COBORDISM CALCULATIONS WITH ADAMS AND JAMES SPECTRAL SEQUENCES

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By

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I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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Let $\xi_n : \mathbb{Z}/p \to U(n)$ be an n-dimensional faithful complex representation of $\mathbb{Z}/p$ and $i_n : U(n) \to O(2n)$ be inclusion for $n \geq 1$. Then the compositions $i_n \circ \xi_n$ and $j_n \circ i_n \circ \xi_n$ induce fibrations on $B\mathbb{Z}/p$ where $j_n : O(2n) \to O(2n+1)$ is the usual inclusion. Let $(B\mathbb{Z}/p, f)$ be a sequence of fibrations where $f_{2n} : B\mathbb{Z}/p \to BO(2n)$ is the composition $Bi_n \circ B\xi_n$ and $f_{2n+1} : B\mathbb{Z}/p \to BO(2n+1)$ is the composition $Bj_n \circ Bi_n \circ B\xi_n$. By Pontrjagin-Thom theorem the cobordism group $\Omega_m(B\mathbb{Z}/p, f)$ of m-dimensional $(B\mathbb{Z}/p, f)$ manifolds is isomorphic to $\pi_m^s(M\mathbb{Z}/p, *)$ where $M\mathbb{Z}/p$ denotes the Thom space of the bundle over $B\mathbb{Z}/p$ that pullbacks to the normal bundle of manifolds representing elements in $\Omega_m(B\mathbb{Z}/p, f)$. We will use the Adams and James Spectral Sequences to get information about $\Omega_m(B\mathbb{Z}/p, f)$, when $p = 3$.

Keywords: Cobordism, (B, f)-structures, Group representation, Lens space .

ÖZET

ADAMS VE JAMES SPEKTRAL DİZİLERİYLE KOBORDİZM HESAPLARI

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$\xi_n : \mathbb{Z}/p \to U(n)$, $\mathbb{Z}/p$ grubunun n boyutlu birebir karmaşık bir temsili ve $i_n : U(n) \to O(2n)$, her $n \geq 1$ için bir kapsama olsun. O zaman $j_n : O(2n) \to O(2n+1)$ fonksiyonunun bilindik kapsama olduğu durumdaki $i_n \circ \xi_n$ ve $j_n \circ i_n \circ \xi_n$ bileşkeleri $B\mathbb{Z}/p$ gruplarının üzerlerinde liflemelerin oluşmasına sebep olurlar. $(B\mathbb{Z}/p, f)$, $f_{2n} : B\mathbb{Z}/p \to BO(2n)$ fonksiyonunun $Bi_n \circ B\xi_n$ bileşkesi ve $f_{2n+1} : B\mathbb{Z}/p \to BO(2n+1)$ fonksiyonunun $Bj_n \circ Bi_n \circ B\xi_n$ bileşkesi oldugu durumdaki bir lifleme dizisi olsun. $M\mathbb{Z}/p$; $B\mathbb{Z}/p$ grubunun üzerindeki, $\Omega_m(B\mathbb{Z}/p, f)$'nin içerisindeki elemanları temsil eden manifoldların normal demetlerini geri çeken vektör demetine ait Thom uzayını ifade etsin. Pontrjagin-Thom teoremi sayesinde $\Omega_m(B\mathbb{Z}/p, f)$ ile gösterilen m boyutlu $(B\mathbb{Z}/p, f)$ manifoldlarının kobordizm grubu $\pi_m^s(M\mathbb{Z}/p, *)$ ile eş yapılıdır. Biz $p = 3$ durumunda, Adams ve James Spektral dizilerini kullanarak $\Omega_m(B\mathbb{Z}/p, f)$ hakkında bilgi edinmeye çalışacağız.

Anahtar sözcükler: Kobordizm, (B, f)-yapıları, Grup temsili, Lens uzayı.

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Contents

Chapter 1

Introduction

Like many other theories in topology, the cobordism theory aims to classify objects in a particular category via an equivalence relation, called being cobordant. The main purpose of this thesis is to calculate Cobordism groups, $\Omega(B, f)$, of manifolds with a particular (B, f)-structure on the normal bundle by the help of Pontrjagin-Thom isomorphism. Since (B, f) is a structure on the normal bundle, the cobordism theory has its objects in the category of the differentiable manifolds. Roughly speaking, two differentiable manifold are called cobordant, if their disjoint union is a boundary of some manifold with (B, f)-structure. However, every manifold M is the boundary of $M \times [0, \infty)$, so it is wiser to shrink the category differentiable manifolds to compact differentiable manifolds, in order not to have a trivial theory.

The foundations of cobordism theory go back to year 1895, a paper by H.Poincaré: Analysis Situs, Journal de l'École Polytechnique. However, Poincaré did not use the term "Cobordism", in fact, his view of homology is similar to cobordism used today. The first applications of cobordism theory was due to L. S. Pontrjagin, in his paper: Smooth manifolds and their applications in homotopy theory. Combined with the efforts of R. Thom cobordism theory became a problem of homotopy theory. In fact, the paper: Quelques propriétés des variétés differentiables, Comm. Math. Helv.,(1954), by R. Thom, is widely accepted as the first major development of the cobordism theory since it provides a homotopy theoretical method for solving the problems of cobordism theory.

In the second Chapter, we discuss a general approach to cobordism with details and notations as in ([1], Chapter I,II). We introduce cobordism categories in both general and special senses. We introduce (B, f)-structures and category of (B, f)-manifolds. Then we define the cobordism groups associated with a (B, f)-structure and introduce a method to calculate these groups, which is known as "Pontrjagin-Thom isomorphism theorem". In consideration of this theorem, the problem of determining $\Omega(B, f)$ turns into a problem in homotopy theory. In the final section of this Chapter, we introduce a particular (B, f)-structure, for which we make our computations.

In Chapter 3, we introduce spectra and discuss homotopy theoretical definition of homology with notations as in [3, 4]. Since our problem turned into a problem in homotopy theory, we introduce a well known tool for homotopy theory problems; spectral sequences; in particular Adams spectral sequence [2, 3] and James spectral sequence [5].

In Chapter 4, we start spectral sequence computations. Firstly, we find the image of Thom class under Steenrod operations [7] by using the characteristic classes described in [6]. Once we found the image of Thom class under these operations, we start computing the E_2 pages of our spectral sequences for some (B, f)-structures associated with representations of the cyclic group $\mathbb{Z}/3$. First, we construct E_2 pages of Adams spectral sequences, which are equivalent to computation of some Ext groups as modules over Steenrod algebra. Thus, we construct projective resolutions and observe what happens in the homologies of dual complexes. After constructing the E_2 pages of Adams spectral sequences, we observe what happens in the pages of the James spectral sequences by looking at Adams spectral sequences.

In the final Chapter, we exemplify the results of the spectral sequences we compute. In particular, we look at the base lines of our results and comprehend which manifolds represent the classes appeared in our cobordism groups.

Chapter 2

Cobordisms

In this Chapter we will give a general notion of cobordism with using similar notations as in the book [1].

2.1 Cobordism category

Properties of cobordism categories indeed hold for the category of compact differentiable manifolds, but it may be useful to think in more abstract sense.

Definition 2.1.1 *A cobordism category (ζ, ∂, i) is a triple with ζ is a category and the following axioms holds*

1. *ζ has finite sums $(+)$, and an initial object, $\emptyset$.*
2. *$\partial : \zeta \to \zeta$ is an additive functor such that for each object X in ζ, $\partial\partial(X)$ is an initial object.*
3. *$i : \partial \to I$ is a natural transformation of additive functors from ∂ to the identity functor I.*
4. *There is a small subcategory ζ_0 of ζ such that each object of ζ is isomorphic to an object of ζ_0.*

In the manifold category; the addition in the first axiom corresponds to disjoint union, ∂ is usual boundary operator and i in the third axiom is the inclusion of the boundary. Whitney embedding theorem asserts that every manifold M can be embedded in $\mathbb{R}^n$ for some n, so that it can be embedded in $\mathbb{R}^\infty$. The fourth axiom regarding this assertion. The subcategory ζ_0 may be considered as the category of the isomorphism classes of objects in ζ, which is equivalent to the category of submanifolds of $\mathbb{R}^\infty$.

2.2 Cobordism semigroup

In order to construct cobordism theory, one must have a relation, so called cobordism relation.

Definition 2.2.1 *Let (ζ, ∂, i) be a cobordism category and X and Y be objects in ζ. X and Y are said to be cobordant, written $X \equiv Y$, if there exist U and V in ζ such that $X + \partial U$ is isomorphic to $Y + \partial V$.*

It is easily seen that this relation is reflexive, symmetric and transitive; thus an equivalence relation. This relation makes more sense when we consider ζ_0 instead of ζ, since we don't want to deal with set theoretical difficulties, i.e. we want the equivalence classes form a set. So for the rest, we mean ζ_0 by ζ.

Definition 2.2.2 *An object X of ζ is said to be closed if ∂X is an initial object.*

Proposition 2.2.3 (see [1]) *The set of equivalence classes under $\equiv$ of the closed objects of the category ζ has an operation induced by the sum in ζ.*

Since the sum in Definition 2.1.1 is associative, commutative, and has an identity element, so does the induced one. As a result, we can impose an algebraic structure for this set.

Definition 2.2.4 *The cobordism semigroup of (ζ_0, ∂, i) is the set of equvalence classes of closed objects in ζ_0. This semigroup is denoted by $\Omega(\zeta_0, \partial, i)$.*

2.3 Manifolds with (B, f)-structure

We denote the classifying space of the orthogonal group of degree n over $\mathbb{R}$, by $BO(n)$ and denote the universal n-plane bundle over $BO(n)$ by γ^n, which may be identified by $EO(n) \times_{O(n)} \mathbb{R}^n$ action of $O(n)$ over $EO(n) \times \mathbb{R}^n$ is given by $A \cdot (x, y) = (Ax, yA^{-1})$; $A \in O(n)$ and $(x, y) \in EO(n) \times \mathbb{R}^n$. We define a category smaller than the one defined in the previous section, which is based on an additional structure on the normal bundle of manifolds in the category ζ_0.

Definition 2.3.1 *Let $f_n : B_n \to BO(n)$ be a fibration and μ be a vector bundle over a space X classified by the map $\xi : X \to BO(n)$. A (B_n, f_n)-structure on μ is a homotopy class of liftings of ξ to B_n, i.e. a homotopy class of maps $\hat{\xi}$ such that the diagram*

$$\begin{tikzcd} & B_n \arrow[d, "f_n"] \\ X \arrow[ur, dashed, "\hat{\xi}"] \arrow[r, "\xi"] & BO(n) \end{tikzcd}$$

commutes.

Let $i : M^m \to \mathbb{R}^{m+n}$ be an embedding. The normal bundle $\eta(i)$ of i is defined as the quotient of the pullback of the tangent bundle, that is; $i^*\tau(R^{m+n})$, by the subbundle $\tau(M)$. We need the following lemma in order to see, (B_n, f_n)-structure on the normal bundle of a manifold is well defined.

Lemma 2.3.2 (for proof see [1], page 15) *If n is sufficiently large and $i_1, i_2 : M^m \to \mathbb{R}^{m+n}$ are two embeddings, then there is a 1-1 correspondence between the (B_n, f_n)-structures for the normal bundles of i_1, i_2.*

Now, we can give the definition of (B, f)-structure on manifolds, which assess the new category we will define.

Definition 2.3.3 *Suppose that* (B, f) *is a sequence of fibrations*

$$f_n : B_n \to BO(n) \quad \text{and maps} \quad g_n : B_n \to B_{n+1}$$

such that the diagram

$$\begin{tikzcd} B_n \arrow[r, "g_n"] \arrow[d, "f_n"] & B_{n+1} \arrow[d, "f_{n+1}"] \\ BO(n) \arrow[r, "Bj_n"] & BO(n+1) \end{tikzcd}$$

commutes with j_n *being the inclusion defined by*

$$j_n(A) = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

for any $A \in BO(n)$ *and* Bj_n *stand for the map induced by* j_n *on classifying spaces. A* (B_n, f_n)*-structure on the normal bundle* μ *of* M^m *in* $\mathbb{R}^{m+n}$ *defines a unique* (B_{n+1}, f_{n+1})*-structure on the normal bundle of* M *in* $\mathbb{R}^{m+n+1}$ *via the inclusion* $\mathbb{R}^{m+n} \hookrightarrow \mathbb{R}^{m+n+1}$*. A* (B, f)*-structure on* M *is an equivalence class of sequence of* (B_n, f_n)*-structures on the normal bundle* μ_n *of* M *in* $\mathbb{R}^{m+n}$*. Two* (B, f)*-structures is equivalent if they agree on a sufficiently large* n*.*

The category of manifolds with (B, f)-structure is a cobordism category.

Definition 2.3.4 *The cobordism category of* (B, f) *manifolds is the category whose objects are differentiable manifolds with* (B, f)*-structure and whose maps are boundary preserving differentiable embeddings with trivialized normal bundle such that the* (B, f)*-structure induced by the map is the same as the* (B, f)*-structure on the domain manifold. The functor* ∂ *applied to a* (B, f) *manifold* M *is the manifold* ∂M *with* (B, f)*-structure induced by the inner normal trivialization, and* ∂ *applied to maps is the restriction on* ∂M*. The map* i *is the inclusion of the boundary with inner normal trivialization.*

The cobordism semigroup of this category is denoted as $\Omega(B, f)$, and subsemigroup of equivalence class of n-dimensional closed manifolds is denoted by $\Omega_n(B, f)$. Note that $\Omega(B, f)$ is the direct sum of the $\Omega_n(B, f)$.

