PRICING AND HEDGING OF CONTINGENT CLAIMS IN INCOMPLETE MARKETS

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a dissertation for the degree of doctor of philosophy.

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ABSTRACT

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In this thesis, we analyze the problem of pricing and hedging contingent claims in the multi-period, discrete time, discrete state case. We work on both European and American type contingent claims.

For European contingent claims, we analyze the problem using the concept of a "λ gain-loss ratio opportunity". Pricing results which are somewhat different from, but reminiscent of, the arbitrage pricing theorems of mathematical finance are obtained. Our analysis provides tighter price bounds on the contingent claim in an incomplete market, which may converge to a unique price for a specific value of a gain-loss preference parameter imposed by the market while the hedging policies may be different for different sides of the same trade. The results are obtained in the simpler framework of stochastic linear programming in a multi-period setting. They also extend to markets with transaction costs.

Until now, determining the buyer's price for American contingent claims (ACC) required solving an integer program unlike European contingent claims for which solving a linear program is sufficient. We show that a relaxation of the integer programming problem which is a linear program, can be used to get the buyer's price for an ACC. We also study the problem of computing the lower hedging price of an American contingent claim in a market where proportional transaction costs exist. We derive a new mixed-integer linear programming formulation for calculating the lower hedging price. We also present and discuss an alternative, aggregate formulation with similar properties. Our results imply that it might be optimal for the holder of several identical American claims to exercise portions of the portfolio at different time points in the presence of proportional transaction costs while this incentive disappears in their absence.

We also exhibit some counterexamples for some new ideas based on our work.

We believe that these counterexamples are important in determining the direction of research on the subject.

Keywords: Contingent Claim, Option Pricing, Hedging, Arbitrage, Transaction Cost, Stochastic Linear Programming, Mixed Integer Programming.

ÖZET

KOŞULLU YÜKÜMLÜLÜKLERİN EKSİK PİYASALARDA FİYATLANDIRILMASI

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Bu tez çalışmasında koşullu yükümlülükler için çok periyotlu, ayrık zamanlı ve ayrık durumlu modellerde korunma ve fiyatlandırma problemlerini incelenmiştir. Hem Avrupa hem de Amerikan tipi koşullu yükümlülükler üzerinde çalışmalar yapılmıştır.

Avrupa tipi koşullu yükümlülükler için problem "λ kazanç-kayıp oranı fırsatı" kavramı kullanılarak analiz edilmiştir. Arbitraj kavramı kullanılarak yapılan çalışmalarda elde edilen sonuçları anımsatan ama bu sonuçlardan farklı fiyatlandırma sonuçları türetilmiştir. Yapılan çalışmalar sonucunda eksik piyasalarda Avrupa tipi yükümlülükler için arbitraj fiyatlamasına göre daha dar fiyat sınırları elde edilmiştir. Kazanç-kayıp önceliği parametresinin özel bir değeri için bu fiyat sınırlarının, alıcı ve satıcının korunma politikaları birbirinden farklı olsa bile, tek bir fiyata yakınsayabileceği gösterilmiştir. Sonuçlar stokastik doğrusal programlama yaklaşımıyla çok periyotlu modellerde elde edilmiştir. Bunların yanında, benzer sonuçlar işlem maliyetlerini hesaba katılarak da elde edilmiştir.

Daha önce yapılan çalışmalar sonucunda, Avrupa tipi bir koşullu sözleşmenin alıcı fiyatını elde etmek için bir doğrusal eniyileme probleminin çözülmesi yeterliydi. Bunun aksine, Amerikan tipi bir koşullu sözleşmenin alıcı fiyatını elde etmek için ise bir karışık tamsayı eniyileme problemi çözülmesi gerekiyordu. Çalışmamızda bu karışık tamsayı eniyileme probleminin doğrusal eniyileme problemi olan bir gevşetmesinin Amerikan tipi koşullu sözleşmenin alıcı fiyatını belirlemek için kullanılabileceği gösterilmiştir. Amerikan tipi koşullu sözleşmelerin alt korunma fiyatı problemi için ayrıca orantısal işlem maliyetlerinin yer aldığı bir piyasada çalışmalar yapılmıştır. Alt korunma fiyatını elde etmek için bir karışık tamsayı doğrusal programlama modeli türetilmiştir. Bu modele alternatif olarak, benzer özellikler gösteren ama daha bütünsel bir model geliştirilmiştir. Sonuçlar, piyasada orantısal işlem maliyetleri bulunması durumunda, birden fazla özdeş Amerikan tipi koşullu sözleşmeye sahip olan yatırımcının, sahip olduğu bu koşullu sözleşmelerin bazılarını farklı zamanlarda uygulayacağını göstermektedir.

Bu tezde ayrıca, çalışmaların devamı olabilecek bazı konularda karşıt örnekler sunularak gelecekte yapılacak çalışmalar için yön belirlenmesine çalışılmıştır.

Anahtar sözcükler: Koşullu Yükümlülük, Opsiyon Fiyatlandırma, Korunma, Arbitraj, İşlem Maliyeti, Stokastik Doğrusal Eniyileme, Karışık Tamsayı Programlama.

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Chapter 1

Introduction

A *derivative security* is a financial instrument whose payoff depends on the value of some underlying instrument. This underlying instrument can be a traded asset such as a stock or currency; or a measurable variable such as the temperature of a certain location. Derivative securities are also categorized according to the conditions of the agreement between the seller and the buyer of the derivative security. A *futures contract*, which is a derivative security, is a contract between two parties, where one of the parties agrees to buy (sell) the underlying instrument from (to) the other side in a future date, with a price which is fixed at the agreement date. This future date is called the *maturity date* and the price is called the *delivery (exercise) price* of the contract. An option which is also a type of a derivative security, differs from a futures contract in the sense that the holder (buyer) of the option is not obliged to fulfill the conditions of the contract. In other words, the holder of the option does not necessarily buy (sell) the underlying security from (to) the seller of the option. Besides, the holder of the option can buy or sell the underlying security (i.e. exercise the option) at or before the maturity date of the option. An option which can only be exercised at the maturity date is called a *European option*, while an option which can be exercised before or at the maturity date is called an *American option*. A *call (put) option* gives the holder the right to buy (sell) the underlying instrument. If the price of the underlying instrument is greater than the strike price at the exercise date, the buyer of a call option can buy the underlying security at the strike price and sell it from its prevailing price in the market resulting with an instant profit. We call this profit as the payoff of the claim for the buyer.

In this thesis, we work on options for which the strike price is not defined. We call such options as contingent claims. The payoff for the holder of a contingent claim can be defined in any sort of correspondence with the value of the underlying instrument at the time of the agreement. Hence, contingent claims are a more generalized version of options. Under this general setting many different types of options can be modelled as special cases of our definition of a contingent claim. European call and put options can be presented by a European contingent claim when we set the payoff of the claim according to its strike price and the price of the underlying security at the maturity date and by setting its payoff to zero for the dates other than the maturity date. American call and put options can be presented by an American contingent claim by setting the payoff of the claim according to its strike price and the price of the underlying security for all dates before its maturity. A Bermudan option is a type of American option for which the holder can exercise the option at one of the specified dates until its maturity. By setting an American contingent claim's payoff to zero for the dates that the Bermudan option could not be exercised and setting its payoffs suitably elsewhere we can obtain a Bermudan option. Some of the options have their payoffs calculated not only using the price of the underlying security at the exercise date but according to the path followed by the price of the underlying security until maturity. Such options are called *path dependent options*. Russian and lookback options are examples of path dependent options. We can also obtain path dependent options by setting the payoff of American contingent claims appropriately.

Options have not been traded in the markets in a significant way until 1973, when Chicago Board of Exchange (CBOE) started trading options. Since then, options started to play a very important role in financial markets. This rise has also showed its reflection in the theory of finance. Most of the literature on derivative securities is based on the question of determining the price of an option. Black and Scholes [7] have given the first widely accepted answer to this question. Their work is based on a no-arbitrage framework. Arbitrage is defined as the profit of an investor without taking any risk. In other words, if a portfolio strategy, which does not require an initial wealth and for which there is no intermediary exogenous infusion (which is self-financing), has no probability of loss but has a positive probability of profit in the end, it is said to create an arbitrage opportunity for the investor following it. The idea is once such an opportunity exists in the market every investor would try to make profit out of that. Hence, the price of the portfolio would increase until it would provide no arbitrage opportunity for the investors. Black and Scholes [7] works on a simple model including a bond, a European option and an underlying stock. They work on a continuous time framework where they assume the stock price process to follow a geometric Brownian motion. They derive the price of the option by determining the price of the portfolio which hedges the option to be priced. Their results were generalized in Merton [45]. These two pioneering works have many extensions in the literature. Leland [42] worked under the setting where transaction costs exist. Broadie *et al.* [10] worked on the model with some portfolio constraints. All these works are done in a complete market setting (a market in which every option can be replicated) where the price for the option is unique. However, the markets are almost never complete due to market imperfections as discussed in Carr *et al.* [14]. When the markets are incomplete not every option can be replicated, hence it is not possible to obtain the price of an option by a replicating portfolio. El Karoui and Quenez [24] developed a different idea for this problem. They considered the replication problem from buyer's and seller's sides separately. The seller's problem involved constructing a portfolio strategy which requires a minimum initial wealth and for which the portfolio has a value at least as large as the payoff of the option for any possible outcome of the stock price process at the maturity date. This problem is called the *super-replication problem* and its optimal value is called the *seller's price.* Conversely, the buyer's problem involved constructing a maximum initial value portfolio strategy which is dominated by the payoff of the claim at the maturity date. The *buyer's price* is the optimal value of buyer's problem. They obtain an interval instead of a unique value for the price of the option. However, this interval might be very large in practice and determining the exact price of the option is still a problem. In order to overcome this problem another pricing approach which has its roots from both arbitrage pricing theory and expected utility theory has been developed.

Expected utility theory assumes that preferences of investors can be represented by expected utility functions which satisfy a set of axioms. The pricing approach is based on equating the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor; see e.g. [16, 30, 36] for related recent work. The combination of expected utility theory with arbitrage pricing resulted with several definitions of performance criteria for a portfolio strategy. Opportunities satisfying these performance criteria are called as *good-deals* or *acceptable investments* in the literature. Cochrane and Saa-Requejo [18] defines a good-deal as a portfolio strategy having a high Sharpe ratio and derives the price bounds for an option in a market which does not allow any such good-deal. Carr *et al.* [14], Roorda *et al.* [55], and Kallsen [36] work under different definitions of good-deals in order to price an option in an incomplete market. Bernardo and Ledoit [5] defines a good-deal as a portfolio strategy having a high gain-loss ratio. In Chapter 2, we study the pricing problem under their framework.

The literature on American option pricing has the same roots as the European option pricing literature [45]. The owner of the American option has the right to exercise the option at any time until maturity. Hence, the pricing problem consists the optimal exercise strategy problem. The first expectation representation for the price of an American option was shown in Harrison and Kreps [28]. There is a vast literature building upon their work, e.g. [8], [17]. Pennanen and King [48] worked on pricing American options in incomplete markets. Their results imply that relaxing the feasible exercise set for the buyer of the option does not make any impact on the pricing interval of the option. We build on their results by correcting one of their proofs in Chapter 3. In Chapter 4 we revisit the problem of pricing American contingent claims while incorporating transaction costs in the model.

In the second chapter of this thesis we work on the problem of pricing European contingent claims under the condition of no λ gain-loss ratio opportunity exists in the market. The λ gain-loss ratio opportunity criterion is a performance measure for a portfolio strategy and it is based on the gain-loss criterion defined by Bernardo and Ledoit [5]. Under this setting we derive conditions for which there is no λ gain-loss ratio opportunity in the market. Bernardo and Ledoit [5] derive same conditions in a one period model consisting of a bond and a stock. They work on both finite and infinite state models. In our setting the market may consist of several stocks in addition to a bond. Our model is a discrete time, finite state model with finite number of periods. We also make studies on the limiting values of the parameter λ which could be helpful in understanding the function of the parameter. Then we work on the pricing problem for European contingent claims both from buyer's and writer's sides to derive martingale expressions representing the pricing interval for the claim. We extend our results to markets with transaction costs. We have published the findings of this chapter in [50].

In the third chapter of this thesis we work on the American contingent claim pricing problem. We work on the same setting as Pennanen and King [48]. We give a correct proof of a theorem which was proposed in [48]. The implication of this theorem is that we need to solve a linear programming problem instead of a mixed-integer programming problem in order to find the buyer's price of an American contingent claim. We obtain pricing results in the form of martingales. We show that our results remain valid under the existence of dividends. We have published the findings of Chapter 3 in [13].

In Chapter 4 we work on the American contingent claim pricing problem in a market in which transaction costs exist. We derive integer programming problems in order to determine the lower bound for the price of an American contingent claim. We show by a counterexample that linear relaxation problems of the derived integer programming models cannot be used to determine the buyer's price of the contingent claim. We also prove the result of Chapter 3 again using the models derived in this chapter. We believe this part of the thesis reveals that the research on American contingent claim pricing in a discrete time, finite state model has to involve the deeper study of integer programming models. We have published the findings of Chapter 4 in [51].

Finally, we outline our contributions, exhibit some counterexamples and future research directions in Chapter 5.

Chapter 2

Expected Gain-Loss Pricing and Hedging of European Contingent Claims by Linear Programming

An important class of pricing theories in financial economics are derived under no-arbitrage conditions. In complete markets, these theories yield unique prices without any assumptions about individual investor's preferences. In other words, the pricing of assets relies on the availability and the liquidity of traded assets that span the full set of possible future states. Ross [56, 57] proves that the no-arbitrage condition is equivalent to the existence of a linear pricing rule and positive state prices that correctly value all assets. This linear pricing rule is the risk neutral probability measure in the Cox-Ross option pricing model. For example Harrison and Kreps [28] showed that the linear pricing operator is an expectation taken with respect to a martingale measure. However, when markets are incomplete state prices and claim prices are not unique. Since markets are almost never complete due to market imperfections as discussed in Carr *et al.* [14], and characterizing all possible future states of economy is impossible, alternative incomplete pricing theories have been developed.

In an incomplete financial market with no arbitrage opportunities, a noticeable feature of the set of risk neutral measures is that the value of the cheapest portfolio to dominate the pay-off at maturity of a European contingent claim (ECC) coincides with the maximum expected value of the (discounted) pay-off of the claim with respect to this set. This value, which may be called the writer's price, allows the writer to assemble a hedge portfolio that achieves a value at least as large as the pay-off to the claim holder at the maturity date of the claim in all non-negligible events. The writer's price is the natural price to be asked by the writer (seller) of a European contingent claim and, together with the bid price obtained by considering the analogous problem from the point of view of the buyer, forms an interval which is sometimes called the "no-arbitrage price interval" for the claim in question.

A writer may nevertheless be induced for various reasons to settle for less than the above price to sell a claim with pay-off F_T; see e.g., chapters 7 and 8 of [26] for a discussion and examples showing that the writer's price may be too high. In such a case, he/she will not be able to set up a portfolio dominating the claim pay-off almost surely, which implies that he/she will face a positive probability of "falling short", i.e., his/her hedge portfolio will take values V_T smaller than those of the claim on a non-negligible event. Thus, the writer will need to choose his/her hedge portfolio (and selling price) according to some optimality criterion to be decided. The gain-loss pricing criterion of our study inspired by the gain-loss ratio criterion of Bernardo and Ledoit [5] suggests to choose the portfolio which gives the best value of the difference of expected positive final positions and a parameter λ (greater than one) times the expected negative final positions, $\mathbb{E}[(V_T - F_T)_+] - \lambda\mathbb{E}[(V_T - F_T)_-]$, aimed at weighting "losses" more than "gains". This criterion gives rise to a new concept different from the ordinary arbitrage, the "λ *gain-loss ratio opportunity*", i.e., a portfolio which can be set up at no cost but yields a positive value for the difference between gains and "λ-losses". In this chapter, we show that the price processes in a multiple period, discrete time, finite state financial market do not admit a λ gain-loss ratio opportunity if and only if there exists an equivalent martingale measure with an additional restriction. As for the maximum and minimum no-arbitrage prices, we determine the maximum and minimum prices which do not introduce λ gain-loss opportunities in the market. Thus, a new price interval (the "λ gain-loss price interval") is determined, generally contained in the no-arbitrage interval (thus more significant from an economical point of view since it is more restrictive). These prices converge to the no-arbitrage bounds in the limit as the gain-loss preference parameter goes to infinity (and hence, the investor essentially looks for an arbitrage). On the other extreme, our results show that the market may actually arrive at a consensus about the pricing rule, i.e., as the gain-loss preference parameter goes down to the smallest value not allowing a λ gain-loss ratio opportunity, the writer and buyer's no-λ gain-loss ratio opportunity prices of a European contingent claim may converge to a single value, hence potentially providing a unique price for the contingent claim in an incomplete market. However, in the incomplete market setting, the same pricing rule leads to different hedging policies for different sides of the same trade. This is an important finding as it will result in different demand and supply schemes for the replicating assets. An attractive feature of our results is that all derivations and computations are carried out using linear programming models derived from simple stochastic programming formulations, which offer a propitious framework for adding additional variables and constraints into the models as well as the possibility of efficient numerical processing; see the book [6] for a thorough introduction to stochastic programming.

Our concept of λ gain-loss ratio opportunity is akin to the notion of a good-deal that was developed in a series of papers by various authors [15, 18, 34, 61]. For example in Cochrane and Saa-Requejo [18], the absence of arbitrage is replaced by the concept of a good deal, defined as an investment with a high Sharpe ratio. While they do not use the term "good-deal", Bernardo and Ledoit [5] replace the high Sharpe ratio by the gain-loss ratio. These earlier studies are carried out using duality theory in infinite dimensional spaces in [15, 34, 61], usually in single period models. Working with single period models is not necessarily a limitation since dynamic models with a fixed terminal date can be viewed as one-period models with investment choices taking values in suitable spaces. Recent work on risk measures and portfolio optimization, e.g. [26], adopts this approach to formulate single period problems using function spaces rich enough to be extended to multiperiod or continuous time markets; see section 8 of Staum [61] for a discussion. In this regard, the contribution of this chapter is to make explicit which consequences can general single period results have when applied to multiperiod discrete space markets.

We note that a second class of pricing theories relies on the Expected Utility framework which posits that if preferences satisfy a number of axioms, then they can be represented by an expected utility function. This framework requires the specification of investor preferences through usually non-linear utility functions; see Chapter 1 of [31]. This model equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the representative investor; see e.g. [16, 30, 36] for related recent work. Recent papers by Cochrane and Saa-Requejo [18], Bernardo and Ledoit [5], Carr *et al.* [14] and Roorda *et al.* [55] and Kallsen [36] unify these two classes of pricing theories and value options in an incomplete market setting. In this chapter, we work with linear programming models, and avoid the non-linearities encountered with utility functions. Our notion of gain-loss ratio opportunity is also related to prospect theory of Kahneman and Tversky [35] proposed as an alternative to expected utility framework. In prospect theory, it is presumed based on experimental evidence that gains and losses have asymmetric effects on the agents' welfare where welfare, or utility, is defined not over total wealth but over gains and losses; see Grüne and Semmler [27] and Barberis *et al.* [1] for details on the use of the gain-loss function as a central part of welfare functions in asset pricing.

The organization of this chapter is as follows. In section 2.1 we review the stochastic process governing the asset prices and we lay out the basics of our analysis. Section 2.2 gives a characterization of the absence of a λ gain-loss ratio opportunity in terms of martingale measures. We consider a related problem in section 2.3 where the investor in search of a λ gain-loss ratio opportunity would also like to find the λ gain-loss ratio opportunity with the limiting value of the parameter λ. Here we re-obtain a duality result which turns out be essentially the duality result of Bernardo and Ledoit in a multi-period but finite probability state space setting. In section 2.4 we analyze the pricing problems of writers and buyers of European contingent claims under the λ gain-loss ratio opportunity viewpoint. We extend our results to markets with transaction costs in section 2.5. We use simple numerical examples to illustrate our results.

2.1 The Stochastic Scenario Tree, Arbitrage and Martingales

Throughout our work we follow the general probabilistic setting of [40] where we model the behavior of the stock market by assuming that security prices and other payments are discrete random variables supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms ω are sequences of real-valued vectors (asset values) over the discrete time periods $t = 0, 1, \dots, T$. For a general reference on mathematical finance in discrete time, finite state markets the reader is referred to Pliska [52]. We assume the market evolves as a discrete, non-recombinant scenario tree (hence, suitable for incomplete markets) in which the partition of probability atoms $\omega \in \Omega$ generated by matching path histories up to time t corresponds one-to-one with nodes $n \in \mathcal{N}_t$ at level t in the tree. The set $\mathcal{N}_0$ consists of the root node $n = 0$, and the leaf nodes $n \in \mathcal{N}_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$. In the scenario tree, every node $n \in \mathcal{N}_t$ for $t = 1, \dots, T$ has a unique parent denoted $\pi(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, 1, \dots, T-1$ has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. The set of all ascendant nodes and all descendant nodes of a node n are denoted $\mathcal{A}(n)$, and $\mathcal{D}(n)$, respectively, in both cases including node n itself. We denote the set of all nodes in the tree by $\mathcal{N}$. The probability distribution P is obtained by attaching positive weights p_n to each leaf node $n \in \mathcal{N}_T$ so that $\sum_{n \in \mathcal{N}_T} p_n = 1$. For each non-terminal (intermediate level) node in the tree we have, recursively,

$$p_n = \sum_{m \in \mathcal{C}(n)} p_m, \quad \forall\, n \in \mathcal{N}_t,\ t = T-1, \dots, 0. \tag{2.1}$$

Hence, each intermediate node has a probability mass equal to the combined mass of the paths passing through it. The ratios $p_m/p_n, m \in \mathcal{C}(n)$ are the conditional probabilities that the child node m is visited given that the parent node $n = \pi(m)$

has been visited. This setting is chosen as it accommodates multi-period pricing for future different states and time periods at the same time, employing realization paths in the valuation process. It is a framework that allows to address the valuation problem with incomplete markets and heterogeneous beliefs which are very stringent assumptions in the classical valuation theory. In this respect, it improves our understanding of valuation in a simple, yet complete fashion.

