} \subset \mathcal{F}$. Then F is trivially self-selective relative to $\mathcal{A}$. On the other hand, if we let $\mathcal{A} = \mathcal{N}$ then, by Koray [2000], F is not self-selective relative to $\mathcal{A}$ since it is a non-dictatorial SCF. So, we conclude that there exists a maximal finite nonempty subset $\mathcal{A}$ of $\mathcal{N}$ such that F is self-selective relative to $\mathcal{A}$.

Definition 9. Given any $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, $q \in [0,1]$, an alternative $a \in I_m$ is said to be a *q-Condorcet winner* at R if $\mid \{i \in N \mid aR^i b\} \mid \geq nq$ for all $b \in I_m \setminus \{a\}$.

We denote the set of all q-Condorcet winners at $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$ by $CW_q(R)$. An SCR C_q is called the *q-Condorcet rule* if it selects all q-Condorcet winners at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$.

Remark 2. Take any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$. For $q = 0$, $C_q(R) = I_m$. For $q = 1$ we have $C_q(R) = \{a\}$ if $L(a, R^i) = I_m$ for each $i \in N$ and $C_q(R) = \emptyset$ otherwise.

We only consider societies with odd number of individuals, i.e., $n = 2k+1$ where $k \geq 1$ is an integer. Moreover, for any $m \in \mathbb{N}$, we fix the usual ordering on I_m, so we have $1 < 2 < \cdots < m$.

Definition 10. Given $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ the *q-Condorcet function*, C_q, is defined by:

$$C_q(R) = \begin{cases} 1 & \text{if } CW_q(R) = \emptyset \\ \min\{CW_q(R)\} & \text{if } CW_q(R) \neq \emptyset \end{cases}$$

Basically, for $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)$, if the set of q-Condorcet winners is empty, then the q-Condorcet function chooses the minimal alternative of I_m relative to the ordering defined above. If the winner set is non-empty, then the q-Condorcet function chooses the minimal alternative of the winner set relative to the ordering that we defined.

For any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$, let $CW_q(L)$ be the set of all q-Condorcet winners at $L \in \mathcal{L}(\mathcal{A}, R)$. Now, given $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, $\mathcal{A} \subseteq \mathcal{F}$, the self-selectivity of the q-Condorcet function relative to $\mathcal{A}$ is defined as follows:

- When $CW_q(L) = \emptyset$ for some $L \in \mathcal{L}(\mathcal{A}, R)$, C_q is self-selective at R relative to $\mathcal{A}$.

- When $CW_q(L) \neq \emptyset$ for each $L \in \mathcal{L}(\mathcal{A}, R)$, C_q is self-selective at R relative to $\mathcal{A}$ if $C_q \in CW_q(L)$ for some $L \in \mathcal{L}(\mathcal{A}, R)$.

Note that there always is a bijection $\mu_{\mathcal{A}} : \mathcal{A} \to I_k$, where $k = |\mathcal{A}|$, such that $\mu_{\mathcal{A}}(C_q)$ is minimal in $\mu_{\mathcal{A}}(CW_q(L))$. Thus, the definition is consistent with our general definition of self-selectivity at R relative to $\mathcal{A}$.

CHAPTER 3

GENERALIZED CONDORCET FUNCTIONS AND SELF-SELECTIVITY DEGREE

We, first, test the self-selectivity of C_q relative to $\mathcal{A} = \{C_q, C_{q'}\}$ for each $q' \in (0, 1]$ and obtain some useful properties of the family of particular singleton-valued refinements of generalized Condorcet rules. Then, we define the notion of *self-selectivity degree* of an SCF relative to q-Condorcet rules to measure self-selectivity of SCFs.

Before proceeding further, it will be illuminating to see how the self-selectivity of C_q differs relative to $\mathcal{A}' = \{C_q, C_{q'}\}$ where $q, q' \in (0, 1]$ are such that $q < q'$, $CW_{q'}(R) \subseteq CW_q(R)$, and $\mathcal{A}'' = \{C_q, C_{q''}\}$ where $q, q'' \in (0, 1]$ are such that $q'' < q$, $CW_q(R) \subsetneqq CW_{q''}(R)$ at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$.

3.1 Example

Consider a society $N = \{\alpha, \beta, \gamma, \delta, \zeta\}$ consisting of five individuals. Take $C_{\frac{1}{2}}, C_{\frac{2}{3}}, C_{\frac{1}{3}} \in \mathcal{F}$, which are all unanimous. Now let us consider the following linear order profile $R \in I_4$:

R^α	R^β	R^γ	R^δ	R^ζ
2	2	3	3	4
1	1	2	2	3
4	4	1	1	2
3	3	4	4	1

First consider the case where $C_{\frac{1}{2}}$ is tested only against $C_{\frac{2}{3}}$, i.e., the set of available SCFs is $\mathcal{A}' = \{C_{\frac{1}{2}}, C_{\frac{2}{3}}\}$. We have $CW_{\frac{1}{2}}(R) = CW_{\frac{2}{3}}(R) = \emptyset$ implying that $C_{\frac{1}{2}}(R) = C_{\frac{2}{3}}(R) = 1$. The complete preorder $R_{\mathcal{A}'}$ on $\mathcal{A}'$ induced by R is represented in the following table with a comma separating alternatives indicating an indifference class:

$R^\alpha_{\mathcal{A}'}$	$R^\beta_{\mathcal{A}'}$	$R^\gamma_{\mathcal{A}'}$	$R^\delta_{\mathcal{A}'}$	$R^\zeta_{\mathcal{A}'}$
$C_{\frac{1}{2}}, C_{\frac{2}{3}}$	$C_{\frac{1}{2}}, C_{\frac{2}{3}}$	$C_{\frac{1}{2}}, C_{\frac{2}{3}}$	$C_{\frac{1}{2}}, C_{\frac{2}{3}}$	$C_{\frac{1}{2}}, C_{\frac{2}{3}}$

