SMOOTHNESS OF THE GREEN FUNCTION FOR SOME SPECIAL COMPACT SETS

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ABSTRACT

SMOOTHNESS OF THE GREEN FUNCTION FOR SOME SPECIAL COMPACT SETS

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Smoothness of the Green functions for some special compact sets, which are sequences of closed intervals with certain parameters, is described in terms of the function $\varphi(\delta) = \frac{1}{\log \frac{1}{\delta}}$ that gives the logarithmic measure of sets. As a tool, we use the so-called nearly Chebyshev polynomials and Lagrange interpolation. Moreover, some concepts of potential theory are explained with illustrative examples.

Keywords: Smoothness of the Green Function, Potential Theory.

ÖZET

BAZI ÖZEL KOMPAKT KÜMELER İÇİN GREEN FONKSİYONUNUN PÜRÜZSÜZLÜĞÜ

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Belirli parametrelere sahip kapalı aralıkların dizisi olarak tanımlanan bazı özel kompakt kümeler için Green fonksiyonlarının pürüzsüzlüğü, $\varphi(\delta) = \frac{1}{\log \frac{1}{\delta}}$ fonksiyonu aracılığıyla betimledik, öyle ki bu fonksiyon kümelerin logaritmik sığasını verir. Yöntem olarak, yaklaşık Chebyshev dediğimiz polinomları ve Lagrange interpolasyonunu kullandık. Ayrıca, potansiyel teorisinin bazı kavramlarını örneklerle açıkladık.

Anahtar sözcükler: Green Fonksiyonunun Pürüzsüzlüğü, Potansiyel Teorisi.

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Contents

Chapter 1

Introduction

In the statement of Newton's laws, the only forces considered were between two material points. These forces are proportional to $m_1 m_2$ and inversely proportional to d^2, where m_1 and m_2 are masses of the point materials and d is the distance between these two particles. After Newton's achievements, Lagrange found a field of gravitational forces that is called a potential field now and introduced a potential function. At present, the achievements of Newton's and Lagrange's works are included in classical mechanics courses.

Later, Gauss discovered the method of potentials which can be applied not only to solve problems in the gravitation theory, but also to solve many problems in mathematical physics including electrostatics and magnetism. Hence, potentials were considered not only for the physical problems that concerns the attraction between positive masses, but also for problems with masses with arbitrary sign. The principal boundary value problems were defined, such as the Dirichlet problem, the electrostatic problem of distribution of charges and the Robin problem. In order to solve the problems mentioned above on domains with sufficiently smooth boundaries, some kind of potentials became efficient such as logarithmic potentials and Green potentials. At the end of the 19th century, studies in potential theory about different potentials have gained significant importance.

In the first half of the 20th century, generalization of the principal problems was based on the general concepts of a capacity and potential functions. Modern potential theory, which is related to analytic function theory, harmonic and subharmonic functions, has many applications on approximation theory, complex analysis and modern physics.

There are several ways to introduce the Green function for a given domain. In this thesis, we consider the geometric function theory approach. Our aim is to find lower and upper bounds of the values of the Green function near the boundary of a special compact set K which is a sequence of closed intervals with certain parameters. As a method, we use the so-called nearly Chebyshev polynomials for K. In this way, we find a modulus of continuity of the Green function. It should be noted that we get a nontrivial smoothness of the Green function for K (that means that $g_K(z)$ is not of class $Lip1$ or $Lip\frac{1}{2}$).

In Chapter 2, we introduce some concepts of potential theory as equilibrium measure, the minimal energy, the logarithmic capacity of a set, the transfinite diameter and the Chebyshev constant and some simple illustrative examples are given. Then we show the relations between these concepts. The concepts introduced in this chapter will be used in the following chapters intensively.

In Chapter 3, we give the definition of the Green function. After this, we consider another approach to give the Green function by using the methods of the geometric function theory. Then, we give some results which characterize the continuity and optimal smoothness of the Green function. In the last part of this chapter, we consider model examples for smoothness of the Green function.

Chapters 4, 5 and 6 contain new results. In Chapter 4, we define a special compact set $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ which is a sequence of closed intervals. We consider the extended Chebyshev polynomials $T_{n_k k}$ on any interval I_k, where the degrees n_k are chosen in a such way that the polynomial $P(x) = x \prod_{k=1}^{m-1} \frac{T_{n_k k}(x)}{T_{n_k k}0}$ is "nearly" Chebyshev polynomial on the set K. After that, we give some relations between the n_k's that will be used in the following chapters.

In Chapter 5, by means of Berstein-Walsh theorem, we find a lower bound on the Green function $g_K(z)$ by using the polynomial that was defined in Chapter 4.

Chapter 6 contains an upper bound on $g_K(z)$ for $z = -\delta$ that realizes the modulus of continuity $w(g_k, \delta)$. We use the methods of approximation theory, namely the possibility to represent every polynomial as a Lagrange interpolation polynomial. We show that Lagrange basis polynomials have the desired bound from above.

Chapter 2

Elements of Potential Theory

In this chapter, we consider the basic concepts of potential theory and relations between these concepts.

2.1 Logarithmic Energy and Capacity

Definition 2.1.1 *Let (X, T) be a topological space; let Borel(X) denote Borel σ-algebra on X, i.e. the smallest σ-algebra on X that contains all open sets $U \in T$. Let μ be a measure on $(X, Borel(X))$. Then the support of μ is defined to be the set of all points $x \in X$ for which every open neighborhood of x has a positive measure:*

$$\operatorname{supp}(\mu) = \{x \in X | x \in N_x \in T \Rightarrow \mu(N_x) > 0\}.$$

We say that μ is a probability measure if $\mu(\operatorname{supp}(\mu)) = 1$.

Definition 2.1.2 *Let μ be a Borel measure with compact support on $\mathbb{C}$. Then its logarithmic energy is defined by*

$$I(\mu) = \iint \log \frac{1}{|z-t|} d\mu(t) d\mu(z).$$

A measure μ is said to be of finite logarithmic energy if $I(\mu) < \infty$.

Definition 2.1.3 *Let $K \subseteq \mathbb{C}$ be a compact set, then we set*

$$V(K) = \inf\{I(\mu) | \operatorname{supp}(\mu) \subseteq K, \mu \geq 0, \mu(K) = 1\}. \tag{2.1}$$

That is, the infimum is taken for all probability Borel measures supported on K. In the case of finite infimum above, it is called the equilibrium energy.

Definition 2.1.4 *The logarithmic capacity of compact set K is defined as*

$$\operatorname{Cap}(K) := e^{-V(K)}.$$

We say that a compact set K is polar if $\operatorname{Cap}(K)$*=0.*

Definition 2.1.5 *A property is said to hold quasi-everywhere (q.e.) if it holds outside a set of zero capacity.*

Definition 2.1.6 *The logarithmic potential $p(\mu; z)$ is defined by*

$$p(\mu; z) = \int \log \frac{1}{|z-t|} d\mu(t).$$

Definition 2.1.7 *If X is a metric space and $f : X \to [-\infty, \infty)$, then f is upper semicontinuous if for every c in $[-\infty, \infty)$, the set $\{x \in X : f(x) < c\}$ is an open subset of X. Similarly, $f : X \to (-\infty, \infty]$ is lower semicontinuous if for every c in $(-\infty, \infty]$, the set $\{x \in X : f(x) > c\}$ is open.*

If G is an open subset of $\mathbb{C}$, a function $f : G \to [-\infty, \infty)$ is subharmonic if f is upper semicontinuous and for every closed disc $\bar{B}(a; r)$ contained in G, we have the inequality

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta.$$

A function $f : G \to \mathbb{R} \cup \{+\infty\}$ is superharmonic if $-f$ is subharmonic.

Logarithmic potentials are superharmonic functions on $\mathbb{C}$. Because, for any holomorphic function f, $\log |f(z)|$ is subharmonic. Thus, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z + re^{i\theta}|} d\theta \leq \log \frac{1}{|z-t|}$$

$\forall z, t \in \mathbb{C}$. If we apply Fubini-Tonelli theorem, then we have

$$\int \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{|z + re^{i\theta}|} d\theta d\mu(t) \leq \int \log \frac{1}{|z-t|} d\mu(t).$$

Lower semi-continuity of $p(\omega; z)$ is obvious from the representation

$$p(\omega; z) = \lim_{M \to \infty} \int \min(M, \log \frac{1}{|z-t|}) d\mu(t).$$

Moreover, logarithmic potentials are harmonic in $\mathbb{C} \setminus K$, where K is a compact set. Because $\log \frac{1}{(z-t)}$ is analytic in $\mathbb{C} \setminus K$, and $\log \frac{1}{|z-t|}$ is the real part of $\log \frac{1}{(z-t)}$. Therefore, $\log \frac{1}{|z-t|}$ is harmonic in $\mathbb{C} \setminus K$. Since harmonic functions can be written in the form of Taylor expansion, the primitive of a harmonic function is again a harmonic function. Hence, logarithmic potentials are harmonic in $\mathbb{C} \setminus K$.

Definition 2.1.8 *Let K be a compact subset of $\mathbb{C}$ with positive capacity. Then there exists a unique measure ω_K for which the infimum in (2.1) is attained (see the theorem below). This measure is called the equilibrium measure for K.*

The corresponding equilibrium potential $p(\omega_k; z)$ where ω_k is the equilibrium measure has the following important properties.

Theorem 2.1.9 (Frostman, see e.g. [3]) *Let E be a bounded F_σ Borel subset of $\mathbb{C}$ of positive capacity. Then there exists a unique probability measure ω_K with the following properties:*

- $p(\omega_K; z) \leq \log \frac{1}{\mathrm{Cap}(E)}$ *for $z \in \mathbb{C}$.*
- $p(\omega_K; z) = \log \frac{1}{\mathrm{Cap}(E)}$ *for quasi-everywhere $z \in K$.*

Example 1 *Let $K = \overline{B(0, R)}$, then the equilibrium measure for K is the uniform measure on ∂K, so $d\omega_k = \frac{dl}{2\pi R}$. Since $dl = R d\theta$, we have*

$$p(\omega_k; z) = \int_K \log \frac{1}{|z-t|} d\omega_k = \int_K \log \frac{1}{|z-t|} \frac{d\theta}{2\pi} = \begin{cases} \log \frac{1}{R}, & \text{if } |z| \leq R, \\ \log \frac{1}{|z|} & \text{if } |z| > R. \end{cases}$$

Hence, the potential integral is constant on K. Therefore, by Frostman theorem, ω_k is the equilibrium measure and $p(\omega_k; z)$ is the equilibrium potential. We see also that, $V(K) = \log \frac{1}{R}$. Hence $\mathrm{Cap}(K) = e^{-V(K)} = e^{\log R} = R$.

Example 2 *Let $K = [-1, 1]$ in $\mathbb{C}$ and $\omega(t) = \frac{1}{\pi} \arcsin t$. Let us check if ω gives the equilibrium measure for this compact set K, and let us find* $\mathrm{Cap}(K)$. *Since $\omega(t) = \frac{1}{\pi} \arcsin t$, $d\omega(t) = \frac{dt}{\pi\sqrt{1-t^2}}$. So the logarithmic potential of K is*

$$p(\omega; z) = \int_K \log \frac{1}{|z-t|} d\omega(t) = \int_{-1}^{1} \log \frac{1}{|z-t|} \frac{dt}{\pi\sqrt{1-t^2}}.$$

Let $t = \cos\tau$, then $dt = -\sin\tau$ and $\sqrt{1-t^2} = \sin\tau$. Hence,

$$p(w; z) = -\frac{1}{\pi} \int_{\pi}^{0} \log|z - \cos\tau| \frac{-\sin\tau dt}{\sin\tau} = -\frac{1}{\pi} \int_0^{\pi} \log|z - \cos\tau| dt.$$

Let $z = \cos\varphi$, $0 \le \varphi \le \pi$. Then we have $|z - cos\tau| = |\sin\frac{\varphi+\tau}{2} \sin\frac{\varphi-\tau}{2}|$. Thus,

$$p(w; z) = -\frac{1}{\pi}\Big(\int_0^{\pi} \log 2 d\tau + \int_0^{\pi} \log|\sin(\frac{\varphi+\tau}{2})|) d\tau + \int_0^{\pi} \log|\sin(\frac{\varphi-\tau}{2})|) d\tau.$$

Let $I_1 = \int_0^{\pi} \log 2 d\tau$, $I_2 = \int_0^{\pi} \log|\sin(\frac{\varphi+\tau}{2})|) d\tau$ and $I_3 = \int_0^{\pi} \log|\sin(\frac{\varphi-\tau}{2})|) d\tau$. Then $I_1 = \pi \log 2$. For I_2; let $x = \frac{\varphi+\tau}{2}$, then

$$\begin{aligned} I_2 = 2\int_{\frac{\varphi}{2}}^{\frac{\varphi}{2}+\frac{\pi}{2}} \log|\sin x| dx =& 2\Big(\int_0^{\frac{\pi}{2}} \log|\sin x| dx + \int_{\frac{\pi}{2}}^{\frac{\varphi}{2}+\frac{\pi}{2}} \log|\sin x| dx \\ & - \int_0^{\frac{\varphi}{2}} \log|\sin x| dx\Big). \end{aligned}$$

For I_3; let $x = \frac{\tau-\varphi}{2}$,(since $|\sin(x)| = |\sin(-x)|$), then

$$\begin{aligned} I_3 = \int_{\frac{-\varphi}{2}}^{\frac{\pi}{2}-\frac{\varphi}{2}} \log|\sin x| dx =& 2\Big(\int_0^{\frac{\pi}{2}} \log|\sin x| dx + \int_{-\frac{\varphi}{2}}^{0} \log|\sin x| dx \\ & - \int_{\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{\pi}{2}} \log|\sin x| dx\Big). \end{aligned}$$

Note that

$$\int_{\frac{\pi}{2}-\frac{\varphi}{2}}^{\frac{\pi}{2}} \log|\sin x| dx = \int_{\frac{\pi}{2}}^{\frac{\varphi}{2}+\frac{\pi}{2}} \log|\sin x| dx$$

and

$$\int_0^{\frac{\varphi}{2}} \log|\sin x|dx = \int_{-\frac{\varphi}{2}}^{0} \log|\sin x|dx$$

because of the functional property of $|\sin x|$. *Additionally,*

$$\int_0^{\frac{\pi}{2}} \log|\sin x|dx = -\frac{\pi}{2}\log 2.$$

Hence;

$$\begin{aligned} p(w;z) &= \frac{-1}{\pi}(I_1 + I_2 + I_3) = \frac{-1}{\pi}(\pi\log 2 + 2(-\frac{\pi}{2}\log 2 - \frac{\pi}{2}\log 2)) \\ &= \frac{-1}{\pi}(-\pi\log 2) = \log 2. \end{aligned}$$

Therefore, by Frostman theorem, since the equilibrium potential is constant on K*, the measure* ω *is the equilibrium measure. Clearly,* $V(K) = \log 2$*. Hence* $\mathrm{Cap}(K) = e^{-V(K)} = \frac{1}{2}$.