A random variable X is a real valued function defined on Ω. It can be *lifted* to the nodes of a partition $\mathcal{N}_t$ of Ω if each level set $\{X^{-1}(a) : a \in \mathbb{R}\}$ is either the empty set or is a finite union of elements of the partition. In other words, X can be lifted to $\mathcal{N}_t$ if it can be assigned a value on each node of $\mathcal{N}_t$ that is consistent with its definition on Ω [40]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of $\mathcal{N}_t$. A stochastic process $\{X_t\}$ is a time-indexed collection of random variables such that each X_t is measurable with respect to $\mathcal{N}_t$. The expected value of X_t is uniquely defined by the sum

$$\mathbb{E}^P[X_t] := \sum_{n \in \mathcal{N}_t} p_n X_n.$$

The conditional expectation of X_{t+1} on $\mathcal{N}_t$ is a random variable taking values over the nodes $n \in \mathcal{N}_t$, given by the expression

$$\mathbb{E}^P[X_{t+1}|\mathcal{N}_t] := \sum_{m \in \mathcal{C}(n)} \frac{p_m}{p_n} X_m.$$

Under the light of the above definitions, the market consists of $J+1$ tradable securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \ldots, S_n^J)$. We assume as in [40] that the security indexed by 0 has strictly positive prices at each node of the scenario tree. Furthermore, the price of the security indexed by 0 grows by a given factor in each time period. This asset corresponds to the risk-free asset in the classical valuation framework. Choosing this security as the numéraire, and using the discount factors $\beta_n = 1/S_n^0$ we define $Z_n^j = \beta_n S_n^j$ for $j = 0, 1, \ldots, J$ and $n \in \mathcal{N}$, the security prices discounted with respect to the numéraire. Note that $Z_n^0 = 1$ for all nodes $n \in \mathcal{N}$, and β_n is a constant, equal to, β_t, for all $n \in \mathcal{N}_t$, for a fixed $t \in [0, \ldots, T]$.

The amount of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted

θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$. We refer to the collection of vectors θ_n for all $n \in \mathcal{N}$ as Θ. The value of the portfolio at state n (discounted with respect to the numéraire) is

$$Z_n \cdot \theta_n = \sum_{j=0}^{J} Z_n^j \theta_n^j.$$

We will work with the following definition of arbitrage: an arbitrage is a sequence of portfolio holdings that begins with a zero initial value (note that short sales are allowed), makes self-financing portfolio transactions throughout the planning horizon and achieves a non-negative terminal value in each state, while in at least one terminal state it achieves a positive value with non-zero probability. The self-financing transactions condition is expressed as

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{\pi(n)},\ n > 0.$$

The stochastic programming problem used to seek an arbitrage is the following optimization problem (P1):

$$\begin{aligned} \max \quad & \sum_{n \in \mathcal{N}_T} p_n Z_n \cdot \theta_n \\ \text{s.t.} \quad & Z_0 \cdot \theta_0 = 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n \geq 0,\ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

If there exists an optimal solution (i.e., a sequence of vectors θ_n for all $n \in \mathcal{N}$) which achieves a positive optimal value, this solution can be turned into an arbitrage as demonstrated by Harrison and Pliska [29].

We need the following definitions.

Definition 1. *If there exists a probability measure $Q = \{q_n\}_{n \in \mathcal{N}_T}$ (extended to intermediate nodes recursively as in (2.1)) such that*

$$Z_t = \mathbb{E}^Q[Z_{t+1}|\mathcal{N}_t]\ (t \leq T-1) \tag{2.2}$$

then the vector process $\{Z_t\}$ is called a vector-valued martingale under Q, and Q is called a martingale probability measure for the process. If one has coordinate-wise $Z_t \geq \mathbb{E}^Q[Z_{t+1}|\mathcal{N}_t], (t \leq T-1)$ (respectively, $Z_t \leq \mathbb{E}^Q[Z_{t+1}|\mathcal{N}_t], (t \leq T-1)$ the process is called a super-martingale (sub-martingale, respectively).

Definition 2. *A discrete probability measure $Q = \{q_n\}_{n\in\mathcal{N}_T}$ is equivalent to a (discrete) probability measure $P = \{p_n\}_{n\in\mathcal{N}_T}$ if $q_n > 0$ exactly when $p_n > 0$.*

King proved the following (c.f. Theorem 1 of [40]):

Theorem 1. *The discrete state stochastic vector process $\{Z_t\}$ is an-arbitrage free market price process if and only if there is at least one probability measure Q equivalent to P under which $\{Z_t\}$ is a martingale.*

The above result is the equivalent of Theorem 1 of Harrison and Kreps [28] in our setting.

2.2 Gain-Loss Ratio Opportunities and Martingales

In our context a λ gain-loss ratio opportunity is defined as follows. For $n \in \mathcal{N}_T$ let $Z_n \cdot \theta_n = x_n^+ - x_n^-$ where x_n^+ and x_n^- are non-negative numbers, i.e., we express the final portfolio value at terminal state n as the sum of positive and negative positions (x_n^+ denotes the gain at node n while x_n^- stands for the loss at node n). Assume that there exist vectors vectors θ_n for all $n \in \mathcal{N}$ such that

$$Z_0 \cdot \theta_0 = 0$$

$$Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0, \; \forall\, n \in \mathcal{N}_t, t \geq 1$$

and

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] > 0,$$

for $\lambda > 1$, where $X^+ = \{x_n^+\}_{n\in\mathcal{N}_T}$, and $X^- = \{x_n^-\}_{n\in\mathcal{N}_T}$. This sequence of portfolio holdings is said to yield a λ gain-loss ratio opportunity (for a fixed value of λ). This formulation is similar to Bernardo and Ledoit [5] gain-loss ratio, and the Sharpe ratio restriction of Cochrane and Saa-Requejo [18]. Yet, it makes the problem easier to tackle within the framework of linear programming. Moreover, the parameter λ can be interpreted as the gain-loss preference parameter of the individual investor. As λ gets bigger, the individual's aversion to loss is becoming more and more pronounced, since he/she begins to prefer near-arbitrage positions. As λ gets closer to 1, the individual weighs the gains and losses equally. In the limiting case of λ being equal to 1 the pricing operator (equivalent martingale measure) is unique if it exists. In fact, the pricing operator may become unique at a value of λ larger than one, which is what we expect in a typical pricing problem.

Consider now the perspective of an investor who is content with the existence of a λ gain-loss ratio opportunity although an arbitrage opportunity does not exist. Such an investor is interested in the solution of the following stochastic linear programming problem that we refer to as (SP1):

$$\begin{aligned} \max \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\ \text{s.t.} \quad & Z_0 \cdot \theta_0 = 0 \\ & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0, \; \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \; \forall\, n \in \mathcal{N}_T, \\ & x_n^+ \geq 0, \; \forall\, n \in \mathcal{N}_T, \\ & x_n^- \geq 0, \; \forall\, n \in \mathcal{N}_T. \end{aligned}$$

If there exists an optimal solution (i.e., a sequence of vectors θ_n for all $n \in \mathcal{N}$) to the above problem that yields a positive optimal value, the solution is said to give rise to a λ gain-loss ratio opportunity (the expected positive terminal wealth outweighing λ times the expected negative final wealth). If there exists a λ gain-loss ratio opportunity in SP1, then SP1 is unbounded. We note that by the fundamental theorem of linear programming, when it is solvable, SP1 has always a basic optimal solution in which no pair x_n^+, x_n^-, for all $n \in \mathcal{N}_T$, can be positive at the same time.

We will say that the discrete state stochastic vector process $\{Z_t\}$ does not admit a λ gain-loss ratio opportunity (at a fixed value of λ) if the optimal value of the above stochastic linear program is equal to zero. Clearly, if λ tends to infinity we essentially recover King's problem P1. It is a well-accepted phenomenon that every rational investor is ready to lose if the benefits of the gains outweigh the costs of the losses [35]. It is also reasonable to assume that the rational investor will try to limit losses. This type of behavior excluded by the no-arbitrage setting is easily modeled by the Expected Utility approach and in prospect theory. Our formulation allows investors to take reasonable risks without explicitly specifying a complicated utility function while it converges to the no-arbitrage setting in the limit. It is easy to see that an arbitrage opportunity is also a λ gain-loss ratio opportunity, and that absence of a λ gain-loss ratio opportunity (at any level λ) implies absence of arbitrage. It follows from Theorem 1 that if the market price process does not admit a λ gain-loss ratio opportunity then there exists an equivalent measure that makes the price process a martingale.

Definition 3. *Given $\lambda > 1$ a discrete probability measure $Q = \{q_n\}_{n\in\mathcal{N}_T}$ is λ-compatible to a (discrete) probability measure $P = \{p_n\}_{n\in\mathcal{N}_T}$ if it is equivalent to P (Definition 2) and satisfies*

$$\max_{n\in\mathcal{N}_T} p_n/q_n \leq \lambda \min_{n\in\mathcal{N}_T} p_n/q_n.$$

Theorem 2. *The process $\{Z_t\}$ does not admit λ gain-loss ratio opportunity (at a fixed level $\lambda > 1$) if and only if there exists a probability measure Q λ-compatible to P which makes the discrete vector price process $\{Z_t\}$ a martingale.*

Proof. We prove the necessity part first. We begin by forming the dual problem to SP1. Attaching unrestricted-in-sign dual multiplier y_0 with the first constraint, multipliers $y_n, (n > 0)$ with the self-financing transaction constraints, and finally multipliers $w_n, (n \in \mathcal{N}_T)$ with the last set of constraints we form the Lagrangian function:

$$\begin{aligned} L(\Theta, X^+, X^-, y, w) &= \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\ &\quad + y_0 Z_0 \cdot \theta_0 + \sum_{t=1}^{T} \sum_{n\in\mathcal{N}_t} y_n Z_n \cdot (\theta_n - \theta_{\pi(n)}) \\ &\quad + \sum_{n\in\mathcal{N}_T} w_n (Z_n \cdot \theta_n - x_n^+ + x_n^-) \end{aligned}$$

that we maximize over the variables Θ, X^+, and X^- separately. From these separate maximizations we obtain the following:

$$y_0 Z_0 = \sum_{n \in \mathcal{C}(0)} y_n Z_n \tag{2.3}$$

$$y_m Z_m = \sum_{n \in \mathcal{C}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \leq t \leq T-1, \tag{2.4}$$

$$p_n \leq y_n \leq \lambda p_n, \ \forall n \in \mathcal{N}_T, \tag{2.5}$$

where we got rid of the dual variables w_n in the process by observing that maximizations over $\theta_n, (n \in \mathcal{N}_T)$ yield the equations

$$(w_n - y_n) Z_n = 0, \forall n \in \mathcal{N}_T,$$

and since the first component $Z_n^0 = 1$ for all states n, we have $y_n = w_n, (n \in \mathcal{N}_T)$. Therefore, we have obtained the dual problem that we refer to SD1 with an identically zero objective function and the constraints given by (2.3)–(2.4)–(2.5).

Now let us observe that problem SP1 is always feasible (the zero portfolio in all states is feasible) and if there is no λ gain-loss ratio opportunity, the optimal value is equal to zero. Therefore, by linear programming duality, the dual problem is also solvable (in fact, feasible since the dual is only a feasibility problem). Let us take any feasible solution $y_n, (n \in \mathcal{N})$ of the dual system given by (2.3)–(2.4)–(2.5). Since the first component, Z_n^0 is equal to 1 in each state n, we have that

$$y_m = \sum_{n \in \mathcal{C}(m)} y_n, \ \forall\, m \in \mathcal{N}_t, 1 \leq t \leq T-1. \tag{2.6}$$

Since $y_n \geq p_n$, it follows that y_n is a strictly positive process such that the sum of y_n over all states $n \in \mathcal{N}_t$ in each time period t sums to y_0. Now, define the process $q_n = y_n / y_0$, for each $n \in \mathcal{N}$. Obviously, this defines a probability measure Q over the leaf (terminal) nodes $n \in \mathcal{N}_T$. Furthermore, we can rewrite (2.4) with the newly defined weights q_n as

$$q_m Z_m = \sum_{n \in \mathcal{C}(m)} q_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \leq t \leq T-1,$$

with $q_0 = 1$, and all $q_n > 0$. Therefore, by constructing the probability measure Q we have constructed an equivalent measure which makes the price process $\{Z_t\}$

a martingale according to Definition 1. By definition of the measure q_n, we have using the inequalities (2.5)

$$p_n \leq q_n y_0 \leq \lambda p_n, \forall n \in \mathcal{N}_T,$$

or equivalently,

$$p_n/q_n \leq y_0 \leq \lambda p_n/q_n, \forall n \in \mathcal{N}_T,$$

which implies that $q_n, n \in \mathcal{N}_T$ constitute a λ-compatible martingale measure. This concludes the necessity part.

Suppose Q is a λ-compatible martingale measure for the price process $\{Z_t\}$. Therefore, we have

$$q_m Z_m = \sum_{n \in \mathcal{C}(m)} q_n Z_n, \ \forall \ m \in \mathcal{N}_t, 1 \leq t \leq T-1,$$

with $q_0 = 1$, and all $q_n > 0$, while the condition $\max_{n\in\mathcal{N}_T} p_n/q_n \leq \lambda \min_{n\in\mathcal{N}_T} p_n/q_n$ holds. If the previous inequality holds as an equality, choose the right-hand (or, the left-hand) of the inequality as a factor y_0 and set $y_n = q_n y_0$ for all $n \in \Omega$. If the inequality is not tight, any value y_0 in the interval $[\max_{n\in\mathcal{N}_T} p_n/q_n, \lambda \min_{n\in\mathcal{N}_T} p_n/q_n]$ will do. It is easily verified that y_n, $n \in \mathcal{N}$ so defined satisfy the constraints of the dual problem SD1. Since the dual problem is feasible, the primal SP1 is bounded above (in fact, its optimal value is zero) and no λ gain-loss ratio opportunity exists in the system. □

As a first remark, we can immediately make a statement equivalent to Theorem 2: The price process (or the market) does not have a λ gain-loss ratio opportunity (at fixed level λ) if and only if there exists an equivalent measure Q to P such that:

$$\frac{\max_{n\in\mathcal{N}_T} p_n/q_n}{\min_{n\in\mathcal{N}_T} p_n/q_n} \leq \lambda \tag{2.7}$$

or, equivalently

$$\frac{\max_{n\in\mathcal{N}_T} q_n/p_n}{\min_{n\in\mathcal{N}_T} q_n/p_n} \leq \lambda \tag{2.8}$$

or,

$$\frac{\max_\omega \frac{dQ}{dP}(\omega)}{\min_\omega \frac{dQ}{dP}(\omega)} \leq \lambda \tag{2.9}$$

using the Radon-Nikodym derivative, and that Q makes the price process a martingale. Clearly, posing the condition as such introduces a nonlinear system of inequalities, whereas our equivalent dual problem SD1 is a linear programming problem. We observed that a similar observation for single period problems was made in a technical note [44] although the language and notation of this reference is very different from ours.

As a second remark, we note that if we allow λ to tend to infinity we find ourselves in King's framework at which point Theorem 1 is valid. Therefore, this theorem is obtained as a special case of Theorem 2.

Example 1. Let us now consider a simple single-period numerical example. Let us assume for simplicity that the market consists of a riskless asset with zero growth rate, and of a stock. The stock price evolves according a trinomial tree as follows. Assume the riskless asset has price equal to one throughout. At time $t = 0$, the stock price is 10. Hence $Z_0 = (1\ 10)^T$. At the time $t = 1$, the stock price can take the values 20, 15, 7.5 with equal probability. Therefore, at node 1 one has $Z_1 = (1\ 20)^T$; at node 2 $Z_2 = (1\ 15)^T$ and finally at node 3 $Z_3 = (1\ 7.5)^T$. In other words, all β factors are equal to one. It is easy to see that the market described above is arbitrage free because we can show the existence of an equivalent martingale measure, e.g., $q_1 = q_2 = 1/8$ and $q_3 = 3/4$. Now, setting up and solving the problems SP1 and/or SD1, we observe that for all values of $\lambda \geq 6$, no λ gain-loss ratio opportunity exists in the market. However, for values of λ strictly between one and six, the primal problem SP1 is unbounded and the dual problem SD1 is infeasible. Therefore, λ gain-loss ratio opportunities exist.

As λ gets smaller, eventually the feasible set of the dual problem reduces to a singleton, at which point an interesting pricing result is observed as we shall see in section 2.4. First, we investigate the problem of finding the smallest λ not allowing λ gain-loss ratio opportunities in the next section.

2.3 Seeking out The Highest Possible λ in a Gain-Loss Ratio Opportunity Framework

We have assumed thus far that the parameter λ was decided by the agent (writer or buyer) before the solution of the stochastic linear programs of the previous section. However, once a λ gain-loss ratio opportunity is found at a certain level of λ it is legitimate to ask whether λ gain-loss ratio opportunities at higher levels of λ continue to exist. In fact, it is natural to wonder how far up one can push λ before λ gain-loss ratio opportunities cease to exist. Therefore, it is relevant, while seeking λ gain-loss ratio opportunities, to consider the following optimization problem LamP1:

$$\begin{aligned}
\sup \quad & \lambda \\
\text{s.t.} \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- > 0 \\
& Z_0 \cdot \theta_0 = 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ \geq 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \geq 0,\ \forall\, n \in \mathcal{N}_T.
\end{aligned}$$

Notice that problem LamP1 is a non-convex optimization problem, and as such is potentially very hard. However, it can be posed in a form suitable for numerical processing as we claim by the next proposition.

Proposition 1. *LamP1 is equivalent to the following problem LamPr under the assumption that a λ gain-loss ratio opportunity exists for some $\lambda > 1$*

$$\begin{aligned}
\sup \quad & \frac{\sum_{n\in\mathcal{N}_T} p_n x_n^+}{\sum_{n\in\mathcal{N}_T} p_n x_n^-} \\
s.t. \quad & Z_0 \cdot \theta_0 = 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ \geq 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \geq 0,\ \forall\, n \in \mathcal{N}_T.
\end{aligned}$$

Proof. We should first note that the assumption of the existence of a λ gain-loss ratio opportunity for some $\lambda > 1$ implies that LamP1 and LamPr have both non-empty feasible sets and their optimal values are greater than 1. We can see this fact by the problem SP1 and the definition of a λ gain-loss ratio opportunity (see problem SP1 and the paragraph following it) based on SP1. Assume that the optimal value of LamP1 is the finite number $\bar{\lambda}$ and the optimal value of LamPr is greater than $\bar{\lambda}$. Then, problem LamPr must have a feasible solution Θ, X^+, X^- which has an objective value λ' that is greater than $\bar{\lambda}$ by the definition of a supremum. Then we see that $\Theta, X^+, X^-, \lambda' - \epsilon$ with $\epsilon < \lambda' - \bar{\lambda}$ constitute another feasible solution to LamP1 with the objective value $\lambda' - \epsilon$. But, this contradicts with the assumption that $\bar{\lambda}$ is the optimal value of LamP1 since $\lambda' - \epsilon > \bar{\lambda}$. Hence, if LamP1 has a finite optimal value, LamPr cannot have an optimal value greater than that. Conversely, assume that the optimal value of LamPr is the finite number $\bar{\lambda}$ and the optimal value of LamP1 is greater than that. Then, LamP1 must have a feasible solution $\Theta, X^+, X^-, \lambda'$ which has an objective value λ' that is greater than $\bar{\lambda}$. Then, Θ, X^+, X^- constitute another feasible solution to LamPr with the objective value greater than λ' thus greater than $\bar{\lambda}$. Again, this contradicts with our assumption that $\bar{\lambda}$ is the optimal value of LamPr. Hence, if LamPr has a finite optimal value, LamP1 cannot have an optimal value greater than that. Using these facts we conclude that, if one of the problems has a finite optimal value the other one also has the same optimal value and if one of them is unbounded, the other one is also unbounded. It proves that they are equivalent when there is a λ gain-loss ratio opportunity. □

Notice that as a result of the homogeneity of the equalities and inequalities defining the constraints of problem LamPr, if Θ, X^+, X^- is feasible for LamPr, then so is $\kappa(\Theta, X^+, X^-)$ for any $\kappa > 0$, and the objective function value is constant along such rays.

Under the assumption

Assumption 1. *The price process* $\{Z_t\}$ *is arbitrage-free, i.e., there does not exist feasible* Θ, X^+, X^- *with* $\mathbb{E}^P[X^+] > 0$ *and* $\mathbb{E}^P[X^-] = 0$*,* we can now take one step further and say that problem LamPr is equivalent to problem LamPL:

$$
\begin{aligned}
\max \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ \\
\text{s.t.} \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^- = 1 \\
& Z_0 \cdot \theta_0 = 0 \\
& Z_n \cdot (\theta_n - \theta_{\pi(n)}) = 0,\ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^+ \geq 0,\ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \geq 0,\ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

This equivalence can be established using the technique described on pp. 151 in [9] as follows. Let us take a solution Θ, X^+, X^- to LamPr, with $\xi^- = \sum_{n\in\mathcal{N}_T} p_n x_n^-$. It is easy to see that the point $\frac{1}{\xi^-}(\Theta, X^+, X^-)$ is feasible in LamPL with equal objective function value. For the converse, let $\Psi = (\Theta, X^+, X^-)$ be a feasible solution to LamPr, and let $\Xi = (\bar{\Theta}, \bar{X}^+, \bar{X}^-)$ be a feasible solution to LamPL. It is again immediate to see that $\Psi + t\Xi$ is feasible in LamPr for $t \geq 0$. Furthermore, we have

$$\lim_{t\to\infty} \frac{\mathbb{E}^P[X^+ + t\bar{X}^+]}{\mathbb{E}^P[X^- + t\bar{X}^-]} = \mathbb{E}^P[\bar{X}^+],$$

which implies that we can find feasible points in LamPr with objective values arbitrarily close to the objective function value at Ξ.

We can now construct the linear programming dual of LamPL using Lagrange duality technique which results in the dual linear program (HD1) in variables $y_n, (n \in \mathcal{N})$ and V:

$$
\begin{aligned}
\min \quad & V \\
\text{s.t.} \quad & y_m Z_m = \sum_{n\in\mathcal{C}(m)} y_n Z_n,\ \forall\, m \in \mathcal{N}_t, 0 \leq t \leq T-1 \\
& p_n \leq y_n \leq V p_n,\ \forall n \in \mathcal{N}_T.
\end{aligned}
$$

Ley $Y(V)$ denote the set of $\{y_n\}$ that are feasible in the above problem for a given V. Notice that, for $V_1 < V_2$, one has $Y(V_1) \subseteq Y(V_2)$, assuming the respective sets to be non-empty. Hence, the optimal value of V is the minimum value such that the associated set $Y(V)$ is non-empty.

The dual can also be re-written as (HD2):

$$\begin{aligned} \min \quad & \max_{n\in\mathcal{N}_T} \frac{y_n}{p_n} \\ \text{s.t.} \quad & y_m Z_m = \sum_{n\in\mathcal{C}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1 \\ & p_n \le y_n, \ \forall n \in \mathcal{N}_T. \end{aligned}$$

Let Y denote the set of feasible solutions to the above problem. We summarize our findings in the proposition below.

Proposition 2. *Under Assumption 1 we have*

1. *Problem LamP1 is equivalent to problem LamPL.*
2. *When optimal solutions exist, for any optimal solution $\Theta^*, (X^+)^*, (X^-)^*, \lambda^*$ of LamP1, we have that $\frac{1}{\mathbb{E}^P[(X^-)^*]}(\Theta^*, (X^+)^*, (X^-)^*)$ is optimal for LamPL.*
3. *When optimal solutions exist, for any optimal solution $\Theta^*, (X^+)^*, (X^-)^*$ of LamPL and any $\kappa > 0$, we have that $\kappa(\Theta^*, (X^+)^*, (X^-)^*), \frac{\mathbb{E}^P[(X^+)^*]}{\mathbb{E}^P[(X^-)^*]}$ is optimal for LamP1.*
4. *The supremum λ^* of λ is equal to $\min_{y\in Y} \max_{n\in\mathcal{N}_T} \frac{y_n}{p_n}$.*

The last item of the above proposition is essentially the duality result of Bernardo and Ledoit (c.f. Theorem 1 on page 151 of [5]) which they prove for single period investments but using an infinite-state setup.