Thus, we have 2^4 linear order profiles compatible with the above complete preorder profile in each component. The linear order profile L' is a member of $\mathcal{L}(\mathcal{A}', R)$:

L'^α	L'^β	L'^γ	L'^δ	L'^ζ
$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$
$C_{\frac{2}{3}}$	$C_{\frac{2}{3}}$	$C_{\frac{2}{3}}$	$C_{\frac{2}{3}}$	$C_{\frac{2}{3}}$

Since $C_{\frac{1}{2}}(L') = C_{\frac{1}{2}}$, we conclude that $C_{\frac{1}{2}}$ is self-selective at R relative to $\mathcal{A}'$. Roughly speaking, $C_{\frac{1}{2}}$ is self-selective at R when it is tested against a less *generous* SCF, namely $C_{\frac{2}{3}}$.

Now consider the case where the set of available SCFs, $\mathcal{A}''$, consists of only $C_{\frac{1}{2}}$ and $C_{\frac{1}{3}}$, i.e., $\mathcal{A}'' = \{C_{\frac{1}{2}}, C_{\frac{1}{3}}\}$. Since $CW_{\frac{1}{3}}(R) = \{2, 3\}$, we have $C_{\frac{1}{3}}(R) = 2$. Thus, $\mathcal{L}(\mathcal{A}'', R)$ consists of one member L'' only, where:

L''^{α}	L''^{β}	L''^{γ}	L''^{δ}	L''^{ζ}
$C_{\frac{1}{3}}$	$C_{\frac{1}{3}}$	$C_{\frac{1}{3}}$	$C_{\frac{1}{3}}$	$C_{\frac{1}{3}}$
$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$	$C_{\frac{1}{2}}$

Now, $C_{\frac{1}{2}}(L'') = C_{\frac{1}{3}} \neq C_{\frac{1}{2}}$. Since $\mathcal{L}(\mathcal{A}'', R) = \{L''\}$, this means that $C_{\frac{1}{2}}$ is not self-selective at R relative to $\mathcal{A}''$. That is, $C_{\frac{1}{2}}$ is not self-selective at R when it is tested against a more *generous* SCF $C_{\frac{1}{3}}$.

In the following proposition, we generalize the result that we provide in the above example and thereby show that q-Condorcet functions exhibit a well-behaved pattern in terms of self-selectivity. That is, any q-Condorcet function chooses itself whenever it is tested against a less generous Condorcet function and fails to choose itself whenever it is tested against a more generous Condorcet function.

3.2 Results

Proposition 1. *Let N be a finite nonempty set of individuals and $q \in (0, 1]$ be given.*

1. *C_q is self-selective relative to $\mathcal{A} = \{C_q, C_{q'}\}$, where $q' \in (0, 1]$ is such that $q < q'$ and $CW_{q'}(R) \subseteq CW_q(R)$ at any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$.*

2. *C_q is not self-selective relative to $\mathcal{A} = \{C_q, C_{q'}\}$, where $q' \in (0, 1]$ is such that $q' < q$ and $CW_q(R) \subsetneq CW_{q'}(R)$ at any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$.*

Proof. First, note that, given $m \in \mathbb{N}, R \in \mathcal{L}(I_m)^n, CW_q(R) = CW_{\frac{l+1}{n}}(R)$ for any $q \in (\frac{l}{n}, \frac{l+1}{n}]$, where l is an integer from the set $\{0, 1, \ldots, n-1\}$. Now take any $q \in (0, 1]$, and let $\mathcal{A} = \{C_q, C_{q'}\}$ for some $q' \in (0, 1]$.

Case 1. Let $q' \in (0, 1]$ be such that $q < q'$ and $CW_{q'}(R) \subseteq CW_q(R)$ at any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$. Now, take any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$. If $CW_q(R) = \emptyset$, then $CW_{q'}(R) = \emptyset$. Thus, $C_q(R) = C_{q'}(R) = \{1\}$. Hence, C_q is self-selective at R relative to $\mathcal{A}$. If $CW_q(R) \neq \emptyset$, then $C_q \in CW_q(L)$ for any $L \in \mathcal{L}(\mathcal{A}, R)$. Therefore, C_q is self-selective at R relative to $\mathcal{A}$.

Case 2. Let $q' \in (0,1]$ be such that $q' < q$ and $CW_q(R) \subsetneq CW_{q'}(R)$ at any $R \in \cup_{m\in\mathbb{N}}\mathcal{L}(I_m)^n$. Then we have $\lceil nq' \rceil < \lceil nq \rceil$ as $CW_q(R) \subsetneq CW_{q'}(R)$ at any $R \in \cup_{m\in\mathbb{N}}\mathcal{L}(I_m)^n$. Set $r = \frac{n}{\lceil nq' \rceil}$, and consider $\lfloor r \rfloor$. Now let $m = \lfloor r \rfloor + 2$, and construct a preference profile $\tilde{R} \in \mathcal{L}(I_m)^n$ as follows: For $i \in \{(\lfloor r \rfloor - t)\lceil nq' \rceil + 1, \ldots, (\lfloor r \rfloor - t + 1)\lceil nq' \rceil\}$, let $L(m-t, \tilde{R}^i) = I_m$ where $t \in \{1, 2, \ldots, \lfloor r \rfloor\}$, $(m-s)\tilde{R}^i(m-s-1)$ for any $s \in \{0, 1, \ldots, m-2\}$ and $1\tilde{R}^i m$. For $i \in \{\lfloor r \rfloor \lceil nq' \rceil + 1, \ldots, n\}$, let $L(m, R^i) = I_m$, and $(m-s)\tilde{R}^i(m-s-1)$ for any $s \in \{0, 1, \ldots, m-2\}$. Pictorially, $\tilde{R}$ is defined as follows:

