EXTENSION OF SYMMETRIC CONNECTIONS MODEL

A Master's Thesis

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by

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I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

EXTENSION OF SYMMETRIC CONNECTIONS MODEL

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In this thesis, we study an extension of symmetric connections model, which is presented by Matthew O. Jackson and Asher Wolinsky in their paper entitled, "A Strategic Model of Social and Economic Networks". We examined the network formation process through a dynamic framework where self-interested individuals can form links, then we analyze to which network structures this process converges.

Keywords: Symmetric connections model, static model, dynamic model.

ÖZET

SİMETRİK BAĞLANTI MODELİNİN GENİŞLETİLMESİ

Mehpare Şule Taşcıer

Yüksek Lisans, Ekonomi Bölümü

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Eylül, 2010

Bu tezde Matthew O. Jackson ve Asher Wolinsky tarafından, "Sosyal ve Ekonomik Ağların Stratejik bir Modeli" adlı makalelerinde sunulan simetrik bağlantı modelinin genişletilmesini çalıştık. Ağların oluşum sürecini dinamik bir çerçevede inceledik öyleki oyundaki kendi çıkarlarını düşünen bireyler kendi aralarında bağ yapabiliyorlardı. Sonra bu ağ yapılarının nasıl bir yapıya yaklaşacaklarını analiz ettik.

Anahtar Kelimeler: Simetrik Bağlantı Modeli, Durağan Model, Dinamik Model.

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CHAPTER 1

INTRODUCTION

Economic situations where network structures play a significant role is wide and varied. For example, personal contacts play critical roles in obtaining information about job opportunities, political party networks can influence election results, organization of workers within a firm influences the firm's efficiency. Given both the presence of economic situations where network structures play a role and their importance in determining the outcome of the interaction, it is essential to have theories about how such network structures form and why do they matter. In his survey, Jackson emphasizes the importance of network structures through some examples. One example, which is discussed to underlie the reasons that why we may care about network structures, is social connections in labor economics. Jackson underlies that the structure of the social network turns out to be the key determinant of

i) who gets which jobs, which has implications for social mobility,

ii) how patterns of unemployment related to ethnicity, education, geography and other variables and for instance why there might be persistent differences in employment between races,

iii) whether or not jobs are being efficiently filled,

iv) the incentives that individuals have to educate themselves and participate in the workforce.

Since network structures affect economic outcomes, it is crucial to know which network configurations will arise.

As Jackson mentioned in his survey, it is useful to crudely divide situations where networks are important into two distinct categories, to be specific about what the scope of this thesis will be: "In one category, the network structure is a distribution or service network that is the choice of a single actor". "In the other category of situations where networks are critical, the network structure connects different individuals and the formation of the network depends on the decisions of many participants". The model that we analyze in this thesis falls into the second category.

We examine the process of network formation in a dynamic framework, where self interested individuals can form or sever links. We determine which network structures the formation process converges to. In our model, there is a set of players who are initially not connected to each other. Over time, pairs of agents meet and decide whether or not to form or sever links. In every period, two players randomly meet to be updated with uniform probability. Agents can sever a link unilaterally but formation of a link requires the agreement of both players. Agents are myopic and their payoffs are determined through a function which is similar to the one that is presented in Jackson and Wolinskys'symmetric connections model.

The remainder of this thesis proceeds as follows. In chapter two, we will first give definitions related to network structures then we will remind Jackson and Wolinkys' connections model. In the third chapter, we will review Watts' results about dynamic network formation. In chapter four, we will present our extensions of symmetric connections model and analyze conditions for complete graph formation and then we will perform the stability analysis for these extensions.

CHAPTER 2

THE CONNECTIONS MODEL

In this chapter, We start by explaining the Connections Model which is presented by Matthew O. Jackson and Asher Wolinsky in their paper entitled, "A Strategic Model of Social and Economic Networks".