Proposition 2.3.5 (for proof see [1], page 17) *The semigroup $\Omega(B, f)$ is an abelian group.*

2.4 Computing $\Omega(B, f)$, Pontrjagin-Thom theorem

If μ is an n-plane bundle over X, classified by the map $\mu : X \to BO(n)$, we can induce a metric on μ from the metric on the universal n-plane bundle, γ^n over $BO(n)$, which is obtained from usual inner product on the subspace of $\mathbb{R}^\infty$ consisting of vectors with only finitely many non-zero component. Thus, we have the following definition for a given vector bundle:

Definition 2.4.1 *If $\mu : X \to BO(n)$ is an n-plane bundle over X, then the Thom space, $M\mu$, of the bundle μ is the space obtained from the total space of μ by collapsing all vectors of length at least 1 to a point $*$.*

We have the following theorem which is known as the generalized Pontrjagin-Thom theorem (see [1], page 18). This theorem gives us a calculation method for the cobordism groups.

Theorem 2.4.2 (Pontrjagin-Thom theorem) *The cobordism group of m-dimensional manifolds with (B, f)-structure, $\Omega_m(B, f)$, is isomorphic to $\lim\limits_{n\to\infty} \pi_{m+n}(MB_n, *)$.*

For the proof see ([1], page 19). With this theorem, one can have a more concrete view of cobordism theory; the problem of computing the cobordism groups becomes a problem of homotopy theory, in particular; computing stable homotopy groups.

2.5 (B, f)-structure associated with a representation of $\mathbb{Z}/p$

Given a cyclic group G of order p, where p is an odd prime, generated by σ, let $\xi_{l_1,l_2,..,l_{p-1}} : G \to U(r)$ be the r-dimensional representation defined as follows

$$\xi_{l_1,l_2,..,l_{p-1}}(\sigma) = \begin{pmatrix} A_{l_1} & 0 & \cdots & 0 \\ 0 & A_{l_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{l_{p-1}} \end{pmatrix}$$

where $A_{l_k} \in U(l_k)$ with

$$A_{l_k} = \begin{pmatrix} \omega^k & 0 & \cdots & 0 \\ 0 & \omega^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^k \end{pmatrix}$$

and $l_1 + l_2 + .. + l_{p-1} = r$

If $i_r : U(r) \to O(2r)$ is inclusion, then $g_{2r} = i_r \circ \xi_{l_1,l_2,..,l_{p-1}} : \mathbb{Z}/p \to O(2r)$ will be a $2r$-dimensional real representation. This representation induces a real vector bundle over $B\mathbb{Z}/p$ which fits into the pullback diagram

$$\begin{tikzcd} E\mathbb{Z}/p \times_{\mathbb{Z}/p} \mathbb{R}^{2r} \arrow[r] \arrow[d] & ESO(2r) \times_{SO(2r)} \mathbb{R}^{2r} \arrow[d] \\ B\mathbb{Z}/p \arrow[r, "f_{2r}"] & BSO(2r) \end{tikzcd}$$

where the action of $\mathbb{Z}/p$ over $E\mathbb{Z}/p \times \mathbb{R}^{2r}$ is given by $\sigma \cdot (\bar{x}, \bar{y}) = (\sigma\bar{x}, \bar{y}\sigma^{-1})$; $(x, y) \in E\mathbb{Z}/p \times \mathbb{R}^{2r}$.

Let M^m be a compact, differentiable m-manifold and μ is the normal bundle of M in R^{m+n}, classified by the map $\mu : M \to BO(n)$. If the map μ has a lifting to $B\mathbb{Z}/p$, i.e. $\exists\mu$ such that the diagram

$$\begin{tikzcd} & B\mathbb{Z}/p \arrow[d, "f_n"] \\ M \arrow[ur, dashed, "\hat{\mu}"] \arrow[r, "\mu"] & BO(n) \end{tikzcd}$$

commutes, then we have a $(B\mathbb{Z}/p, f_n)$-structure on μ; which is a homotopy class of liftings of μ to $B\mathbb{Z}/p$.

Let $f_n : \mathbb{Z}/p \to O(n)$ be a sequence of representations of $\mathbb{Z}/p$, where each f_n is induced by trivial representation when $n < 2r$, f_{2r} is defined as above and $f_{n+1} = Bj_n \circ f_n$ whenever $n > 2r$ where Bj_n is as before.

Each f_n induces a fibration on $B\mathbb{Z}/p$, so that we have a sequence of fibrations. Together with identity maps $\mathbb{Z}/p \to \mathbb{Z}/p$, we have a commutative diagram

$$\begin{tikzcd} B\mathbb{Z}/p \arrow[r, "id"] \arrow[d, "f_n"] & B\mathbb{Z}/p \arrow[d, "f_{n+1}"] \\ BO(n) \arrow[r, "Bj_n"] & BO(n+1) \end{tikzcd}$$

whenever $n \geq 2r$. As a result, we get a $(B\mathbb{Z}/p, f)$-structure on M.

Chapter 3

Spectral Sequences

In this section we introduce spectral sequences as a well-known technique to compute stable homotopy groups. But we first introduce the notion of spectra.

3.1 Spectra

For this section our focus is to define spectra, maps between spectra, homotopy of the maps between spectra and homology-cohomology of spectra. For more details, we refer to [3] and [4].

Definition 3.1.1 *A spectrum $E = \{E_n, e_n\}$ is a sequence $\{E_n : n \in \mathbb{Z}\}$ of a space with base point, together with a sequence of maps, $e_n : \Sigma E_n \to E_{n+1}$, where Σ denotes the reduced suspension.*

Here are some examples of spectra, letting $E = \{E_n, e_n\}$:

Example 1 *The first example is the sphere spectrum, that is; all $E_n = S^n$ and all $e_n = id_{S^{n+1}}$ are the identity maps. We denote this spectrum by S. More generally we can mention the suspension spectrum, that is E_0 is a given space with base* *point and $E_{n+1} = \Sigma E_n$ and maps are identity maps. S is the suspension spectrum of S^0.*

Example 2 *Let G be an abelian group, choose $E_n = K(G,n)$ and $e_n : \Sigma K(G,n) \to K(G,n+1)$ is the adjoint of $K(G,n) \to \Omega K(G,n+1)$, so that we get a spectra, which is called Eilenberg-MacLane Spectra and denoted by KG.*

Example 3 *Let (B,f) and $j_n : BO(n) \to BO(n+1)$ be as in Definition 2.3.3. The map j_n induces a vector bundle, $j_n^*(\gamma^{n+1})$, on $BO(n)$ which may be seen as the Whitney sum of γ^n and a trivial line bundle, so that the Thom space of $j_n^*(\gamma^{n+1})$, $Mj_n^*(\gamma^{n+1})$, becomes the suspension of $M\gamma^n$, then we have the following diagram*

$$\begin{tikzcd} \Sigma MB_n \arrow[r, "Mg_n"] \arrow[d, "\Sigma Mf_n"] & MB_{n+1} \arrow[d, "Mf_{n+1}"] \\ \Sigma MBO(n) \arrow[r, "Mj_n"] & MBO(n+1) \end{tikzcd}$$

thus $MB = \{MB_n, Mg_n\}$ is a spectrum, which is known as the Thom spectrum of the family (B,f).

Definition 3.1.2 *A spectrum $E = \{E_n, e_n\}$ is connective if $\exists k \geq 0$ such that $E_n = \{*\}$ whenever $n < k$.*

Definition 3.1.3 *Let $E = \{E_n, e_n\}$ be a spectrum. F is said to be a subspectrum if the sequence $F_n \subseteq E_n$ is also a spectrum, where each $f_n : \Sigma F_n \to F_{n+1}$ is the restriction of e_n on ΣF_n, i.e. $f_n = e_n|_{\Sigma F_n}$.*

Since we generally work with CW-complexes, it seems we would better mention what a CW-spectrum is.

Definition 3.1.4 *A CW-spectrum is a spectrum of CW-complexes E_n, with maps e_n are inclusions of subcomplexes. The base points of E_n's are 0-cells.*

A CW-spectrum is of finite type if it has finitely many cells in each E_n. A subspectrum F of a CW-spectrum E is defined as above definition of subspectrum with additional condition that each $F_n \subseteq E_n$ is a subcomplex.

3.1.1 Map of spectra

There is a difference between the meaning of a function from one spectrum to another and the meaning of a map between spectra. A function is a sequence of maps between spaces, while a map is an equivalence class of functions under some equivalence relation. Let us give the following definition in order to make things more clear.

Definition 3.1.5 *A function of degree r between two given spectra $E = \{E_n, e_n\}$ and $F = \{F_n, f_n\}$ is a sequence of maps $g_n : E_n \to F_{n-r}$ such that the diagrams*

$$\begin{tikzcd} \Sigma E_n \arrow[r, "e_n"] \arrow[d, "g_n"] & E_{n+1} \arrow[d, "g_{n+1}"] \\ \Sigma F_{n-r} \arrow[r, "f_n"] & F_{n-r+1} \end{tikzcd}$$

are strictly commutative for each n.

The condition “strictly commutative” means not up to homotopic, and this condition is imposed in order to avoid difficulties in further constructions, since otherwise we should know what the homotopies are.

Definition 3.1.6 *A subspectrum $E' = \{E'_n, e'_n\}$ of $E = \{E_n, e_n\}$ is said to be cofinal in E if for each n and finite subcomplex $K \subseteq E_n$ there is an m such that $\Sigma^m K$ maps into E'_{m+n} under the composition*

$$\Sigma^m E_n \xrightarrow{\Sigma^{m-1} e_n} \Sigma^{m-1} E_n \xrightarrow{\Sigma^{m-2} e_{n+1}} \cdots \xrightarrow{\Sigma e_{m+n-2}} \Sigma E_{m+n-1} \xrightarrow{e_{m+n-1}} E_{m+n}$$

Here, m depends on K and n. Now, we construct following equivalence relation in order to give the definition of a map between two spectra.

Definition 3.1.7 *Consider all cofinal subspectra $E' \subseteq E$ and all functions $f' : E' \to F$. Say two functions $f' : E' \to F$ and $f'' : E'' \to F$ are related, $f' \equiv f''$, if there is a cofinal subspectrum E''' contained in both E' and E'' such that the restrictions of f' and f'' to E''' coincide.*

This relation is in fact an equivalence relation(see [4], pages 142-143).

Definition 3.1.8 *A map $g : E \to F$ between two CW-spectra is an equivalence class of above equivalence relation.*

3.1.2 Homology and Cohomology of spectra

Let I^+ be the union $I \cup \{*\}$ where I is unit interval and $*$ is the basepoint not belonging I and $E = \{E_n, e_n\}$ is a spectrum. Let $\wedge$ denotes the smash product of spaces with base points, i.e. if U and V such spaces;

$$U \wedge V = U \times V/((U \times *) \cup (* \times V)).$$

We define cylinder spectrum of E by $Cyl(E) = \{Cyl(E)_n, 1 \wedge e_n\}$ with

$$Cyl(E)_n = I^+ \wedge E_n$$

and

$$1 \wedge e_n : (I^+ \wedge E_n) \wedge S^1 \to I^+ \wedge E_{n+1}.$$

Note that smash product with a circle is equivalent to suspension. We define homotopy of spectra by a map of cylinder as an analogue of homotopy of maps.

Definition 3.1.9 *Given spectra $E = \{E_n, e_n\}$ and $F = \{F_n, f_n\}$, and maps $f, f' : E \to F$, we say f and f' are homotopic if there is a map $h : Cyl(E) \to F$*

such that $f = hi_0$ and $f' = hi_1$ where i_0 and i_1 are natural embeddings corresponding to both ends of $Cyl(E)$. We denote the set of homotopy classes of maps of degree r with $[E, F]_{-r}$.

For the definition of "smash product of spectra" you may see ([4], Part 3, Chapter 4).

Definition 3.1.10 *Let E and F be two spectra, we can define E-homology and E-cohomology of the spectrum F by*

- $E_n(F) = [S, E \wedge F]_{-n}$
- $E^n(F) = [F, E]_n$

3.2 Adams Spectral Sequence

We describe the Adams Spectral Sequence in this section. For notations and more details, we refer to [2, 3]. Before introducing the spectral sequence, we need the following definitions.

3.2.1 Steenrod Algebra $\mathcal{A}_p$

We state the axiomatic development of $\mathcal{A}_p$ as in ([7], pages 76-77). For details and proofs we refer to ([7], Chapters VI,VII,VIII).

Definition 3.2.1 *Let p be an odd prime as before and X is a space, we have the following axioms:*

1. *$\beta : H^n(X, \mathbb{Z}/p) \to H^{n+1}(X, \mathbb{Z}/p)$ is the Bockstein operator associated with the exact sequence*

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

2. *For all integers $i \geq 0$ and $n \geq 0$ there is a natural homomorphism*

$$P^i : H^n(X, \mathbb{Z}/p) \to H^{n+2i(p-1)}(X, \mathbb{Z}/p)$$

with $P^0 = 1$.

3. $\dim x = 2k$ *implies* $P^k x = x^p$

4. $\dim x > 2k$ *implies* $P^k x = 0$

5. *Cartan formula:* $P^k(xy) = \sum_i P^i x P^{k-i} y$

6. *Adem relations: If $a < pb$ then*

$$P^a P^b = \sum_{t=0}^{\lfloor a/p \rfloor} -1^{a+t} \binom{(p-1)(b-t)-1}{a-pt} P^{a+b-t} P^t$$

If $a \geq b$ then

$$P^a \beta P^b = \sum_{t=0}^{\lfloor a/p \rfloor} -1^{a+t} \binom{(p-1)(b-t)}{a-pt} \beta P^{a+b-t} P^t$$

$$+ \sum_{t=0}^{\lfloor a-1/p \rfloor} -1^{a+t-1} \binom{(p-1)(b-t)-1}{a-pt-1} P^{a+b-t} \beta P^t$$

The mod-p Steenrod Algebra, $\mathcal{A}_p$ is the graded associative algebra generated by the elements P^i of degree $2i(p-1)$ and β of degree 1; subject to the conditions; $P^0 = 1$, $\beta^2 = 0$ and Adem relations.