$\tilde{R}^1 \cdots \tilde{R}^{\lceil nq' \rceil}$	$\tilde{R}^{\lceil nq' \rceil+1} \cdots \tilde{R}^{2\lceil nq' \rceil}$	
2	3	
1	2	
m	1	$\cdots$
$m-1$	m	
$\vdots$	$\vdots$	
3	4	

	$\tilde{R}^{(\lfloor r \rfloor -1)\lceil nq' \rceil+1} \cdots \tilde{R}^{\lfloor r \rfloor \lceil nq' \rceil}$	$\tilde{R}^{\lfloor r \rfloor \lceil nq' \rceil+1} \cdots R^n$
	$m-1$	m
	$m-2$	$m-1$
$\cdots$	$\vdots$	$\vdots$
	2	3
	1	2
	m	1

Now for any $a \in I_m \setminus \{1\}$, we have $\mid \{i \in N \mid a\tilde{R}^i(a+1)\} \mid = \lceil nq' \rceil < \lceil nq \rceil$. Therefore, $a \notin CW_q(\tilde{R})$, in particular $C_q(\tilde{R}) \neq a$. Moreover for each $i \in N$ $2\tilde{R}^i 1$, thus $1 \notin CW_q(\tilde{R})$. Hence $CW_q(\tilde{R}) = \emptyset$, so $C_q(\tilde{R}) = 1$. On the other hand, $2 \in CW_{q'}(\tilde{R})$ and $1 \notin CW_{q'}(\tilde{R})$ implying that $C_{q'}(\tilde{R}) = 2$.

So, we have $C_{q'}L^iC_q$ for each $i \in N$, where $\mathcal{L}(\mathcal{A}, \tilde{R}) = L$, which implies that $C_q(L) = C_{q'} \neq C_q$. Hence, C_q is not self-selective at $\tilde{R}$ relative to $\mathcal{A}$, thus it is not self-selective relative to $\mathcal{A}$. □

Definition 11. An SCF F is said to be of *degree (1-q)* if it is self-selective relative to $\mathcal{A} = \{F, C_{q'}\}$ for any $q' \in (q, 1]$, and it is not self-selective relative to $\mathcal{A} = \{F, C_{q'}\}$ for some $q' \in (0, q]$.

Remark 3. By previous proposition, given $\mid N \mid = n$, C_q has degree $\frac{n-l}{n}$ where $q \in (\frac{l}{n}, \frac{l+1}{n}]$ for some integer $l \in \{0, 1, \dots, n-1\}$.

An immediate corollary to the above proposition shows the maximal subset, $\mathcal{A}_r$, of the set of rival SCFs such that C_q is self-selective relative to $\mathcal{A} = \{C_q\} \cup \mathcal{A}_r$.

Corrolary 1. *Let N be a finite nonempty set of individuals and $q \in (0, 1]$ be such that $q \in (\frac{l}{n}, \frac{l+1}{n}]$ for some integer $l \in \{0, 1, \dots, n-1\}$. Now, $\mathcal{A}_r = \{C_{q'} \mid q' \in (\frac{l}{n}, 1]\}$ is the maximal subfamily of $\{C_q \mid q \in (0, 1]\}$ such that C_q is self-selective relative to $\mathcal{A} = \{C_q\} \cup \mathcal{A}_r$.*

Proof. First note that by previous proposition, $C_{q'} \notin \mathcal{A}_r$ for any $q \in (0, \frac{l}{n}]$. Let $m \in \mathbb{N}, R \in \mathcal{L}(I_m)^n$ be given. If $CW_q(R) = \emptyset$ then for any $q' \in (\frac{l}{n}, 1]$, $CW_{q'}(R) = \emptyset$. So, $C_q(R) = C_{q'}(R) = 1$ for any $C_{q'} \in \mathcal{A}_r$, implying that C_q is self-selective at R relative to $\mathcal{A} = \{C_q\} \cup \mathcal{A}_r$. If $CW_q(R) \neq \emptyset$, then $C_q(R) \in CW_q(L)$ for any $L \in \mathcal{L}(\mathcal{A}, R)$. Therefore, C_q is self-selective at R relative to $\mathcal{A}$. Hence, C_q is self-selective relative to $\mathcal{A}_r$. □

Remark 4. If the self-selectivity degree of an SCF F increases, then F becomes more self-selective.

The above corollary provides a useful property of the family of generalized Condorcet functions. By the previous proposition, a Condorcet function, C_q, is not self-selective when it is tested against a more *generous* Condorcet function $C_{q'}$. So, the corollary implies that, C_q fails to choose itself among any set of rival SCFs including $C_{q'}$. Furthermore, if a Condorcet function chooses itself in pairwise tests with other Condorcet functions, then it also chooses itself after the aggregation of the test SCFs.

CHAPTER 4

SELF-SELECTIVITY DEGREES OF SOME FAMILIES OF SOCIAL CHOICE FUNCTIONS

4.1 p-Qualified Majority Functions

Now, given $m \in \mathbb{N}, \lambda \in \mathcal{L}(I_m)$, write $\tau(\lambda) = a$ if and only if $L(a, \lambda) = I_m$ for some $a \in I_m$. For any $m \in \mathbb{N}, R \in \mathcal{L}(I_m)^n$ let $T(R) = \{\tau(R^i) : i \in N\}$.

Definition 12. Let $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$ be given. An alternative $a \in T(R)$ is said to be a *p-qualified majority winner* for some $p \in [0, 1]$ if $\mid \{i \in N : L(a, R^i) = I_m\} \mid \geq np$.

