CONSTRUCTION OF MODULAR FORMS WITH POINCARÉ SERIES

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By
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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

CONSTRUCTION OF MODULAR FORMS WITH POINCARÉ SERIES

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July, 2010

In this thesis, we construct holomorphic modular forms of integral weight $k > 2$ for the principle congruence subgroup $\bar{\Gamma}(N)$ by means of Poincaré series. We start by providing the necessary background information on modular forms. Then, we show that Poincaré series are in fact holomorphic modular forms and we obtain explicit formulas for their Fourier coefficients. For the special case when Poincaré series are Eisenstein series, their Fourier coefficients become relatively simple. We give Fourier coefficients of the Eisenstein series belonging to the principle congruence subgroup. Finally, as an application of what has been studied, we construct Eisenstein series for the Hecke congruence supgroup.

Keywords: Poincaré series, Eisenstein series, modular forms, cusp forms, modular group, congruence subgroups.

ÖZET

MODÜLER FORMLARIN POINCARÉ SERİLERİ KULLANILARAK OLUŞTURULMASI

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Matematik, Yüksek Lisans
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Bu tezde, Poincaré serileri vasıtasıyla esas denklik altgrubu $\bar{\Gamma}(N)$ için 2'den büyük tam sayı ağırlıklı analitik modüler formlar inşa ediyoruz. Modüler formlarla ilgili gerekli temel bilgileri temin ederek başlıyoruz. Daha sonra, Poincaré serilerinin aslında analitik modüler formlar olduğunu gösteriyoruz ve onların Fourier katsayıları için açık formüller elde ediyoruz. Poincaré serilerinin Eisenstein serilerine dönüştüğü özel durum için Fourier katsayıları oldukça basitleşiyor. Esas denklik altgrubuna dahil olan Eisenstein serilerinin Fourier katsayılarını veriyoruz. En sonunda, çalışılanların bir uygulaması olarak, Hecke denklik altgrubu için bir Eisenstein serisi inşa ediyoruz.

Anahtar sözcükler: Poinkaré Serileri, Eisenstein Serileri, modüler formlar, cusp formlar, modüler grup, denklik altgrupları .

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Construction of Modular Forms with Poincaré Series

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July 30, 2010

Chapter 1

Introduction

Let Γ be a subgroup of the full modular group

$$\Gamma(1) = SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \; a, b, c, d \in \mathbb{Z}, \; ad - bc = 1 \right\},$$

of finite index and υ be a multiplier system of real weight k on Γ (see definition 3.1.1). An unrestricted modular form is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ on the upper half plane $\mathbb{H}$ satisfying

$$f(Tz) = \upsilon(T)(cz + d)^k f(z)$$

for all $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, and all $z \in \mathbb{H}$. The concept of modular forms first arose in connection with the theory of elliptic functions in the first period of the nineteenth century. The theory was further developed by Felix Klein in 1980s as the concept of automorphic forms for one variable became understood. The term modular form as a systematic description is usually attributed to Eric Hecke [6] whose many contributions to the subject showed that modular forms have far reaching applications in number theory.

There are many ways of constructing modular forms. Our special interest shall be the one in which the Poincaré Series are employed as building blocks.

This method is particularly convenient for modular forms of real weight $k > 2$ and with arbitrary multiplier systems.

The foundation of the general theory of Poincaré series were laid by Petersson whose work also applies more generally to automorphic forms on any horocyclic group having a finite number of generators. Formulae for the Fourier coefficients of Poincaré series are given in Petersson [15, 16, 17] and Selberg [22]. An alternative method applied to the full modular group $\Gamma(1)$ is given by Schwandt [21].

Eisenstein series are special cases of Poincaré series. For many subgroups of $\Gamma(1)$ that are of interest to us, the Fourier coefficients of the Eisenstein series are quite simple. The properties of Eisenstein series were first studied by Hecke [5], who showed that if f is an entire modular form of weight $k > 2$ there exist a linear combination F of Eisenstein series such that $f - F$ is a cusp form. He also gave the explicit formula for the Fourier coefficients of Eisenstein series in [5].

The main purpose of this thesis is to construct modular forms of integral weight $k > 2$ for the congruence subgroup $\bar{\Gamma}(N)$ by means of Poincaré series belonging to $\bar{\Gamma}(N)$. We aim to obtain explicit formulae for the Fourier coefficients of the Eisenstein series on $\bar{\Gamma}(N)$. For that purpose we first calculate formulae for the Fourier coefficients of the Poincaré series belonging to $\bar{\Gamma}(N)$ and apply these results to the particular case when Poincaré series are Eisenstein series. We also give an application in which we construct an Eisenstein series for the Hecke congruence supgroup $\Gamma_0(N)$ with $N > 2$.

The content of this thesis is organized as follows:

In Chapter 2, we study necessary definitions and facts about the full modular group $\Gamma(1)$. We mainly follow [18], [4] and [20]. Special attention is given to the subgroups of finite index in $\Gamma(1)$, particularly to the congruence subgroups. Next, mapping properties for the elements of $\hat{\Gamma}(1)$ are closely investigated so that a complete classification of linear fractional transformations in $\hat{\Gamma}(1)$ is given.

In chapter 3, we explain the general theory of modular forms of arbitrary real weight by using the notations and results found in [18] and [8]. We present a general approach to automorphic factors and multiplier systems defined for the modular group and its subgroups. Then, we study unrestricted modular forms, holomorphic modular forms, entire modular forms and cusp forms.

In chapter 4, we introduce Poincaré Series as in the form defined by Rankin in [18] and use these series to construct modular forms of real weight $k > 2$ on a subgroup Γ of $\Gamma(1)$. We obtain Fourier expansions of the Poincaré series on Γ. Then we restrict our attention to the case when $\Gamma = \bar{\Gamma}(N)$ and calculate the explicit formula for the Fourier coefficients of the Poincaré series belonging to $\bar{\Gamma}(N)$. Our ultimate goal, in this chapter, is to evaluate the Fourier coefficients of the Eisenstein series belonging to $\bar{\Gamma}(N)$ with $N > 1$ by applying the results that we found in previous sections.

In Chapter 5, we present an application which exemplifies the important results emphasized in foregoing chapters by constructing Eisenstein series for the Hecke congruence subgroup $\Gamma_0(N)$.

Chapter 2

Modular Group

This chapter is concerned with the group of linear fractional transformations $\hat{\Gamma}(1)$ which are associated by the matrices belonging to full modular group $\Gamma(1)$. The groups of particular interest shall be those on which modular functions and modular forms defined. In this chapter, we shall first present necessary definitions, properties and results about the modular group $\Gamma(1)$ and its congruence subgroups as an introduction to succeeding sections. Next, we shall introduce fundamental region $\mathfrak{R}$ for the modular group $\hat{\Gamma}(1)$ and use this fundamental region $\mathfrak{R}$ to construct fundamental regions $\mathfrak{R}_{\hat{\Gamma}}$ for congruence subgroups whose coset representations are known. Then we shall classify the linear fractional transformations in $\hat{\Gamma}(1)$ and analyze parabolic and elliptic transformations in greater detail. This classification give rise to a classification of the fixed points in $\mathbb{C}$. We shall use the standard notations and some facts specialized in [4], [8], and [18]. The content of this chapter is standard and can be found in any books on modular forms (see for example,[1] and [20]).

2.1 The Modular Group and Congruence Subgroups

In this section we restrict our attention to the modular group and review some of its basic properties. We start with the definition of the modular group.

Definition 2.1.1. The homogeneous modular group, denoted by $\Gamma(1)$, is the group of 2×2 matrices defined by

$$\Gamma(1) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d, \in \mathbb{Z} \quad \text{and} \quad ad - bc = 1 \right\} = SL_2(\mathbb{Z})$$

Definition 2.1.2. The inhomogeneous modular group, denoted by $\hat{\Gamma}(1)$, is the group of linear fractional transformations T,

$$T : z \rightarrow \frac{az+b}{cz+d}, \quad ad - bc = 1,$$

where $a, b, c, d, \in \mathbb{Z}$ and $z \in \bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We can identify each transformation T by the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$ clearly determine the same linear fractional transformation. Therefore the group of distinct linear fractional transformations is the quotient group $\hat{\Gamma}(1) \cong \Gamma(1)/\Lambda$ where Λ denotes the subgroup consisting of I and $-I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$.

With each $T \in \Gamma(1)$ we associate a linear fractional mapping $T(z) = \frac{az+b}{cz+d}$ defined on $\bar{\mathbb{C}}$ and we write for brevity Tz in place of $T(z)$. By writing Γ for the homogeneous group and $\hat{\Gamma}$ for the associated inhomogeneous group we indicate that we regard the latter as being determined by the former. This point of view is especially convenient when we are concerned with algebraic properties of groups, in particular,with multiplier systems.

It is well known that $\Gamma(1)$ is generated by the matrices

$$U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.1}$$

The modular group $\Gamma(1)$ has many subgroups of special interest in number theory. An important class of subgroups of modular group consist of what are called congruence subgroups. Let N be a positive integer, then

$$\Gamma(N) = \{T \in \Gamma(1) | \quad T \equiv I \quad (mod\ N)\}$$

is called the *principle congruence group of level* N. We also write

$$\bar{\Gamma}(N) = \{T \in \Gamma(1) | \quad T \equiv \pm I \quad (mod\ N)\}$$

These two homogeneous groups $\Gamma(N)$ and $\bar{\Gamma}(N)$ give rise to the same inhomogeneous group which we denote $\hat{\Gamma}(N)$. Both $\Gamma(N)$ and $\bar{\Gamma}(N)$ are normal subgroups of the modular group $\Gamma(1)$ and $\hat{\Gamma}(N)$ is a normal subgroup of $\hat{\Gamma}(1)$. Any subgroup of the modular group $\Gamma(1)$ which contains $\Gamma(N)$ is called a *congruence subgroup of level* N. The following examples are of interest to us.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | \quad c \equiv 0 \quad (mod\ N) \right\}$$

$$\Gamma^0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | \quad b \equiv 0 \quad (mod\ N) \right\}$$

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) | \quad c \equiv 0 \quad (mod\ N), \quad a \equiv d \equiv 1 \quad (mod\ N) \right\}$$

The first one called the *Hecke congruence group.* We note

$$\Gamma(1) = \Gamma_0(1) = \Gamma^0(1) = \Gamma_1(1) = \Gamma(1),$$

$$\Gamma(1) \supset \Gamma_0(N) \supset \Gamma_1(N) \supset \Gamma(N).$$

Since $-I \in \Gamma(N)$ if and only if $n = 1$ or $n = 2$ we have

$$\hat{\Gamma}(N) \cong \Gamma(N)/\Lambda \cong \bar{\Gamma}(N)/\Lambda \qquad (n = 1, 2)$$

$$\hat{\Gamma}(N) \cong \Gamma(N) \cong \bar{\Gamma}(N)/\Lambda \qquad (n \geq 3).$$

Let n be a positive integer, since the number of incongruent matrices T modulo n is clearly less than or equal to n^4, $[\hat{\Gamma}(1) : \hat{\Gamma}(n)]$ is clearly finite. Next theorem gives the exact formula for $[\hat{\Gamma}(1) : \hat{\Gamma}(n)]$.

Theorem 2.1.3. ***[4, Theorem 2.1.4]***

$$[\hat{\Gamma}(1) : \hat{\Gamma}(n)] = \begin{cases} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & \text{if} \quad n = 1, 2 \\ \frac{1}{2} n^3 \prod_{p|n} \left(1 - \frac{1}{p^2}\right) & \text{if} \quad n \geq 3 \end{cases} \tag{2.2}$$

Lemma 2.1.4.

$$[\hat{\Gamma}_0(n) : \hat{\Gamma}(n)] = \begin{cases} n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) & \text{if} \quad n = 1, 2 \\ \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) & \text{if} \quad n \geq 3 \end{cases}$$

Proof. See [18], page 26. □

Now we are able calculate index of $\hat{\Gamma}_0(n)$ in $\hat{\Gamma}(1)$. By Lemma (2.1.4) and Theorem 2.1.3, we deduce

$$[\hat{\Gamma}(1) : \hat{\Gamma}_0(n)] = \frac{[\hat{\Gamma}(1) : \hat{\Gamma}(n)]}{[\hat{\Gamma}_0(n) : \hat{\Gamma}(n)]} = n \prod_{p|n} \left(1 + \frac{1}{p}\right) \tag{2.3}$$

2.2 Fundamental Regions

In this section a fundamental region $\mathfrak{R}$ of $\hat{\Gamma}(1)$ shall be constructed and the connection between $\mathfrak{R}$ and standard fundamental region $\mathfrak{R}_{\hat{\Gamma}}$ of a subgroup $\hat{\Gamma}$ of $\hat{\Gamma}(1)$ shall be emphasized. For that connection, a coset representation set for $\hat{\Gamma}(1)$ over $\hat{\Gamma}$ shall be employed.

Definition 2.2.1. Let Γ be any subgroup of $\Gamma(1)$ and $\mathbb{H}$ be the upper half plane.

Two points $z_1, z_2 \in \mathbb{H}$ are said to be *equivalent* or *congruent* modulo Γ if there exists $T \in \hat{\Gamma}$ such that $Tz_1 = z_2$.

This is clearly an equivalence relation and we write

$$z_1 \equiv z_2 \, (mod \; \Gamma).$$

This equivalence relation divides the upper half plane $\mathbb{H}$ into a disjoint collection of equivalence classes called orbits. The orbit $\hat{\Gamma}z$ is the set of all complex number of the form Tz where $T \in \hat{\Gamma}$.

Definition 2.2.2. Let $\hat{\Gamma} \subseteq \hat{\Gamma}(1)$ and $\mathfrak{R}_{\hat{\Gamma}}$ be an open subset of $\mathbb{H}$. $\mathfrak{R}_{\hat{\Gamma}}$ is called a fundamental region of $\hat{\Gamma}$ if it has the following two properties;
(i) No two distinct point of $\mathfrak{R}_{\hat{\Gamma}}$ are equivalent under $\hat{\Gamma}$.
(ii) If $z \in \mathbb{H}$ then, there exists $z_1 \in \bar{\mathfrak{R}}_{\hat{\Gamma}}$, in the closure of $\mathfrak{R}_{\hat{\Gamma}}$ such that z_1 is equivalent to z under Γ.

Theorem 2.2.3. *A fundamental domain for $\hat{\Gamma}(1)$ is given by*

$$\mathfrak{R} = \{z \in \mathbb{H} | \quad |Re z| < \frac{1}{2} \quad and \quad |z| > 1\}$$

Proof. See [8], page 15. □

Remark 2.2.4. *Once a fundamental region $\mathfrak{R}_{\hat{\Gamma}}$ of $\hat{\Gamma} \subseteq \hat{\Gamma}(1)$ is given, $T(\mathfrak{R}_{\hat{\Gamma}})$ is again a fundamental region of $\hat{\Gamma}$ for any $T \in \hat{\Gamma}$.*

Proof. See [18], page 50. □

We now construct fundamental regions for subgroups $\hat{\Gamma} \subset \hat{\Gamma}(1)$. Suppose that $\hat{\Gamma}$ is a subgroup of $\hat{\Gamma}(1)$ with $[\hat{\Gamma}(1) : \hat{\Gamma}] = n$ so that $\hat{\Gamma}(1)$ can be written as a disjoint union of n cosets

$$\hat{\Gamma}(1) = \bigcup_{i=1}^{n} \hat{\Gamma} A_i$$

where the union is taken over a coset representation $\{A_1, A_2, \ldots, A_n\}$.

Once we know a set of coset representatives of $\hat{\Gamma}(1)$ over a subgroup $\hat{\Gamma}$ of finite index, we can construct a fundamental region $\mathfrak{R}_{\hat{\Gamma}}$ for the subgroup $\hat{\Gamma}$. The following theorem gives the connection between the fundamental region $\mathfrak{R}$ of $\hat{\Gamma}(1)$ and the fundamental region $\mathfrak{R}_{\hat{\Gamma}}$ of the subgroup $\hat{\Gamma}$.

Theorem 2.2.5. ***[9, Theorem 12]*** *Suppose $\hat{\Gamma}$ is a subgroup of $\hat{\Gamma}(1)$ and $[\hat{\Gamma}(1) : \hat{\Gamma}] = n$ with $\hat{\Gamma}(1) = \bigcup_{i=1}^{n} \hat{\Gamma} A_i$. Then $\mathfrak{R}_{\hat{\Gamma}} = \bigcup_{i=1}^{n} A_i \mathfrak{R}$ is a fundamental region of $\hat{\Gamma}$ which we call the standard fundamental region.*

Theorem 2.2.6. ***[9, Corollary 14]*** *Suppose $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ are two conjugate subgroups of $\hat{\Gamma}(1)$ of finite index n with $\hat{\Gamma}_2 = B\hat{\Gamma}_1 B^{-1}$. Then,*

$$\hat{\Gamma}(1) = \bigcup_{i=1}^{n} \hat{\Gamma}_1 A_i \qquad \textit{if and only if} \qquad \hat{\Gamma}(1) = \bigcup_{i=1}^{n} \hat{\Gamma}_2 (BA_i)$$

Theorem 2.2.5 together with the Theorem 2.2.6 allows us to deduce the following corollary.