By way of illustration, setting up and solving the problem LamPL for the trinomial numerical example of the previous section, one obtains the largest value of λ as six, as the optimal value of the problem LamPL. This is the smallest value of λ that does not allow a λ gain-loss ratio opportunity. Put in other words, it is the supremum of all values of λ allowing a λ gain-loss ratio opportunity.

2.4 Financing of European Contingent Claims and Gain-Loss Ratio Opportunities: Positions of Writers and Buyers

Now, let us take the viewpoint of a writer of European contingent claim F which is generating pay-offs $F_n, (n > 0)$ to the holder (liabilities of the writer), depending on the states n of the market (hence the adjective contingent). The following is a legitimate question on the part of the writer: what is the minimum initial investment needed to replicate the pay-outs F_n using securities available in the market with no risk of positive expected terminal wealth falling short of λ times the expected negative terminal wealth? King [40] posed a similar question in the context of no-arbitrage pricing, hence for preventing the risk of terminal positions being negative at any state of nature. Here, obviously we are working with an enlarged feasible set of replicating portfolios, if not empty.

Let us now pose the problem of financing of the writer who opts for the λ gain-loss ratio opportunity viewpoint rather than the classical arbitrage viewpoint. The writer is facing the stochastic linear programming problem WP1

$$
\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T, \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0 \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T,
\end{aligned}
$$

as opposed to King's financing problem

$$
\begin{aligned}
\min \quad & Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

Let us assume that a price of F_0 is attached to a contingent claim F. The following definition is useful.

Definition 4. *A contingent claim F with price F_0 is said to be λ-attainable if there exist vectors θ_n for all $n \in \mathcal{N}$ satisfying:*

$$Z_0 \cdot \theta_0 \leq \beta_0 F_0,$$

$$Z_n \cdot (\theta_n - \theta_{\pi(n)}) = -\beta_n F_n, \; \forall \; n \in \mathcal{N}_t, t \geq 1$$

and

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] = 0.$$

Proposition 3. *At a fixed level $\lambda > 1$, assume the discrete vector price process $\{Z_t\}$ does not have a λ gain-loss ratio opportunity. Then the minimum initial investment W_0 required to hedge the claim with no risk of expected positive terminal wealth falling short of λ times the expected negative terminal wealth satisfies*

$$W_0 = \frac{1}{\beta_0} \max_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}$$

where $Y(\lambda)$ is the set of all $y \in \mathbb{R}^{|\mathcal{N}|}$ satisfying the conditions (2.3)– (2.4)–(2.5), i.e., the feasible set of SD1.

Proof. Let us begin by forming the linear programming dual of problem WP1. Forming the Lagrangian function after attaching multipliers $v_n, (n > 0)$, $w_n, (n \in \mathcal{N}_T)$ (all unrestricted-in-sign) and $V \geq 0$ we obtain

$$\begin{aligned} L(\Theta, X^+, X^-, v, w, V) \;&=\; Z_0 \cdot \theta_0 + V(\lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- - \sum_{n \in \mathcal{N}_T} p_n x_n^+) \\ &\quad + \sum_{t=1}^{T} \sum_{n \in \mathcal{N}_t} v_n \left(Z_n \cdot (\theta_n - \theta_{\pi(n)}) + \beta_N F_n \right) \\ &\quad + \sum_{n \in \mathcal{N}_T} w_n (Z_n \cdot \theta_n - x_n^+ + x_n^-) \end{aligned}$$

that we maximize over the variables Θ, X^+, and X^- separately again. This results in the dual problem WD2.1

$$\begin{aligned} \max \quad & \sum_{n>0} v_n \beta_n F_n \\ \text{s.t.} \quad & Z_0 = \sum_{n \in \mathcal{C}(0)} v_n Z_n \\ & v_m Z_m = \sum_{n \in \mathcal{C}(m)} v_n Z_n, \ \forall\, m \in \mathcal{N}_t, 1 \le t \le T-1 \\ & V p_n \le v_n \le V \lambda p_n, \ \forall n \in \mathcal{N}_T, \\ & V \ge 0. \end{aligned}$$

We observe that no feasible solution to WD2.1 could have a V-component equal to zero as this would lead to infeasibility in the v-component. Therefore, it is easy to see that the dual is equivalent to the linear-fractional programming problem (that we refer to as WD2.2) using the equivalences $V = 1/y_0$ and $v_n = y_n/y_0$:

$$\begin{aligned} \max \quad & \frac{\sum_{n>0} y_n \beta_n F_n}{y_0} \\ \text{s.t.} \quad & y_m Z_m = \sum_{n \in \mathcal{C}(m)} y_n Z_n, \ \forall\, m \in \mathcal{N}_t, 0 \le t \le T-1 \\ & p_n \le y_n \le \lambda p_n, \ \forall n \in \mathcal{N}_T. \end{aligned}$$

However, the feasible set of the previous problem is identical to the feasible set $Y(\lambda)$ of the dual SD1 in Proposition 1. Therefore, if the price process $\{Z_t\}$ does not admit a λ gain-loss ratio opportunity, then there exists a feasible solution to the dual SD1, and hence, a feasible solution to the dual problems WD2.2 and WD2.1. Since WD2.1 is feasible and bounded above, the primal problem WP1 is solvable by linear programming duality theory. Hence, the result follows. □

Notice that in the previous proof we obtained two equivalent expressions for the dual problem of WP1, namely the dual problem in the statement of the Proposition 3 or WD2.2, which is a linear-fractional programming problem, and the linear programming problem WD2.1 that is used for numerical computation. For future reference, we refer to the feasible set of WD2.1 as $Q(\lambda)$, and to its projection on the set of v's as $\bar{Q}(\lambda)$. It is not difficult to verify that $\bar{Q}(\lambda)$ is the set of martingale measures λ-compatible to P. Since we observed that no optimal (in fact, feasible) solution to WD2.1 could have a V-component equal to zero as this would lead to infeasibility in the v-component, by the complementary slackness property of optimal solutions to the primal and the dual problems in linear programming, we should have in all optimal solutions (Θ, X^+, X^-) to the primal:

$$\mathbb{E}^P[X^+] - \lambda \mathbb{E}^P[X^-] = 0.$$

We immediately have the following.

Corollary 1. *At fixed level $\lambda > 1$, assume the discrete vector price process $\{Z_t\}$ does not allow λ gain-loss ratio opportunity. Then, contingent claim F priced at F_0 is λ-attainable if and only if*

$$\beta_0 F_0 \geq \max_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}.$$

In the light of the above, the minimum acceptable price to the writer of the contingent claim F is given by the expression

$$F_0^w = \frac{1}{\beta_0} \max_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}. \tag{2.10}$$

Let us now look at the problem from the viewpoint of a potential buyer. The buyer's problem is to decide the maximum price he/she should pay to acquire the claim, with no risk of expected positive terminal wealth falling short of λ times the expected negative terminal wealth. This translates into the problem

$$\begin{aligned}
\max \quad & -Z_0 \cdot \theta_0 \\
\text{s.t.} \quad & Z_n \cdot (\theta_n - \theta_{\pi(n)}) = \beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T, \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0, \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T, \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T.
\end{aligned}$$

The interpretation of this problem is the following: find the maximum amount needed for acquiring a portfolio replicating the proceeds from the contingent claim without the risk of expected negative wealth magnified by a factor λ exceeding the expected positive terminal wealth. By repeating the analysis done for the writer (that we do not reproduce here), we can assert that the maximum acceptable price F_0^b to the buyer in our framework is given by the following, provided that the price process $\{Z_t\}$ does not admit λ gain-loss ratio opportunity (at fixed level λ):

$$F_0^b = \frac{1}{\beta_0} \min_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}. \tag{2.11}$$

Therefore, for fixed $\lambda > 1$ and P, we can conclude that the writer's minimum acceptable price and the buyer's maximum acceptable price in a market without λ gain-loss ratio opportunity constitute a λ gain-loss price interval given as

$$[\frac{1}{\beta_0} \min_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}; \frac{1}{\beta_0} \max_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}].$$

We could equally express this interval as

$$[\frac{1}{\beta_0} \min_{v, V \in Q(\lambda)} \mathbb{E}^v[\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{v, V \in Q(\lambda)} \mathbb{E}^v[\sum_{t=1}^{T} \beta_t F_t]]$$

where the optimization is over all martingale measures λ-compatible to P. This is the interval of prices which do not induce either the buyer or writer to engage in buying or selling the contingent claim. They can also be thought of as bounds on the price of the contingent claim. Let us recall that the no-arbitrage pricing interval obtained by King [40] corresponds to

$$[\frac{1}{\beta_0} \min_{q \in \bar{Q}} \mathbb{E}^q[\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q \in \bar{Q}} \mathbb{E}^q[\sum_{t=1}^{T} \beta_t F_t]];$$

where $\bar{Q}$ is the set of $q \in \mathbb{R}^{|\mathcal{N}|}$ satisfying

$$Z_0 = \sum_{n \in \mathcal{C}(0)} q_n Z_n$$

$$q_m Z_m = \sum_{n \in \mathcal{C}(m)} q_n Z_n, \; \forall\, m \in \mathcal{N}_t, 1 \le t \le T-1$$

and

$$q_n \ge 0 \; \forall n \in \mathcal{N}_T.$$

Clearly, for fixed λ we have the inclusion $\bar{Q}(\lambda) \subset \bar{Q}$ using the positivity of V. Hence, the pricing interval obtained above is a smaller interval in width in comparison to the arbitrage-free pricing interval of [40]. Notice that the two intervals will become indistinguishable as λ tends to infinity. The more interesting question is the behavior of the interval as λ is decreased. Before we examine this issue we consider some numerical examples.

Example 2. Consider the same simple market model of Example 1 in Section 2.2. We assume a contingent claim on the stock, of the European Call type with a strike price equal to 9 is available. Therefore, we have the following pay-off structure: $F_1 = 11, F_2 = 6, F_3 = 0$, corresponding to nodes 1, 2 and 3, respectively. Computing the no-arbitrage bounds using linear programming, one obtains the interval of prices $[2.0; 2.2]$ corresponding to the buyer and to the writer's problems respectively. For $\lambda = 8$, the price interval for no λ gain-loss ratio opportunity is $[2.09; 2.14]$. For $\lambda = 7$, the interval becomes $[2.10; 2.13]$. Finally, for $\lambda = 6$, which is the smallest allowable value for λ below which the above derivations lose their validity, the interval shrinks to a single value of 2.125, since both the buyer and the writer problems return the same optimal value. Therefore, for two investors that are ready to accept an expected gain prospect that is at least six times as large as an expected loss prospect, it is possible to agree on a common price for the contingent claim in question. In this particular example, the problem HD1 for $\lambda^* = 6$ which is the optimal value for λ, possesses a single feasible point $y = (2.66, 0.33, 0.33, 2)^T$. Dividing the components by 2.66 which is the component y_0, we obtain the unique equivalent martingale measure $(1/8, 1/8, 3/4)^T$ (which is also λ-compatible) leading to the unique price of the contingent claim.

Interestingly, the hedging policies of the buyer and the writer at level $\lambda^* = 6$ need not be identical. For the writer an optimal hedging policy is to short 6.75 units of riskless asset at $t = 0$ and buy 0.887 units of the stock. If node 1 were to be reached, the hedging policy dictates to liquidate the position in both the bond and the stock. In case of node 2, the position in the stock is zeroed out, and a position of 0.562 units in the bond is taken. Finally at node 3, the position in the stock is zeroed out, but a short position of 0.094 units remains in the riskless asset. For the buyer an optimal hedging policy is to buy 5.625 units of riskless asset at $t = 0$ and short 0.775 units of the stock. At time $t = 1$ if node 1 were to be reached, the hedging policy dictates to pass to a position of 1.125 units in the bond, and to a zero position in the stock. In case of node 2, all positions are zeroed out. At node 3, the position in the stock is zeroed out while a short position of 0.187 units remains in the riskless asset.

Example 3. Let us now consider a two-period version of the previous example. The market is again described through a trinomial structure. Let the asset price be as in Example 1 and 2 for time $t = 1$. At time $t = 2$, from node 1 at which the price is 20, the price can evolve to 22, 21 and 19 with equal probability, thereby giving the asset price values at nodes 4, 5 and 6. From node 2 at which the price takes value equal to 15, the price can go to 17 or 14 or 13 with equal probability, resulting in the asset price values at nodes 7, 8 and 9. Finally, from node 3, we have as children nodes the node 10, node 11 and node 12, with equally likely asset price realizations equal to 9, 8 and 7, respectively. Therefore, the trinomial tree contains 9 paths, each with a probability equal to 1/9. The riskless asset is assumed to have value one throughout. It can be verified that this market is arbitrage free.

Solving for the supremum of λ values allowing a λ gain-loss ratio opportunity, we obtain 14.5.

Now, let us assume we have a European Call option F on the stock with strike price equal to 14, resulting in pay-off values $F_4 = 8$, $F_5 = 7$, $F_6 = 5$ and $F_7 = 3$ where the index corresponds to the node number in the tree (all other values F_n are equal to zero). The no-arbitrage bounds yield the interval $[0.33, 1.2]$ for this contingent claim. The no-λ gain-loss ratio opportunity intervals go as follows: for $\lambda = 17$ one has $[0.86; 1.00]$, for $\lambda = 16$, $[0.9; 0.99]$, for $\lambda = 15$ $[0.94; 0.98]$. For the limiting value of $\lambda^* = 14.5$ the bounds again collapse to a single price of 0.9718 attained at the same λ-compatible martingale measure $q_4 = q_5 = 0.028$, $q_6 = 0.085$, $q_7 = 0.042$, $q_8 = q_9 = q_{10} = 0.028$, $q_{11} = 0.324$ and $q_{12} = 0.408$.

Node	B	S
0	−4.056	0.503
1	−14	1
2	7.13	−0.243
3	−4.563	0.57
8	3.729	
9	3.972	
10	0.57	
12	−0.57	

Table 2.1: The writer's optimal hedge policy for $\lambda = 14.5$.

Node	B	S
0	−0.915	−0.006
1	−80.465	3.972
2	14	−1
3	−15.324	1.915
4	14.915	
5	9.944	
9	1	
10	1.915	
12	−1.915	

Table 2.2: The buyer's optimal hedge policy for $\lambda = 14.5$.

Two tables, Table 2.1 and Table 2.2, summarize the optimal hedge policies of the writer and the buyer, respectively, when the single price is reached. We only report the results for nodes where non-zero portfolio positions are held. The symbols B and S stand for the riskless asset and the stock, respectively. Again, the hedge policies are quite different, but result in an identical price.

Returning to the issue of the behavior of the price interval when λ decreases, consider solving the problem LamPL or its dual HD1 (or HD2) for computing the smallest λ which does not allow gain-loss ratio opportunities, i.e., λ^* which is the supremum of values of λ yielding a λ gain-loss ratio opportunity. If one solves the dual problem HD1 to obtain as optimal solutions V^*, y^*, and if this solution is the unique feasible solution to the linear program HD1, i.e., if the set of equations and inequalities defining the constraints of HD1 for the fixed value of V^* admit a unique solution vector y^*, then this immediately implies that the no-λ gain-loss ratio opportunity pricing bounds at level $\lambda = V^*$, i.e., the bounds $\frac{1}{\beta_0} \min_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}$ and $\frac{1}{\beta_0} \max_{y \in Y(\lambda)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}$ coincide since both problems possess the common single feasible point y^*. However, the following example shows that the bounds do not have to coincide for the smallest λ value for which there are no λ gain-loss ratio opportunities in the market.

Example 4. Let us assume that the market consists of a riskless asset with zero growth rate, and two stocks. The stock price evolves according to a quadrinomial tree with one period as follows. At time $t = 0$, the stock price is 10 for both of the stocks. Hence $Z_0 = (1\ 10\ 10)^T$. At the time $t = 1$, the first stock's price can take the values 10, 10, 15, 5 and the second stock's price can take values 14, 2, 9, 11 with probabilities 0.25, 0.2, 0.5 and 0.05, respectively. Therefore, at node 1 one has $Z_1 = (1\ 10\ 14)^T$ with $p_1 = 0.25$; at node 2 $Z_2 = (1\ 10\ 2)^T$ with $p_2 = 0.2$; at node 3 $Z_3 = (1\ 15\ 9)^T$ with $p_3 = 0.5$ and finally at node 4 $Z_4 = (1\ 5\ 11)^T$ with $p_4 = 0.05$. The payoff structure of the contingent claim to be valued is $F_1 = 10, F_2 = 0, F_3 = 0, F_4 = 0$ We find that the minimum λ value which does not allow λ gain-loss ratio opportunities in the market is 10. However, for $\lambda = 10$, the price interval of the option for no λ gain-loss ratio opportunity is $[2.5; 5.26]$.

The above example shows that pricing interval does not necessarily reduce to a single point for the smallest λ. Then, we pose the question for a market in which there is only one bond and one risky asset. Example 5 shows that there is no unique price even under this simple setting.

Example 5. Let us assume that the market consists of a riskless asset with zero growth rate, and a stock. There are 2 periods and the stock price evolves irregularly for both periods. At the first period the tree branches into 2 nodes and at the second period the tree branches into 3 nodes for both of the nodes at $t = 1$, i.e., node 1 branches into nodes 3, 4, 5 and node 2 branches into nodes 6, 7, 8 at period 2. At time $t = 0$, the stock price is 8. Hence $Z_0 = (1\ 8)^T$. At the time $t = 1$, the stock's price can take the values 5, 10. Therefore, at node 1 one has $Z_1 = (1\ 5)^T$ and at node 2 $Z_2 = (1\ 10)^T$. At time $t = 2$, the stock's price can take the values 2, 6, 10 with probabilities 0.2, 0.1 and 0.1, respectively, given that its price was 5 at time $t = 1$ and 13, 11, 8 with probabilities 0.05, 0.05 and 0.5, respectively, given that its price was 10 at time $t = 1$. Therefore, at node 3 one has $Z_3 = (1\ 2)^T$ with $p_3 = 0.2$; at node 4 $Z_4 = (1\ 6)^T$ with $p_4 = 0.1$; at node 5 $Z_5 = (1\ 10)^T$ with $p_5 = 0.1$; at node 6 $Z_6 = (1\ 13)^T$ with $p_6 = 0.05$; at node 7 $Z_7 = (1\ 11)^T$ with $p_7 = 0.05$; and at node 8 $Z_8 = (1\ 8)^T$ with $p_8 = 0.5$. The payoff structure of the claim to be valued is $F_3 = 3, F_8 = 3$ and 0 elsewhere. We find that the minimum λ value which does not allow λ gain-loss ratio opportunities in the market is 5. However, for $\lambda = 5$, the price interval of the option for no λ gain-loss ratio opportunity is $[1.38; 1.56]$.

The natural question at this point is what happens if we work with a simpler setting. The following theorem shows that the martingale measure is unique for the smallest λ when there is only a bond and a risky stock in the market with just one period (no intermediary trading is allowed) under a minimal structural assumption on the stochastic scenario tree.

Theorem 3. *Assume that there is a bond and a risky stock in the market consisting of one period such that for all $n \in \mathcal{N}_1$ (the leaf nodes) $Z_n^1 \neq Z_{\pi(n)}^1$ (or $Z_n^1 \neq Z_0^1$). Then, at the smallest value λ^*, $Y(\lambda)$ is a singleton.*

Proof. Let $L = |\mathcal{N}_1|$ be the number of leaf nodes. Let us view the problem of computing the smallest λ such that $Y(\lambda)$ has a solution, as a parametric feasibility problem with parameter λ. In other words, for fixed $\lambda \geq 1$ we are interested to determine whether the restriction A^L onto the L-dimensional space composed of y_n for all $n \in \mathcal{N}_1$ (i.e., $\mathbb{R}^L$) of the set $A = \{y_n : y_0 Z_0 = \sum_{n \in \mathcal{C}(0)} y_n Z_n\}$, has non-empty intersection with the L-dimensional box $H_\lambda = \{y_n : p_n \leq y_n \leq \lambda p_n,\ \forall n \in \mathcal{N}_1.\}$.

Notice that A^L defines an affine set in the L-dimensional space of "leaf variables".

If the smallest value λ^* of λ, such that $A^L \cap H_\lambda$ is not empty, is equal to one, the theorem clearly holds because the set of solutions is necessarily a singleton in this case. So, we assume $\lambda^* > 1$. Let us fix some $\lambda > 1$ such that $A^L \cap H_\lambda$ is non-empty and is not a singleton. There are two cases to consider.

Case 1 There exist two "distinct", meaning all components different, L-vectors, y^1 and y^2, in $A^L \cap H_\lambda$. In this case, λ can be reduced since $A^L \cap H_\lambda$ is a convex set and any convex combination of y^1 and y^2 is also in the set.

Case 2 There are no "distinct" L-vectors y^1 and y^2 say, in $A^L \cap H_\lambda$. For this case, we first observe that there must be $i \in \mathcal{N}_1$ such that $y_i^1 = y_i^2, \forall y^1, y^2 \in A^L \cap H_\lambda$. Otherwise, we would be able to find a set of vectors $\{y^1, y^2, \ldots, y^k : \nexists i \in \mathcal{N}_1, y_i^a = y_i^b, \forall a, b \in \{1, \ldots, k\}\}$. Then, we could take a convex combination of these vectors in $A^L \cap H_\lambda$, which is a distinct vector with $\{y^1, y^2, \ldots, y^k\}$. This contradicts with the assumption of case 2. Our second observation is there must be $i \in \mathcal{N}_1$ such that $y_i = p_i, \forall y \in A^L \cap H_\lambda$. Otherwise, we would find a set of vectors $\{y^1, y^2, \ldots, y^k : \nexists i \in \mathcal{N}_1, y_i^a = p_i, \forall a \in \{1, \ldots, k\}\}$ and we could get a convex combination of these vectors y' such that $\nexists i \in \mathcal{N}_1, y_i' = p_i$. One can see that $\bar{y} : \bar{y}_i = 0, \forall \in \mathcal{N}_1$ is a feasible solution to the equations defining the set A. Then, we could take a convex combination of y' and $\bar{y}$ which is distinct with y' and which is in $A^L \cap H_\lambda$, contradicting the assumption of case 2. After these two observations we need to analyze the system of equations defining the set A. For a risky asset and a bond there are just two equations. The first one is $y_0 = \sum_{n \in \mathcal{C}(0)} y_n$. The second one is $y_0 Z_0^1 = \sum_{n \in \mathcal{C}(0)} y_n Z_n^1$. A solution of these two equations satisfies $\sum_{n \in \mathcal{C}(0)} y_n (Z_n^1 - Z_0^1) = 0$. Let $\alpha_n = (Z_n^1 - Z_0^1); \forall n \in \mathcal{N}_1$. Note that our structural assumption implies that $\alpha_n \neq 0; \forall n \in \mathcal{N}_1$. Let us say that $i \in \mathcal{N}_1$ is such that $y_i = p_i, \forall y \in A^L \cap H_\lambda$ and y be any vector in $A^L \cap H_\lambda$. First assume that $\alpha_i > 0$. Consider any $j \in \mathcal{N}_1$. If $\alpha_j > 0$ then $y_j = p_j$. Otherwise, we could find ϵ small enough such that when we decrease y_j by ϵ and increase y_i by $\alpha_j \epsilon / \alpha_i$ resulting in another solution in $A^L \cap H_\lambda$ with $y_i \neq p_i$, which is a contradiction. Conversely, if $\alpha_j < 0$ then $y_j = \lambda p_j$. Otherwise, we could find ϵ small enough such that increasing y_j by ϵ and increasing y_i by $-\alpha_j \epsilon / \alpha_i$ we could get another solution in $A^L \cap H_\lambda$ with $y_i \neq p_i$ which is again a contradiction. A similar argument follows for the case $\alpha_i < 0$. Therefore there can only be a unique solution for this case contradicting with the assumption $A^L \cap H_\lambda$ is not a singleton.