In the same way, one can show that $\mathrm{Cap}([a,b]) = \frac{b-a}{4}$.

Example 3 *Let* $K = [-1,1]$ *and let* ϑ *be the uniform measure on* K*, that is* $d\vartheta = \frac{1}{2}dx$*. It is obvious from Frostman theorem that this measure is not the equilibrium measure for* K*, because we found in the previous example that the equilibrium measure for* K *is* $\frac{1}{\pi}\arcsin(t)$ *and it is unique according to Frostman theorem. But let us see this fact by some calculations.*
The logarithmic potential for K *is*

$$p(\vartheta;z) = \int_{-1}^{1} \log\frac{1}{|z-t|}\frac{dt}{2} = -\frac{1}{2}\int_{-1}^{1}\log\frac{1}{|z-t|}dt.$$

Let substitute $z - t = \tau$*, then*

$$p(\vartheta;z) = -\frac{1}{2}\int_{z-1}^{z+1}\log|\tau|d\tau = 1 - \frac{1}{2}[(1+z)\log(1+z) + (1-z)log(1-z)].$$

Here $p(\vartheta;z)$ *is not constant on* K*, because* $p(\vartheta;0) = 1$ *and* $p(\vartheta;1) = p(\vartheta;-1) = 1 - \log 2$*. Hence,* ϑ *is not equilibrium measure for* K *by Frostman theorem.*

2.2 Transfinite Diameter

Let E denote a closed and bounded infinite set of points in the z-plane. For n points $z_1, \ldots, z_n \in E$, let V be the Vandermonde determinant of the numbers $z_1, \ldots, z_n$. So

$$V(z_1, \ldots, z_n) = \begin{vmatrix} 1 & z_1 & {z_1}^2 & \ldots & {z_1}^{n-1} \\ 1 & z_2 & {z_2}^2 & \ldots & {z_2}^{n-1} \\ 1 & . & . & \ldots & . \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ 1 & . & . & \ldots & {z_n}^{n-1} \end{vmatrix} = \prod_{k,l=1;\, k<l}^{n} (z_k - z_l), \quad n \geq 2. \quad (2.2)$$

Let $V_n = V_n(E)$ denote the maximum value of $|V(z_1, \ldots, z_n)|$ as $z_1, \ldots, z_n$ range over all n distinct points of the set E. Here such a maximum exists, since $V(z_1, \ldots, z_n)$ is a continuous function on the compact set E^n, the cartesian product of E with itself n times. The points $z_1, \ldots, z_n$ for which the maximum is attained are called the Fekete points. Now, let us define

$$d_n = {V_n}^{\frac{2}{n(n-1)}} = {V_n}^{\frac{1}{\binom{n}{2}}}.$$

The value of d_n is the geometric mean of the distances between $\binom{n}{2}$ pairs of points formed by this set of n points for which $V(z_1, \ldots, z_n)$ achieves its maximum.

Proposition 2.2.1 *[4] For any natural number $n \geq 2$ and compact set $E \subset \mathbb{C}$, we get*

$$d_{n+1}(E) \leq d_n(E).$$

That is, $d_2(E), d_3(E) \ldots$ is a decreasing sequence.

Proof: Let $k_1, \ldots, k_{n+1}$ denote a system of points of the set E such that

$$|V(k_1, \ldots, k_{n+1})| = V_{n+1}.$$

Since $V(k_1, \ldots, k_{n+1}) = (k_1 - k_2) \cdot (k_1 - k_3) \ldots (k_1 - k_{n+1}) \cdot V(k_2, \ldots, k_{n+1})$, we obtain

$$V_{n+1} \leq |k_1 - k_2||k_1 - k_3| \ldots |k_1 - k_{n+1}| \cdot V_n.$$

Similarly,

$$\begin{aligned} V_{n+1} &\leq |k_2 - k_1||k_2 - k_3| \ldots |k_2 - k_{n+1}| \cdot V_n \\ V_{n+1} &\leq |k_3 - k_1||k_3 - k_2| \ldots |k_3 - k_{n+1}| \cdot V_n \\ &\vdots \\ V_{n+1} &\leq |k_{n+1} - k_1||k_{n+1} - k_1| \ldots |k_{n+1} - k_n| \cdot V_n. \end{aligned}$$

After multiplying these inequalities, we obtain

$${V_{n+1}}^{n+1} \leq {V_{n+1}}^2 \cdot {V_n}^{n+1}.$$

Divide both sides of the last inequality by ${V_{n+1}}^2$ (note that it is positive)

$${V_{n+1}}^{n-1} \leq {V_n}^{n+1}.$$

Now take, $\frac{2}{(n+1)(n-1)n}$ power of both sides, we obtain

$${V_{n+1}}^{\frac{2}{n(n+1)}} \leq {V_n}^{\frac{2}{n(n-1)}}.$$

So, we have

$$d_{n+1} = {V_{n+1}}^{\frac{2}{n(n+1)}} \leq {V_n}^{\frac{2}{n(n-1)}} = d_n.$$

Proposition 2.2.2 *[4] The value d_n does not exceed the diameter of the set E for any $n \in \mathbb{N}$.*

Proof: **Case 1:** $n = 2$

For $n = 2$, $d(E) = \text{Diam}(E)$, because, let z_1 and z_2 be Fekete points, then $V(z_1, z_2) = (z_1 - z_2)$, then clearly $V(z_1, z_2)$ attains its maximum value when $|z_1 - z_2|$ is maximum, i.e, $\text{Diam}(E) = |z_1 - z_2|$.

Case 2: $n > 2$

Let $z_1, \ldots, z_n$ be the Fekete points of the set E. Let $p \in \mathbb{N}$, $l \in N$ and $|z_p - z_l| = \max_{i \neq j \ i,j \leq n}\{|z_i - z_j|\}$ then

$$\begin{aligned} d_n(z_1, \ldots, z_n\ ) &= V_n{}^{\frac{2}{n(n-1)}} \\ &= |z_1 - z_2|^{\frac{2}{n(n-1)}} \cdot |z_1 - z_3|^{\frac{2}{n(n-1)}} \ldots |z_1 - z_n|^{\frac{2}{n(n-1)}} \\ &\cdot |z_2 - z_3|^{\frac{2}{n(n-1)}} \ldots |z_{n-1} - z_n|^{\frac{2}{n(n-1)}} \\ &\leq |z_p - z_l|^{\frac{n(n-1)}{2} \frac{2}{n(n-1)}} \\ &= |z_p - z_l| \\ &\leq \mathrm{Diam}(E). \end{aligned}$$

So by Proposition 2.2.1, we see that d_n approaches a finite limit as $n \to \infty$. This limit is called the **transfinite diameter** of the set E and is denoted by $d = d(E)$.

Corollary 2.2.3 *If E consists of finite number of points, then $d(E) = 0$.*

Theorem 2.2.4 *[3] If K is a compact set, then the transfinite diameter of K equals its logarithmic capacity.*

Corollary 2.2.5 *[4] Let K be a compact set, then* $\mathrm{Cap}(K) = \mathrm{Cap}(\partial K)$.

Proof: The Fekete points lie on ∂K by maximum principle, so we have $d(K) = d(\partial K)$. By Theorem 2.2.4, we have $\mathrm{Cap}(K) = \mathrm{Cap}(\partial K)$.

Corollary 2.2.6 *[4] Logarithmic capacity has the following properties.*

a) *Monotonicity : If $E \subseteq F$ then* $\mathrm{Cap}(E) \leq \mathrm{Cap}(F)$.

b) *Homogeneity : If $z^* = az + b$ maps E onto E^*, then* $\mathrm{Cap}(E) = |a| \, \mathrm{Cap}(E)$.

c) *Contraction property : If $|\gamma(z) - \gamma(z')| \leq |z - z'|$ for $z, z' \in E$ then* $\mathrm{Cap}(\gamma(E)) \leq \mathrm{Cap}(E)$.

Proof: **a)** Let $E \subset F$, $z_1, \ldots, z_n$ be the Fekete points for E and $z'_1, \ldots, z'_n$ be the Fekete points for F. Then

$$V_n(z_1, \ldots, z_n) = \prod_{k,l=1,k<l}^{n} |z_k - z_l| \leq \prod_{k,l=1,k<l}^{n} |z'_k - z'_l| = V_n(z'_1, \ldots, z'_n)$$

If $\frac{2}{n(n-1)}$-th power is taken on both sides;

$$(V_n(z_1, \ldots, z_n))^{\frac{2}{n(n-1)}} \leq (V_n(z'_1, \ldots, z'_n))^{\frac{2}{n(n-1)}} \Rightarrow d_n(z_1, \ldots, z_n) \leq d_n(z'_1, \ldots, z'_n).$$

Letting $n \to \infty$, we have $d(E) \leq d(F)$. By Theorem 2.2.4, $\mathrm{Cap}(E) \leq \mathrm{Cap}(F)$.
b) Let $z_1, \ldots, z_n$ be the Fekete points for E. Then, $z^*_1, \ldots, z^*_n$ are the Fekete points for E^*. Then we have

$$\begin{aligned} V_n(z^*_1, \ldots, z^*_n) = \prod_{k,l=1,k<l}^{n} |z^*_k - z^*_l| &= \prod_{k,l=1,k<l}^{n} |az_k + b - az_l + b| = \prod_{k,l=1,k\,l}^{n} |a||z_k - z_l| \\ &= |a|^{\frac{n(n-1)}{2}} \prod_{k,l=1,k<l} n|z_k - z_l| = |a|^{\frac{n(n-1)}{2}} V_n(z_1, \ldots, z_n). \end{aligned}$$

If $\frac{2}{n(n-1)}$-th power of first and last terms of the equation above is taken, and letting $n \to \infty$, we get

$$d(E^*) = |a| d(E);$$

so, by Theorem 2.2.4, $\mathrm{Cap}(E^*) = |a| \, \mathrm{Cap}(E)$.
c) Let $z_1, \ldots, z_n$ be the Fekete points for E. Then, by following very similar ways in proofs of a) and b), we get $d(\gamma(E)) \leq d(E)$. Then again by Theorem 2.2.4, $\mathrm{Cap}(\gamma(E)) \leq \mathrm{Cap}(E)$.

Example 4 *Let us find the capacity of a closed circle with radius R by using transfinite diameter.*
Let us work on unit circle and let's denote it by D. By symmetry, the Fekete points on ∂D are uniformly distributed, that is the points are equally placed around the unit circle in the shape of a regular n-gon.
Hence, if $z_1, \ldots, z_n$ are the Fekete points for D, then $z_k = e^{\frac{i2\pi(k-1)}{n}}$, for $k = 1, \ldots, n$. Therefore, every Fekete point for unit circle is a root of the equation

$$x^n - 1 = 0.$$

We know that $x^n - 1 = (x-1)(x - e^{\frac{2\pi i}{n}}) \ldots (x - e^{\frac{2\pi i(n-1)}{n}})$.
We also have $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \ldots + x + 1)$. *If we divide both sides by* $(x-1)$, *we get*

$$(x - e^{\frac{2\pi i}{n}}) \ldots (x - e^{\frac{2\pi i(n-1)}{n}}) = (x^{n-1} + x^{n-2} + \ldots + x + 1).$$

Substitute 1 for x, *then we have*

$$|(1 - e^{\frac{2\pi i}{n}})| \ldots |(1 - e^{\frac{2\pi i(n-1)}{n}})| = (1 + 1 + \ldots + 1) = n.$$

Similarly, by the symmetry property of the Fekete points on unit circle, we have

$$|e^{\frac{2\pi i}{n}} - 1||e^{\frac{2\pi i}{n}} - e^{\frac{4\pi i}{n}}| \ldots |e^{\frac{2\pi i}{n}} - e^{\frac{2\pi i(n-1)}{n}}| = n.$$
$$\vdots$$
$$|e^{\frac{2\pi i(n-1)}{n}} - 1| \ldots \ldots |e^{\frac{2\pi i(n-1)}{n}} - e^{\frac{2\pi i(n-2)}{n}}| = n.$$

Hence, $(V_n(z_1, \ldots, z_n))^2 = n^n$. *If we take the* $\frac{1}{n(n-1)}$*-th power of both sides, then we get*

$$d_n(z_1, \ldots, z_n) = n^{\frac{1}{n-1}}.$$

By letting $n \to \infty$, *we have* $d(U) = 1$. *Therefore, the transfinite diameter of unit disc is 1. Now, we apply Corollary 2.2.6 and Theorem 2.2.4. Since the mapping* $z^* = Rz$ *maps unit circle to a circle with radius* R, *then capacity of a circle with radius* R *is* R.

2.3 Chebyshev Polynomials and the Chebyshev Constant

Definition 2.3.1 *The polynomial* $T_n(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ *with the least maximum modulus on a compact subset* K *of* $\mathbb{C}$ *is called the Chebysev polynomial for* K. *(For more information about Chebyshev polynomials, please look at the Appendix.)*

Let $T_n(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ be the Chebyshev polynomial for a compact subset K of $\mathbb{C}$. Let $M_n = ||T_n||_K$.

Now define $\tau_n = (M_n)^{\frac{1}{n}}$.

Lemma 2.3.2 *[4] $\tau_n = (M_n)^{1/n}$, $n = 1, 2, \ldots$ is bounded and converges.*

Definition 2.3.3 *The number τ to which the sequence $\{\tau_n\}$ converges is called the Chebyshev constant of the set K.*

Lemma 2.3.4 *[4] We have $M_n \leq \frac{V_{n+1}}{V_n} \leq (n+1)M_n$ for all $n \in \mathbb{N}$.*

Theorem 2.3.5 *[4] The Chebyshev constant τ of the set K is equal to the transfinite diameter of the set K.*

Proof: By lemma 2.3.4, we have

$$M_n \leq \frac{V_{n+1}}{V_n} \leq (n+1)M_n.$$

It can be written as

$$\tau_n^n \leq \frac{V_{n+1}}{V_n} \leq (n+1)\tau_n^n.$$

So, we have

$$\begin{aligned} \tau_2^2 &\leq \frac{V_3}{V_2} \leq 3\tau_2^2. \\ \tau_3^3 &\leq \frac{V_4}{V_3} \leq 4\tau_3^3. \\ \vdots \quad & \quad \vdots \quad \quad \vdots \\ \tau_n^n &\leq \frac{V_{n+1}}{V_n} \leq (n+1)\tau_n^n. \end{aligned}$$

If we multiply the expressions above consequently, we get

$$(\tau_2^2 \tau_3^3 \ldots \tau_n^n)V_2 \leq V_{n+1} < [(n+1)!](\tau_2^2 \tau_3^3 \ldots \tau_n^n)V_2. \tag{2.3}$$

Now take $\frac{2}{n(n+1)}$-th power of all sides of equation (2.3) we get

$$\begin{aligned}(\tau_2^2\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}}(V_2)^{\frac{2}{n(n+1)}} &\leq d_{n+1}\\ &< [(n+1)!]^{\frac{2}{n(n+1)}}(\tau_2^2\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}}(V_2)^{\frac{2}{n(n+1)}}.\end{aligned}$$

<u>Claim :</u> $(V_2)^{\frac{2}{n(n+1)}} \to 1$ and $[(n+1)!]^{\frac{2}{n(n+1)}} \to 1$ as $n \to \infty$.