We denote set of all p-qualified majority winners by $MW_p(R)$ at each $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$. An SCR M_p is said to be a *p-qualified majority rule* if it selects all p-qualified majority winners at each $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$.

Definition 13. Given $m \in \mathbb{N}, R \in \mathcal{L}(I_m)^n$, the *p-qualified majority function*, M_p, is defined by:

$$M_p(R) = \begin{cases} 1 & \text{if} \quad MW_p(R) = \emptyset \\ \min\{MW_p(R)\} & \text{if} \quad MW_p(R) \neq \emptyset \end{cases}$$

Now, let $MW_p(L)$ be the set of all p-qualified majority winners where

$L \in \mathcal{L}(\mathcal{A}, R)$ for $R \in \mathcal{L}(I_m)^n$. Given $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, the self-selectivity of the *p-qualified majority function* relative to $\mathcal{A}$ is defined as follows[1] :

- When $MW_p(L) = \emptyset$ for some $L \in \mathcal{L}(\mathcal{A}, R)$, then M_p is trivially self-selective at R relative to $\mathcal{A}$.

- When $MW_p(L) \neq \emptyset$ for each $L \in \mathcal{L}(\mathcal{A}, R)$, M_p is self-selective at R relative to $\mathcal{A}$ if $M_p \in MW_p(L)$ for some $L \in \mathcal{L}(\mathcal{A}, R)$.

Proposition 2. *Let N be a finite nonempty society with $n \geq 3$.*

1. *M_p is self-selective relative to $\mathcal{A} = \{M_p, C_q\}$ for every $q \in (\frac{n-1}{n}, 1]$ when $p \in (\frac{1}{n}, 1]$.*

2. *M_p is self-selective relative to $\mathcal{A} = \{M_p, C_q\}$ for every $q \in (0, 1]$ when $p \in [0, \frac{1}{n}]$.*

Proof. (1) Take any $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, $q \in (\frac{n-1}{n}, 1]$ and let $\mathcal{A} = \{M_p, C_q\}$, where $p \in (\frac{1}{n}, 1]$. First, consider the case where $CW_q(R) \neq \emptyset$. Then we have $MW_p(R) \neq \emptyset$, and in particular $C_q(R) = M_p(R)$. Thus, M_p is self-selective at R relative to $\mathcal{A}$. Now, consider the case where $CW_q(R) = \emptyset$. Then we have either $MW_p(R) = \emptyset$ or $MW_p(R) \neq \emptyset$. If the former holds, we have $C_q(R) = M_p(R) = 1$. If the latter holds, $M_p \in MW_p(L)$, where $L \in \mathcal{L}(\mathcal{A}, R)$. Therefore, M_p is self-selective at R relative to $\mathcal{A}$.

Now, let $\mathcal{A} = \{M_p, C_q\}$ for some $q \in (0, \frac{n-1}{n}]$, where $p \in (\frac{1}{n}, 1]$. Set $m = n + 2$, and define $\tilde{R} \in \mathcal{L}(I_m)^n$ as follows: An alternative $a \in I_m$ is most preferred by individual $i \in N$ if $a - i = 2$, $\mid \{i \in N \mid L(2, \tilde{R}^i = m - 1\} \mid = n$, and $1 \in I_m$ is bottom ranked by all individuals. That is we have:

[1] Note that here and in the definitions of self-selectivity for other classes of SCRs in the sequel, the note closing chapter 2 applies.

$\tilde{R}^1$	$\tilde{R}^2$	$\dots$	$\tilde{R}^n$
3	4	$\dots$	$n+2$
2	2	$\dots$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	1	$\dots$	1

So, $MW_p(R) = \emptyset$ implying that $M_p(\tilde{R}) = 1$. On the other hand, $CW_q(\tilde{R}) \neq \emptyset$ and $1 \notin CW_q(\tilde{R})$. Hence, $\mathcal{L}(\mathcal{A}, \tilde{R})$ consists of only one element L where C_q is top ranked by all individuals. Thus, $M_p(L) = C_q \neq M_p$. Therefore, M_p is not self-selective relative to $\mathcal{A}$.

(2) Take any $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, $q \in (0, 1]$ and let $\mathcal{A} = \{M_p, C_q\}$ where $p \in [0, \frac{1}{n}]$. Clearly, $MW_p(R) = T(R)$. Now, take any $L \in \mathcal{L}(\mathcal{A}, R)$, then we have $M_p \in MW_p(L)$. Hence M_p is self-selective relative to $\mathcal{A}$ whenever $p \in [0, \frac{1}{n}]$. □

Corrolary 2. *Let N be a finite nonempty society with $n \geq 3$.*

1. *For $p \in (\frac{1}{n}, 1]$, M_p has degree $\frac{1}{n}$.*
2. *For $p \in [0, \frac{1}{n}]$, M_p has degree 1.*

Proof. Follows from the definition of self-selectivity degree. □

Corrolary 3. *Let N be a finite nonempty society with $n \geq 3$.*

1. *For $p \in (\frac{1}{n}, 1]$, $\mathcal{A}_r = \{C_q \mid q \in (\frac{n-1}{n}, 1]\}$ is the maximal subfamily of $\{C_q \mid q \in (0, 1]\}$ such that M_p is self-selective relative to $\mathcal{A} = \{M_p\} \cup \mathcal{A}_r$.*
2. *For $p \in [0, \frac{1}{n}]$, $\mathcal{A}_r = \{C_q \mid q \in (0, 1]\}$ is the maximal family such that M_p is self-selective relative to $\mathcal{A} = \{M_p\} \cup \mathcal{A}_r$.*

Proof. (1) Note that by above proposition, $C_q \notin \mathcal{A}_r$ for any $q \in (0, \frac{n-1}{n}]$. Let $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ be given. We now that $C_q = C_{q'}$ for any $q, q' \in (\frac{n-1}{n}, 1]$. So we have either $CW_q(R) = \emptyset$ or $CW_q(R) \neq \emptyset$. Thus, as we discussed in the above proposition, both cases imply that M_p is self-selective relative to $\mathcal{A}$.