2.1 Definitions

Definition 2.1.1 *Let N be the finite set of players i.e. $\{1, 2, 3, 4, \ldots, N\}$. The network relations among these players are represented by graphs such that the nodes of the graph stand for players and its arcs represent pairwise relationships.*

Definition 2.1.2 *The complete graph, denoted g^N, is the set of all subsets of N with size two. The set of all possible graphs of on N is $\{g|g \subset g^N\}$. Let ij denote subset of N which contains the players i and j. We will refer to this set simply as the link ij. Let $g + ij$ denotes the graph, when the link ij is added to graph g and $g - ij$ denotes the graph where the link ij is deleted from the graph g.*

Definition 2.1.3 *Let $N(g) = \{i|\exists j \text{ s.t. } ij \in g\}$ and $n(g)$ be the cardinality of the set $N(g)$. A path in graph g connecting the players i_1 and i_n is the set of distinct nodes i.e. $\{i_1, i_2, i_3, \ldots, i_n\} \subset N(g)$ such that $\{i_1 i_2, i_3 i_4, \ldots, i_{n-1} i_n\} \subset g$.*

Definition 2.1.4 *The graph $g' \subset g$ is a component of the graph g, if $\forall i \in N(g')$ and $\forall j \in N(g')$ such that $i \neq j$ there exists a path in g' connecting nodes i and j, and $\forall i \in N(g)$ and $\forall j \in N(g')$, $ij \in g$ implies $ij \in g'$.*

Definition 2.1.5 *The value of a graph is represented by a value function, $v : \{g | g \subset g^N\} \mapsto \mathbb{R}$. The set of all such functions is denoted by V.*

Definition 2.1.6 *An allocation rule $Y : \{g | g \subset g^N\} \times V \mapsto \mathbb{R}^N$ identifies how the value generated by a graph is distributed across players. $Y_i(g, v)$ is the payoff of player i from graph g under the value function v.*

Definition 2.1.7 *A graph $g \subset g^N$ is strongly efficient if $v(g) \geq v(g')$, $\forall g' \subset g^N$.*

Definition 2.1.8 *The graph g is pairwise stable with respect to v and Y if*

i) $\forall\ ij \in g$, $Y_i(g, v) \geq Y_i(g - ij, v)$ *and* $Y_j(g, v) \geq Y_j(g - ij, v)$

ii) $\forall\ ij \notin g$, $Y_i(g, v) > Y_i(g + ij, v)$ *then* $Y_j(g, v) < Y_j(g + ij, v)$

2.2 The Connections Model

The connections model is constructed by Jackson and Wolinsky. It models social communication among individuals. Individuals directly communicate with those to whom they are linked. By the help of these direct connections, they also benefit from those to whom their adjacent nodes are linked. However, the value of communication decreases by distance. Moreover, direct communication is costly so that individuals must weigh the benefits of a link against its cost.

Let $w_{ij} \geq 0$ denotes the intrinsic value of player j to player i and c_{ij} denotes the cost of maintaining the link ij to player i. The utility of each player i from graph g is then

$$u_i(g) = w_{ii} + \sum_{j \neq i} \delta^{t_{ij}} w_{ij} - \sum_{j : ij \in g} c_{ij},$$

where t_{ij} is the number of links in the shortest path between players i and j (if there is no path between two players then $t_{ij} = \infty$) and $0 < \delta < 1$ reflects the idea that the value which individual i gets from being connected to individual j is proportional to the proximity of j to i. Here, I should note that,

$$v(g) = \Sigma_{i \in \mathbb{N}} u_i(g).$$

In what follows, we will consider the symmetric version of the connections model that is, $c_{ij} = c$, $\forall ij$ and $w_{ij} = 1$, $\forall j \neq i$ and $w_{ii} = 0$, $\forall i \in \mathbb{N}$.

2.2.1 Stability in the Connections Model without Side Payments

Here, we will remind some implications of stability for connections model without side payments, where the allocation rule is $Y_i(g) = u_i(g)$.

Proposition 2.2.1 *The unique strongly efficient network in the symmetric connections model is*

i) the complete graph g^N *if* $c < \delta - \delta^2$

ii) a star[1] *encompassing everyone if* $\delta - \delta^2 < c < \delta + \frac{(N-2)}{2}\delta^2$

iii) no links if $\delta + \frac{(N-2)}{2}\delta^2 < c$.

