STABILITY ANALYSIS AND CONTROLLER DESIGN FOR THE HEAT EQUATION WITH TIME DELAYED FEEDBACK

A THESIS

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By

Sina Yamaç Çalışkan

August 2010

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Hitay Özbay(Supervisor)

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Ömer Morgül

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Prof. Dr. Mehmet Önder Efe

Approved for the Institute of Engineering and Sciences:

Prof. Dr. Levent Onural
Director of Institute of Engineering and Sciences

ABSTRACT

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Sina Yamaç Çalışkan
M.S. in Electrical and Electronics Engineering
Supervisor: Prof. Dr. Hitay Özbay
August 2010

In this thesis, the stability analysis for the system defined by the heat equation with time delayed feedback is performed. In the first part of the thesis, stability conditions in terms of LMI conditions which are obtained from the analysis in time domain, are explained. Necessary and sufficient conditions for stability are obtained using a frequency domain analysis. In the second part of the thesis, robust stability conditions are obtained for the system with parametric uncertainty. In the third part, an H_∞ controller design procedure is given for this type of plants described by the heat equation with time delayed feedback. Finally, the results are illustrated with simulations.

Keywords: Stability Analysis, Time Delay, H_∞ Control, Parametric Uncertainty, Heat Equation, Infinite Dimensional System

ÖZET

ZAMAN GECİKMELİ GERİ BİLDİRİM İÇEREN ISI DENKLEMİ İÇİN KARARLILIK ANALİZİ VE KONTROLÖR TASARIMI

Sina Yamaç Çalışkan
Elektrik ve Elektronik Mühendisliği Bölümü Yüksek Lisans
Tez Yöneticisi: Prof. Dr. Hitay Özbay
Ağustos 2010

Bu tezin kapsamında zaman gecikmeli geri bildirim içeren ısı denklemi tarafından tanımlanan sistemin kararlılık çözümlemesi gerçekleştirilmiştir. Tezin ilk kısmında zaman alanında yapılan çözümleme sonucunda doğrusal matris eşitsizlikleri cinsinden elde edilen kararlılık koşulları açıklanmıştır. Frekans alanında çözümleme yapılarak kararlılık için gerekli ve yeterli koşullar elde edilmiştir. Tezin ikinci kısmında sistemi tanımlayan parametrelerde belirsizlik olması durumunda sistemin gürbüz kararlılığını sağlayacak koşullar bulunmuştur. Tezin üçüncü kısmında zaman gecikmeli ısı denklemiyle tanımlanan sistem için bir H-sonsuz kontrolör tasarım prosedürü verilmiştir. Son olarak bulunan sonuçlar simulasyonlarla örneklendirilmiştir.

Anahtar Kelimeler: Kararlılık Analizi, Zaman Gecikmesi, H-sonsuz Kontrol, Parametrik Belirsizlik, Isı Denklemi, Sonsuz Boyutlu Sistemler

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Dedicated to my grandmother, Behice
(9^{th} April 1928 - 6^{th} October 2009)

Chapter 1

Introduction

Partial differential equations (PDEs) are useful tools for modeling natural phenomena. Many of the fundamental laws of physics are expressed as partial differential equations. Therefore, a broad range of problems which arises in science and engineering can be solved using PDEs. The heat equation is one of the well-known example for the partial differential equations; it can be written as

$$\frac{\partial}{\partial t} z(x,t) = \frac{\partial^2}{\partial x^2} z(x,t), \tag{1.1}$$

where x is the spatial variable, t is the time and $z(x,t)$ is the temperature at a point x at time t. This equation is used for describing the heat conduction phenomena. It is a less general form of the diffusion equation. The reader is referred to [8] and [22] for a general review of the heat equation, diffusion equation, linear PDEs and nonlinear PDEs. The heat equation can be modeled using a one dimensional rod. Any point on the one dimensional rod has a corresponding spatial position $x \in [0, \pi]$. Inputs to the system are taken to be the temperatures at end points (i.e $x = 0$ and $x = \pi$) of a one dimensional rod. Output is denoted by $z(x_0, t)$ where $x_0 \in (0, \pi)$ is an interior point of the one dimensional rod. Transfer function for the heat equation system with no feedback is stable with negative real poles $p_n = -n^2$ where $n \in \mathbb{Z} \cup \{0\}$ [4].

The heat equation with feedback is in the form

$$\frac{\partial}{\partial t}z(x,t) = a\frac{\partial^2}{\partial x^2}z(x,t) + f(z), \tag{1.2}$$

where f is the feedback term and a is the conductivity of the material. In case of the existence of the feedback, system may be destabilized. In practice, a feedback term may arise in the equation for several reasons. One such reason is the heat generation or loss [2]. There are various researchers who investigated the systems represented by the heat equations with different linear and nonlinear feedback structures, i.e. $f(z)$, using various types of control techniques [1, 2, 5, 6, 7, 20, 21]. In [1] and [5], feedback is taken to be $f(z) = a_0 z(x,t)$. The term $f(z)$ is considered to be the disturbance for the system, which is a bounded continuous function in [2]. Tracking and disturbance rejection problem for this system is solved using interior point control. In [20], feedback is in the form $f(z) = a_0 z(x,t) + \frac{\partial}{\partial x}z(x,t)$. Motion of the heat carrier and heat dissipation in the surface is considered also considered in [21] by choosing an appropriate $f(z)$ function. Then sliding mode control is investigated for this system. Nonlinear forms for the feedback $f(z)$ are also considered in [6] and [7].

In this thesis, input-output stability of the system represented by the heat equation with time delayed feedback is investigated. When there is a time-delay in the feedback, resulting system dynamics can be expressed as a PDE with time delay. Uniqueness, existence and asymptotic behaviour of the solutions of the PDEs with discrete state-dependent feedbacks are first considered in [23]. In [24], discrete state-dependent delay in [23] is approximated by a series of distributed delay terms. Then it is proved that, approximated system has a global attractor. Delay-dependent stability of ordinary differential equations and PDEs with distributed delays are also considered in [17]. One can refer to these papers for a general discussion of PDEs with time delays. Fridman and Orlov investigated the heat equation with linear time-delayed feedback in [11, 12, 13]. They found Linear Matrix Inequality (LMI) conditions for the stability of the time-delayed heat equation system with time-varying delays [13]. The analysis in [13] is in time domain. Caliskan and Ozbay, [3], analyzed the same system in frequency domain and obtained new necessary and sufficient conditions for the stability of the system. Analytical bound for the upper bound of the time delay for which the system stays stable is found in this work [3]. Another LMI condition is obtained using modified Lyapunov functionals in a recent paper [29].

The thesis is organized as follows. In Section 2, results of the [13] and [3] is explained. Necessary and sufficient conditions obtained by Caliskan and Ozbay in [3] are presented in this section. In Section 3, parametric uncertainty is applied to the parameters of the system represented by the heat equation with time-delayed feedback. Robust stability conditions are obtained in this section. In Section 4, H_∞ controllers are designed for robustly stabilizing the system. In Section 5, simulations are performed to illustrate the theoretical results.

Chapter 2

Stability Analysis of the Heat Equation with Time-Delayed Feedback and Constant Parameters

The one dimensional heat equation is a partial differential equation which models the heat distribution on a one dimensional rod:

$$\frac{\partial}{\partial t}z(x,t) = a\frac{\partial^2}{\partial x^2}z(x,t). \tag{2.1}$$

In this equation, $z(x,t)$ stands for the temperature at time instant $t \geq t_0$ at the point $x \in [0,\pi]$ on a one dimensional rod of length π. The parameter $a > 0$ is the heat conductivity parameter, which depends on the conductivity of the medium. The one dimensional heat equation has been investigated with different initial conditions and boundary values, see e.g. [4], [19].

The heat equation with feedback is in the form

$$\frac{\partial}{\partial t}z(x,t) = a\frac{\partial^2}{\partial x^2}z(x,t) + f(z). \tag{2.2}$$

In the thesis, a linear feedback with time delay is considered:

$$f(z) = a_0 z(x,t) - a_1 z(x, t-\tau), \tag{2.3}$$

where $a, a_1, a_0 \in \mathbb{R}$ and $\tau > 0$.

With the linear time delayed feedback, heat equation is in the following form:

$$\frac{\partial}{\partial t} z(x,t) = a \frac{\partial^2}{\partial x^2} z(x,t) + a_0 z(x,t) - a_1 z(x, t-\tau). \tag{2.4}$$

In [11], [12] and [13] the heat equation with linear time delayed feedback is analyzed in time domain. In [3], frequency domain analysis of the same system is performed. In this chapter time domain analysis and frequency domain analysis for the heat equation with linear time delayed feedback and constant parameters will be explained.

Initial conditions for the partial differential equation (2.1) are assumed to be zero:

$$z(x,\theta) = 0 \qquad \forall \;\; x \in (0 \;,\; \pi) \;\; \text{and} \;\; \theta \in [t_0 - \tau \;,\; t_0]. \tag{2.5}$$

Consider two inputs, applied from the end points of the rod. Input u_1 is applied from $x = 0$ and input u_2 is applied from $x = \pi$:

$$z(0,t) = u_1(t) \;\;,\;\; \text{with} \;\; u_1(t) = 0, \;\; \text{for} \;\; t < t_0, \tag{2.6}$$

$$z(\pi,t) = u_2(t) \;\;,\;\; \text{with} \;\; u_2(t) = 0, \;\; \text{for} \;\; t < t_0. \tag{2.7}$$

Let the output $y(t)$ of the system be the temperature at a point $x_o \in (0 \;,\; \pi)$, i.e. $y(t) = z(x_o, t)$. Transfer functions from u_1 and u_2 to y are represented with $G_1(s)$ and $G_2(s)$ respectively.

In this chapter, first LMI conditions for the stability of the system (2.4) will be reviewed [13]. In time domain analysis, time delay τ in the system is assumed to be a function of time t. Then, frequency domain analysis for the heat equation with linear time delayed feedback and constant parameters will be explained. In frequency domain analysis, time delay $\tau \in \mathbb{R}_+$ is fixed. Transfer functions

$G_1(s)$ and $G_2(s)$ are derived in the frequency domain analysis section. Then by analyzing the pole locations of G_1 and G_2 we obtain necessary and sufficient conditions for stability of this system in terms of the parameters (a, a_0, a_1, τ).

2.1 Analysis In Time Domain

A set of LOI (Linear Operator Inequality) conditions for the exponential stability of the infinite dimensional linear systems in the form of

$$\frac{\partial}{\partial t} z(t) = Az(t) + A_1 z(t - \tau(t)) \quad t \geq t_0 \tag{2.8}$$

is derived in [13]. In (2.8), the state of the system $z(t) \in \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space. Moreover (2.8) satisfies three assumptions:

Assumption (A1): The operator A generates a strongly continuous semigroup $T(t)$ and the domain of the operator $\mathcal{D}(A)$ is dense in the Hilbert space $\mathcal{H}$.

Assumption (A2): The operator A_1 that acts on the delayed state is bounded in the Hilbert space $\mathcal{H}$.

Assumption (A3): The function $\tau(t)$ is piecewise continuous. In each closure of continuity, $\tau(t)$ is in the class C^1 where C^1 denotes continuously differentiable functions from $\mathbb{R}_+$ to $\mathcal{H}$. Moreover $\tau(t)$ satisfies

$$\inf_t \tau(t) > 0 \quad \text{and} \quad \sup_t \tau(t) < h, \tag{2.9}$$

where $h > 0$ and $h \in \mathbb{R}$ for every $t > t_0$.

Let the initial conditions for the linear infinite dimensional system be

$$x_{t_0} = \phi(\theta) \quad \theta \in [-h, 0], \ \phi \in W, \tag{2.10}$$

where $W = C([-h, 0], \mathcal{D}(A)) \cap C^1([-h, 0], \mathcal{H})$ and $\mathcal{D}(A)$ is the domain of the operator A. The initial value problem defined by (2.8) and (2.10) is well posed and it can be defined as the integral initial value problem with initial condition (2.10) and the integral equation

$$x(t) = T(t-t_0)x(t_0) + \int_{t_0}^{t} T(t-s)A_1 x(s-\tau(s))ds. \tag{2.11}$$

Definition 1. The system (2.8) is said to be exponentially stable with a decay rate $\delta > 0$ if there exists a constant $K > 1$ such that

$$|x(t,t_0,\phi)| \leq Ke^{-2\delta(t-t_0)}||\phi||_W^2 \tag{2.12}$$

for every $t \geq t_0$. $\square$

Theorem 1. ([13]) Let (A1)-(A3) be satisfied for the system (2.8) with $\sup_t |\dot{\tau}(t)| = d < 1$. Given $\delta > 0$, consider that there exists linear operators $P > 0$, $R \geq 0$, $S \geq 0$ and $Q \geq 0$ subject to

$$\beta\langle x, x\rangle \leq \langle x, Px\rangle \leq \gamma_P[\langle x, x\rangle + \langle Ax, Ax\rangle], \quad \langle x, Qx\rangle \leq \gamma_Q\langle x, x\rangle, \tag{2.13}$$

$$\langle x, Rx\rangle \leq \gamma_R\langle x, x\rangle, \quad \langle x, Sx\rangle \leq \gamma_S\langle x, x\rangle, \tag{2.14}$$

for every $x \in \mathcal{D}(A)$ and positive constants β,γ_Q,γ_R,γ_S,γ_P. $\square$

Consider the LOI

$$\begin{bmatrix} \Phi_{11} & 0 & PA_1 \\ 0 & 0 & 0 \\ A_1^* & 0 & 0 \end{bmatrix} + h^2 \begin{bmatrix} A^*RA & 0 & A^*RA_1 \\ 0 & 0 & 0 \\ A_1^*RA & 0 & A_1^*RA_1 \end{bmatrix}$$

$$-e^{-2\delta h} \begin{bmatrix} R & 0 & -R \\ 0 & (S+R) & -R \\ -R & -R & 2R+(1-d)Q \end{bmatrix} \leq 0, \tag{2.15}$$

$\Phi_{11} = A^*P + PA + 2\delta P + Q + S.$

If the LOI (2.15) holds in the space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$, then the system defined by (2.8) is exponentially stable with the decay rate δ for all differentiable delays $d < 1$. Moreover, under these sufficient conditions, the inequality (2.12) is satisfied with $K = \max(\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2))/\beta$. If the LOI (2.15) is feasible with Q $= 0$, then (2.8) is exponentially stable for all fast varying delays, i.e d is not bounded in the interval $0 < \tau < h$.