Proof: Here V_2 is a finite number, so it is clear that

$$\lim_{n\to\infty}(V_2)^{\frac{2}{n(n+1)}} = (V_2)^0 = 1.$$

For $[(n+1)!]^{\frac{2}{n(n+1)}}$, let $k = \lim_{n\to\infty}[(n+1)!]^{\frac{2}{n(n+1)}}$ then $\ln k = \lim_{n\to\infty}\frac{2}{n(n+1)} \cdot \ln[(n+1)!]$. If we apply L'Hópital's rule, we have

$$\begin{aligned}\lim_{n\to\infty} k &= 2\lim_{n\to\infty}\frac{(\ln[(n+1)!])'}{(n(n+1))'}\\ &= 2\lim_{n\to\infty}\frac{[(n+1)!]'}{(n+1)!(n(n+1))'} \to 0.\end{aligned}$$

so $k = 1$.

Hence, we just need to prove that $(\tau_1\cdot\tau_2^2\cdot\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}} \to \tau$ as $n \to \infty$. Let $k = \lim_{n\to\infty}(\tau_1\cdot\tau_2^2\cdot\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}}$. Then

$$\begin{aligned}\ln k &= \lim_{n\to\infty}\frac{2}{n(n+1)}(\ln(\tau_1\cdot\tau_2^2\cdot\tau_3^3\ldots\tau_n^n))\\ &= \lim_{n\to\infty}\frac{2}{n(n+1)}(\ln\tau_1 + 2\ln\tau_2 + 3\ln\tau_3\ldots + n\ln\tau_n).\end{aligned}$$

We know from calculus that if a sequence of real numbers $a_1,\ldots,a_n$ converges to a, then $\frac{(a_1+\ldots+a_n)}{n}$ also converges to a as $n \to \infty$. Additionally, we know that $\lim_{n\to\infty}\tau_n = \tau$, so $\lim_{n\to\infty}\ln\tau_n = \ln\tau$.

Hence, if $a_1 = \log\tau_1$, $a_2 = \log\tau_2$, $a_3 = \log\tau_2$, $a_4 = \log\tau_3,\ldots,a_{\frac{n(n+1)}{2}} = \log\tau_n$ and $a = \tau$, then we have $\lim_{n\to\infty}\frac{(a_1+\ldots+a_n)}{\frac{n(n+1)}{2}}\frac{(n+1)}{2} = a$, so $\tau = k$. Hence, $(\tau_1\tau_2^2\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}} \to \tau$ as $n \to \infty$ is proven. Hence, we have

$$\begin{aligned}\lim_{n\to\infty}(\tau_2^2\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}}(V_2)^{\frac{2}{n(n+1)}} &\leq \lim_{n\to\infty} d_{n+1}\\ &\leq \lim_{n\to\infty}[(n+1)!]^{\frac{2}{n(n+1)}}(\tau_2^2\tau_3^3\ldots\tau_n^n)^{\frac{2}{n(n+1)}}(V_2)^{\frac{2}{n(n+1)}}.\end{aligned}$$

So $\tau \leq d \leq \tau$, hence $\tau = d$.

Corollary 2.3.6 *For any compact set K, $\tau(K) = d(K) = \mathrm{Cap}(K)$.*

This corollary is a direct result of Theorem 2.3.5 and Theorem 2.2.4.

Example 5 *In Example 2, we found that* $\mathrm{Cap}[-1,1] = \frac{1}{2}$. *Let us find the Chebyshev constant for this interval.*

Note that Chebyshev polynomial of degree n on $[-1,1]$ is

$$T_n(x) = 2^{n-1}\prod_{1}^{n}(x - \xi_j),$$

where $\xi_j, j = 1, \ldots, n$ are zeros of the Chebyshev polynomial. Then, $M_n = ||T_n||_{[-1,1]} = \frac{1}{2^{n-1}}$. Thus,

$$\tau([-1,1]) = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}}\right)^{\frac{1}{n}} = \frac{1}{2}.$$

Hence, this example illustrates the equality of the Capacity and the Chebyshev constant.

Remark 2.3.7 *As it is seen from this chapter, the logarithmic capacity, transfinite diameter and the Chebyshev constant of a nonpolar compact set are the same, but each has some advantages that other concepts do not. For example; the advantage of transfinite diameter over the logarithmic capacity is that transfinite diameter is more geometric. Note that, the definition of the logarithmic capacity of a set is given by measures, while the definition of transfinite diameter is given by distances. Thus, if we know the equilibrium measure of a compact set, it is useful to use the logarithmic capacity. On the other hand, for some compact sets, if their corresponding Chebyshev polynomials are known, in other words, the polynomials on these compact sets which have the least deviation, then using the Chebyshev constant is much more advantageous.*

Chapter 3

The Green Function

3.1 Green Function

Definition 3.1.1 *Let K be a compact subset of $\mathbb{C}$ with positive capacity. If U is the unbounded component of K, then we define the Green function for K with the pole at ∞ as*

$$g_K(z) = g_U(z, \infty) = V(K) - p(\omega_k; z),$$

where $p(\omega_k; z)$ is the equilibrium potential with equilibrium measure ω_k and $V(K)$ is the equilibrium energy .

Here it is obvious that $g_K(z)$ is nonnegative because $p(\omega_k; z)$ is smaller than or equal to the equilibrium energy.

Under this definition of the Green function, we have three very important properties of the Green function.

Proposition 3.1.2 *The Green Function $g_K(z)$ has following properties.*

i) *The function $g_K(z)$ is subharmonic on $\mathbb{C}$ and harmonic on $\mathbb{C} \setminus K$.*

ii) *The function $g_K(z) - \log|z|$ is harmonic in a neighborhood of infinity and remains bounded as z goes to infinity.*

iii) $g_K(z) = 0$ *quasi-everywhere on K.*

Proof: **(i)** We know that $p(\omega_k; z)$ is superharmonic on K and harmonic on $\mathbb{C} \setminus K$. Since $V(K)$ is a constant (we are working with equilibrium measure ω_k), and $-p(\omega_k; z)$ is subharmonic function on K, then $g_K(z)$ is subharmonic function on K. Similarly, $p(\omega_z; z)$ is harmonic on $\mathbb{C} \setminus K$, so $-p(\omega_z; z)$ is harmonic on $\mathbb{C} \setminus K$. Hence, $g_K(z)$ is harmonic on $\mathbb{C} \setminus K$.

(2) Here, we should observe that if ω_k is the equilibrium measure then $p(\omega_k; z) \sim \log \frac{1}{|z|}$ because

$$\int_K \log \frac{1}{|z|} d\omega_k(t) = \log \frac{1}{|z|}, \quad \text{since } \omega_k(K) = 1.$$

$$\text{So,} \qquad p(\omega_k, z) - \log \frac{1}{|z|} = \int_K \log \frac{|z|}{|z-t|} d\mu(t).$$

If $z \to \infty$, then $\log \frac{|z|}{|z-t|} \to 0$ uniformly with respect to $t \in K$. Therefore, $p(\omega_k; z) \sim \log \frac{1}{|z|}$. Moreover,

$$g_K(z) - \log|z| = -p(\omega_k, z) + V(K) - \log|z| \sim V(K).$$

Since K has positive capacity, then $V(K)$ is finite. Thus $g_K(z) - \log|z|$ remains bounded as z goes to infinity and harmonic in the neighborhood of infinity.

The third property directly comes from the definition of the Green function with pole at ∞.

Theorem 3.1.3 *[11] If B is a Borel set such that $\mathbb{C} \setminus B$ is bounded and of positive capacity, then the Green function g_B with properties (i)-(iii) which are defined at previous proposition exist and is uniquely defined.*

Note that, the idea of proof of this theorem comes from the minimum principle. In more details, if g'_B is another function satisfying these properties, then $g_B - g'_B$ is bounded and harmonic on $\mathbb{C}$. Then, the result follows from minimum principle.

Example 6 *In Example 1, for compact set $K = \overline{B(0,R)}$, we found that*

$$p(\omega_k; z) = \int_K \log \frac{1}{|z-t|} d\omega_k = \int_K \log \frac{1}{|z-t|} \frac{d\theta}{2\pi} = \begin{cases} \log \frac{1}{R}, & \text{if } |z| \le R, \\ \log \frac{1}{|z|} & \text{if } |z| > R. \end{cases}$$

Hence, the Green function for K is

$$g_K(z) = \begin{cases} 0, & \text{if } |z| \le R, \\ \log \frac{|z|}{|R|} & \text{if } |z| > R. \end{cases}$$

Definition 3.1.4 *Let K be a compact subset of $\mathbb{C}$. Then the Robin constant for K is defined as $\lim_{z\to\infty} (g_K(z) - \log|z|)$ and is denoted by $\mathrm{Rob}(K)$.*

Second condition of Proposition 3.1.2 gives that $\mathrm{Rob}(K) = V(K)$. Hence, for a compact set K, the capacity is also equal to

$$\mathrm{Cap}(K) = e^{-\mathrm{Rob}(K)}.$$

Definition 3.1.5 (The Green Function with Pole at $\alpha \neq \infty$) *For an open set Ω of $\mathbb{C}_\infty$, a Green Function with pole at $\alpha \neq \infty$ is a function $G : \Omega \times \Omega \to (-\infty, \infty]$ having these properties:*

i) *For each $\alpha \in \Omega$, the function $G(z, \alpha, \Omega) = G_\Omega(\alpha)$ is positive and harmonic.*

ii) *For each $\alpha \in \Omega$, $z \mapsto G_\Omega(\alpha) + \log|z - \alpha|$ is harmonic in a neighborhood of α.*

iii) *$G_\Omega(\alpha)$ is the smallest function from $\Omega \times \Omega$ into $(-\infty, \infty]$ satisfying properties (i) and (ii).*

Theorem 3.1.6 *[3] Let $\Omega_1 \in \mathbb{C}_\infty$, let G be the Green function of Ω_1 and let $\Omega_2 \in \mathbb{C}_\infty$ be another region with pole α. f we have a conformal mapping $T : \Omega_2 \to \Omega_1$, then*

$$H(z, \alpha, \Omega_2) = G(T(z), T(\alpha), \Omega_1),$$

where H is the Green function for Ω_2.

Example 7 *In Example 6, for $K = \overline{B(0,R)}$, we have*

$$g(z,\infty,\mathbb{C}\setminus K) = \log\frac{|z|}{R}.$$

Now, let take $\Omega_1 = \{|z| > 1 : z \in \mathbb{C}\}$ and $\Omega_2 = \mathbb{C}\setminus[-1,1]$, and take the conformal mapping $\psi : \Omega_1 \to \Omega_2$ such that

$$\psi(z) = \frac{1}{2}\Big(z + \frac{1}{z}\Big),$$

which is a continuous mapping. Moreover, $\psi(\infty) = \infty$. Thus, by Theorem 3.1.6 with $w = \psi(z)$

$$g(w,\infty,\Omega_2) = g(z,\infty,\Omega_1) = \log|z| = \log|w + \sqrt{w^2-1}|,$$

where $\sqrt{w^2-1} \geq 0$. Therefore,

$$\mathrm{Rob}([-1,1]) = \lim_{|w|\to\infty}[g(w,\infty,\Omega_2) - \log|w|] = \lim_{|w|\to\infty}\log\Big|1 + \sqrt{1 - \frac{1}{w^2}}\Big| = \log 2.$$

Hence, $\mathrm{Cap}[-1,1] = \frac{1}{2}$.

Example 8 *Let $K = [a,b]$ where a and b are real numbers and $b > a$. Let $\Omega_1 = \mathbb{C}\setminus\overline{B\Big(\frac{a+b}{2},\frac{b-a}{2}\Big)}$, and $\Omega_2 = \mathbb{C}\setminus K$ and take the mapping $\phi : \Omega_1 \to \Omega_2$ such that*

$$\phi(z) = \frac{b-a}{4}\Big(\frac{2z-a-b}{b-a} + \frac{b-a}{2z-a-b}\Big) + \frac{a+b}{2},$$

where $\phi(\infty) = \infty$.

If $w = \phi(z)$, then

$$\begin{aligned} &\frac{b-a}{4}\Big(\frac{2z-a-b}{b-a} + \frac{b-a}{2z-a-b}\Big) + \frac{a+b}{2} \\ &\Rightarrow \frac{4w-2(a+b)}{b-a} = \frac{t^2+(b-a)^2}{t(b-a)} \\ &\Rightarrow t^2 - 2(2w-(a-b))t + (b-a)^2 = 0, \end{aligned}$$

where $t = 2z - a - b$. Hence,

$$\begin{aligned} t &= \frac{2(2w-(a+b)) + 2\sqrt{(2w-(a+b))^2-(b-a)^2}}{2} \\ &\Rightarrow 2z - a - b = (2w-(a+b)) + \sqrt{(2w-(a+b))^2-(b-a)^2} \\ &\Rightarrow z = w + \frac{\sqrt{(2w-(a+b))^2-(b-a)^2}}{2}. \end{aligned}$$

Now, by Example 6 and Theorem 3.1.6, we have

$$\begin{aligned} g(z,\infty,\Omega_1) &= \log\frac{2|z|}{b-a} = \log\left|\frac{2w}{b-a} + \frac{\sqrt{(2w-(a+b))^2-(b-a)^2}}{b-a}\right| \\ &= g(w,\infty,\Omega_2). \end{aligned}$$

Thus,

$$\begin{aligned} \mathrm{Rob}([a,b]) &= \lim_{|w|\to\infty} g(w,\infty,\Omega_2 - \log|w|) \\ &= \lim_{|w|\to\infty} \log\left|\frac{2}{b-a} + \frac{\sqrt{(2-\frac{a+b}{w})^2-(\frac{b-a}{w})^2}}{b-a}\right| \\ &= \log\left|\frac{2}{b-a}+\frac{2}{b-a}\right| = \log\left|\frac{4}{b-a}\right|. \end{aligned}$$

Hence,

$$\mathrm{Cap}(K) = \frac{b-a}{4}.$$

Now for $I=[a,b]$, the Chebyshev polynomial is

$$\tilde{T}_{nI}(x) = 2^{n-1}\prod_{j=1}^{n}\left(\frac{x-\frac{a+b}{2}}{\frac{b-a}{2}} - \xi_j\right) = \frac{2^{2n-1}}{(b-a)^2}\prod_{j=1}^{n}\left(x-\frac{a+b}{2}-\xi_j\frac{b-a}{2}\right).$$

Hence, by the definition of the Chebyshev constant, we get

$$\tau(K) = \lim_{n\to\infty}\left(\frac{(b-a)^n}{2^{2n-1}}\right)^{\frac{1}{n}} = \frac{b-a}{4}.$$

Therefore, for $K=[a,b]$ with $a<b$, we have

$$\tau(K) = \mathrm{Cap}(K) = \frac{b-a}{4}.$$

3.2 Some Additional Properties of Green Fucntion

Theorem 3.2.1 *[3] Let $\{\Omega_n\}$ be a sequence of open sets such that $\Omega_n \subseteq \Omega_{n+1}$ and $\Omega = \bigcup_n \Omega_n$. If G_n is the Green function for Ω_n and G is the Green function for Ω, then for each $\alpha\in\Omega$, $G_n(z,\alpha)\uparrow G(z,\alpha)$ uniformly on compact subsets of $\Omega\setminus\{\alpha\}$.*

Corollary 3.2.2 *[3] Let $\{D_n\}$ be a sequence of open sets such that $D_{n+1} \subseteq D_n$ and let $K = \bigcap_{j=1}^{\infty} \overline{D_j}$, so $\overline{D_j} \downarrow K$. Then*

$$g_K(z) = \lim_{j\to\infty} g_{\overline{D_j}}(z)$$

uniformly on compact set from $\overline{\mathbb{C}} \setminus K$.