(2) Obvious. $\square$

4.2 Convex and Concave Scoring Functions

Given any $m \in \mathbb{N}$, consider a vector $s = (m, m-1, \dots, 1) \in \mathbb{R}^m$. For any $i \in N, a \in I_m$ denote a_i with $[a_i = s_k$ if and only if $\mid \{b \in I_m \mid bR^i a\} \mid= k-1]$.

Definition 14. Given any $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, an alternative $a \in I_m$ is said to be a *scoring winner at* R if $\sum_{i\in N} a_i \geq \sum_{i\in N} b_i$ for any $b \in I_m$.

We denote the set of all scoring winners by $SW(R)$ at each $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$. Now, an SCR S is called as a *scoring rule* if it selects all scoring winners at each $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$.

Definition 15. Let $m \in \mathbb{N}$ be given.

i. An SCR $S \in \mathcal{N}$ is called a *concave scoring rule* if $s_i \geq s_{i+1}$ for any $i \in \{1, 2, \dots, m-1\}$ and $s_1 - s_2 \leq s_2 - s_3 \leq \dots \leq s_{m-1} - s_m$.

ii. An SCR $S \in \mathcal{N}$ is called a *convex scoring rule* if $s_i \geq s_{i+1}$ for any $i \in \{1, 2, \dots, m-1\}$ and $s_1 - s_2 \geq s_2 - s_3 \geq \dots \geq s_{m-1} - s_m$.

Definition 16. Given $m \in \mathbb{N}, R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$, an SCF $S \in \mathcal{F}$ is called a *scoring function* if $S(R) = \min\{SW(R)\}$.

A scoring function is said to be self-selective relative to a set, $\mathcal{A}$, containing itself if, for any $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$, there exists $L \in \mathcal{L}(\mathcal{A}, R)$ such that $S \in SW(L)$.

Proposition 3. *1. Given $n \geq 3$, a concave scoring function S is not self-selective relative to $\mathcal{A} = \{S, C_q\}$ for any $q \in (0, 1]$.*

2. Given $n \geq 5$, a convex scoring function S is not self-selective relative to $\mathcal{A} = \{S, C_q\}$ for any $q \in (0, 1]$.

Proof. First consider the case where $s_1 - s_2 = s_2 - s_3 = \ldots = s_{m-1} - s_m$ for any $m \in \mathbb{N}$. Now let $m = n + 1$, and define $\tilde{R} \in \mathcal{L}(I_m)^n$ as follows: For the first $n-1$ individual, let $L(1, \tilde{R}^i) = I_m$ and, $t\tilde{R}^i(t+1)$ for every $t \in \{1, 2, .., m-1\}$. For the last individual, let $L(2, \tilde{R}^i) = I_m$, $L(1, \tilde{R}^i) = \{1\}$, and $t\tilde{R}^i(t+1)$ for every $t \in \{2, 3, \ldots, m-1\}$. Pictorially, $\tilde{R}$ is defined as:

$\tilde{R}^1$	$\tilde{R}^2$	$\cdots$	$\tilde{R}^{n-1}$	$\tilde{R}^n$
1	1	$\cdots$	1	2
2	2	$\cdots$	2	3
3	3	$\cdots$	3	4
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$m-1$	$m-1$	$\cdots$	$m-1$	m
m	m	$\cdots$	m	1

For any $q \in (0, 1]$, we have either $CW_q(\tilde{R}) = \emptyset$ or $1 \in CW_q(\tilde{R})$. Thus, $C_q(\tilde{R}) = 1$. On the other hand, $\sum_{i \in N} 2_i > \sum_{i \in N} a_i$ for any $a \in I_m \setminus \{2\}$. Therefore, $S(\tilde{R}) = 2$. Thus, $S(L) = C_q \neq S$ as $|\{i \in N | C_q L^i S\}| = n - 1$, where $\mathcal{L}(\mathcal{A}, \tilde{R}) = L$. Hence, S is not self-selective relative to $\mathcal{A} = \{S, C_q\}$ for any $q \in (0, 1]$.

Now, consider the cases where we have at least one strict inequality between $s_j - s_{j+1}$ and $s_{j+1} - s_{j+2}$ for some $j \in \{1, \ldots, m-2\}$.

Let S be a concave scoring function. Set $m = n$ and let $\tilde{R}$ be defined as above. Then, for any $q \in (0, 1]$, either $CW_q(\tilde{R}) = \emptyset$, or $1 \in CW_q(\tilde{R})$. Thus, $C_q(\tilde{R}) = 1$. Moreover, we have $\sum_{i \in N} 2_i > \sum_{i \in N} a_i$ for any $a \in I_m \setminus \{2\}$. Hence, $S(\tilde{R}) = 2$. As $|\{i \in N | C_q L^i S\}| = n - 1$, where $\mathcal{L}(\mathcal{A}, \tilde{R}) = L$, $S(L) = C_q \neq S$. Thus, a concave scoring function S is not self-selective relative to $\mathcal{A} = \{S, C_q\}$ for any $q \in (0, 1]$.

Now, consider a convex scoring function S. Take any $m \in \mathbb{N}$. Define $R' \in \mathcal{L}(I_m)^n$ as follows: For $i \in \{1, \ldots, \frac{n-1}{2}\}$, $L(2, R'^i) = I_m$ and $L(1, R'^i) = \{1\}$. For $i \in \{\frac{n+1}{2}, \ldots, n\}$, $L(1, R'^i) = I_m$ and $L(2, R'^i) = I_m \setminus \{1\}$.