Therefore, Case 2 cannot occur, i.e., we are always in Case 1 i.e., λ can be reduced, if $A^L \cap H_\lambda$ is not a singleton.

A consequence of the above reasoning is that if λ cannot be reduced, i.e., $\lambda = \lambda^*$, then $A^L \cap H_\lambda$ must be a singleton. □

Notice that the analysis of the writer's and buyer's hedging problems can also be done using a simple utility function and the conjugate duality framework of convex optimization [53]. The utility function corresponding to no-arbitrage is given as

$$u_w(v) = v - I_{v \geq 0}(v)$$

where $I_{v \geq 0}$ is the indicator function of convex analysis which equals zero if $v \geq 0$, and $+\infty$ otherwise. Our problems involving the gain-loss objective function (and/or constraint) could alternatively be modeled using the equally simple piecewise-linear utility function

$$u(v) = \begin{cases} v & \text{if } v \geq 0 \\ \lambda v & \text{if } v < 0. \end{cases}$$

Then, all our results could be obtained using the concave conjugate function u^* given by

$$u^*(y) = \inf_v (yv - u(v))$$

which is finite in our case (in fact, zero) provided that $1 \leq y \leq \lambda$, which are exactly the constraints showing up in our dual problems where the argument of the u^* function is precisely y_n/p_n. However, the path taken in the present work through linear programming duality is simpler and more accessible.

In closing this section we point out that Bernardo and Ledoit's gain-loss ratio results that were obtained in a single-period, non-linear optimization framework are very similar to the approach described above. We showed that similar results can be obtained in a multi-period (finite probability), linear optimization setting, which is simpler yet much more intuitive.

2.5 Proportional Transaction Costs

The problem of hedging and pricing contingent claims in the presence of transaction costs was investigated in e.g. [23, 30, 33]. In [23], it was assumed that the cost of trading a stock (excluding the numéraire) is proportional to the price. We assume that the proportional transaction costs for buying and selling a stock are different, and there is no transaction cost for the numéraire. An investor who buys one share of stock j when the stock price (discounted with respect to the numéraire) is Z_n^j pays $Z_n^j(1+\eta)$ whereas upon selling the investor gets $Z_n^j(1-\zeta)$, where η and ζ are both in $[0,1)$. Let us now denote the components of Z_n corresponding to the indices from 1 to J, as the vector $\bar{Z}_n$. Similarly, we refer to the components of Z_n corresponding to the indices from 1 to J, as the vector $\bar{Z}_n$, and as $\bar{\theta}_n$ to the portfolio positions corresponding to all these stocks excluding the numéraire, for node n of the scenario tree. Then, the arbitrage problem which will be referred as TC1 becomes the following:

$$
\begin{aligned}
\max \quad & \sum_{n\in\mathcal{N}_T} p_n Z_n \cdot \theta_n \\
\text{s.t.} \quad & \theta_0^0 + \bar{Z}_0 \cdot \bar{\theta}_0 + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- = 0 \\
& \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = 0, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T, \\
& \bar{\theta}_0 = t_0^+ - t_0^- \\
& \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N}.
\end{aligned}
$$

where t_n^+ and t_n^- are vectors in $\Re_+^J$ denoting number of shares bought and sold, respectively at node n. The following theorem, which is equivalent to Theorem 4 of [40] states the conditions for no-arbitrage in a market with transaction costs.

Theorem 4. *The discrete state stochastic vector process $\{Z_t\}$ is an arbitrage free market price process if and only if there is at least one probability measure Q equivalent to P, which, extended to intermediate nodes recursively as in (2.1), makes the process $\{Z_t\}$ fulfill the condition*

$$(1-\zeta)\bar{Z}_t \leq \mathbb{E}^Q[\bar{Z}_T|\mathcal{N}_t] \leq (1+\eta)\bar{Z}_t, \quad \forall t \leq T-1. \tag{2.12}$$

The proof is omitted. It is not hard to see that for $\eta = \zeta = 0$ one recovers the statement of Theorem 1.

The λ gain-loss ratio opportunity seeking investor (at a fixed λ) is interested in solving the problem TC2:

$$\begin{aligned}
\max \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\
\text{s.t.} \quad & \theta_0^0 + \bar{Z}_0 \cdot \bar{\theta}_0 + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- = 0 \\
& \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = 0, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& \bar{\theta}_0 = t_0^+ - t_0^- \\
& \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T.
\end{aligned}$$

The counterpart of Theorem 2 in this case becomes the following.

Theorem 5. *The discrete state stochastic vector process $\{Z_t\}$ is a λ gain-loss ratio opportunity free market price process at level λ if and only if there is at least one probability measure Q, λ-compatible to P, which, extended to intermediate nodes recursively as in (2.1), makes the process $\{Z_t\}$ fulfill condition (2.12).*

Proof. We prove the necessity part first. Assume that the market is λ gain-loss ratio opportunity free . We see that the fourth and the fifth constraints can be used to get rid of variables $\bar{\theta}$ in the formulation of TC2. Since $\bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-$, $\forall\, n \in \mathcal{N}_t, t \geq 1$ and $\bar{\theta}_0 = t_0^+ - t_0^-$, it becomes $\bar{\theta}_n = t_n^+ - t_n^- + t_0^+ - t_0^-$, $\forall\, n \in \mathcal{N}_1$. Using the same reasoning we have $\bar{\theta}_n = \sum_{m\in\mathcal{A}(n)}(t_m^+ - t_m^-)$, $\forall\, n \in \mathcal{N}$. Then TC2 becomes:

$$
\begin{aligned}
\max \quad & \sum_{n\in\mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n\in\mathcal{N}_T} p_n x_n^- \\
\text{s.t.} \quad & \theta_0^0 + \bar{Z}_0 \cdot (t_0^+ - t_0^-) + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- = 0 \\
& \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (t_n^+ - t_n^-) + \eta \bar{Z}_n \cdot t_n^+ + \zeta \bar{Z}_n \cdot t_n^- = 0, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \theta_n^0 + \bar{Z}_n \cdot \sum_{m\in\mathcal{A}(n)} (t_m^+ - t_m^-) - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

The dual of this problem is the following feasibility problem:

$$
\begin{aligned}
\min \quad & 0 \\
\text{s.t.} \quad & v_n = \sum_{m\in S(n)} v_m, \ \forall\, n \in \mathcal{N}_t, 0 \leq t \leq T-1 \\
& (1+\eta) v_n \bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} v_m \bar{Z}_m \geq 0, \ \forall\, n \in \mathcal{N} \\
& (1-\zeta) v_n \bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} v_m \bar{Z}_m \leq 0, \ \forall\, n \in \mathcal{N} \\
& p_n \leq v_n \leq \lambda p_n, \ \forall n \in \mathcal{N}_T.
\end{aligned}
$$

If there is no λ gain-loss ratio opportunity, the optimal value of TC2 is equal to zero. Therefore, by linear programming duality, the dual problem is also solvable (in fact, feasible since the dual is only a feasibility problem). Let us take any feasible solution $v_n, (n \in \mathcal{N})$ of the dual problem. Since $v_n \geq p_n$, it follows that v_n is a strictly positive process such that the sum of v_n over all states $n \in \mathcal{N}_t$ in each time period t sums to v_0. Now, define the process $q_n = v_n/v_0$, for each $n \in \mathcal{N}$. Obviously, this defines a probability measure Q over the leaf (terminal) nodes $n \in \mathcal{N}_T$ and it extends to intermediate nodes recursively as in (2.1) as an implication of the first constraint in the dual problem. Furthermore, we can rewrite the second and the third constraints of the dual problem with the newly defined weights q_n as

$$(1+\eta)q_n\bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} q_m\bar{Z}_m \geq 0,\ \forall\, n\in\mathcal{N}$$

$$(1-\zeta)q_n\bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} q_m\bar{Z}_m \leq 0,\ \forall\, n\in\mathcal{N}$$

with $q_0 = 1$, and all $q_n > 0$. Therefore, by constructing the probability measure Q we have constructed an equivalent measure which makes the process $\{Z_t\}$ fulfill condition (2.12). By definition of the measure q_n, we have using the last constraint of the dual problem

$$p_n \leq q_n v_0 \leq \lambda p_n, \forall n \in \mathcal{N}_T,$$

or equivalently,

$$p_n/q_n \leq v_0 \leq \lambda p_n/q_n, \forall n \in \mathcal{N}_T,$$

which implies that $q_n, n \in \mathcal{N}_T$ constitute a measure λ-compatible to P . This concludes the necessity part.

Suppose Q is a probability measure λ-compatible to P, which extends to intermediate nodes recursively as in (2.1) and which makes the process $\{Z_t\}$ fulfill condition (2.12). Therefore, we have

$$(1+\eta)q_n\bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} q_m\bar{Z}_m \geq 0,\ \forall\, n\in\mathcal{N}$$

$$(1-\zeta)q_n\bar{Z}_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} q_m\bar{Z}_m \leq 0,\ \forall\, n\in\mathcal{N}$$

with $q_0 = 1$, and all $q_n > 0$, while the condition $\max_{n\in\mathcal{N}_T} p_n/q_n \leq \lambda \min_{n\in\mathcal{N}_T} p_n/q_n$ holds. If the previous inequality holds as an equality, choose the right-hand (or, the left-hand) of the inequality as a factor v_0 and set $v_n = q_n v_0$ for all $n \in \mathcal{N}$. If the inequality is not tight, any value v_0 in the interval $[\max_{n\in\mathcal{N}_T} p_n/q_n, \lambda \min_{n\in\mathcal{N}_T} p_n/q_n]$ will do. It is easily verified that such defined v_n, $n \in \mathcal{N}$ satisfy the constraints of the dual problem. Since the dual problem is feasible, the primal TC2 is bounded above (in fact, its optimal value is zero) and no λ gain-loss ratio opportunity exists in the system.

□

For $\eta = \zeta = 0$ one recovers Theorem 2.

Now, the no-arbitrage price bounds of the previous section are computed by solving

$$\begin{array}{lr} \min & \theta_0^0 + \bar{Z}_0 \cdot \bar{\theta}_0 + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- \\ \text{s.t.} & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\ & \bar{\theta}_0 = t_0^+ - t_0^- \\ & \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N}, \end{array}$$

for the writer, and

$$\begin{array}{lr} \max & -\theta_0^0 - \bar{Z}_0 \cdot \bar{\theta}_0 - \eta \bar{Z}_0 \cdot t_0^+ - \zeta \bar{Z}_0 \cdot t_0^- \\ \text{s.t.} & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = \beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & Z_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\ & \bar{\theta}_0 = t_0^+ - t_0^- \\ & \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\ & t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N}, \end{array}$$

for the buyer. These bounds are also obtained using the dual expressions:

$$[\frac{1}{\beta_0} \min_{q \in \tilde{Q}(\eta,\zeta)} \mathbb{E}^Q[\sum_{t=1}^T \beta_t F_t]; \frac{1}{\beta_0} \max_{q \in \tilde{Q}(\eta,\zeta)} \mathbb{E}^Q[\sum_{t=1}^T \beta_t F_t]].$$

where $\tilde{Q}(\eta, \zeta)$ is the (closure of) set of measures Q equivalent to P such that the process $\{\bar{Z}_t\}$ satisfies condition (2.12). The proofs are omitted for these results since they are similar to the proof of our next result.

Now, let us consider the no λ gain-loss ratio opportunity bounds obtained from the perspective of the buyer and the writer by going through the usual problems in the hedging space:

$$
\begin{aligned}
\min \quad & \theta_0^0 + \bar{Z}_0 \cdot \bar{\theta}_0 + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- \\
\text{s.t.} \quad & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0 \\
& \bar{\theta}_0 = t_0^+ - t_0^- \\
& \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T,
\end{aligned}
$$

for the writer, and

$$
\begin{aligned}
\max \quad & -\theta_0^0 - \bar{Z}_0 \cdot \bar{\theta}_0 - \eta \bar{Z}_0 \cdot t_0^+ - \zeta \bar{Z}_0 \cdot t_0^- \\
\text{s.t.} \quad & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (\bar{\theta}_n - \bar{\theta}_{\pi(n)}) + \eta \bar{Z}_0 \cdot t_n^+ + \zeta \bar{Z}_0 \cdot t_n^- = \beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0 \\
& \bar{\theta}_0 = t_0^+ - t_0^- \\
& \bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T,
\end{aligned}
$$

for the buyer. We see that the fourth and the fifth constraints can be used to get rid of variables $\bar{\theta}$ in the formulation of the above problem. Since $\bar{\theta}_n - \bar{\theta}_{\pi(n)} = t_n^+ - t_n^-, \ \forall\, n \in \mathcal{N}_t, t \geq 1$ and $\bar{\theta}_0 = t_0^+ - t_0^-$, it becomes $\bar{\theta}_n = t_n^+ - t_n^- + t_0^+ - t_0^-, \ \forall\, n \in \mathcal{N}_1$. Using the same reasoning we have $\bar{\theta}_n = \sum_{m \in \mathcal{A}(n)} (t_m^+ - t_m^-), \ \forall\, n \in \mathcal{N}$. Then we obtain the following linear program:

$$
\begin{aligned}
\min \quad & \theta_0^0 + \bar{Z}_0 \cdot (t_0^+ - t_0^-) + \eta \bar{Z}_0 \cdot t_0^+ + \zeta \bar{Z}_0 \cdot t_0^- \\
\text{s.t.} \quad & \theta_n^0 - \theta_{\pi(n)}^0 + \bar{Z}_n \cdot (t_n^+ - t_n^-) + \eta \bar{Z}_n \cdot t_n^+ + \zeta \bar{Z}_n \cdot t_n^- = -\beta_n F_n, \ \forall\, n \in \mathcal{N}_t, t \geq 1 \\
& \theta_n^0 + \bar{Z}_n \cdot \sum_{m \in \mathcal{A}(n)} (t_m^+ - t_m^-) - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0 \\
& t_n^+, t_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& x_n^+ \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

The dual problem of this program is

$$
\begin{aligned}
\max \quad & \sum_{n>0} v_n \beta_n F_n \\
\text{s.t.} \quad & v_0 = 1 \\
& v_n = \sum_{m \in C(n)} v_m, \ \forall\, n \in \mathcal{N}_t, 0 \leq t \leq T-1 \\
& (1+\eta) v_n \bar{Z}_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} v_m \bar{Z}_m \geq 0, \ \forall\, n \in \mathcal{N} \\
& (1-\zeta) v_n \bar{Z}_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} v_m \bar{Z}_m \leq 0, \ \forall\, n \in \mathcal{N} \\
& V p_n \leq v_n \leq V \lambda p_n, \ \forall n \in \mathcal{N}_T, \\
& V \geq 0.
\end{aligned}
$$

Denote the feasible set of the above dual problem by $\tilde{\mathcal{Q}}(\lambda, \eta, \zeta)$, i.e., the set of probability measures v_n and positive V such that

$$(1-\zeta)\bar{Z}_t \leq \mathbb{E}^v[\bar{Z}_T | \mathcal{N}_t] \leq (1+\eta)\bar{Z}_t, \ \ \forall t \leq T-1$$

and $V p_n \leq v_n \leq V \lambda p_n, \ \forall n \in \mathcal{N}_T$.

By setting $y_0 = 1/V$ and $y_n = v_n/V$, and simplifying we obtain the following equivalent program:

$$
\begin{aligned}
\max \quad & \frac{\sum_{n>0} y_n \beta_n F_n}{y_0} \\
\text{s.t.} \quad & y_n = \sum_{m \in C(n)} y_m, \ \forall\, n \in \mathcal{N}_t, 0 \le t \le T-1 \\
& (1+\eta) y_n \bar{Z}_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} y_m \bar{Z}_m \ge 0, \ \forall\, n \in \mathcal{N} \\
& (1-\zeta) y_n \bar{Z}_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} y_m \bar{Z}_m \le 0, \ \forall\, n \in \mathcal{N} \\
& p_n \le y_n \le \lambda p_n, \ \forall n \in \mathcal{N}_T.
\end{aligned}
$$

Denote the feasible set of the previous problem $\tilde{\mathcal{Y}}(\lambda, \eta, \zeta)$. Going through a similar derivation for the buyer's case (omitted for brevity) we have proved the following result.

Proposition 4. *The price interval of a contingent claim for no λ gain-loss ratio opportunity at level λ is*

$$
[\frac{1}{\beta_0} \min_{q, V \in \tilde{\mathcal{Q}}(\lambda,\eta,\zeta)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]; \frac{1}{\beta_0} \max_{q, V \in \tilde{\mathcal{Q}}(\lambda,\eta,\zeta)} \mathbb{E}^Q[\sum_{t=1}^{T} \beta_t F_t]]
$$

or, equivalently

$$
[\frac{1}{\beta_0} \min_{y \in \tilde{\mathcal{Y}}(\lambda,\eta,\zeta)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}; \frac{1}{\beta_0} \max_{y \in \tilde{\mathcal{Y}}(\lambda,\eta,\zeta)} \frac{\sum_{n>0} y_n \beta_n F_n}{y_0}].
$$

Obviously, the no λ gain-loss ratio opportunity bounds are tighter compared to the no-arbitrage bounds. Notice that $\tilde{\mathcal{Q}}(\lambda, 0, 0)$ and $\tilde{\mathcal{Y}}(\lambda, 0, 0)$ coincide with $Q(\lambda)$ and $Y(\lambda)$, respectively.

Example 6. Considering the same problem as in example 2 with $\eta = \zeta = 0.1$, the supremum of the values of λ allowing a λ gain-loss ratio opportunity opportunity is computed to 3.715 (notice the drop from 6 in the case of no transaction costs). The no-arbitrage interval for the contingent claim is found to be $[1.2; 3.08]$. At $\lambda = 4$, the no λ gain-loss ratio opportunity interval is $[2.83; 2.98]$. At $\lambda = 3.715$ which is the limiting value, the common bound is equal to 2.97. The unique measure leading to this common price is given as $q_1 = q_2 = 0.175$ and $q_3 = 0.65$.

2.6 Conclusion

In this chapter, we studied the problem of pricing and hedging contingent claims in incomplete markets in a multi-period linear optimization (discrete-time, finite probability space) framework. We developed an extension of the concept of no-arbitrage pricing (λ gain-loss ratio opportunity) based on expected positive and negative final wealth positions, which allow to obtain arbitrage only in the limit as a gain-loss preference parameter tends to infinity. We analyzed the resulting optimization problems using linear programming duality. We showed that the pricing bounds obtained from our analysis are tighter than the no-arbitrage pricing bounds. This result, in line with the Bernardo and Ledoit [5] single period results, was also obtained for a multi-period model in the computationally more tractable linear programming environment. Our results indicated that for a limiting value of risk aversion parameter that can be computed easily, a unique price for a contingent claim in incomplete markets may be found (although this is not guaranteed) while different hedging schemes exist for different sides of the same trade. We also extended our results to markets with transaction costs.

Chapter 3

Pricing American Contingent Claims by Stochastic Linear Programming

Mathematical programming tools, especially stochastic programming (see [59] for a recent survey) are becoming increasingly useful as an entry point for studying the specialized methods of mathematical finance [25, 40, 48]. In this chapter, we are interested in the pricing of American Contingent Claims (ACC) as well as their special cases, in a multi-period, discrete time, discrete state space framework.

In the area of pricing contingent claims research concentrates mainly on defining and characterizing the range of contingent claim prices consistent with the absence of arbitrage. This range is determined by the upper hedging and the lower hedging prices, also known as the superreplication and subreplication bounds as we discuss in Chapter 2. In the absence of arbitrage, the upper hedging price is the value of the least costly self-financing portfolio strategy composed of market instruments whose pay-off is at least as large as the contingent claim pay-off. This price can also be interpreted from the perspective of a writer (seller) of the contingent claim as the smallest initial wealth required to replicate the contingent claim pay-off at expiration in a self-financed manner. Hence, we refer to the upper hedging price as the writer's price as well. Similarly, the lower hedging price is the value of the most precious self-financing portfolio strategy composed of market instruments whose pay-off is dominated by the contingent claim pay-off at expiration. The lower hedging price can also be interpreted as the largest amount the contingent claim buyer can borrow (in the form of cash or by short-selling stocks) to acquire the claim while paying off his/her debt in a self-financed manner using the contingent claim pay-off at expiration [17]. Hence, we refer to this price as the buyer's price as well as the lower hedging price. For European contingent claims, which can only be exercised at expiration, the upper and lower hedging prices are usually expressed as supremum and infimum, respectively, of the expectation of the discounted contingent claim pay-off (at expiration) over all probability measures that make the underlying stock price a martingale. We direct the reader to the book by Föllmer and Schied [26] for an in-depth treatment of pricing contingent claims in discrete time.

Similar expectation expressions were developed by Harrison and Kreps [28] and Chalasani and Jha [17] for American contingent claims, which can be exercised at any time until expiration. However, the possibility of early exercise complicates the expressions where one has to take supremums over all stopping times which represent potential exercise strategies of the contingent claim buyer. In particular, the upper hedging price is the supremum of the expectation of the discounted contingent claim pay-off (at some time between now and expiration) over all stopping times and all probability measures that make the underlying stock-price process a martingale. While the upper hedging price can be cast as a linear programming problem in discrete time [17, 48], the lower hedging price is harder to compute. It is the supremum over all stopping times of the infimum of the expected discounted contingent claim pay-off (at some time between now and expiration) over all probability measures that make the underlying stock price process a martingale. More precisely, the lower hedging price of an American contingent claim is given by an expression of the form

$$\max_{\tau \in \mathcal{T}} \min_{P \in \mathcal{P}} \mathbb{E}^P[F_\tau]$$

where $\mathcal{T}$ is the set of stopping times, $\mathcal{P}$ is the set of all martingale measures, and F_τ is the discounted contingent claim pay-off at time τ; see e.g., Theorem 12.4 of

[17].