3.2.2 Construction of the spectral sequence

Let E be a finite connective CW-spectrum. Consider $H^*(E, \mathbb{Z}/p) = K\mathbb{Z}/p^*(E)$ as a free $\mathcal{A}_p$-module, with at most finitely many generators for each $H^k(E, \mathbb{Z}/p) = [E, K\mathbb{Z}/p]_k$. These generators determine a map $E \to K_0$, where K_0 is a wedge of Eilenberg-MacLane spectra and has finite type. Since any map is homotopic to an inclusion, replacing the map $E \to K_0$ with an inclusion, we can form the quotient $E_1 = K_0/E_0$. Repeating the same argument for E_i for $i > 0$ we get the diagram

$$\begin{tikzcd}
E \arrow[rr] & & K_0 \arrow[rr] \arrow[dr] & & K_1 \arrow[rr] \arrow[dr] & & K_2 \arrow[rr] \arrow[dr] & & \cdots \\
& & & K_0/E = E_1 \arrow[ur] & & K_1/E_1 = E_2 \arrow[ur] & & K_2/E_2 = E_3 \arrow[ur] &
\end{tikzcd}$$

Proposition 3.2.2 (for proof, see [2, 3]) *The natural map*

$$[E, \bigvee_i K(G, n_i)] \to \prod_i [E, K(G, n_i)]$$

whose coordinates are obtained by composing with the projections of $\bigvee_i K(G, n_i)$ *onto its factors, is an isomorphism if* E *is a connective* CW*-spectrum of finite type and* $n_i \to \infty$ *as* $i \to \infty$

With the proposition in hand, the associated diagram of cohomology

$$\begin{tikzcd}
0 & H^*(E) \arrow[l] & H^*(K_0) \arrow[l] & & H^*(K_1) \arrow[ll] & & H^*(K_2) \arrow[ll] & & \cdots \arrow[ll] \arrow[dl] \\
& & & H^*(E_1) \arrow[ul] \arrow[dl] & & H^*(E_2) \arrow[ul] \arrow[dl] & & H^*(E_3) \arrow[ul] \arrow[dl] & \\
& & 0 & & 0 \arrow[ul] & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

gives a free resolution of $H^*(E, \mathbb{Z}/p)$ by $\mathcal{A}_p$-modules.

Let E be a CW-spectrum of finite type and let the functor $\pi_t^s(\_)$ denote the stable homotopy. Since it is a homology theory, when applying to the cofibrations (see [2, 3])

$$E_d \to K_d \to E_{d+1},$$

we get long exact sequences forming a staircase diagram

$$
\begin{tikzcd}
 & \pi^s_{t+1}(E_d) \arrow[r] \arrow[d] & \pi^s_{t+1}(K_d) \arrow[r] & \pi^s_{t+1}(E_{d+1}) \arrow[r] \arrow[d] & \pi^s_{t+1}(K_{d+1}) \arrow[r] & \pi^s_{t+1}(E_{d+2}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^s_{t}(E_{d-1}) \arrow[r] \arrow[d] & \pi^s_{t}(K_{d-1}) \arrow[r] & \pi^s_{t}(E_d) \arrow[r] \arrow[d] & \pi^s_{t}(K_d) \arrow[r] & \pi^s_{t}(E_{d+1}) \arrow[r] \arrow[d] & {} \\
{} \arrow[r] & \pi^s_{t-1}(E_{d-2}) \arrow[r] & \pi^s_{t-1}(K_{d-2}) \arrow[r] & \pi^s_{t-1}(E_{d-1}) \arrow[r] & \pi^s_{t-1}(K_{d-1}) \arrow[r] & \pi^s_{t-1}(E_d) \arrow[r] & {}
\end{tikzcd}
$$

So we have a spectral sequence [2], which is called Adams spectral sequence. Now, since K_d is a wedge of Eilenberg-MacLane spectra, $\pi^s(K_d)$ is a direct sum of $\mathbb{Z}$ for each $K(\mathbb{Z}/p, n_i)$ summand in K_d by Proposition 3.2.2. Since $H^*(K_d)$ is free over $\mathcal{A}_p$ then the natural map

$$\pi^s_t(K_d) \to \mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{K_d}), \mathbb{Z}/\mathrm{p})$$

is an isomorphism, where $\mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}$ denotes the group of homomorphisms that lowers degree by t. Consequently we get,

$$E_1^{t,s} = \mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{K_d}), \mathbb{Z}/\mathrm{p}) = \pi^{\mathrm{s}}_{\mathrm{t}}(\mathrm{K_d}).$$

The differential

$$d_1 : \pi^s_t(K_d) \to \pi^s_t(K_{d+1})$$

is induced by the map $K_d \to K_{d+1}$ in the resolution of E constructed above. This implies the E_1 page of the spectral sequence consist of the complexes

$$0 \to \mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{K_0}), \mathbb{Z}/\mathrm{p}) \to \mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{K_1}), \mathbb{Z}/\mathrm{p}) \to ...$$

so that the homology groups of this complex, which give us the E_2 page of the Adams spectral sequence, are

$$E_2^{s,t} = \mathrm{Ext}^{\mathrm{s,t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{E}; \mathbb{Z}/\mathrm{p}), \mathbb{Z}/\mathrm{p}).$$

Theorem 3.2.3 ([2, 3]) *Let E be a connective CW-spectrum of finite type. The Adams spectral sequence for E converges to $\pi_*^s(E)/\langle p - torsion\rangle$ or equivalently has the following properties:*

1. *For fixed s and t, and r is sufficiently large, the groups $E_r^{s,t}$ are independent of r, and the stable groups $E_\infty^{s,t}$ are isomorphic to $F^{s,t}/F^{s+1,t+1}$ for the filtration of $\pi_{t-s}^s(E)$ by the images of the maps $\pi_t^s(E_s) \to \pi_{t-s}^s(E)$.*
2. *$\bigcap_i F^{s+i,t+i}$ is equal to $\pi_*^s(E)/\langle p' - torsion\rangle$.*

3.3 James Spectral Sequence

In this section we show that there exist a spectral sequence, called James Spectral Sequence, with notations as in the paper of P. Teichner, [5]. For more details, you may want to see [5]. Let $f : E \to B$ be a π^s-orientable fibration whose fibers are F and $\vartheta : B \to BSO$ be a stable vector bundle. Since $SO = \bigcup_{n\in\mathbb{N}} SO(n)$ and $BSO(n) \subseteq BSO(n+1)$ for each n, then there are inclusions

$$\imath_n : BSO(n) \to BSO$$

and

$$\imath_n^{n+1} : BSO(n) \to BSO(n+1)$$

defined as similar fashion with $\jmath_n$ above. By the pullback

$$\begin{tikzcd} E_n \arrow[r, "e_n"] \arrow[d, "\vartheta_n"] & E \arrow[d, "\vartheta"] \\ BSO(n) \arrow[r, "\imath_n"] & BSO \end{tikzcd}$$

we construct a sequence of fiber bundles over E from the stable vector bundle ϑ. If we compose the maps e_n with the fibration f, we get a sequence of fibrations $f_n : E_n \to B$ with fibers F_n and a sequence of vector bundles ϑ_n over E_n such that the following diagrams commute

$$\begin{tikzcd} F_n \arrow[r] \arrow[d] & E_n \arrow[r, "f_n"] \arrow[d] & B \arrow[d, "id"] \\ F_{n+1} \arrow[r] & E_{n+1} \arrow[r, "f_{n+1}"] & B \end{tikzcd} \qquad \begin{tikzcd} E_n \arrow[r, "\vartheta_n"] \arrow[d] & BSO(n) \arrow[d, "\imath_n^{n+1}"] \\ E_{n+1} \arrow[r, "\vartheta_{n+1}"] & BSO(n+1) \end{tikzcd}$$

Now, consider the disk-sphere bundle pair $(D(\vartheta_n), S(\vartheta_n))$ which is a relative fibration over E_n with relative fiber (D^n, S^{n-1}). If we compose with f_n, we get a relative fibration over B with relative fiber $(D(\vartheta_n|F_n), S(\vartheta_n|F_n))$ which is π^s-orientable since f, so that f_n's are orientable. Thus, there is a relative Atiyah-Hirzebruch-Serre spectral sequence(see [8], Chapter 15)

$${}^nE : H_s(B; \pi_t^s(D(\vartheta_n|F_n), S(\vartheta_n|F_n))) \Rightarrow \pi_{t+s}^s(D(\vartheta_n), S(\vartheta_n)) \cong \pi_{t+s}^s(M\vartheta_n)$$

for each n. Thus, James spectral sequence is defined as direct limit of ${}^nE$, that is;

$$E : E_{s,t}^i := \lim_{n\to\infty} {}^nE_{s,t}^i$$

which will converge to

$$\lim_{n\to\infty} \pi_{t+s}^s(M\vartheta_n)$$

as the direct limit functor is exact. The differential of this spectral sequence d_r^i is obtained by direct limits, i.e. $d_r^i := \lim\limits_{n\to\infty} {}^nd_r^i$.

Theorem 3.3.1 (see [5]) *Let $f : E \to B$ be a π^s-orientable fibration whose fibers are F and $\vartheta : E \to BSO$ be a stable vector bundle. Then there exist a spectral sequence*

$$E_{r,t}^2 \cong H_r(B; \pi_t^s(M\vartheta|F)) \Rightarrow \pi_{r+t}^s(M\vartheta),$$

which is called the James spectral sequence for the fibration f.

Chapter 4

Some Spectral Sequence Computation

4.1 Image of the Thom class under Steenrod operations

This section primarily deals with finding the image of the Thom class under mod-p Steenrod operations by the help of characteristic classes. For more details about the Steenrod operations and characteristic classes, you may see [6] and [7].

4.1.1 The Thom class

Theorem 4.1.1 (Thom Isomorphism Theorem [9]) *If (B, f) is as in Definition 2.3.3, then there is a cohomology class $\bar{U}_r \in \tilde{H}^r(MB_r, \mathbb{Z})$ for each r such that the map*

$$\theta : \tilde{H}^n(B_r, \mathbb{Z}) \to \tilde{H}^{n+r}(MB_r, \mathbb{Z})$$

defined by $\theta(\sigma) = \bar{U}_r \cup \sigma$ is an isomorphism. The class $\bar{U}_r$ is called Thom class.

The class $\bar{U} \in [MB, K\mathbb{Z}] = \tilde{H}^0(MB, \mathbb{Z})$ with $\bar{U} = \{\bar{U}_r\}$ is called the total Thom class, or simply the Thom class and we write $H^*(MB, \mathbb{Z}) \cong \bar{U} \cup H^*(B, \mathbb{Z})$ with isomorphism defined as cup product, as is the theorem above, so that an element of $H^*(MB, \mathbb{Z})$ is of the form $\bar{U} \cup \sigma$ with $\sigma \in H^*(B, \mathbb{Z}/p)$. Returning to our more specific (B, f)-structure as in Section 2.5, we can view $H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$ as the product $U \cup H^*(B\mathbb{Z}/p, \mathbb{Z}/p)$ where U is the mod-p reduction of the integral cohomology class $\bar{U} \in H^*(M\mathbb{Z}/p, \mathbb{Z})$. To construct the E_2 page of the Adams spectral sequence, we must construct projective resolution of $H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$, as $\mathcal{A}_p$-modules. So we need to know the image of U under mod-p Steenrod operations, which we can find by using characteristic classes.

4.1.2 Method of computation

We have $H^*(\mathbb{Z}/p, \mathbb{Z}) = \mathbb{Z}[\bar{x} : p\bar{x} = 0]$ with $\bar{x}$ being the Chern class of the 1-dimensional representation $\xi_{1,0,..,0}$ from the Borel-Hirzebruch description of characteristic classes (see [10]), so that $H^*(\mathbb{Z}/p, \mathbb{Z}/p) = \mathbb{Z}/p[x] \otimes \bigwedge(a)$ where x is the image of $\bar{x}$ under mop-p reduction and a is defined via $\beta(a) = x$, with β being the Bockstein associated with the exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow \mathbb{Z}/p^2 \longrightarrow \mathbb{Z}/p \longrightarrow 0.$$

Consider the diagram formed by two exacts sequences

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z} \arrow[r, "\times p"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}/p \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}/p \arrow[r] & \mathbb{Z}/p^2 \arrow[r] & \mathbb{Z}/p \arrow[r] & 0
\end{tikzcd}$$

this diagram induces a diagram on cohomology

$$\begin{tikzcd}
\arrow[r] & H^t(M\mathbb{Z}/p, \mathbb{Z}/p^2) \arrow[r, "red"] & H^t(M\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r] & H^{t+1}(M\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r] & {} \\
\arrow[r] & H^t(M\mathbb{Z}/p, \mathbb{Z}) \arrow[r, "red"] \arrow[u, "red"] & H^t(M\mathbb{Z}/p, \mathbb{Z}/p) \arrow[r] \arrow[u, "id"] & H^{t+1}(M\mathbb{Z}/p, \mathbb{Z}) \arrow[r] \arrow[u, "red"] & {}
\end{tikzcd}$$

Composition of two bottom horizontal maps is zero, so $\beta(U) = 0$. We can find the image of the Thom class U under other mod-p Steenrod operations, i.e. $P^i(U)$, by using Wu classes (see [6]). First we need to have information about Chern classes. If c_i is the i'th Chern class of a representation ϕ, then p_i; the i'th Pontrjagin class of ϕ is equal to,

$$p_i = c_i^2 - 2c_{i-1}c_{i+1} + \cdots + (-1)^i 2c_{2i}$$

[6] and once we know Pontrjagin classes, we can compute Wu classes.