R'^1	$\dots$	$R'^{\frac{n-1}{2}}$	$R'^{\frac{n+1}{2}}$	$\dots$	R'^n
2	$\dots$	2	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	2	$\dots$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
1	$\dots$	1	$\vdots$	$\vdots$	$\vdots$

So, for any $q \in (0,1]$ we have $C_q(R') = 1$. If $\sum_{i \in N} 2_i > \sum_{i \in N} 1_i$ holds then S is not self-selective relative to $\mathcal{A}$. This situation occurs if and only if the following inequality holds:

$$(\frac{n-1}{2})(s_2 - s_m) > (s_1 - s_2)$$

Now, define $R'' \in \mathcal{L}(I_m)^n$ as follows: For $i \in \{1, \dots, \frac{n-1}{2}\}$, $L(2, R'^i) = I_m$ and $L(1, R'^i) = I_m \setminus \{2\}$. For $i \in \{\frac{n+1}{2}, \dots, n\}$, $L(a, R''^i) = I_m$ if $a - i = \frac{5-n}{2}$ for some $a \in I_m$, 1 is the second choice and 2 is the third choice of each $i \in \{\frac{n+1}{2}, \dots, n\}$.

R''^1	$\dots$	$R''^{\frac{n-1}{2}}$	$R''^{\frac{n+1}{2}}$	$\dots$	R''^n
2	$\dots$	2	3	$\dots$	$3 + (\frac{n-1}{2})$
1	$\dots$	1	1	$\dots$	1
$\vdots$	$\vdots$	$\vdots$	2	$\dots$	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Then, we have $C_q(R'') = 1$ for each $q \in (0,1]$. Again, if $\sum_{i \in N} 2_i > \sum_{i \in N} 1_i$ holds then S is not self-selective relative to $\mathcal{A}$. But this situation requires the following inequality:

$$s_1 - s_2 > (\frac{n+1}{n-1})(s_2 - s_3)$$

Combining the above two inequalities imply that for $n \geq 5$, a convex scoring function S is not self-selective relative to $\mathcal{A} = \{S, C_q\}$ for any $q \in (0,1]$. □

We say that a SCF F has degree $-\infty$ if it is not self-selective relative to

$\mathcal{A} = \{F, C_q\}$ for any $q \in (0, 1]$. So, we have an immediate corollary to the above proposition:

Corrolary 4. *Let N be a finite nonempty set of individuals.*

1. *For any $n \geq 3$, a concave scoring function S has degree $-\infty$.*
2. *For any $n \geq 5$, a convex scoring function S has degree $-\infty$.*

Proof. By definition. □

4.3 k-Plurality Functions and Majoritarian Compromise

Now, consider a different type of scoring rule, namely the *k-plurality rule.* In this method, each individual gives exactly one point to each of the k-alternatives which she likes best, and then k-plurality rule chooses the alternative which gets the most points. Given $m \in \mathbb{N}$, the scoring vector of a k-plurality rule, $1 \leq k \leq m - 1$[2], assigns 1 to the first k-components and 0 to the rest, i.e. $s = (1, \ldots, 1, 0, \ldots, 0)$. We denote the set of all k-plurality winners by $PW_k(R)$ at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$, and define an SCR P_k as a k-plurality rule if it selects all k-plurality winners at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$.

Definition 17. Given $m \in \mathbb{N}, R \in \mathcal{L}(I_m)^n$, an SCF $P_k \in \mathcal{N}$ is said to be a *k-plurality function* if $P_k(R) = \min\{PW_k(R)\}$.

A k-plurality function is said to be self-selective relative to a set, $\mathcal{A}$, containing itself if, for any $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$, there exists $L \in \mathcal{L}(\mathcal{A}, R)$ such that $P_k \in PW_k(L)$.

A k-plurality function is a convex scoring function for $k = 1$. Therefore, from previous proposition, it is known that a 1-plurality function, P_1, is not

[2] Given $m \in \mathbb{N}$, k-plurality rule, when $k = m$, is trivially self-selective relative to any set of test functions $\mathcal{A}_r = \{C_q \mid q \in (0, 1]\}$ with $\mid \mathcal{A}_r \mid \geq k - 1$.

self-selective relative to $\mathcal{A} = \{P_1, C_q\}$ for any $q \in (0,1]$ whenever $n \geq 5$. The following remark gives a preference profile over a set of alternatives when there are exactly 3 individuals such that P_1 is not self-selective relative to $\mathcal{A} = \{P_1, C_q\}$ for any $q \in (0,1]$.

Remark 5. Let $n = 3$, and consider P_1. Set $m = 4$ and define $R \in \mathcal{L}(I_m)^n$ as follows:

R^1	R^2	R^3
2	3	4
1	1	1
3	2	2
4	4	3

Clearly, for any $q \in (0,1]$, $C_q(R) = 1$. On the other hand we have $P_1(R) = 2$. So, C_q is top ranked by individuals 2 and 3, and P_1 is top ranked by individual 1 over the linear order profile L, where $\mathcal{L}(\mathcal{A}, R) = \{L\}$. So, $P_1(L) = C_q$ implying that 1-plurality function is not self-selective relative to $\mathcal{A}$ for any $n \geq 3$.

Thus, a 1-plurality function has degree $-\infty$ for $n \geq 3$. However, if we test P_k, for $k > 1$, against only one SCF, then P_k is not well-defined over the preference profile on the set of SCFs since we only have two functions as alternatives over the induced preference profile on the set of SCFs. Therefore, for $k > 1$, the self-selectivity degree of a k-plurality function is not well-defined. The following remark shows that whenever we test a k-plurality function, $k > 1$, against any set of q-Condorcet functions, so that P_k is well-defined over the induced preference profile on the set of SCFs, P_k is never self-selective relative to the set of rival SCFs. Thus we need to test a k-plurality function against any set of q-Condorcet functions with $\mid \{C_q \mid q \in (0,1]\} \mid \geq k$.