Against this background, Pennanen and King [48] showed that the above expression for the lower hedging price can also be cast as

$$\min_{P\in\mathcal{P}}\max_{\tau\in\mathcal{T}}\mathbb{E}^P[F_\tau]$$

by interchanging the order of the max and min after observing that the outer maximization over the set $\mathcal{T}$ of stopping times can be replaced by maximization over a set of randomized stopping times, a central notion in [17] (see also the definition of the sets E and $\tilde{E}$ just before Theorem 7 in this chapter) and convex duality theory. From an optimization point of view, Pennanen and King's characterization of the set of the lower hedging price for ACCs follows from a representation of the buyer's price as the optimal value of a linear programming problem in the hedging space of the buyer, instead of posing the same hedging problem over integer valued variables. This important observation opens the way to harnessing the well-developed linear programming algorithms and software for the calculation of the buyer's price for ACCs. However, while their result is correct, their proof has a serious gap that we shall explain in section 3.2 through a counterexample. In this chapter we present an alternative proof of this result. After defining the buyer's problem similarly to the one in [48] we formulate an integer programming problem for the buyer's price. Then, we prove that the bound from the buyer's perspective can be computed by solving a linear program. This result gives a correct alternative proof of Theorem 3 of [48]. Independently, Flåm [25] proves a similar result for the contingent claim writer's price using considerations of total unimodularity. However, as discussed above the computation of the lower and upper hedging prices leads to different problems where it appears that the buyer's problem is harder to analyze. In fact, Pennanen and King [48] also give an analysis of the writer's pricing problem. Hence, we concentrate on the buyer's problem in our work. Our proof uses direct construction of an integral optimal solution from a fractional solution. The result remains valid for dividend paying stocks as well. The significance of the result stems from the fact that there exist linear programming algorithms with a computational complexity bounded above by a low order polynomial in the number of variables and constraints for computing a solution to ϵ-accuracy; see Section 6 of [3]. In practice, one has access to numerous software packages capable of handling very large instances of linear programs with dimensions reaching hundred thousand variables and constraints. Based on our experiences with European index options [49], multi-period hedging problems with approximately 70,000 variables and 22,000 equality, and 40,000 inequality constraints can be solved very quickly using the GAMS/CPLEX solver [11, 19].

3.1 The Stochastic Scenario Tree and American Contingent Claims

We will use all the concept and the notation which is described in Section 2.1. At this point we need to define an ACC in our framework. Besides, additional notation will be defined. An ACC F is a financial instrument generating a real-valued stochastic (cash-flow) process $(F_t)_{t=0,\dots,T}$. At any stage $t = 0,\dots,T$, the holder of an ACC may decide to take F_t in cash and terminate the process. Using this definition, an American call option on a stock S with strike price K corresponds to $F = S - K$. American put is obtained by reversing the sign of F. We can define a European call option with maturity T by setting $F_t = 0$ for $t \neq T$. Bermudan call options having exercise date set $G \subset \{1,\dots,T\}$ can be defined by setting $F_t = 0$ for $t \notin G$.

The market consists of $J + 1$ tradable securities indexed by $j = 0, 1, \dots, J$ with prices at node n given by the vector $S_n = (S_n^0, S_n^1, \dots, S_n^J)$. We assume as in [48] that the security indexed by 0 has strictly positive prices at each node of the scenario tree. This asset might also be considered as the risk-free asset in the classical valuation framework, in which case its price would be same at each node belonging to the same time period.

The number of shares of security j held by the investor in state (node) $n \in \mathcal{N}_t$ is denoted θ_n^j. Therefore, to each state $n \in \mathcal{N}_t$ is associated a vector $\theta_n \in \mathbb{R}^{J+1}$.

The value of the portfolio at state n is

$$S_n \cdot \theta_n = \sum_{j=0}^{J} S_n^j \theta_n^j.$$

In our finite probability space setting an American contingent claim F generates payoff opportunities F_n, $(n \geq 0)$ to its holder depending on the states n of the market.

We use Figure 3.1 to illustrate the stochastic scenario tree. In this example there are only three periods. At the first period, which is denoted by $t = 0$, stock prices are known, so there is only one node at this period. The index of this node is 0. This node branches to three nodes at the second period. The three possible states for the second period are respresented by three nodes: the upper node is indexed by 1, the middle node is indexed by 2 and the lower node is indexed by 3. Then, each node in the second period branches to three nodes at the third period. Hence, there are nine nodes at the third period. These nodes are indexed in the same fashion from 4 to 12. We assume that there are only two financial instruments in the market: a stock and a bond. Bond price is assumed to be 1 for each node, which means that the risk free interest rate is zero. The number inside each node represents the price of the stock at that node. The number next to a node represents the payoff of some fictitious contingent claim at that node. There is not a fixed delivery price for this contingent claim. Hence, its payoff is greater at node 6 than its payoff at node 7 although the price of the stock is greater at node 6. We will use this toy scenario tree and contingent claim as a counterexample below after the proof of Theorem 6.

For further details on arbitrage-free pricing of European and American contingent claims using stochastic linear programming we refer the reader to [25, 40, 48].

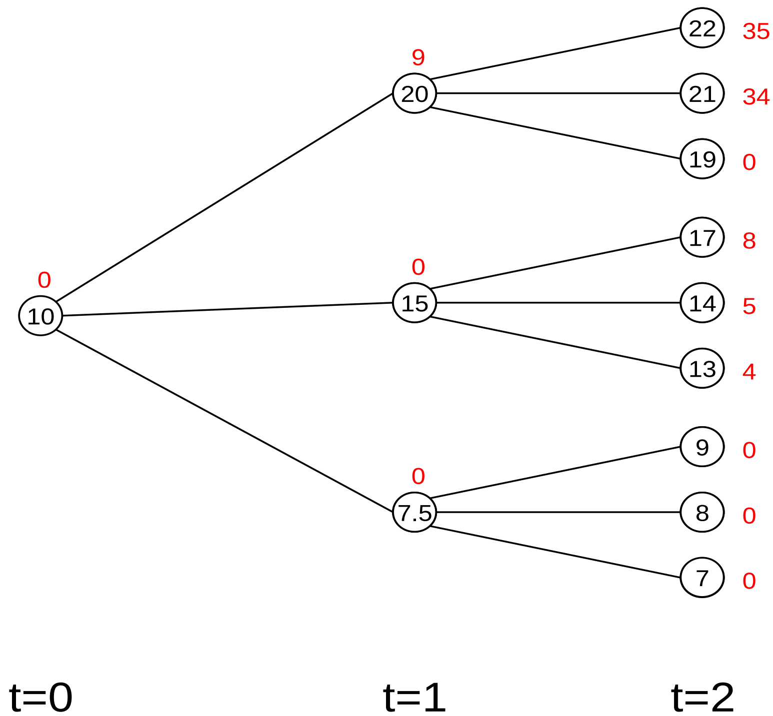

Figure 3.1: The tree representing the counterexample to the proof in [48].

3.2 The Main Result

We will now give a new proof of Theorem 3 of [48]. An arbitrage seeking buyer's problem can be formulated as the following problem that we will refer as AP1.

$$\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\
& S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}.
\end{aligned}$$

The definition of variables e_n is as follows:

$$e_n = \left\{ \begin{array}{ll} 1, & \text{if the ACC is exercised at node n} \\ 0, & \text{o.w.} \end{array} \right\}$$

The optimal value of V is the largest amount that a potential buyer is willing to disburse for acquiring a given American contingent claim F. The computation of this quantity via the above integer programming problem is carried out by construction of a least costly (adapted) portfolio process replicating the proceeds from the contingent claim by self-financing transactions using the market-traded securities in such as way to avoid any terminal losses. The integer variables and related constraints represent the one-time exercise of the American contingent claim; see [48] for further details.

A linear programming relaxation of AP1 is the following problem AP2:

$$\begin{aligned} \max \quad & V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\ & S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\ & \sum_{m \in \mathcal{A}(n)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \\ & e_n \geq 0, \ \forall\, n \in \mathcal{N}. \end{aligned}$$

Theorem 6. *There exists an optimal solution to AP2 with* $e_n \in \{0, 1\}$, $\forall\, n \in \mathcal{N}$.

Proof. Assume that AP2 has an optimal solution V^*, e^* and θ^* such that $e_n^* \notin \{0, 1\}$ for some $n \in \mathcal{N}$.

Case 1: We will first consider the case where e^* has a value not equal to 0 or 1 for the root, which is the starting node of the tree (i.e. $e_0^* \notin \{0, 1\}$). In order to deal with this case, we will form the Lagrangian function for AP2. That is

$$\begin{aligned} L(V, e, \theta, x, y, z) \;\; = \;\; & V - y_0[S_0 \cdot \theta_0 - F_0 e_0 + V] - \sum_{n \in \mathcal{N} \setminus \{0\}} y_n[S_n \cdot (\theta_n - \theta_{\pi(n)}) - F_n e_n] \\ & + \sum_{n \in \mathcal{N}_T} x_n S_n \cdot \theta_n - \sum_{n \in \mathcal{N}_T} z_n[\sum_{m \in \mathcal{A}(n)} e_m - 1]. \end{aligned}$$

After rearranging the above function we have

$$\begin{aligned} L(V, e, \theta, x, y, z) &= (1 - y_0)V + \sum_{n\in\mathcal{N}_T}(x_n - y_n)S_n \cdot \theta_n + \sum_{n\in\mathcal{N}\setminus\mathcal{N}_T} \theta_n \cdot [\sum_{m\in\mathcal{C}(n)} y_m S_m - y_n S_n] \\ &\quad + \sum_{n\in\mathcal{N}}[y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m]e_n + \sum_{n\in\mathcal{N}_T} z_n. \end{aligned}$$

Then the dual problem of AP2 can be formulated as

$$\begin{aligned} \min \quad & \sum_{n\in\mathcal{N}_T} z_n \\ \text{s.t.} \quad & y_0 = 1 \\ & [x_n - y_n]S_n = 0, \ \forall\, n \in \mathcal{N}_T \\ & \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N}\setminus\mathcal{N}_T \\ & y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{N} \\ & x_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

Since $S_n \neq 0$, second constraint implies that $x_n = y_n, \forall\, n \in \mathcal{N}_T$. Thus the dual problem can be rearranged as

$$\begin{aligned} \min \quad & \sum_{n\in\mathcal{N}_T} z_n \\ \text{s.t.} \quad & y_0 = 1 \\ & \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N}\setminus\mathcal{N}_T \\ & y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{N} \\ & y_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

We have an optimal solution to AP2 with $e_0^* \notin \{0, 1\}$. Then complementary slackness implies that the third constraint of the above program should be satisfied as an equality for the corresponding optimal solution of the dual problem (i.e., $y_0 F_0 - \sum_{m\in\mathcal{N}_T} z_m = 0$). Since $y_0 = 1$, we have $F_0 = \sum_{m\in\mathcal{N}_T} z_m$. Thus, the optimal solution to the dual problem is found to be F_0. Then, by strong duality we know that F_0 is the optimal value of AP2. One can easily show that a feasible solution to AP2 is $e_0 = 1$, $V = F_0$ and all the other variables as zeros (each

θ_n as a zero vector) with objective value F_0. This is an optimal solution with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$, thus the proof for the first case is complete.

Case 2: Now assume that optimal solution e^* is such that $e_0^* = 0$ and $e_n^* \notin \{0,1\}$ for some $n \in \mathcal{N}$. Let $I = \{i | e_i^* \notin \{0,1\}, i \in \mathcal{N}\}$. Let $G = \{g | g \in I, \mathcal{A}(g) \cap I = \{g\}\}$. Let w be the element with the smallest time index (that is closest to the root) in G. Note that $e_n^* = 0$, $\forall\, n \in \mathcal{A}(w) \backslash \{w\}$ in this case. Also, let k denote the time index for node w.

Claim: One can always find an optimal solution to AP2 with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w) \backslash \{w\}$.
To prove the claim we will consider the following two linear programs to which we will refer as AR1 and AR2 respectively:

$$\begin{aligned} \max \quad & e_w \\ \text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\ & S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w) \backslash \{w\} \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\ & \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\ & e_n \geq 0, \ \forall\, n \in \mathcal{D}(w), \end{aligned}$$

$$\begin{aligned} \min \quad & e_w \\ \text{s.t.} \quad & S_w \cdot (\theta_w - \theta^*_{\pi(w)}) = F_w e_w \\ & S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{D}(w) \backslash \{w\} \\ & S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\ & \sum_{m \in \mathcal{A}(n) \cap \mathcal{D}(w)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w) \\ & e_n \geq 0, \ \forall\, n \in \mathcal{D}(w). \end{aligned}$$

Let us denote the optimal solution of AR1 as $\bar{\theta}_{\mathcal{D}(w)}$, $\bar{e}_{\mathcal{D}(w)}$ and to AR2 as $\tilde{\theta}_{\mathcal{D}(w)}, \tilde{e}_{\mathcal{D}(w)}$. If the optimal value of AR1 is 1, then we see that $(\bar{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N} \backslash \mathcal{D}(w)})$, $(\bar{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N} \backslash \mathcal{D}(w)})$ form another optimal solution of AP2 with $e_w = 1$. For this optimal solution we have $e_w = 1$ and $e_i = 0$, $\forall\, i \in \mathcal{A}(w) \backslash \{w\}$ (we have also $e_i = 0$, for all $i \in \mathcal{D}(w)\backslash\{w\}$ for this solution). Similarly, if the optimal value of AR2 is 0, then $(\tilde{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\backslash\mathcal{D}(w)})$, $(\tilde{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\backslash\mathcal{D}(w)})$ form another optimal solution of AP2 with $e_w = 0$. Then, for this optimal solution we have $e_i = 0$, for all $i \in \mathcal{A}(w)$. So, our claim will be proved if we can show that AR2's having an optimal value greater than 0 implies that the optimal value of AR1 is 1. To show that we will consider the dual problems of AR1 and AR2. The dual problems DAR1 and DAR2 of AR1 and AR2, respectively, are

$$\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n \geq 1 \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& y_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w),
\end{aligned}$$

$$\begin{aligned}
\max \quad & -\sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n - y_w S_w \cdot \theta^*_{\pi(w)} \\
\text{s.t.} \quad & \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{D}(w)\backslash\mathcal{N}_T \\
& -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n \geq -1 \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{D}(w)\backslash\{w\} \\
& y_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T \cap \mathcal{D}(w).
\end{aligned}$$

We will denote the optimal value of AR2 by α, which is equal to the optimal value of DAR2. We know that $\alpha \leq 1$. Assume that $\alpha > 0$. Then by complementary slackness we know that the second constraint of DAR2 must be satisfied as an equality at the corresponding optimal solution, since $e_w \neq 0$ at the optimal solution of AR2. Then at the optimal solution of DAR2, we have

$$0 > \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n = -1. \tag{3.1}$$

Moreover, we must have $y_w \geq 0$ for any feasible solution of DAR1 and DAR2. This follows from the following fact. We have $y_n \geq 0$, $\forall n \in \mathcal{N}_T \cap \mathcal{D}(w)$. Then, since $S_n^0 > 0$ for all n, we have $y_n \geq 0$, $\forall n \in \mathcal{N}_{T-1} \cap \mathcal{D}(w)$ by the first constraints of DAR1 and DAR2. Similarly, we can show the same successively for $(T-2), (T-3), \ldots, k$. So, we have $y_w \geq 0$. Then, using the second inequality of (3.1) we have

$$\sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n + y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n$$

$$y_w S_w \cdot \theta^*_{\pi(w)} \geq -y_w F_w$$

$$S_w \cdot \theta^*_{\pi(w)} \geq -F_w$$

where the last step follows from $y_w \geq 0$. Then, for DAR1 at any feasible solution we have

$$1 \leq -y_w F_w + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n \leq y_w S_w \cdot \theta^*_{\pi(w)} + \sum_{n \in \mathcal{N}_T \cap \mathcal{D}(w)} z_n$$

whence we see that the optimal solution of DAR1 cannot be less than 1. It is easy to see by AR1 that optimal value of DAR1 cannot be greater than 1 either. Hence, we conclude that the optimal value of DAR1 and therefore that of AR1, is 1. This completes the proof of our claim.
Using the claim we see that there always exists an optimal solution to AP2 with $e_w \in \{0, 1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w)$. So, one can eliminate all the nodes having time index k in I by applying the above procedure. Then, proceeding successively with the nodes in $(k+1)^{st}, (k+2)^{nd} \ldots (T)^{th}$ time indices one can find an optimal solution for AP2 with $e_n \in \{0, 1\}$, $\forall\ n \in \mathcal{N}$. We note that, at each step the size of I might increase, but no nodes with a time index less than or equal to that of the node eliminated at that particular step can show up in I at the next step. This completes the proof of the theorem. □

In their proof Pennanen and King [48] claim that for an optimal solution of AP2 if the contingent claim is exercised partially at a node, then there is another optimal solution in which the contingent claim is fully exercised at that node. However, we have discovered counterexamples to this claim by computer experimentation. For some special cases, one can show, contrary to this claim, that there is another optimal solution where the claim is not exercised at that node, but no optimal solution exists in which the claim is fully exercised at that node.

For a counterexample to the claim of [48] let us return to the example of the fictitious contingent claim in Figure 3.1 at the end of Section 3.1. We wrote a simple GAMS code to construct and solve the buyer's problem (the linear programming relaxation of it) using CPLEX Version 9.0.2 with the data given in the example. The optimal value, i.e., the buyer's price, of this problem is 2. CPLEX 9.0.2 reports a fractional optimal solution of this problem where we have $e_1 = 0.625$. We show the non-zero variables of this solution in Table 3. Here, θ_{nj} denotes the number of shares of security j ($j = 0$ for the bond and $j = 1$ for the stock) held by the investor at node n. Besides, e_n is the variable for the execution time of the contingent claim.

Opt. Value	θ_{00}	θ_{01}	θ_{10}	θ_{11}	θ_{20}	θ_{21}	θ_{50}	e_1	e_4	e_5	e_7	e_8	e_9
2	6	-0.8	83.125	-4.375	9	-1	4	0.625	0.375	0.375	1	1	1

Table 3.1: The optimal values of variables in the counterexample (the remaining variables have value zero).

If the proof in [48] were correct, according to their argument, we would have another optimal solution to this problem with $e_1 = 1$. However when we add the constraint $e_1 = 1$ and solve the same problem again, we see that the optimal solution becomes 1.8. This is contradicting the argument in [48]. While this example is based on a fictitious contingent claim, it illustrates the difficulty of defining an optimal "rational" exercise policy. These difficulties are also discussed in [17]. In this example, it appears that the buyer could exercise early at node 1, and take away 9 units since there is a possibility of not getting anything should the process end at node 6. However, such an early exercise is not optimal as the example shows. Such examples (one can find others that are similar) remain difficult to construct, but they clearly demonstrate the gap in the proof of [48].

Returning to the consequences of Theorem 6, this result shows that one can always find a feasible solution to AP1 that gives the optimal value of the relaxed problem AP2. Then, since the optimal value of a problem cannot be better than the optimal value of its relaxation we say that optimal value of AP1 can be found by solving AP2.

One major implication of this result is the passage to a linear programming problem from an NP-hard integer programming problem that is potentially very difficult to solve in practice. Linear programming algorithms with a computational complexity bounded above by a low order polynomial in the number of variables and constraints for computing a solution to ϵ-accuracy are well known and well studied; see Section 6 of [3]. For practical computation, the problem AP2 has $|\mathcal{N}|(J+2)+1$ variables and $|\mathcal{N}|+2|\mathcal{N}_T|$ constraints in addition to $|\mathcal{N}|$ non-negativity constraints. In practice, the state-of-the-art linear programming solvers can easily handle instances where the cardinality of $\mathcal{N}$ is $22,200$ and the cardinality of $\mathcal{N}_T$ is $20,000$ [49].

A second implication is that one can use duality to get expressions for the buyer's price of the ACC in terms of martingale measures and stopping times as pointed out in the introduction. These aforementioned two results are given in [48]. Here we re-iterate the second major implication in detail, for the sake of completeness. For simplicity, we assume w.l.o.g. that $S_n^0 = 1, \forall n = 1, \ldots, T$. We assume an interest-free environment. However, the more general case is easy to implement using the discounted price process of [40]. We will need the buyer's price of a ECC in order to find that of an ACC. The buyer's price of an ECC is derived in [40]. We will briefly show the derivation here. Under the assumption of an interest-free environment, the buyer's problem for an ECC with payoffs F_n is

$$\begin{aligned} \max \quad & V \\ \text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 - V \\ & S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n, \ \forall\, n \in \mathcal{N}_t, 1 \le t \le T \\ & S_n \cdot \theta_n \ge 0, \ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

The dual problem of this program is

$$\begin{aligned} \min \quad & \sum_{n\in\mathcal{N}} y_n F_n \\ \text{s.t.} \quad & y_0 = 1 \\ & \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n,\ \forall\, n \in \mathcal{N}\backslash\mathcal{N}_T. \\ & y_n \geq 0,\ \forall\, n \in \mathcal{N}_T. \end{aligned}$$

Then, the buyer's price of an ECC can be expressed as

$$\min_{Q\in\tilde{\mathcal{Q}}} \sum_{n\geq 0} q_n F_n \tag{3.2}$$

where $\tilde{\mathcal{Q}}$ denotes the closure of the set of all martingale measures equivalent to P, i.e., the set

$$\tilde{\mathcal{Q}} = \{q \mid q_0 = 1,\, q_n S_n = \sum_{m\in\mathcal{C}(n)} q_m S_m,\, \forall n \in \mathcal{N}\backslash\mathcal{N}_T;\, 0 \leq q_n,\, \forall n \in \mathcal{N}_T\}.$$

Define the sets

$$E = \{e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \in \{0,1\}\ P\text{-a.s.}\},$$

$$\tilde{E} = \{e \mid e \text{ is } (\mathcal{F}_t)_{t=0}^T\text{-adapted, } \sum_{t=0}^T e_t \leq 1 \text{ and } e_t \geq 0\ P\text{-a.s.}\}.$$

One common way to describe exercise strategies of ACCs is by stopping times. These are functions $\tau : \Omega \to \{0,\ldots,T\} \cup \{+\infty\}$ such that $\{\omega \in \Omega \mid \tau(\omega) = t\} \in \mathcal{F}_t$, for each $t = 0,\ldots,T$. The relation $e_t = 1 \Leftrightarrow \tau = t$ defines a one-to-one correspondence between stopping times and decision processes $e \in E$. The set of stopping times will be denoted by $\mathcal{T}$. The set $\tilde{E}$ corresponds to the set of randomized stopping times discussed extensively in [17].