Let R be a commutative ring with 1 and $A^* = (A^0, A^1, A^2, \cdots)$ be a graded-commutative R-algebra and A^Π consists of the elements $a = 1 + a_1 + \cdots$ where each $a_i \in A^i$.

Definition 4.1.2 *Let $\{K_n(x_1, .., x_n)\}$ be a sequence of homogeneous polynomials of degree n and x_i has degree i for each i. For $a = a_0 + a_1 + \cdots \in A^\Pi$ with $a_0 = 1$, let $K(a) = 1 + K_1(a_1) + K_2(a_1, a_2) + \cdots \in A^\Pi$. We say K_n form a multiplicative sequence of polynomials, if the equality $K(ab) = K(a)K(b)$ holds for all $a, b \in A^\Pi$ with $a_0 = 1$ and all graded commutative R-algebras.*

Lemma 4.1.3 ([6], page 221) *Given a formal power series $f(t) = 1 + r_1 t + r_2 t^2 + \cdots$ with coefficients in R, there exist a unique multiplicative sequence $\{K_n\}$ with coefficients in R satisfying $K(1 + t) = f(t)$.*

Such a $\{K_n\}$ is called the multiplicative sequence belonging to the power series $f(t)$. Now, let $p = 2s + 1$, then the n'th Wu class of ϕ will be equal to

$$q_n = K_{sn}(p_1, .., p_{sn})$$

reduced modulo p, where $K = \{K_i\}$ is the multiplicative sequence belonging to the power series $f(t) = 1 + t^s$(see [6], page 221).

Theorem 4.1.4 (see [6], page 229) *If $p = 2s + 1$, then the n'th $\mathrm{mod} - \mathrm{p}$ Wu class of a representation ϕ is $q_n = \theta^{-1} P^n \theta(1)$, where θ is the Thom isomorphism. As a result, we get $P^n\theta(1) = \theta(q_n)$ which implies $P^n(U) = U \cup q_n$.*

1-dimensional representations

Let $\xi_{0,..,1,..,0}$ be the representation as in Section 2.5 with 1 sits on the j'th place. Denote $\xi_{0,..,1,..,0}$ by ξ_j for simplicity, so that $\xi_j(\sigma) = \omega^j\sigma$, then we say $c_0(\xi_j) = 1$, $c_1(\xi_j) = c_1(\xi_1^{\otimes j}) = jc_1(\xi_1)$ and $c_n(\xi_j) = 0$ whenever $n > 1$. From the Chern classes, the Pontrjagin classes of ξ_j can be found that; $p_0(\xi_j) = 1$, $p_1(\xi_j) = (j\bar{x})^2$ and $p_n(\xi_j) = 0$ when $n > 1$. Thus, $q_1(\xi_j) = K_s(p_1, 0, .., 0)$ with mod-p coefficients, so that $q_1(\xi_j) = (jx)^{2s} = j^{2s}x^{2s} = x^{2s}$ (see Lemma 4.1.3) and $q_n(\xi_j) = 0$ when $n > 1$. As a result, we get

$$P^n(U) = \begin{cases} U & \text{if } n = 0 \\ U \cup x^{2s} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

where U is the Thom class.

$p = 3$ Cases:

Let $p = 3$, then the Pontrjagin classes will be equal to the Wu classes reduced modulo p.

Case 1 *For both 1-dimensional representations $\xi_{1,0}$ and $\xi_{0,1}$, above computations imply that Wu classes is $q_1(\xi_{1,0}) = q_1(\xi_{0,1}) = x^2$ and $q_n(\xi) = 0$ when $n \neq 1$, so that we have*

$$P^n(U) = \begin{cases} U & \textit{if } n = 0 \\ U \cup x^2 & \textit{if } n = 1 \\ 0 & \textit{otherwise} \end{cases}$$

Case 2 *Consider the 2-dimensional representation $\xi_{1,1}$. The total Chern class is, $c = (1 + \bar{x})(1 + 2\bar{x}) = 1 + 3\bar{x} + 2\bar{x}^2$ which implies $c_0(\xi_{1,1}) = 1$, $c_1(\xi_{1,1}) = 3\bar{x}$, $c_2(\xi_{1,1}) = 2\bar{x}^2$ and $c_n(\xi_{1,1}) = 0$ when $n > 2$. The Pontrjagin classes $p_0(\xi_{1,1}) = 1$, $p_1(\xi_{1,1}) = 2\bar{x}^2$, $p_2(\xi_{1,1}) = (2\bar{x}^2)^2 = 4\bar{x}^4$ and $p_n(\xi_{1,1}) = 0$ when $n \neq 1$ or $n \neq 2$. Consequently; $q_0(\xi_{1,1}) = 1$, $q_1(\xi_{1,1}) = 2x^2$, $q_2(\xi_{1,1}) = x^4$ and $q_n(\xi_{1,1}) = 0$*

otherwise, thus we get

$$P^n(U) = \begin{cases} U & \text{if } n = 0 \\ 2U \cup x^2 & \text{if } n = 1 \\ U \cup x^4 & \text{if } n = 2 \\ 0 & \text{otherwise} \end{cases}$$

which is the same as for the representations $\xi_{2,0}$ *and* $\xi_{0,2}$.

Case 3 *Consider the* 3*-dimensional representation* $\xi_{3,0}$. *Then the total Chern class of this representation will be* $(1+\bar{x})^3 = 1+3\bar{x}+3\bar{x}^2+\bar{x}^3$, *so that* $c_0(\xi_{3,0}) = 1$, $c_1(\xi_{3,0}) = 3\bar{x}$, $c_2(\xi_{3,0}) = 3\bar{x}^2$, $c_3(\xi_{3,0}) = \bar{x}^3$ *and* $c_n(\xi_{3,0}) = 0$ *otherwise. The Pontrjagin classes* $p_0(\xi_{3,0}) = 1$, $p_1(\xi_{3,0}) = 9\bar{x}^2$, $p_2(\xi_{3,0}) = 3\bar{x}^4$, $p_3(\xi_{3,0}) = \bar{x}^6$ *and* $p_n(\xi_{3,0}) = 0$ *when* $n > 3$. *Thus, Wu classes are equal to the Pontrjagin classes reduced modulo* 3*, so;* $q_0(\xi_{3,0}) = 1$, $q_3(\xi_{3,0}) = x^6$ *and* $q_n(\xi_{3,0}) = 0$ *otherwise. As a result, we get*

$$P^n(U) = \begin{cases} U & \text{if } n = 0 \\ U \cup x^6 & \text{if } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

which is the same as for the representations $\xi_{2,1}$, $\xi_{1,2}$ *and* $\xi_{0,3}$.

4.2 Construction of E_2-pages

In this section, we will construct the E_2-pages of the spectral sequences for the cases mentioned previously; but first, we need to fix a basis for $\mathcal{A}_p$.

4.2.1 Basis for $\mathcal{A}_p$

Definition 4.2.1 *A monomial in* $\mathcal{A}_p$ *is called admissible if it is in the form*

$$\beta^{\varepsilon_0} P^{s_1} \beta^{\varepsilon_1} \beta^{\varepsilon_{k-1}} P^{s_1} ... P^{s_k} \beta^{\varepsilon_k}$$

with $\varepsilon_\jmath \in \{-1, 1\}$ *and* s_i *are positive integers, satisfying the inequality*

$$s_i \geq ps_{i+1} + \varepsilon_i$$

whenever $i \in \mathbb{Z}^+$.

Example 4 *The admissible monomials of degree less than* 12 *when* $p = 3$ *are:* β, P^i, βP^i, $P^i\beta$, $\beta P^i\beta$ *for* $i = 1, 2$.

Proposition 4.2.2 (see [6]) *Admissible monomials form a basis for* $\mathcal{A}_p$.

The proof follows from Adem relations (see [7], page 77-78), any monomial which is not admissible can be decomposed into admissible monomials, while admissible monomails cannot.

4.2.2 Construction of free resolutions

We said that we need to construct a projective resolution for $H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$ in order to calculate the E_2 page of the Adams spectral sequence. In fact, we construct a minimal free resolution of $H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$, that is, a free resolution

$$\cdots \to F_2 \to F_1 \to F_0 \to H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$$

where at each step we choose minimum number of free generators for all F_i in each degree.

Lemma 4.2.3 ([3]) *If we have a minimal free resolution*

$$\cdots \to F_2 \to F_1 \to F_0 \to H^*(M\mathbb{Z}/p, \mathbb{Z}/p)$$

then all boundary maps of the complex

$$\cdots \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(\mathrm{F}_2, \mathbb{Z}/\mathrm{p}) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(\mathrm{F}_1, \mathbb{Z}/\mathrm{p}) \leftarrow \mathrm{Hom}_{\mathcal{A}_p}(\mathrm{F}_0, \mathbb{Z}/\mathrm{p}) \leftarrow 0$$

are zero. As a result, we get

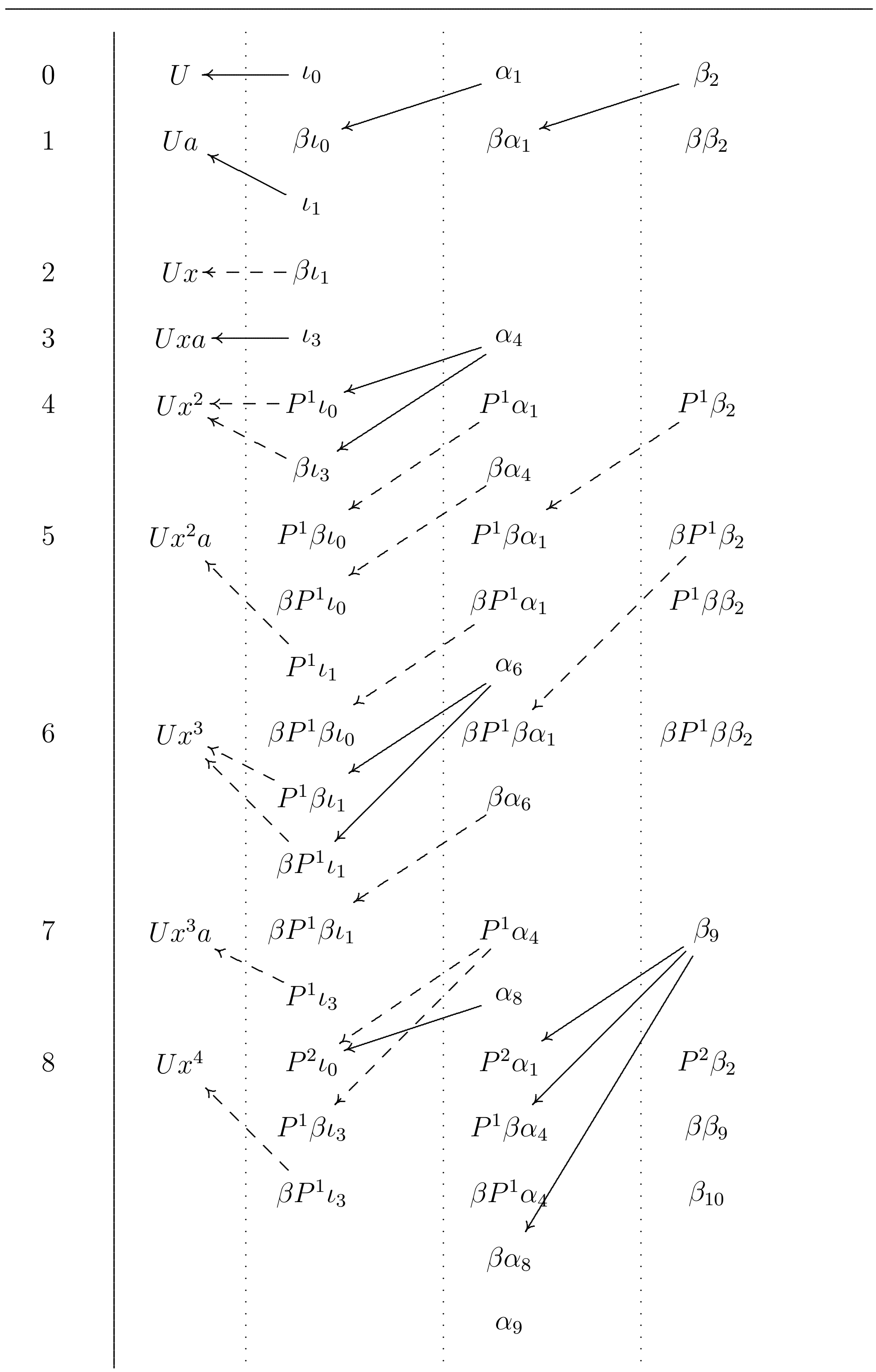
$$E_2^{s,t} = \mathrm{Ext}^{\mathrm{s,t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{H}^*(\mathrm{M}\mathbb{Z}/\mathrm{p};\mathbb{Z}/\mathrm{p}),\mathbb{Z}/\mathrm{p}) = \mathrm{Hom}^{\mathrm{t}}_{\mathcal{A}_\mathrm{p}}(\mathrm{F_s},\mathbb{Z}/\mathrm{p}).$$

Remark 4.2.4 *We construct our resolutions with the notations as in [3], page 24.*