Remark 6. Take any finite nonempty set of individuals N with $n \geq 3$. Consider any k-plurality function, P_k, for $k \geq 3$. Take any $\mathcal{A}_r = \{C_q \mid q \in (0,1]\}$

with $\mid \mathcal{A}_r \mid \geq k$, and let $\mathcal{A} = \{P_k\} \cup \mathcal{A}_r$. Set $m = 4 + (k-3)n$, and define $R \in \mathcal{L}(I_m)^n$ as follows: For $i \in \{1, \ldots, \frac{n-1}{2}\}$, $2R^i 3R^i 4R^i(4+i)$. For $i \in \{\frac{n+1}{2}, \ldots, n-1\}$, $1R^i 2R^i 4R^i(4+i)$. For $i = n$, $3R^i 1R^i 2R^i(4+i)$. Finally, for each $i \in N$, $[(4+i)+tn]R^i[(4+i)+(t+1)n]$. That is, we have the following preference profile:

R^1	$\ldots$	$R^{\frac{n-1}{2}}$	$R^{\frac{n+1}{2}}$	$\ldots$	R^{n-1}	R^n
2	$\ldots$	2	1	$\ldots$	1	3
3	$\ldots$	3	2	$\ldots$	2	1
4	$\ldots$	4	4	$\ldots$	4	2
5	$\ldots$	$\frac{n+7}{2}$	$\frac{n+9}{2}$	$\ldots$	$n+3$	$n+4$
$n+5$	$\ldots$	$\frac{3n+7}{2}$	$\frac{3n+9}{2}$	$\ldots$	$2n+3$	$2n+4$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

So, for any $q \in (0,1]$, we have $C_q(R) = 1$. On the other hand, for any $k \geq 3$, $P_k = 2$. It is given that $\mid \mathcal{A}_r \mid \geq k$. Thus, for any $L \in \mathcal{L}(\mathcal{A}, R)$, $P_k(L) \in \mathcal{A} \setminus \{P_k\}$ since $\mid \{i \in N \mid 1R^i 2\} \mid = \frac{n+1}{2}$. Hence, P_k is not self-selective relative to $\mathcal{A}$ for $k \geq 3$.

Let $n = 3$, and consider P_2. Take any $\mathcal{A}_r$ as defined above with $\mid \mathcal{A}_r \mid \geq 2$, and let $\mathcal{A} = \{P_2\} \cup \mathcal{A}_r$. Set $m = 3$ and define $R \in \mathcal{L}(I_m)$ as follows: $3R^1 2R^1 1$, $1R^2 2R^2 3$, and $1R^3 2R^3 3$. Clearly $C_q(R) = 1$ for each $q \in (0,1]$, however $P_2(R) = 2$. So, for any $L \in \mathcal{L}(\mathcal{A}, R)$, $P_2(L) \in \mathcal{A} \setminus \{P_2\}$. Therefore, P_2 is not self-selective relative to $\mathcal{A}$ for $n = 3$. Now, let $n \geq 5$, $m = 3$ and define $R \in \mathcal{L}(I_m)$ as follows: For $i \in \{1, \ldots, \frac{n-1}{2}\}$, $2R^i 3R^i 1$. For $i \in \{\frac{n+1}{2}, \ldots, n-1\}$, $1R^i 2R^i 3$. Finally, for $i = n$, $3R^i 1R^i 2$.

$R^1 \ldots R^{\frac{n-1}{2}}$	$R^{\frac{n+1}{2}} \ldots R^{n-1}$	R^n
2	1	3
3	2	1
1	3	2

Thus, we have $P_2(R) = 2$ and, for each $q \in (0,1]$, $C_q(R) = 1$ implying that $P_2(L) \in \mathcal{A} \setminus \{P_2\}$. Hence, a 2-plurality function is not self-selective relative to $\mathcal{A}$.

As we have seen, in k-plurality rule, an alternative does not need to have the majority of the votes to get chosen. Moreover, number k is exogenous for each preference profile over the set of alternatives. The next SCR, *majoritarian compromise*[3], basically differs from k-plurality rule within these two situations. Firstly, in majoritarian compromise rule, an alternative needs to have at least a majority of the votes to get chosen, which is more restrictive than a plurality rule. Secondly, the number k is endogenously determined for each preference profile over the set of alternatives, which is less restrictive than a plurality rule. We provide self-selectivity degree of majoritarian compromise rule and conclude that it inherits almost the same self-selectivity properties with any k-plurality rule.

We define a *majoritarian compromise rule* as follows [4]: We start by examining the first row of the preference profile. If an alternative gets a majority of votes, then this alternative is referred as a *majoritarian compromise winner.* If there is no majoritarian compromise winner at the first row, we start considering alternatives at the first two rows of the preference profile. If a majority of the individuals prefers an alternative as either their first best or second best, then that alternative is chosen by the majoritarian compromise rule. If there is no majoritarian compromise winner in the first two rows, then we move on to the third row and apply the same procedure. We stop when an alternative receives a majority support. We denote the set of all majoritarian compromise winners by $MCW(R)$ at each preference profile $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$, and define an SCR MC as a *majoritarian compromise rule* if it selects all majoritarian compromise winners at each $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)$.