By taking $S = R = 0$, quasi delay-independent conditions are obtained. This condition becomes delay independent as $\delta \to 0$. Resulting LOI condition is stated in Theorem 2.

Theorem 2. ([13]) Let the assumptions (A1)-(A3) be satisfied for the system (2.8) with $\sup_t |\dot{\tau}(t)| = d < 1$. Given $\delta > 0$, assume there exists linear operators $P > 0$,and $Q \geq 0$ subject to (2.13). Consider the LOI

$$\begin{bmatrix} (A+\delta)^*P + P(A+\delta) + Q & PA_1 \\ A_1^*P & -(1-d)Qe^{-2\delta h} \end{bmatrix} \leq 0. \tag{2.16}$$

If the LOI (2.16) holds in the space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$, then the system defined by (2.8) is exponentially stable with the decay rate δ for all differentiable delays $d < 1$. Moreover, under these conditions, the inequality (2.12) is satisfied with $K = \max(\gamma_P, h\gamma_Q/\beta)$. □

Since the matrix that multiplies h^2 in LOI (2.15) depends on the operator A, which is unbounded, the feasibility of strict LOIs (2.15) and (2.16) for $h = 0$ (or $\delta = 0$) does not necessarily imply the feasibility of the LOIs (2.15) and (2.16) for small enough h (or δ). In order to avoid the unbounded multiplication of h, another LOI condition is derived in [13] with the help of the descriptor method explained in [10]. This LOI condition is stated in Theorem 3.

Theorem 3. ([13]) Let the assumptions (A1)-(A3) are satisfied for the system (2.8) with $\sup_t |\dot{\tau}(t)| = d < 1$. Given $\delta > 0$, consider that there exists indefinite operators $P_2, P_3 \in \mathcal{L}(\mathcal{H})$ and linear operators $P > 0$, $R \geq 0$, $S \geq 0$

and $Q \geq 0$ subject to (2.13) and (2.14). Consider the LOI

$$\begin{bmatrix} \Phi_{d11} & \Phi_{d12} & 0 & P_2^* A_1 + Re^{-2\delta h} \\ * & \Phi_{d22} & 0 & P_3^* A_1 \\ * & * & -(S+R)e^{-2\delta h} & Re^{-2\delta h} \\ * & * & * & -[2R + (1-d)Q]e^{-2\delta h} \end{bmatrix} \leq 0, \qquad (2.17)$$

$$\Phi_{d11} = A^* P_2 + P_2^* A + 2\delta P + Q + S - Re^{-2\delta h},$$

$$\Phi_{d12} = P - P_2^* + A^* P_3, \quad \Phi_{d13} = -P_3 - P_3^* + h^2 R.$$

Here * denotes the symmetric terms in the matrix. If the LOI (2.17) holds in the space $\mathcal{D}(A) \times \mathcal{D}(A) \times \mathcal{D}(A)$, then the system defined by (2.8) is exponentially stable with the decay rate δ for all differentiable delays $d < 1$. Moreover, under these conditions, the inequality (2.12) is satisfied with $K = \max(\gamma_P, h(\gamma_Q + \gamma_S + h^2\gamma_R/2)/\beta)$. If the LOI is feasible with Q = 0, then (2.8) is exponentially stable for all fast varying delays $0 < \tau < h$. Feasibility of the LOI (2.17) for $h = 0$ implies the feasibility of the LOI for small h. $\square$

Consider the bounded operator $A_1 = -a_1$ and the operator $A = a\frac{\partial^2}{\partial t^2} + a_0$ with the dense domain $\mathcal{D}\left(\frac{\partial^2}{\partial t^2}\right) = \{z \in W^{2,2}([0,\pi], R) : \quad z(0) = z(\pi) = 0\}$. With this setting for the operators, system (2.8) becomes our original system (2.4). Operator $A = a\frac{\partial^2}{\partial t^2} + a_0$ generates an exponentially stable semigroup [4]. Consider that inputs for the system is zero:

$$u_1(t) = u_2(t) = 0 \quad t \geq t_0. \qquad (2.18)$$

This results in the boundary conditions

$$z(0,t) = z(\pi,t) = 0 \quad t \geq t_0. \qquad (2.19)$$

The boundary value problem with the partial differential equation (2.4) and boundary conditions (2.19) is in the Hilbert space $\mathcal{H} = L_2(0,\pi)$.

Equivalent LMI conditions are obtained from the LOI conditions by defining bounded operators $P = p$ and $Q = q$ where $p, q > 0$ are constants.

Theorem 4. ([13]) Consider the LMI

$$\Psi_\delta \triangleq \begin{bmatrix} q - 2\Big(\frac{\pi^2}{l^2}a - a_0 - \delta\Big)p & -a_1 p \\ -a_1 p & -(1-d)q(1-e^{-2\delta h}) \end{bmatrix} < 0. \tag{2.20}$$

Given $\delta > 0$, if LMI (2.20) holds for some scalars $p > 0$ and $q > 0$, then the boundary value problem, with the partial differential equation (2.4) and boundary conditions (2.19), is exponentially stable for all differentiable delays (2.9) with $\sup_t |\dot{\tau}(t)| = d < 1$. If $\Psi_0 < 0$, then $\Psi_\delta < 0$ for small enough δ since $\Psi_\delta = \Psi_0 + \mathrm{diag}\{2\delta p, (1-d)qe^{-2\delta h}\}$ □

Using Schur complements formula [13], it is stated that LMI (2.20) with $\delta = 0$ is feasible if and only if for some scalars $p > 0$ and $q > 0$, the following inequality holds:

$$q^2 - 2\Big(\frac{\pi^2}{l^2}a - a_0\Big)pq + \frac{a_1^2 p^2}{1-d} < 0. \tag{2.21}$$

Left side of the inequality has its minimum value at $q = (\frac{\pi^2}{l^2}a - a_0)p$. Thus, LMI (2.20) with $\delta = 0$ is feasible if and only if

$$\frac{\pi^2}{l^2}a - a_0 > 0, \quad a_1^2 < \Big(\frac{\pi^2}{l^2}a - a_0\Big)^2(1-d). \tag{2.22}$$

In frequency domain analysis part, it is assumed that $l = \pi$ and time delay is fixed i.e. $d = 0$. Thus (2.22) becomes

$$|a_1| < a - a_0, \tag{2.23}$$

which is the same with the stability independent of delay condition that will be obtained in the frequency domain analysis section. In other words, stability independent of delay condition in frequency domain analysis section matches with the result of [13].

Theorem 5. ([13]) Consider the LMI

$$\begin{bmatrix} \phi_{11} & \phi_{12} & 0 & \phi_{14} \\ * & -2p_3 + h^2 r & 0 & -p_3 a_1 \\ * & * & -(s+r)e^{-2\delta h} & re^{-2\delta h} \\ * & * & * & \phi_{44} \end{bmatrix} < 0, \tag{2.24}$$

$\phi_{11} = -2(a - a_0)p_2 + 2\delta p_1 + q + s - re^{-2\delta h}$,

$\phi_{12} = p_1 - p_2 - (a - a_0)p_3$,

$\phi_{14} = -p_2 a_1 + re^{-2\delta h}$,

$\phi_{44} = -(2r + (1 - d)q)e^{-2\delta h}$.

Given $\delta > 0$, if LMI (2.24) and $p_2 - \delta p_3 \geq 0$ holds for some scalars $p_1 > 0$, $p_2 > 0$, $p_3 > 0$, $s > 0$, $p_1 > 0$, and $q \geq 0$, then the boundary value problem with the partial differential equation (2.4) and boundary conditions (2.19) with $l = \pi$ is exponentially stable for all differentiable delays (2.9) with $\sup_t |\dot{\tau}(t)| = d < 1$. Additionally, if LMI (2.24) is feasible with $q = 0$, then the boundary value problem with the partial differential equation (2.4) and boundary conditions (2.19) with $l = \pi$ is exponentially stable for all fast varying delays (2.9) with no restriction on $\dot{\tau}$. If LMI (2.24) is feasible with $\delta = 0$, then the boundary value problem with the partial differential equation (2.4) and boundary conditions (2.19) with $l = \pi$ is exponentially stable with a sufficiently small decay rate. □

In [13], it is stated that the LMIs (2.20) and (2.24) guarantees the stability of the ODE

$$\dot{y}(t) + (a - a_0)y(t) + a_1 y(t - \tau(t)) = 0. \tag{2.25}$$

This is the first modal dynamics in the modal representation of the Dirichlet boundary value problem with the partial differential equation (2.4) and boundary conditions (2.19) with $l = \pi$:

$$\dot{y}(t) + (ak^2 - a_0)y(t) + a_1 y(t - \tau(t)) = 0 \quad k = 1, 2, \ldots \; . \tag{2.26}$$

Stability of the boundary value problem implies the stability of (2.26). It is stated that stability of (2.25) is the necessary condition for the stability of the boundary value problem. In frequency domain analysis part, it will be shown that the stability of (2.25) is also the sufficient condition for the stability of the boundary value problem for fixed time delay $\tau(t) = \tau$.

2.2 Analysis In Frequency Domain

Note that in the previous section, it is assumed that time delay is a bounded and differentiable function of time, that is $0 < \tau(t) \leq h$ and $\sup_t |\dot{\tau}(t)| = d < 1$. In this section, it is assumed that $\tau(t) = \tau$ is fixed. In other words, in terms of the parameters of Section 2.1, frequency domain analysis is performed for the case where $d = 0$ and $h = \tau$.

After taking the Laplace transform with respect to time of the equation (2.4) with initial conditions (2.5), following differential equation is obtained in s-domain:

$$(s - a_0 + a_1 e^{-\tau s})\ Z(x,s) = a\frac{\partial^2}{\partial x^2} Z(x,s). \tag{2.27}$$

In the equation (2.27), $s \in \mathbb{C}$ is the Laplace transform variable and $Z(x,s)$ is the Laplace transform of the function $z(x,t)$. Boundary conditions in the time domain can be translated in to s-domain as $Z(x,0) = U_1(s)$ and $Z(x,\pi) = U_2(s)$. With the functions $U_1(s)$ and $U_2(s)$, the solution of the partial differential equation can be written as

$$Z(x,s) = G_1(x,s)U_1(s) + G_2(x,s)U_2(s),$$

where

$$G_1(x,s) = \left(e^{-(x-\pi)\ \lambda(s)} - e^{(x-\pi)\ \lambda(s)}\right)/\Delta(s), \tag{2.28}$$

$$G_2(x,s) = \left(e^{x\ \lambda(s)} - e^{-x\ \lambda(s)}\right)/\Delta(s), \tag{2.29}$$

with

$$\Delta(s) = e^{\pi\ \lambda(s)} - e^{-\pi\ \lambda(s)}, \tag{2.30}$$

and

$$\lambda(s) = \sqrt{\frac{s - a_0 + a_1 e^{-\tau s}}{a}}. \qquad (2.31)$$

Note that $G_1(0,s) = G_2(\pi,s) = 1$, $G_1(\pi,s) = G_2(0,s) = 0$ and $G_1(\frac{\pi}{2},s) = G_2(\frac{\pi}{2},s)$. Poles of the transfer functions $G_1(s)$ and $G_2(s)$ are the same. The poles can be obtained as the solution of the $\Delta(s) = 0$, which is equivalent to

$$e^{-2\pi\lambda(s)} = 1 = e^{j2\pi n}, \quad n = 0, \pm 1, \pm 2, \dots ,$$

or

$$\lambda(s) = \pm jn \quad n = 0, \pm 1, \pm 2, \dots .$$

The case $n = 0$ is not admissible, because in this case (2.27) implies

$$\frac{\partial^2}{\partial x^2} Z(x,s) = 0,$$

and the solution is

$$z(x,t) = u_1(t) + x\ (u_2(t) - u_1(t))/\pi.$$

Thus (2.4) imposes conditions on free inputs u_1 and u_2 which makes the system ill posed. As a result of this discussion, the poles of this system are the solutions $s \in \mathbb{C}$ satisfying

$$\frac{s - a_0 + a_1 e^{-\tau s}}{a} = -n^2 \quad , \quad n = 1, 2, 3, \dots . \qquad (2.32)$$

Location of the zeros of the system depends on the output point x_0. Zeros for the transfer function $G_1(s)$ are computed from the following equation:

$$e^{-2(x_0-\pi)\lambda(s)} = 1 = e^{j2\pi n}, \quad n = 0, \pm 1, \pm 2, \dots ,$$

which is equivalent to

$$\lambda(s) = \pm j \frac{\pi}{x_0 - \pi} n \quad n = 0, \pm 1, \pm 2, \dots .$$

If we take the square of both sides, zeros of the transfer function $G_1(s)$ is the solution of the following equations for $s \in \mathbb{C}$. The case $n = 0$ is proved to be inadmissible. Therefore $n = 0$ case will be disregarded in the final equation:

$$\frac{s - a_0 + a_1 e^{-\tau s}}{a} = -n^2\Big(\frac{\pi}{x_0 - \pi}\Big)^2 \quad , \quad n = 1, 2, 3, \dots \; . \tag{2.33}$$

From a similar discussion, zeros of the transfer function $G_2(s)$ are the solutions of the following equations for $s \in \mathbb{C}$

$$\frac{s - a_0 + a_1 e^{-\tau s}}{a} = -n^2\Big(\frac{\pi}{x_0}\Big)^2 \quad , \quad n = 1, 2, 3, \dots . \tag{2.34}$$