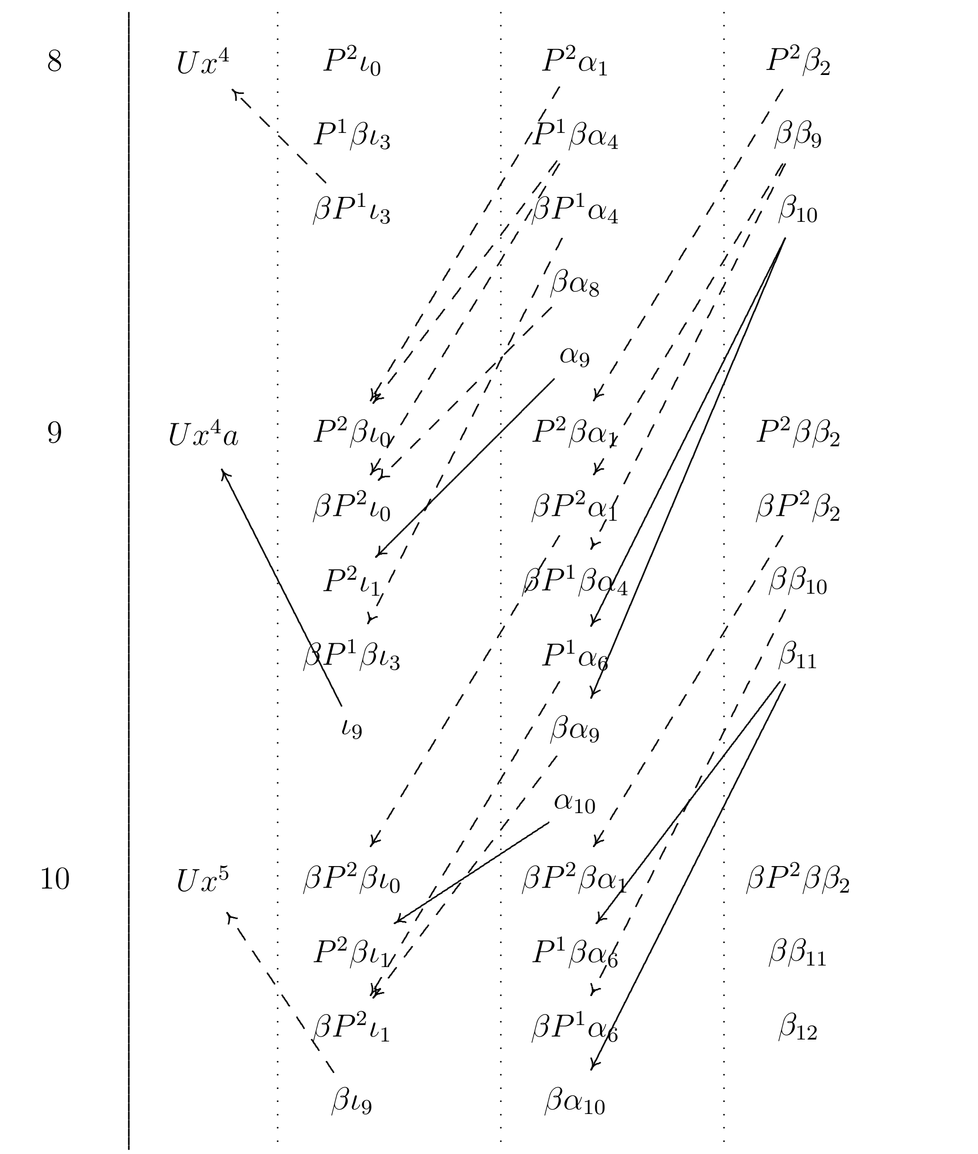
Remark 4.2.5 *We compare p-torsion part of James spectral sequences with Adams Spectral sequences.*

Now, lets start constructing the free resolution for the case 1.

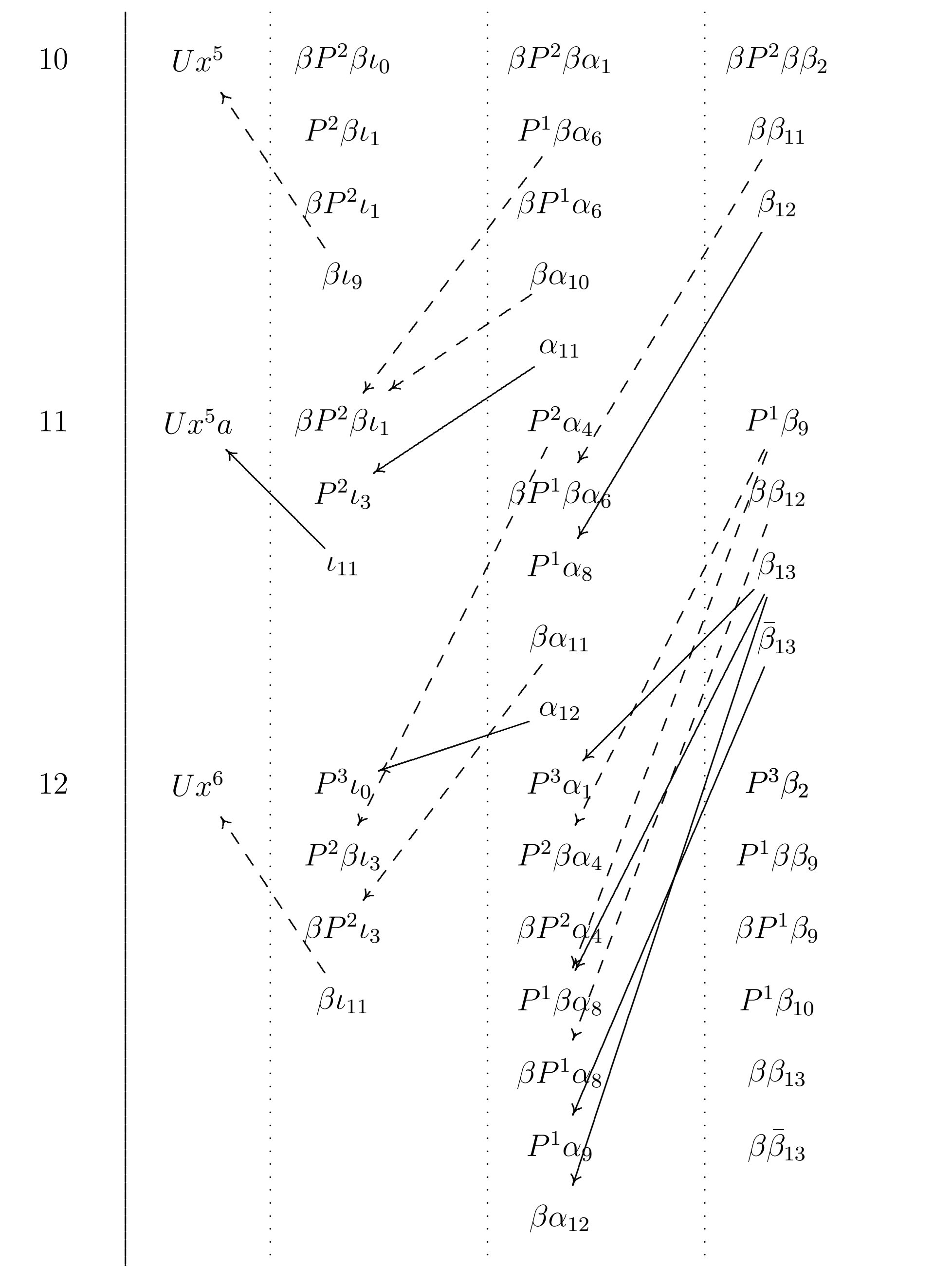
$$H^*(\mathbb{Z}/p, \mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$



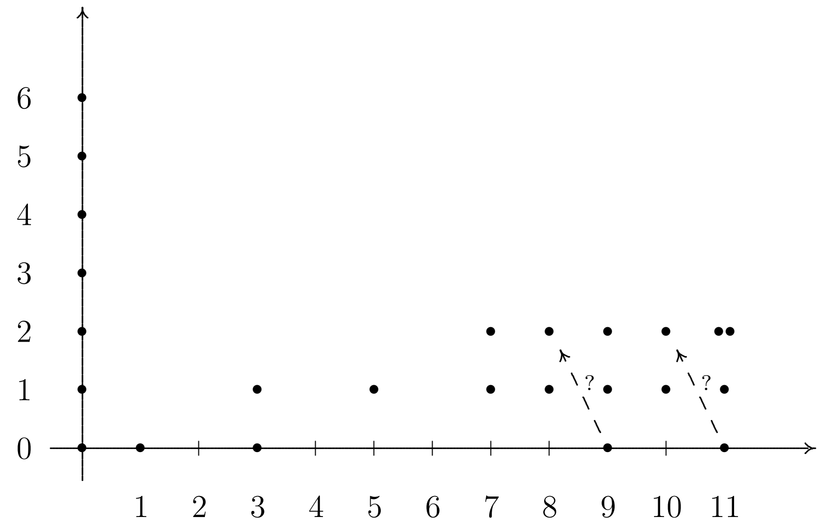
$$H^*(\mathbb{Z}/p, \mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$



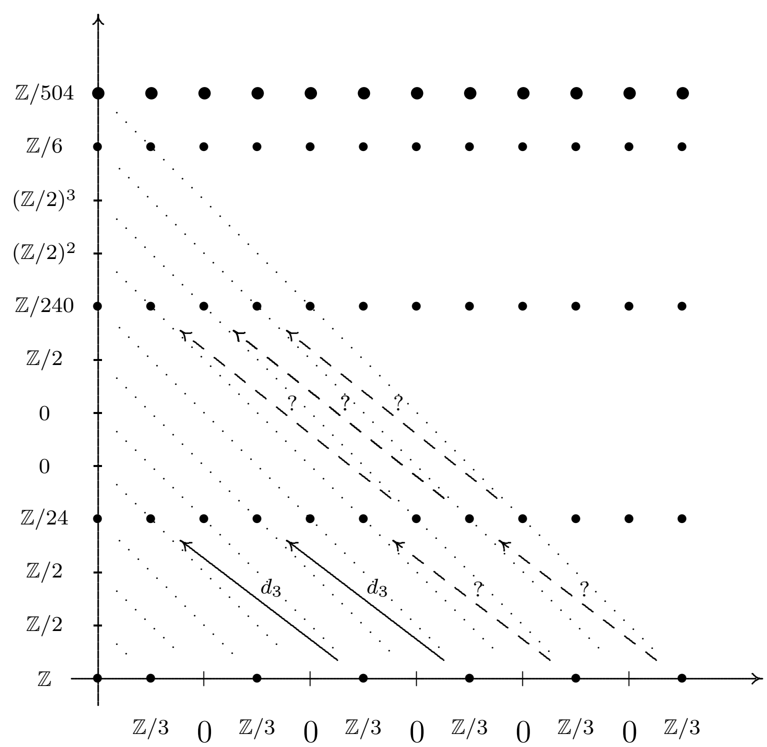
$$H^*(\mathbb{Z}/p,\mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$



We deduce from the above calculations that the E_2 page of the Adams Spectral sequence for case 1 is

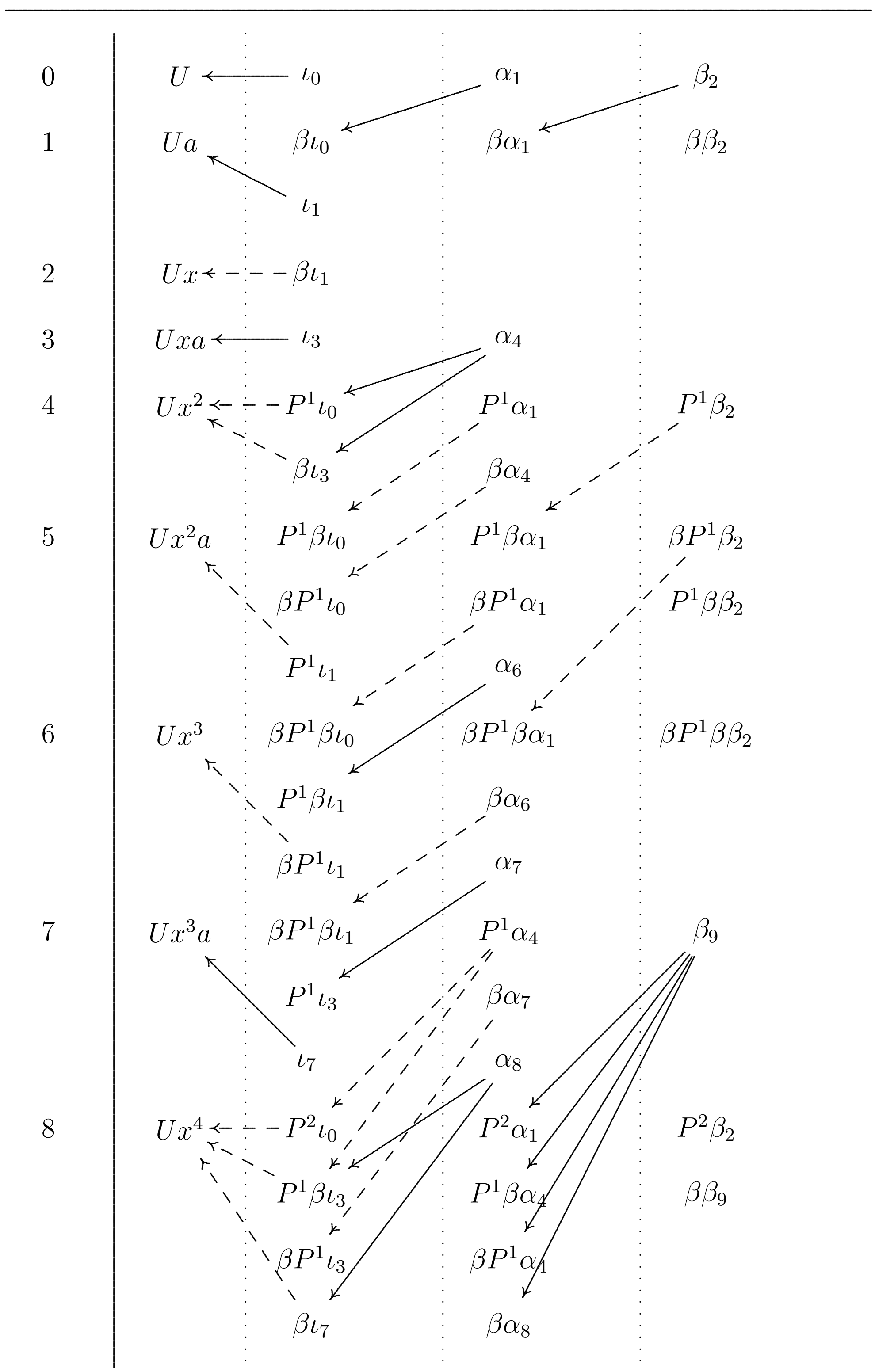


If we look at the James spectral sequence for the fibration $* \longrightarrow B\mathbb{Z}/3 \longrightarrow B\mathbb{Z}/3$ where our stable vector bundle is induced by $\xi_{1,0}$, we get