Therefore, for any given K, the function $g_K(z)$ can be found as $\lim_j g_{K_j}$ where $K = \bigcap_j K_j$.

Corollary 3.2.3 *[3] If $\{K_n\}$ is a sequence of compact sets such that $K_n \supseteq K_{n+1}$ for all n and $\bigcap_n K_n = K$, the $\mathrm{Cap}(K_n) \to \mathrm{Cap}(K)$.*

Proof: For K_n,

$$\mathrm{Rob}(K_n) = \lim_{|z|\to\infty} (g_{K_n}(z) - \log|z|),$$
$$\mathrm{Cap}(K_n) = e^{-Rob(K_n)}.$$

By the previous corollary, we know that $g_K(z) = \lim_{n\to\infty}(g_{K_n}(z) - \log|z|)$, then $\lim_{n\to\infty} \mathrm{Rob}(K_n) = \mathrm{Rob}(K)$. Thus,

$$\lim_{n\to\infty} \mathrm{Cap}(K_n) = \mathrm{Cap}(K).$$

Example 9 *Let $D_n = B\Big(0, 1 + \frac{1}{n}\Big)$, it is clear that $D_{n+1} \subseteq D_n$. Let $K_n = \overline{D_n}$, then we know that*

$$g_{K_n}(z) = \log\frac{|z|}{1+\frac{1}{n}}.$$

Moreover, $K = \bigcap_{j=1}^{\infty} K_j = \overline{B(0,1)}$. We also know that

$$g_K(z) = \log|z|.$$

Then, let us check the results of previous corollaries for this specific example:

$$\lim_{n\to\infty} g_{K_n}(z) = \lim_{n\to\infty} \log\frac{|z|}{1+\frac{1}{n}} = \log|z| = g_K(z).$$

Additionally,

$$\lim_{n\to\infty} \mathrm{Cap}(K_n) = \lim_{n\to\infty} (1 + \frac{1}{n}) = 1 = \mathrm{Cap}(K).$$

Example 10 *Let* $K_n = \Big[-1 - \frac{1}{n}, 1 + \frac{1}{n}\Big]$, *it is clear that* $K_n \supseteq K_{n+1}$. *Let* $\psi(z)$ *be a mapping such that*

$$\psi : \mathbb{C} \setminus \overline{B(0,1)} \to \mathbb{C} \setminus K_n$$
$$\psi(z) = \Big(\frac{1+\frac{1}{n}}{2}\Big)\Big(z + \frac{1}{z}\Big).$$

If $w = \psi(z)$, *then*

$$\Big(\frac{1+\frac{1}{n}}{2}\Big)\Big(z + \frac{1}{z}\Big) = w \Rightarrow (n+1)z^2 - 2nwz + (n+1)^2 = 0$$
$$\Rightarrow z = \frac{2nw + \sqrt{4n^2w^2 - 4(n+1)^2}}{2(n+1)} = \frac{nw + n\sqrt{w^2 - (\frac{n+1}{n})}}{n+1}$$
$$\Rightarrow z = \frac{w + \sqrt{w^2 - (1+\frac{1}{n})^2}}{1+\frac{1}{n}}.$$

Then, by Theorem 3.1.6,

$$g_{\overline{B(0,1)}} = \log|z| = \log\left|\frac{w + \sqrt{w^2 - (1+\frac{1}{n})^2}}{1+\frac{1}{n}}\right| = g_{K_n}(w).$$

Note that, $K = \bigcap K_n = [-1, 1]$, *and* $g_K(w) = \log|w + \sqrt{w^2 - 1}|$, *so*

$$\lim_{n\to\infty} g_{K_n} = \lim_{n\to\infty} \log\left|\frac{w + \sqrt{w^2 - (1+\frac{1}{n})^2}}{1+\frac{1}{n}}\right| = \log|w + \sqrt{w^2 - 1}| = g_K(z).$$

Moreover, $\mathrm{Cap}(K_n) = \frac{1+\frac{1}{n}-(-1-\frac{1}{n})}{4} = \frac{1}{2} + \frac{1}{2n}$, *and* $\mathrm{Cap}(K) = \frac{1}{2}$, *so*

$$\lim_{n\to\infty} \mathrm{Cap}(K_n) = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} = \mathrm{Cap}(K).$$

Proposition 3.2.4 *Let* K *be a compact set such that* $K = \{|P(z)| \le 1 : z \in \mathbb{C}\}$, *where* $P(z) = \alpha_n z^n + \alpha_{n-1}z^{n-1} + \ldots + \alpha_0$, *then*

$$g_K(z) = \frac{1}{n}\log|P(z)|.$$

Moreover,

$$\operatorname{Cap}(K) = \alpha_n^{-\frac{1}{n}}.$$

Proof: Here, $\frac{1}{n}\log|P(z)| = \log|z| + \frac{\log\alpha_n}{n} + o(z)$. Now, we should check whether $\frac{1}{n}\log|P(z)|$ is the Green function or not.

On ∂K, $|P(z)| = 1$, so $g_{\partial K}(z) = 0$. Near ∞, $\frac{1}{n}\log|P(z)| - \log|z| = \frac{\log|\alpha_n|}{n} + o(z)$ which is bounded. Hence, $\frac{1}{n}\log|P(z)| - \log|z| \in H(\infty)$.

Moreover, polynomials are analytic functions and logarithm of analytic functions are harmonic functions. Hence, $\frac{1}{n}\log|P(z)| \in H(\mathbb{C}\setminus K)$.

Therefore, by the uniqueness of the Green function, $\frac{1}{n}\log|P(z)| = g_K(z)$. Now,

$$\begin{aligned}
&\operatorname{Rob}(K) = \lim_{|z|\to\infty}(g_K(z) - \log|z|) = \lim_{|z|\to\infty}\left(\frac{\log\alpha_n}{n} + o(z)\right) = \log\alpha_n^{\frac{1}{n}} \\
&\Rightarrow \operatorname{Cap}(K) = e^{-\operatorname{Rob}(K)} = e^{\log\alpha_n^{-\frac{1}{n}}} = \alpha_n^{-\frac{1}{n}}.
\end{aligned}$$

Theorem 3.2.5 (Bernstein-Walsh Theorem, [13]) *Let $K \in \mathbb{C}$ be a non-polar compact set. Then, for any polynomial P of degree n, we have*

$$|P(z)| \leq \exp\left(ng_K(z)\right)\|P\|_K,$$

$\forall z \in \mathbb{C}$, *where* $\|P\|_K = \sup_{z\in K}|P(z)|$.

From the theorem above, we have the following representation of the Green function:

Corollary 3.2.6

$$g_K(z) = \sup\left\{\frac{\log|P(z)|}{\deg P} : P \in \Pi, \deg P \geq 1, \|P\|_K \leq 1\right\},$$

where $K \in \mathbb{C}$ is a non-polar compact set, Π_n denotes the set of all polynomials of degree at most n, $\Pi = \bigcup_{n=0}^{\infty}\Pi_n$.

3.3 Smoothness of the Green Function

Definition 3.3.1 *Let U be a bounded nonempty subset of $\mathbb{R}^k$, let f be a real-valued function defined on the boundary of U, then the classical Dirichlet problem on U is to find a harmonic function h on U such that $\lim_{y\to x} h(y) = f(x), \forall x \in \partial U$. The point p is said to be a regular point with respect to the Dirichlet problem if the classical Dirichlet problem has a solution for each continuous function at p. Also, the set U is said to be regular with respect to the Dirichlet problem if every boundary point of U is a regular point in the Dirichlet sense.*

Theorem 3.3.2 (Wiener Theorem, see e.g. [12]) *Let K be a compact set, $\Omega = \mathbb{C} \setminus K$, and let $0 \le \lambda \le 1$ and set*

$$A_n(z) = \{y | y \notin \Omega, \lambda^n \le |y - z| \le \lambda^{n-1}\}.$$

Then $z \in \partial\Omega$ is a regular boundary point of Ω if and only if

$$\sum_{n=1}^{\infty} \frac{n}{\log\left(\frac{1}{\operatorname{Cap}(A_n(z))}\right)} = \infty.$$

Theorem 3.3.3 (see e.g. [12]) *Let K be a compact set. Then $g_K(z)$ is continuous on the whole plane if and only if K is regular with respect to the Dirichlet problem.*

Example 11 *Let $K = [0, 1]$, then K is a regular set. To see this, let us show that 0 is a regular point of $\mathbb{C}\setminus K$. Let us choose $\lambda = \frac{1}{2}$. Then, we have $\operatorname{Cap}(A_n) = \frac{1}{2^{n+2}}$. Thus, $\log \frac{1}{\operatorname{Cap}(A_n(z))} = (n+2)\log 2$. Since, $\sum_{n=0}^{\infty} \frac{n}{(n+2)\log 2} = \infty$, then 0 is a regular point. Similarly, all other points of K are regular points of $\mathbb{C} \setminus K$. Thus K is regular. By the theorem above, $g_K(z)$ is continuous on the whole plane.*

On the other hand, in Example 8, we see that $g_K(z)$ is continuous, thus the set K is regular.

Example 12 *[12] Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ such that $I_k = [a_k, b_k]$, where $b_k = e^{-M^k}$ and $a_k = b_k - b_{k+1}$ with $M \geq 0$. Then by Theorem 3.3.2, $g_K(z)$ is continuous if and only if $\sum_{k=1}^{\infty} \frac{M_k}{M_{k+1}} = \infty$. In this case $\frac{M_k}{M_{k+1}} = \frac{1}{M}$. Thus, $g_K(z)$ is continuous.*

Definition 3.3.4 *Let f be a real or complex valued function on Euclidean space. Then f is called Hölder continuous if there exists nonnegative real constants C and α such that*

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

for all x and y in the domain of f. In this case, f satisfies Hölder condition of order $\alpha > 0$ or f belongs to Lipschitz class α, denoted by $f \in Lip(\alpha)$. If $\alpha = \frac{1}{2}$ and $K \subset [a, b]$, this is an optimal smoothness for g_K.

Definition 3.3.5 *Given a function f, the modulus of continuity of f is a function $w(f, \delta) = \sup_{|x-y| \leq \delta} |f(x) - f(y)|$, where $x, y \in \mathrm{Dom}(f)$.*

Let K be a regular compact set. Since $g_K(z) = 0$, $\forall z \in K$, and $g_K(z)$ is continuous on $\mathbb{C} \setminus K$, it is interesting to figure out what kind of continuity $g_K(z)$ has near the boundary of K. Given regular compact set K and $\delta > 0$. We say that the point $p = p(\delta)$ realizes the modulus of continuity of g_K if $\mathrm{dist}(p, K) \leq \delta$, and $g_K(z) \leq g_K(p)$ for all z with $\mathrm{dist}(z, K) \leq \delta$. Here, we get $w(g_k, \delta) = g_K(p) - g_K(\phi)$ for some $\phi \in K$.

Let us start with some known examples.

Example 13 *Let $K = [-1, 1]$, then $g_K(z)$ admits a modulus of continuity at the points $-1 - \delta$ and $1 + \delta$. From Example 8, we know that $g_K(z) = |z + \sqrt{z^2 - 1}|$. Thus, $g_K(-1 - \delta)$, $g_K(1 + \delta) \leq \sqrt{3\delta}$, and $g_K(\delta i) \leq \delta$, and for all remained z such that $\mathrm{dist}(z, K) \leq \delta$, $g_K(z) \leq C\delta^{\alpha}$, where $\frac{1}{2} < \alpha < 1$. Thus, $g_K(z) \in Lip(\frac{1}{2})$.*

Let $K = \overline{B(0, 1)}$, then by Example 6, we have

$$g_K(z) = \begin{cases} 0, & \text{if } |z| \leq 1, \\ \log|z| & \text{if } |z| > 1. \end{cases}$$

Then, $\forall z \in B(0, 1+\delta) \setminus \overline{B(0,1)}$*, we have* $g_K(z) \leq \delta$*, hence* $g_K(z) \in Lip(1)$*.*

Now, let us introduce some recent results.