Put $s = \sigma + j\omega$ into the equation (2.32). Here the variables $\sigma \in \mathbb{R}$ and $\omega \in \mathbb{R}$. After separating real and imaginary parts, two equations are obtained for two unknowns σ and ω. These equations are given below:

$$-an^2 = \sigma - a_0 + a_1 e^{-\tau\sigma}\cos(\tau\omega) \quad n = 1, 2, 3 \dots \; , \tag{2.35}$$

$$0 = \omega - a_1 e^{-\tau\sigma}\sin(\tau\omega). \tag{2.36}$$

Similarly, put $s = \sigma + j\omega$ into the equation (2.33) and separate real and imaginary parts to obtain following equations for the zeros of the transfer function $G_1(s)$:

$$-an^2\Big(\frac{\pi}{x_0 - \pi}\Big)^2 = \sigma - a_0 + a_1 e^{-\tau\sigma}\cos(\tau\omega) \quad n = 1, 2, 3 \dots \; , \tag{2.37}$$

$$0 = \omega - a_1 e^{-\tau\sigma}\sin(\tau\omega). \tag{2.38}$$

From the same discussion, real part σ and imaginary part ω of the zeros $s = \sigma + j\omega$ of the transfer function $G_2(s)$ is obtained as follows:

$$-an^2\Big(\frac{\pi}{x_0}\Big)^2 = \sigma - a_0 + a_1 e^{-\tau\sigma}\cos(\tau\omega) \quad n = 1, 2, 3 \dots \; , \tag{2.39}$$

$$0 = \omega - a_1 e^{-\tau\sigma}\sin(\tau\omega). \tag{2.40}$$

Remark 1. Assume that $a_1 = 0$. In this case, the only solution for (2.36), (2.38) and (2.40) is $\omega = 0$. Thus poles of the both transfer functions $G_1(s)$ and $G_2(s)$ are $(a_0 - a\ n^2)$, zeros of the transfer function $G_1(s)$ are $a_0 - a\ n^2\Big(\frac{\pi}{x_0 - \pi}\Big)^2$ and zeros of the transfer function $G_2(s)$ are $a_0 - a\ n^2\Big(\frac{\pi}{x_0}\Big)^2$ for $n = 1, 2, \dots$. In this case, the system is stable if and only if $a > a_0$. In other words, system is stable if and only if heat diffusion is stronger than the feedback. □

For the rest of the thesis, it will be assumed that $a_1 \neq 0$ and $\tau > 0$. When $\tau = 0$, the terms with a_1 and a_0 can be combined and the discussion in Remark 1 becomes valid.

Remark 2. Since a particular solution of (2.36) is $\omega = 0$, all real poles $s = \sigma$ satisfy

$$\sigma + a_1 e^{\tau\sigma} = a_0 - a\, n^2 \ , \qquad n = 1, 2, 3, \ldots .$$

Assume that $a > 0$. Then this equation has a solution for $\sigma \geq 0$ if and only if

$$a - a_0 \leq -a_1 \tag{2.41}$$

Thus system is unstable if (2.41) holds. $\square$

For the rest of the thesis, it will be assumed that $a > 0$ and $-a_1 < a - a_0$, which is a necessary condition for the stability of the system.

Consider the equation (2.32). It has roots in $\mathbb{C}_-$ for all $n = 1, 2, \ldots$ if and only if the equation

$$1 + \frac{a_1 e^{-\tau s}}{s + (an^2 - a_0)} = 0 \tag{2.42}$$

has its roots in $\mathbb{C}_-$ for all $n = 1, 2, \ldots$.

This condition is equivalent to the stability of the feedback systems shown in Figure 2.1, where

$$G_n(s) = \frac{a_1 e^{-\tau s}}{s + (an^2 - a_0)}.$$

Small gain theorem asserts that system is stable independent of delay if $(an^2 - a_0) > |a_1|$ for all $n = 1, 2, 3, \ldots$. Previously it is assumed that $a > 0$. Therefore, if this condition is satisfied for $n = 1$, then the condition is satisfied for all $n \geq 2$. In conclusion, the system is stable independent of delay if

$$|a_1| < a - a_0. \tag{2.43}$$

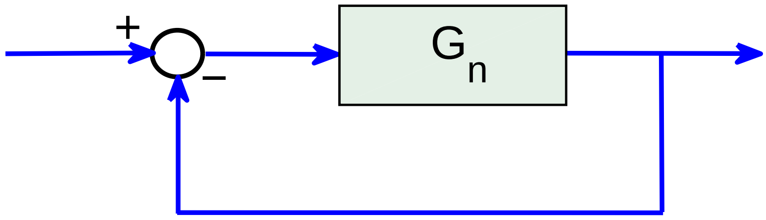

Figure 2.1: Equivalent Feedback System.

If $a_1 < 0$ and $a - a_0 > -a_1$, then the system is stable independent of delay. If $a_1 > 0$ and $a - a_0 > a_1$, then the system is stable independent of delay. In Remark 2, it is showed that the system is unstable if $a - a_0 < -a_1$. In order to make a stability analysis dependent on delay, assume that

$$a_1 > 0 \quad \text{and} \quad -a_1 < a - a_0 < a_1.$$

Define

$$\omega_n := \sqrt{a_1^2 - (an^2 - a_0)^2}. \tag{2.44}$$

Here ω_n is the frequency where we have

$$|G_n(j\omega_n)| = 1.$$

The crossover frequency ω_n will be used in phase margin computations in the analysis. For a fixed n, consider the two cases for the stability analysis of the feedback system shown in Figure 2.1.

Case 1. In this case assume $0 < an^2 - a_0 < a_1$. This results in a stable G_n. Stability of the feedback system shown in Figure 2.1 is equivalent to:

$$\pi - \tan^{-1}\left(\frac{\omega_n}{an^2 - a_0}\right) - \tau\omega_n > 0.$$

where the left hand side is the phase margin of the system. For $\omega > 0$, we take $\theta = \tan^{-1}(\omega)$ in $0 < \theta < \pi/2$.

The above condition can be expressed as

$$\tau_n^{\max} := \frac{\pi - \tan^{-1}\sqrt{x_n^2 - 1}}{(an^2 - a_0)\sqrt{x_n^2 - 1}} > \tau, \tag{2.45}$$

where $x_n := a_1/(an^2 - a_0)$.

Claim: $\tau_n^{\max} < \tau_m^{\max}$ when $n < m$.

Proof. First note that $n < m$ implies $x_m < x_n$ because

$$\frac{x_n}{x_m} = \frac{am^2 - a_0}{an^2 - a_0} > 1.$$

Then we have

$$\frac{\tau_n^{\max}}{\tau_n^{\max}} = \frac{(\pi - \theta_n)/\sin(\theta_n)}{(\pi - \theta_m)/\sin(\theta_m)}$$

where $\theta_n \in (0, \pi/2)$, $\theta_n := \tan^{-1}\sqrt{x_n^2 - 1}$ and $\theta_m < \theta_n$ whenever $n < m$. Since the function $(\pi - \theta)/\sin(\theta)$ is a decreasing function of θ on $\theta \in (0, \pi/2)$, we conclude that

$$\frac{(\pi - \theta_n)/\sin(\theta_n)}{(\pi - \theta_m)/\sin(\theta_m)} < 1 \quad \text{when} \quad \theta_m < \theta_n.$$

Thus $\tau_n^{\max} < \tau_m^{\max}$ whenever $n < m$. □

Lemma 1. The system (2.4) with initial conditions (2.5), $a_1 > 0$ and $0 < a - a_0 < a_1$ is stable if and only if

$$a_1\tau < (\pi - \theta)/\sin(\theta) \text{ , where } \theta := \cos^{-1}(\frac{a - a_0}{a_1}). \tag{2.46}$$

□

Case 2. In this case assume $-a_1 < an^2 - a_0 < 0$. This results in an unstable G_n with a single pole in $\mathbb{C}_+$. Stability of the feedback system shown in Figure 2.1 is equivalent to

$$\tan^{-1}\left(\frac{\omega_n}{a_0 - an^2}\right) - \tau\omega_n > 0.$$

where the left hand side is the phase margin of the system.

The above condition can be expressed as

$$\tau_n^{\max} := \frac{\tan^{-1}\sqrt{x_n^2 - 1}}{(a_0 - an^2)\sqrt{x_n^2 - 1}} > \tau, \tag{2.47}$$

where $x_n := a_1/(a_0 - an^2)$.

Claim: $\tau_n^{\max} < \tau_m^{\max}$ when $n < m$.

Proof. First note that $n < m$ implies $x_m > x_n$ because

$$\frac{x_n}{x_m} = \frac{a_0 - am^2}{a_0 - an^2} > 1.$$

Then we have

$$\frac{\tau_n^{\max}}{\tau_n^{\max}} = \frac{\theta_n/\sin(\theta_n)}{\theta_m/\sin(\theta_m)}$$

where $\theta \in (0, \pi/2)$, $\theta_n := \tan^{-1}\sqrt{x_n^2 - 1}$ and $\theta_n < \theta_m$ whenever $n < m$. Since the function $(\theta)/\sin(\theta)$ is an increasing function of θ on $\theta \in (0, \pi/2)$ we conclude that

$$\frac{\theta_n/\sin(\theta_n)}{\theta_m/\sin(\theta_m)} < 1 \quad \text{when} \quad \theta_n < \theta_m.$$

Thus $\tau_n^{\max} < \tau_m^{\max}$ whenever $n < m$. $\square$

Lemma 2. The system (2.4) with initial conditions (2.5), $a_1 > 0$ and $-a_1 < a - a_0 < 0$ is stable if and only if

$$a_1\tau < \theta/\sin(\theta) \ , \quad \text{where} \ \ \theta := \cos^{-1}(\frac{a_0 - a}{a_1}). \tag{2.48}$$

$\square$

The results of Lemma 1 and Lemma 2 can be combined as follows:

The system (2.4) with initial conditions (2.5), $a_1 > 0$ and $-a_1 < a - a_0 < a_1$ is stable if and only if $a_1\tau < h(a - a_0)$ where

$$h(a - a_0) := \begin{cases} \theta/\sin(\theta) & \text{if } -a_1 < a - a_0 < 0 \\ (\pi - \theta)/\sin(\theta) & \text{if } 0 < a - a_0 < a_1 \end{cases},$$

and $\theta = \cos^{-1}(|a - a_0|/a_1)$. When $a = a_0$ we have $\theta = \theta_o = \pi/2$ and $\theta_o/\sin(\theta_o) = (\pi - \theta_o)/\sin(\theta_o) = \dfrac{\pi}{2}$. Therefore, the function $h(a - a_0)$ is continuous around $a - a_0 = 0$. Figure 2.2 captures all the stability conditions derived here.

Summary of the Results: Consider the system represented by the heat equation with time delayed feedback (2.4), with initial conditions (2.5), and boundary conditions (2.6) and (2.7)

(i) If $a - a_0 < -a_1$, then the system is unstable independent of delay.

(ii) If $-a_1 < a - a_0 < a_1$, then the system is stable if and only if $a_1\tau < h(a - a_0)$.

(iii) If $|a_1| < a - a_0$, then the system is stable independent of delay.

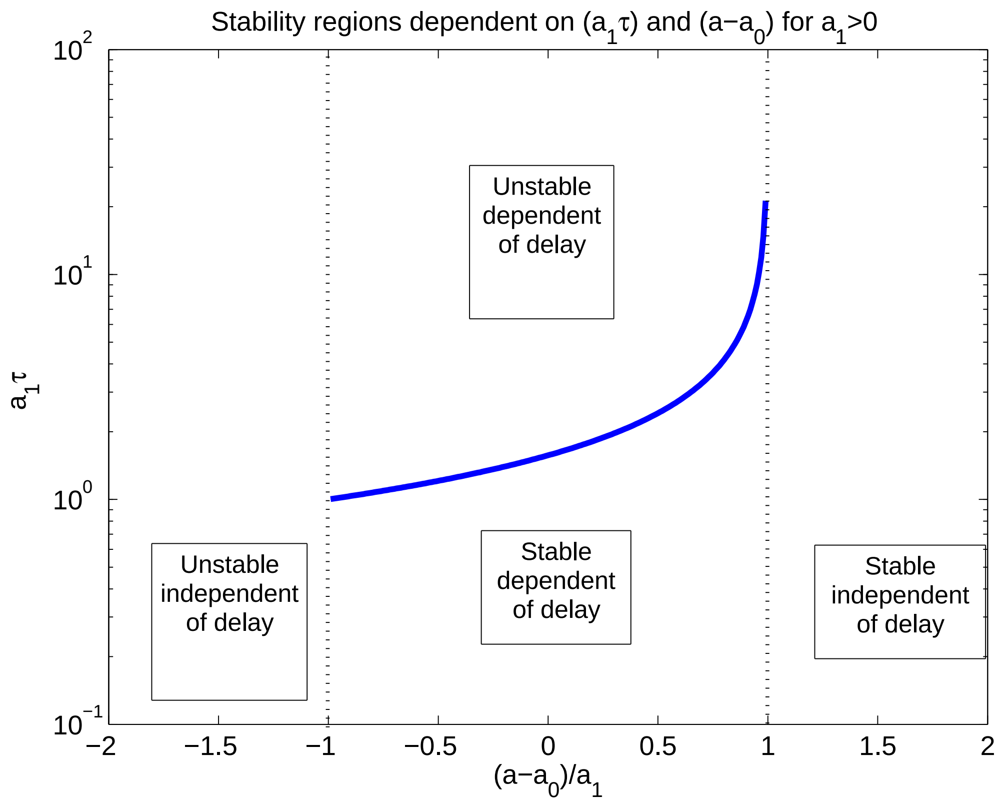

Figure 2.2: Stable and Unstable Regions in the Parameter Space.

2.3 Analysis of Pole Locations

For the function $h(a - a_0)$ defined in the previous chapter, we have

$$\lim_{\eta \to -a_1^+} h(\eta) = 1, \tag{2.49}$$

and $h(a - a_0) > 1$ for every $a - a_0 > -a_1$. Thus if $a_1\tau < 1$ then it is guaranteed that $a_1\tau < h(a - a_0)$ for every $a - a_0 > -a_1$. In other words if $a_1\tau < 1$ then the system is stable for every $a - a_0 > -a_1$.