Corollary 2.2.7. *Let $\hat{\Gamma}_1$ and $\hat{\Gamma}_2$ be two conjugate subgroups of $\hat{\Gamma}(1)$ of finite index n with $\hat{\Gamma}_2 = B\hat{\Gamma}_1 B^{-1}$ and $\hat{\Gamma}(1) = \bigcup_{i=1}^{n} \hat{\Gamma}_1 A_i$, then*

$$\mathfrak{R}_{\hat{\Gamma}_2} = \bigcup_{i=1}^{n} BA_i(\mathfrak{R}_{\hat{\Gamma}(1)}) \quad \textit{is a fundamental region of } \hat{\Gamma}_2. \tag{2.4}$$

2.3 Mapping Properties

In this section, we classify the linear fractional transformations in $\hat{\Gamma}(1)$ as elliptic, parabolic and hyperbolic transformations. This classification of mappings give rise to the classification of the fixed points of $\bar{\mathbb{C}}$. We show that $\hat{\Gamma}_z(1)$, the group consisting of all $T \in \hat{\Gamma}(1)$ that fixes z, is a cyclic group.

2.3.1 Fixed Points and Classification of Transformations

A point $z \in \bar{\mathbb{C}}$ is called a *fixed point* of a mapping $M \in \hat{\Gamma}(1)$ if and only if $Mz = z$. The following theorem will be very useful in classification of fixed points and transformations.

Theorem 2.3.1. *If $M \in \Gamma(1)$ and $trM = t$, then there exists an $L \in \Gamma(1)$ such that if $L^{-1}ML = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then*

$$|\alpha - \frac{1}{2}t| \leq \frac{1}{2}|\gamma|, \qquad |\delta - \frac{1}{2}t| \leq \frac{1}{2}|\gamma|, \qquad |\gamma| \leq |\beta|, \qquad 3\gamma^2 \leq |t^2 - 4|.$$

Proof. See, [18], page 9. □

From Theorem 2.3.1, we deduce that

Corollary 2.3.2. *Let U, S be given by (2.1) and $M \in \Gamma(1)$, then*
(i) If $|trM| = 0$, then $M = \pm L^{-1}SL$ for some $L \in \Gamma(1)$.
(ii) If $|trM| = 1$, then $M = \pm L^{-1}(SU)^r L$ for some $L \in \Gamma(1)$ and $r = 1$ or 2.
(ii) If $|trM| = 2$, then $M = \pm L^{-1}U^k L$ for some $L \in \Gamma(1)$ and $k \in \mathbb{Z}$.

Proof. See, [18], page $43 - 45$. □

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and consider the equation $Mz = z$. Observe that $M\infty = \infty$ if and only if $M = U^k$ for some $k \in \mathbb{Z}$. We assume in the first place that $c \neq 0$ so that $z \neq \infty$. The equation $Mz = z$ is equivalent to

$$cz^2 + (d - a)z - b = 0,$$

which has two, not necessarily distinct, roots, namely

$$z_1, z_2 = \frac{(a-d) \pm [(a+d)^2 - 4bc]^{1/2}}{2c} = \frac{(a-d) \pm [(a+d)^2 - 4]^{1/2}}{2c}$$

where in the last equation we use the fact $ad - bc = 1$. It is clear that the nature of the roots z_1, z_2 depends upon the sign of the integer $(a + d)^2 - 4$.

Case 1: If $|trM| > 2$ then, z_1 and z_2 are distinct real numbers. In this case M is called a *hyperbolic transformation* and z_1 and z_2 are called *hyperbolic fixed points*. The fixed points of such transformations are less important in the theory. It is easy to see that that they are all irrational numbers.

Case 2: If $|trM| = 2$ then, $z_1 = z_2$ and we have one real fixed point. In this case M is called a *parabolic transformation* and z_1 is called a *parabolic fixed pint* or a *cusp*.

Here $trM = \pm 2$ and by Corollary 2.3.2, $M = \pm L^{-1}U^kL$ for some $L \in \Gamma(1)$ and $k \in \mathbb{Z}(k \neq 0)$. Thus $Mz_1 = z_1$ is equivalent to $U^k(Lz_1) = Lz_1$, that is

$$Lz_1 = \infty \quad \text{or} \quad z_1 = L^{-1}\infty.$$

Since $c \neq 0$, z_1 is a finite rational number. So far we have assumed that $c \neq 0$. Now let $c = 0$, then $M = \pm U^k$ fore some $k \in \mathbb{Z}$ and $trM = \pm 2$. If $k = 0$ we obtain the identity transforation under which every point is fixed. Since $U^k\infty = \infty$, $z_1 = \infty$ is also considered as parabolic fixed point. The set of all parabolic fixed points is denoted by $\mathbb{P}$. Let $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, then

$$\mathbb{P} = \{z \in \bar{\mathbb{C}} | z = L^{-1}\infty,\ L \in \hat{\Gamma}(1)\}$$

Now it becomes clear that for a standard fundamental region all cusps are rational (we assume $\infty = \frac{1}{0}$). This result allows us to give an alternative definition of parabolic points (cusps) of a standard fundamental region.

Definition 2.3.3. Let $\hat{\Gamma}$ be a subgroup of $\hat{\Gamma}(1)$ and $\mathfrak{R}_{\hat{\Gamma}}$ be a fundamental region of $\hat{\Gamma}$. *A parabolic point (or a cusp) of* $\hat{\Gamma}$ is any rational point q or $q = \infty$ such that $q \in \bar{\mathfrak{R}}_{\hat{\Gamma}}$

Case 3: If $|trM| < 2$ then z_1 and z_2 are conjugate complex numbers, one of which, say z_1, lies in $\mathbb{H}$. M is then called an *elliptic transforation* and z_1 and z_2 are called *elliptic fixed points*. There are two possibilities here:

(i) If $trM = 0$ then by Corollary 2.3.2, $M = \pm L^{-1}SL$ for some $L \in \Gamma(1)$.

(ii) If $trM = 1$ then by Corollary 2.3.2, $M = \pm L^{-1}(SU)^r L$ for $r = 1, 2$ and some $L \in \Gamma(1)$.

In the case (i), $Mz = z$ is equivalent to $S(Lz) = Lz$, which means Lz is a fixed point for S and therefore $Lz_1 = i$, $Lz_2 = -i$. Hence we have

$$z_1 = L^{-1}i, \quad z_2 = L^{-1}(-i) = \bar{z}_1$$

Here the bar denotes the complex conjugate. These points are called *elliptic fixed points of order* 2. We denote by

$$\mathbb{E}_2 = \{z \in \mathbb{C} | z = L^{-1}i,\ L \in \hat{\Gamma}(1)\}$$

the set of all elliptic fixed points of order 2 in $\mathbb{C}$.
In the case (ii), $Mz = z$ is equivalent to $(SU)^r(Lz) = Lz$ where $r = 1$ or 2.That means Lz is a fixed point for SU or $(SU)^2$. An elementary calculation shows that SU and $(SU)^2$ have fixed points

$$\rho = e^{2\pi i/3} \text{ and } \rho^2$$

so that $Lz_1 = \rho$ and $Lz_2 = \rho^2$. Hence

$$z_1 = L^{-1}\rho,\ z_2 = L^{-1}\rho^2$$

These points are called *elliptic fixed points of order* 3. We denote by

$$\mathbb{E}_3 = \{z \in \mathbb{C} | z = L^{-1}\rho,\ L \in \hat{\Gamma}(1)\}$$

the set of all elliptic fixed points of order 3 in $\mathbb{C}$. We also write

$$\mathbb{E} = \mathbb{E}_2 \cup \mathbb{E}_3$$

We now assume Γ is a subgroup of $\Gamma(1)$, The mappings $T \in \hat{\Gamma}$ can be divide into four classes similarly but some of these classes may be empty. For $m = 2, 3$ we define the set of all elliptic fixed points in $\mathbb{H}$ of elliptic transformations of order

m belonging to $\hat{\Gamma}$ as follows,

$$\mathbb{E}_2(\Gamma) = \{z \in \mathbb{H} | z = L^{-1}i, \text{ for some } L \in \hat{\Gamma}(1), \quad L^{-1}SL \in \hat{\Gamma}\}$$

$$\mathbb{E}_3(\Gamma) = \{z \in \mathbb{H} | z = L^{-1}\rho, \text{ for some } \; L \in \hat{\Gamma}(1), \quad L^{-1}SUL \in \hat{\Gamma}\}$$

We write

$$\mathbb{E}(\Gamma) = \mathbb{E}_2(\Gamma) \cup \mathbb{E}_3(\Gamma)$$

2.3.2 Generators of the Stabilizer Groups

Suppose $z \in \mathbb{H}' = \mathbb{H} \cup \mathbb{P}$. The *stabilizer* of $z\,(mod\,\Gamma)$ is defined to be the subset Γ_z of Γ consisting of all $T \in \Gamma$ for which $Tz = z$. Clearly Γ_z is a subgroup of Γ. For $\Gamma = \Gamma(1)$ we write $\Gamma_z(1)$. The corresponding inhomogeneous groups are denoted by $\hat{\Gamma}_z$ and $\hat{\Gamma}_z(1)$. Evidently $\hat{\Gamma}_z$ is a subgroup of $\hat{\Gamma}_z(1)$. Observe that if $L \in \hat{\Gamma}(1)$ then

$$L^{-1}\hat{\Gamma}_{Lz}L = (L^{-1}\hat{\Gamma}L)_z. \tag{2.5}$$

In particular with $\hat{\Gamma} = \hat{\Gamma}(1)$,

$$L^{-1}\hat{\Gamma}_{Lz}(1)L = \hat{\Gamma}_z(1) \tag{2.6}$$

We note that $\hat{\Gamma}_\infty(1) = \langle U \rangle$, $\hat{\Gamma}_i(1) = \langle S \rangle$ and $\hat{\Gamma}_\rho(1) = \langle SU \rangle$. If $z = L^{-1}\infty$ is a parabolic fixed point(a cusp), then by equation (2.6)

$$\hat{\Gamma}_z(1) = L^{-1}\hat{\Gamma}_\infty(1)L = \langle L^{-1}UL \rangle.$$

If $z = L^{-1}i \in \mathbb{E}_2$, then by equation (2.6)

$$\hat{\Gamma}_z(1) = L^{-1}\hat{\Gamma}_i(1)L = \langle L^{-1}SL \rangle.$$

If $z = L^{-1}i \in \mathbb{E}_3$, then by equation (2.6)

$$\hat{\Gamma}_z(1) = L^{-1}\hat{\Gamma}_\rho(1)L = \langle L^{-1}SUL \rangle.$$

We can summarize what we did above as follows,

$$\hat{\Gamma}_z(1) = \begin{cases} \langle L^{-1}UL \rangle & if\, z = L^{-1}\infty \\ \langle L^{-1}SL \rangle & if\, z = L^{-1}i \\ \langle L^{-1}SUL \rangle & if\, z = L^{-1}\rho \\ \hat{\Lambda} = \{I\} & otherwise \end{cases}$$

where $L \in \hat{\Gamma}(1)$ and $z \in \mathbb{H}' = \mathbb{H} \cup \mathbb{P}$. As it is summarized above, in all cases $\hat{\Gamma}_z(1)$ is a cyclic group.

We denote the *order* of $z\,(mod\; \Gamma)$ by $n(z, \Gamma)$ and define it to be

$$n(z, \Gamma) := [\hat{\Gamma}_z(1) : \hat{\Gamma}_z].$$

Now we develop a similar result for a smaller stabilizer group, namely the group $\hat{\Gamma}_z$ consisting of all transformations $M \in \hat{\Gamma}$ that fixes z where $\hat{\Gamma}$ is a subgroup of $\hat{\Gamma}(1)$ and z is a cusp of $\hat{\Gamma}$. For that purpose we will use next lemma which shows that $\hat{\Gamma}_z$ is also a cyclic group where $z \in \mathbb{Q}$ or $z = \infty$.

Lemma 2.3.4. *[9, Lemma 2] Let $z \in \bar{\mathbb{Q}} = \mathbb{Q} \cup \infty$ and $\hat{\Gamma}$ be a subgroup of $\hat{\Gamma}(1)$. Assume that $\hat{\Gamma}_z = \{M \in \Gamma |\, Mz = z\}$. Then $\hat{\Gamma}_z$ is a nontrivial cyclic subgroup of $\hat{\Gamma}$.*

Proof. Clearly $\hat{\Gamma}_z \subseteq \hat{\Gamma}$. Since $[\hat{\Gamma}(1) : \hat{\Gamma}] < \infty$, there exists $m \in \mathbb{N}^+$ such that $U^m \in \hat{\Gamma}$ (or else $U, U^2, U^3, \ldots$ represents infinitely many distinct cosets.) First assume $z = \infty$. Let $m \in \mathbb{N}$ be minimal such that $U^m \in \hat{\Gamma}$, then $U^m \in \hat{\Gamma}_\infty$ and so $\hat{\Gamma}_\infty \neq \langle I \rangle$. We claim that $\hat{\Gamma}_\infty = \langle U^m \rangle$. Given any $M \in \hat{\Gamma}_\infty$ we have $M\infty = \infty$ and therefore $M = U^k$ for some $k \in \mathbb{Z}$. Assume without loss of generality that $k > 0$ and write $k = sm + r$ where $s \in \mathbb{Z}^+$ and $0 \leq r < m$. Then $U^r = U^{k-sm}$ which is an element of $\hat{\Gamma}_\infty$. This contradicts with the minimality of m unless $r = 0$.

Now we suppose $z \in \mathbb{Q}$ and write $q = \frac{a}{b} \neq \infty$ where $(a, b) = 1$. Pick integers x, y

so that $-ax - by = 1$. Then

$$L = \begin{pmatrix} x & y \\ b & -a \end{pmatrix} \in \Gamma(1) \quad \text{and} \quad Lz = \infty$$

Observe that $L\hat{\Gamma}_z L^{-1}$ is the subgroup of $L\hat{\Gamma}L^{-1}$ leaving ∞ fixed. By the first part,

$$L\hat{\Gamma}_z L^{-1} = \langle U^m \rangle$$

where m is the least positive integer such that $U^m \in L\hat{\Gamma}L^{-1}$. It follows that

$$\hat{\Gamma}_z = \langle L^{-1}U^m L \rangle. \tag{2.7}$$

Since $U^j \neq I$ for $j \in \mathbb{Z}^+$, $\hat{\Gamma}_z$ is actually infinite. □

It is straightforward to verify that the number m above is independent of L, that is, if there exists L_1, L_2 such that $L_1\infty = L_2\infty = \infty$, then the smallest integers m_1 and m_2 such that $U^{m_1} \in L_1\hat{\Gamma}L_1^{-1}$ and $U^{m_2} \in L_1\hat{\Gamma}L_1^{-1}$, respectively, are same.

From (2.7), we easily verify that

$$m = [\hat{\Gamma}_z(1) : \hat{\Gamma}_z] = n(z, \Gamma)$$

Therefore we conclude that with $n_L := n(L^{-1}\infty, \Gamma)$, we have

$$\hat{\Gamma}_z = \langle L^{-1}U^{n_L} L \rangle \tag{2.8}$$

where n_L is the least possible integer such that

$$U^{n_L} \in L\hat{\Gamma}L^{-1}.$$

This number n_L is also called the *width of the cusp* $z\,(mod\ \Gamma)$. By the remark above $n_{L_1} = n_{L_2}$ if $L_1 z = L_2 z = \infty$.

Chapter 3

General Theory of Modular forms

The main object of this chapter is to present basic definitions and fundamental facts about modular forms. We start with a discussion of automorphic factors and multiplier systems. Next we define modular forms by means of their Fourier expansions. Most of the content of this chapter is taken from [18]. For additional details about the materials in this chapter the reader is referred to [9], [13] or [18].

3.1 Automorphic Factors and Multiplier Systems

In this section we shall be concerned with the properties of automorphic factors and multiplier systems. These properties will be needed when we construct modular forms by means of Poincaré series. We start with the following basic notation. For any $T \in \Gamma(1)$ and $z \in \mathbb{C}$ we define

$$T : z = cz + d \quad \text{where} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

By an elementary calculation we can obtain the identity

$$LT : z = (L : Tz)(T : z) \tag{3.1}$$

for all $L, T \in \Gamma(1)$ and $z \in \mathbb{C}$. Throughout this section, we shall suppose that k is a fixed real number, not necessarily an integer. For a nonzero $z \in \mathbb{C}$, we adopt

$$z^k := |z|^k e^{ik \arg z},$$

where $-\pi \leq \arg z < \pi$.