Example 14 *[12] Let K be a compact subset of an open disc with positive capacity, then*

$$g_K(z) \leq C|z|^{\frac{1}{2}} \exp\Big(D \int_{|z|}^{1} \frac{\Theta^2(u)}{u^3} du\Big) \log \frac{2}{Cap(K)}$$

for $|z| \leq 1$*, where C and D are constants and* $\Theta = \Theta_K$ *is a function that measures the density near the origin of the circular projection of K onto the positive real axis. More specifically, let* $\tilde{K}$ *be the set of* $a \in [0,1]$ *such that K intersects the circle* $|z| = a$*, then* $\Theta(t) = d([0,t] \setminus \tilde{K})$*, where d is the Lebesgue measure.*

Example 15 *[6] Let* K^α *be a Cantor-type set, such that* $1 < \alpha < 2$*. Then* $K^\alpha = \bigcap_{s=0}^{\infty} E_s$*, where* $E_0 = I_{1,0} = [0,1]$*,* E_s *is a union of* 2^s *closed basic intervals* $I_{j,s}$ *of length* $l_s = l_{s-1}^{\alpha}$ *with* $2l_1^{\alpha-1} < 1$ *and where* E_{s+1} *is obtained by deleting the open concrete subinterval of length* $h_s := l_s - 2l_{s+1}$ *from each* $I_{j,s}$ *with* $j = 1, 2, \ldots, 2^s$*. Then, for every* $0 \leq \epsilon \leq \gamma$*, there exists constants* δ_0, C_0*, depending on* α *and* ϵ*, such that*

$$g_{K^\alpha}(z) \leq C_0 \varphi^{\gamma-\epsilon}(\delta)$$

for $z \in \mathbb{C}$ *with* $\mathrm{dist}(z, K^\alpha) = \delta \leq \delta_0$*, where* $\varphi(\delta) = (\log \frac{1}{\delta})^{-1}$*, and* $\gamma = \frac{\log \frac{2}{\alpha}}{\log \alpha}$*. Here, the smoothness of the Green function is described in terms of the function* $\varphi(\delta)$*. Moreover,*

$$g_{K^\alpha}(-\delta) \geq C\varphi^{\gamma}(\delta).$$

Chapter 4

Nearly Chebyshev Polynomials

We will consider a special compact set, see also [5]. Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$ such that $I_k = [a_k, b_k]$. Now fix $m \in \mathbb{Z}_+$ and define

$$P(x) = x \prod_{k=1}^{m-1} \gamma_k \tilde{T}_{n_k k}(x)$$

on K, where $\gamma_k = \frac{1}{\tilde{T}_{n_k k}(0)}$, n_k denotes the degree of the Chebyshev polynomial defined in I_k and $\tilde{T}_{n_k k}(x) = \tilde{T}_{n_k}(\frac{x - c_k}{\delta_k})$, where $c_k = \frac{b_k + a_k}{2}$ and $\delta_k = \frac{b_k - a_k}{2}$.

Notation 1 *For simplicity of notation, from now on let $T_{n_k}(x) = \tilde{T}_{n_k k}(x)$. Thus, $T_{n_k}(x)$ represents the extended Chebyshev polynomial which is defined on a given I_k with degree n_k.*

First of all, we need to approximate the degrees n_k in order to make $P(x)$ the polynomial which has almost the least deviation on a given compact set K, in other words, in order to make $P(x)$ a nearly Chebyshev polynomial for K.

4.1 Determination of the Degrees of the Chebyshev Polynomials

Theorem 4.1.1 *Let $b_k = e^-2^k$, $b_k^2 = b_{k+1}$, and $b_k - b_{k+1} = a_k$. We can determine all n_k values for $k < m$ so that $|P(x)| \leq b_m$ $\forall x \in K$ and $x > b_m$.*

Before the proof of the theorem, let us prove the following two lemmas.

Lemma 4.1.2 $\left|\frac{T_{n_k}(b_j)}{T_{n_k}(0)}\right| \leq \left(\frac{b_j}{b_k}\right)^{n_k}$*, where $j < k$.*

Proof: Let $\{x_1, x_2, \ldots, x_{n_k}\}$ be the zeros of $T_{n_k}(x)$. Then,

$$\left|\frac{T_{n_k}(b_j)}{T_{n_k}(0)}\right| = \prod_{k=1}^{n_k} \frac{b_j - x_k}{x_k}.$$

Now, take any arbitrary zero x_p of the Chebyshev polynomial. Since $x_p > a_k$, we get

$$\frac{b_j - x_p}{x_p} \leq \frac{b_j - a_k}{a_k} = \frac{b_j - b_k + b_{k+1}}{b_k - b_{k+1}} = \frac{b_j}{b_k}\frac{(1 - b_j^{(2^{k-j}-1)} + b_j^{(2^{k-j+1}-1)})}{(1 - b_k)}.$$

Now, since $1 - b_j^{2^{k-j}} \geq b_j$, then $1 - b_j^{(2^{k-j}-1)} + b_j^{(2^{k-j+1}-1)} \leq 1 - b_j^{2^{k-j}}$. Thus, $\frac{b_j - x_p}{x_p} \leq \frac{b_j}{b_k}$. Since x_p is arbitrary, we get

$$\left|\frac{T_{n_k}(b_j)}{T_{n_k}(0)}\right| = \prod_{k=1}^{n_k} \frac{b_j - x_k}{x_k} \leq \left(\frac{b_j}{b_k}\right)^{n_k}.$$

Lemma 4.1.3 $|T_{n_k}(0)| \geq \frac{1}{2}4^{n_k}(b_k^{(\epsilon_k - 1)})^{n_k}$*, where $\epsilon_k = \frac{1}{e^{2^k}2^k}$.*

Proof: By definition, $|T_{n_k}(0)| \geq \frac{1}{2}4^{n_k}\prod_{j=1}^{n_k}\frac{a_k}{b_{k+1}} = \frac{1}{2}4^{n_k}\prod_{j=1}^{n_k}(b_k^{-1} - 1)$. Now let

$$(b_k^{-1})^{1-\epsilon_k} \leq b_k^{-1} - 1 \Rightarrow (1 - \epsilon_k)2^k \leq 2^k \log{(1 - e^{-2^k})}.$$

By the power series expansion of $\log(1-x)$, it is known that $\log(1-x) < -x$.

$$(1-\epsilon_k)2^k \leq 2^k \log(1-e^{-2^k}) \leq -e^{-2^k} \Rightarrow \epsilon_k \geq \frac{1}{e^{2^k}2^k}.$$

Thus,

$$|T_{n_k}(0)| \geq \frac{1}{2}4^{n_k}(b_k^{(\epsilon_k-1)})^{n_k}$$

with $\epsilon_k = \frac{1}{e^{2^k}2^k}$.

Proof: [*of Theorem* 4.1.1] Let $x = b_p$ is fixed where $1 \leq p \leq m-1$. Then

$$|P(b_p)| = b_p \prod_{k=1}^{m-1} \gamma_k \tilde{T}_{n_k k}(b_p).$$

Let us divide this representation of the polynomial above into two parts in a way that

$$L_p = b_p \prod_{k=p}^{m-1} \gamma_k \tilde{T}_{n_k k}(b_p),$$

$$R_p = \prod_{k=1}^{p-1} \gamma_k \tilde{T}_{n_k k}(b_p).$$

Note that for any $1 \leq p \leq m-1$, the value of R_p is less than 1. However, the value of L_p is huge when it is compared to R_p. Hence, we want to have

$$\begin{aligned}
&\frac{b_{m-1}}{b_m} \leq |T_{n_{m-1}}(0)| \\
&\frac{b_{m-2}}{b_m}\left|\frac{T_{n_{m-1}}(b_{m-2})}{T_{n_{m-1}}(0)}\right| \leq |T_{n_{m-2}}(0)| \\
&\frac{b_{m-3}}{b_m}\left|\frac{T_{n_{m-1}}(b_{m-3})}{T_{n_{m-1}}(0)}\right|\left|\frac{T_{n_{m-2}}(b_{m-3})}{T_{n_{m-2}}(0)}\right| \leq |T_{n_{m-3}}(0)| \\
&\vdots \\
&\frac{b_{m-j}}{b_m}\left|\frac{T_{n_{m-1}}(b_{m-j})}{T_{n_{m-1}}(0)}\right| \cdots \left|\frac{T_{n_{m-j+1}}(b_{m-j})}{T_{n_{m-j+1}}(0)}\right| \leq |T_{n_{m-j}}(0)| \\
&\vdots
\end{aligned}$$

However, usage of the lemmas above guarantees the correctness of the above expression. They also give us an easier calculation opportunity. Thus, let us calculate degrees according to

$$\frac{b_{m-1}}{b_m} \leq \frac{1}{2} 4^{n_{m-1}} (b_{m-1}^{(\epsilon_{m-1}-1)})^{n_{m-1}} \tag{4.1}$$

$$\frac{b_{m-2}}{b_m} \Big(\frac{b_{m-2}}{b_{m-1}}\Big)^{n_{m-1}} \leq \frac{1}{2} 4^{n_{m-2}} (b_{m-2}^{(\epsilon_{m-2}-1)})^{n_{m-2}} \tag{4.2}$$

$$\frac{b_{m-3}}{b_m} \Big(\frac{b_{m-3}}{b_{m-1}}\Big)^{n_{m-1}} \Big(\frac{b_{m-3}}{b_{m-2}}\Big)^{n_{m-2}} \leq \frac{1}{2} 4^{n_{m-3}} (b_{m-3}^{(\epsilon_{m-3}-1)})^{n_{m-3}} \tag{4.3}$$

$$\vdots$$

$$\frac{b_{m-j}}{b_m} \Big(\frac{b_{m-j}}{b_{m-1}}\Big)^{n_{m-1}} \cdots \Big(\frac{b_{m-j}}{b_{m-j+1}}\Big)^{n_{m-j+1}} \leq \frac{1}{2} 4^{n_{m-j}} (b_{m-j}^{(\epsilon_{m-j}-1)})^{n_{m-j}} \tag{4.4}$$

$$\vdots$$

From (4.1), we get

$$\begin{aligned}
& b_{m-1}^{-1} \leq 2^{2n_{m-1}-1} (b_{m-1})^{n_{m-1}(\epsilon_{m-1}-1)} \\
& \Rightarrow b_{m-1}^{(n_{m-1}(1-\epsilon_{m-1})-1)} \leq 2^{2n_{m-1}-1} \\
& \Rightarrow 2^{m-1} - 2^{m-1} n_{m-1}(1-\epsilon_{m-1}) \leq 2n_{m-1} \log 2 - \log 2 \\
& \Rightarrow n_{m-1} \geq \frac{2^{m-1} + \log 2}{2^{m-1}(1-\epsilon_{m-1}) + 2\log 2} = 1 - \frac{\log 2 - \frac{1}{e^{2^{m-1}}}}{2^{m-1} - \frac{1}{e^{2^{m-1}}} + \log 4}.
\end{aligned}$$

Hence, we can take $n_{m-1} = 1$.

From (4.2), we get

$$\begin{aligned}
& b_{m-2}^{-3} b_{m-2}^{-1} \leq 2^{2n_{m-2}-1} (b_{m-2})^{n_{m-2}(\epsilon_{m-2}-1)} \\
& \Rightarrow b_{m-2}^{(n_{m-2}(1-\epsilon_{m-2})-4)} \leq 2^{2n_{m-2}-1} \\
& \Rightarrow n_{m-2}(2\log 2 + 2^{m-2}(1-\epsilon_{m-2})) \geq 2^{m-2} 4 + \log 2 \\
& \Rightarrow n_{m-2} \geq \frac{2^{m-2} 4 + \log 2}{2^{m-2}(1-\epsilon_{m-2}) + \log 4} = 4 - \frac{4\log 4 - \log 2 - \frac{4}{e^{2^{m-2}}}}{2^{m-2} - \frac{1}{e^{2^{m-2}}} + \log 4}.
\end{aligned}$$

Hence, we can take $n_{m-2} = 4$.

From (4.3), by similar calculations, we get

$$n_{m-3} \geq 14 - \frac{14\log 4 - \log 2 - \frac{14}{e^{2^{m-2}}}}{2^{m-2} - \frac{1}{e^{2^{m-2}}} + \log 4}.$$

Thus, we can take n_{m-3}=14.

Finally, if

$$\zeta = \sum_{k=1}^{j-1}(n_{m-k}(2^{j-k}-1)) + (2^j - 1),$$

then, from (4.4), by similar calculations, we get

$$n_{m-j} \geq \zeta - \frac{\zeta\log 4 - \log 2 - \frac{\zeta}{e^{2^{m-2}}}}{2^{m-2} - \frac{1}{e^{2^{m-2}}} + \log 4}.$$

Hence, for any $j \leq m-1$, we have

$$n_{m-j} = \sum_{k=1}^{j-1}(n_{m-k}(2^{j-k}-1)) + (2^j - 1)$$

and by construction, we have $|P(x)| \leq b_m$.

Corollary 4.1.4 *$n_{m-j-1} = 4n_{m-j} - 2n_{m-j+1}$ with $n_{m-1} = 1$ and $n_{m-2} = 4$.*

Proof: We prove this corollary by induction. First of all, note that, $4n_{m-2} - 2n_{m-1} = 4\cdot 4 - 2\cdot 1 = 14 = n_{m-3}$, by calculations which are done in the proof of the previous theorem. Now, this recursive relation holds up to n_{m-j}. Then;

$$\begin{aligned}
n_{m-j-1} &= n_{m-1}(2^j-1) + n_{m-2}(2^{j-1}-1) + \ldots + 7n_{m-j+2} + 3n_{m-j+1} + n_{m-j} \\
&+ (2^{j+1}-1) = n_{m-j} + 3[n_{m-j+1} + 3n_{m-j+2} + 7n_{m-j+3} + \ldots \\
&+ (2^{j-1}-1)n_{m-1} + (2^j-1)] - 2[n_{m-j+2} + 3n_{m-j+3} + \ldots \\
&+ (2^{j-2}-1)n_{m-1} + (2^{j-1}-1)] = 4n_{m-j} - 2n_{m-j+1}.
\end{aligned}$$

Corollary 4.1.5 *For any $k < m-1$, $n_k = \frac{1}{2\sqrt{2}}((2+\sqrt{2})^{m-k} + (2-\sqrt{2})^{m-k})$.*

Proof: We prove this corollary by induction too. First of all, $\frac{1}{2\sqrt{2}}((2+\sqrt{2})^{m-(m-1)} + (2-\sqrt{2})^{m-(m-1)}) = 1 = n_{m-1}$. Now, assume this corollary holds up to n_q. Then by previous theorem, we have;

$$\begin{aligned}
n_{q+1} &= 4n_q - 2n_{q-1} \\
&= \sqrt{2}((2+\sqrt{2})^{m-q} + (2-\sqrt{2})^{m-q}) - \frac{1}{\sqrt{2}}((2+\sqrt{2})^{m-q} + (2-\sqrt{2})^{m-q}) \\
&= \sqrt{2}(2+\sqrt{2})^{m-q-1}(\frac{3}{2} + \sqrt{2}) - \sqrt{2}(2-\sqrt{2})^{m-q-1}(\frac{3}{2} - \sqrt{2}) \\
&= \frac{1}{2\sqrt{2}}((2+\sqrt{2})^{m-q+1} + (2-\sqrt{2})^{m-q+1}).
\end{aligned}$$

4.2 Some Properties of the Degrees

In this section, we give some additional properties of the degrees found in the previous section that will be used intensively in Chapter 5 and Chapter 6.