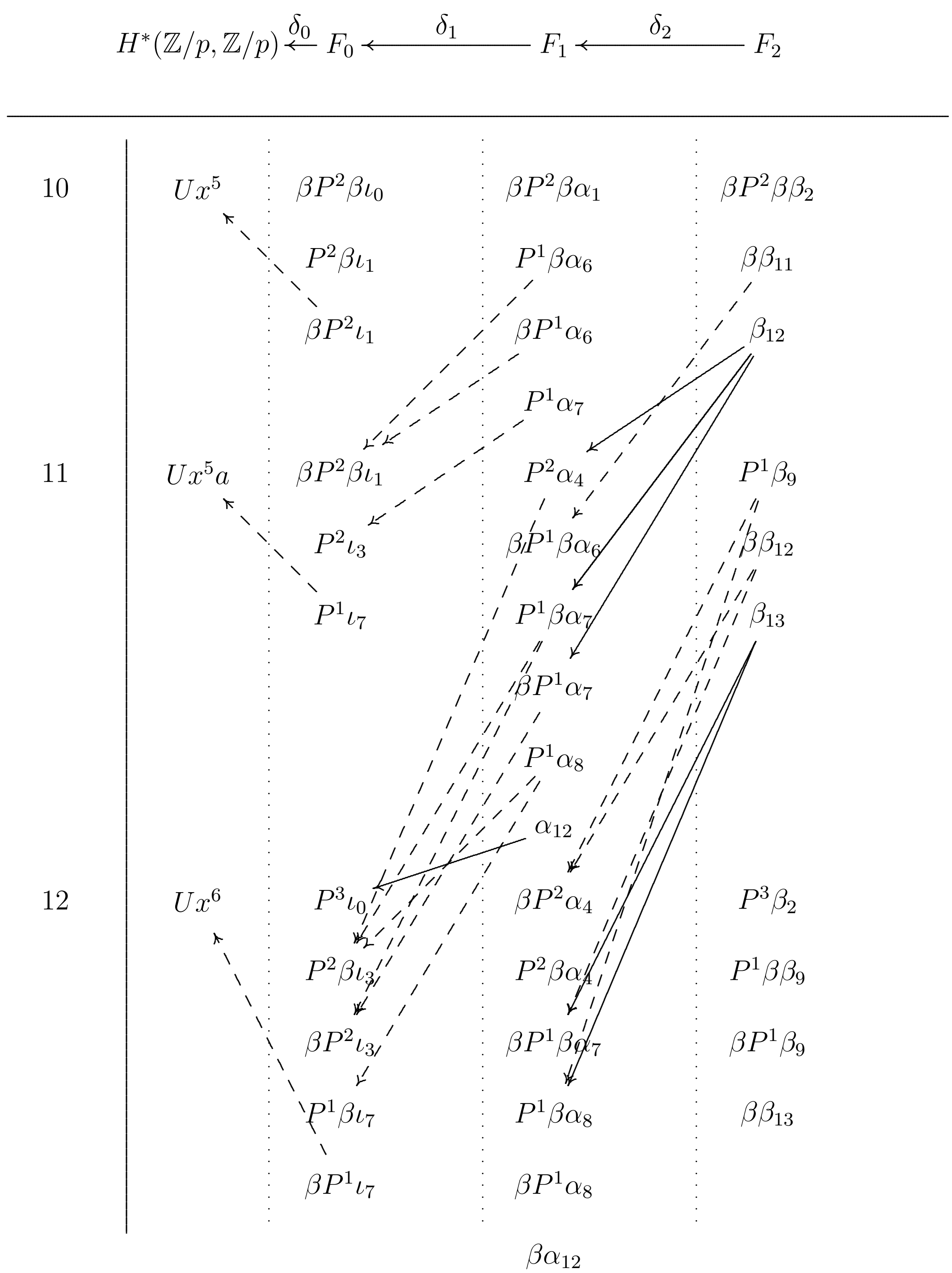
Now, start our computation of free resolution for the case 2.

$$H^*(\mathbb{Z}/p,\mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\quad\delta_1\quad} F_1 \xleftarrow{\quad\delta_2\quad} F_2$$

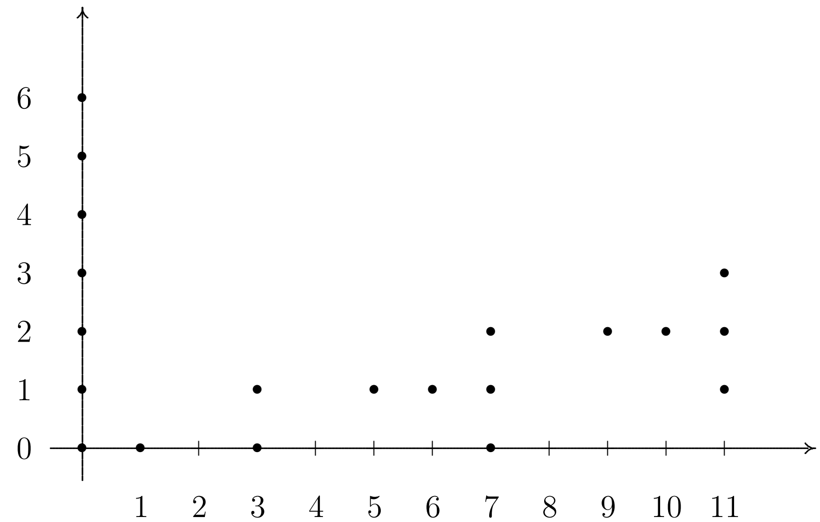


$$H^*(\mathbb{Z}/p, \mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$

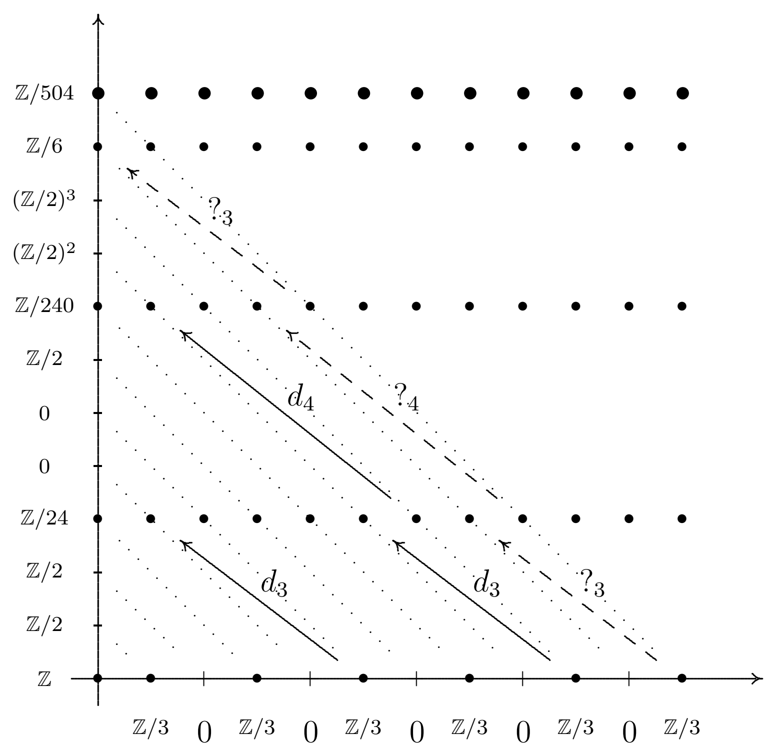
8	Ux^4	$P^2\iota_0$	$P^2\alpha_1$	$P^2\beta_2$
		$P^1\beta\iota_3$	$P^1\beta\alpha_4$	$\beta\beta_9$
		$\beta P^1\iota_3$	$\beta P^1\alpha_4$	
		$\beta\iota_7$	$\beta\alpha_8$	
9	Ux^4a	$P^2\beta\iota_0$	$P^2\beta\alpha_1$	$P^2\beta\beta_2$
		$\beta P^2\iota_0$	$\beta P^2\alpha_1$	$\beta P^2\beta_2$
		$P^2\iota_1$	$\beta P^1\beta\alpha_4$	β_{11}
		$\beta P^1\beta\iota_3$	$P^1\alpha_6$	
10	Ux^5	$\beta P^2\beta\iota_0$	$\beta P^2\beta\alpha_1$	$\beta P^2\beta\beta_2$
		$P^2\beta\iota_1$	$P^1\beta\alpha_6$	$\beta\beta_{11}$
		$\beta P^2\iota_1$	$\beta P^1\alpha_6$	β_{12}
			$P^1\alpha_7$	


$H^*(\mathbb{Z}/p, \mathbb{Z}/p)$
δ_0
F_0
δ_1
F_1
δ_2
F_2
10
Ux^5
$\beta P^2\beta\iota_0$
$\beta P^2\beta\alpha_1$
$\beta P^2\beta\beta_2$
$P^2\beta\iota_1$
$P^1\beta\alpha_6$
$\beta\beta_{11}$
$\beta P^2\iota_1$
$\beta P^1\alpha_6$
β_{12}
$P^1\alpha_7$
11
Ux^5a
$\beta P^2\beta\iota_1$
$P^2\alpha_4$
$P^1\beta_9$
$P^2\iota_3$
$\beta P^1\beta\alpha_6$
$\beta\beta_{12}$
$P^1\iota_7$
$P^1\beta\alpha_7$
β_{13}
$\beta P^1\alpha_7$
$P^1\alpha_8$
α_{12}
12
Ux^6
$P^3\iota_0$
$\beta P^2\alpha_4$
$P^3\beta_2$
$P^2\beta\iota_3$
$P^2\beta\alpha_4$
$P^1\beta\beta_9$
$\beta P^2\iota_3$
$\beta P^1\beta\alpha_7$
$\beta P^1\beta_9$
$P^1\beta\iota_7$
$P^1\beta\alpha_8$
$\beta\beta_{13}$
$\beta P^1\iota_7$
$\beta P^1\alpha_8$
$\beta\alpha_{12}$

From above free resolution, we see that the E_2 page of the Adams Spectral sequence for case 2 is



James spectral sequence for the fibration $* \longrightarrow B\mathbb{Z}/3 \longrightarrow B\mathbb{Z}/3$ where our stable vector bundle is induced by $\xi_{1,1}$ has E_2 page



Free resolution for the final case is like:

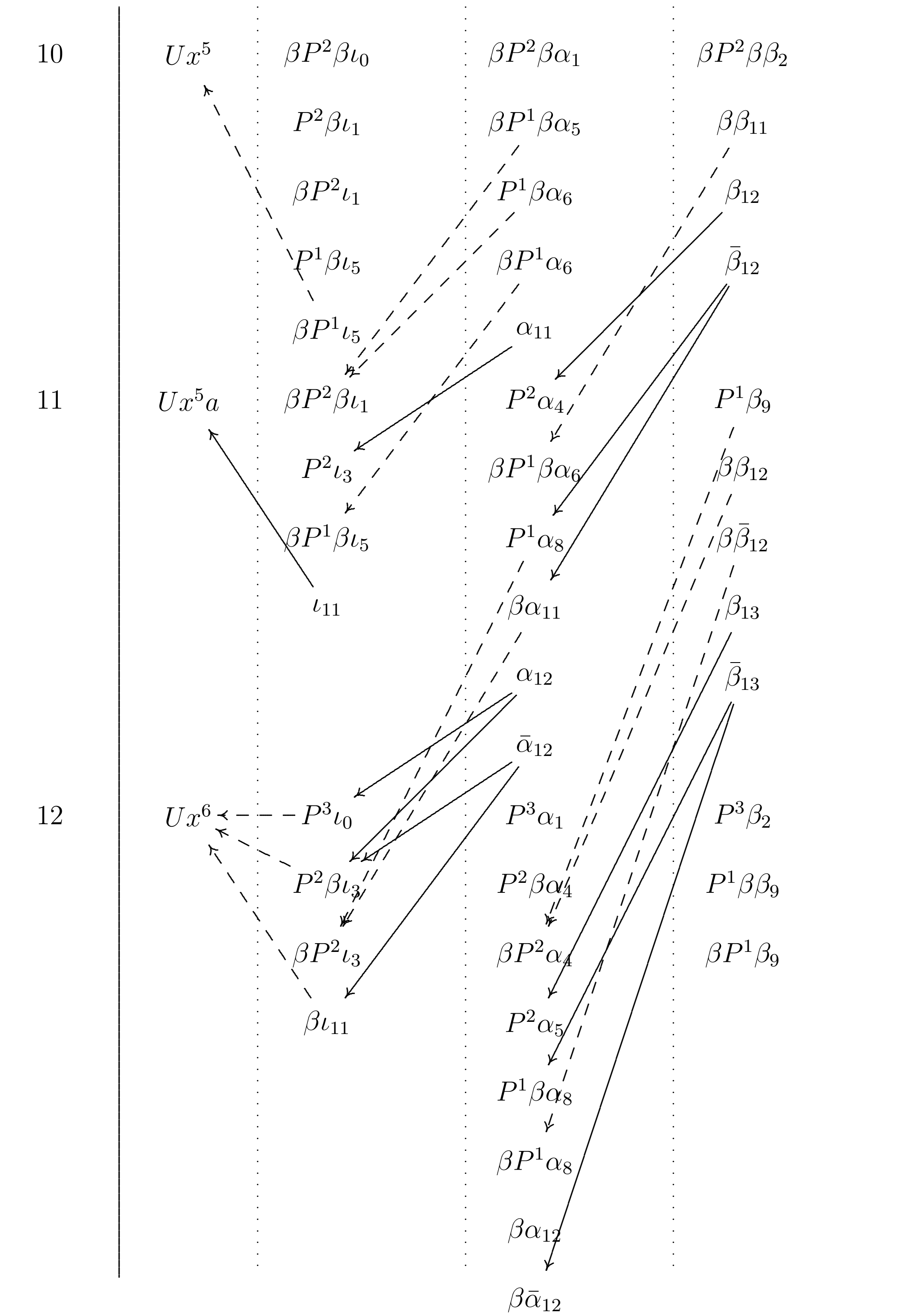
$$H^*(\mathbb{Z}/p,\mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$