Definition 3.1.1. Let Γ be a subgroup of $\Gamma(1)$. A function ν defined on $\Gamma \times \mathbb{H}$ is called an automorphic factor (AF) of weight k on Γ if the following four conditions are satisfied:

(i) For each $T \in \Gamma$, $\nu(T, z)$ is a holomorphic function of $z \in \mathbb{H}$.

(ii) For all $z \in \mathbb{H}$ and $T \in \Gamma$,

$$|\nu(T, z)| = |T : z|^k.$$

(iii) For all $L, T \in \Gamma$ and $z \in \mathbb{H}$,

$$\nu(LT, z) = \nu(L, Tz)\nu(T, z). \tag{3.2}$$

(iv) If $-I \in \Gamma$, then, for all $T \in \Gamma$ and all $z \in \mathbb{H}$,

$$\nu(-T, z) = \nu(T, z). \tag{3.3}$$

The last condition indicates that ν can be regarded as a function on $\hat{\Gamma} \times \mathbb{H}$ so that T can be treated as a mapping. In this case the condition (iv) can be omitted as being obvious. If we take $L = T = I$ in (3.2), we find

$$\nu(I, z) = 1 \quad \text{for all } z \in \mathbb{H}.$$

We note that the equation (3.1) is very similar to (3.2). Accordingly we define for all $T \in \Gamma$ and all $z \in \mathbb{H}$,

$$\mu(T, z) := (T : z)^k. \tag{3.4}$$

First observation is that if k is an even integer, then $\mu(-T, z) = \mu(T, z)$ and in that case it is clear that $\mu(T, z)$ is an AF of weight k on Γ. Now consider the function

$$\frac{\nu(T, z)}{\mu(T, z)}$$

which has constant modulus and is holomorphic on $\mathbb{H}$. Since a holomorphic function of constant modulus on $\mathbb{H}$ have to be constant, we deduce

$$\nu(T, z) = \upsilon(T)\mu(T, z) \tag{3.5}$$

for all $T \in \Gamma$ and all $z \in \mathbb{H}$, where $\upsilon(T)$ depends only on the matrix T and

$$|\upsilon(T)| = 1.$$

We call $\upsilon(T)$ a *multiplier* and the function υ defined by (3.5) on Γ is called a *multiplier system* (MS) of weight k. Observe that if υ_1 and υ_2 are multiplier systems of weight k_1 and k_2 for Γ, then $\upsilon_1\upsilon_2$ is a multiplier system of weight $k_1 + k_2$ for Γ. If we take $T = I$ in (3.5), we find that

$$\upsilon(I) = 1.$$

Moreover, if $-I \in \Gamma$, we have by (3.3)and (3.5) that

$$\upsilon(-I) = e^{-\pi i k}.$$

Therefore unlike the AF ν, the MS υ is defined on Γ but not, in general on $\hat{\Gamma}$. If we insert (3.5) into (3.2), we obtain for any $T, L \in \Gamma$ that

$$\upsilon(TL) = \sigma(L,T)\upsilon(L)\upsilon(T), \tag{3.6}$$

where

$$\sigma(L,T) := \frac{\mu(L,Tz)\mu(T,z)}{\mu(LT,z)}. \tag{3.7}$$

By (3.4), we obtain

$$|\sigma(L,T)| = 1.$$

If $k \in \mathbb{Z}$, by (3.1) and (3.4), for all $L, T \in \Gamma$

$$\sigma(L,T) = 1.$$

Therefore if $k \in \mathbb{Z}$, for any $T, L \in \Gamma$, (3.6) reduces to

$$\upsilon(TL) = \upsilon(L)\upsilon(T).$$

It follows that a multiplier system of weight $k \in \mathbb{Z}$ is just a unitary character on the matrix group Γ which satisfies the consistency condition $\upsilon(-I) = e^{-\pi i k}$.

For $L \in \Gamma(1)$ let $\Gamma^L := L^{-1}\Gamma L$. It is easy to show that

$$\nu^L(L^{-1}TL, z) = \frac{\nu(T, Lz)\mu(L,z)}{\mu(L, L^{-1}TLz)}$$

is an AF on $\Gamma^L \times \mathbb{H}$ which we call the conjugate AF. We denote the associated multiplier system to υ by υ^L. Observe that

$$\upsilon^L(L^{-1}TL) = \upsilon(T)/\sigma(L, L^{-1}TL).$$

In particular, if $k \in \mathbb{Z}$,

$$\upsilon^L(L^{-1}TL) = \upsilon(T)$$

for $L \in \Gamma(1)$ and $T \in \Gamma$. Note that for any $L_1, L_2 \in \Gamma(1)$

$$\upsilon^{L_1 L_2} = (\upsilon^{L_1})^{L_2}.$$

3.1.1 Cusp Parameter

Throughout this section we assume that Γ is a subgroup of $\Gamma(1)$ of finite index containing $-I$ and that ν is an AF of weight k on Γ. Suppose further that $L \in \Gamma(1)$ and $\zeta = L\infty$ is a cusp. In this section we shall investigate $\nu^L(U^{n_L}, z)$ where $n_L = n(L\infty, \Gamma)$ and $z \in \mathbb{H}$. This leads us to the definition of the cusp parameter κ_L associated with the cusp $L\infty$.

Recall that the conjugate AF ν^L is defined on $\Gamma^L \times \mathbb{H}$ and n_L is the least positive integer such that $U^{n_L} \in L^{-1}\Gamma L$. Since $U^{n_L} \in \Gamma^L$ we have

$$\nu^L(U^{n_L}, z) = \upsilon^L(U^{n_L})(U^{n_L} : z)^k = \upsilon^L(U^{n_L}) = \upsilon(LU^{n_L}L^{-1}) \tag{3.8}$$

The cusp parameter $\kappa_L = \kappa(L\infty, \Gamma, \upsilon)$ associated with the cusp $L\infty$ and the MS υ is defined by

$$\upsilon(LU^{n_L}L^{-1}) = \nu^L(U^{n_L}, z) =: e^{2\pi i \kappa_L} \tag{3.9}$$

where $0 \leq \kappa_L < 1$ and $n_L = n(L\infty, \Gamma)$. By (3.8) and (3.9) we have for any $m \in \mathbb{Z}$,

$$e^{2\pi m \kappa_L} = \nu^L(U^{mn_L}, z) = \upsilon^L(U^{mn_L}) = \upsilon(LU^{mn_L}L^{-1}). \tag{3.10}$$

3.2 Modular Forms

The aim of this section is to explain the general theory of modular forms. Throughout this section, we assume that Γ is a subgroup of $\Gamma(1)$, ν is an automorphic factor of weight $k \in \mathbb{R}$ in Γ and υ is the associated multiplier system. Suppose further that $-I \in \Gamma$.

Definition 3.2.1. *An unrestricted modular form of weight k* for the group Γ is a function $f(z)$ defined on $\mathbb{H}$ which satisfies the following two properties:
(i) f is a meromorphic function on $\mathbb{H}$.
(ii) For all $T \in \Gamma$ and all $z \in \mathbb{H}$

$$f(Tz) = \nu(T,z)f(z) = \upsilon(T)(T:z)^k f(z)$$

where the multiplier $\upsilon(T)$ is a complex number of unit modulus independent of z.

The set of all unrestricted modular forms of weight k for the group Γ with multiplier system υ is denoted by $M'(\Gamma,k,\upsilon)$. If $f \in M'(\Gamma,k,\upsilon)$ and $L \in \Gamma(1)$, the L-transform f_L of f is defined by

$$f_L(z) = f(z)|L = (L:z)^{-k}f(Lz). \tag{3.11}$$

Next theorem gives some basic properties satisfied by the function f_L.

Theorem 3.2.2. ***[18, Theorem 4.1.1]*** *Suppose that $f \in M'(\Gamma,k,\upsilon)$ and that $L, L_1, L_2 \in \Gamma(1)$. Then we have*
(i) $f_L \in M'(L^{-1}\Gamma L, k, \upsilon^L)$ where

$$\upsilon^L(T) = \upsilon(LTL^{-1})\frac{\sigma(LTL^{-1},L)}{\sigma(L,T)} \quad \textit{for} \quad T \in L^{-1}\Gamma L.$$

(ii) $f|(L_1L_2) = \sigma(L_1,L_2)(f|L_1)|L_2$ where $\sigma(L_1,L_2)$ is defined by (3.7).
(ii) $T \in \Gamma$, $f_{TL} = \sigma(T,L)\upsilon(T)f_L$; in particular $f_T = \upsilon(T)f$ and $f_{-L} = e^{\pm\pi ik}f_L$
(iv) If $\zeta = L\infty$, then

$$f_L(z+n_L) = e^{2\pi i\kappa_L}f_L(z) \tag{3.12}$$

for all $z \in \mathbb{H}$, where $n_L = n(L\infty,\Gamma)$ is the width of the cusp ζ (mod Γ) and κ_L is its parameter.

We now impose a further restriction on the behavior of unrestricted modular form f near each cusp. Let us write

$$t = t_L = e^{2\pi i z/n_L}$$

where n_L is the width of the cusp $\zeta = L\infty\ (mod\,\Gamma)$ and define the function $F_L(t)$ by

$$F_L(t) = e^{-2\pi i \kappa_L z/n_L} f_L(z) \tag{3.13}$$

By (3.12), $F_L(t)$ is well defined for $0 < |t| < 1$ and is a meromorphic function of t. If, in particular f_L is holomorphic on $\{z \in \mathbb{H}|\ Im(z) > y\}$ where $y \geq 0$, then F_L becomes holomorphic for all t such that $0 < |t| < e^{-2\pi y/n_L}$. Therefore, F_L has a convergent Laurent series expansion at $t = 0$, valid for $0 < |t| < e^{-2\pi y/n_L}$, i.e. there exist $\alpha_L > 0$ such that

$$F_L(t) = \sum_{m=-\infty}^{\infty} a_m(L) t^m$$

for $0 < |t| < \alpha_L$. Hence, by (3.13)

$$f_L(z) = e^{2\pi i \kappa_L z/n_L} \sum_{m=-\infty}^{\infty} a_m(L) e^{2\pi i m z/n_L} = \sum_{m=-\infty}^{\infty} a_m(L) e^{2\pi i z(m+\kappa_L)/n_L}$$

for $Im\, z > y_L$ where $y_L = (n_L/2\pi)\log(1/\alpha_L)$ which we call the Fourier series expansion of $f_L(z)$ at point ∞ or the Fourier series expansion of f at the cusp $L\infty$. Additionally, if $F_L(t)$ is a meromorphic function at $t = 0$, i.e, $f_L(z)$ is a meromorphic function at the point ∞ then there exist an integer N_L such that

$$f_L(z) = e^{2\pi i \kappa_L z/n_L} \sum_{m=N_L}^{\infty} a_m(L) e^{2\pi i m z/n_L} \tag{3.14}$$

where $Im\, z > y_L$ for some $y_L > 0$. This expression determines the behavior of f_L near the point ∞.

Definition 3.2.3. Let $f \in M'(\Gamma, k, \upsilon)$, f is called a modular form of weight k for the group Γ with MS υ if it satisfies the following additional condition

(iii)f is meromorphic at each cusp of the standard fundamental region of Γ.

The class of all modular forms of weight k for the group Γ with MS υ is denoted by $M(\Gamma, k, \upsilon)$. We observe in particular that if $f = 0$ then $f \in M(\Gamma, k, \upsilon)$. Let now $f \in M(\Gamma, k, \upsilon)$ be such that $f \neq 0$, then the Fourier series of $f_L(z)$ starts with the term $a_{N_L}(L)t^{N_L+\kappa_L}$ where $t = e^{2\pi iz/n_L}$. The number $\kappa_L + N_L$ is called *the order of f at the cusp $L\infty$* $(mod\ \Gamma)$ and write

$$ord(f, L\infty, \Gamma) := \kappa_L + N_L$$

We define for $z \in \mathbb{H}$

$$ord(f, z, \Gamma) := \begin{cases} \frac{1}{m} ord(f, z) & \text{if} \quad z \in \mathbb{E}_m(\Gamma) \\ ord(f, z) & \text{if} \quad z \notin \mathbb{E}_m(\Gamma) \end{cases}$$

Definition 3.2.4. Let $f \in M(\Gamma, k, \upsilon)$, f is called an entire modular form if f is regular in $\mathbb{H}$ and f is regular at each parabolic point z $(mod\ \Gamma)$, i.e. $ord(f, z, \Gamma) \geq 0$ for all $z \in \mathbb{H}' = \mathbb{H} \cup \mathbb{P}$. If, in addition, f has a zero of positive order at each parabolic point z $(mod\ \Gamma)$, f is called a *cusp form*, i.e. $ord(f, z, \Gamma) > 0$ for all $z \in \mathbb{P}$ or $f = 0$.

We denote by $H(\Gamma, k, \upsilon)$ the subset of $M(\Gamma, k, \upsilon)$ consisting of all forms f that are holomorphic on $\mathbb{H}$. The class of all entire modular forms of weight k for the group Γ with MS υ is denoted by $\{\Gamma, k, \upsilon\}$ and the class of all such cusp forms is denoted by $\{\Gamma, k, \upsilon\}_0$. We note that

$$\{\Gamma, k, \upsilon\}_0 \subseteq \{\Gamma, k, \upsilon\} \subseteq H(\Gamma, k, \upsilon) \subseteq M(\Gamma, k, \upsilon) \subseteq M'(\Gamma, k, \upsilon)$$

Definition 3.2.5. If f is a modular form on Γ with $k = 0$ and $\upsilon(T) = 1$ for all $T \in \Gamma$ then f is called a modular function on Γ.

We close this chapter with the following well known results about the modular functions.

Theorem 3.2.6. *Every entire modular function is constant.*

Proof. See [1], page 115. □

Corollary 3.2.7. *[9, **Corollry 9**] If f is a modular function on Γ and f is bounded in $\mathbb{H}$, then f is constant.*

Proof. Since f is meromorphic and and bounded in $\mathbb{H}$ it is actually regular in $\mathbb{H}$. Moreover by the equation (3.14), the expansion of the function $f_L(z)$ at the point ∞ has the form

$$f_L(z) = e^{2\pi i\kappa_L/n_L} \sum_{m=N_L}^{\infty} a_m(L) e^{2\pi imz/n_L}$$

for all z with $Im\, z > y_L$ for some $y_L > 0$. Since $f \in M(\Gamma, 0, 1)$, $f_L(z) = f(Lz)$ and it follows that the expansions of the function $f(z)$ at each cusp $L\infty$

$$f(z) = e^{2\pi i\kappa_L/n_L} \sum_{m=N_L}^{\infty} a_m(L) e^{2\pi imL^{-1}z/n_L}$$

If a term with $m < 0$ actually appeared in the expansion at a cusp $L\infty$, then $f(z)$ would not be bounded as $z \to L\infty$ from within the fundamental region of Γ. Hence the expansion is of the form,

$$f(z) = e^{2\pi i\kappa_L/n_L} \sum_{m=0}^{\infty} a_m(L) e^{2\pi imL^{-1}z/n_L}$$

Therefore f is an entire modular function and hence, by theorem above, is constant. $\square$

Chapter 4

Construction of Modular Forms with Poincaré Series

Let Γ be a subgroup of finite index in $\Gamma(1)$ with a multiplier system υ of weight $k > 2$ and assume $m \in \mathbb{Z}$. In this chapter, we shall be concerned with the Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$ studied by Rankin in [18]. We shall first define the Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$ for any $m \in \mathbb{Z}$ and show that they are indeed holomorphic modular forms of weight k on Γ. Then we study a decomposition theorem which is due in its simplest form to Hecke [5] and which asserts a holomorphic modular form can be written as a sum of a cusp form and linear combination of Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$ with $m \leq 0$. Next we shall obtain an explicit formulae for the Fourier coefficients of the Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$. Then our particular interest shall be the Poincaré series belonging to $\bar{\Gamma}(N)$. We shall apply the results about the Fourier coefficients of $G_L(z, m, \Gamma, k, \upsilon)$ to the particular case when $\Gamma = \bar{\Gamma}(N)$ and k is an integer and obtain explicit formulae for the Fourier coefficients of $G_L(z, m, \bar{\Gamma}(N), k, \upsilon)$ which is indeed the main purpose and therefore the main result of this chapter. Finally we shall consider the series $G_L(z, m, \bar{\Gamma}(N), k, \upsilon)$ with $m = 0$, the Eisenstein series, in greater detail and conclude this chapter by evaluating the explicit formulae for the Fourier coefficients of the Eisenstein series belonging to $\bar{\Gamma}(N)$.

4.1 Poincaré Series

The main purpose of this section is to construct a modular form belonging to $H(\Gamma, k, \upsilon)$ as sum of an infinite series, namely the Poincaré Series. The theorems and results of this section are taken form [18]. We start with a preliminary result.