Corollary 4.2.1

$$1 + \sum_{k=q+1}^{m-1} n_k = n_{q-1} - 3n_q.$$

Proof: We prove this corollary by induction too. First of all, $1 + n_{m-1} + n_{m-2} = 1+1+4 = 6 = 48 - 3\cdot 14 = n_{m-4} - 3n_{m-3}$. Now assume $1 + \ldots + n_{q+2} = n_q - 3n_{q+1}$. Then

$$\begin{aligned}
1 + \ldots + n_{q+1} = n_q - 3n_{q+1} + n_{q+1} &= n_q - 2n_{q+1} \\
&= 4n_q - 2n_{q+1} - 3n_q - n_{q-1} - 3n_q.
\end{aligned}$$

Note that the last equality comes from Corollary 4.1.4.

Corollary 4.2.2

$$\sum_{k=q}^{p} n_k = \frac{1}{2}[(2+\sqrt{2})^{m-q} + (2-\sqrt{2})^{m-q} - (2+\sqrt{2})^{m-p-1} - (2-\sqrt{2})^{m-p-1}],$$

where $1 \leq q \leq p \leq m-1$.

Proof:

$$\begin{aligned}
\sum_{k=q}^{p} n_k &= \frac{1}{2\sqrt{2}} \sum_{k=q}^{p} [(2+\sqrt{2})^{m-k} - (2-\sqrt{2})^{m-k}] \\
&= \frac{1}{2\sqrt{2}} \Big[(2+\sqrt{2})^{m-p} \sum_{k=0}^{p-q} (2+\sqrt{2})^{k} - (2-\sqrt{2})^{m-p} \sum_{k=0}^{p-q} (2-\sqrt{2})^{k}\Big] \\
&= \frac{1}{2\sqrt{2}} \Big[(2+\sqrt{2})^{m-p} \Big(\frac{\sqrt{2}((2+\sqrt{2})^{p-q+1}-1)}{2+\sqrt{2}}\Big) \\
&- (2-\sqrt{2})^{m-p} \Big(\frac{\sqrt{2}(1-(2+\sqrt{2})^{p-q+1})}{2-\sqrt{2}}\Big)\Big] \\
&= \frac{1}{2}[(2+\sqrt{2})^{m-q} + (2-\sqrt{2})^{m-q} - (2+\sqrt{2})^{m-p-1} - (2-\sqrt{2})^{m-p-1}].
\end{aligned}$$

Corollary 4.2.3

$$1 + \sum_{k=1}^{m-1} n_k = \frac{1}{2}[(2+\sqrt{2})^{m-1} + (2-\sqrt{2})^{m-1}].$$

Proof: Apply Corollary 4.2.2 with $q = 1$ and $p = m - 1$.

Corollary 4.2.4

$$b_m b_{m-1} b_{m-2}^{n_{m-2}} \dots b_{q+1}^{n_{q+1}} = b_q^{1+1+\dots+n_q} = b_q^{n_{q-2}-3n_{q-1}}.$$

Proof: We prove this theorem by induction.

First of all, $b_m b_{m-1} = b_{m-2}^4 b_{m-2}^2 = b_{m-2}^6 = b_{m-2}^{1+1+4}$. Now assume that $b_m b_{m-1} b_{m-2}^{n_{m-2}} \dots b_{q+2}^{n_{q+2}} = b_{q+1}^{1+1+\dots+n_{q+1}}$. Then

$$\begin{aligned}
&b_m b_{m-1} b_{m-2}^{n_{m-2}} \dots b_{q+1}^{n_{q+1}} = b_{q+1}^{1+1+\dots+n_{q+1}} b_{q+1}^{n_{q+1}} = b_q^{2+2n_{m-1}+\dots+2n_{q+2}+4n_{q+1}} \\
&= b_q^{1+\dots+n_{q+1}+n_q-3n_{q+1}+3n_{q+1}} = b_q^{1+n_{m-1}+\dots+n_q} = b_q^{n_{q-2}-3n_{q-1}}.
\end{aligned}$$

Corollary 4.2.5 *We have the following inequalities:*

$$n_q b_q \le \sum_{k=q}^{p} n_k b_k \le 2 n_q b_q,$$
$$\frac{n_p}{b_p} \le \sum_{k=q}^{p} \frac{n_k}{b_k} \le 2\frac{n_p}{b_p}.$$

Proof: Lower bounds in both inequalities are obvious. Let us find the upper bounds. Note that

$$\sum_{k=q}^{p} \frac{n_k}{b_k} = n_q b_q\Big(1 + b_q\Big(\frac{1}{2+\sqrt{2}} + \frac{1}{(2+\sqrt{2})^2 + \ldots + \frac{1}{(2+\sqrt{2})^{p-q}}}\Big)\Big) \le n_q b_q(1+b_q)$$
$$\le 2 n_q b_q.$$

Moreover,

$$\sum_{k=q}^{p} \frac{n_k}{b_k} \le \frac{n_p}{b_p}\Big(b_p(2+\sqrt{2}) + b_p^3(2+\sqrt{2})^2 + \ldots + b_p^{2^{q-p}-1}(2+\sqrt{2})^{q-p}\Big) \le 2\frac{n_p}{b_p}.$$

Corollary 4.2.6

$$\frac{4+2\sqrt{2}}{3+2\sqrt{2}}\frac{n_q}{2^q} - \epsilon_q \le \sum_{k=q}^{m-1} \frac{n_k}{2^k} \le \frac{4+2\sqrt{2}}{3+2\sqrt{2}}\frac{n_q}{2^q},$$

where $\epsilon_q = (1+\sqrt{2})\frac{(2-\sqrt{2})^{m-q}}{2^q}$.

Proof: We use some properties of geometric series in this proof.

$$\sum_{k=q}^{m-1} \frac{n_k}{2^k} = \frac{1}{2\sqrt{2}}\Big(\sum_{k=0}^{m-1} \frac{(2+\sqrt{2})^{m-k}}{2^k} - \sum_{k=0}^{q-1} \frac{(2+\sqrt{2})^{m-k}}{2^k} -$$
$$\Big(\sum_{k=0}^{m-1} \frac{(2-\sqrt{2})^{m-k}}{2^k} - \sum_{k=0}^{q-1} \frac{(2-\sqrt{2})^{m-k}}{2^k}\Big)\Big)$$
$$= \frac{(2+\sqrt{2})^m}{2\sqrt{2}}\Big(\sum_{k=0}^{m-1} \frac{1}{(2(2+\sqrt{2}))^k} - \sum_{k=0}^{q-1} \frac{1}{(2(2+\sqrt{2}))^k}\Big)$$
$$- \frac{(2-\sqrt{2})^m}{2\sqrt{2}}\Big(\sum_{k=0}^{m-1} \frac{1}{(2(2-\sqrt{2}))^k} - \sum_{k=0}^{q-1} \frac{1}{(2(2-\sqrt{2}))^k}\Big)$$

$$\begin{aligned}
&= \frac{(2+\sqrt{2})^m}{2\sqrt{2}}\Big(\frac{1+2(2+\sqrt{2})+\ldots+(2(2+\sqrt{2}))^{m-q-2}}{(2+\sqrt{2})^{m-1}2^{m-1}}\Big) \\
&- \frac{(2-\sqrt{2})^m}{2\sqrt{2}}\Big(\frac{1+2(2-\sqrt{2})+\ldots+(2(2-\sqrt{2}))^{m-q-2}}{(2-\sqrt{2})^{m-1}2^{m-1}}\Big) \\
&= \frac{1}{2\sqrt{2}}\frac{4+2\sqrt{2}}{3+2\sqrt{2}}\Big(\frac{(2+\sqrt{2})^{m-q}}{2^q} - \frac{1}{2^m}\Big) \\
&- \frac{1}{2\sqrt{2}}\frac{4-2\sqrt{2}}{3-2\sqrt{2}}\Big(\frac{(2-\sqrt{2})^{m-q}}{2^q} - \frac{1}{2^m}\Big) \\
&= \frac{4+2\sqrt{2}}{3+2\sqrt{2}}\frac{n_q}{2^q} - (1+\sqrt{2})\frac{(2-\sqrt{2})^{m-q}}{2^q} + \frac{1}{2^{m+1}}.
\end{aligned}$$

Corollary 4.2.7 *$b_q^2 \le b_c b_{c-1}\ldots b_q \le b_q$, where $q \le c$.*

Proof: It is obvious that $b_c b_{c-1}\ldots b_q \le b_q$. For the other inequality,

$$b_c b_{c-1}\ldots b_q = b_q^{2^{c-q}}\ldots b_q = b_q^{1+2+\ldots+2^{c-q}} \ge b_q^2.$$

Chapter 5

A Lower Bound for the Green Function

In this chapter, we will find a lower bound for the Green function for compact sets defined in previous chapter for $-\delta = -b_s$ value which is close to the boundary of these compact sets. By Corollary 3.2.6, we know that any polynomial on these compact sets gives us a lower bound for the Green function. However, the majority of them gives us useless lower bounds. The polynomial which was introduced in the previous chapter with calculated degrees gives us a nearly Chebyshev polynomial, so we use a slightly modified version of this polynomial to have a good lower bound for the Green function.

Note that

$$P(x) = x \prod_{k=1}^{m-1} \frac{T_{n_k}(x)}{T_{n_k}(0)} = x \prod_{k=1}^{m-1} \frac{Q_{n_k}(x)}{Q_{n_k}(0)},$$

where if $\{x_1, \ldots, x_{n_k}\}$ are the zeros of the Chebyshev polynomial T_{n_k}, then $Q_{n_k} = \prod_{k=1}^{n_k}(x - x_k)$, since $T_{n_k} = 2^{n_k - 1}(\frac{2}{b_k - a_k})^{n_k} \prod_{k=1}^{n_k}(x - x_k)$. Now, let $H(x) = \frac{P(x)}{b_m}$. We use this polynomial to have a lower bound for the Green function at $-\delta$ value, since by the construction of degrees $|H(x)| \leq 1$ on compact sets, whereas $|P(x)| \leq b_m$.

Theorem 5.0.8 *Let $K = \{0\} \cup \bigcup_{k=1}^{\infty} I_k$, where $I_k = [a_k, b_k]$, $b_k = e^{-2^k}$, $b_k^2 = b_{k+1}$ and $a_k = b_k - b_{k+1}$. Then*

$$g_K(-\delta) \geq 3\varphi^{\gamma},$$

where $\varphi = \left(\log \frac{1}{\delta}\right)^{-1}$, $\gamma = \frac{\log(1+\frac{\sqrt{2}}{2})}{2}$, and $\delta = b_s = e^{-2^s}$. Note that $s \in \mathbb{Z}$, $s \leq m-1$, and s is big enough so that b_s is close to the boundary of K.

Proof: If we evaluate $|H(x)|$ at $-\delta$, we get

$$|H(-\delta)| = \frac{\delta}{b_m} \prod_{k=1}^{m-1} \left| \frac{Q_{n_k}(-\delta)}{Q_{n_k}(0)} \right| = \frac{\delta}{b_m} \frac{(\delta + x_1)(\delta + x_2) \ldots (\delta + x_N)}{x_1 x_2 x_3 \ldots x_N},$$

where $\{x_1, x_2, \ldots, x_N\}$ are all zeros of the corresponding Chebyshev polynomials and N is the total degree. Let x_p be an arbitrary zero in I_k, then

$$\frac{\delta + x_p}{x_p} \geq \begin{cases} \frac{\delta}{b_k}, & \text{if } k > s, \\ 2 & \text{if } k = s, \\ 1 & \text{if } k < s. \end{cases}$$

Thus,

$$\begin{aligned} |H(-\delta)| &\geq \frac{\delta}{b_m} \frac{\delta}{b_{m-1}} \left(\frac{\delta}{b_{m-2}}\right)^{n_{m-2}} \cdots \left(\frac{\delta}{b_{s+1}}\right)^{n_{s+1}} 2^{n_s} \\ &= \frac{\delta^{1+n_{m-1}+n_{m-2}+\ldots+n_{s+1}} 2^{n_s}}{b_m b_{m-1}^{n_{m-1}} b_{m-2}^{n_{m-2}} b_{s+1}^{n_{s+1}}} \ldots = \frac{\delta^{n_s - 3n_{s+1}} 2^{n_s}}{\delta^{n_{s-2} - 3n_{s-1}}}. \end{aligned}$$

The last equality is from Corollary 4.2.4 and Corollary 4.2.1.

Note that $n_s - 3n_{s+1} - n_{s-2} + 3n_{s-1} = -n_{s+1} + 3n_s - 3n_{s+1} = -n_s - n_{s+1}$. Thus,

$$|H(-\delta)| \geq \left(\frac{2}{\delta}\right)^{n_s} \left(\frac{1}{\delta}\right)^{n_{s+1}}.$$

Therefore,

$$\log |H(-\delta)| \geq n_s \log \frac{2}{\delta} + n_{s+1} \log \frac{1}{\delta}.$$

By Corollary 4.2.3, if we denote the total degree in this calculation by N, then

$$N = 2 + \sum_{k=1}^{m-1} n_k = \frac{1}{2}((2+\sqrt{2})^{m-1} + (2-\sqrt{2})^{m-1}) + 1.$$

Note that $\frac{n_s}{N} \geq \frac{1}{\sqrt{2}}(2+\sqrt{2})^{1-s}$, and $\frac{n_{s+1}}{N} \geq \frac{1}{\sqrt{2}}(2+\sqrt{2})^{-s}$. Thus,

$$\begin{aligned}\frac{\log|H(-\delta)|}{N} &\geq \frac{1}{\sqrt{2}}(2+\sqrt{2})^{1-s}\log\frac{2}{\delta} + \frac{1}{\sqrt{2}}(2+\sqrt{2})^{-s}\log\frac{1}{\delta} \\ &\geq \frac{2+\sqrt{2}}{\sqrt{2}}(2-\sqrt{2})^s + \frac{1}{\sqrt{2}}(2-\sqrt{2})^s = \frac{3+\sqrt{2}}{\sqrt{2}}(2-\sqrt{2})^s.\end{aligned}$$

Moreover, $2^s = \log\frac{1}{\delta}$. Note that $(2-\sqrt{2})^s = \left(\log\frac{1}{\delta}\right)^{\frac{\log(2-\sqrt{2})}{2}} = \varphi^\gamma$, where $\varphi = \left(\log\frac{1}{\delta}\right)^{-1}$, and $\gamma = \frac{\log(1+\frac{\sqrt{2}}{2})}{2}$. So,

$$g_K(-\delta) \geq \frac{3+\sqrt{2}}{\sqrt{2}}\varphi^\gamma \geq 3\varphi^\gamma.$$

Chapter 6

An Upper Bound for the Green Function

In this chapter, first of all, we are going to find an upper bound for the Green function for compact sets defined in Chapter 4 for $-\delta = -b_s$ value which is close to the boundary of these compact sets. Theorem 3.2.5 states that

$$g_K(-\delta) = \sup\left\{\frac{\log|P(-\delta)|}{\deg P} : P \in \Pi, \deg P \geq 1, |P|_K \leq 1\right\}.$$

Let $F(-\delta)$ be the function that realizes the supremum above. By the tools of Lagrange interpolation, we can write $F(x)$ in term of the Lagrange basis polynomials (see the Appendix for more explanation), as

$$F(-\delta) = \sum_{k=0}^{N} F(x_k) l_k(-\delta).$$

We choose the interpolating points as the zeros of the Chebyshev polynomials on $I_{m-1}, I_{m-2}, \ldots, I_3$ with $x_0 = 0$ and $x_1 = b_m$. Then, we have two ways to determine the upper bound of the Green function on K. The first way is to determine the basis polynomial which has maximal absolute value. Let's say $|l_k(-\delta)|$ is the maximal one, then with the condition $|F(x_k)| \leq 1$, we have

$$\frac{\log|F(-\delta)|}{N} \leq \frac{\log\sum_{k=0}^{N}|F(x_k)l_k(-\delta)|}{N} \leq \frac{\log N}{N} + \frac{\log|l_k(-\delta)|}{N},$$

where N denotes the total degree.