0	U	ι_0	α_1	β_2
1	Ua	$\beta\iota_0$	$\beta\alpha_1$	$\beta\beta_2$
		ι_1		
2	Ux	$\beta\iota_1$		
3	Uxa	ι_3	α_4	
4	Ux^2	$P^1\iota_0$	$P^1\alpha_1$	$P^1\beta_2$
		$\beta\iota_3$	$\beta\alpha_4$	
			α_5	
5	Ux^2a	$P^1\beta\iota_0$	$P^1\beta\alpha_1$	$P^1\beta\beta_2$
		$\beta P^1\iota_0$	$\beta P^1\alpha_1$	$\beta P^1\beta_2$
		$P^1\iota_1$	$\beta\alpha_5$	
		ι_5	α_6	
6	Ux^3	$\beta P^1\beta\iota_0$	$\beta P^1\beta\alpha_1$	$\beta P^1\beta\beta_2$
		$P^1\beta\iota_1$	$\beta\alpha_6$	
		$\beta P^1\iota_1$		
		$\beta\iota_5$		
7	Ux^3a	$\beta P^1\beta\iota_1$	$P^1\alpha_4$	β_9
		$P^1\iota_3$	α_8	

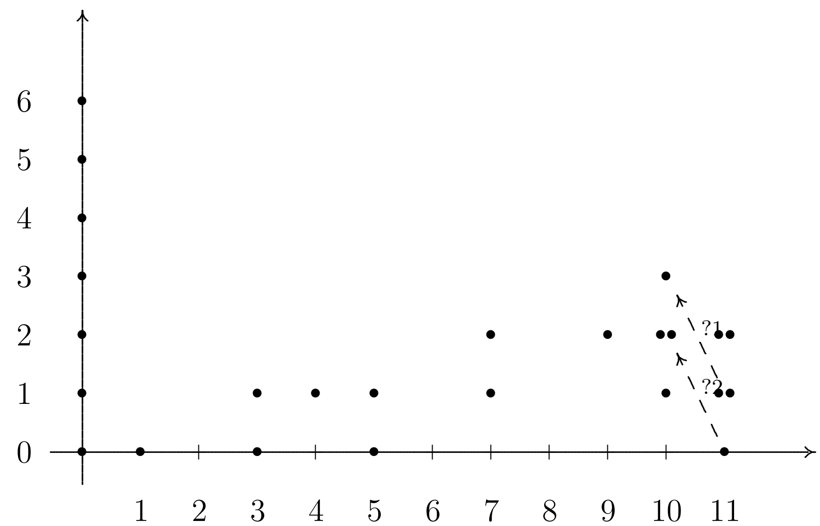
$$H^*(\mathbb{Z}/p,\mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$

7	Ux^3a	$P^1\iota_3$	$P^1\alpha_4$	β_9
		$\beta P^1\beta\iota_1$	α_8	
8	Ux^4	$P^2\iota_0$	$P^2\alpha_1$	$P^2\beta_2$
		$P^1\beta\iota_3$	$P^1\beta\alpha_4$	$\beta\beta_9$
		$\beta P^1\iota_3$	$\beta P^1\alpha_4$	
			$P^1\alpha_5$	
			$\beta\alpha_8$	
9	Ux^4a	$P^2\beta\iota_0$	$P^2\beta\alpha_1$	$P^2\beta\beta_2$
		$\beta P^2\iota_0$	$\beta P^2\alpha_1$	$\beta P^2\beta_2$
		$P^2\iota_1$	$\beta P^1\beta\alpha_4$	β_{11}
		$\beta P^1\beta\iota_3$	$P^1\beta\alpha_5$	
		$P^1\iota_5$	$\beta P^1\alpha_5$	
			$P^1\alpha_6$	
10	Ux^5	$\beta P^2\beta\iota_0$	$\beta P^2\beta\alpha_1$	$\beta P^2\beta\beta_2$
		$P^2\beta\iota_1$	$\beta P^1\beta\alpha_5$	$\beta\beta_{11}$
		$\beta P^2\iota_1$	$P^1\beta\alpha_6$	β_{12}
		$P^1\beta\iota_5$	$\beta P^1\alpha_6$	$\bar{\beta}_{12}$
		$\beta P^1\iota_5$	α_{11}	

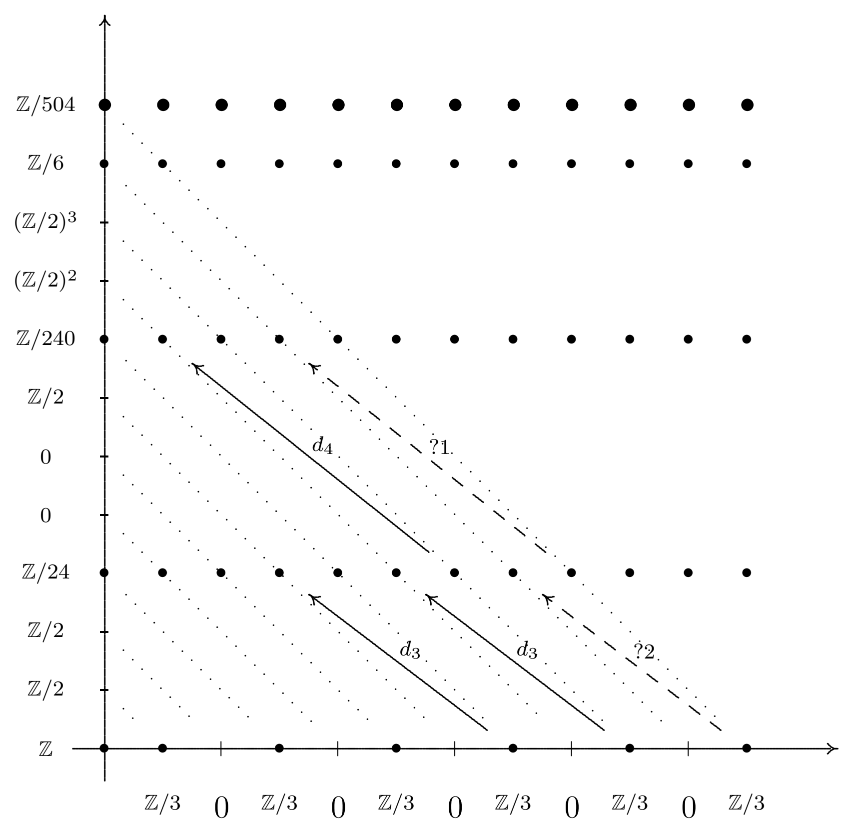
$$H^*(\mathbb{Z}/p, \mathbb{Z}/p) \xleftarrow{\delta_0} F_0 \xleftarrow{\delta_1} F_1 \xleftarrow{\delta_2} F_2$$



For case 3 the E_2 page of the Adams Spectral sequence is



Looking at the E_2 page of the James spectral sequence for the final case.



Chapter 5

Examples and Edge-Homomorphisms

In this Chapter, we will discuss some examples for the representatives of the classes appeared in our computations.

5.1 Lens spaces

Consider the lens space

$$S^{2k-1}/\mathbb{Z}/p = L(p; q_1, q_2, .., q_k),$$

which is the orbit space of the action of $\mathbb{Z}/p$ defined as $\omega \cdot (z_1, .., z_k) = (\omega^{q_1} z_1, .., \omega^{q_k} z_k)$, with $\omega \in \mathbb{Z}/p \subset S^1$ and $S^{2k-1} \subseteq \mathbb{C}^k$ as $S(\mathbb{C}^k)$. Lens spaces are classified by the following theorem:

Theorem 5.1.1 ([11], page 100) *Let $L = L(p : q_1, .., q_k)$ and $\acute{L} = L(p : \acute{q}_1, .., \acute{q}_k)$, there are numbers $a \neq 0(mod - p)$ and $i_1, ..., i_k$ such that $(q_1, .., q_k)$ is a permutation of $((-1)^{i_1} a\acute{q}_1, .., (-1)^{i_1} a\acute{q}_k)$ (mod-p), if and only if L is diffeomorphic to $\acute{L}$.*

From the theorem, we can deduce that when $p = 3$ all lens spaces having same dimension are always diffeomorphic, since such a permutation always exists.

5.2 K-Theory groups of lens spaces when $p = 3$

For details and notations we refer to [12] for this section. Let P^n denote the complex projective space of $\mathbb{C}^{n+1}$, which can be defined as the quotient of the action of S^1 on S^{2n+1} which is defined as follows; if $\zeta \in S^1$ and $(z_1, .., z_{n+1}) \in S^{2n+1}$ then $\zeta \cdot (z_1, .., z_{n+1}) = (\zeta z_1, .., \zeta z_{n+1})$. Let ψ denote the canonical line bundle (see [12], page 6) over P^n, which may be identified with the quotient of $S^{2n+1} \times \mathbb{C}$ by the S^1 action $\zeta \cdot (x, z) = (\zeta x, \zeta^{-1} z)$, so that $\psi^{\otimes 3}$ may be identified with the quotient of $S^{2n+1} \times \mathbb{C}$ by the S^1 action $\zeta \cdot (x, z) = (\zeta x, \zeta^{-3} z)$. It is seen that $S(\psi^{\otimes 3})$ (with the metric on $\psi^{\otimes 3}$ is the usual Euclidean metric in $\mathbb{C}$), is the lens space $L_{3,n} = S^{2n+1}/\mathbb{Z}/3$, where the action is via the third root of unity, by the identification $[(x, z)] = [x, z] \equiv [z^{1/3} x]$. Thus, we have a sphere bundle $\pi' : L_{3,n} \to P_n$. From the Gysin exact sequence belonging the sphere bundle π' (see [12], page 187), we get an exact sequence,

$$0 \to K(P_n) \xrightarrow{\alpha} K(P_n) \xrightarrow{\pi'^*} K(L_{3,n}) \longrightarrow 0$$

where α is multiplication by the Euler class of $\psi^{\otimes 3}$, i.e. $1 - [\psi^{\otimes 3}]$(see [12], page 188), and $K = K_{\mathbb{C}}$ is the complex K-theory. From ([12], Theorem 2.5) we have $K(P_n) \cong \mathbb{Z}[u]/(u^{n+1})$ with $u = 1 - [\psi]$. From the cube of pullback diagrams

$$\begin{tikzcd}
 & S^{2n+1} \times_{\mathbb{Z}/3} \mathbb{C} \arrow[rr] \arrow[dl] \arrow[dd] & & S^{2n+1} \times_{S^1} \mathbb{C} \arrow[dl] \arrow[dd, "\psi"] \\
E\mathbb{Z}/3 \times_{\mathbb{Z}/3} \mathbb{C} \arrow[rr] \arrow[dd] & & ES^1 \times_{S^1} \mathbb{C} \arrow[dd] & \\
 & L_{3,n} \arrow[rr, "\pi'"] \arrow[dl, "c_1"] & & P^n \arrow[dl, "c"] \\
B\mathbb{Z}/3 \arrow[rr, "B\xi_{1,0}"] & & BS^1 &
\end{tikzcd}$$

it is seen that $\pi'^* \circ c \backsimeq B\xi_{1,0} \circ c_1$ where c and c_1 are classifying maps. The actions above is described as follows:

- $\langle\sigma\rangle = \mathbb{Z}/3$ acts on $S^{2n+1} \times \mathbb{C}$ by $\sigma \cdot (x, z) = (\sigma x, \sigma^{-1} z)$, where $\sigma x = (\omega z_1, .., \omega z_{n+1})$, with $\omega = e^{2\pi i/3}$ and $(z_1, .., z_{n+1}) \in S^{2n+1} \subseteq \mathbb{C}^{n+1}$.
- S^1 acts on $S^{2n+1} \times \mathbb{C}$ by $\zeta \cdot (x, z) = (\zeta x, \zeta^{-1} z)$, $\zeta \in S^1$.
- Other actions are as described in Section 2.5.

Consider the identification of $S^{2n+1} \times_{\mathbb{Z}/3} \mathbb{C}$ by $\pi'^*(S^{2n+1} \times_{S^1} \mathbb{C})$ via the map $([x, z'], [x, z]) \to [z'^{1/3}x, z'^{-1/3}z]$ where $([x, z], [x, z']) \in (S^{2n+1} \times_{S^1} \mathbb{C}) \times L_{3,n}$, so that we get a pullback diagram. The other sides of the cube are also pullback diagrams. Consequently, the above cube is cube of pullback diagrams.