When $a_1\tau \geq 1$, an alternative method for determining the stability of the system can be applied. In this method, possible poles located in $\mathbb{C}_+$ satisfying (2.35) and (2.36) are searched. For the case $\omega \neq 0$, the equation (2.36) becomes

$$e^{\tau\sigma} = (a_1\tau)\frac{\sin(\tau\omega)}{\tau\omega}. \tag{2.50}$$

If $|a_1\tau| \geq 1$, then a solution for (2.50) with $\sigma \geq 0$ may exists. The solution is

$$\tau\sigma = \ln\Big((a_1\tau)\frac{\sin(\tau\omega)}{\tau\omega}\Big). \tag{2.51}$$

The solution (2.51) is valid when

$$(a_1\tau)\frac{\sin(\tau\omega)}{\tau\omega} > 0. \tag{2.52}$$

Combine (2.35) and (2.36) to obtain

$$\tau\sigma = (a_0\tau) - (a\tau)n^2 - (\tau\omega)\cot(\tau\omega). \tag{2.53}$$

Use equation (2.51) in equation (2.53):

$$\ln\Big((a_1\tau)\frac{\sin(\tau\omega)}{\tau\omega}\Big) + (\tau\omega)\cot(\tau\omega) = (a_0\tau) - (a\tau)n^2, \tag{2.54}$$

for $n = 1, 2, 3, \dots$.

Note that for every solution ω_0 of (2.54), $-\omega_0$ is also a solution, i.e. poles appears in complex conjugate pairs. The variables can be scaled with τ. Define $\widehat{a} = a\tau$, $\widehat{a}_0 = a_0\tau$, $\widehat{a}_1 = a_1\tau$, $\widehat{\sigma} = \sigma\tau$, $\widehat{\omega} = \omega\tau$.

Consider the system described by the transfer function $G_1(x_0, s)$ and $G_2(x_0, s)$ where $x_0 \in (0, \pi)$. Assume that $\widehat{a}_1 \geq 1$, $a - a_0 > -a_1$ and $a_0 \neq an^2$ for $n = 1, 2, 3, \ldots$. There is at least one pole of the system (2.4) in $\bar{\mathbb{C}}_+$ if and only if among all solutions $\widehat{\omega}_{n,k}$, $k = 1, 2, 3\ldots$ of the equations

$$h(\widehat{\omega}) = \widehat{a}_0 - \widehat{a}n^2 \qquad n = 1, 2, 3, \ldots \ , \tag{2.55}$$

where

$$h(\widehat{\omega}) := \begin{cases} \ln\Big(\widehat{a}_1 \frac{\sin(\widehat{\omega})}{\widehat{\omega}}\Big) + \widehat{\omega}\cot(\widehat{\omega}) & \text{if } \widehat{a}_1 \frac{\sin(\widehat{\omega})}{\widehat{\omega}} \geq 0 \\ 0 & \text{if } \widehat{a}_1 \frac{\sin(\widehat{\omega})}{\widehat{\omega}} \leq 0 \end{cases},$$

there exists at least one $\widehat{\omega}_{n,k}$ such that

$$\widehat{a}_1 \frac{\sin(\widehat{\omega}_{n,k})}{\widehat{\omega}_{n,k}} \geq 1. \tag{2.56}$$

2.4 Numerical Examples

Example 1. Consider the one dimensional heat equation example in [13]. The parameters in the one dimensional heat equation is set as $a = 1$, $a_0 = r$ where $|r| \leq 1.9$ and $a_1 = 1$. This yields to the following partial differential equation:

$$\frac{\partial}{\partial t} z(x, t) = \frac{\partial^2}{\partial x^2} z(x, t) + rz(x, t) - z(x, t - \tau). \tag{2.57}$$

In [13], LMIs that are stated in Section 2.1 are solved using LMI toolbox of MATLAB. The maximum time delay in the system in which the system stays stable is found to be $\tau_{\max} = 1.025$ seconds. With the given $\tau_{\max}$, the critical solution of the equation (2.55), i.e. the $\widehat{\omega}$ value which satisfies the equation (2.55) and maximizes the expression (2.56), is obtained as $\widehat{\omega} = 0.4495$. In this case, maximum value for the equation (2.56) is obtained as

$$1 \times 1.025 \times \frac{\sin(0.4495)}{0.4495} = 0.99083 < 1.$$

Using the frequency domain stability analysis performed in the thesis, it can be shown that as $a - a_0$ value gets smaller, stability region shrinks. Therefore the worst case occurs at $r = 1.9$. For $a_0 = r = 1.9$, $a - a_0 = -0.9$. Thus $\tau < \tau_{\max} = h(-0.9) = 1.0347$ seconds. The critical solution of the equation (2.55) for $\tau_{\max} = 1.0347$ seconds is obtained as $\widehat{\omega} = 0.451$. In this case, maximum value for the equation (2.56) is obtained as

$$1 \times 1.0347 \times \frac{\sin(0.451)}{0.451} = 0.9999785 < 1.$$

In the frequency domain, a higher value is obtained for $\tau_{\max}$ for which the system stays stable. In other words, maximum value of the $\tau_{\max}$ can be improved. In [13], LMI toolbox tries to make a certain matrix strictly negative. This results in numerical errors when finding the solution of the LMI. In frequency domain analysis, analytical solution for the stability bound of the time delay τ is obtained. Thus, a higher value for the maximum value of time delay is obtained in frequency domain analysis.

Example 2. Take $a = 2$, $a_0 = 1.5$, $a_1 = 1$ and $\tau = 2$. The graph of $h(\widehat{\omega})$ versus $\widehat{\omega}$ is as shown in Figure 2.3, where the intersection points with $(\widehat{a}_0 - \widehat{a}\ n^2)$ are shown as the roots $\widehat{\omega}_{n,k}$ for $n = 1, 2, 3$, and $k = 0, 1, 2, 3$.

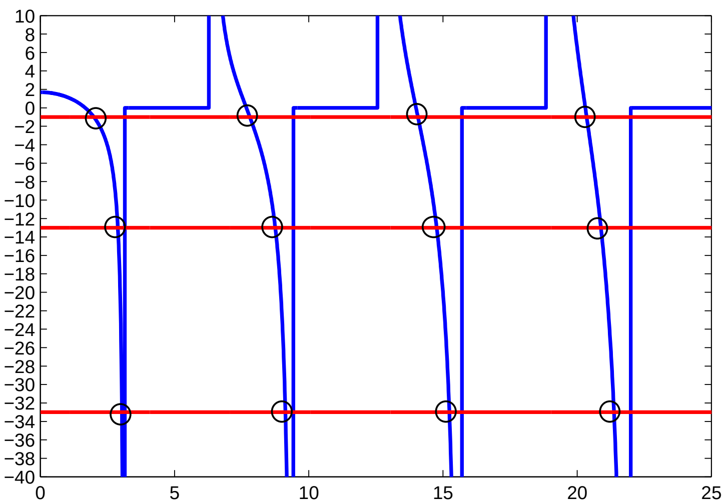

Figure 2.3: Function $h(\hat{\omega})$.

First few roots are listed in the table below.

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	1.997	7.808	14.069	20.355
$n=2$	2.890	8.755	14.768	20.888
$n=3$	3.041	9.131	15.240	21.373

Table 2.1: The roots of $h(\omega) = a_0 - a\ n^2$.

	$k=0$	$k=1$	$k=2$	$k=3$
$n=1$	0.4560	0.1279	0.0709	0.0490
$n=2$	0.0861	0.0709	0.0547	0.0427
$n=3$	0.0330	0.0317	0.0296	0.0271

Table 2.2: Corresponding $sin(\widehat{\omega})/\widehat{\omega}$ values

The maximum of the values $\sin(\widehat{\omega})/\widehat{\omega}$ is 0.456. Check the stability condition.

$$\widehat{a}_1 = 2 < 0.456^{-1} = 2.193$$

Therefore the system is stable.

Chapter 3

Stability Analysis of the Heat Equation with Time-Delayed Feedback and Parametric Uncertainty

In the previous chapter, frequency domain analysis is performed for the system with fixed parameters a, a_0, a_1 and τ. Stability conditions in terms of parameters (a, a_0, a_1, τ) are obtained. In this section, robust stability of the system under parametric uncertainty will be examined. In parametric uncertainty, the parameters a, a_0, a_1 and τ are not fixed. Instead it is known that parameters belong to a finite interval on the real line. In other words, $a \in [a^-, a^+],\ a_0 \in [a_0^-,\ a_0^+],\ a_1 \in [a_1^-, a_1^+],\ \tau \in [\tau^-, \tau^+]$.

In order to obtain the robust stability condition for this system with parametric uncertainty in their parameters, a method called zero exclusion principle will be used. Zero exclusion principle is used for determining the Hurwitz stability of uncertain quasipolynomials [14].

In this chapter, first, zero exclusion principle is explained. Then, zero exclusion principle is applied on the one dimensional heat equation with time-delayed feedback and parametric uncertainty. Finally, a stable system is picked and parametric uncertainty is applied. Then using the method discussed in this section, robust stability is investigated for this system.

3.1 Zero Exclusion Principle

Consider a generic linear time invariant system with concentrated delays. This system can be expressed as follows:

$$\sum_{k=0}^{N}(A_k\dot{x}(t-\tau_k)+B_kx(t-\tau_k))=0 \quad det(A_0)\neq 0. \tag{3.1}$$

Here $0=\tau_0<\tau_1<\cdots<\tau_N$ are delays in the system. This system is exponentially stable if and only if all zeros of the characteristic equation is in left half plane. Characteristic equation for this system is given below:

$$f(s)=\det\left(\sum_{k=0}^{N}(A_ks+B_k)e^{-\tau_k s}\right)=\sum_{k=0}^{n}\sum_{i=0}^{m}a_{ki}s^{n-k}e^{-r_i s}. \tag{3.2}$$

Coefficients of the exponential terms are real and ordered as $0=r_0<r_1<\cdots<r_m$. Characteristic equation $f(s)$ for time delay systems are not polynomials. They are quasipolynomials, which are generalizations for polynomials.

Characteristic quasipolynomial $f(s)$ can be written as

$$f(s)=\sum_{i=0}^{m}p_i(s)e^{-r_i s}=\sum_{k=0}^{n}\psi_k(s)s^{n-k}. \tag{3.3}$$

Here $p_i(s)$ terms are in the form

$$p_i(s)=a_{0i}s^n+a_{1i}s^{n-1}+\cdots+a_{ni}, \qquad i=0,1,\ldots,m. \tag{3.4}$$

and $\psi_k(s)$ terms are quasipolynomials in the form

$$\psi_k(s)=a_{k0}e^{r_0 s}+a_{k1}e^{r_1 s}+\cdots+a_{km}e^{r_m s}, \qquad i=0,1,\ldots,n. \tag{3.5}$$

Define the following sets

$$F = \{f(s, \mathbf{a}, \mathbf{r}) \mid (\mathbf{a}, \mathbf{r}) \in Q_F\}, \tag{3.6}$$

$$Q_F = \{(\mathbf{a}, \mathbf{r}) \mid f(s, \mathbf{a}, \mathbf{r}) \in F\}. \tag{3.7}$$

In these definitions, the coefficient vector $\mathbf{a}$ is defined as

$$\mathbf{a} = (a_{00}, \ldots, a_{0n}, a_{10}, \ldots, a_{1n}, \ldots, a_{mn}). \tag{3.8}$$

Similarly, the exponent coefficient vector $\mathbf{r}$ is defined as

$$\mathbf{r} = (r_1, r_2, \ldots, r_n). \tag{3.9}$$

Consider the following assumptions:

Assumption (A1). Every member of F has a non-zero principal term. According to (3.3), this assumption may be stated as $\deg(p_0) \geq \deg(p_i)$ for $i = 1, 2, \ldots$ for every $f(s) \in F$. □

Assumption (A2). The exponent coefficient vector of every member of the uncertain quasipolynomial has only positive components, that is, $r_i > 0,\ i = 1, 2, \ldots, m$ for every $f(s) \in F$. □

Assumption (A3). There exists $R > 0$ and $\varepsilon > 0$ such that for every $f(s) \in F$, the corresponding quasipolynomial $\psi_0(s)$ has no zeros of magnitude greater than R (if any) with real part greater than $-\varepsilon$. □

Assumption (A4). The set Q_F is compact and pathwise connected. □

Using the assumptions stated above, zero exclusion principle can be stated.