The second way is to find a function which is an upper bound for $|l_k(-\delta)|$ for $k = 0, 1, \ldots, N$, and calculate the above expression for this upper bound function. In our case, we prefer to use the second way as the first way requires more calculations and theorems.

However, in order to find the upper bound of the Green function for K, we do not use the degrees of the Chebyshev polynomials defined on intervals I_k which were used at evaluating the lower bound of the Green function for K, since if we take n_k as in the previous chapter, then the Lagrange basis polynomials do not give us the desired bound. The reason for this problem is that the degrees of the Chebyshev polynomials which are defined on first few intervals are so big when compared to the length of these intervals. For this reason, we have to reduce the degrees of the basis polynomials for the first few intervals. Let m_k denote the new degrees such that $m_k = n_k - v_k$, where $v_k = \lfloor \frac{n_k \log 8}{2^k} \rfloor$, where $\lfloor x \rfloor$ denotes the floor function of x.

The theorem below gives us an upper bound of a basis polynomial.

Theorem 6.0.9 *Let $x_k \in I_q$, then*

$$\log |l_k(-\delta)| \leq n_s + 2^s n_s,$$

where $3 \leq q \leq m - 1$.

Proof: In this proof, we use Corollary 4.2.1, Corollary 4.2.4, Corollary 4.2.6 and Corollary 4.2.7. First of all let $x_k \in I_q$ where $3 \leq q \leq s$, then

$$|l_k(-\delta)| \leq \frac{\delta(\delta + b_m)(\delta + b_{m-1})(\delta + b_{m-2})^{m_{m-2}} \ldots (\delta + b_q)^{m_q}}{a_q^2(a_q - b_{m-1})(a_q - b_{m-2})^{m_{m-2}} \ldots (a_q - b_{q+1})^{m_{q+1}} m_q (\frac{b_q^2}{4})^{m_q - 1}}$$
$$\prod_{y=3}^{q-1} \prod_{x_a \in I_y} \left(1 + \frac{\delta + x_k}{x_a - x_k}\right) \leq \frac{\delta^{1+1+m_{m-1}+\ldots+m_{s+1}} (2\delta)^{m_s} b_{s-1}^{m_{s-1}} \ldots b_q^{m_q} 4^{m_q}}{b_q^{1+1+1+m_{m-1}+\ldots+m_{q+1}} 4 m_q (b_q)^{2m_q - 2}}$$
$$\prod_{k=s+1}^{m} \left(1 + \frac{b_k}{\delta}\right)^{n_k} \prod_{k=q}^{s-1} \left(1 + \frac{\delta}{b_k}\right)^{n_k} \left(\prod_{k=q+1}^{m} \left(1 - b_q - \frac{b_k}{b_q}\right)^{n_k} \right)^{-1}$$

$$\prod_{b=3}^{q-1}\prod_{x_a\in I_b}\Big(1+\frac{\delta+x_k}{x_a-x_k}\Big)\le\frac{AB_qD_q}{C_q}\frac{\delta^{2+n_{m-1}+\ldots+n_{s+1}}(2\delta)^{n_s}b_{s-1}^{n_{s-1}}\ldots b_q^{n_q}4^{n_q}}{b_q^{1+1+1+n_{m-1}+\ldots+n_{q+1}+2n_q-2}}$$
$$\frac{b_q^{v_{m-1}+\ldots+v_{q+1}+2v_q}}{\delta^{v_{m-1}+\ldots+v_{s+1}}(2\delta)^{v_s}b_{s-1}^{v_{s-1}}\ldots b_q^{v_q}4^{v_q}},$$

where $A=\prod_{k=s+1}^{m}\Big(1+\frac{b_k}{\delta}\Big)^{n_k}$, $B_q=\prod_{k=q}^{s-1}\Big(1+\frac{\delta}{b_k}\Big)^{n_k}$, $C_q=\prod_{k=q+1}^{m}\Big(1-b_q-\frac{b_k}{b_q}\Big)^{n_k}$ and $D_q=\prod_{b=1}^{q-1}\prod_{x_a\in I_b}\Big(1+\frac{\delta+x_k}{x_a-x_k}\Big)$.

Now, note that $\frac{n_k\log 8}{2^k}-1\le v_k\le\frac{n_k\log 8}{2^k}$, so $\Big(\frac{1}{8}\Big)^{n_q}\le b_q^{v_q}\le\Big(\frac{1}{8}\Big)^{n_q}b_q^{-1}$, and by Corollary 4.2.6, we have

$$\Big(\frac{1}{8}\Big)^{\frac{4+2\sqrt2}{3+2\sqrt2}n_q}\le b_q^{\sum_{k=q}^{m-1}v_q}\le\Big(\frac{1}{8}\Big)^{\frac{4+2\sqrt2}{3+2\sqrt2}n_q}8^{(1+\sqrt2)(2-\sqrt2)^{m-q}}b_q^{-(m-q)}.$$

Thus,

$$|l_k(-\delta)|\le\frac{AB_qD_q}{C_q}\frac{\delta^{1+n_{s-1}-3n_s}2^{n_s}b_q^{n_{q-2}-3n_{q-1}}b_s^{3n_{s-1}-n_{s-2}}b_q^{n_q}4^{n_q}}{b_q^{2+n_{q+1}-3n_q}b_q^{2n_q-2}}$$
$$\frac{\Big(\frac{1}{8}\Big)^{\frac{7+4\sqrt2}{3+2\sqrt2}n_q}8^{(1+\sqrt2)(2-\sqrt2)^{m-q}}b_q^{-(m-q-1)}}{\Big(\frac{1}{8}\Big)^{\frac{4+2\sqrt2}{3+2\sqrt2}n_s}\Big(\frac{1}{8}\Big)^{\frac{1}{2}[(2+\sqrt2)^{m-q}+(2-\sqrt2)^{m-q}-(2+\sqrt2)^{m-s}-(2-\sqrt2)^{m-s}]}}$$
$$=\frac{AB_qD_qF_q}{C_qG_q}\frac{4^{n_q}2^{n_s}}{\delta^{n_s-1}}8^{\Upsilon},$$

where $F_q=8^{(1+\sqrt2)(2-\sqrt2)^{m-q}}$, $G_q=b_q^{(m-q-1)}$, and $\Upsilon=\frac{1}{2}[(2+\sqrt2)^{m-q}+(2-\sqrt2)^{m-q}-(2+\sqrt2)^{m-s}-(2-\sqrt2)^{m-s}]-\frac{7+4\sqrt2}{3+2\sqrt2}n_q+\frac{4+2\sqrt2}{3+2\sqrt2}n_s$. Now note that, since $(2-\sqrt2)^{s-q}\le 1$. we have

$$\begin{aligned}\Upsilon&\le\frac{(2+\sqrt2)^{m-q}}{2}\Big(1-\frac{7+4\sqrt2}{4+3\sqrt2}\Big)+\frac{(2+\sqrt2)^{m-s}}{2}\Big(\frac{4+2\sqrt2}{4+3\sqrt2}-1\Big)\\&=-\frac{1}{2\sqrt2}\frac{2+3\sqrt2}{4+3\sqrt2}(2+\sqrt2)^{m-q}-\frac{1}{2\sqrt2}\frac{2}{4+3\sqrt2}(2+\sqrt2)^{m-s}\\&\le-\frac{2+3\sqrt2}{4+3\sqrt2}n_q-\frac{2}{4+3\sqrt2}n_s.\end{aligned}$$

Thus,

$$|l_k(-\delta)|\le\frac{AB_qD_qF_q}{C_qG_q}\frac{4^{n_q}2^{n_s}}{\delta^{n_s-1}}\frac{1}{8^{\frac{2+3\sqrt2}{4+3\sqrt2}n_q+\frac{2}{4+3\sqrt2}n_s}}.$$

Now we have

$$\log D_q = \sum_{b=3}^{q-1} \sum_{x_a \in I_b} \log(1 + \frac{\delta + x_k}{x_a - x_k}) \leq (\delta + b_q) \sum_{k=3}^{q-1} \frac{n_k}{b_k} \leq (2 + \frac{1}{6}) n_{q-1} b_{q-1}$$

$$-\log C_q = -\sum_{k=q+1}^{m} n_k \log(1 - b_q - \frac{b_k}{b_q}) \leq \sum_{k=q+1}^{m} n_k \Big(b_q + \frac{b_k}{b_q} \Big) \leq 4 n_{q+1} b_q$$

$$\leq \frac{1}{6} n_{q-1} b_{q-1}$$

$$\log A = \sum_{k=s+1}^{m} n_k \log 1 + \frac{b_k}{\delta} \leq \frac{1}{\delta} \sum_{k=s+1}^{m} n_k b_k \leq 2 n_{s+1} b_s \leq \frac{1}{6} n_{q-1} b_{q-1}$$

$$\log B_q = \sum_{k=q}^{s-1} n_k \log 1 + \frac{\delta}{b_k} \leq \delta \sum_{k=q}^{s-1} \frac{n_k}{b_k} \leq 2 n_{s-1} b_{s-1} \leq \frac{1}{6} n_{q-1} b_{q-1}$$

$$\log F_q = 3 \log 2 (1 + \sqrt{2})(2 - \sqrt{2})^{m-q} \leq 6 (2 - \sqrt{2})^{m-q} \leq \frac{1}{6} n_{q-1} b_{q-1}$$

$$\log G_q = 2^q (m - q - 1) \leq \frac{1}{6} n_{q-1} b_{q-1}.$$

Thus,

$$\log \frac{A B_q D_q F_q}{C_q G_q} \frac{4^{n_q} 2^{n_s}}{\delta^{n_s - 1}} \frac{1}{8^{\frac{2+3\sqrt{2}}{4+3\sqrt{2}} n_q + \frac{2}{4+3\sqrt{2}} n_s}} \leq 3 n_{q-1} b_{q-1} + n_q \log 2 (2 - 3 \frac{2 + 3\sqrt{2}}{4 + 3\sqrt{2}})$$
$$+ n_s \log 2 (1 - 3 \frac{2}{4 + 3\sqrt{2}}) + (n_s - 1) 2^s.$$

Note that, $3 n_{q-1} b_{q-1} + n_q \log 2(2 - 3\frac{2+3\sqrt{2}}{4+3\sqrt{2}}) + n_s \log 2(1 - 3\frac{2}{4+3\sqrt{2}}) - 2^s \leq 0$, since $q \geq 3$. Thus,

$$\log |l_k(-\delta)| \leq n_s 2^s.$$

Now let $x_k \in I_q$, where $s \leq q \leq m-1$, then by similar calculations above and by Corollary 4.2.7, we have

$$|l_k(-\delta)| \leq \frac{A D_s}{C_q E_q} \frac{2^{n_s} 4^{n_q} b_q}{\delta^{n_s}} \frac{b_q^{v_{m-1} + \ldots + v_{q+1} + 2 v_q} b_{q-1}^{v_{q-1}} \ldots b_s^{v_s}}{b_s^{v_{m-1} + \ldots + v_s}}$$

$$\leq \frac{A D_s}{C_q E_q} \frac{\left(\frac{1}{8}\right)^{\frac{7+4\sqrt{2}}{3+2\sqrt{2}} n_q} 8^{(1+\sqrt{2})(2-\sqrt{2})^{m-q}} b_q^{-(m-q-1)}}{\left(\frac{1}{8}\right)^{\frac{4+2\sqrt{2}}{3+2\sqrt{2}} n_s} 8^{\frac{1}{2}[(2+\sqrt{2})^{m-s} + (2-\sqrt{2})^{m-s} - (2+\sqrt{2})^{m-q} - (2-\sqrt{2})^{m-q}]} \delta^2}$$

$$\frac{2^{n_s} 4^{n_q} b_q}{\delta^{n_s}} \leq \frac{A D_s}{C_q E_q G_q} \frac{2^{n_s} 4^{n_q} b_q}{\delta^{2 + n_s}} 8^{\Upsilon},$$

where $E_q = \prod_{k=s}^{q-1}\left(1 - b_k - \frac{b_q}{b_k}\right)^{n_k}$, and $\Upsilon = \frac{1}{2}[(2+\sqrt{2})^{m-q} + (2-\sqrt{2})^{m-q} - (2+\sqrt{2})^{m-s} - (2-\sqrt{2})^{m-s}] - \frac{7+4\sqrt{2}}{3+2\sqrt{2}}n_q + \frac{4+2\sqrt{2}}{3+2\sqrt{2}}n_s + (1+\sqrt{2})(2-\sqrt{2})^{m-q}$. Note that in this case

$$\Upsilon \leq -\frac{2+3\sqrt{2}}{4+3\sqrt{2}}n_q - \frac{2}{4+3\sqrt{2}}n_s + (2\sqrt{2}-\frac{3}{2})(2-\sqrt{2})^{m-q} + \frac{9-5\sqrt{2}}{4}(2-\sqrt{2})^{m-s}.$$

Thus,

$$|l_k(-\delta)| \leq \frac{AD_sH_q}{C_qE_qG_q}\frac{2^{n_s}4^{n_q}b_q}{\delta^{2+n_s}}\frac{1}{8^{\frac{2+3\sqrt{2}}{4+3\sqrt{2}}n_q + \frac{2}{4+3\sqrt{2}}n_s}},$$

where $H_q = 8^{(2\sqrt{2}-\frac{3}{2})(2-\sqrt{2})^{m-q} + \frac{9-5\sqrt{2}}{4}(2-\sqrt{2})^{m-s}}$. In this case, like the calculations above, we have

$$\log\frac{AD_sH_q}{C_qE_qG_q} \leq 7n_{s-1}b_{s-1}.$$

Hence,

$$\log|l_k(-\delta)| \leq 7n_{s-1}b_{s-1} + n_s\log 2(1 - \frac{6}{4+3\sqrt{2}}) - 2^q + n_q\log 2(2 - 3\frac{2+3\sqrt{2}}{4+3\sqrt{2}}) + 2^{s+2} + n_s2^s.$$