Theorem 7. *([48])If there is no arbitrage in the market price process, the buyer's price for American contingent claim F can be expressed as*

$$\max_{\tau\in\mathcal{T}} \min_{Q\in\tilde{\mathcal{Q}}} \mathbb{E}^Q[F_\tau] = \min_{Q\in\tilde{\mathcal{Q}}} \max_{\tau\in\mathcal{T}} \mathbb{E}^q[F_\tau]. \tag{3.3}$$

Proof. If we set e fixed in AP1 and maximize with respect to θ, we have a European contingent claim with payoffs $F_t e_t$ for $t = 0, 1, \ldots, T$. Then, by (3.2), for the buyer's price of this ECC, we have

$$\min_{Q \in \tilde{\mathcal{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t].$$

Then, maximizing with respect to e, for the buyer's price of the ACC we have

$$\max_{e \in E} \min_{Q \in \tilde{\mathcal{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t].$$

The correspondence between stopping times and the process $e \in E$ implies that the buyer' s price for the ACC can be expressed as the left hand side of equation (3.3) since maximization over $\mathcal{T}$ is equivalent to maximization over E after making the appropriate change in the objective function. By Theorem 6, instead of last expression we can use

$$\max_{e \in \tilde{E}} \min_{Q \in \tilde{\mathcal{Q}}} \mathbb{E}^Q[\sum_{t=0}^{T} F_t e_t]. \tag{3.4}$$

Since $\tilde{E}$ and $\tilde{\mathcal{Q}}$ are bounded convex sets, by Corollary 37.6.1 of [53] we can change the order of max and min without changing the value. Then, for each fixed $Q \in \tilde{\mathcal{Q}}$, the objective in (3.4) is linear in e. So the maximum over $\tilde{E}$ is attained at an extreme point of $\tilde{E}$. We know that the extreme points of $\tilde{E}$ are the elements of the set E. Thus, we reach the expression on the right hand side in (3.3). □

We can extend our result for stocks that pay dividends or interest. We assume that there is no dividend associated with S^0. We have the following corollary (proven here for the first time, to the best of the authors' knowledge).

Corollary 2. *If each security $j = 1, \ldots, J$ pays dividend payments D_n^j in node n, under the assumption of no arbitrage in the market price process, the buyer's price F_b for an American contingent claim F can be expressed as*

$$F_b = \max_{\tau \in \mathcal{T}} \min_{Q \in \tilde{\mathcal{Q}}'} \mathbb{E}^Q[F_\tau] = \min_{Q \in \tilde{\mathcal{Q}}'} \max_{\tau \in \mathcal{T}} \mathbb{E}^q[F_\tau]$$

where

$$\tilde{\mathcal{Q}}' = \{q \mid q_0 = 1,\ q_n S_n = \sum_{m \in \mathcal{C}(n)} q_m(S_m + D_m),\ \forall n \in \mathcal{N} \backslash \mathcal{N}_T;\ 0 \leq q_n,\ \forall n \in \mathcal{N}_T\}.$$

Proof. If dividends are paid, self-financing constraints of AP1 becomes

$$S_n \cdot (\theta_n - \theta_{\pi(n)}) - D_n \cdot \theta_{\pi(n)} = F_n e_n, \; \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T.$$

The rest of the argument, including the proof of Theorem 6 follows as it is in the case of stocks without dividends. □

3.3 Conclusion

In this chapter, we presented an alternative proof of an interesting and important result announced by Pennanen and King [48] on the computation of the buyer's price of an American contingent claim by linear programming instead of 0-1 integer programming. We included a numerical example that helps illustrate some important arguments related to our proof. We also showed that the result is unaffected by dividend payments. While European contingent claim prices were known to be computable using linear programming, the result opens the way to computing the prices of American contingent claims also by linear programming, which allows the numerical solution of very large multi-period hedging problems.

Chapter 4

Integer Programming Models for Pricing American Contingent Claims under Transaction Costs

The purpose of this chapter is to examine, using integer programming, the problem of computing a fair price (in the sense of not allowing arbitrage) for the holder (buyer) of an American contingent claim in a discrete-time finite state incomplete market model where the stock trades incur transaction costs proportional to the magnitude of the trade. Since American contingent claims allow the holder to exercise the claim at any point during its lifetime as opposed to their European counterparts which can only be exercised at maturity, the computation of a fair price also involves the choice of an optimal exercise strategy, which opens the way to modeling with binary variables. King [40] showed the connections between linear programming and modern techniques of contingent claim pricing in mathematical finance in the context of European claims. The main contribution of this chapter is to further the bond between finite dimensional optimization and mathematical finance by adding two integer programming models to the list of finite-dimensional optimization approaches useful for pricing contingent claims in financial markets.

It is well-known that a fair price for the buyer (lower hedging price) of a European contingent claim in frictionless markets can be found by computing the minimum value of the expectation of the discounted option pay-off at maturity with respect to probability measures that make the underlying stock price process a martingale. The fair price to the seller (upper hedging price) is then found by calculating the maximum value of the above expectation over the same set of measures. When the market is complete, i.e., when the martingale measure is unique, the buyer and seller prices coincide. This phenomenon also occurs when the pay-off from the contingent claim at maturity can be perfectly replicated by the existing instruments in the market. These results are the main building blocks of mathematical finance and go back to Harrison and Kreps [28], and Harrison and Pliska [29]. In continuous trading models, the replication argument is at the heart of the celebrated Black-Scholes formula; see Black and Scholes [7] and Merton [45].

Similar expectation representations for American claims have been given also for the first time in Harrison and Kreps [28]. These expressions involve the maximization over a set of stopping times of the minimum of discounted expected pay-off at the point of stopping over all martingale measures for the buyer of the American claim, and the maximization over a set of stopping times of the maximum of discounted expected pay-off at the point of stopping over all martingale measures for the seller of the American claim. No arbitrage pricing of American claims was first studied by Bensoussan [4] and Karatzas [37] for complete markets in continuous time. A good reference for continuous time pricing of American contingent claims is Detemple [22]; see also the survey by Myeni [46]. The book by Föllmer and Schied [26] contains a thorough discussion of pricing and hedging American claims in discrete time but infinite state space setting. A derivation of these formulae in a discrete-time, finite state probability context can be found in Chalasani and Jha [17] and King [40].

In the presence of transaction costs proportional to the magnitude of the stock trades it is usually the case that perfect replication is impossible, and therefore the markets become incomplete. Furthermore, it was shown by Soner *et al.* [60] and Levental and Skorohod [43] that for a European call option written on a stock in continuously trading markets the seller's price is equal to the initial stock price, and the hedging strategy is a simple buy-and-hold strategy. However, in discrete-time trading under proportional transaction costs hedging strategies that are non-trivial can be found. The papers by Jouini and Kallal [33], Cvitanic and Karatzas [20], El Karoui and Quenez [24] concentrate on the computation of the no-arbitrage prices in continuous time for European claims under transaction costs, while Koehl, Pham and Touzi [41], Jaschke [32] and Ortu [47] obtain similar results in discrete time, and Edirisinghe *et al.* [23] give a dynamic programming algorithm for European option pricing under different forms of trading frictions. Karatzas and Kou [38] study no-arbitrage pricing and hedging of ACCs in continuous time under portfolio constraints, and Buckdahn and Hu [12] consider jump diffusions for the stock price process in a similar context. Davis and Zariphopoulou [21] study utility maximization for pricing American claims. Bouchard and Temam [8] extend and generalize the discrete-time results of Chalasani and Jha for the upper hedging price to general discrete time markets in an infinite state space setting. In a separate line of work, Tokarz and Zastawniak [62] develop efficient dynamic programming algorithms for pricing American options in discrete time under small transaction costs, and Roux and Zastawniak [58] extend previous work by removing the restriction on transactions costs. It is important to note that Roux and Zastawniak [58] allow a revision of portfolio positions before new prices are revealed. This feature of their formulation enables them to work with path independent portfolio and exercise strategies. However, as illustrated and discussed in [23], path independent strategies can be sub-optimal hedging strategies in the presence of transaction costs. Our models in this chapter allow a revision of the portfolio (and exercise) only after new prices are revealed, and are based on path dependent strategies.

In Chalasani and Jha [17], Bouchard and Temam [8] and Pennanen and King [48], the seller price (the upper hedging price) is thoroughly studied. In this chapter, we focus on the lower hedging problem and give a new (to the best of our knowledge) integer programming formulation for computing the lower hedging price, departing from a max-min expression of Chalasani and Jha for the lower hedging price. Then we exhibit a numerical example showing that a linear relaxation might lead to a non-zero duality gap. This result implies that it might be optimal for the holder of several identical ACCs to exercise them partially at different time points. We also prove that for frictionless markets, the linear programming relaxation is exact. Hence, there is no incentive for the holder of ACCs not facing transaction costs to exercise them partially. We also give an alternative, aggregated, formulation which relaxes an assumption of Chalasani and Jha, and has properties similar to those of the former while it has a reduced number of variables. The two formulations are, in general not equivalent unless the market is frictionless. All formulations and results of this chapter are easily extended to allow dividend paying stocks.

4.1 Preliminaries

Throughout this chapter, we refer to the optimal value of an optimization problem P as $opt(P)$. All the notation and properties of the stochastic tree described in section 2.1 will be used in this chapter. Additional notation is defined below.

We denote the set of all nodes except the root by $\mathcal{N}^1$, and the set of all nodes except the root node and the leaf nodes by $\bar{\mathcal{N}}$. In this chapter the set $\mathcal{A}(n)$ denotes the collection of ascendant nodes or the unique path leading to node n (excluding itself) from node 0. In section 2.1 node n was included in $\mathcal{A}(n)$. We also use the notation $t(n)$ to denote the time period that the node n belongs to, $\mathcal{D}(n)$ for all descendants of node n (including node n itself), and $\mathcal{D}(n,t) := \mathcal{D}(n) \cap \mathcal{N}_t$ to mean the period t descendants of node n for $t > t(n)$.

The market consists of a riskless asset (cash account) and a risky security with prices at node n given by the scalar S_n. We assume the cash account appreciates in value by a factor $R \geq 1$ in each period. Transaction costs are modeled as follows: at node n, selling one share of stock the investor gets $S_n(1-\mu)$, and has to disburse $S_n(1+\lambda)$ upon acquisition of one share of stock. Our choice of two instruments is by no means a limitation of our models, and all the development in this chapter can be re-iterated for a financial market with several risky securities and a claim with pay-off contingent on the values of several securities.

All the information given for an ACC in section 3.1 and for stopping times in section 3.2 is also valid in this chapter. We also need the following definition:

Definition 5. *For any probability measure $\mathbb{P}$ and exercise strategy (stopping time) τ, we say that $\mathbb{P}$ is a (λ, μ, τ)-approximate martingale measure, if $\mathbb{P}$-almost surely,*

$$S_t^*(1-\mu) \leq \mathbb{E}^{\mathbb{P}}[S_\tau^*|\mathcal{N}_t] \leq S_t^*(1+\lambda) \ \forall t < \tau \tag{4.1}$$

where S_t^* denotes the discounted stock price $S_t R^{-t}$. We use $\mathcal{P}(\lambda, \mu, \tau)$ to denote the set of all (λ, μ, τ)-approximate martingale measures.

The buyer's objective is to compute the largest amount it can borrow against the ownership of the claim while picking a suitable exercise time for the claim and covering this debt by self-financing portfolio transactions in the financial market (here represented by cash and the risky asset) using the proceeds from the claim at the chosen date of exercise. In other words, the buyer's strategy is to find the maximum amount, x^* say, he/she can borrow (by short selling stock) to acquire the claim and with the remaining cash to initiate a self-financing, adapted portfolio trading strategy and a stopping time (exercise strategy) τ such that at time τ the value of the portfolio and the pay-off from the claim are sufficient to close all short positions to avoid any losses. The buyer has to enforce this strategy over all paths. It is clear (see also Theorem 8.2 [17]) that if the buyer can acquire the claim for a price inferior to x^*, then this constitutes an arbitrage opportunity for the buyer as follows. The buyer still borrows x^*, acquires the claim for a price $p < x^*$, ending up with the difference $x^* - p$ at time 0, follows the optimal self-financing portfolio strategy and the exercise strategy to repay the debt in all states of the world. Since the details are worked out in [17], we direct the reader to section 8 of that reference.

Since for a fixed exercise strategy, the valuation of the claim can be expressed as an expectation using convex duality theory, the following max-min expression for the lower hedging price $h_{low}(\lambda, \mu, F)$ of an ACC F was given in Theorem 12.2

of Chalasani and Jha [17]:

$$h_{low}(\lambda, \mu, F) = \max_{\tau \in \mathcal{T}} \min_{\mathbb{P} \in \mathcal{P}(\lambda,\mu,\tau)} \mathbb{E}^{\mathbb{P}}[F_\tau^*] \tag{4.2}$$

where F_t^* denotes the discounted ACC pay-off $F_t R^{-t}$. This price is finite if and only if the market is arbitrage free in the sense of Chalasani and Jha (see definition on p. 53 of [17] and Theorem 13.1), which we assume to be the case in sections 4.2, 4.3, and 4.4.

In closing this section, we note three assumptions present in [17]: (a) debt must be repaid in cash, (b) no transaction cost is incurred when a portfolio is liquidated to settle a debt, and (c) no new portfolio positions are taken at period T. While not stated explicitly in [17] it is clearly the case that Chalasani and Jha are interested in path dependent portfolio and exercise strategies which we also adopt. The numerical example at the opening of section 4.2 below illustrates the importance of this point.

4.2 The Formulation

Before we go into the derivation of a new mixed-integer programming formulation for computing the lower hedging price, we shall consider a small numerical example. Consider a two period example in Figure 4.1 where we assume for simplicity that the cash account does not generate any interest. The numbers inside the circles are the node numbers. The numbers next to nodes in the tree are the stock prices. The stock price is initially 10 at $t = 0$. It either goes up to 15 or down 7 at $t = 1$ with some probabilities. If it is equal to 15 at $t = 1$, then either it goes up to 18 or down to 14 at $t = 2$. If it is equal to 7 at $t = 1$, then either it goes up to 13 or down to 4 at $t = 2$. This gives a non-recombinant stochastic tree with node 0 as the root, node 1 (up to 15) and node 2 (down to 7) at $t = 1$. At $t = 2$, from node 1, the tree evolves to either node 3 (up to 18 from 15) or to node 4 (down to 14 from 15); from node 2 it evolves to either node 5 (up to 13 from 7) or to node 6 (down to 4 from 7). We assume $\lambda = \mu = 0.01$. We want to calculate the lower hedging price of an American call option with strike price

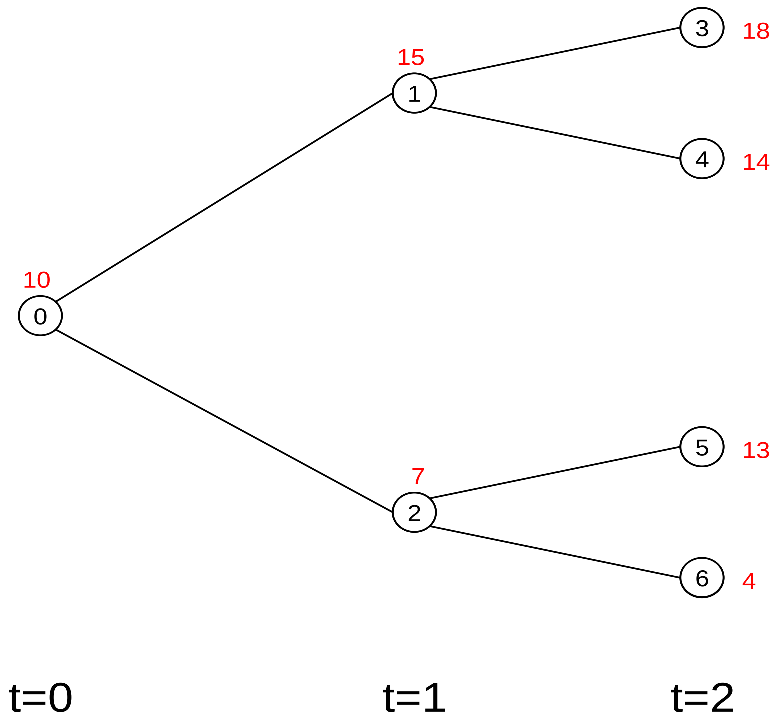

Figure 4.1: A numerical example for $P^1(0.01, 0.01)$.

equal to 10 (an at-the-money American call).

Using expression (4.2) and evaluating different possibilities, one can find that the optimal value accurate to six digits is 2.435125 and attained using the following optimal exercise strategy: exercise if the stock price evolves to node 1 at $t = 1$, exercise at $t = 2$ if the stock price evolves to node 5. Notice that the optimal strategy is a path dependent exercise strategy. In fact, the two path independent exercise strategies that are of interest in this example, e.g., exercise only at $t = 1$ or only at time $t = 2$ are both sub-optimal with objective function values 1.812500 and 2.415296, respectively. This example is contradicting with a well known result in the literature which shows its never optimal to exercise an American option in the absence of dividend payments. However, the example shows that, in the existence of transaction costs this result remains no more valid.

Now, we are ready to derive a formulation for the lower hedging price. First let us deal with the inner minimization for a fixed exercise strategy that is treated as a constant. We use binary variables e_n to denote exercise decisions, i.e., the ACC is exercised at node n if $e_n = 1$, and is not exercised at node n if $e_n = 0$. Since the ACC can only be exercised once over each path (scenario) in the tree, one has to enforce the restriction:

$$\sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T. \tag{4.3}$$

The above is in one-to-one correspondence with the stopping time definitions in section 3.2. We use E to denote the set of all binary valued e_n, $n \in \mathcal{N}$ satisfying (4.3).

Now, for a given set of fixed values e_n^* for e_n, $n \in \mathcal{N}$ respecting the above restriction (4.3), since the optimal exercise strategy is a not necessarily a path independent strategy, we must allow for the possibility that all time periods $1, \ldots, T$ are eligible to be picked as the stopping time τ over a given path as long as there is at most one exercise period over all paths. Therefore, we express the inner minimization problem in (4.2) taking into account all exercise possibilities as:

$$\min_{q_n, n \in \mathcal{N}} \sum_{n \in \mathcal{N} \setminus \{0\}} q_n e_n^* F_n^* + e_0^* F_0$$

subject to the restrictions

$$q_n S_n^*(1-\mu) \leq \sum_{m \in \mathcal{D}(n,t')} q_m S_m^* \leq q_n S_n^*(1+\lambda) \forall n \in \mathcal{N}_t, \forall t < t', \text{and } t' \in [1, \ldots, T],$$

$$q_n = \sum_{m \in \mathcal{C}(n)} q_m \forall n \in \mathcal{N}_t, \forall t \in [0, \ldots, T-1],$$

$$q_0 = 1,$$

$$q_n \geq 0, \forall n \in \mathcal{N}_T.$$

Let $\mathcal{Q}(\lambda, \mu)$ denote the set of probability measures $\mathbb{Q} = \{q_n\}_{n \in \mathcal{N}}$ satisfying the above constraints. Hence, we can rewrite expression (4.2) as:

$$\max_{e \in E} \min_{\mathbb{Q} \in \mathcal{Q}(\lambda,\mu)} \sum_{n \in \mathcal{N} \setminus \{0\}} q_n e_n F_n^* + e_0 F_0 \tag{4.4}$$

Now, attaching Lagrange multipliers b_0 to the last constraint, b_n to each of the second set of constraints for $n \in \mathcal{N} \setminus \{0\}$, and (non-negative) $d_n^{t'}$ and $u_n^{t'}$ to each of the first set of constraints, we obtain the Lagrange function

$$\begin{aligned} L(q_n, b_n, u_n^{t'}, d_n^{t'}) \;=\; & \sum_{n \in \mathcal{N}\setminus\{0\}} q_n e_n^* F_n^* + e_0^* F_0 + \\ & \sum_{t'=1}^{T} \sum_{t<t'} \sum_{n \in \mathcal{N}_t} d_n^{t'} [q_n S_n^* (1-\mu) - \sum_{m \in \mathcal{D}(n,t')} q_m S_m^*] + \\ & \sum_{t'=1}^{T} \sum_{t<t'} \sum_{n \in \mathcal{N}_t} u_n^{t'} [\sum_{m \in \mathcal{D}(n,t')} q_m S_m^* - q_n S_n^* (1+\lambda)] + \\ & \sum_{n \in \mathcal{N}\setminus\{0\}} b_n (q_n - \sum_{m \in \mathcal{C}(n)} q_m) + b_0 (\sum_{m \in \mathcal{C}(0)} q_m - 1). \end{aligned}$$

and are ready to compute the dual problem through

$$\max_{b_n, u_n^{t'}, d_n^{t'}} \min_{q_n} L(q_n, b_n, u_n^{t'}, d_n^{t'}).$$

After rearranging and minimizing the Lagrange function separately over each $q_n \geq 0$ for all $n \in \mathcal{N}$ we obtain the Lagrange dual problem

$$\begin{aligned} \max \quad & -b_0 + \sum_{t=1}^{T} (d_0^t S_0 (1-\mu) - u_0^t S_0 (1+\lambda)) + e_0^* F_0 \\ \text{s.t.} \quad & b_n \leq R b_{\pi(n)} + e_n^* F_n + \sum_{m \in \mathcal{A}(n)} S_n (u_m^{t(n)} - d_m^{t(n)}) + \\ & \qquad \sum_{t=t(n)+1}^{T} S_n ((1-\mu) d_n^t - (1+\lambda) u_n^t), \forall n \in \bar{\mathcal{N}}, \\ & 0 \leq R b_{\pi(n)} + e_n^* F_n + \sum_{m \in \mathcal{A}(n)} S_n (u_m^T - d_m^T), \; \forall \, n \in \mathcal{N}_T \end{aligned}$$

with the non-negativity constraints on all the variables u_n^t, d_n^t, for all $n \in \mathcal{N}$ and all $t \in [0, 1, \ldots, T]$.

The above problem combined with the outer maximization over $e \in E$ yields the problem $P^1(\lambda, \mu)$

$$\begin{aligned}
\max \quad & -b_0 + \sum_{t=1}^{T}(d_0^t S_0(1-\mu) - u_0^t S_0(1+\lambda)) + e_0 F_0 \\
\text{s.t.} \quad & b_n \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(n)} S_n(u_m^{t(n)} - d_m^{t(n)}) + \\
& \qquad \sum_{t=t(n)+1}^{T} S_n((1-\mu)d_n^t - (1+\lambda)u_n^t), \forall n \in \bar{\mathcal{N}}, \\
& 0 \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(n)} S_n(u_m^T - d_m^T), \ \forall\, n \in \mathcal{N}_T \\
& 1 \geq \sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m \ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}
\end{aligned}$$

and the non-negativity constraints on all the variables u_n^t, d_n^t, for all $n \in \mathcal{N}$ and all $t \in [0, 1, \dots, T]$.

Hence, we have proved the following.

Theorem 8. *$h_{low}(\lambda, \mu, F) = opt(P^1(\lambda, \mu))$.*

This problem has a very clear hedging interpretation. We view the non-negative variable u_n^t as a long position in the risky asset acquired at node n for liquidation at time period t. Similarly we let non-negative variable d_n^t denote a short position in the risky asset open at node n to be closed at time period t. We view b_n as the cash position at node n. The first set of constraints express the following balance requirement for each "interior" (non-leaf nodes also excluding the root node) node: cash available from the parent node (magnified by the interest) plus pay-off from the option in case of exercise and proceeds from short sales after accounting for transaction costs, and proceeds from liquidation of earlier long positions (without incurring transaction costs) should be sufficiently large to balance new long positions destined for liquidation in future time points (with transaction costs) and closing of short positions earlier established at no transaction cost. A similar interpretation holds for the leaf nodes where no transaction costs are involved, since no new positions are acquired. These hedging constraints are in one-to-one correspondence with the hedging strategy of the buyer as announced on p. 52–53 of [17]: the buyer starts out by borrowing a certain amount at time 0 to acquire the ACC, and chooses a path dependent exercise strategy from which he/she obtains a certain pay-off with which to close his/her initial debt.