By abuse of notation we will denote $[\pi'^* \circ \psi] = 1 - u \in K(L_{3,n})$, then $1-(1-u)^3 = 0$ in $K(L_{3,n})$ (see [12], page 192). As a result, we get

$$K(L_{3,n}) = \mathbb{Z}[u]/(u^{n+1}, 1 - (1 - u)^3).$$

Now, we have

$$u^{n+1} = 0 = 3u - 3u^2 + u^3$$

in $K(L_{3,n})$. Multiplying $3u - 3u^2 + u^3$ by u^{n-1}, we get $3u^n = 0$, as $u^3 = 3u^2 - 3u$. Then $3u^n = 3u^{n-3}u^3 = 3u^{n-3}(3u^2 - 3u) = 9u^{n-1} - 9u^{n-2} = 0$, so multiplying with u we get $9u^{n-1} = 0$ which implies that $9u^{n-2} = 0$ either. If $n \geq 5$, then writing $9u^{n-2} = 9u^{n-5}u^3 = 9u^{n-5}(3u^2 - 3u) = 27u^{n-3} - 27u^{n-4} = 0$, multiplying by u we get $27u^{n-3} = 0$ so that $27u^{n-4} = 0$. Applying same process inductively; we get $3^{n/2}u^2 = 0 = 3^{n/2}u$, when n is even and $3^{(n-1)/2}u^2 = 0 = 3^{(n+1)/2}u$, when n is odd. As a result, we have

$$K(L_{3,n}) = \begin{cases} \mathbb{Z}[u]/(u^{n+1}, 3^k u^2, 3^{k+1}u) & \text{if } n = 2k+1 \\ \mathbb{Z}[u]/(u^{n+1}, 3^k u^2, 3^k u) & \text{if } n = 2k \end{cases}$$

The tangent bundle $\tau(L_{3,n})$, of $L_{3,n}$ fits into the pullback diagram

$$\begin{tikzcd}
\tau(L_{3,n}) \oplus \varepsilon \arrow[r, "\tilde{i}"] \arrow[d] & E\mathbb{Z}/3 \times_{\mathbb{Z}/3} \mathbb{C}^{n+1} \arrow[d] \\
L_{3,n} \arrow[r, "i"] & B\mathbb{Z}/3
\end{tikzcd}$$

where ε is a trivial bundle over $L_{3,n}$. The action of $\langle\sigma\rangle = \mathbb{Z}/3$ over $E\mathbb{Z}/3 \times \mathbb{C}^{n+1}$ is given by $\sigma \cdot (x, z) = (\sigma x, \sigma^{-1} z)$ with $z = (z_1, .., z_k) \in \mathbb{C}^{n+1}$, and $\sigma^{-1} z = \sigma^{-1}(z_1, .., z_k) = (\sigma^{-1} z_1, .., \sigma^{-1} z_{n+1})$, and the map $\tilde{i}$ comes from the map $\tau(S^{2n+1}) \oplus \varepsilon \to E\mathbb{Z}/3 \times \mathbb{C}^{n+1}$ defined by $(x, \gamma'(0), v) \mapsto (x, \overline{\gamma'(0) + xv})$ where γ is a smooth curve on S^{2n+1} with $\gamma(0) = x$. Thus we have $[\tau(L_{3,n})] = (n+1)[\psi]$ in $K(L_{3,n})$. As a result; we have the following information on the K-theory representatives of tangent and normal bundles of $L_{3,n}$, i.e. $[\tau(L_{3,n})]$ and $[\eta(L_{3,n})]$ respectively, for $n = 0, .., 4$ up to stably equivalence ($\cong$).

1. $K(L_{3,0}) = \mathbb{Z}[u]/(u) \cong \mathbb{Z}$, so $\tilde{K}(L_{0,3}) = 0$
   Both tangent and normal bundles are stably equivalent to trivial line bundle, i.e. $[\tau] \cong 0 \cong [\eta]$.

2. $K(L_{3,1}) = \mathbb{Z}[u]/(u^2, 3u) \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/3$, so $\tilde{K}(L_{3,1}) = \mathbb{Z}/3$
   $[\tau] = 2(1-u) = 2 - 2u$ then $[\tau] \cong u$ and $[\eta] \cong 2u$.

3. $K(L_{3,2}) = \mathbb{Z}[u]/(u^3, 3u^2, 3u) \cong \mathbb{Z} \oplus \mathbb{Z}/3$, so $\tilde{K}(L_{3,2}) = \mathbb{Z}/3 \oplus \mathbb{Z}/3$
   $[\tau] = 3(1-u) = 3 - 3u = 3$ then $[\tau] \cong 0 \cong [\eta]$.

4. $K(L_{3,3}) = \mathbb{Z}[u]/(u^4, 3u^2, 9u) \cong \mathbb{Z} \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/3$, so $\tilde{K}(L_{3,3}) = \mathbb{Z}/9 \oplus \mathbb{Z}/3$
   $[\tau] = 4(1-u) = 4 - 4u$ then $[\tau] \cong 5u$ and $[\eta] \cong 4u$.

5. $K(L_{3,4}) = \mathbb{Z}[u]/(u^5, 9u^2, 9u) \cong \mathbb{Z} \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/9$, so $\tilde{K}(L_{3,4}) = \mathbb{Z}/9 \oplus \mathbb{Z}/9$
   $[\tau] = 5(1-u) = 5 - 5u$ then $[\tau] \cong 4u$ and $[\eta] \cong 5u$.

5.3 Examples

We can get information on the manifolds that may appear on the base line of the our James spectral sequences, in consideration of the following theorem:

Theorem 5.3.1 (see [5]) *Let E, B, M, ϑ and f be as in Theorem 3.3.1 and let $[\tilde{\nu} : M \to E] \in \Omega_n(\vartheta) \cong \pi_n^s(M\vartheta)$, i.e. the composition $\vartheta \circ \tilde{\nu}$ classifies the stable normal bundle. Then the edge-homomorphism coming from the base line $\mathfrak{ed} : \Omega_n(\vartheta) \to H_n(B)$ is*

$$\mathfrak{ed}[\tilde{\nu} : M \to E] = f_* \circ \tilde{\nu}_*[M] \in H_n(B)$$

where $[M] \in H_n(M)$ is the fundamental class given by the orientation determined by $\tilde{\nu}$.

Let $\varpi \in \Omega_m(B\xi)$ is a class which appears on the base line of the our James spectral sequence. Our candidate for the manifold that represents ϖ is the lens space $L_{3,n}$.

Looking back to James spectral sequence: Let $\nu : L_{3,n} \to BSO$ classifies the normal bundle of $L_{3,n}$:

- The manifolds at $(1,0)$ is trivially $L_{3,0} = S^1$ for all cases.

For case 1: i.e. 1-dimensional representation

- The map $L_{3,1} \xrightarrow{c_1} B\mathbb{Z}/p$ is a $(B\mathbb{Z}/p, f)$-structure on $L_{3,1}$ which generates the image of the edge homomorphism in $H_3(B\mathbb{Z}/3)$, i.e. $c_{1*}([L_{3,1}])$ generates

$H_3(B\mathbb{Z}/3)$ and we have the diagram

$$\begin{tikzcd}
L_{3,1} \arrow[r, "c_1"] \arrow[dr, "\eta"] \arrow[ddr, "\nu"'] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
& BU(1) \arrow[d, "\imath"] \\
& BSO
\end{tikzcd}$$

where $\vartheta' = B\xi_{1,0}$ and the map $\imath$ is usual inclusion. Say $\vartheta = \imath \circ \vartheta'$. We have $[\eta] \cong 2u \cong 1 - u \cong [\vartheta' \circ c_1]$ in $\tilde{K}_{\mathbb{C}}(L_{3,1})$ which implies $[\nu] \cong [\vartheta \circ c_1]$ in $\tilde{K}_{\mathbb{R}}(L_{3,1})$. As a result, the manifold at $(3, 0)$ is the lens space $L_{3,1}$.

- If there is a manifold at $(9, 0)$, it may or may not be the lens space $L_{3,4}$, since $[\eta] \cong 5u \ncong 1 - u \cong [\vartheta' \circ c_1]$ and $[\eta] \ncong (1 - u)^2 = [\vartheta' \circ B\varphi \circ c_1]$ in $\tilde{K}_{\mathbb{C}}(L_{3,4})$, where $\varphi : \mathbb{Z}/3 \to \mathbb{Z}/3$ is the map defined by $\sigma \mapsto \sigma^2$; $\langle\sigma\rangle = \mathbb{Z}/3$.

For case 2: i.e. 2-dimensional representation

- Let $\vartheta' = B\xi_{1,1}$ and $\tilde{\vartheta} : B\mathbb{Z}/3 \to BU(2)$ be the map induced by $\xi_{0,2}$ on classifying spaces, i.e. $\tilde{\vartheta} = B\xi_{0,2}$. Since we can view $B\mathbb{Z}/p$ as $EU/(\mathbb{Z}/p)$, then there is a map $\rho : B\mathbb{Z}/3 \to B\mathbb{Z}/3$ such that the diagram (composed with $\imath$)

$$\begin{tikzcd}
B\mathbb{Z}/3 \arrow[r, "\rho"] \arrow[dr, "\tilde{\vartheta}"] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
& BU(2)
\end{tikzcd}$$

commutes. Say $\rho \circ c_1 = \widetilde{c_1}$, the map $L_{3,1} \xrightarrow{\widetilde{c_1}} B\mathbb{Z}/p$ is a $(B\mathbb{Z}/p, f)$-structure on $L_{3,1}$ which generates the image of the edge homomorphism in $H_3(B\mathbb{Z}/3)$, i.e. $\widetilde{c_1}_*([L_{3,1}])$ generates $H_3(B\mathbb{Z}/3)$ and from the diagram

$$\begin{tikzcd}
L_{3,n} \arrow[r, "c_1"] \arrow[drr, "\eta"] \arrow[ddrr, "\nu"'] & B\mathbb{Z}/3 \arrow[r, "\rho"] \arrow[dr, "\tilde{\vartheta}"] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
 & & BU(2) \arrow[d, "\imath"] \\
 & & BSO
\end{tikzcd}$$

we have $[\eta] \cong 2u \cong 2(1-u)^2 \cong [\vartheta' \circ \widetilde{c_1}]$ in $\tilde{K}_{\mathbb{C}}(L_{3,1})$ which implies $[\nu] \cong [\vartheta \circ \widetilde{c_1}]$ in $\tilde{K}_{\mathbb{R}}(L_{3,1})$. As a result, the manifold at $(3,0)$ is the lens space $L_{3,1}$,

- The manifold at $(7,0)$ may or may not be the lens space $L_{3,3}$, since we cannot find such $\rho : B\mathbb{Z}/3 \to B\mathbb{Z}/3$ satisfying $[\eta] \cong [\vartheta' \circ \rho \circ c_1]$ in $\tilde{K}_{\mathbb{C}}(L_{3,3})$.

For case 3: i.e. 3-dimensional representation

- Let $\vartheta' = B\xi_{3,0}$ and $\tilde{\vartheta} : B\mathbb{Z}/3 \to BU(3)$ be the map induced by $\xi_{2,1}$ on classifying spaces, i.e. $\tilde{\vartheta} = B\xi_{2,1}$. Then there is a map $\rho : B\mathbb{Z}/3 \to B\mathbb{Z}/3$ such that the diagram(composed with $\imath$)

$$\begin{tikzcd}
B\mathbb{Z}/3 \arrow[r, "\rho"] \arrow[dr, "\tilde{\vartheta}"] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
 & BU(3)
\end{tikzcd}$$

commutes. Say $\rho \circ c_1 = \widetilde{c_1}$, then the map $L_{3,1} \xrightarrow{\widetilde{c_1}} B\mathbb{Z}/p$ is a $(B\mathbb{Z}/p, f)$-structure on $L_{3,1}$ which generates the image of the edge homomorphism in

$H_3(B\mathbb{Z}/3)$, and from the diagram

$$\begin{tikzcd}
L_{3,1} \arrow[r, "c_1"] \arrow[drr, "\eta"] \arrow[ddrr, "\nu"'] & B\mathbb{Z}/3 \arrow[r, "\rho"] \arrow[dr, "\tilde{\vartheta}"] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
 & & BU(3) \arrow[d, "\imath"] \\
 & & BSO
\end{tikzcd}$$

we see $[\eta] \cong 2u \cong 2(1-u) + (1-u)^2 \cong [\vartheta' \circ \widetilde{c_1}]$ in $\tilde{K}_{\mathbb{C}}(L_{3,1})$ which implies $[\nu] \cong [\vartheta \circ \widetilde{c_1}]$ in $\tilde{K}_{\mathbb{R}}(L_{3,1})$. As a result, the manifold at $(3,0)$ is the lens space $L_{3,1}$,

- The map $L_{3,2} \xrightarrow{c_1} B\mathbb{Z}/p$ is a $(B\mathbb{Z}/p, f)$-structure on $L_{3,2}$ which generates the image of the edge homomorphism in $H_5(B\mathbb{Z}/3)$, i.e. $c_{1*}([L_{3,2}])$ generates $H_5(B\mathbb{Z}/3)$ and consider the diagram

$$\begin{tikzcd}
L_{3,2} \arrow[r, "c_1"] \arrow[dr, "\eta"] \arrow[ddr, "\nu"'] & B\mathbb{Z}/3 \arrow[d, "\vartheta'"] \\
 & BU \arrow[d, "\imath"] \\
 & BSO
\end{tikzcd}$$

where $\vartheta' = B\xi_{3,0}$. We have $[\eta] \cong 0 \cong [\vartheta' \circ c_1]$ in $\tilde{K}_{\mathbb{C}}(L_{3,2})$ which implies $[\nu] \cong [\vartheta \circ c_1]$ in $\tilde{K}_{\mathbb{R}}(L_{3,2})$. As a result, the manifold at $(5,0)$ is the lens space $L_{3,2}$,

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