For each $a \in MCW(R)$ at a given preference profile $R \in \cup_{m \in \mathbb{N}} \mathcal{L}(I_m)^n$, we

[3]Introduced by Murat Sertel.
[4]Sanver [2009]

denote the set of individuals supporting that alternative by $Supp(a)$. Then, we define the set of majoritarian compromise winners with highest support, $MCW^*(R)$, by

$$MCW^*(R) = \{a \in MCW(R) \mid \forall b \in MCW(R)\colon \mid Supp(a) \mid \geq \mid Supp(b) \mid\}$$

Definition 18. Given $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$, an SCF MC is called a *majoritarian compromise function* if $MC(R) = \min\{MCW^*(R)\}$.

The majoritarian compromise function is said to be self-selective relative to a set, $\mathcal{A}$, containing itself if for any $R \in \cup_{m\in\mathbb{N}} \mathcal{L}(I_m)^n$, there exists $L \in \mathcal{L}(\mathcal{A}, R)$ such that $MC \in MCW^*(L)$.

Proposition 4. *Let N be a finite nonempty set of individuals with $n = 3$. MC is self-selective relative to $\mathcal{A} = \{MC, C_q\}$ for every $q \in (0, 1]$.*

Proof. Suppose, on the contrary, that there exist $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ such that MC is not self-selective at R relative to $\mathcal{A} = \{MC, C_q\}$ for some $q \in (0, 1]$. We have either $CW_q(R) = \emptyset$ or $CW_q(R) \neq \emptyset$. First consider the case where $CW_q(R) = \emptyset$, so $C_q(R) = 1$. Since MC is not self-selective at R relative to $\mathcal{A}$, we must have $MC(R) \in I_m \setminus \{1\}$, and also $\mid \{i \in N \mid 1R^iMC(R)\} \mid\geq 2$. However, this contradicts with $MC(R) \in I_m \setminus \{1\}$. Now, consider the case where $CW_q(R) \neq \emptyset$, and let $C_q(R) = a$. Then we must have $MC(R) \in I_m \setminus \{a\}$, and $\mid \{i \in N \mid aR^iMC(R)\} \mid\geq 2$ again contradicting with $MC(R) \in I_m \setminus \{a\}$. Hence, MC is self-selective relative to $\mathcal{A} = \{MC, C_q\}$ for every $q \in (0, 1]$ whenever $n = 3$. □

The above proposition implies that for $n = 3$, the majoritarian compromise function has degree 1. The following corollary shows the maximal set of rival SCF such that majoritarian compromise funtcion is relatively self-selective when $n = 3$.

Corrolary 5. *Let N be a finite set of individuals with $n = 3$. $\mathcal{A}_r = \{C_q \mid q \in (0,1]\}$ is the maximal set such that MC is self-selective relative to $\mathcal{A} = \{MC\} \cup \mathcal{A}_r$.*

Proof. Suppose that there exist $m \in \mathbb{N}$, $R \in \mathcal{L}(I_m)^n$ such that for every $L \in \mathcal{L}(\mathcal{A}, R)$ we have $MC \notin MCW(L)$. Thus for some $q \in (0,1]$ we must have $MC(R) \neq C_q(R)$ and also $\mid \{i \in N \mid C_q R^i MC(R)\} \mid \geq 2$,contradicting with $MC(R) \in MCW(R)$. □

Proposition 5. *Let N be a finite nonempty set of individuals with $n \geq 5$. MC is not self-selective relative to $\mathcal{A} = \{MC, C_q\}$ for every $q \in (0,1]$.*

Proof. Let $m = 4$ and define $R \in \mathcal{L}(I_m)$ as follows: For $i \in \{1, \ldots, \frac{n-1}{2}\}$, $2R^i 3R^i 1R^i 4$. For $i \in \{\frac{n+1}{2}, \ldots, n-1\}$, $1R^i 2R^i 3R^i 4$. For $i = n$, $3R^i 1R^i 2R^i 4$. So we have:

$R^1 \ldots R^{\frac{n-1}{2}}$	$R^{\frac{n+1}{2}} \ldots R^{n-1}$	R^n
2	1	3
3	2	1
1	3	2
4	4	4

Thus, $C_q(R) = 1$ for every $q \in (0,1]$ and $MC(R) = 2$. Hence, $MC(L) = C_q$ where L is the only preference profile over $\mathcal{A}$ induced by R. □

Thus, by definition, for every $n \geq 5$, the majoritarian compromise function has degree $-\infty$.

CHAPTER 5

CONCLUSION

In this thesis, we localize the notion of self-selectivity. For this purpose, we restrict the set of rival SCFs to particular singleton-valued refinements of generalized Condorcet rules. First, we characterize the self-selectivity of generalized Condorcet functions and, then, show that this family of SCFs has some useful properties. Well-behaved pattern with respect to self-selectivity exhibited by this family allows us to define the concept of self-selectivity degree of SCFs. Combining the self-selectivity degree of SCFs and the aggregation property of test SCFs enable us to find the maximal set of SCFs relative to which an SCF is self-selective. Hence, we show that self-selectivity degree can be used to compare strength of self-selectivity of SCFs.

We test self-selectivity of some family of SCFs and obtain non-dictatorial self-selective SCFs. However, for a given society, these non-dictatorial self-selective SCFs are equal to either a 1-Condorcet function or a $\frac{1}{n}$-Condorcet function. Hence, except the generalized Condorcet functions, there is not a continuous change in the self-selectivity degree of non-dictatorial SCFs that we test. That is, we observe sharp changes in self-selectivity degrees within some families of SCFs. However, we still do not know due to which properties of these SCFs there exist such a change in self-selectivity degree. Thus, a full characterization of self-selective SCFs with this restricted set of test SCFs may shed some light on this problem. On the other hand, in our study, we only consider SCFs. However, allowing social choice rules to be set-valued and defining the self-selectivity degree accordingly are yet to be dealt with. Finally, SCCs enable us to use algebraic operations. Thus, the change in self-selectivity degree under algebraic operations is an open problem.

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