Proof:

i) Given that $c < \delta - \delta^2$, any two agents who are not directly connected will improve their utilities and thus the total value by forming a link.

ii) and iii) Consider g' a component of g containing m individuals. Let $k-1 \geq m$ be the number of links in this component. The values of these direct links are

[1] *The term star describes a component in which all players are linked to a one central player and there are no other links:* $g \subset g^N$ *is a star if* $g \neq \emptyset$ *and there exists* $i \in N$ *such that* $jk \in g$*, then either* $j = i$ *or* $k = i$*. Individual* i *is the center of the star.*

$k(2\delta - 2c)$. This leaves at most $\frac{m(m-1)}{2} - k$ indirect links. The value of each indirect link is at most $2\delta^2$. Therefore, the overall value of the component is at most

$$k(2\delta - 2c) + (m(m-1) - 2k))\delta^2 \tag{2.1}$$

If this component is star then its value would be

$$(m-1)(2\delta - 2c) + (m-1)(m-2)\delta^2 \tag{2.2}$$

Notice that, (2.1) − (2.2) = $(k - (m-1))(2\delta - 2c - 2\delta^2)$, which is at most 0 since $k \geq m-1$ and $c > \delta - \delta^2$ and less than 0 if $k > m-1$. The value of this component can equal the value of the star only when $k = m-1$. Any graph with $k = m-1$, which is not a star, must have an indirect connection which has a path longer than two, getting value less than $2\delta^2$. Therefore, the value of the indirect links will be below $(m-1)(m-2)\delta^2$, which is what one get with star. Jackson and Wolinsky have shown that if $c > \delta - \delta^2$, then any component of a strongly efficient graph must be a star. Note that, any component of a strongly efficient graph must have nonnegative value. In that case, a direct calculation using (2.2) shows that a single star of $m+n$ individuals is greater in value than separate stars of m and n individuals. Thus, if the strongly efficient graph is nonempty, it must consist of a single star. Again, it follows from (2.2) that if a star of n individuals has nonnegative value, then a star of $n+1$ individuals has higher value. Finally, to complete *ii)* and *iii)* notice that, a star encompassing everyone has positive value only when $\delta + \frac{N-2}{2}\delta^2 > c$.

Proposition 2.2.2 *In the symmetric connections model with $Y_i(g) = u_i(g)$:*

i) A pairwise stable network has at most one(non empty) component.

ii) For $c < \delta - \delta^2$, the unique pairwise stable network is the complete graph, g^N.

iii) For $\delta - \delta^2 < c < \delta$, a star encompassing all players is pairwise stable, but not necessarily the unique pairwise stable graph.

iv) For $\delta < c$, any pairwise stable network which is nonempty is such that each player has at least two links thus is inefficient.

Proof:

i) Suppose that g is pairwise stable and has two or more non-trivial components. Let u^{ij} denotes the utility which accrues to i from the link ij, given the rest of g: so $u^{ij} = u_i(g+ij) - u_i(g)$ if $ij \notin g$ and $u^{ij} = u_i(g) - u_i(g-ij)$ if $ij \in g$. Consider $ij \in g$, then $u^{ij} \geq 0$. Let kl belong to a different component. Since i is already in a component with j, but k is not, it follows that $u^{kj} > u^{ij} \geq 0$, since k also receives δ^2 in value for the indirect connection to i, which is not included in u^{ij}. For similar reasons, $u^{jk} > u^{lk} \geq 0$. This contradicts with pairwise stability, $jk \notin g$.

ii) It follows from the fact that, in this cost range, any two agents who are not directly connected benefits from forming a link.

iii) It is straightforward to verify that the star is stable. It is the unique stable graph in this cost range if $N = 3$, it is never the unique stable graph if $N = 4$. (If $\delta - \delta^3 < c < \delta$, then a line is also stable.)

iv) In this cost range, pairwise stability precludes loose ends, so that every connected agent has at least two links. This means that the star is not stable and so by the previous proposition, any non empty pairwise stable graph must be inefficient.

CHAPTER 3

A DYNAMIC MODEL OF NETWORK FORMATION

In his paper entitled, "A Dynamic Model of Network Formation" Alison Watts extends Jackson and Wolinskys' connections model to a dynamic framework.