Theorem 4.1.1. *Let A be a nonnegative constant, k a real number greater than 2, and suppose that for each pair of integers μ, ν with $\mu \neq 0$, a function $f_{\mu,\nu}$ is defined on $\mathbb{H}$ and that*

$$|f_{\mu,\nu}(z)| \leq e^{Ay/|\mu z+\nu|^2}$$

for all $z \in \mathbb{H}$, where $y = Im\, z$ then the double series

$$\sum_{\substack{\mu=-\infty \\ \mu\neq 0}}^{\infty} \sum_{\nu=-\infty}^{\infty} \frac{f_{\mu,\nu}(z)}{|\mu z+\nu|^k} \tag{4.1}$$

is absolutely convergent for all $z \in \mathbb{H}$ and absolutely uniformly convergent on every compact subset of $\mathbb{H}$. Further, for every $\varepsilon > 0$, there exist a positive number B, depending only on A, k and ε, such that if $F(z)$ is the sum of the series (4.1), then

$$|F(z)| \leq Be^{A/|z|}(|z|^{-k} + |z|^{-\frac{1}{2}k}) \tag{4.2}$$

for all $z \in \mathbb{A}_\varepsilon$, where

$$\mathbb{A}_\varepsilon := \{z \in \mathbb{H} | \varepsilon \leq \arg z \leq \pi - \varepsilon\}.$$

Proof. See [18], page 136. □

Now, let Γ be a subgroup of finite index in $\Gamma(1)$ and assume ν is an AF of wight k on Γ and υ is the associated MS. Moreover let $-I \in \Gamma$ and $\zeta = L^{-1}\infty$ be any point in $\mathbb{P}$ where $L \in \Gamma(1)$. For convenience we put $M = L^{-1}$ and for brevity we write

$$n := n(L^{-1}\infty, \Gamma) = n_M \quad \text{and} \quad \kappa := \kappa(L^{-1}\infty, \Gamma, \upsilon) = \kappa_M, \tag{4.3}$$

where n_M is the width of the cusp ζ and κ_M is its parameter. It follows that $U^n \in L\Gamma L^{-1}$ and hence $L^{-1}U^nL \in \Gamma$. Then by (3.9),

$$e^{2\pi i\kappa} = \nu^M(U^n, z) = \upsilon^M(U^n) = \upsilon(MU^nM^{-1}) = \upsilon(L^{-1}U^nL)$$

for all $z \in \mathbb{H}$. Let

$$\hat{\Gamma} = \hat{\Gamma}_\zeta \cdot \mathfrak{R}_L \tag{4.4}$$

where $\mathfrak{R}_L$ is a set of right coset representatives of $\hat{\Gamma}$ modulo $\hat{\Gamma}_\zeta$ which is not necessarily finite.

We now own the tools which we need to define Poincaré series. For any $m \in \mathbb{Z}$, $L \in \Gamma(1)$ and $k > 2$ the *Poincaré series* is defined by

$$G_L(z, m, \Gamma, k, \upsilon) = G_L(z, m) := \sum_{T \in \mathfrak{R}_L} \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n} LTz\right)}{\mu(L, Tz)\nu(T, z)}. \tag{4.5}$$

The modular properties of Poincaré series are given by next theorem.

Theorem 4.1.2. ***[18, Theorem 5.1.2]*** *The series* (4.5) *defines G_L as a holomorphic function on $\mathbb{H}$, when $k > 2$. The series absolutely convergent on $\mathbb{H}$ and absolutely uniformly convergent on every compact subset of $\mathbb{H}$. Its sum $G_L(z, m)$ does not depend upon the choice of transversal $\mathfrak{R}_L$, and $G_L(z, m) \in H(\Gamma, k, \upsilon)$. More generally for any $S \in \Gamma(1)$,*

$$G_L(z, m, \Gamma, k, \upsilon)|S = \{\sigma(L, S)\}^{-1} G_{LS}(z, m, \Gamma^S, k, \upsilon^S). \tag{4.6}$$

(a) If $m + \kappa > 0$, then $G_L \in \{\Gamma, k, \upsilon\}_0$ and may vanish identically; here κ is defined by (4.3).
(b) If $m + \kappa = 0$ (so that $m = \kappa = 0$), then $G_L \in \{\Gamma, k, \upsilon\}$ and is called an Eisenstein Series; it does not vanish identically and ord$(G_L, L^{-1}\infty, \Gamma) = 0$.

(c) If $m+\kappa<0$ (so that $m\leq -1$), G_L does not vanish identically and $ord(G_L, L^{-1}\infty, \Gamma) = m+\kappa$
In both cases (b) and (c), $ord(G_L, \zeta, \Gamma) > 0$ at every cusp $\zeta \not\equiv L^{-1}\infty \ (mod\,\Gamma)$.

Proof. Let n, κ be defined as in (4.3). We present the proof in three parts. First we establish the analytic properties of the sum (4.5). Next we examine that the definition of $G_L(z, m, \Gamma, k, \upsilon)$ does not depend on the particular choice of $\mathfrak{R}_L$ and lastly we prove $G_L(z, m, \Gamma, k, \upsilon)$ is a modular form satisfying the given properties.

Part I: We claim that there is at most one term in the series (4.5) for which the matrix LT has a given second row. In order to prove this claim, let $T, T' \in \mathbb{R}_L$ and assume LT and LT' are two matrices with the same second row, then it is easy to see that $LT' = U^s LT$ for some $s \in \mathbb{Z}$. It follows that $T'T^{-1} = L^{-1}U^s L$. Since $TT' \in \Gamma$, we have $s \leq n$, therefore $TT' \in \Gamma_z$ which implies $T = T'$ and this proves the claim. In particular, there is at most one term in the series (4.5) for which $LT \in \hat{\Gamma}_U$ where $\hat{\Gamma}_U = \langle U \rangle$. Let this term, if it exists, be removed from the series (4.5) and observe for the remaining series that

$$\left| \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n} LTz\right)}{\mu(L, Tz)\nu(T, z)} \right| = \frac{\exp\left(\frac{-2\pi(m+\kappa)y}{n|LT:z|^2}\right)}{|LT:z|^k} \tag{4.7}$$

where $y = Im\, z$ and $LT : z = \mu z + \nu$ with $\mu \neq 0$. Then the remaining series is of the form (4.1) with

$$f_{\mu,\nu}(z) = \begin{cases} \exp\left(\frac{-2\pi(m+\kappa)y}{n|LT:z|^2}\right) & \text{if } (\mu, \nu) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then

$$|f_{\mu,\nu}(z)| \leq e^{Ay/n|LT:z|^2} \quad \text{where} \quad A = \max\{0, -2\pi(m+\kappa)/n\}$$

Then by Theorem (4.1.1), the series (4.5) is absolutely convergent on $\mathbb{H}$ and absolutely and uniformly convergent on every compact subset of $\mathbb{H}$. Moreover since each term of the series is holomorphic on $\mathbb{H}$, $G_L(z, m, \Gamma, k, \upsilon)$ is holomorphic on $\mathbb{H}$.

Part II: It suffices to show that in any term of the series (4.5), T can be replaced by $L^{-1}U^nLT$. Let $L^{-1}U^nL = R$, we investigate what kind of the changes occur in the denominator of each term in (4.5), if T is replaced by RT.

$$\begin{aligned}\mu(L, RTz)\nu(RT, z) &= \mu(L, RTz)\mu(RT, z)\upsilon(RT)\\ &= \upsilon(R)\upsilon(T)\sigma(L, T)\mu(L, Tz)\\ &= \upsilon(R)\mu(L, Tz)\nu(T, z).\end{aligned}$$

By (3.9) we have $\upsilon(R) = \upsilon(L^{-1}U^nLT) = e^{2\pi i\kappa}$, hence

$$\mu(L, RTz)\nu(RT, z) = e^{2\pi i\kappa}\mu(L, Tz)\nu(T, z).$$

Now we observe the changes in the nominator of each term in (4.5) when T is replaced by RT

$$\begin{aligned}\exp\left(\frac{2\pi i(m+\kappa)}{n}L(RT)z\right) &= \exp\left(\frac{2\pi i(m+\kappa)}{n}(LTz+n)\right)\\ &= e^{2\pi i\kappa}\exp\left(\frac{2\pi i(m+\kappa)}{n}LTz\right)\end{aligned}$$

Therefore no change occurs in any term of the series if T is replaced by RT. This proves part II.

Part III: Let $S \in \Gamma(1)$ and consider the transform

$$\begin{aligned}G_L(z, m, \Gamma, k, \upsilon)|S &= \mu(L, S)^{-1}G_L(Sz, m, \Gamma, k, \upsilon)\\ &= \sum_{T\in\mathfrak{R}_L}\frac{\exp\left(\frac{2\pi i(m+\kappa)}{n}LTSz\right)}{\mu(S, z)\mu(L, TSz)\nu(T, Sz)}.\end{aligned}$$

Let $\zeta' := S^{-1}\zeta = S^{-1}L^{-1}\infty$ and $\mathfrak{R}'_{LS} := S^{-1}\mathfrak{R}_LS$. Then by (2.5) and (4.4),

$$\hat{\Gamma}^S = S^{-1}\hat{\Gamma}S = S^{-1}\hat{\Gamma}_\zeta S\cdot\mathfrak{R}'_{LS} = \hat{\Gamma}^S_{\zeta'}\cdot\mathfrak{R}'_{LS}. \tag{4.8}$$

As T runs through $\mathfrak{R}_L$, $T' = S^{-1}TS$ runs through $\mathfrak{R}'_{LS}$ so that $TS = ST'$.

Therefore if we write T in place of T'

$$G_L(z,m,\Gamma,k,\upsilon)|S = \sum_{T\in\mathfrak{R}'_{LS}} \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n}LSTz\right)}{\mu(S,z)\mu(L,STz)\nu(STS^{-1},Sz)}. \tag{4.9}$$

Moreover it is easy to observe that

$$\mu(S,z)\mu(L,STz)\nu(STS^{-1},Sz) = \mu(LS,Tz)\nu^S(T,z)\sigma(L,S).$$

Hence,

$$G_L(z,m,\Gamma,k,\upsilon)|S = \{\sigma(L,S)\}^{-1} \sum_{T\in\mathfrak{R}'_{LS}} \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n}LSTz\right)}{\mu(LS,Tz)\nu^S(T,z)}. \tag{4.10}$$

Since $\zeta' = S^{-1}LS^{-1}\infty$, $n(\zeta',\Gamma^S) = n$. Further

$$\kappa' := \kappa(\zeta',\Gamma^S,\upsilon^S) = \kappa(S^{-1}M\infty,\Gamma^S,\upsilon^s)$$

so that

$$e^{2\pi i\kappa'} = \upsilon^{SS^{-1}M}(U^n) = \upsilon^M(U^n) = e^{2\pi i\kappa},$$

which means $\kappa' = \kappa$. Then from (4.10),

$$G_L(z,m,\Gamma,k,\upsilon)|S = \{\sigma(L,S)\}^{-1}G_{LS}(z,m,\Gamma^S,k,\upsilon^S),$$

which is (4.6).
In particular, if $S\in\Gamma$ that is $\Gamma = \Gamma^S$, we have by (4.8), $\hat{\Gamma} = \hat{\Gamma}_\zeta \cdot \mathfrak{R}'_{LS}$. Further

$$\mu(S,z)\mu(L,STz)\nu(STS^{-1},Sz) = \frac{\mu(L,STz)\nu(S,Tz)}{\upsilon(S)}.$$

Therefore, by (4.9), when $S\in\Gamma$,

$$\begin{aligned} G_L(z,m,\Gamma,k,\upsilon)|S &= \upsilon(S)\sum_{T\in\mathfrak{R}'_{LS}} \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n}LSTz\right)}{\mu(L,STz)\nu(S,Tz)} \\ &= \upsilon(S)G_L(z,m,\Gamma,k,\upsilon), \end{aligned}$$

which proves that G_L is an unrestricted modular form. In order to conclude that G_L is a modular form, we need to analyze the behavior of $G_L|S$ at ∞. Taking into consideration (4.6), we need to consider the behavior of $G_{LS}(z, m, \Gamma^S, k, \upsilon^S)$ near ∞. We write

$$n_S = (S\infty, \Gamma) \quad \text{and} \quad \kappa_S = (S\infty, \Gamma, \upsilon).$$

It is shown in Part I that $LT \in \hat{\Gamma}_U$ for at most one term in the series. If $LT = U^s$ for some $T \in \mathfrak{R}_L$ and $s \in \mathbb{Z}$, then the corresponding term in the series is

$$\frac{\exp\left(2\pi i(m+\kappa)(z+s)/n\right)}{\mu(L, L^{-1}U^s z)\nu(L{-}1U^s, z)} =: \delta_L \exp(2\pi i(m+\kappa)z/n),$$

where δ_L is given by

$$\delta_L = \delta_L(\Gamma, m, \upsilon) = \frac{\exp(2\pi is(m+\kappa)/n)}{\mu(L, L^{-1}U^s z)\nu(L{-}1U^s, z)} = \frac{\exp(2\pi is(m+\kappa)/n)}{\upsilon(L^{-1}U^s)\sigma(L, L^{-1})}. \quad (4.11)$$

We now define δ_L to be zero if $LT \notin \hat{\Gamma}_U$ for all $T \in \Gamma$. Hence, we have the following definition for δ_L

$$\delta_L(\Gamma, m, \upsilon) = \begin{cases} \frac{\exp(2\pi is(m+\kappa)/n)}{\upsilon(L^{-1}U^s)\sigma(L,L^{-1})} & LT \in \hat{\Gamma}_U \text{ for some } T \in \Gamma \\ 0 & LT \notin \hat{\Gamma}_U \text{ for all } T \in \Gamma \end{cases} \quad (4.12)$$

Further by (4.2),

$$|G_L(z, m, \Gamma, k, \upsilon) - \delta_L \exp(2\pi i(m+\kappa)z/n)| \leq Be^{A/|z|}\left(|z|^{-k} + |z|^{\frac{-1}{2}k}\right). \quad (4.13)$$

for all $z \in \mathbb{H}$ such that $0 < \varepsilon \leq \arg z \leq \pi - \varepsilon$, where B is a nonnegative number depending on ε, k and m. Now let $\delta'_{LS} = (\Gamma^S, m, \upsilon^S)$. Then by definition of δ_L, $\delta'_{LS} = 0$ except when $LTS \in \hat{\Gamma}_U$ for some $T \in \hat{\Gamma}^S$ and $|\delta'_{LS}| = 0$. This implies $\delta'_{LS} \neq 0$ if and only if $LST \in \hat{\Gamma}_U$ for some $T \in \hat{\Gamma}$, that is $TS\infty = L^{-1}\infty$. It follows that $\delta'_{LS} \neq 0$ if and only if $S\infty \equiv L^{-1}\infty \ (mod\, \Gamma)$ which means $n_S = n = n(\zeta', \Gamma^S)$ and $\kappa_S = \kappa = \kappa'$.

Now we are ready to examine the behavior of $G_{LS}(z, m)$ near ∞. We make a similar analysis as it is done in (3.13). Note that $G_{LS}(z, m)\exp(-2\pi i\kappa_S z/n_S)$ is periodic with period n_S for all $S \in \Gamma(1)$ so that we can assume $0 \leq Re\, z \leq n_S$ and $|z| \geq 1$. Then we have $\varepsilon \leq \arg z \leq \pi - \varepsilon$. It follows by (4.13) that

$$\left| G_{LS}(z, m, \Gamma^S, k, \upsilon^S) - \delta'_{LS} \exp(2\pi i(m + \kappa_S)z/n_S) \right| \leq B_\varepsilon |z|^{\frac{-1}{2}k}.$$

Let $t = e^{2\pi i z/n_S}$, then $G_{LS}t^{-\kappa_s} - \delta'_{LS}$ can be expressed as a Laurent series in powers of t where $0 < |t| < 1$ so that we have

$$G_{LS}(z, m, \Gamma^S, k, \upsilon^S) = t^{\kappa_s} \left(\delta'_{LS} t^m + \sum_{j=0}^{\infty} g_j t^j \right) \tag{4.14}$$

where $g_0 = 0$ if $\kappa_S = 0$. We therefore conclude that $G_L \in M(\Gamma, k, \upsilon)$. Since $\delta'_{LS} \neq 0$ if and only if $S\infty \equiv L^{-1}\infty \ (mod\, \Gamma)$, all the results of the theorem follow.

□

We now present a theorem which formalize the relation between Poincaré series on a group and on one of its normal subgroups.