Now, let us return to the numerical example introduced at the beginning of this section. When we solve the problem as a mixed-integer programming problem we obtain the following hedging strategy: short sell 0.502917 shares of stock at time $t = 0$ to be closed (without transaction costs) at time $t = 1$, with the proceeds of this short sale (0.502917×9.9) acquire the American call for 2.435125, and keep the remaining 2.54375 in the cash account. If the stock price moves up at time $t = 1$, exercise the option to collect 5, and using the cash position coming from node 0, close the short position. If the stock moves down, do not exercise, close the short position from node 0, and acquire a new short position in the stock of the order of 1/3 shares to be closed at time $t = 2$. This leaves $1\frac{1}{3}$ in cash. If the stock moves up to 14, exercise the option, and with the total cash close the short position in the stock. If the stock price moves down to 4, just close the short position using the available cash.

Suppose that the stock makes dividend payments D_n at node n. Then model $P^1(\lambda, \mu)$ is modified as follows:

$$
\begin{aligned}
\max \quad & -b_0 + \sum_{t=1}^{T} (d_0^t S_0(1-\mu) - u_0^t S_0(1+\lambda)) + e_0 F_0 \\
\text{s.t.} \quad & b_n \le R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(n)} (S_n + D_n)(u_m^{t(n)} - d_m^{t(n)}) + \\
& \qquad \sum_{t=t(n)+1}^{T} S_n((1-\mu)d_n^t - (1+\lambda)u_n^t), \forall n \in \bar{\mathcal{N}}, \\
& 0 \le R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(n)} (S_n + D_n)(u_m^T - d_m^T), \ \forall\, n \in \mathcal{N}_T \\
& 1 \ge \sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m \ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}
\end{aligned}
$$

and the non-negativity constraints on all the variables u_n^t, d_n^t, for all $n \in \mathcal{N}$ and all $t \in [0, 1, \ldots, T]$. Now, for a given set of fixed values for e_n, $n \in \mathcal{N}$ the inner minimization problem in (4.2) becomes:

$$\min_{q_n, n \in \mathcal{N}} \sum_{n \in \mathcal{N} \setminus \{0\}} q_n e_n F_n^* + e_0 F_0$$

subject to the restrictions

$$q_n S_n^*(1-\mu) \leq \sum_{m \in \mathcal{D}(n,t')} q_m (S_m^* + D_m^*) \leq q_n S_n^*(1+\lambda) \forall n \in \mathcal{N}_t, \forall t < t', \quad \text{and } t' \in [1, \ldots, T],$$

$$q_n = \sum_{m \in \mathcal{C}(n)} q_m \forall n \in \mathcal{N}_t, \forall t \in [0, \ldots, T-1],$$

$$q_0 = 1,$$

$$q_n \geq 0, \forall n \in \mathcal{N}_T.$$

Let $\mathcal{Q}_D(\lambda, \mu)$ denote the set of probability measures $\mathbb{Q}$ satisfying the above constraints. Hence, in the presence of dividend payments we can modify the expression (4.2) for the lower hedging price, now referred to as $h_{low}^d(\lambda, \mu, F)$. Let $\mathcal{P}_D(\lambda, \mu, \tau)$ denote the set of all measures such that $\mathbb{P}$-almost surely we have

$$S_t^*(1-\mu) \leq \mathbb{E}^{\mathbb{P}}[S_\tau^* + D_\tau^* | \mathcal{N}_t] \leq S_t^*(1+\lambda) \ \forall t < \tau. \tag{4.5}$$

Hence, we state the following theorem without proof.

Theorem 9.

$$\begin{aligned} h_{low}^d(\lambda, \mu, F) &= \max_{e \in E} \min_{\mathbb{Q} \in \mathcal{Q}_D(\lambda,\mu)} \sum_{n \in \mathcal{N} \setminus \{0\}} q_n e_n F_n^* + e_0 F_0 \\ &= \max_{\tau \in \mathcal{T}} \min_{\mathbb{P} \in \mathcal{P}_D(\lambda,\mu,\tau)} \mathbb{E}^{\mathbb{P}}[F_\tau^*]. \end{aligned}$$

In the next section we investigate a relaxation of $P^1(\lambda, \mu)$ in connection with randomized stopping times.

4.3 Randomized Stopping Times and Relaxation

Chalasani and Jha [17] (section 9) and Pennanen and King [48] obtained pricing expressions for the seller of an ACC in terms of randomized stopping times. A randomized stopping time [2, 17] is a non-negative adapted process (in our case, node function) Z with the property that on every path ω one has

$$\sum_{t=0}^{T} Z(\omega_t) = 1.$$

That is, the sum of random variables $Z_0, Z_1, \ldots, Z_T$ is equal to 1 on every path. When a randomized stopping time Z is used to describe an exercise strategy, we can think of the value Z_n at node n as the probability of exercise at node n given that node n has been reached.

Stopping times are degenerate randomized stopping times. A stopping time τ corresponds to the randomized stopping time Z^τ whose values are restricted to lie in the set $\{0, 1\}$ and defined as follows for any $\omega \in \Omega$, and $t \in \{0, 1, \ldots, T\}$:

$$Z^\tau(\omega_t) = \begin{cases} 1 & \text{if } \tau(\omega) = t, \\ 0 & \text{otherwise.} \end{cases}$$

The ordinary (or pure) stopping times are extreme points of the convex set of randomized stopping times, or the set $\mathcal{Z}$ of randomized stopping times is the convex hull of the set $\mathcal{T}$ stopping times.

In our setting the set $\tilde{E}$ of randomized stopping times corresponds to the set of e_n such that $e_n \in [0, 1]$ for all $n \in \mathcal{N}$ satisfying the inequalities (4.3). The practical meaning of passing from stopping times to randomized stopping times as allowable exercise strategies is the possibility of different exercise times for a portfolio of identical ACCs. For a single ACC, a randomized stopping time based exercise strategy can be interpreted as the probabilities of exercise at nodes n with a fractional e_n value.

Chalasani and Jha also proposed in Remark 12.3 of [17] a formula for the lower hedging price using randomized stopping times. The use of randomized stopping times in the hedging policy as advocated by Chalasani and Jha [17] implies the following linear programming relaxation $P^2(\lambda,\mu)$ of $P^1(\lambda,\mu)$:

$$\begin{aligned}
\max \quad & -b_0 + \sum_{t=1}^{T}(d_0^t S_0(1-\mu) - u_0^t S_0(1+\lambda)) + e_0 F_0 \\
\text{s.t.} \quad & b_n \le R b_{\pi(n)} + e_n F_n + \sum_{m\in\mathcal{A}(n)} S_n(u_m^{t(n)} - d_m^{t(n)}) + \\
& \qquad \sum_{t=t(n)+1}^{T} S_n((1-\mu)d_n^t - (1+\lambda)u_n^t), \forall n \in \bar{\mathcal{N}}, \\
& 0 \le R b_{\pi(n)} + e_n F_n + \sum_{m\in\mathcal{A}(n)} S_n(u_m^T - d_m^T),\ \forall\, n \in \mathcal{N}_T \\
& 1 \ge \sum_{m\in\mathcal{A}(n)\cup\{n\}} e_m\ \forall\, n \in \mathcal{N}_T \\
& e_n \in [0,1],\ \forall\, n \in \mathcal{N}
\end{aligned}$$

and the non-negativity constraints on all the variables u_n^t, d_n^t, for all $n \in \mathcal{N}$ and all $t \in [0,1,\ldots,T]$. In other words, the relaxation $P^2(\lambda,\mu)$ leads to a new price $h'_{low}(\lambda,\mu,F) := opt(P^2(\lambda,\mu))$. Chalasani and Jha in Remark 12.3 of [17] hinted that a relaxation of $h_{low}(\lambda,\mu,F)$ based on randomized stopping times yields the same value as $h_{low}(\lambda,\mu,F)$. They did not give an explicit formulation nor a proof of this statement. However, in our relaxation using randomized stopping times, one cannot in general expect to find an integer optimal hedge policy by solving the relaxed problem, i.e., $h_{low}(\lambda,\mu,F)$ can be smaller than $h'_{low}(\lambda,\mu,F)$. To see this it suffices to go back to the small example of section 4.2. When we solve this example as a linear program, we obtain an optimal value equal to 2.450000, which is higher than the value we obtained earlier. This higher value is obtained by the following fractional exercise policy: 2/3 exercise at node 1, and 1/3 exercise at node 3 or node 4, and full exercise at node 5 as before.

On the other hand, in all computational experience, the linear programming relaxation is either exact, or leads to very small duality gaps that are easily closed by off-the-shelf state-of-the-art solvers.

It is clear from the example above that it may be beneficial to the holder of a portfolio of identical ACCs to exercise portions of the portfolio at different time points.

4.4 The Frictionless Case

We know from Chapter 3 that when $\lambda = \mu = 0$ (the frictionless case), the linear programming relaxation model AP2 of AP1 yields the same optimal value as AP1. We prove a similar result under this setting here.

Theorem 10. *The optimal value of* $P^2(0,0)$ *is equal to the optimal value of* $P^1(0,0)$*. Furthermore, there exists an optimal solution to* $P^2(0,0)$ *with* $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$.

Proof. We prove directly the second statement which implies the first one. Assume that $P^2(0,0)$ has an optimal solution with the component e^* of the form $e^*_n \notin \{0,1\}$ for some $n \in \mathcal{N}$.

<u>**Case 1**</u>**:** We will first consider the case where e^* has a value not equal to 0 or 1 for the root node of the tree (i.e. $e^*_0 \notin \{0,1\}$). In order to deal with this case, we will form the dual problem of $P^2(0,0)$ which can be formulated as

$$
\begin{aligned}
\min \quad & \sum_{n \in \mathcal{N}_T} z_n \\
\text{s.t.} \quad & \sum_{m \in \mathcal{C}(0)} y_m = 1/R \\
& \sum_{m \in \mathcal{C}(n)} y_m = y_n/R,\ \forall\, n \in \mathcal{N} \setminus \mathcal{N}_T \cup \{0\} \\
& \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n,\ \forall\, n \in \mathcal{N} \setminus \mathcal{N}_T \\
& y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \le 0,\ \forall\, n \in \mathcal{N} \\
& y_n, z_n \ge 0,\ \forall\, n \in \mathcal{N}_T.
\end{aligned}
$$

We have an optimal solution to $P^2(0,0)$ with $e^*_0 \notin \{0,1\}$. Then complementary slackness implies that the fourth constraint of the above program corresponding to the root node should be satisfied as an equality for the corresponding optimal solution of the dual problem (i.e., $y_0 F_0 - \sum_{m \in \mathcal{N}_T} z_m = 0$). Since $y_0 = 1$, we have $F_0 = \sum_{m \in \mathcal{N}_T} z_m$. Thus, the optimal solution to the dual problem is found to be

F_0. Then, by strong duality we know that F_0 is the optimal value of $P^2(0,0)$. One can easily show that a feasible solution to $P^2(0,0)$ is $e_0 = 1$, and all the other variables as zeros with objective value F_0. This is an optimal solution with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$, thus the proof for the first case is complete.

Case 2: Now assume that optimal solution e^* is such that $e_0^* = 0$ and $e_n^* \notin \{0,1\}$ for some $n \in \mathcal{N}$. Let $I = \{i | e_i^* \notin \{0,1\}, i \in \mathcal{N}\}$. Let $G = \{g | g \in I, \mathcal{A}(g) \cap I = \{g\}\}$. Let w be the element with the smallest time index (that is closest to the root) in G. Note that $e_n^* = 0$, $\forall\, n \in \mathcal{A}(w)$ in this case. Also, let k denote the time index for node w.

Claim: One can always find an optimal solution to $P^2(0,0)$ with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w)$.
To prove the claim we will consider the following two linear programs to which we will refer as AR^1 and AR^2 respectively, where we define $\mathcal{N}_T^w = \mathcal{N}_T \cap \mathcal{D}(w)$, and the symbols with * refer to variables that are treated as constants (within the confines of the proof) :

$$\max e_w$$

subject to

$$b_w \leq R b^*_{\pi(w)} + e_w F_w + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{t(w)*} - d_m^{t(w)*}) + \sum_{t=t(w)+1}^{T} S_w (d_w^t - u_w^t),$$

$$b_n \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{t(n)*} - d_m^{t(n)*}) + \sum_{m \in \mathcal{A}(n) \setminus \mathcal{A}(w)} S_n (u_m^{t(n)} - d_m^{t(n)}) + \sum_{t=t(n)+1}^{T} S_n (d_n^t - u_n^t),$$

$$\forall n \in \mathcal{D}(w) \setminus \{w\}$$

$$0 \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{T*} - d_m^{T*}) + \sum_{m \in \mathcal{A}(n) \setminus \mathcal{A}(w)} S_n (u_m^T - d_m^T), \ \forall\, n \in \mathcal{N}_T^w$$

$$1 \geq \sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m \ \forall\, n \in \mathcal{N}_T^w$$

$$e_n \geq 0, \ \forall\, n \in \mathcal{D}(w),$$

$$\min e_w$$

subject to

$$b_w \leq R b^*_{\pi(w)} + e_w F_w + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{t(w)*} - d_m^{t(w)*}) + \sum_{t=t(w)+1}^{T} S_w (d_w^t - u_w^t),$$

$$b_n \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{t(n)*} - d_m^{t(n)*}) + \sum_{m \in \mathcal{A}(n) \setminus \mathcal{A}(w)} S_n (u_m^{t(n)} - d_m^{t(n)}) + \sum_{t=t(n)+1}^{T} S_n (d_n^t - u_n^t),$$

$$\forall n \in \mathcal{D}(w) \setminus \{w\}$$

$$0 \leq R b_{\pi(n)} + e_n F_n + \sum_{m \in \mathcal{A}(w)} S_n (u_m^{T*} - d_m^{T*}) + \sum_{m \in \mathcal{A}(n) \setminus \mathcal{A}(w)} S_n (u_m^T - d_m^T), \ \forall\, n \in \mathcal{N}_T^w$$

$$1 \geq \sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m \ \forall\, n \in \mathcal{N}_T^w$$

$$e_n \geq 0, \ \forall\, n \in \mathcal{D}(w).$$

For convenience we collect the values of b_n, u_n^t and d_n^t in a vector θ. Let us denote the optimal solution of AR^1 as $\bar{\theta}_{\mathcal{D}(w)}$, $\bar{e}_{\mathcal{D}(w)}$ and the optimal solution of AR^2 as $\tilde{\theta}_{\mathcal{D}(w)}, \tilde{e}_{\mathcal{D}(w)}$. If the optimal value of AR^1 is 1, then we see that $(\bar{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\setminus\mathcal{D}(w)})$, $(\bar{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\setminus\mathcal{D}(w)})$ form another optimal solution of $P^2(0,0)$ with $e_w = 1$. For this optimal solution we have $e_w = 1$ and $e_i = 0$, $\forall\, i \in \mathcal{A}(w)$ (we have also $e_i = 0$, for all $i \in \mathcal{D}(w)\setminus\{w\}$ for this solution). Similarly, if the optimal value of AR^2 is 0, then $(\tilde{\theta}_{\mathcal{D}(w)}, \theta^*_{\mathcal{N}\setminus\mathcal{D}(w)})$, $(\tilde{e}_{\mathcal{D}(w)}, e^*_{\mathcal{N}\setminus\mathcal{D}(w)})$ form another optimal solution of $P^2(0,0)$ with $e_w = 0$. Then, for this optimal solution we have $e_i = 0$, for all $i \in \mathcal{A}(w)$. So, our claim will be proved if we can show that AR^2's having an optimal value greater than 0 implies that the optimal value of AR^1 is 1. To show that we will consider the dual problems of AR^1 and AR^2. The dual problems DAR^1 and DAR^2 of AR^1 and AR^2, respectively, are

$$\begin{aligned}
\min \quad & \sum_{n\in\mathcal{N}_T^w} z_n + y_w\left(Rb^*_{\pi(w)} + S_w \sum_{m\in\mathcal{A}(w)} \sum_{t=t(w)}^{T} (u_m^{t*} - d_m^{t*})\right) \\
\text{s.t.} \quad & R(\sum_{m\in\mathcal{C}(n)} y_m) = y_n,\ \forall\, n \in \mathcal{D}(w) \setminus \mathcal{N}_T^w \\
& \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n,\ \forall\, n \in \mathcal{D}(w) \setminus \mathcal{N}_T^w \\
& -y_w F_w + \sum_{n\in\mathcal{N}_T^w} z_n \geq 1 \\
& y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0,\ \forall\, n \in \mathcal{D}(w) \setminus \{w\} \\
& z_n \geq 0,\ \forall\, n \in \mathcal{N}_T^w, \\
& y_n \geq 0,\ \forall\, n \in \mathcal{D}(w),
\end{aligned}$$

$$
\begin{aligned}
\max \quad & -\sum_{n\in\mathcal{N}_T^w} z_n - y_w\left(Rb^*_{\pi(w)} + S_w \sum_{m\in\mathcal{A}(w)}\sum_{t=t(w)}^{T}(u_m^{t*}-d_m^{t*})\right)\\
\text{s.t.} \quad & R(\sum_{m\in\mathcal{C}(n)} y_m) = y_n,\ \forall\, n\in\mathcal{D}(w)\setminus\mathcal{N}_T^w\\
& \sum_{m\in\mathcal{C}(n)} y_m S_m = y_n S_n,\ \forall\, n\in\mathcal{D}(w)\setminus\mathcal{N}_T^w\\
& -y_w F_w + \sum_{n\in\mathcal{N}_T^w} z_n \geq -1\\
& y_n F_n - \sum_{m\in\mathcal{D}(n)\cap\mathcal{N}_T} z_m \leq 0,\ \forall\, n\in\mathcal{D}(w)\setminus\{w\}\\
& z_n \geq 0,\ \forall\, n\in\mathcal{N}_T^w,\\
& y_n \geq 0,\ \forall\, n\in\mathcal{D}(w).
\end{aligned}
$$

We will denote the optimal value of AR^2 by α, which is equal to the optimal value of DAR^2. We know that $\alpha \leq 1$. Assume that $\alpha > 0$. Then by complementary slackness we know that the third constraint of DAR^2 must be satisfied as an equality at the corresponding optimal solution, since $e_w \neq 0$ at the optimal solution of AR^2. Then at the optimal solution of DAR^2, we have

$$0 > \sum_{n\in\mathcal{N}_T^w} z_n + y_w(Rb^*_{\pi(w)} + S_w\theta^*_{\pi(w)}) \geq -y_w F_w + \sum_{n\in\mathcal{N}_T^w} z_n = -1, \tag{4.6}$$

where we denote by $\theta^*_{\pi(w)}$ the term $\sum_{m\in\mathcal{A}(w)}\sum_{t=t(w)}^{T}(u_m^{t*}-d_m^{t*})$. Then, using the second inequality of (4.6) we have

$$
\begin{aligned}
\sum_{n\in\mathcal{N}_T^w} z_n + y_w(Rb^*_{\pi(w)} + S_w\theta^*_{\pi(w)}) &\geq -y_w F_w + \sum_{n\in\mathcal{N}_T^w} z_n\\
y_w(Rb^*_{\pi(w)} + S_w\cdot\theta^*_{\pi(w)}) &\geq -y_w F_w\\
Rb^*_{\pi(w)} + S_w\cdot\theta^*_{\pi(w)} &\geq -F_w
\end{aligned}
$$

where the last step follows from $y_w \geq 0$. Then, for DAR^1 at any feasible solution we have

$$1 \leq -y_w F_w + \sum_{n\in\mathcal{N}_T^w} z_n \leq y_w(Rb^*_{\pi(w)} + S_w\theta^*_{\pi(w)}) + \sum_{n\in\mathcal{N}_T^w} z_n$$

whence we see that the optimal solution of DAR^1 cannot be less than 1. It is easy to see by AR^1 that optimal value of DAR^1 cannot be greater than 1 either. Hence, we conclude that the optimal value of DAR^1 and therefore that of AR^1, is 1. This completes the proof of our claim.
Using the claim we see that there always exists an optimal solution to $P^2(0,0)$ with $e_w \in \{0,1\}$ and $e_i = 0$ for all $i \in \mathcal{A}(w)$. So, one can eliminate all the nodes having time index k in I by applying the above procedure. Then, proceeding successively with the nodes in $(k+1)^{st}, (k+2)^{nd} \dots (T)^{th}$ time indices one can find an optimal solution for $P^2(0,0)$ with $e_n \in \{0,1\}$, $\forall\, n \in \mathcal{N}$. We note that at each step the cardinality of I might increase, but no nodes with a time index less than or equal to that of the node eliminated at that particular step can appear again in I at the next step. This completes the proof of the theorem. □

The above theorem implies the formula

$$h_{low}(0,0,F) = \max_{Z \in \mathcal{Z}} \min_{\mathbb{Q} \in \mathcal{Q}(0,0)} \mathbb{E}^{\mathbb{Q}}[F_Z^*]. \tag{4.7}$$

Following the same proof technique as in Theorem 4 of [48] we can also interchange the max and the min in the above expression, and replace randomized stopping times with ordinary stopping times as a result of the theorem above. Notice that $\mathcal{Q}(0,0)$ coincides with the set of measures $\mathcal{M}$ that make the stock price process a martingale [17, 40, 48], i.e., the set of $\{q_n\}$, for all $n \in \mathcal{N}$ such that

$$q_n S_n^* = \sum_{m \in \mathcal{C}(n)} S_m^* q_m \forall n \in \mathcal{N}_t, \forall t \in [0, \dots, T-1],$$

$$q_n = \sum_{m \in \mathcal{C}(n)} q_m \forall n \in \mathcal{N}_t, \forall t \in [0, \dots, T-1],$$

$$q_0 = 1,$$

$$q_n \geq 0, \forall n \in \mathcal{N}_T.$$

Hence, in an arbitrage free market we re-obtain the well-known expressions

$$\begin{aligned} h_{low}(0,0,F) &= \max_{Z\in\mathcal{Z}} \min_{\mathbb{Q}\in\mathcal{Q}(0,0)} \mathbb{E}^{\mathbb{Q}}[F_Z^*] \\ &= \max_{Z\in\mathcal{Z}} \min_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[F_Z^*] \\ &= \max_{\tau\in\mathcal{T}} \min_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}^{\mathbb{Q}}[F_\tau^*] \\ &= \min_{\mathbb{Q}\in\mathcal{M}} \max_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}[F_\tau^*] \\ &= \min_{\mathbb{Q}\in\mathcal{M}} \max_{Z\in\mathcal{Z}} \mathbb{E}^{\mathbb{Q}}[F_Z^*]. \end{aligned}$$

We note that the proof of the previous theorem also gives a procedure for constructing an integer optimal hedge policy by solving a series of smaller linear programs. Finally, the theorem remains valid in the presence of dividend payments as can be routinely verified.

4.5 Another Formulation

In an unpublished manuscript [48], Pennanen and King proposed another, more compact (with a reduced number of continuous variables), mixed-integer programming formulation for computing the buyer's price to an American claim in a frictionless market. This is the formulation that we have used in Chapter 3. In the present section we extend their formulation to include proportional transaction costs.