Theorem 1. ([14]) Let F satisfy the assumptions (A1)-(A4). Then all members of F are Hurwitz stable (i.e. all zeros are on left half plane) if and only if

i) At least one member of F is Hurwitz stable,

ii) For every point $s = j\omega$ on the imaginary axis, the value set $V_F(j\omega) = \{f(j\omega) \mid f \in F\}$ computed at this point does not contain the origin of the complex plane. □

Theorem 2. ([14]) The second condition (ii) of Theorem 1 is equivalent to the following two conditions:

i) At least for one point $j\omega_0$, of the imaginary axis, $0 \notin V_F(j\omega_0)$,

ii) $0 \notin \partial V_F(j\omega)$ for all other point of the imaginary axis. Here $\partial V_F(j\omega)$ stands for the boundary of the value set $V_F(j\omega)$. □

3.2 Robust Stability of the System

For the stability of one dimensional heat equation with time delayed feedback system, the quasipolynomials

$$f(s) = s - a_0 + a_1 e^{-\tau s} + an^2 \quad n = 1, 2, \ldots \tag{3.10}$$

must be stable. Although infinitely many number of quasipolynomials must be Hurwitz stable for the stability of the system, in the previous section, it is showed that if the quasipolynomial for the case $n = 1$ is stable, we can conclude the quasipolynomials with $n > 1$ is also stable. The quasipolynomial for the case $n = 1$ is given below:

$$f(s) = s - a_0 + a_1 e^{-\tau s} + a. \tag{3.11}$$

This quasipolynomial can be decomposed as

$$f(s) = \sum_{i=0}^{1} p_i(s) e^{-r_i s} = \sum_{k=0}^{1} \psi_k(s) s^{n-k}, \tag{3.12}$$

$$p_0(s) = s - a_0 + a, \tag{3.13}$$

$$p_1(s) = a_1, \tag{3.14}$$

$$\psi_0(s) = 1, \tag{3.15}$$

$$\psi_1(s) = a_1 e^{-\tau s} + a - a_0. \tag{3.16}$$

Consider that parameters in our system is not fixed but has a a parametric uncertainty such that $a \in [a^-, a^+]$, $a_0 \in [a_0^-, a_0^+]$, $a_1 \in [a_1^-, a_1^+]$, $\tau \in [\tau^-, \tau^+]$. Now the sets F and Q_F, that are defined in (3.6) and (3.7) respectively, become as follows

$$F = \{f(s, a, a_0, a_1, \tau) \mid (a, a_0, a_1, \tau) \in Q_F\} \tag{3.17}$$

where

$$Q_F = \{(a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 \in [a_0^-, a_0^+], a_1 \in [a_1^-, a_1^+], \tau \in [\tau^-, \tau^+]\} \tag{3.18}$$

Now check the validity of the assumptions A1-A4 which are stated in previous section

Validity of Assumption (A1): $\deg(s - a_0 + a) = 1 \geq \deg(a_1) = 0$. Assumption 1 is valid for the quasipolynomial (3.11). □

Validity of Assumption (A2): $r_1 = \tau > 0$ is assumed in problem definition. Assumption 2 is valid for the quasipolynomial 3.11. □

Validity of Assumption (A3): $\psi_0(s) = 1$ has no zeros. Assumption 3 is valid for the quasipolynomial 3.11. □

Validity of Assumption (A4): Q_F is a finite and connected set in R^4 thus it is compact and pathwise connected [25]. Assumption 4 is valid for the quasipolynomial 3.11. □

Since all the assumptions are satisfied, Theorem 1 and Theorem 2 can be applied to the quasipolynomial 3.11. In order to apply Theorem 1, first the value set for our system must be found. The value set $V_F(j\omega) = \{f(j\omega) \mid f \in F\}$ for the quasipolynomial 3.11 is given below:

$$V_F(j\omega) = \Big\{ j\Big(\omega - a_1 \sin(\omega\tau)\Big) + \Big(a_1 \cos(\omega\tau) + a - a_0\Big) \mid (a, a_0, a_1, \tau) \in Q_F \Big\}. \tag{3.19}$$

By combining Theorem 1 and Theorem 2, we can state that system is robustly stable if the following conditions hold:

(i) At least one member of F is Hurwitz stable.

(ii) At least for one point $j\omega_0$, of the imaginary axis, $0 \notin V_F(j\omega_0)$.

(iii) $0 \notin \partial V_F(j\omega)$ for all other point of the imaginary axis. Here $\partial V_F(j\omega)$ stands for the boundary of the value set $V_F(j\omega)$.

Assume that at least one member of F is Hurwitz stable. In order to check the conditions (ii) and (iii), all $\omega \in [0, \infty]$ values should be considered. Divide the set into subsets $S_1 = (a_1^+, \infty]$ and $S_2 = [0, a_1^+]$ such that $[0, \infty] = S_1 \cup S_2$ and consider each subset separately.

Case 1. Consider that $\omega \in S_1$. This implies $\omega - a_1 \sin(\omega\tau) \geq \omega - a_1^+ > 0$. In other words, when $\omega \in S_1$, for all $z \in V_F$, $\mathrm{Im}(z) \neq 0$. Thus origin of the complex plane is not an element of the value set for $\omega \in S_1$. Also by choosing an $\omega_0 \in S_1$, condition (ii) is satisfied.

Case 2. Consider that $\omega \in S_2$. It is assumed that condition (i) is satisfied. An ω_0 value which satisfies the condition (ii) is found in Case 1. Thus in this case, we only need to check the validity of the condition (iii). Since Q_F is compact and mapping from Q_F to the complex plane is continuous, the boundary of the value set V_F can be found as follows:

$$\partial V_F(j\omega) = \{ j\Big(\omega - a_1 \sin(\omega\tau)\Big) + \Big(a_1 \cos(\omega\tau) + a - a_0\Big) \mid (a, a_0, a_1, \tau) \in B_F \}, \tag{3.20}$$

where B_F is the boundary of Q_F. Note that $B_F \in R^3$. Boundary of Q_F can be written as the sum of 8 regions:

$$B_F = \bigcup_{i=1}^{8} R_i. \tag{3.21}$$

These regions are given below

$$R_1 = \{((a, a_0, a_1, \tau) \mid a = a^-, a_0 \in [a_0^-, a_0^+], a_1 \in [a_1^-, a_1^+], \tau \in [\tau^-, \tau^+]\}, \tag{3.22}$$

$$R_2 = \{((a, a_0, a_1, \tau) \mid a = a^+, a_0 \in [a_0^-, a_0^+], a_1 \in [a_1^-, a_1^+], \tau \in [\tau^-, \tau^+]\}, \tag{3.23}$$

$$R_3 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 = a_0^-, a_1 \in [a_1^-, a_1^+], \tau \in [\tau^-, \tau^+]\}, \tag{3.24}$$

$$R_4 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 = a_0^+, a_1 \in [a_1^-, a_1^+], \tau \in [\tau^-, \tau^+]\}, \tag{3.25}$$

$$R_5 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 \in [a_0^-, a_0^+], a_1 = a_1^-, \tau \in [\tau^-, \tau^+]\}, \tag{3.26}$$

$$R_6 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 \in [a_0^-, a_0^+], a_1 = a_1^+, \tau \in [\tau^-, \tau^+]\}, \tag{3.27}$$

$$R_7 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 \in [a_0^-, a_0^+], a_1 \in [a_1^-, a_1^+], \tau = \tau^-\}, \tag{3.28}$$

$$R_8 = \{((a, a_0, a_1, \tau) \mid a \in [a^-, a^+], a_0 \in [a_0^-, a_0^+], a_1 \in [a_1^-, a_1^+], \tau = \tau^+\}. \tag{3.29}$$

Note that if there are constant parameters, some of these regions become redundant. Consider that $a = a_c$, $a_0 = a_{0c}$ and $a_1 = a_{1c}$ are constant, $\tau \in [\tau^-, \tau^+]$. In this case, the boundary of Q_F can be written as follows:

$$B_F = R_{new1} \cup R_{new2}, \tag{3.30}$$

$$R_{new1} = \{((a, a_0, a_1, \tau) \mid a = a_c, a_0 = a_{0c}, a_1 = a_{1c}, \tau = \tau^-\}, \tag{3.31}$$

$$R_{new2} = \{((a, a_0, a_1, \tau) \mid a = a_c, a_0 = a_{0c}, a_1 = a_{1c}, \tau = \tau^+\}. \tag{3.32}$$

3.3 Numerical Examples

Example 1. Take $a = 2$, $a_0 = 1.5$, $a_1 = 1$ and $\tau \in [2, \tau^+]$ where $\tau_+ > 2$. In this example, boundary of the value set for different values of τ^+ will be found. From the discussion on Chapter 2, it can be stated that system is stable for

$a = 2$, $a_0 = 1.5$, $a_1 = 1$ and $\tau < \frac{\pi - cos^{-1}(0.5)}{sin(cos^{-1}(0.5))} = 2.4184$. Thus for at least one member of F, which is $F_1 = \{f(s, a, a_0, a_1, \tau) \mid (a, a_0, a_1, \tau) = (2, 1.5, 1, 2)\} \in F$, the polynomial F_1 is Hurwitz stable. Condition (i) is satisfied. Value set for the quasipolynomial $f(s) = s + 1 + e^{-\tau s}$ is found to be

$$V_F(j\omega) = \Big\{ j\Big(\omega - \sin(\omega\tau)\Big) + \Big(\cos(\omega\tau) + 0.5\Big) \mid \tau \in [2, \tau^+] \Big\}.$$

Chose $\omega_0 = 2 > a_1^+ = a_1 = 1$. Then for the point $j\omega_0$ on the imaginary axis, $0 \notin V_F(j\omega_0)$. Condition (ii) is also satisfied. We only need to check Condition (iii) for $\omega \in [0, a_1] = [0, 1]$. Boundary of the set Q_F is two points, $\tau = 2$ and $\tau = \tau^+$. Boundary of the value set, $\partial V_F(j\omega)$ for several τ^+ values and for the frequency range $\omega \in [0, 1]$ is plotted in Figure (3.1). There is no need to plot the boundary of the value set for frequencies $\omega > 1$ because it is clear that origin is not contained for these frequency values.

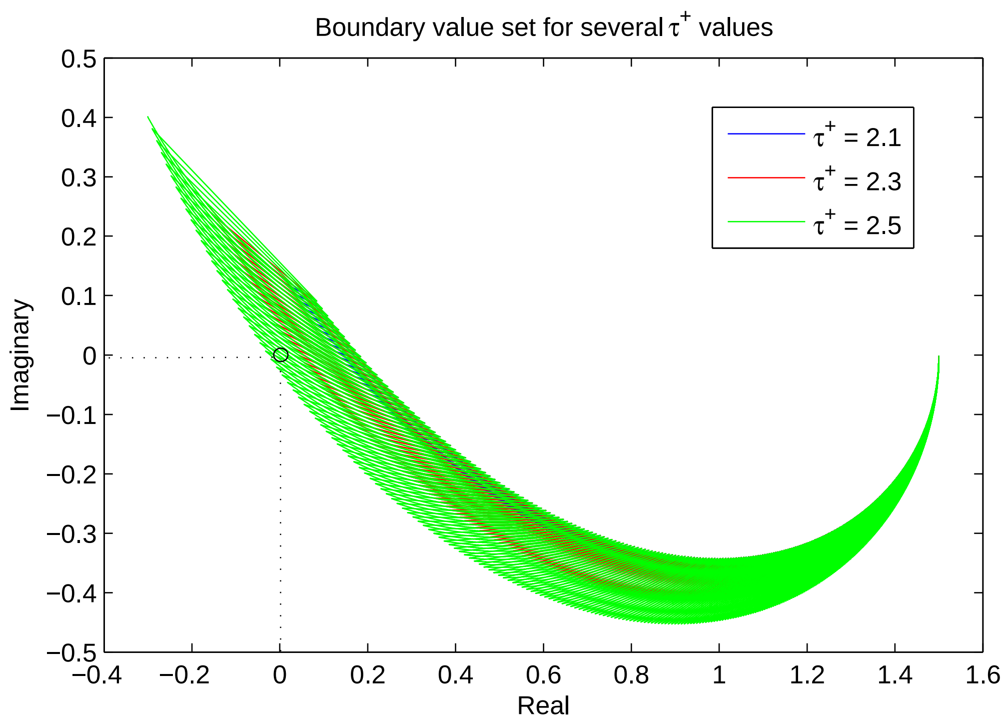

Figure 3.1: Boundary value set for different values of τ^+

From the plot, it can be observed that the origin is contained for $\tau^+ = 2.5$. Origin is not the element of the boundary value set when $\tau^+ = 2.1$ and $\tau^+ = 2.3$. In other words, system is robustly stable when $\tau^+ = 2.1$ and $\tau^+ = 2.3$. System is not robustly stable when $\tau^+ = 2.5$.

Example 2. Take $a = 2$, $a_0 = 1.5$, $a_1 = [0.95, 1.05]$ and $\tau \in [1.95, 2.05]$ In Chapter 2, it is found that the system is stable for the parameters $(a, a_0, a_1, \tau) = (2, 1.5, 1, 2)$ in Example 2. Thus for at least one member of F, which is $F_1 = \{f(s, a, a_0, a_1, \tau) \mid (a, a_0, a_1, \tau) = (2, 1.5, 1, 2)\} \in F$, the polynomial F_1 is Hurwitz stable. Condition (i) is satisfied. Value set for the quasipolynomial $f(s) = s + 1 + a_1 e^{-\tau s}$ is found to be

$$V_F(j\omega) = \Big\{ j\Big(\omega - a_1 \sin(\omega\tau)\Big) + \Big(a_1 \cos(\omega\tau) + 0.5\Big) \mid a_1 \in [0.95, 1.05], \tau \in [1.95, 2.05] \Big\}.$$

Chose $\omega_0 = 2 > a_1^+ = 1.05$. Then for the point $j\omega_0$ on the imaginary axis, $0 \notin V_F(j\omega_0)$. Condition (ii) is also satisfied. We only need to check Condition (iii) for $\omega \in [0, a_1^+] = [0, 1.05]$. Boundary of the set Q_F is the union of 4 sets.

$$B_F = \bigcup_{i=1}^{4} R_i,$$

$$R_1 = \{a_1 = 0.95, \tau \in [1.95, 2.05]\},$$

$$R_2 = \{a_1 = 1.05, \tau \in [1.95, 2.05]\},$$

$$R_3 = \{a_1 \in [0.95, 1.05], \tau = 1.95\},$$

$$R_4 = \{a_1 \in [0.95, 1.05], \tau = 2.05\}.$$

Boundary of V_F is

$$\partial V_F(j\omega) = \Big\{ j\Big(\omega - a_1 \sin(\omega\tau)\Big) + \Big(a_1 \cos(\omega\tau) + 0.5\Big) \mid (a_1, \tau) \in B_F \Big\}.$$

After checking the values of the complex variables in the set ∂V_F, it is observed that the set ∂V_F does not contain the origin of the complex plane. Thus the system is robustly stable. Boundary value set ∂V_F is given in Figure (3.2).