Theorem 4.1.3. ***[18, Theorem 5.1.5]*** *Suppose that $k > 2$ and that $-I \in \Delta \subseteq \Gamma$, where Δ is normal in Γ and let $\mu = [\hat{\Gamma} : \hat{\Delta}]$. Let υ be a MS on Γ (and therefore on Δ) of weight k. Define n and κ by (4.3), where $L \in \Gamma(1)$ and $\zeta = L^{-1}\infty$, and put*

$$n' = n(\zeta, \Delta) \quad \textit{and} \quad \kappa' = \kappa(\zeta, \Delta, \upsilon).$$

Then $n' = nl$ and $\kappa' = \{l\kappa\}$ (fractional part), where l is a positive integral divisor of μ. Let

$$\hat{\Delta} = \hat{\Delta}_\zeta \cdot \mathfrak{R}$$

where $\hat{\Delta}_\zeta$ is the stabilizer of ζ modulo Δ, then there exist a set $\mathfrak{L}$ of μ/l matrices $L_j (1 \leq j \leq \mu/l)$ in Γ such that

$$\hat{\Gamma} = \hat{\Gamma}_\zeta \cdot \mathfrak{R} \cdot \mathfrak{L},$$

and for any $m \in \mathbb{Z}$,

$$G_L(z,m,\Gamma,k,\upsilon) = \sum_{j=1}^{\mu/l} \frac{G_{LL_j}(z,lm+[l\kappa],\Delta,k,\upsilon)}{\upsilon(L_j)\sigma(L,L_j)}$$

Proof. It follows from the definitions of n and n' that $n' = nl$ where l is a positive integral divisor of μ. If we put $m = l$ in (3.10) we get

$$e^{2\pi il\kappa} = \upsilon(L^{-1}U^{nl}L) = \upsilon(L^{-1}U^{n'}L) = e^{2\pi il\kappa'}$$

which indicates that $\kappa' = \{\kappa l\}$ since $0 \leq \kappa' < 1$. For the existence of the set $\mathfrak{L}$ we refer reader to the Theorem 1.1.3 in [18]. Then by (4.5),

$$G_L(z,m,\Gamma,k,\upsilon) = \sum_{j=1}^{\mu/l} \sum_{T\in\mathfrak{R}} \frac{\exp\left(\frac{2\pi i(m+\kappa)}{n} LTL_jz\right)}{\mu(L,TL_jz)\nu(TL_j,z)} \tag{4.15}$$

we observe that

$$\frac{m+\kappa}{n} = \frac{lm+l\kappa}{ln} = \frac{lm+[l\kappa]+\{l\kappa\}}{n'}$$

and by using the properties of the function ν, we have

$$\mu(L,TL_jz)\nu(TL_j,z) = \mu(L,TL_jz)\nu(T,L_jz)\upsilon(L_j)\mu(L_j,z).$$

Therefore by (4.6) and (4.15),

$$\begin{aligned} G_L(z,m,\Gamma,k,\upsilon) &= \sum_{j=1}^{\mu/l} \frac{G_L(L_jz,lm+[l\kappa],\Delta,k,\upsilon)}{\upsilon(L_j)\mu(L_j,z)} \\ &= \sum_{j=1}^{\mu/l} \frac{G_L(z,lm+[l\kappa],\Delta,k,\upsilon)|L_j}{\upsilon(L_j)} \\ &= \sum_{j=1}^{\mu/l} G_{LL_j}(z,lm+[l\kappa],\Delta^{L_j},k,\upsilon^{L_j}) \\ &= \sum_{j=1}^{\mu/l} \frac{G_{LL_j}(z,lm+[l\kappa],\Delta,k,\upsilon)}{\upsilon(L_j)\sigma(L,L_j)}, \end{aligned}$$

where in the last equation we use $\Delta^{L_j} = \Delta$ and $\upsilon^{L_j} = \upsilon$ since Δ is normal in Γ.

$\square$

We now study a decomposition theorem introduced by Hecke in [5] which shows that a holomorphic modular form can be written as a sum of a cusp form and a linear combination of Poincaré series $G_L(z,m)$ with $m \leq 0$. For that, we need the following definition. Let $L \in \Gamma(1)$, if

$$L \neq -U^r, \quad \forall r \in \mathbb{Z}$$

then L is called a *regular* matrix.

Theorem 4.1.4. *Let $f \in H(\Gamma, k, \upsilon)$, where $k > 2$ and let Ω be a set of λ regular matrices such that the λ cusps $L\infty$ $(L \in \Omega)$ are incongruent modulo Γ. Further suppose that for each $L \in \Omega$,*

$$f_L(z) = e^{2\pi i \kappa_L z/n_L} \sum_{m=-\infty}^{\infty} a_m(L) e^{2\pi i m z/n_L} \quad (z \in \mathbb{H}),$$

where only a finite number of coefficients $a_m(L)$ for $m \leq 0$ are, of course, nonzero. Let

$$H(z) := f(z) - \sum_{S\in\Omega} \sum_{m+\kappa_S \leq 0} a_m(S) G_{S^{-1}}(z, m, \Gamma, k, \upsilon).$$

Then $H \in \{\Gamma, k, \upsilon\}_0$. In particular if $f \in \{\Gamma, k, \upsilon\}$, then

$$H(z) = f(z) - \sum_{\substack{S\in\Omega \\ \kappa_S = 0}} a_0(S) G_{S^{-1}}(z, 0, \Gamma, k, \upsilon)$$

and $H \in \{\Gamma, k, \upsilon\}_0$.

Proof. Let $S, L \in \Omega$ and $t = e^{2\pi i z/n_L}$ and assume $S \neq L$, then by (4.6) and

(4.14),

$$\begin{aligned} G_{S^{-1}}(z,m,\Gamma,k,\upsilon)|L &= \{\sigma(S^{-1},L)\}^{-1}G_{S^{-1}L}(z,m,\Gamma^L,k,\upsilon^L) \\ &= \{\sigma(S^{-1},L)\}^{-1}t^{\kappa_L}\left(\delta'_{S^{-1}L}t^m+\sum_{j=0}^{\infty}g_jt^j\right) \\ &= \{\sigma(S^{-1},L)\}^{-1}\left(\delta'_{S^{-1}L}t^{m+\kappa_L}+\sum_{j+\kappa_L>0}^{\infty}g_jt^{j+\kappa_L}\right). \end{aligned}$$

Since $S, L \in \Omega$, the cusps $S\infty$ and $L\infty$ are incongruent modulo Γ it follows that $\delta'_{S^{-1}L} = 0$ as discussed earlier. Therefore,

$$G_{S^{-1}}(z,m,\Gamma,k,\upsilon)|L = \sum_{j+\kappa_L>0} c_j(S,L)t^{j+\kappa_L}$$

where $c_j = \{\sigma(S^{-1},L)\}^{-1}g_j$. In the case when $S = L$ we have

$$\begin{aligned} G_{L^{-1}}(z,m,\Gamma,k,\upsilon)|L &= \{\sigma(L^{-1},L)\}^{-1}G_I(z,m,\Gamma^L,k,\upsilon^L) \\ &= \{\sigma(L^{-1},L)\}^{-1}\left(\delta'_I t^{m+\kappa_L}+\sum_{j+\kappa_L>0}^{\infty}g_jt^{j+\kappa_L}\right). \end{aligned}$$

Since L is regular, $\sigma(L^{-1},L) = 1$ and we also have $\delta'_I = 1$, hence

$$G_{L^{-1}}(z,m,\Gamma,k,\upsilon)|L = t^{m+\kappa_L}\sum_{j+\kappa_L>0} c_j(L^{-1},L)t^{j+\kappa_L}.$$

It follows that $H \in M(\Gamma,k,\upsilon)$ and for all $L \in \Omega$

$$H_L(z) = \sum_{j+\kappa_L>0} h_j(L)t^{j+\kappa_L}.$$

which means H is a cusp form. □

It is obvious that the families $M(\Gamma,k,\upsilon)$, $H(\Gamma,k,\upsilon)$, $\{\Gamma,k,\upsilon\}$ and $\{\Gamma,k,\upsilon\}_0$ are vector spaces over the field of complex numbers. If S denotes any one of these families, and $f, g \in S$ then $\alpha f + \beta g \in S$ for any complex numbers α, β.

Theorem 4.1.5. ***[18, Theorem 5.2.3]*** *Let $k > 2$. The vector space $\{\Gamma,k,\upsilon\}_0$ is spanned by the set of Poincaré series $G_L(z,m,\Gamma,k,\upsilon)$ with $m + \kappa_M > 0$.*

By this Theorem and Theorem 4.1.4, it is straightforward to deduce the following theorem.

Theorem 4.1.6. *The set of Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$ spans the space $H(\Gamma, k, \upsilon)$ with $k > 2$.*

4.2 The Fourier Coefficients of Poincaré Series

The object of this section is to obtain an explicit formulae for the Fourier coefficients of the Poincaré series $G_L(z, m, \Gamma, k, \upsilon)$ where $m \in \mathbb{Z}$. For that, we employ certain standard formulae for Gamma and Bessel functions. First observation here is that since the space of holomorphic modular forms $H(\Gamma, k, \upsilon)$ is spanned by Poincaré series with $m \in \mathbb{Z}$, if we have explicit formulae for the Fourier coefficients of Poincaré series with $m \in \mathbb{Z}$ then we can have information about the Fourier coefficients of any $f \in H(\Gamma, k, \upsilon)$. We start with the following formula for the gamma function $\Gamma(k)$ given by Whittaker and Watson in [26] (§12.2).

$$\int_{-\infty+ic}^{\infty+ic} w^{-k} e^{\pi i \mu w} dw = (2\pi)^k \frac{\mu^{k-1} e^{-\frac{1}{2}k\pi i}}{\Gamma(k)} \tag{4.16}$$

where $\mu, c \in \mathbb{R}^+$ and $k > 1$ and the integral is taken along the line $Im(w) = c$. Here no confusion should arise between the gamma function $\Gamma(k)$ and the group $\Gamma(k)$. We also require certain properties of the Bessel functions J_{k-1} and I_{k-1} which are defined by the absolutely convergent infinite series

$$J_{k-1}(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2m+k-1}}{m!\, \Gamma(m+k)}$$

and

$$I_{k-1}(z) = \sum_{m=0}^{\infty} \frac{(\frac{1}{2}z)^{2m+k-1}}{m!\, \Gamma(m+k)}$$

for all $z \in \mathbb{C}$. The following integral representation of J_{k-1} can be found in [25](§6.2),

$$J_{k-1}(z) = \frac{1}{2\pi i}\left(\frac{1}{2}z\right)^{k-1}\int_{-\infty}^{(0+)} w^{-k}e^{\left(w-\frac{z^2}{4w}\right)}dw \tag{4.17}$$

for all nonzero $z \in \mathbb{C}$ and $k \in \mathbb{R}$. Here $\int_{-\infty}^{(0+)}$ means the path of integration starts starts at $-\infty$ on the negative real axis, encircles the origin in a counterclockwise direction and returns to the starting point. In the case when $k > 1$, from the formula given in (4.17), by making a contour deformation and a change of variable, we can deduce the following two formulae (see [26], §6.2),

$$\int_{-\infty+ic}^{\infty+ic} w^{-k}e^{-2\pi i(\mu_1 w+\mu_2 w^{-1})}dw = 2\pi\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{2}(k-1)} e^{-\frac{1}{2}k\pi i}J_{k-1}(4\pi\sqrt{\mu_1\mu_2}). \tag{4.18}$$

and

$$\int_{-\infty+ic}^{\infty+ic} w^{-k}e^{-2\pi i(\mu_1 w+\mu_2 w^{-1})}dw = 2\pi\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{2}(k-1)} e^{-\frac{1}{2}k\pi i}I_{k-1}(4\pi\sqrt{\mu_1\mu_2}). \tag{4.19}$$

where μ_1 , μ_2 and c are positive real numbers.
The following theorem which uses the results above shall be required to obtain Fourier coefficients of the Poincaré series $G_L(z, m, \Gamma, k, v)$.

Theorem 4.2.1. *Suppose that $z \in \mathbb{H}$, $k > 1$ and that κ and λ are real numbers. Write*

$$F_k(z, \kappa, \lambda) = \sum_{h=-\infty}^{\infty} (z+h)^{-k}\exp\left(-2\pi i\left(\kappa h + \frac{\lambda}{z+h}\right)\right). \tag{4.20}$$

The series is absolutely uniformly convergent on every compact subset of $\mathbb{H}$ and defines $F_k(z)$ as a holomorphic function on $\mathbb{H}$. Further, $F_k(z, \kappa, \lambda)$ can be expressed as a fourier series

$$F_k(z, \kappa, \lambda) = \sum_{r+\kappa>0} g_r e^{2\pi i(r+\kappa)z}$$

which is absolutely and uniformly convergent on every compact subset of $\mathbb{H}$. The

Fourier coefficients g_r are given by the formulae

$$g_r = \frac{(2\pi)^k}{\Gamma(k)} e^{-\frac{1}{2}k\pi i}(r+\kappa)^{k-1} \quad \textit{when} \quad \lambda = 0, \tag{4.21}$$

$$g_r = 2\pi e^{-\frac{1}{2}k\pi i}\left(\frac{r+\kappa}{\lambda}\right)^{\frac{1}{2}(k-1)} J_{k-1}(4\pi\sqrt{\lambda(r+\kappa)}\,) \quad \textit{when} \quad \lambda > 0, \tag{4.22}$$

$$g_r = 2\pi e^{-\frac{1}{2}k\pi i}\left(\frac{r+\kappa}{|\lambda|}\right)^{\frac{1}{2}(k-1)} I_{k-1}(4\pi\sqrt{|\lambda|(r+\kappa)}\,) \quad \textit{when} \quad \lambda < 0. \tag{4.23}$$

Proof. It is obvious that the series given by (4.20) is absolutely uniformly converges on every compact subset of $\mathbb{H}$. Therefore it defines $F_k(z,\kappa,\lambda)$ as a holomorphic function of z on $\mathbb{H}$. We observe that

$$F_k(z+1,\kappa,\lambda) = e^{2\pi i\kappa}F_k(z,\kappa,\lambda).$$

Write $t = e^{2\pi iz}$ and define

$$G(t) := e^{-2\pi i\kappa z}F_k(z,\kappa,\lambda).$$

As in (3.13) and the paragraph following it, $G(t)$ is well defined for all t such that $0 < |t| < 1$. Since $F_k(z,\kappa,\lambda)$ is holomorphic on $\mathbb{H}$, $G(t)$ is holomorphic on $\{t : 0 < |t| < 1\}$ and has a convergent Laurent series on this punctured neighborhood of origin, i.e.

$$G(t) = \sum_{r=-\infty}^{\infty} g_r t^r$$

by Cauchy integral formula, we have

$$g_r = \frac{1}{2\pi i}\oint_{|t|=\rho} \frac{G(t)}{t^{r+1}}\,dz$$

where $0 < \rho < 1$. We choose $\rho = e^{-2\pi c}$ with $c > 0$, then

$$g_r = \int_{ic}^{1+ic} e^{-2\pi i(r+\kappa)}F_k(z,\kappa,\lambda)\,dz.$$

Since the series in (4.20) is uniformly convergent, we can interchange the places of integration and summation so that we have

$$\begin{aligned} g_r &= \sum_{h=-\infty}^{\infty} \int_{ic}^{1+ic} (z+h)^{-k} e^{-2\pi i(r+\kappa)z} \exp\left[-2\pi i \left(\kappa h + \frac{\lambda}{z+h}\right)\right] dz \\ &= \sum_{h=-\infty}^{\infty} \int_{ic+h}^{1+ic+h} (z+h)^{-k} \exp\left[-2\pi i\{(r+\kappa)z + \lambda/z\}\right] dz \\ &= \int_{ic+h}^{1+ic+h} (z+h)^{-k} \exp\left[-2\pi i\{(r+\kappa)z + \lambda/z\}\right] dz. \end{aligned}$$

If $r+\kappa \leq 0$, then $Re(-2\pi i(r+\kappa)z) = 2\pi(r+\kappa)Im(z) \leq 0$ and the path of integration can be changed by a large semicircle in $\mathbb{H}$. Then, by using the fact that $k > 1$, we can conclude $g_r = 0$. We may therefore assume that $r+\kappa > 0$. If $\lambda = 0$, then by taking $\mu = r+\kappa$ in (4.16), we obtain (4.21). If $\lambda \neq 0$, then by taking $\mu_1 = r+\kappa$ and $\mu_2 = |\lambda|$ in (4.18) and in (4.19), we obtain (4.22) and (4.22). $\square$

This theorem shall be required when we investigate the Fourier coefficients of $G_L(z, m, \Gamma, k, \upsilon)$ with $k > 2$. For the rest of this section we use the following notation

$$n_1 := n_I = n(\infty, \Gamma), \qquad \kappa_1 := \kappa_I = \kappa(\infty, \Gamma, \upsilon) \tag{4.24}$$

and

$$n_2 := n_M = n(L^{-1}\infty, \Gamma), \qquad \kappa_2 := \kappa_M = \kappa(L^{-1}\infty, \Gamma, \upsilon). \tag{4.25}$$

We also write

$$\hat{\Gamma}_1 := \hat{\Gamma}_{\zeta_1} = \langle U^{n_1} \rangle, \qquad \hat{\Gamma}_2 := \hat{\Gamma}_{\zeta_2} = \langle L^{-1} U^{n_2} L \rangle \tag{4.26}$$

where $\zeta_1 = \infty$ and $\zeta_2 = L^{-1}\infty$. Then, as before,

$$\hat{\Gamma} = \hat{\Gamma}_2 \cdot \mathfrak{R}_L.$$

The group $\hat{\Gamma}$ can be expressed as a disjoint union of double cosets $\hat{\Gamma}_2 T \hat{\Gamma}_1$ for $T \in \hat{\Gamma}$. If we denote a representative set of these double cosets by $\mathfrak{T}$, we have

$$\hat{\Gamma} = \hat{\Gamma}_2 \mathfrak{T} \hat{\Gamma}_1$$

so that for any $S \in \hat{\Gamma}$ there exists a unique $T \in \mathfrak{T}$ such that $S \in \hat{\Gamma}_2 T \hat{\Gamma}_1$. Now we write

$$\mathfrak{T}_L := L\mathfrak{T} - \hat{\Gamma}_U.$$

It is convenient to consider $\mathfrak{T}_L$ as a set of matrices rather then transformations. As it is explained in great detail in [18], the set $\mathfrak{T}_L$ can be taken to be the following disjoint union

$$\mathfrak{T}_L = \bigcup_{\gamma=1}^{\infty} \mathfrak{T}_L(\gamma)$$

where $\mathfrak{T}_L(\gamma)$ is the set of all matrices $S \in L\Gamma$ of the form $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ such that

$$0 \leq \delta < \gamma n_1, \qquad 0 \leq \alpha < \gamma n_2. \tag{4.27}$$

We define the generalized Kloosterman sum to be

$$W(r, m, \gamma) := \sum_{S \in \mathfrak{T}_L(\gamma)} \frac{\exp\left[\frac{2\pi i}{\gamma}\left(\frac{(m+\kappa_2)\alpha}{n_2} + \frac{(r+\kappa_1)\delta}{n_1}\right)\right]}{\upsilon(MS)\sigma(L, M)} \sigma(M, S). \tag{4.28}$$

We now have the material required for the following theorem which indeed achieves the main goal of this section.