Note that, $7n_{s-1}b_{s-1} + n_s\log 2(1 - \frac{6}{4+3\sqrt{2}}) - 2^q + n_q\log 2(2 - 3\frac{2+3\sqrt{2}}{4+3\sqrt{2}}) + 2^{s+2} \leq n_s$. Thus if $x_k \in I_q$, where $3 \leq q \leq m-1$, we have

$$\log|l_k(-\delta)| \leq n_s + 2^sn_s.$$

Theorem 6.0.10 *Let $x_1 = b_m$, then*

$$\log|l_1(-\delta)| \leq 3n_s + 2^sn_s.$$

Proof: If we make the calculations as in the proof of the previous theorem, we get

$$|l_1(-\delta)| \leq \frac{AD_s}{E_m}\frac{2^{n_s}}{\delta^{n_s}}\frac{b_{m-1}^{v_{m-1}}\cdots b_s^{v_s}}{b_s^{v_{m-1}+\ldots+v_s}} \leq \frac{AD_s}{E_m}\frac{2^{n_s}}{\delta^{n_s}}\frac{8^{[n_s(3+\frac{4+2\sqrt{2}}{3+2\sqrt{2}})-n_{s-1}+1]}}{\delta^2}.$$

Since, $n_s(3+\frac{4+2\sqrt{2}}{3+2\sqrt{2}})-n_{s-1}+1\leq(5-3\sqrt{2})n_s$, we get

$$|l_1(-\delta)|\leq\frac{AD_s}{E_m}\frac{2^{n_s}}{\delta^{2+n_s}}8^{(5-3\sqrt{2})n_s}.$$

By similar calculations at previous theorem, we get

$$\log|l_1(-\delta)|\leq 5n_{s-1}b_{s-1}+n_s\log 2(16-9\sqrt{2})+2^{s+2}+2^s n_s.$$

Note that $5n_{s-1}b_{s-1}+n_s\log 2(16-9\sqrt{2})+2^{s+2}\leq 3n_s$. Thus,

$$\log|l_1(-\delta)|\leq 3n_s+2^s n_s.$$

Remark 6.0.11 *Let $x_0=0$, then the corresponding basis polynomials are expanded, we get*

$$|l_0(-\delta)|\leq|l_1(-\delta)|.$$

Thus, there is no need to find an upper bound for $\log|l_0(-\delta)|$.

Theorem 6.0.12 *Let $\tilde{N}=2+\sum_{k=3}^{m-1}m_k$ and let $x_0=0,x_1=b_m$ and $x_k\in I_q$, where $3\leq q\leq m-1$. Then*

$$\log|l_k(-\delta)|\leq 3n_s+2^s n_s$$

for $k=0,1,\ldots,\tilde{N}-1$.

Proof: This theorem is a result of Theorem 6.0.9 and Theorem 6.0.10.

Theorem 6.0.13

$$g_K(-\delta)\leq 45\varphi^{\gamma},$$

where $\varphi=\left(\log\frac{1}{\delta}\right)^{-1}$, $\gamma=\frac{\log(1+\frac{\sqrt{2}}{2})}{2}$, and $\delta=b_s=e^{-2^s}$.

Proof: Let $F(-\delta)$ realize the supremum value at Theorem 3.2.5, then we have

$$g_K(-\delta) \le \frac{\log|F(-\delta)|}{\tilde{N}} \le \frac{\log \tilde{N}}{\tilde{N}} + \frac{\log|l_k(-\delta)|}{\tilde{N}} \le \frac{\log \tilde{N}}{\tilde{N}} + \frac{3n_s + 2^s n_s}{\tilde{N}}.$$

Note that $\frac{\log \tilde{N}}{\tilde{N}} + \frac{3n_s}{\tilde{N}} \le \frac{7}{10}(2-\sqrt{2})^s$. Moreover, $\frac{9}{10} \le (1-\frac{\log 8}{2^k})$, for $k = 3, 4, \dots, m-1$. Thus, $\frac{9}{10}N \le \tilde{N} \le N$, where $N = \sum_{k=3}^{m-1} n_k$. Hence,

$$g_K(-\delta) \le \frac{7}{10}(2-\sqrt{2})^s + \frac{10}{9}\frac{2^s n_s}{N}.$$

Additionally, $\frac{n_s}{N} \le \frac{1}{\sqrt{2}}(2+\sqrt{2})^{3-s}$. Therefore,

$$\begin{aligned} g_K(-\delta) &\le \frac{7}{10}(2-\sqrt{2})^s + \frac{10}{9}(2+\sqrt{2})^3(2-\sqrt{2})^s \\ &= (2-\sqrt{2})^s(\frac{10}{9}(2+\sqrt{2})^3 + \frac{7}{10}). \end{aligned}$$

Note that $(2-\sqrt{2})^s = \left(\log\frac{1}{\delta}\right)^{\frac{\log(2-\sqrt{2})}{2}} = \varphi^\gamma$, where $\varphi = \left(\log\frac{1}{\delta}\right)^{-1}$, and $\gamma = \frac{\log(1+\frac{\sqrt{2}}{2})}{2}$. So,

$$g_K(-\delta) \le (\frac{10}{9}(2+\sqrt{2})^3 + \frac{7}{10})\varphi^\gamma \le 45\varphi^\gamma.$$

Theorem 6.0.14 *Let* $0 < \delta << 1$ *be fixed. Then*

$$3\varphi(\delta)^\gamma \le g_K(z) \le 45\varphi(\delta)^\gamma,$$

where $z \in A$, $A = \{z | dist(z, K) \le \delta\}$ *and* φ *and* γ *are as defined in Theorem 6.0.13.*

Proof: Since $\delta << 1$, the there exist s such that $b_{s+1} \le \delta \le b_s$. Then we have, $\varphi(b_{s+1}) = \varphi(b_s^2) = \frac{1}{2}\varphi(b_s)$. Thus, $\frac{1}{2}\varphi(b_s) \le \varphi(\delta) \le \varphi(b_s)$. By the structure of the set K, the modulus of continuity is attained at $-\delta$ for $g_K(z)$ with $dist(z, K) \le \delta$. Thus,

$$g_K(z) \le g_K(-\delta) \le 45\varphi(\delta)^\gamma,$$

where φ and γ are same as defined in Theorem 6.0.13. Note that, by Theorem 5.0.8, we have

$$3\varphi(\delta)^{\gamma} \leq g_K(z) \leq 45\varphi(\delta)^{\gamma}.$$

Appendix A

Chebyshev Polynomials

Definition A.0.15 *The polynomial $T_n(z) = z^n + c_1 z^{n-1} + \ldots + c_n$ with the least maximum modulus on a compact subset K of $\mathbb{C}$ is called the Chebysev polynomial of degree n for K.*

The theorem below guarantees the existence of the Chebyshev polynomials for any compact set K.

Theorem A.0.16 *[4] For any n, there exists a polynomial of degree n whose maximum modulus is minimal on a compact set K.*

Since, in this thesis, Chebyshev polynomials on closed intervals are used more often, some properties of it should be given.

Definition A.0.17 *Let Π_n denote all polynomials of degree at most n. The Chebyshev polynomials on the interval $[-1, 1]$ are usually denoted by $T_n(x)$ and uniquely defined by the condition*

$$\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} T_r(x) T_s(x) dx = 0, \ r \neq s, \tag{A.1}$$

where $T_r \in \Pi_n$ and $T_r(1) = 1$ for $r \geq 0$.

From the definition above, we can conclude that

$$T_n(x) = \cos n(\arccos x) \tag{A.2}$$

because if we apply the integration in (A.1) with $T_n(x)$ as in (A.2) with $x = \cos\theta$, substitution $0 \leq \theta \leq \pi$, we get

$$\int_0^\pi \cos r\theta \cos s\theta d\theta.$$

Due to the orthogonality of trigonometric functions, the above integral is 0. Moreover, for $\theta = 0$, we get $\cos n(\arccos(\cos 0)) = \cos n(\arccos(1)) = 1$. Now we will obtain some properties of Chebyshev polynomials.

Property 1 $T_0(x) = 1$ *and* $T_1(x) = x$, *and* $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

Proof: If we apply the substitution $x = \cos\theta$ to (A.2), we get

$$T_n(\cos\theta) = \cos n\theta.$$

For $n = 0$, we get $T_0(x) = 1$, and $T_1(x) = x$. Additionally we know by trigonometric identities that $\cos(n+1)\theta + \cos(n-1)\theta = 2\cos n\theta\cos\theta$, which implies $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$.

By using the recurrence relation in Property 1, we can find all Chebyshev polynomials defined on $[-1, 1]$.

$$\begin{aligned}
&T_0(x) = 1\\
&T_1(x) = x\\
&T_2(x) = 2x^2 - 1\\
&T_3(x) = 4x^3 - 3x\\
&T_4(x) = 8x^4 - 8x^2 + 1\\
&T_5(x) = 16x^5 - 20x^3 + 5x\\
&T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1\\
&\vdots
\end{aligned}$$

By using known trigonomeric properties we can find the following three properties of Chebyshev polynomials.

Property 2 *$T_m(T_n(x)) = T_{mn}(x)$ for all nonnegative integers m and n.*

Property 3 *$T_m(x)T_n(x) = \frac{1}{2}(T_{m+n}(x) + T_{|m-n|}(x))$ for all nonnegative integers m and n.*

Property 4 *$T_n(-x) = (-1)^n T_n(x)$ for all nonnegative integers n.*

Property 5 *(**Zeros and Extrema of Chebyshev Polynomials**) Let $\xi_j = \cos(\frac{(2j-1)\pi}{2n})$ and $\eta_j = \cos(\frac{j\pi}{n})$, then ξ_j are the zeros of $T_n(x)$ for $j = 1, 2, \ldots, n$ and η_j are extrema of $T_n(x)$ for $j = 0, 1, \ldots, n$.*

The above property can easily be proved by using $T_n(x) = \cos n\theta$ with $x = \cos\theta$ substitution.

Extension of Chebyshev Polynomials

Chebyshev polynomials can be extended to the whole real line by using DeMoivre's theorem.

$$\begin{aligned} T_n(\cos\theta) = \cos(n\theta) &= \frac{e^{in\theta} + e^{-in\theta}}{2} \\ &= \frac{1}{2}[(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n] \\ &= \frac{1}{2}[(x + i\sqrt{1-x^2})^n + (x - i\sqrt{1-x^2})^n] \\ &= \frac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n]. \end{aligned}$$

Hence,

$$T_n(x) = \frac{1}{2}[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n]$$

for all $|x| \geq 1$.

Let's denote the extended Chebyshev polynomial by $\tilde{T}_n(x)$. By using the zeros of Chebyshev polynomials, we also have

$$\tilde{T}_n(x) = 2^{n-1} \prod_{j=1}^{n} (x - \xi_j)$$

for all $x \in \mathbb{R}$. Here ξ_j's are zeros of $T_n(x)$ on $[-1, 1]$.

Chebyshev Polynomials on any Closed Interval on $\mathbb{R}$

Let $I = [a, b]$ be any closed interval on $\mathbb{R}$. Then consider the mapping

$$\begin{aligned} \chi : &[a, b] \to [-1, 1], \\ x \mapsto &\frac{x - \frac{b+a}{2}}{\frac{b-a}{2}}. \end{aligned}$$

Note that, the mapping is 1-1 and onto. Moreover, it carries b to 1 and a to -1. Hence, the Chebyshev polynomial on any compact interval I on $\mathbb{R}$, denote by $\tilde{T}_{nI}(x)$, is

$$\tilde{T}_{nI}(x) = \tilde{T}_n(\frac{x - \frac{b+a}{2}}{\frac{b-a}{2}}).$$

Appendix B

Lagrange Interpolation

Let $\{x_0, x_1, \ldots, x_n\} \subset [a, b]$ be such that $x_k \neq x_l$ for $k \neq l$. Let $y_k = f(x_k)$.

Theorem B.0.18 *For all y_k, where $k = 0, 1, \ldots, n$, there exists a unique polynomial P_n such that $P_n(x_k) = y_k$ for $k = 0, 1, \ldots, n$.*

Proof: Let $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, then by $P(x_k) = y_k$, we get

$$\begin{aligned}
&a_0 + a_1 x_0 + \ldots + a_n x_0^n = y_0 \\
&a_0 + a_1 x_1 + \ldots + a_n x_1^n = y_1 \\
&\vdots \\
&a_0 + a_1 x_n + \ldots + a_n x_n^n = y_n.
\end{aligned}$$

Note that this system forms a Vandermonde matrix, and the determinant of this matrix is not equal to zero as $x_k \neq x_l$ for $k \neq l$. Thus, the system has unique solution.

Let's define $w(x) = \prod_{k=0}^{n}(x - x_k)$, then clearly $w \in \Pi_{n+1}$. If we define the basis polynomials by $l_k(x) = \frac{w(x)}{(x-x_k)w'(x_k)}$ for $k = 0, 1, \ldots, n$, then $l_k \in \Pi_n$. The most important property of the basis polynomials is $l_k(x_i) = \delta_{ki}$ for $i = 0, 1, \ldots, n$, where

$$\delta_{ki} = \begin{cases} 1, & \text{if } k = i, \\ 0, & \text{if } k \neq i. \end{cases}$$

This method is called the *Lagrange interpolation* $formula$. We write it as

$$L_n f(x) = \sum_{k=0}^{n} f(x_k) l_k(x).$$

The uniqueness property allows us to regard the interpolation process as an operator from $C[a, b]$ to Π_n, which depends on the choice of the fixed points $\{x_0, x_1, \ldots, x_n\}$. Moreover, the operator is linear in and independent of f.

Proposition B.0.19 *Lagrange interpolation have the following properties.*

i) *Let* $P \in \Pi_n$, *then* $L_n(P, x) = P(x)$.

ii) $\sum_{k=0}^{n} l_k(x) = 1$.

iii) $\sum_{k=0}^{n} (n - x_k)^p l_k(x) = 0$, *where* $p \leq n$.

Proof: The first property follows from the definition. For the second property, let $P(x) = 1$, then $L_n(1, x) = 1 = \sum_{k=0}^{n} l_k(x)$. For the third property, let $(x - n)^p \in \Pi_n$, then $(x - n)^p = \sum_{k=0}^{n} (x_k - n) l_k(x)$. Now let $x = n$, then result follows.

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