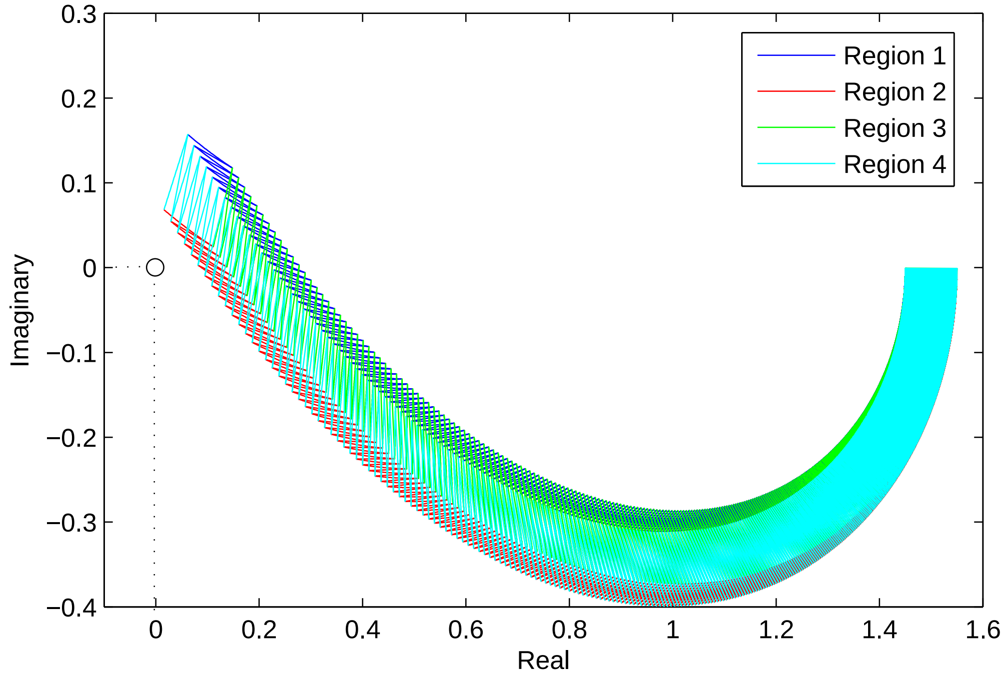

Figure 3.2: Boundary value set for Example 2

Chapter 4

$\mathbf{H}_\infty$ Controller for One Dimensional Heat Equation with Time-Delayed Feedback

In this chapter, the method for obtaining the H_∞ controller which solves the mixed sensitivity problem for the plant described by the heat equation with time delayed feedback will be explained. The chapter begins with the explanation of the mixed sensitivity problem. Then, a methodology for solving the mixed sensitivity problem for the plant described by the heat equation with time delayed feedback is given. Finally, an example is given in order to show how to apply the procedure for finding the H_∞ controller.

4.1 Mixed Sensitivity Problem

4.1.1 Robust Stability Problem

Consider the feedback system shown in Figure 4.1. In this figure, P_0 is the nominal plant and C is the controller. Consider the set of all possible plants

$$P \in \mathcal{P} = \{P_0 + \Delta \mid P_0 \text{ and } P_0 + \Delta \text{ has the same number of poles in right half}$$

$$\text{plane } C_+ \text{ and } |\Delta(j\omega)| < |W(j\omega)| \text{ for all } \omega \text{ values}\}$$

The open loop transfer function for the feedback system in Figure 4.1 is $G_0(s) = C(s)P_0(s)$. Number of the right half plane poles of $G_0(s)$ is denoted as N_p. Assume that $G_0(s)$ is strictly proper. Also assume that there is no unstable pole zero cancellation. Consider that controller $C(s)$ is designed so that the feedback system in Figure 4.1 is stable. This is equivalent to the fact that when Nyquist plot is plotted for $G_0(j\omega)$, the point -1 in complex plane is encircled N_p times in counter-clockwise direction.

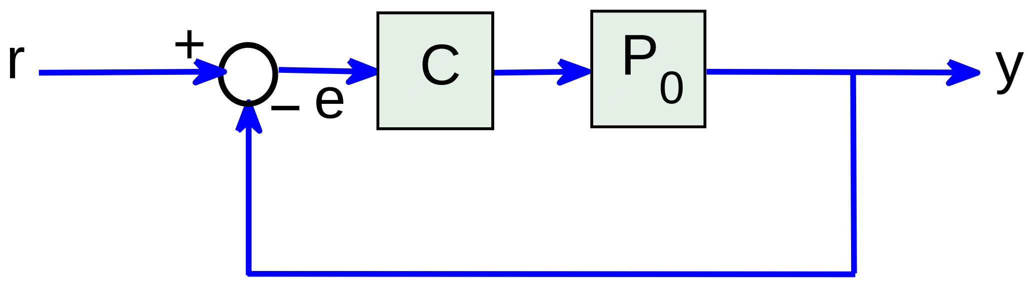

Figure 4.1: Feedback system with controller and nominal plant P_0

Robust stability problem is to find the condition in terms of the nominal plant P_0, controller C and uncertainty weight W such that for all possible plants $P_0+\Delta$ with additive uncertainty Δ, system is stable. Here note that uncertainty weight $W(j\omega)$ must satisfy that $|\Delta(j\omega)| < |W(j\omega)|$ for every ω. Solution of this problem comes from the Nyquist plot. Note that, assuming C is stabilizing the nominal plant P_0, for the nominal plant, the distance from the point on Nyquist plot at an arbitrary point ω_0 to the point -1 can be found as follows:

$$d(\omega_0) = |G_0(j\omega_0) - (-1)|. \tag{4.1}$$

Keep the controller same and add uncertainty to the plant so that plant equation becomes $P = P_0 + \Delta$. New open loop transfer function becomes as follows:

$$G(j\omega) = C(j\omega)P(j\omega) < G_0(j\omega) + W(j\omega)C(j\omega). \tag{4.2}$$

This means that if the distance from -1 to $G_0(j\omega)$ is greater than $W(j\omega)C(j\omega)$ for every ω, number of encirclements do not change with the additional uncertainty in plant dynamics.

$$\begin{aligned}
|G(j\omega)+1| &> |W(j\omega)C(j\omega)| \quad \forall\omega,\\
|W(j\omega)C(j\omega)(1+P_0(j\omega)C(j\omega))^{-1}| &< 1 \quad \forall\omega,\\
\sup_{\omega} |W(j\omega)C(j\omega)(1+P_0(j\omega)C(j\omega))^{-1}| &< 1,\\
||WC(1+P_0C)^{-1}||_\infty &< 1. \qquad (4.3)
\end{aligned}$$

Here $||.||$ denotes the ∞ norm. In order to have robust stability, the controller must stabilize the nominal plant P_0 and $||WC(1+P_0C)^{-1}||_\infty < 1$. Equivalently, for robust stability, the controller must stabilize the nominal plant P_0 and satisfy $||W_2P_0C(1+P_0C)^{-1}||_\infty < 1$ where

$$|W_2(j\omega)| = \frac{|W(j\omega)|}{|P_0(j\omega)|}.$$

4.1.2 Nominal Performance Problem

Consider the same feedback system which is given in Figure 4.1. Consider the set of all reference signals of interest

$$r \in \mathcal{R} = \{W_1 R_1 \mid 0 < ||r_1||_2 < 1\},$$

where $||.||_2$ is the 2-norm. Also consider the set of all stabilizing controllers

$$\mathscr{C}(P_0) = \{C \mid C \text{ stabilizes the plant } P_0\}.$$

Our aim is to chose a controller $C \in \mathscr{C}(P_0)$ such that

$$\sup_{\omega} \frac{||e||_2}{||r||_2} = ||W_1(1+P_0C)^{-1}||_\infty = ||W_1(1+P_0C)^{-1}||$$

is minimized.

By minimizing the H_∞ cost γ_{opt}, which is defined as

$$\gamma_{\text{opt}} = \inf_{C \in \mathcal{C}(P_0)} \left\| \begin{bmatrix} W_1(1+P_0C)^{-1} \\ W_2P_0C(1+P_0C)^{-1} \end{bmatrix} \right\|_\infty,$$

robust stability problem and nominal performance problem can be solved simultaneously. Problem of minimizing γ_{opt} is called mixed sensitivity problem [26].

4.2 Finding the H_∞ Controller

Problem in this chapter is to determine the optimal H_∞ controller which stabilizes the system and achieves the minimum H_∞ cost γ_{opt} which is defined in the previous section as

$$\gamma_{\text{opt}} = \inf_{C \in \mathcal{C}(P_0)} \left\| \begin{bmatrix} W_1(1+P_0C)^{-1} \\ W_2P_0C(1+P_0C)^{-1} \end{bmatrix} \right\|_\infty.$$

Consider the system which is defined by the one dimensional heat equation with time delayed feedback which is explained in Chapter 2. In the analysis in

Chapter 2, two ends of the one dimensional rod are selected as input points. These points are $x = 0$ and $x = \pi$. Output point is selected as an arbitrary fixed point $x_0 \in (0, \pi)$. In this chapter, without loss of generality, output point is selected as $x_0 = \pi/2$. With this selection of the output point, transfer functions from $x = 0$ to the output point x_0 and the transfer function from $x = \pi$ to the output point x_0 becomes identical. This identical transfer function is the transfer function of the nominal plant for which the H_∞ controller will be designed. Nominal plant transfer function is

$$P_0 = \frac{e^{\pi\lambda(s)/2} - e^{-\pi\lambda(s)/2}}{e^{\pi\lambda(s)} - e^{-\pi\lambda(s)}}, \tag{4.4}$$

where

$$\lambda(s) = \sqrt{\frac{s - a_0 + a_1 e^{-\tau s}}{a}}.$$

In [15], [16] and [28], a method is proposed for obtaining the optimum H_∞ controller for the SISO plant with time delay, P_0. However, in order to apply this method, several conditions must be satisfied. These conditions are given below.

Condition 1. P_0 has no imaginary axis zeros or poles. This condition can be omitted with the proper selection of the weights W_1 and W_2.

Condition 2. P_0 has finitely many unstable poles.

Condition 3. P_0 can be expressed in the following form

$$P_0 = \frac{m_n N_0}{m_d}, \tag{4.5}$$

where m_n is an inner function, m_d is a finite dimensional inner function, N_0 is an outer function.

Condition 2 and 3 is the results of the restrictions of the Skew-Toeplitz approach to H_∞ controller of the infinite dimensional systems. From complex analysis, number of the unstable roots of the equations

$$e^{\pi\lambda(s)/2} - e^{-\pi\lambda(s)/2} = 0 \tag{4.6}$$

and

$$e^{\pi\lambda(s)} - e^{-\pi\lambda(s)} = 0 \tag{4.7}$$

is known to be finite. Moreover, there is a root-finding algorithm for determining the unstable roots of the equation (4.7) [3]. This algorithm is explained in Chapter 1 and it can be used to detect the unstable roots of the equation

$$e^{\pi\lambda(s)/2} - e^{-\pi\lambda(s)/2} = 0$$

by doubling the value of a in the algorithm.

In summary, P_0 has finitely many poles and zeros. These poles and zeros can be found using the algorithm in [3]. Steps for obtaining the optimal H_∞ controller is given below.

Step 1 - Factorize P_0: Find the unstable poles and zeros of the nominal plant P_0. Assume that there are n unstable poles and m unstable zeros. Let the set of the unstable poles be $\{p_1, \ldots, p_i, \ldots, p_n\}$ and let the set of the unstable zeros be $\{z_1, \ldots, z_i, \ldots, z_m\}$. Choose

$$m_d = \frac{\prod_{i=1}^{n}(s - p_i)}{\prod_{i=1}^{n}(s + p_i)}, \tag{4.8}$$

$$m_n = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{m}(s + z_i)}, \tag{4.9}$$

and

$$N_0 = \frac{m_d P_0}{m_n}. \tag{4.10}$$

With these choices, m_d and m_n become finite dimensional inner (i.e. stable all-pass) functions. N_0 becomes an infinite dimensional outer (i.e. stable minimum phase) function.

Step 2 - Choose the weights: Plot the magnitude Bode plot of the transfer function P_0 or the magnitude plot of the outer function N_0 which is exactly the same plot because of the fact that

$$|P_0(j\omega)| = |N_0(j\omega)|.$$

Determine an operating frequency range $(0, \omega_{BW})$. Find the negative slope of the Bode magnitude plot as $-20n$ dB, where n is the relative degree of the plant near $\omega \approx \omega_{BW}$. Pick W_2 as

$$W_2 = (s + a_w)^n, \tag{4.11}$$

where

$$a_w = \frac{10}{\omega_{BW}}. \tag{4.12}$$

Finally, for tracking of step-like low frequency reference signals, we pick W_1 as

$$W_1 = \frac{\epsilon s + 1}{s + \epsilon}, \tag{4.13}$$

where $\epsilon << 1$.

Step 3 - Find $\gamma_{\mathbf{opt}}$: Assume that $\alpha_i = p_i \geq 0$ are the distinct unstable poles of the system. Number of unstable poles of the system are l. Consider the function

$$E_\rho(s) = \Big(\frac{W_1(s)W_1(-s)}{\rho^2} - 1\Big). \tag{4.14}$$

Let $\beta_1, \ldots, \beta_{2n_1}$ be the distinct zeros of E_ρ. These β_i values can be enumerated into two sets $\{\beta_i, \ldots, \beta_{n_1}\}$ and $\{\beta_{n_1+1}, \ldots, \beta_{2n_1}\}$ where $\beta_i, \ldots, \beta_{n_1}$ are in $\mathbb{C}_+$ and $\beta_{n_1+i} = -\beta_i$. Define

$$F_\rho = G_\rho(s) \prod_{k=1}^{n_1} \frac{s - \eta_k}{s + \eta_k}, \tag{4.15}$$

where $\eta_1, \ldots, \eta_{n_1}$ are the poles of $W_1(-s)$ and $G(s)$ is the minimum phase transfer function which is determined from the following spectral factorization:

$$G_\rho(s)G_\rho(-s) = \Bigg(1 - \Big(\frac{W_1(s)W_1(-s)}{\rho^2} - 1\Big)\Big(\frac{W_2(s)W_2(-s)}{\rho^2} - 1\Big)\Bigg)^{-1}. \tag{4.16}$$

Consider the polynomials $L_1(s)$ and $L_2(s)$ with degrees less than or equal to $n_1 + l$. For an arbitrary real number a, these polynomials satisfy the following interpolation conditions:

$$0 = L_1(\beta_k) + m_n(\beta_k)F_\rho(\beta_k)L_2(\beta_k) \quad k = 1, \ldots, n_1, \tag{4.17}$$

$$0 = L_1(\alpha_k) + m_n(\alpha_k)F_\rho(\alpha_k)L_2(\alpha_k) \quad k = 1, \ldots, l, \tag{4.18}$$

$$0 = L_2(-\beta_k) + m_n(\beta_k)F_\rho(\beta_k)L_1(-\beta_k) \quad k = 1, \dots, n_1, \tag{4.19}$$

$$0 = L_2(-\alpha_k) + m_n(\alpha_k)F_\rho(\alpha_k)L_1(-\alpha_k) \quad k = 1, \dots, l. \tag{4.20}$$

Note that interpolation conditions (4.17), (4.18), (4.19) and (4.20) can be written as

$$M_\rho \Psi = 0, \tag{4.21}$$

where Ψ is a $2(n_1+l) \times 1$ column vector. First $n_1 + l$ column is the coefficients of the polynomial L_1. Second $n_1 + l$ column is the coefficients of the polynomial L_2. M_ρ, which depends on ρ, is a square matrix of size $2(n_1+l) \times 2(n_1+l)$. M_ρ can be decomposed as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \\ M_{31} & M_{32} \\ M_{41} & M_{42} \end{bmatrix}. \tag{4.22}$$

M_{11} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{11}(i,j) = \beta(i)^{n_1+l-j}$. M_{12} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{12}(i,j) = m_n(\beta(i))F_\rho(\beta(i))\beta(i)^{n_1+l-j}$. M_{21} is a matrix of size $l \times (n_1 + l)$ where $M_{21}(i,j) = \alpha(i)^{n_1+l-j}$. M_{22} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{22}(i,j) = m_n(\alpha(i))F_\rho(\alpha(i))\alpha(i)^{n_1+l-j}$. M_{31} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{11}(i,j) = (-\alpha(i))^{n_1+l-j}$. M_{32} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{32}(i,j) = m_n(\beta(i))F_\rho(\beta(i))(-\alpha(i))^{n_1+l-j}$. M_{41} is a matrix of size $l \times (n_1 + l)$ where $M_{41}(i,j) = (-\beta(i))^{n_1+l-j}$. M_{42} is a matrix of size $n_1 \times (n_1 + l)$ where $M_{42}(i,j) = m_n(\alpha(i))F_\rho(\alpha(i))(-\beta(i))^{n_1+l-j}$.