Theorem 4.2.2. *Let $L \in \Gamma(1)$, $m \in \mathbb{Z}$, $k > 2$ and put $M = L^{-1}$. Then*

$$G_L(z, m, \Gamma, \kappa, \upsilon) = \delta_L e^{2\pi i(m+\kappa_1)z/n_1} + \sum_{r+\kappa_1>0} a(r, m, L) e^{2\pi i(r+\kappa_2)z/n_2},$$

where n_1, κ_1 and n_2, κ_2 are defined by (4.24) and (4.25) respectively, and $\delta_L = 0$ except when $MU^s \in \Gamma$ for some $s \in \mathbb{Z}$, in which case $n_1 = n_2$ and $\kappa_1 = \kappa_2$ and

$$\delta_L = \frac{e^{2\pi i s(m+\kappa_1)/n_1}}{\upsilon(MU^s)\sigma(L, M)}.$$

The coefficients are given by the following formulae for $r+\kappa_1>0$:

$$a(r,m,L)=\frac{(2\pi)^k}{\Gamma(k)}e^{-\frac{1}{2}k\pi i}(r+\kappa_1)^{k-1}\sum_{\gamma=1}^{\infty}\frac{W(r,0,\gamma)}{(n_1\gamma)^k}\quad \textit{when}\quad m=\kappa_2=0,$$

$$a(r,m,L)=2\pi e^{-\frac{1}{2}k\pi i}\frac{n_2^{\frac{1}{2}(k-1)}}{n_1^{\frac{1}{2}(k+1)}}\left(\frac{r+\kappa_1}{m+\kappa_2}\right)^{\frac{1}{2}(k-1)}$$
$$\times\sum_{\gamma=1}^{\infty}\frac{W(r,m,\gamma)}{\gamma}J_{k-1}\left(\frac{4\pi}{\gamma}\sqrt{\frac{(r+\kappa_1)(m+\kappa_2)}{n_1n_2}}\right)\quad \textit{when}\quad m+\kappa_2>0,$$

$$a(r,m,L)=2\pi e^{-\frac{1}{2}k\pi i}\frac{n_2^{\frac{1}{2}(k-1)}}{n_1^{\frac{1}{2}(k+1)}}\left|\frac{r+\kappa_1}{m+\kappa_2}\right|^{\frac{1}{2}(k-1)}$$
$$\times\sum_{\gamma=1}^{\infty}\frac{W(r,m,\gamma)}{\gamma}I_{k-1}\left(\frac{4\pi}{\gamma}\sqrt{\frac{(r+\kappa_1)|m+\kappa_2|}{n_1n_2}}\right)\quad \textit{when}\quad m+\kappa_2<0.$$

Proof. See [18], page 162. □

4.3 Poincaré Series Belonging to $\bar{\Gamma}(N)$

In the previous section we obtained the Fourier series expansion of the Poincaré series $G_L(z,m,\Gamma,k,\upsilon)$ with $m\in\mathbb{Z}$ and $k>2$. In this section we shall apply the results of §4.2 to the particular case when $\Gamma=\bar{\Gamma}(N)$, $N\geq 1$, in order to determine an explicit formulae for the Fourier coefficients of the Poincaré series $G_L(z,m,\bar{\Gamma}(N),k,\upsilon)$ where $m\in\mathbb{Z}$ and k is an integer. We start with choosing a multiplier system. Throughout this section we assume that $\upsilon(T)=1$ for all $T\in\Gamma(N)$, it follows that $\upsilon(T)=(-1)^k$ for all $T\in\bar{\Gamma}(N)-\Gamma(N)$. If $N=1$ or 2, we suppose k is even so that $\upsilon(T)=1$ for all $T\in\Gamma(N)$. Therefore, in all cases, $\upsilon(T)=1$ when $T\in\Gamma(N)$. Moreover let $L\in\Gamma(1)$ and put $M=L^{-1}$. As discussed earlier in §2.3, the order of the cusp $M\infty$ $(mod\,\Gamma)$ is the least positive integer n_M such that $U^{n_M}\in M^{-1}\bar{\Gamma}(N)M$. In this case we find that $n_M=N$.

Therefore, we deduce that

$$n(M\infty, \bar{\Gamma}(N)) = n_M = N \quad \text{for all } M \in \Gamma(1)$$

The cusp parameter κ_M is defined by $\upsilon(MU^{n_M}M^{-1}) = e^{2\pi i\kappa_M}$ and since the matrix $MU^{n_M}M^{-1}$ belongs to $\Gamma(N)$, we have $e^{2\pi i\kappa_M} = 1$. Then we conclude that

$$\kappa(M\infty, \bar{\Gamma}(N), \upsilon) = \kappa_M = 0 \quad \text{for all } M \in \Gamma(1)$$

We now consider the Kloosterman sum $W(r, m, \gamma)$ defined by (4.28) for the case $\Gamma = \bar{\Gamma}(N)$. Recall that n_1, n_2 and $\kappa_1\kappa_2$ are defined by (4.24) and (4.25) and we have $n_1 = n_2 = N$ and $\kappa_1 = \kappa_2 = 0$. Write $L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The set $\mathfrak{T}_L(\gamma)$ is empty except when

$$\gamma > 0 \quad \text{and} \quad \gamma \equiv \varepsilon C \,(mod\, N)$$

where $\varepsilon = \pm 1$. We note that, in this case $\mathfrak{T}_L(\gamma)$ consists of all matrices $S \in L\bar{\Gamma}(N)$ for which (4.27) holds. If $S \in L\bar{\Gamma}(N)$ we have $S \equiv \varepsilon L \; (mod\, N)$. Then by (4.27), if $S \in \mathfrak{T}_L(\gamma)$ we have

$$0 \leq \delta < \gamma N, \qquad 0 \leq \alpha < \gamma N, \tag{4.29}$$

$$[\alpha, \delta] \equiv \varepsilon [A, D] \,(mod\, N), \qquad \alpha\delta \equiv 1 + \varepsilon\beta\gamma \,(mod\, N). \tag{4.30}$$

and this determines S uniquely. Moreover since $S \equiv \varepsilon L \; (mod\, N)$, $\upsilon(MS) = \varepsilon^k$ where $\varepsilon = \pm 1$. Then by (4.28) we have in the case when $\Gamma = \bar{\Gamma}(N)$ that

$$W(r, m, \gamma) = \sum \varepsilon^k \exp\left(\frac{2\pi i}{N\gamma}(m\alpha + r\delta)\right) \tag{4.31}$$

where the summation is taken over all α, δ satisfying (4.29) and (4.30).

The next items which needs to be analyzed for the case we are considering are the set $\mathfrak{R}_L$ and δ_L introduced in (4.4) and in (4.12) respectively. One can expect that the structures of $\mathfrak{R}_L$ and δ_L is simplified by taking $\Gamma = \bar{\Gamma}(N)$. For the Poincaré series $G_L(z, m, \bar{\Gamma}, k, \upsilon)$, the matrices in $\mathfrak{R}_L$ belong to $\bar{\Gamma}(N)$. Since each term in the series (4.5) is unaltered when $T \in \mathfrak{R}_L$ is replaced by $-T$, we may take the matrices in $\mathfrak{R}_L$ to belong to $\Gamma(N)$, i.e the set $L\mathfrak{R}_L$ may be taken to consist of all matrices $S = LT$ satisfying

(i) $S \equiv L \ (mod\, N)$

(ii) If S_1 and S_2 are two different matrices in $L\mathfrak{R}_L$ then $[\gamma_1, \delta_1] \neq \pm[\gamma_2, \delta_2]$ where $[\gamma_1, \delta_1]$ and $[\gamma_2, \delta_2]$ are second rows of S_1 and S_2 respectively.

Recall that for our particular case we have $n(M\infty, \bar{\Gamma}(N)) = N$ and $\kappa(M\infty, \bar{\Gamma}(N), \upsilon) = 0$ for all $M \in \Gamma(1)$. Then by (4.5) we have

$$G_L(z, m, \bar{\Gamma}(N), k, \upsilon) = \sum_{S \in L\mathfrak{R}_L} \frac{\exp\left(\frac{2\pi i m}{N} Sz\right)}{\nu(S, z)/\upsilon(L)} \tag{4.32}$$

$$= \sum_{S \equiv L\,(mod\, N)}{}' (S : Z)^{-k} \exp\left(\frac{2\pi i m}{N} Sz\right) \tag{4.33}$$

where the prime indicates that the summation is subject to the conditions (i) and (ii) introduced above. Now we make use of the definition δ_L given in (4.12), in order to determine its form when $\Gamma = \bar{\Gamma}(N)$. By definition $\delta_L = 0$ except when $LT = U^s$ for some $T \in \mathfrak{R}_L$ and $s \in \mathbb{Z}$. Therefore in our case $\delta_L = 0$ except when $L \equiv \varepsilon U^s \ (mod\, N)$ $(\varepsilon = \pm 1)$ for some $s \in \mathbb{Z}$. If we put $n = N$ and $L^{-1}U^s = T$ in (4.11), we get

$$\delta_L = \delta(\bar{\Gamma}(N), m, \upsilon) = \frac{e^{2\pi i s m/N}}{\upsilon(T)\sigma(L, L^{-1})} = \varepsilon^k e^{2\pi i s m/N}$$

It follows since $L \equiv \varepsilon U^s \ (mod\, N)$ that

$$\delta_L = \begin{cases} e^{2\pi i m B/N} & \text{if} \quad C \equiv 0,\ D \equiv 1\,(mod\, N) \\ (-1)^k e^{-2\pi i m B/N} & \text{if} \quad C \equiv 0,\ D \equiv -1\,(mod\, N) \\ 0 & \text{otherwise.} \end{cases} \tag{4.34}$$

We now have covered all the necessary calculations required to restate the Theorem 4.2.2 for the particular case when $\Gamma = \bar{\Gamma}(N)$. Next theorem gives the explicit formulae for the Fourier coefficients of the Poincaré series $G_L(z, m, \bar{\Gamma}(N), k, \upsilon)$.

Theorem 4.3.1. *[18, Theorem 5.5.1] Let $L \in \Gamma(1)$, $m \in \mathbb{Z}$ and let MS υ be subject to the restrictions imposed at the beginning of this section. Then*

$$G_L(z, m, \bar{\Gamma}(N), k, \upsilon) = \delta_L e^{2\pi imz/N} + \sum_{r=1}^{\infty} a(r, m, L) e^{2\pi irz/N} \tag{4.35}$$

where for $r \geq 1$,

$$a(r, 0, L) = \left(\frac{2\pi r}{Ni}\right)^k \frac{1}{r\Gamma(k)} \sum_{\substack{\gamma=1 \\ \gamma \equiv \pm C \, (mod \, N)}}^{\infty} \gamma^{-k} W(r, 0, \gamma),$$

$$a(r, m, L) = \frac{2\pi}{Ni^k} \left(\frac{r}{m}\right)^{\frac{1}{2}(k-1)} \times \sum_{\substack{\gamma=1 \\ \gamma \equiv \pm C \, (mod \, N)}}^{\infty} \gamma^{-1} W(r, m, \gamma) J_{k-1}\left(4\pi \sqrt{(rm)/(N\gamma)}\right) \quad \textit{for} \quad m > 0,$$

$$a(r, m, L) = \frac{2\pi}{Ni^k} \left(\frac{r}{|m|}\right)^{\frac{1}{2}(k-1)} \times \sum_{\substack{\gamma=1 \\ \gamma \equiv \pm C \, (mod \, N)}}^{\infty} \gamma^{-1} W(r, m, \gamma) I_{k-1}\left(4\pi \sqrt{(r|m|)/(N\gamma)}\right) \quad \textit{for} \quad m < 0.$$

Here δ_L defined by (4.34) *and $W(r, m, \gamma)$ by* (4.31).

4.3.1 Eisenstein Series Belonging to $\bar{\Gamma}(N)$

In this section we restrict our attention to the Fourier coefficients of the Poincaré series $G_L(z, m, \bar{\Gamma}(N), k, \upsilon)$ in the case when $m = 0$, i.e. G_L is an Eisenstein series. One can observe from (4.33) that $G_L(z, m, \bar{\Gamma}(N), k, \upsilon)$ depends only on the second row $[C, D]$ of the matrix L, hence we denote the Eisenstein series $G_L(z, 0, \bar{\Gamma}(N), k, \upsilon)$ by $E_k(z, C, D, N)$ so that we have

$$E_k(z, C, D, N) = \sum_{\substack{\gamma \equiv C, \, \delta \equiv D \, (mod \, N) \\ (\gamma, \delta) = 1}} (\gamma z + \delta)^{-k}.$$

where the summation is subject to the condition (ii). Since $[C, D]$ is the second row of the matrix L belonging to $\Gamma(1)$, C and D are relatively prime. Since they arise only congruences modulo N, it is only necessary to assume that $(C, D, N) = 1$. Moreover if we write for the coefficients $a(r, 0, L)$ given in theorem 4.3.1 that

$$a(r, 0, L) = a(r, C, D, N)$$

then we have by (4.35)

$$E_k(z, C, D, N) = \delta_L + \sum_{r=1}^{\infty} a(r, C, D, N) e^{2\pi i r z/N} \tag{4.36}$$

where δ_L is given by (4.34) with $m = 0$.

Now our aim is to obtain a formula for $a(r, C, D, N)$ which is simpler than given in the Theorem (4.3.1). For that purpose, we shall define a related modular form $E_k^*(z, C, D, N)$ and obtain its Fourier coefficients. Then by using the relations which we shall establish between the functions $E_k(z, C, D, N)$ and $E_k^*(z, C, D, N)$, we evaluate the fourier coefficients of $E_k(z, C, D, N)$. We define for any integers C and D

$$E_k^*(z, C, D, N) := \sum_{\substack{m \equiv C,\, n \equiv D (mod\, N) \\ (m,n) \neq (0,0)}} (mz + n)^{-k}$$

where $z \in \mathbb{H}$ and k is an integer greater than 2. We observe that if (C,D,N)=h, then

$$E_k^*(z, C, D, N) = h^{-k} E_k^*(z, C/h, D/h, N/h)$$

and $(C/h, D/h, N/h) = 1$. Therefore, we may assume that $(C, D, N) = 1$ We now state the theorem which provides the desired relations between the functions E_k and E_k^*.