The Pennanen and King formulation uses "position" variables θ_n for the stock as opposed to the "flow" variables u_n^t, d_n^t of $P^3(\lambda,\mu)$. Translating this formulation to our setting we pose the hedging problem of the buyer of ACC as the following problem $P^3(\lambda, \mu)$

$$
\begin{aligned}
\max \quad & -\beta_0 - S_0\theta_0 - S_0\phi_0(\theta_0, \lambda, \mu) + F_0 e_0 \\
\text{s.t.} \quad & F_n e_n = \beta_n - R\beta_{\pi(n)} + S_n(\theta_n - \theta_{\pi(n)}) + \\
& \qquad S_n\phi_n(\theta_n - \theta_{\pi(n)}, \lambda, \mu),\ \forall\, n \in \mathcal{N}^1 \\
& 0 \leq \beta_n + S_n\theta_n, \forall\, n \in \mathcal{N}_T \\
& 1 \geq \sum_{m \in \mathcal{A}(n) \cup \{n\}} e_m,\ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\},\ \forall\, n \in \mathcal{N},
\end{aligned}
$$

where θ_n represents the portfolio position in the stock at node n, β_n represents the cash position at node n, ϕ_n is the transaction cost function:

$$
\phi_n(x, \lambda, \mu) = \begin{cases} \lambda x & \text{if } x \geq 0 \\ -\mu x & \text{otherwise} \end{cases}
$$

and the first set of constraints represent the balance of monetary flow at each node of the tree except the root node, i.e., the self-financing portfolio transactions. The second set of constraints expresses the requirement to finish off with non-zero positions at all leaf nodes. The formulation is consistent with the arbitrage definitions of [48] after the necessary adjustments for transaction costs are made. The buyer price is finite in an arbitrage free market; [48]. Note that this formulation allows to take positions in the final period, and penalizes all changes of portfolio unlike in assumptions (b) and (c) of $P^1(\lambda, \mu)$.

While the above problem involves a nonlinearity, it can be transformed into an equivalent linear integer programming model as in [23]:

$$
\begin{aligned}
\max \quad & -\beta_0 - S_0\theta_0 - S_0(\lambda\zeta_0^+ + \mu\zeta_0^-) + F_0 e_0 \\
\text{s.t.} \quad & F_n e_n = \beta_n - R\beta_{\pi(n)} + S_n(\theta_n - \theta_{\pi(n)}) + \\
& \qquad S_n(\lambda\zeta_n^+ + \mu\zeta_n^-) \forall\, n \in \mathcal{N}^1 \\
& \theta_0 = \zeta_0^+ - \zeta_0^- \\
& \theta_n - \theta_{\pi(n)} = \zeta_n^+ - \zeta_n^-, \ \forall\, n \in \mathcal{N}_t, t \in \{1, \ldots, T\} \\
& 0 \leq \beta_n + S_n\theta_n, \forall\, n \in \mathcal{N}_T \\
& 1 \geq \sum_{m \in \mathcal{A}(n)\cup\{n\}} e_m, \ \forall\, n \in \mathcal{N}_T \\
& \zeta_n^+, \zeta_n^- \geq 0, \ \forall\, n \in \mathcal{N} \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}.
\end{aligned}
$$

The optimal value is the largest amount that a potential buyer can borrow for acquiring a given American contingent claim F. The buyer's strategy is to construct a least costly (adapted) portfolio process under transaction costs to cover his/her debt replicating the proceeds from the contingent claim by self-financing transactions using the market-traded securities in such a way to avoid any terminal losses. The integer variables and related constraints represent the one-time exercise of the American contingent claim as in previous sections. Pennanen and King [48] elaborate on the formulation without transaction costs, and establish that any price lower than the optimal value of $P^3(0,0)$ leads to an arbitrage. Dividend payments can be accommodated by subtracting the term $\theta_{\pi(n)}D_n$ from the right hand side of the first set of constraints.

The two formulations $P^1(\lambda,\mu)$ and $P^3(\lambda,\mu)$ are neither identical nor equivalent. To see this, it suffices to observe that model $P^3(\lambda,\mu)$ does not respect assumptions (b) and (c) of model $P^1(\lambda,\mu)$, namely that no transaction cost is involved in liquidating a position to settle a debt and no positions are taken at the leaf nodes. With model $P^3(\lambda,\mu)$, all changes in the stock positions are penalized through transaction costs. Solving the same valuation example as in section 4.2 using $P^3(\lambda,\mu)$ we obtain a buyer's price of 2.416151 which is smaller than the price 2.435125 we obtained using $P^1(\lambda,\mu)$. Since $P^3(\lambda,\mu)$ removes assumption (b) it leads to bigger losses in transaction fees, and renders the same American call option less valuable. Note that $P^3(\lambda,\mu)$ in its linearized form, has

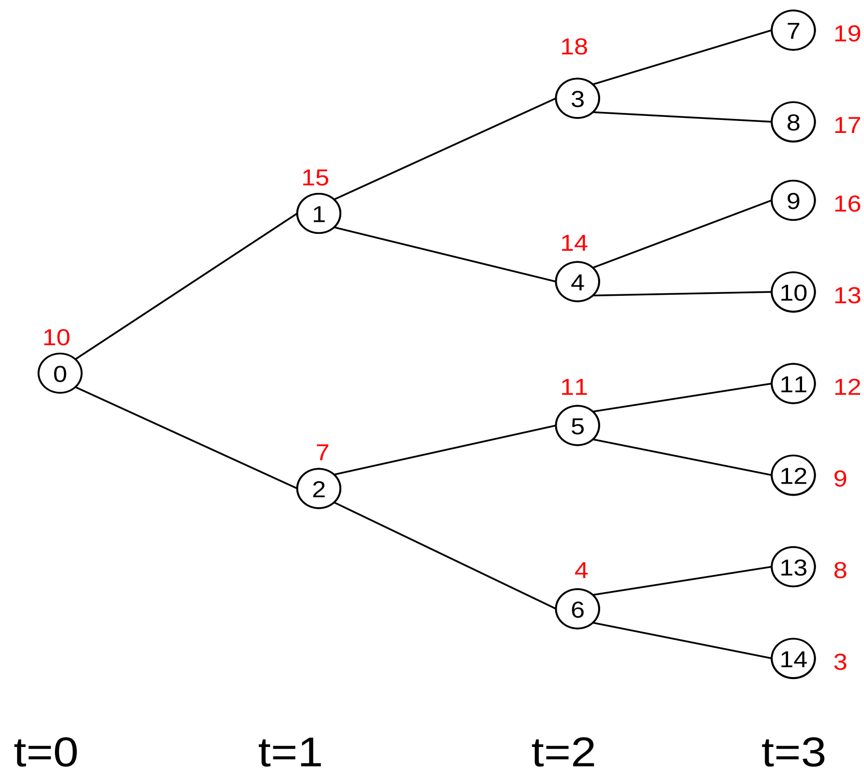

Figure 4.2: A numerical example for $P^3(0.01, 0.01)$.

four variables per node, as opposed to $P^1(\lambda, \mu)$ where each node n contributes $1 + 2(T - t(n))$ continuous variables to the total. However, model $P^3(\lambda, \mu)$ has an additional set of $|\mathcal{N}_T| + |\mathcal{N}|$ constraints.

It is a legitimate question to ask whether the linear programming relaxation $P^4(\lambda, \mu)$ of $P^3(\lambda, \mu)$ is exact in the sense of resulting in the same price as $P^3(\lambda, \mu)$. The answer is negative. The numerical example for the financial market with zero interest rate and stock prices evolving as in Figure 4.2 (the numbers on top of the nodes are the stock prices and the numbers inside the nodes are the node numbers) for an American call option with strike equal to 10 gives a buyer's price of 2.118810 while the LP relaxation gives a fractional optimal solution with value 2.142805.

Another question of interest is the nature of the relationship between $P^3(0,0)$ and $P^1(0,0)$. These models are in fact equivalent, although not identical.

Theorem 11. $\mathit{opt}(P^1(0,0)) = \mathit{opt}(P^2(0,0)) = \mathit{opt}(P^3(0,0)) = \mathit{opt}(P^4(0,0))$.

Proof. From Theorem 6, we know that $\mathit{opt}(P^3(0,0)) = \mathit{opt}(P^4(0,0))$. We also know from Theorem 10 of this chapter that $\mathit{opt}(P^1(0,0)) = \mathit{opt}(P^2(0,0))$. However, after some evident simplifications the linear programming dual of $P^4(0,0)$ is the problem

$$\begin{aligned} \min \quad & \sum_{n \in \mathcal{N}_T} z_n \\ \text{s.t.} \quad & \sum_{m \in \mathcal{C}(0)} y_m = 1/R \\ & R(\sum_{m \in \mathcal{C}(n)} y_m) = y_n, \ \forall\, n \in \mathcal{N} \setminus \mathcal{N}_T \cup \{0\} \\ & \sum_{m \in \mathcal{C}(n)} y_m S_m = y_n S_n, \ \forall\, n \in \mathcal{N} \setminus \mathcal{N}_T \\ & y_n F_n - \sum_{m \in \mathcal{D}(n) \cap \mathcal{N}_T} z_m \leq 0, \ \forall\, n \in \mathcal{N} \\ & y_n, z_n \geq 0, \ \forall\, n \in \mathcal{N}_T, \end{aligned}$$

which is exactly the dual of $P^2(0,0)$. □

In closing the section, we note that the observation $\mathit{opt}(P^3(0,0)) = \mathit{opt}(P^4(0,0))$ was first proposed in [48] and proved in Chapter 3. It was also communicated to us [63] that the algorithms of Roux and Zastawniak [58] yield essentially a similar conclusion for the frictionless case, namely that the frictionless case is computationally "easier" although this is not stated explicitly in their paper. The reader is reminded, however, that they are using path independent strategies.

4.6 Conclusion

In this chapter, departing from a formula in [17] for the lower hedging price of an ACC, we developed an integer programming formulation for computing the price in question as well as an optimal hedge policy for the buyer of the ACC in finite state discrete time markets with transaction costs. The formulation has a linear relaxation which fails to be exact, but which is, at least in our experiments, very close to the integer optimal value. The linear relaxation turns out to be exact in the absence of transaction costs. We also proposed another formulation which relaxes an assumption of [17]. The second formulation has similar properties. A common feature that emerges from these formulations is that in the presence of proportional transaction costs, the holder of a portfolio of identical ACCs might have an incentive to exercise partially his/her claims at different time points whereas this incentive disappears in frictionless markets.

Chapter 5

Conclusion

In this chapter we firstly summarize our findings in our thesis. Then we show counterexamples for some extension of our work. Finally, we point possible further research directions.

5.1 Concluding Remarks

In this thesis we studied the problem of pricing European and American type contingent claims in a multi-period discrete-time, finite probability space framework.

In the second chapter, we studied the problem of pricing European contingent claims under no λ gain-loss ratio opportunity condition. This condition is more restricted than the classical no-arbitrage condition in the sense that it eliminates a greater set of portfolio strategies from the market. We analyzed the resulting optimization problems using linear programming duality and obtained results based on martingales. We showed that the pricing bounds obtained from our analysis are tighter than the no-arbitrage pricing bounds. This result, in line with the Bernardo and Ledoit [5] single period results, was also obtained for a multi-period model in the computationally more tractable linear programming environment. We derived a program in order to obtain the lowest level of the risk aversion parameter for which the option pricing problems would yield a legitimate pricing interval. Besides, we showed that for a limiting value of risk aversion parameter that can be computed easily, a unique price for a contingent claim in incomplete markets may be found although this is not guaranteed. We also extended our results to markets with transaction costs.

In the third chapter we presented an alternative proof of an interesting and important result announced by Pennanen and King [48] on the computation of the buyer's price of an American contingent claim by linear programming instead of 0-1 integer programming. We included a numerical example that helps illustrate some important arguments related to our proof. We obtained the martingale result for the buyer's price of the American option. We also showed that the result is unaffected by dividend payments.

In the fourth chapter, departing from a formula in [17] for the lower hedging price of an ACC, we developed an integer programming formulation for computing the price in question as well as an optimal hedge policy for the buyer of the ACC in finite state discrete time markets with transaction costs. The formulation has a linear relaxation which fails to be exact, but which is, at least in our experiments, very close to the integer optimal value. The linear relaxation turns out to be exact in the absence of transaction costs. We also proposed another formulation which relaxes an assumption of [17]. The second formulation has similar properties. A common feature that emerges from these formulations is that in the presence of proportional transaction costs, the holder of a portfolio of identical ACCs might have an incentive to exercise partially his/her claims at different time points whereas this incentive disappears in frictionless markets.

5.2 Counterexamples

In this section, we will exhibit some counterexamples for some future research directions which are based on our findings in the previous chapters of this thesis.

5.2.1 Expected Gain-Loss Pricing and Hedging of American Contingent Claims

One of the most straightforward research directions to go under the light of our findings in this thesis is combining Chapter 2 and Chapter 3 in order to have an improved result for the problem of pricing ACCs in incomplete markets. In Chapter 2 we consider the problem of pricing an ECC under the no λ gain-loss ratio opportunity condition. In Chapter 3 we deal with the problem of finding the lower bound of the pricing interval of an ACC in an arbitrage-free market. As we claim in Chapter 2, the λ gain-loss ratio opportunity condition is more restrictive than the arbitrage condition, hence using it as a basis for the pricing problem of an ECC, one can obtain a tighter pricing interval for the claim that is to be priced. Actually this is also true for the ACC pricing problem.

The mixed-integer programming formulation that should be used in order to find the buyer's price of an ACC under no λ gain-loss ratio opportunity condition is as follows:

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\
& S_n \cdot \theta_n - x_n^+ + x_n^- = 0, \ \forall\, n \in \mathcal{N}_T, \\
& \sum_{n \in \mathcal{N}_T} p_n x_n^+ - \lambda \sum_{n \in \mathcal{N}_T} p_n x_n^- \geq 0 \\
& \sum_{m \in \mathcal{A}(n)} e_m \leq 1, \ \forall\, n \in \mathcal{N}_T \\
& x_n^+, x_n^- \geq 0, \ \forall\, n \in \mathcal{N}_T, \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}.
\end{aligned}
$$

Note that we use the notation of Chapter 3 in this formulation. The main question to investigate here is (as it was for the problem in Chapter 3) whether the problem obtained by relaxing the variables e_n $n \in \mathcal{N}$. of this mixed-integer program is equivalent (i.e. both problems have the same optimal value) to itself or not. Because, if the relaxation problem, which is a linear programming problem, is equivalent to the original problem, we can obtain similar results to those we have obtained in Chapter 3 for this new setting. However our studies resulted with a counterexample to this idea which shows that the original and the relaxation problems are not equivalent. Let us consider the example in Figure 5.1. In this example there is only one bond and the underlying asset in addition to the ACC to be priced. The price of the bond is equal to 1 at each node. The numbers inside each node in the tree represents the price of the stock at that node. For the non-terminal nodes, the numbers above each node represents the payoff of the ACC (if exercised) at that node. For the terminal nodes, the first number next to the node represents the payoff of the ACC and the second number represents the probability associated with that node.

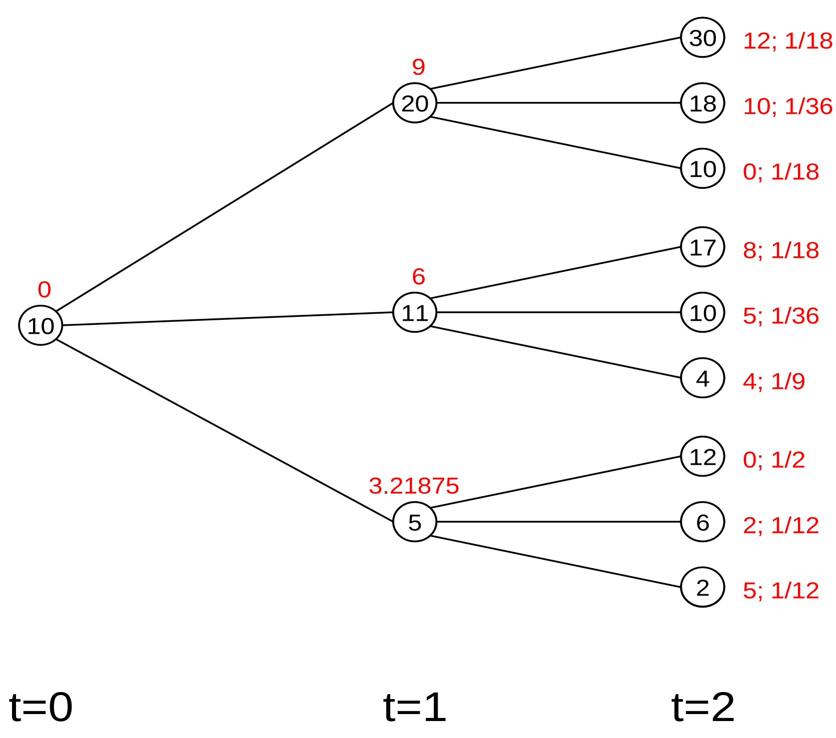

Figure 5.1: A counterexample for the problem of pricing ACCs under no λ gain-loss ratio opportunity condition

If we solve the above optimization model by using the parameter values in

Figure 5.1 and using $\lambda = 17$ we obtain an optimal result of 5.176. Besides, if we solve the linear relaxation of the above optimization problem using the same parameters, we obtain an optimal solution of 5.179. This proves that two optimization problems are not equivalent. Hence, we cannot progress in the manner of Chapter 3 under this setting. However, we discuss a different approach for this case in the next section as a possible future research direction.

5.2.2 Pricing of ACCs with Multiple Exercise Rights

In this thesis we consider the problem of pricing American and European type contingent claims. However there are many types of contingent claims which are traded in the market. Some of these contingent claims give multiple exercise opportunities to the holder of the claim. Swing options, which are mostly used in energy markets are an example of such type of contingent claims. As an introductory step to the pricing problem of these contingent claims we have considered the problem of pricing an ACC with multiple exercise rights under no-arbitrage condition. If the holder of the contingent claim has k (note that $k \leq T + 1$ where T denotes the last period) number of exercise rights until the maturity of an ACC, the buyer's pricing problem becomes the following mixed-integer programming problem:

$$
\begin{aligned}
\max \quad & V \\
\text{s.t.} \quad & S_0 \cdot \theta_0 = F_0 e_0 - V \\
& S_n \cdot (\theta_n - \theta_{\pi(n)}) = F_n e_n, \ \forall\, n \in \mathcal{N}_t, 1 \leq t \leq T \\
& S_n \cdot \theta_n \geq 0, \ \forall\, n \in \mathcal{N}_T \\
& \sum_{m \in \mathcal{A}(n)} e_m \leq k, \ \forall\, n \in \mathcal{N}_T \\
& e_n \in \{0, 1\}, \ \forall\, n \in \mathcal{N}.
\end{aligned}
$$

We use the notation of Chapter 3 in this formulation. We consider the linear relaxation of this problem which is obtained by just removing the last constraint and adding $0 \leq e_n \leq 1; \forall\ n \in \mathcal{N}$ instead of that. We have expected that these two problems would be equivalent. If this was the case we could use our results as a basis for our future studies in swing options. However, the counterexample represented by Figure 5.2 shows that two problems do not necessarily have the same optimal value. In this example there is only one bond and the underlying asset in addition to the ACC to be priced. There are 2 exercise rights for the owner of this ACC. The price of the bond is equal to 1 at each node. The numbers inside each node in the tree represents the price of the stock at that node. The numbers next to the nodes represent the payoff of the ACC (if exercised) at that node.

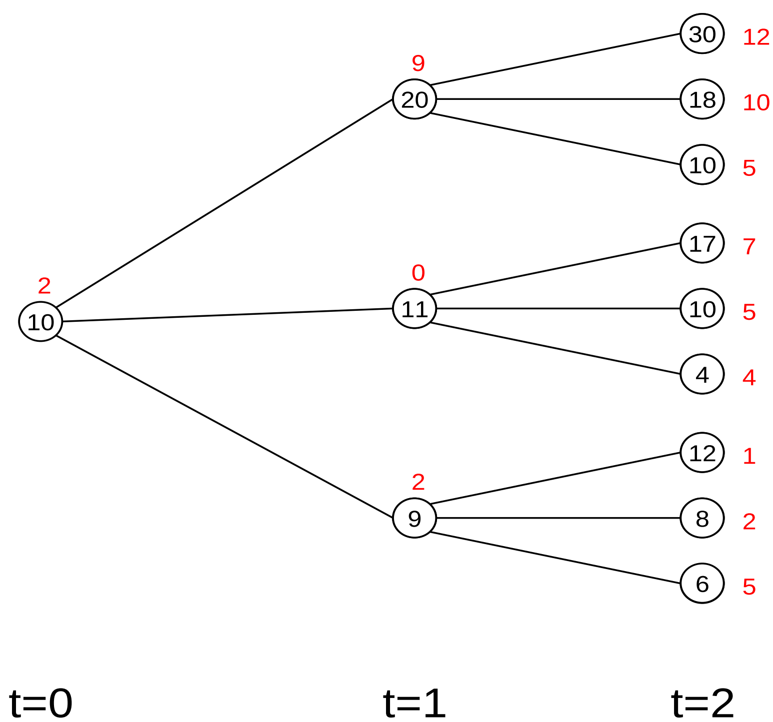

Figure 5.2: A counterexample for the problem of pricing ACCs with multiple exercise rights

The optimal value of the original mixed-integer programming problem is 4.636 and of the linear relaxation problem is 4.882 with the parameter values shown in

Figure 5.2. This proves that the two optimization problems are not equivalent. Therefore, we need to develop the problem in a different direction as it is discussed in the next section.

5.3 Future Research Directions

There are still many aspects to be examined regarding the pricing and hedging problems for ACCs. Under the light of the counterexamples of the previous section we can say that tackling with the difficulties of integer programming seems inevitable for the progress of the research in the area. As a future research, we can examine the problems of pricing ACCs under no λ gain-loss ratio opportunity condition and pricing ACCs with multiple exercise rights in detail, in order to obtain efficient cuts and solution algorithms for the mixed-integer programming problems. Determining the complexity of the problems is another issue that we can examine.

There is another type of contingent claim introduced by Kifer [39] which is called a game (Israeli) option. This contingent claim resembles an ACC. The holder of the claim has the right to exercise and get the payoff of the claim whenever he wants until the maturity of the claim. But for the game options, the writer has the right to terminate the contract at any time until maturity of the claim, whence he pays the payoff of the claim in addition to a penalty cost. They are called game options because the conditions of the contract looks like a game between the buyer and the seller. Kifer [39] examines the problem in both discrete and continuous time settings under no-arbitrage condition however they use one stock and one bond in their model. We can examine the problem in our stochastic scenario tree setting in order to determine the pricing interval for the claim. The first step would be constructing the mixed-integer programming model. Then, we would again work on the relaxations in order to prove that solving a linear programming problem is sufficient to determine the pricing interval. Failure of this step would lead us to the search for efficient cuts and solution algorithms for the problem.

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