In order to find the optimum γ value, first pick a range $[\gamma_{\min}, \gamma_{\max}]$ such that $\gamma_{\text{opt}} \in [\gamma_{\min}, \gamma_{\max}]$. Then for every $\gamma \in [\gamma_{\min}, \gamma_{\max}]$, set $\rho = \gamma$ and calculate the matrix M_ρ. If there exists some cases where the number of zeros of the E_ρ is less than $2n_1$, eliminate this case and redefine the interval $[\gamma_{\min}, \gamma_{\max}]$. After successfully calculating the M_ρ matrix, find its minimum singular value $\sigma_{\min}$. The γ point which yields to the smallest $\sigma_{\min}$ value is the value γ_{opt}.

Step 4 - Determine the optimum H_∞ controller: After determining the minimum H_∞ cost γ_{opt}, find polynomials $L_1(s)$ and $L_2(s)$ which satisfies the interpolation conditions (4.17), (4.18), (4.19) and (4.20). Then define $L = L_2/L_1$. Set $\rho = \gamma_{\text{opt}}$ and calculate $E_\rho = E_{\gamma_{\text{opt}}}$ and $F_\rho = F_{\gamma_{\text{opt}}}$. With the previously defined functions, optimal controller can be written as

$$C_{\text{opt}} = E_{\gamma_{\text{opt}}}(s) m_d(s) \frac{N_0(s)^{-1} F_{\gamma_{\text{opt}}}(s) L(s)}{1 + m_n(s) F_{\gamma_{\text{opt}}}(s) L(s)}. \tag{4.23}$$

4.3 Design Example

The one dimensional heat equation with time-delayed feedback which is previously defined in Chapter 2 is given below:

$$\frac{\partial}{\partial t} z(x,t) = a \frac{\partial^2}{\partial x^2} z(x,t) + a_0 z(x,t) - a_1 z(x, t-\tau). \tag{4.24}$$

Inputs for the system are applied from the two ends of the one dimensional rod. These points are $x = 0$ and $x = \pi$ points. These inputs can be defined as

$$z(0,t) = u_1(t) \;,\; \text{with } u_1(t) = 0, \text{ for } t < t_0, \tag{4.25}$$

$$z(\pi,t) = u_2(t) \;,\; \text{with } u_2(t) = 0, \text{ for } t < t_0. \tag{4.26}$$

For this example, assume that the output point of the system is the midpoint of the rod i.e $x_o = \pi/2$. For this selection of the output, transfer function from u_1 to x_o, $G_1(s)$ and transfer function from u_2 to x_o, $G_2(s)$, becomes the same. With the selection of the parameter set $(a, a_0, a_1, \tau) = (2, 1.5, 1, 2.5)$, transfer functions G_1 and G_2 become unstable. Transfer functions can be obtained as

$$G_1(s) = G_2(s) = P_0(s) = \frac{e^{\pi\lambda(s)/2} - e^{-\pi\lambda(s)/2}}{e^{\pi\lambda(s)} - e^{-\pi\lambda(s)}}, \tag{4.27}$$

where

$$\lambda(s) = \sqrt{\frac{s - 1.5 + e^{-2.5s}}{2}}. \tag{4.28}$$

Right half plane poles of the transfer function $P_0(s)$ are $p_{1,2} = 0.062 \pm j0.8444$. Transfer function $P_0(s)$ has no right half plane zeros. Thus nominal plant transfer function P_0 can be factorized as

$$P_0 = \frac{m_n N_0}{m_d},$$

where

$$m_d(s) = \frac{s^2 - 0.124s + 0.7169}{s^2 + 0.124s + 0.7169} \tag{4.29}$$

$$m_n(s) = 1; \tag{4.30}$$

and

$$N_0 = \frac{m_d(s)P_0(s)}{m_n(s)} = m_d(s)P_0(s). \tag{4.31}$$

Magnitude Bode plot of $N_0(s)$ is the same as the magnitude Bode plot of the nominal plant function $P_0(s)$ since $|N_0(j\omega)| = |P_0(j\omega)|$ for every ω. Weights are selected as

$$W_1 = \frac{0.001s + 1}{s + 0.001}, \tag{4.32}$$

$$W_2 = (s + 0.1)^2 \tag{4.33}$$

Degree of the weight W_2 is 2 because magnitude Bode plot is decreasing approximately with $-40 = 2 \times -20$ dB/dec at the end of the operation range. Operation range for the system is selected as 100 radians/sec. Magnitude Bode plots of P_0 and the weights are given in Figure 4.2.

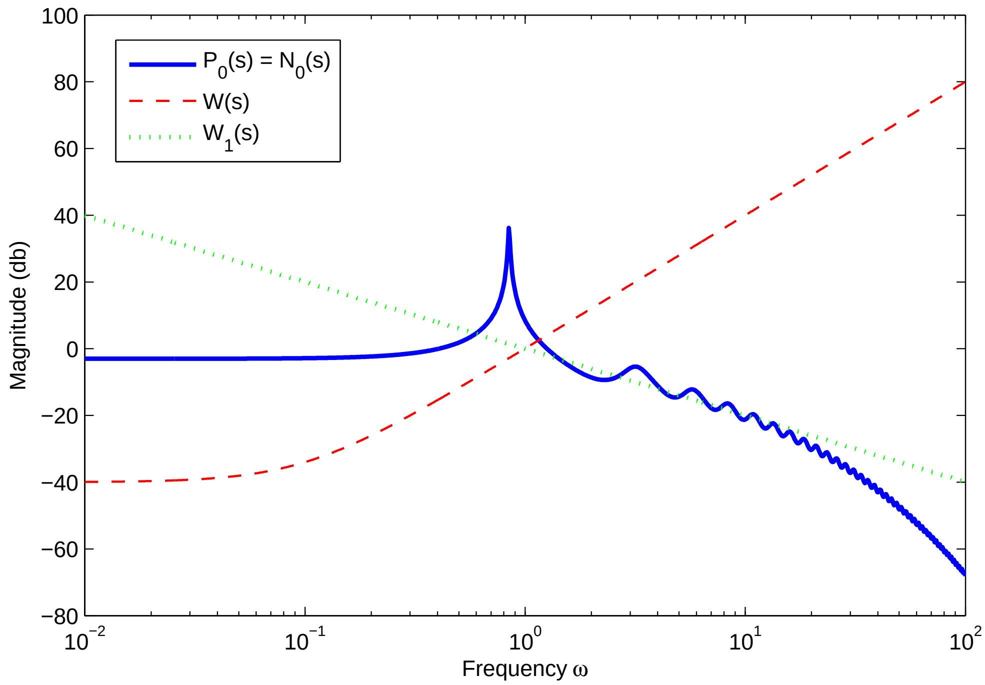

Figure 4.2: Magnitude Bode plots of the nominal plant P_0 and weights

Consider that $\gamma \in [0.01, 10]$. Calculate M matrix for each γ and find the minimum singular value for each case. Find the largest γ value for which the minimum singular value is zero. This value is obtained as $\gamma_{opt} = \gamma = 1.739288854$. This case is plotted in Figure 4.3.

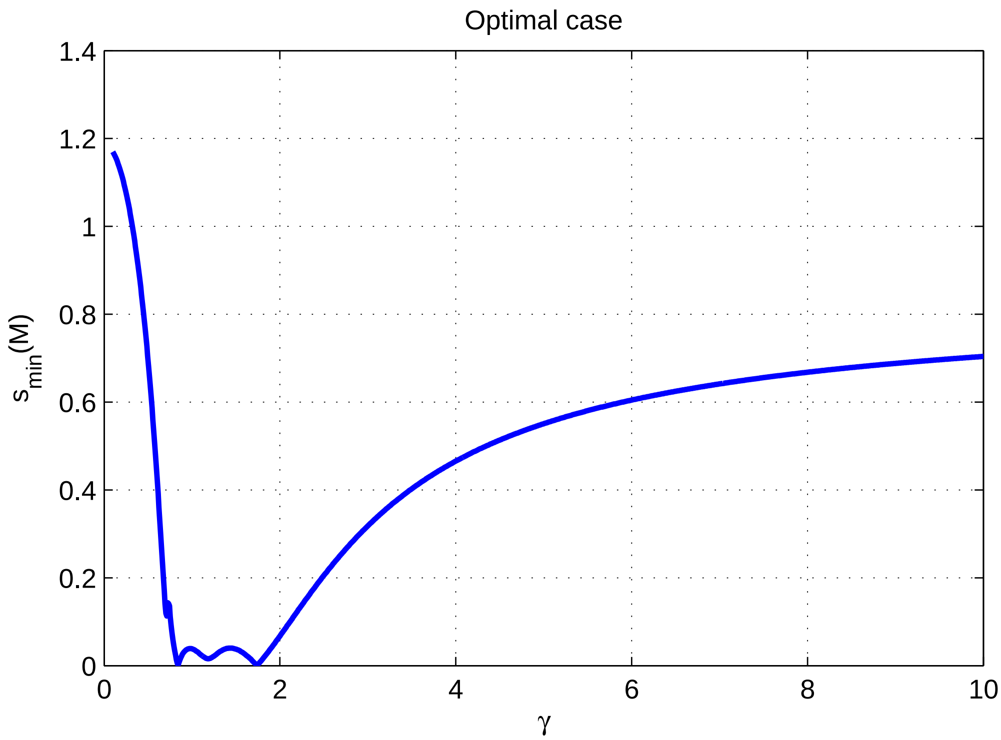

Figure 4.3: Singular value of the M matrix with respect to γ

With this computation of γ_{opt}, the optimal controller is obtained as follows:

$$C_{\text{opt}}(s) = C(s)N_0(s)^{-1} = C_1(s)\Big(\frac{1}{s^2+0.124s+0.7169}P_0(s)\Big), \tag{4.34}$$

where

$$C(s) = 1.739\ \frac{(s+0.027+0.6979j)(s+0.027-0.6979j)}{(s+0.001)(s+2.0631)(s+0.062+0.844j)(s+0.062-0.844j)},$$

$$C_1(s) = 1.739\ \frac{(s+0.027+0.6979j)(s+0.027-0.6979j)}{(s+0.001)(s+2.0631)}.$$

In (4.34), the term $\Big(\frac{1}{s^2+0.124s+0.7169}P_0(s)\Big)$ cancels all left half plane poles and zeros of P_0.

Define

$$T = P_0C(1+P_0C)^{-1}.$$

The graph of $\psi(\omega) = \sqrt{(W_1(j\omega)S(j\omega))^2 + (W_2(j\omega)T(j\omega))}$ with respect to frequency is given in Figure 4.4. It is a constant function at the value $\gamma_{\text{opt}} = 1.7393$

as expected. Magnitude and angle Bode plots for the controller is given in Figure 4.5.