Theorem 4.3.2. *Let $(C, D, N) = 1$ and k be an integer greater than 2. Then if*

$N > 2$,

$$E_k^*(z, C, D, N) = \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \left(\sum_{\substack{m=1 \\ mh \equiv 1 \, (mod\, N)}}^{\infty} m^{-k} \right) E_k(z, Ch, Dh, N)$$

and

$$E_k(z, C, D, N) = \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \left(\sum_{\substack{m=1 \\ mh \equiv 1 \, (mod\, N)}}^{\infty} \frac{\mu(m)}{m^k} \right) E_k^*(z, Ch, Dh, N) \tag{4.37}$$

where $\mu(m)$ is the Möbius function. Further,

$$E_k^*(z) := E_k(z, C, D, 1) = 2\zeta(k) E_k(z, C, D, 1) := 2\zeta(k) E_k(z)$$

and

$$E_k^*(z, C, D, 2) = 2(1 - 2^{-k}) \zeta(k) E_k(z, C, D, 2), \tag{4.38}$$

where $\zeta(k)$ is the Riemann zeta function. Finally, for all $N \geq 1$ and all $T \in \Gamma(1)$

$$E_k^*(z, C, D, N)|S = E_k^*(z, C\alpha + D\gamma, C\beta + D\delta, N). \tag{4.39}$$

Proof. See [18] page 176. □

The theorem above shows that $E_k^*(z, C, D, N)$ are also modular forms. Let

$$E_k^*(z, C, D, N) = \delta_L^* + \sum_{r=1}^{\infty} a^*(r, C, D, N) e^{2\pi i r z / N}.$$

We employ the functions

$$\begin{aligned} \sigma_{k-1}(r) &= \sum_{d|r,\, d>0} d^{k-1} \\ \sigma_{k-1}(r, C, D, N) &= \sum_{\substack{d|r,\, d \in \mathbb{Z} \\ \frac{r}{d} \equiv C \, (mod\, N)}} d^{k-2} |d| e^{2\pi i d D / N} \end{aligned}$$

in the next theorem which gives the formulae for $a^*(r, C, D, N)$.

Theorem 4.3.3. *Let (C,D,N)=1, the constant term in the Fourier expansion of* $E_k^*(z, C, D, N)$ *is*

$$\delta_L^* = \begin{cases} \sum_{r \equiv D\,(mod\,N)} r^{-k} & when\ C \equiv 0\,(mod\,N), \\ 0 & othewise. \end{cases} \tag{4.40}$$

Further for $r \geq 1$*,*

$$a^*(r, C, D, N) = \frac{(2\pi/Ni)^k}{\Gamma(k)} \sigma_{k-1}(r, C, D, N)$$

Proof. We require the Theorem 4.2.1 with $\kappa = \lambda = 0$ and in place of z we put either $(mz + D)/N$ or $-(mz + D)/N$ according to $m > 0$ or $m < 0$ in order not to violate the condition in theorem 4.2.1 that $z \in \mathbb{H}$. Only in the case when $C \equiv 0\,(mod\,N)$ we take $m = 0$ and obtain the constant term δ_L^*. We first observe that

$$F_k\left(\frac{mz + D}{N}, 0, 0\right) = N^k \sum_{h=-\infty}^{\infty} (mz + D + hN)^{-k} = \sum_{n \equiv D\,(mod\,N)} (mz + n)^{-k}$$

where in the last equation we put $n = D + hN$. Similarly we have

$$F_k\left(\frac{-(mz + D)}{N}, 0, 0\right) = N^k \sum_{h=-\infty}^{\infty} (-mz - D + hN)^{-k} = (-1)^k N^k \sum_{n \equiv D\,(mod\,N)} (mz + n)^{-k}$$

where in the last equation we write n in place of $D - hN$. It follows that

$$\begin{aligned} E_k^*(z, C, D, N) &= \sum_{\substack{m \equiv C,\, n \equiv D (mod\,N) \\ (m,n) \neq (0,0)}} (mz + n)^{-k} \\ &= \sum_{\substack{n \equiv D \\ n \neq 0\,(m=0)}} n^{-k} + \sum_{\substack{m \equiv C \\ m > 0}} \sum_{n \equiv D} (mz + n)^{-k} + \sum_{\substack{m \equiv C \\ m < 0}} \sum_{n \equiv D} (mz + n)^{-k} \\ &= \delta_L^* + \sum_{\substack{m \equiv C \\ m > 0}} N^{-k} F_k\left(\frac{mz + D}{N}, 0, 0\right) + (-1)^k \sum_{\substack{m \equiv C \\ m < 0}} N^{-k} F_k\left(\frac{-(mz + D)}{N}, 0, 0\right) \end{aligned}$$

the congruences under the summations above are taken modulo N. According to the Theorem 4.2.1, $F_k(z,0,0)$ is given by

$$F_k(z,0,0) = \sum_{r=1}^{\infty} g_r e^{2\pi i r z}$$

where, since $\lambda = 0$,

$$g_r = \frac{(2\pi)^k}{\Gamma(k)} i^{-k} r^{k-1}$$

Therefore, we have

$$E_k^*(z,C,D,N) = \delta_L^* + \frac{(2\pi/Ni)^k}{\Gamma(k)} \left(\sum_{\substack{m \equiv C \\ m>0}} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n(mz+D)/N} \right. \tag{4.41}$$

$$\left. + (-1)^k \sum_{\substack{m \equiv C \\ m<0}} \sum_{n=1}^{\infty} n^{k-1} e^{-2\pi i n(mz+D)/N} \right) \tag{4.42}$$

where congruences are modulo N. Since both of the series on the right hand side of the above equation are absolutely convergent, we can rearrange them in powers of $e^{2\pi i r z/N}$ with $r = nm$ for the first one and $r = -nm$ for the second. This gives

$$\begin{aligned} a^*(r,C,D,N) &= \frac{2\pi/(Ni)^k}{\Gamma(k)} \left(\sum_{\substack{d|r, d>0 \\ \frac{r}{d} \equiv C}} d^{k-1} e^{2\pi i d D/N} - \sum_{\substack{d|r, d<0 \\ \frac{r}{d} \equiv C}} d^{k-1} e^{2\pi i d D/N} \right) \\ &= \frac{2\pi/(Ni)^k}{\Gamma(k)} \sum_{\substack{d|r \\ \frac{r}{d} \equiv C}} d^{k-2} |d| e^{2\pi i d D/N} \end{aligned}$$

so that we have for $r \geq 1$

$$a^*(r,C,D,N) = \frac{2\pi/(Ni)^k}{\Gamma(k)} \sigma_{k-1}(r,C,D,N)$$

as desired. □

From the Theorem 4.3.3, we immediately deduce the following theorem.

Theorem 4.3.4. *Let $(C, D, N) = 1$ where $N > 1$ and k is an integer greater than 2; k is supposed even when $N = 2$. Then the Fourier coefficients $a(r, C, D, N)$ of the Eisenstein series $E_k(z, C, D, N)$ in (4.36) are given by the following formulae:*
(i)If $N > 2$ and $r \geq 1$,

$$a(r, C, D, N) = \frac{(2\pi/Ni)^k}{\Gamma(k)} \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\,(mod\,N)}}^{N} \frac{\mu(m)}{m^k} \sigma_{k-1}(r, hC, hD, N).$$

(ii)If $N = 2$ and $r \geq 1$,

$$a(r, C, D, 2) = \frac{(\pi/2)^k}{2(1 - 2^{-k})\zeta(k)\Gamma(k)} \sigma_{k-1}(r, C, D, 2).$$

Proof. (i)Let $N > 2$. For $r \geq 1$, by (4.37), we have

$$a(r, C, D, N) = \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} a^*(r, hC, hD, N).$$

We also have by the previous theorem that

$$a^*(r, hC, hD, N) = \frac{2\pi/(Ni)^k}{\Gamma(k)} \sigma_{k-1}(r, hC, hD, N). \tag{4.43}$$

It follows that

$$a(r, C, D, N) = \frac{2\pi/(Ni)^k}{\Gamma(k)} \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} \sigma_{k-1}(r, hC, hD, N).$$

(ii)Now let $N = 2$, by (4.39) we have for all $r \geq 1$

$$a^*(r, C, D, 2) = 2(1 - 2^{-k})\zeta(k)a(r, C, D, 2).$$

so that by (4.43) we get

$$a(r, C, D, 2) = \frac{(\pi/i)^k}{2(1 - 2^{-k})\zeta(k)\Gamma(k)} \sigma_{k-1}(r, C, D, 2)$$

where

$$\begin{aligned}
\sigma_{k-1}(r, C, D, 2) &= \sum_{\substack{d|r,\, d\in\mathbb{Z} \\ \frac{r}{d}\equiv C\,(mod\,2)}} d^{k-2}|d|e^{\pi i d D} \\
&= \sum_{\substack{d|r,\, d>0 \\ \frac{r}{d}\equiv C\,(mod\,2)}} d^{k-2}d(-1)^{dD} + \sum_{\substack{d|r,\, d<0 \\ \frac{r}{d}\equiv C\,(mod\,2)}} d^{k-2}(-d)(-1)^{dD} \\
&= 2\sum_{\substack{d|r,\, d>0 \\ \frac{r}{d}\equiv C\,(mod\,2)}} d^{k-1}(-1)^{dD}.
\end{aligned}$$

□

Chapter 5

Construction of an Eisenstein Series for $\Gamma_0(N)$

This chapter is devoted to the applications of the important results found in chapter 4. In the previous chapter we studied Eisenstein series belonging to the congruence subgroup $\bar{\Gamma}(N)$. In what follows, as an application of the results given in the previous chapter, we shall construct Eisenstein series for the Hecke congruence supgroup $\Gamma_0(N)$ where $N > 2$.

Firstly we shall present two more results form [18] which is necessary to apply Theorem 4.1.3 in our construction. Next, we shall introduce necessary definitions and results about Dirichlet characters, Gauss sums and Dirichlet-L functions which we employ to simplify our calculations. To facilitate the reading let us restate the Theorem 4.1.3.

Theorem 5.0.5. *Suppose that $k > 2$ and that $-I \in \Delta \subseteq \Gamma$, where Δ is normal in Γ and let $\mu = [\hat{\Gamma} : \hat{\Delta}]$. Let υ be a MS on Γ (and therefore on Δ) of weight k. Define n and κ by* (4.3)*, where $L \in \Gamma(1)$ and $\zeta = L^{-1}\infty$, and put*

$$n' = n(\zeta, \Delta) \quad \textit{and} \quad \kappa' = \kappa(\zeta, \Delta, \upsilon).$$

Then $n' = nl$ and $\kappa' = \{l\kappa\}$(fractional part), where l is a positive integral divisor *of* μ. *Let*

$$\hat{\Delta} = \hat{\Delta}_\zeta \cdot \mathfrak{R}$$

where $\hat{\Delta}_\zeta$ *is the stabilizer of* ζ *modulo* Δ, *then there exist a set* $\mathfrak{L}$ *of* μ/l *matrices* $L_j (1 \le j \le \mu/l)$ *in* Γ *such that*

$$\hat{\Gamma} = \hat{\Gamma}_\zeta \cdot \mathfrak{R} \cdot \mathfrak{L},$$

and for any $m \in \mathbb{Z}$,

$$G_L(z, m, \Gamma, k, \upsilon) = \sum_{j=1}^{\mu/l} \frac{G_{LL_j}(z, lm + [l\kappa], \Delta, k, \upsilon)}{\upsilon(L_j)\sigma(L, L_j)}$$

To determine the matrices $L_j (1 \le j \le \mu/l)$ in the Theorem 5.0.5, we employ the following theorem.

Theorem 5.0.6. ***[18, Theorem 1.1.2]*** *Let* Γ_2 *be a subgroup of finite index* μ *in a group* Γ_1 *and let* T *be a fixed member of* Γ_1. *Then there exists a finite number of elements* $L_1, L_2, \ldots, L_m$, *say, in* Γ_1 *and* m *disjoint sets*

$$\mathfrak{S}_i = \bigcup \{L_i T^k : 0 \le k < \sigma_i\} \quad (1 \le i \le m),$$

where

$$\sigma_i = \min\{k : T^k \in L_i^{-1} \Gamma_2 L_i, k \in \mathbb{Z}^+\}, \tag{5.1}$$

such that

$$\mu = \sigma_1 + \sigma_2 + \cdots + \sigma_m \tag{5.2}$$

and

$$\Gamma_1 = \Gamma_2 \cdot \bigcup_{i=1}^{m} \mathfrak{S}_i.$$

Moreover if T *has finite order* σ, *then* σ_i *divides* σ *for* $1 \le i \le m$. *Also, if* Γ_2 *is* *normal in* Γ_1*, then* $\sigma_i = \sigma_0$*, say, for* $1 \leq i \leq m$ *and so*

$$\mu = m\sigma_0.$$

Proof. Take any $L_1 \in \Gamma_1$ and define σ_1 by (5.1); since $L_1^{-1}\Gamma_2 L_1$ has finite index μ in Γ_1, σ_1 is a finite positive number and the member of $\mathfrak{S}_1$ belong to σ_1 different right cosets of Γ_2 in Γ_1. If $\mu = \sigma_1$, this completes the proof and $m = 1$ in his case. If $\mu > \sigma_1$ we take any L_2 not belonging to $\Gamma_2\mathfrak{S}_1$ and define σ_2 by (5.1). The σ_2 elements L_2T^k $(0 \leq k < \sigma_2)$ belong to different right cosets of Γ_2. Moreover $L_2T^k \notin \Gamma_2\mathfrak{S}_1$; for if $L_2T^k \in \Gamma_2\mathfrak{S}_1$ then $L_2 \in \Gamma_2\mathfrak{S}_1T^{-k} = \Gamma_2\mathfrak{S}_1$, which is not true. If $\mu = \sigma_1+\sigma_2$ the theorem follows; if $\mu > \sigma_1+\sigma_2$, we choose an $L_3 \notin \Gamma_2(\mathfrak{S}_1 \bigcup \mathfrak{S}_2)$ and proceed similarly. Since μ is finite and $\sigma_i > 0$ for each i, there exists a positive integer m such that (5.2) holds and the process then terminates, giving the required result. The two final sentences are immediate consequences. □

Theorem 5.0.7. ***[18, Theorem 1.1.3]*** *Let* Γ_2 *be a normal subgroup of finite index* μ *in a group* Γ_1 *and let* T *be a fixed member of* Γ_1*. Let* σ *be the least positive integer such that* $T^\sigma \in \Gamma_2$ *and write*

$$\mathfrak{S} = \bigcup\{T^k : 0 \leq k < \sigma\}.$$

Then there exists $m = \mu/\sigma$ *distinct elements* $L_1, L_2, \ldots, L_m$ *of* Γ_1 *such that*

$$\Gamma_1 = \mathfrak{S} \cdot \Gamma_2 \cdot \mathfrak{L}, \tag{5.3}$$

where $\mathfrak{L} = \bigcup\{L_i : 1 \leq i \leq m\}$*. Also if* Γ_1^* *and* Γ_2^* *are the subgroups of* Γ_1 *generated by* T *and* T^σ*, respectively, and* $\Gamma_2 = \Gamma_2^* \cdot \mathfrak{R}$*, then*

$$\Gamma_1 = \Gamma_1^* \cdot \mathfrak{R} \cdot \mathfrak{L}.$$

Proof. The equation given in (5.3) can be proven similarly to that of theorem

5.0.6. The normality of Γ_2 in Γ_1 comes in when we infer from

$$T^k M_2 L_i = T^l M_2' L_j \quad (0 \leq l \leq k < \sigma;\ M_2, M_2' \in \Gamma_2),$$

that $L_j \in \mathfrak{S}\Gamma_2 L_i$, therefore $L_j = L_i$, $k = l$ and $M_2' = M_2$. Since $\Gamma_1^* = \mathfrak{S} \cdot \Gamma_2^*$, we have

$$\Gamma_1 = \mathfrak{S} \cdot \Gamma_2 \cdot \mathfrak{L} = \mathfrak{S} \cdot \Gamma_2^* \cdot \mathfrak{R} \cdot \mathfrak{L} = \Gamma_1^* \cdot \mathfrak{R} \cdot \mathfrak{L},$$

which completes the proof. $\square$

While we are dealing with the construction of an Eisenstein series for the congruence group $\Gamma_0(N)$ $(N > 2)$, in order to reduce our results to simpler forms, we shall require the following knowledge about Dirichlet characters, Gauss sums an Dirichlet-L functions.

Recall that a reduced residue system modulo k is a set of $\varphi(k)$ integers $\{a_1, a_2, \ldots, a_{\varphi(k)}\}$ incongruent modulo k, each of which is relatively prime k. For each integer a the corresponding residue class $\hat{a}$ is the set of all integers congruent to a modulo k:

$$\hat{a} = \{x :\ x \equiv a\,(mod\,k)\}.$$

Definition 5.0.8. Let G be a group of reduced residue classes modulo k. Corresponding to each character f of G we define an arithmetical function $\chi = \chi_f$ as follows:

$$\chi(n) = f(\hat{n}) \qquad \text{if } (n,k) = 1,$$
$$\chi(n) = 0 \qquad \text{if } (n,k) > 1.$$

The function χ is called *Dirichlet character modulo k. The principal character* χ_1 is defined by

$$\chi_1(n) = \begin{cases} 1 & \text{if } (n,k) = 1 \\ 0 & \text{if } (n,k) > 1. \end{cases}$$

Definition 5.0.9. Let χ be a Dirichlet character mod k and let d be any positive divisor of k. A character χ modulo k is said to be *induced by a character* χ'

modulo d if

$$\chi(n) = \chi'(n \,(mod\, d))$$

A character which is not induced by any other character is called a *primitive character.*

Now we define Gauss sums associated with the Dirichlet character χ and Dirichlet-L functions.