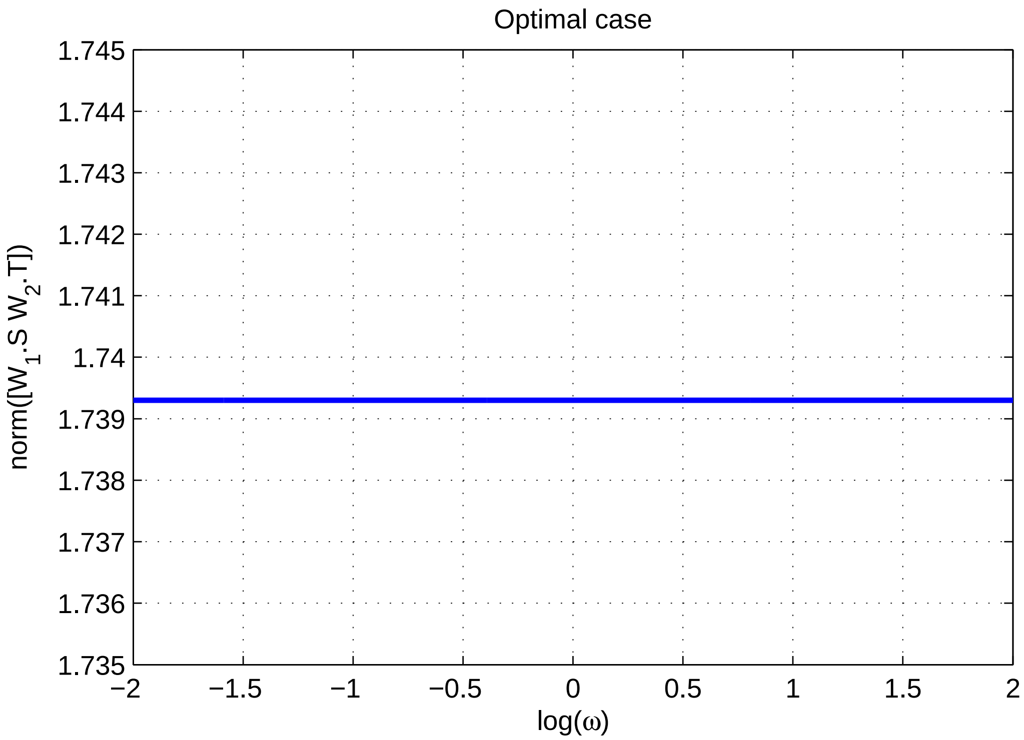

Figure 4.4: $\psi(\omega)$ vs. ω

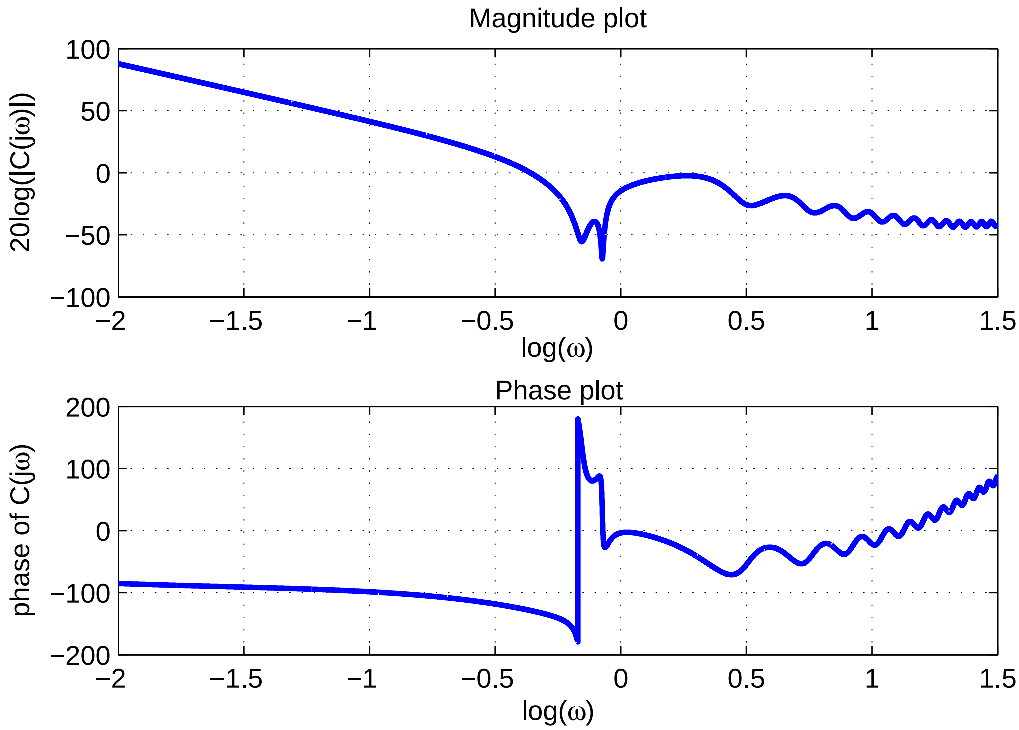

Figure 4.5: Bode plots for the controller C_{opt}

Chapter 5

Simulations

In this chapter, the heat equation with time delayed feedback will be solved using numerical methods. These solutions will be compared with our theoretical results. Presently, there is no commercial software package that can handle heat equations with time delayed feedback. Finite difference method can be used for solving heat equations with no time delay [18]. In this chapter, a numerical discretization algorithm is developed by using the finite difference method. This algorithm will be first explained and then used to simulate the system. Reduced order modeling of the two dimensional heat equation can also be achieved using proper orthogonal decomposition [9].

5.1 Solution Algorithm

Before stating the numerical algorithm, let us recall the one dimensional heat equation with time delayed feedback

$$\frac{\partial}{\partial t}z(x,t) = a\frac{\partial^2}{\partial x^2}z(x,t) + a_0 z(x,t) - a_1 z(x,t-\tau) \tag{5.1}$$

with initial conditions

$$z(x,\theta) = 0 \qquad \forall \;\; x \in (0\;,\;\pi) \;\text{ and }\; \theta \in [t_0-\tau\;,\;t_0]. \tag{5.2}$$

Here $x \in [0, \pi]$ is the spatial position parameter. In this chapter, it is assumed that a step input is applied from $x = 0$ and no input is applied from the other end point $x = \pi$. In other words, we have the following:

$$\begin{aligned} z(0,t) &= u(t) \ \ \forall t > t_0 - \tau \ \ \text{with} \ \ u_1(t) = 0, \ \ \text{for} \ \ t < t_0, \qquad (5.3) \\ z(\pi, t) &= 0 \ \ \forall t. \qquad (5.4) \end{aligned}$$

Assume that $t_0 = 0$. Use the finite difference approximation given below for the second order partial derivative:

$$\frac{\partial^2}{\partial x^2} z(x,t) \approx \frac{z(x+\Delta x, t) - 2z(x,t) + z(x - \Delta x, t)}{(\Delta x)^2}. \qquad (5.5)$$

Use the forward finite difference approximation given below for the first order partial derivative:

$$\frac{\partial}{\partial t} z(x,t) \approx \frac{z(x, t+\Delta t) - z(x,t)}{\Delta t}. \qquad (5.6)$$

Choose $\Delta x = \frac{\pi}{20}$. For convergence select $\Delta t = 0.01 < (\Delta x)^2/2$. Let

$$x_m = m\Delta x \qquad 0 \leq m \leq 20 \qquad (5.7)$$

and

$$t_n = n\Delta t \qquad n \geq 0. \qquad (5.8)$$

Define

$$z(x_m, t_n) = z_{(m,n)}. \qquad (5.9)$$

Use approximation (5.5), (5.6) and definition (5.9) in (5.1). Define

$$\Gamma = a_0(\Delta x)^2 + \frac{(\Delta x)^2}{\Delta t} - 2a. \qquad (5.10)$$

After algebraic manipulations, following equation is obtained:

$$z_{(m,n+1)} = \frac{\Delta t}{(\Delta x)^2}\left(a z_{(m+1,n)} + \Gamma z_{(m,n)} + a z_{(m-1,n)} - a_1 (\Delta x)^2 z_{(m,n-\tau/\Delta t)} \right) \qquad (5.11)$$

for $0 < m < 20$. Since initial conditions are zero, (5.11) reduces to following equation for the time interval $t \in [0, \tau)$:

$$z_{(m,n+1)} = \frac{\Delta t}{(\Delta x)^2}\left(a z_{(m+1,n)} + \Gamma z_{(m,n)} + a z_{(m-1,n)} \right) \qquad (5.12)$$

for $0 < m < 20$.

We can state our algorithm as follows

Set $z_{(m,0)} = 0$ for $0 \leq m \leq 20$.

Set $n = 0$

While $n < (\tau/0.001) - 1$

Set $z_{(0,n+1)} = 1$

Set $z_{(20,n+1)} = 0$

Find $z_{(m,n+1)}$ using (5.12)

Increase n by 1

End of the loop

Set $n = (\tau/0.001)$

While $n < n_{final}$

Set $z_{(0,n+1)} = 1$

Set $z_{(20,n+1)} = 0$

Find $z_{(m,n+1)}$ using (5.11)

Increase n by 1

End of the loop

End of the algorithm

where n_{final} is the n value which corresponds to the time instant at which the simulation ends.

5.2 Results

Consider the one dimensional heat equation with time delayed feedback with parameters $a = 1$, $a_0 = 1.9$, $a = 1$. Step input is applied from point $x = 0$ and no input is applied from point $x = \pi$. Using our method in frequency domain, it is found that system is stable for $\tau < 1.0347$. For $\tau = 1$, after running our algorithm, the result shown in Figure 5.1 is obtained for $z(\pi/2, t)$.

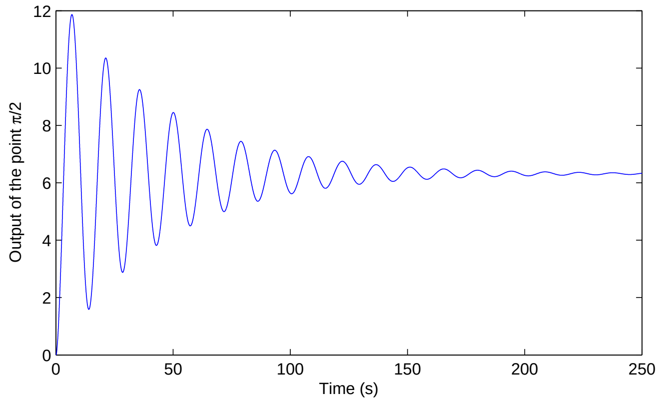

Figure 5.1: Output to the step input for $\tau = 1$

Now consider the case $\tau = 2$. After running our algorithm and plotting the $x = \pi/2$ point, following result is obtained.

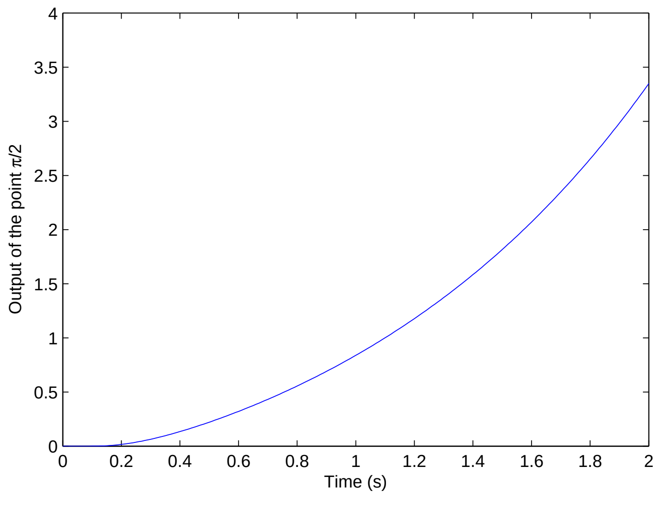

Figure 5.2: Output to the step input for $\tau = 2$

From these plots, it can be stated that our theoretical results are validated by simulation results.

Chapter 6

Conclusions

In this thesis, the one dimensional heat equation with linear time-delayed feedback is investigated. Using an analysis in the frequency domain, necessary and sufficient conditions for the stability of the system are found. Robust stability conditions for the system are obtained in the presence of uncertainty in the system parameters. H_∞ controller for the system is also obtained.

In Chapter 2, we reviewed the results of [13], where LMI conditions are derived using Lyapunov-Krasovksii functionals. After that, we fix the time delay and analyze the same system in the frequency domain using the Laplace transform. Stability conditions are obtained from the Nyquist criterion. These conditions are used to impose an analytical upper bound for the time-delay of the system whenever other parameters of the system are fixed. Then we compared our results with the result of [13] and show that for fixed time-delay case, the results of [13] can be improved.

In Chapter 3, we added parametric uncertainty to the parameters of the system. After that, parameter space which guarantees the robust stability of the uncertain system is found using zero-exclusion principle. Results are illustrated with examples. In Chapter 4, we designed an H_∞ controller for the system.

We considered that two inputs are applied from the end points and input is measured from the middle of the one dimensional rod. Finally, in Chapter 5, method of finite differences are used for designing a computational algorithm which simulates the heat equation with linear time-delayed feedback. Using the computational algorithm, results that are obtained in Chapter 2 are verified.

In the literature, time domain techniques are preferred for obtaining stability conditions for the heat equation with time-delayed feedback. Our thesis shows that frequency domain is an alternative for the time domain analysis. Time-domain analysis techniques give LMI conditions. There may be computational problems when solving these LMI conditions and these problems may result in some conservatism. By using frequency domain analysis, we obtained analytical bounds for system parameters which are exact and easy to calculate. Frequency domain analysis has also other advantages over time domain analysis. These advantages can be exploited for finding the robust stability conditions of the system with parametric uncertainty. We have also shown that it is also possible to design H_∞ controllers using the system dynamics in frequency domain.

APPENDIX A

The Matlab Codes

A.1 Implementation of the Numerical Discretization Algorithm

simulation.m

```
clc;
clear;
%Setting the parameters for time delayed heat equation
fprintf('Enter parameters:\n');
a = input('Enter a : ');
a_0 = input('Enter a_0 : ');
a_1 = input('Enter a_1 : ');
tau = input('Enter tau : ');
u1 = input('Enter step size for input applied at point x = 0 : ');
fprintf('\nSimulating the system...\n')
fprintf('d/dx z(x,t) = %2.3f d^2/dx^2 z(x,t) + %2.3f z(x,t) -
%2.3f z(x,t-%2.3f)\n',a,a_0,a_1,tau);
result = [];
```

```
%Create the initial array
deltat = 0.01;
deltax = pi/20;
previous = [u1 zeros(1,20)];
result = [result;previous];
middle = [0];
for n = [0:tau/0.01]
dummy = zeros(1,21);
    dummy(1) = u1;
    for m = [2:20]
    dummy(m) = previous(m)+a*(deltat/(deltax)^2)*(previous(m+1)
    -2*previous(m)+previous(m-1)) + deltat*a_0*previous(m);
        if(dummy(m)<0)dummy(m)=0;end
    end
    previous = dummy;
    middle = [middle dummy(11)];
    result = [result;dummy];
end
for n = [tau/0.01+1:250*tau/0.01]
    dummy = zeros(1,21);
    dummy(1) = u1;
    for m = [2:20]
    dummy(m) = previous(m)+a*(deltat/(deltax)^2)*(previous(m+1)
    -2*previous(m)+previous(m-1))
    + deltat*(a_0*previous(m)-a_1*result(n-tau/0.01,m));
        if(dummy(m)<0)dummy(m)=0;end
    end
    previous = dummy;
    middle = [middle dummy(11)];
    result = [result;dummy];
end
```

```
t = [0:0.01:250*tau+0.01];
plot(t,middle)
xlabel('Time (s)');
ylabel('Output of the point \pi/2');
```

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