Definition 5.0.10. For any Dirichlet character χ mod k the sum

$$G(n, \chi) = \sum_{m=1}^{k} \chi(m) e^{2\pi imn/k} \tag{5.4}$$

is called *Gauss sum associated with* χ.

Theorem 5.0.11. ***[2, Theorem 8.15]*** *Let χ be a primitive Dirichlet character mod k, then for all n we have*

$$G(n, \chi) = \bar{\chi}(n) G(1, \chi) \tag{5.5}$$

where $\bar{\chi}$ denotes the complex conjugate of χ.

Definition 5.0.12. Let $s \in \mathbb{C}$ with $\Re s > 1$, and χ be a Dirichlet character then the *Dirichlet L-functions* $L(s, \chi)$ is defined by the series

$$L(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{5.6}$$

Here, by applying Möbius inversion formula to the Dirichlet-L function $L(s, \chi)$ (see [2], Theorem 11.5), we have

$$\frac{1}{L(s, \chi)} = \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^s} \tag{5.7}$$

where $\mu(n)$ is the Möbius function of n.

In what follows, we construct the Eisenstein series for the group $\Gamma_0(N)$ corresponding to the cusp $\zeta = \infty$. Each incongruent cusp modulo $\Gamma_0(N)$ give rise to an Eisenstein series belonging to $\Gamma_0(N)$. The explicit formulas of all the Eisentein series for the congruence subgroup $\Gamma_0(N)$ can be found in [3]. For convenience we put $\zeta = I^{-1}\infty$. Firstly, we note that $-I \in \bar{\Gamma}(N)$ and the group $\bar{\Gamma}(N)$ is normal in $\Gamma_0(N)$ so that we can apply the theorem 5.0.5 with $\Gamma = \Gamma_0(N)$ and $\Delta = \bar{\Gamma}(N)$. Now assume υ is a multiplier system on $\Gamma_0(N)$ that is constant on $\Gamma(N)$ of integer weight $k > 2$. We write

$$n := n(\infty, \Gamma_0(N)) \quad \kappa := \kappa(\infty, \Gamma_0(N), \upsilon)$$

and

$$n' := n(\infty, \bar{\Gamma}(N)) \quad \kappa' := \kappa(\infty, \bar{\Gamma}(N), \upsilon).$$

We now calculate these numbers as they are required in theorem 5.0.5. First consider n, the width of the cusp $\zeta = \infty$ $(mod\, \Gamma_0(N))$, as it is explained in §2.3.2, n equals to the least positive integer s such that $U^s \in I\Gamma_0(N)I^{-1} = \Gamma_0(N)$, i.e., we search for the least positive integer s such that

$$\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N).$$

It follows that $n = 1$. Similarly, n' equals to the least positive integer s' such that $U^{s'} \in \bar{\Gamma}(N)$. Therefore $n' = N$. Then the inhomogeneous stabilizer group of the cusp $\zeta = I^{-1}\infty$ modulo $\bar{\Gamma}(N)$ is generated by U^N, i.e.,

$$\hat{\bar{\Gamma}}_\zeta(N) = \langle U^N \rangle$$

Note that $\hat{\bar{\Gamma}}(N) = \hat{\Gamma}(N)$, hence we have

$$\hat{\Gamma}(N) = \hat{\Gamma}_\zeta(N) \cdot \mathfrak{R} = \langle U^N \rangle \cdot \mathfrak{R}. \tag{5.8}$$

where $\mathfrak{R}$ is a set of right coset representatives of $\hat{\Gamma}(N)$ over $\hat{\Gamma}_\zeta(N)$.

To calculate the values of κ and κ', we need to choose a particular MS υ satisfying our assumptions. Observe that any Dirichlet character modulo N with $\chi(-1) = (-1)^k$ satisfies the consistency condition $\upsilon(-I) = (-1)^k$ so that we can set

$$\upsilon(T) = \chi(d) \quad \text{for all} \quad T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

We also assume that χ is nonprinciple and primitive. This MS is constant on $\Gamma(N)$ as it is desired. Then with this MS, we deduce that

$$e^{2\pi i\kappa} = \upsilon(I^{-1}U^n I) = \upsilon(U) = \upsilon\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = \chi(1) = 1.$$

Therefore $\kappa = 0$. Similarly

$$e^{2\pi i\kappa'} = \upsilon(I^{-1}U^{n'} I) = \upsilon(U^N) = \upsilon\left(\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix}\right) = \chi(1) = 1$$

so that $\kappa' = 0$.

We now calculate the index of $\hat{\bar{\Gamma}}(N) = \hat{\Gamma}(N)$ in $\hat{\Gamma}_0(N)$, i.e., the number μ in the theorem 5.0.5. By lemma 2.1.4, we have

$$[\hat{\Gamma}_0(n) : \hat{\Gamma}(n)] = \begin{cases} n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) & \text{if} \quad n = 1, 2 \\ \frac{1}{2} n^2 \prod_{p|n} \left(1 - \frac{1}{p}\right) & \text{if} \quad n \geq 3 \end{cases}$$

It follows since $N > 2$ that,

$$[\hat{\Gamma}_0(N) : \hat{\Gamma}(N)] = \frac{1}{2} N^2 \prod_{p|N} \left(1 - \frac{1}{p}\right) = \frac{1}{2} N\varphi(N).$$

Since $n' = N$ and $n = 1$, the number l in the Theorem 5.0.5 turns out to be N. Then by theorem 5.0.5, with $\Gamma = \Gamma_0(N)$, $\Delta = \bar{\Gamma}(N)$ and $\zeta = \infty$, there exists a set $\mathfrak{L}$ of $\mu/l = \frac{1}{2} N\varphi(N)/N = \frac{\varphi(N)}{2}$ matrices L_j $(1 \leq j \leq \frac{\varphi(N)}{2})$ in $\Gamma_0(N)$ such that

$$\hat{\Gamma}_0(N) = (\hat{\Gamma}_0(N))_\zeta \cdot \mathfrak{R} \cdot \mathfrak{L}$$

where $\mathfrak{R}$ is given by (5.8). These $\frac{\varphi(N)}{2}$ matrices L_j's are found by applying the algorithm explained in theorem 5.0.6 and 5.0.7 with $\Gamma_1 = \hat{\Gamma}_0(N)$, $\Gamma_2 = \hat{\Gamma}(N)$, $T = U$ and $\sigma = N$. We can take the matrices L_j to be

$$\begin{pmatrix} \tilde{d}_j & 1 \\ d_j\tilde{d}_j - 1 & d_j \end{pmatrix}, \quad 1 \leq j \leq \frac{\varphi(N)}{2}$$

where d_j runs through the first $\frac{\varphi(N)}{2}$ coprime integers to N and $\tilde{d}_j$'s are the inverses of d_j's modulo N.

Then by the last part of the Theorem 5.0.5, the Eisenstein series for $\Gamma_0(N)$ corresponding to the cusp $\zeta = I^{-1}\infty$, is given by

$$G_I(z, 0, \Gamma_0(N), k, \upsilon) = \sum_{j=1}^{\frac{\varphi(N)}{2}} \frac{G_{IL_j}(z, l.0 + [l\kappa], \bar{\Gamma}(N), k, \upsilon)}{\upsilon(L_j)\sigma(L, L_j)}$$

Here, as calculated above $\kappa = 0$ and since k is an integer, $\sigma(L, L_j) = 1$. We also set above that $\upsilon(L_j) = \chi(d_j)$ where χ is a primitive character. Therefore we have

$$\begin{aligned} G_I(z, 0, \Gamma_0(N), k, \upsilon) &= \sum_{j=1}^{\frac{\varphi(N)}{2}} \frac{G_{L_j}(z, 0, \bar{\Gamma}(N), k, \upsilon)}{\chi(d_j)} \\ &= \sum_{j=1}^{\frac{\varphi(N)}{2}} \frac{E_k(z, d_j\tilde{d}_j - 1, d_j, N)}{\chi(d_j)} \end{aligned}$$

where the sum is taken over the first $\frac{\varphi(N)}{2}$ integers d_j's that are coprime to N. Here, let us first consider the constant term δ_I of the Eisenstein series $G_I(z, 0, \Gamma_0(N), k, \upsilon)$. By the relation above, we have

$$\delta_I = \sum_{j=1}^{\frac{\varphi(N)}{2}} \frac{\delta_{L_j}}{\chi(d_j)}$$

where δ_{L_j} is the constant term of the Eisenstein series $E_k(z, d_j\tilde{d}_j - 1, d_j, N)$ belonging to $\bar{\Gamma}(N)$. Hence δ_{L_j} is given by 4.34 with $m = 0$, i.e,

$$\delta_{L_j} = \begin{cases} 1 & \text{if} \quad d_j \equiv 1 \ (mod\, N) \\ (-1)^k & \text{if} \quad d_j \equiv -1 \ (mod\, N) \\ 0 & \text{otherwise} \end{cases}$$

Since for all d_j's in the summation we have that $(d_j, N) = 1$ and $1 \leq d_j < N$, δ_{L_j} is nonzero only when $d_j = 1$. Therefore there is only one nonzero term corresponding to $d_j = 1$ in the above summation. Hence we deduce that

$$\delta_I = 1.$$

Now, let us denote $G_I(z, 0, \Gamma_0(N), k, \upsilon)$ by $E_k(z, \Gamma_0(N))$. Since the sum is taken over the first $\frac{\varphi(N)}{2}$ integers d_j's which are coprime to N, the summation can be written as

$$E_k(z, \Gamma_0(N)) = \sum_{\substack{d=1 \\ (d,N)=1}}^{\frac{N-1}{2}} \bar{\chi}(d) E_k(z, d\tilde{d} - 1, d, N) \tag{5.9}$$

Then by 4.37, we have

$$E_k(z, \Gamma_0(N)) = \sum_{\substack{d=1 \\ (d,N)=1}}^{\frac{N-1}{2}} \bar{\chi}(d) \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\, (mod\, N)}}^{\infty} \frac{\mu(m)}{m^k} E_k^*(z, h(d\tilde{d} - 1), hd, N) \tag{5.10}$$

Here, we need to recall the equation 4.41 which asserts

$$E_k^*(z, C, D, N) = \delta_L^* + \frac{(2\pi/Ni)^k}{\Gamma(k)} \left(\sum_{\substack{m\equiv C \\ m>0}} \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n(mz+D)/N} \right.$$
$$\left. + (-1)^k \sum_{\substack{m\equiv C \\ m<0}} \sum_{n=1}^{\infty} n^{k-1} e^{-2\pi i n(mz+D)/N} \right)$$

For brevity, we set $t_N = e^{2\pi i/N}$ and $q = e^{2\pi iz}$ and let $C = c + uN$ where $u, c \in \mathbb{Z}$ with $0 \leq c < N$. In our case, since the matrices L_j's belongs to the group $\Gamma_0(N)$, $c = 0$. Then we have for E_k^* in 5.10 that

$$E_k^* = \delta_L^* + \frac{(2\pi/Ni)^k}{\Gamma(k)} \sum_{\substack{m\geq 1\\ n\geq 1}} n^{k-1} q^{nm} t_N^{nhd} + (-1)^k t_N^{-nhd} \tag{5.11}$$

$$= \delta_L^* + \frac{(2\pi/Ni)^k}{\Gamma(k)} \sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \left(t_N^{nhd} + (-1)^k t_N^{-nhd}\right). \tag{5.12}$$

If we insert 5.12 into 5.10, we get

$$E_k(z, \Gamma_0(N)) - \delta_I = \frac{(2\pi/Ni)^k}{\Gamma(k)} \sum_{\substack{d=1\\(d,N)=1}}^{\frac{N-1}{2}} \bar{\chi}(d) \sum_{\substack{h=1\\(h,N)=1}}^{N} \sum_{\substack{m=1\\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} \times \left(\sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \left(t_N^{nhd} + (-1)^k t_N^{-nhd}\right) \right).$$

We calculate the right hand side of the above equation as follows,

$$\sum_{\substack{d=1\\(d,N)=1}}^{\frac{N-1}{2}} \bar{\chi}(d) \sum_{\substack{h=1\\(h,N)=1}}^{N} \sum_{\substack{m=1\\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} \sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \left(t_N^{nhd} + (-1)^k t_N^{-nhd}\right)$$
$$= \sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \sum_{\substack{h=1\\(h,N)=1}}^{N} \sum_{\substack{m=1\\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} \sum_{\substack{d=1\\(d,N)=1}}^{\frac{N-1}{2}} \bar{\chi}(d) \left(t_N^{nhd} + (-1)^k t_N^{-nhd}\right)$$

Since χ is a primitive Dirichlet character, $\bar{\chi} = 0$ if $(d, N) > 1$ so that we can remove the restriction $(d, N) = 1$ under the last summation above. Furthermore if we write $N - d$ instead of d in the second part of the last summation, we obtain,

$$\sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \sum_{\substack{h=1\\(h,N)=1}}^{N} \sum_{\substack{m=1\\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} \sum_{d=1}^{N} \bar{\chi}(d) t_N^{nhd}$$

Here, we notice that the last summation above is a Gauss sum defined in (5.4), hence the expression above can be written in terms of the Gauss sum $G(nh, \bar{\chi})$ as

$$\sum_{n=1}^{\infty} n^{k-1} \frac{q^n}{1-q^n} \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\,(mod\,N)}}^{\infty} \frac{\mu(m)}{m^k} G(nh, \bar{\chi}) \tag{5.13}$$

Since χ is a primitive Dirichlet character, we have by theorem 5.0.11 that

$$G(nh, \bar{\chi}) = \chi(nh) G(1, \bar{\chi}) := \chi(nh) G(\bar{\chi})$$

Then the expression given in (5.13) takes the form

$$\begin{aligned} G(\bar{\chi}) \sum_{n=1}^{\infty} n^{k-1} \frac{\chi(n) q^n}{1-q^n} & \sum_{\substack{h=1 \\ (h,N)=1}}^{N} \sum_{\substack{m=1 \\ mh\equiv 1\,(mod\,N)}}^{\infty} \bar{\chi}(m) \frac{\mu(m)}{m^k} \\ = G(\bar{\chi}) \sum_{n=1}^{\infty} n^{k-1} \frac{\chi(n) q^n}{1-q^n} & \sum_{m=1}^{\infty} \bar{\chi}(m) \frac{\mu(m)}{m^k}. \end{aligned}$$

Here if we use the identity given in (5.7) for the last summation above, we finally obtain,

$$E_k(z, \Gamma_0(N)) - \delta_I = \frac{(2\pi/Ni)^k}{\Gamma(k)} \frac{G(\bar{\chi})}{L(\bar{\chi}, k)} \sum_{n=1}^{\infty} \chi(n) n^{k-1} \frac{q^n}{1-q^n} \tag{5.14}$$

where $L(s, \chi)$ is the Dirichlet-L function defined by (5.6). Finally if we insert $\delta_I = 1$ into (5.14), we can conclude that the Eisenstein series belonging to the Hecke congruence subgroup $\Gamma_0(N)$ with $N > 2$ associated to the cusp $\zeta = \infty$ is of the form

$$E_k(z, \Gamma_0(N)) = 1 + \frac{(2\pi/Ni)^k}{\Gamma(k)} \frac{G(\bar{\chi})}{L(\bar{\chi}, k)} \sum_{n=1}^{\infty} \chi(n) n^{k-1} \frac{q^n}{1-q^n}.$$

This last equation can be expressed in terms of generalized Bernoulli numbers $B_{n,\chi}$ which are defined as follows:

Definition 5.0.13. Let χ be a Dirichlet character modulo N over $\mathbb{C}$. *The generalized Bernoulli numbers* $B_{n,\chi}$ are defined by

$$\frac{x}{e^{Nx}-1}\sum_{k=1}^{N}\chi(k)e^{kx}=\sum_{n=0}^{\infty}B_{n,\chi}\frac{x^n}{n!}. \tag{5.15}$$

In the case when χ is a nonprinciple Dirichlet character such that $\chi(-1) = (-1)^k$, the generalized Bernoulli numbers $B_{n,\chi}$ defined by (5.15) is calculated in [24] as

$$B_{k,\chi}=(-1)^{k-1}\frac{2k!}{G(\bar{\chi})}\left(\frac{N}{2\pi i}\right)^k L(\bar{\chi},k).$$

By using the last identity above, we conclude that

$$E_k(z,\Gamma_0(N))=1-\frac{2k}{B_{k,\chi}}\sum_{n=1}^{\infty}\chi(n)n^{k-1}\frac{q^n}{1-q^n}.$$

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