3.1 Model

Jackson and Wolinsky analyze the symmetric connections model in a static framework. They decide which networks are stable and which networks are efficient for this model. However, they leave open the question of which stable networks will form. Watts answers this question in his paper.

3.1.1 Static Model and Results

Watts examines the same symmetric connections model developed by Jackson and Wolinsky(1996). However, Watts adopts a stability notion which is slightly different than the one considered by Jackson and Wolinsky.

Definition 3.1.1 *A graph g is stable if*

i) $\forall\ ij \in g,\ u_i(g) \geq u_i(g - ij)$ *and* $u_j(g) \geq u_j(g - ij)$

ii) $\forall\ ij \notin g$, $u_i(g) > u_i(g + ij - ig - jg)$ *then* $u_j(g) < u_j(g + ij - ig - jg)$*,*

where ig *is defined as follows: if agent* i *is directly linked to only agents* $\{k_1, k_2, k_3, \ldots, k_m\}$ *in graph* g*, then* ig *is any subset of* $\{ik_1, \ldots, ik_m\}$*.*

The results stated in the last chapter concerning stability and efficiency still holds under this version of stability.

3.1.2 Dynamic Model

Initially, there are n players who are unconnected. The players meet over time and then have the chance to form links with each other. Time, T is divided into periods and is defined as a countable and an infinite set, $\{1, 2, 3, \ldots, t, \ldots\}$. Let g_t represents the network that exists at the end of period t and let each player receives the payoff $u_i(g_t)$, which is the same payoff function defined in Jackson and Wolinskys' symmetric connections model;

$$u_i(g_t) = \sum_{j \neq i} \delta^{t_{ij}} - \sum_{j: ij \in g} c.$$

In each period, a link is randomly identified to be updated with uniform probability. A link ij which is identified is denoted by $i : j$. If link ij is already formed that is $ij \in g_{t-1}$ then either player i or j can sever this link. If link $ij \notin g_{t-1}$ then players i and j can bilaterally form this link and simultaneously sever any of their other existing links under the condition that they both agree. Here, each player is myopic, so players' decisions are based on whether or not severing or forming a link will increase their payoff in period t. If after some time period t, no additional links are formed or broken, then the network formation process reaches a stable state.

Proposition 3.1.2 *If* $(\delta - c) > \delta^2 > 0$*, then every link forms (as soon as possible) and remains. If* $(\delta - c) < 0$*, then no links ever form.*

Proof: Assume that, $(\delta - c) > \delta^2 > 0$. Since $\delta < 1$, we know that $(\delta - c) > \delta^2 > \delta^3 > \ldots > \delta^{n-1}$. Thus, each agent prefers a direct link to any indirect link. Each period, two agents, say i and j, meet. If players i and j are not directly connected, then they will each gain at least $(\delta - c) - \delta^{t(ij)} > 0$ from forming a direct link, and so the connection will take place. Using the same reasoning as above, if an agent ever breaks a direct link, his payoff will strictly decrease. Therefore, no direct links are ever broken.

Assume $(\delta - c) < 0$ and that initially no agents are linked. In the first time period, two agents, say i and j, meet and have the opportunity to link. If such a link is formed, then each agent will receive a payoff of $(\delta - c) < 0$, since agents are myopic, they will refuse to link. Thus, no links are formed in the first time period. A similar analysis proves that no links are formed in later periods.

CHAPTER 4

EXTENSION OF SYMMETRIC CONNECTIONS MODEL

We slightly change the symmetric connections model. In symmetric connections model, Jackson and Wolinsky do their analysis with the payoff function:

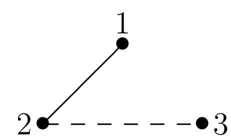
$$u_i(g) = \sum_{j \neq i} \delta^{t_{ij}} - \sum_{j: ij \in g} c.$$

Instead of assigning the same cost c to every direct link, we form a cost function with the convention that when a player has more direct or indirect links, this makes him more valuable for other individuals, so forming a link with him should be more costly than others. Then, we assign weights to these links between two players. We analyze both of these models in a dynamic framework and we impose the condition that once a link is made it cannot be broken. Note that, throughout this thesis, at figures, solid lines denote the links which are already formed, and dash lines denote the links which are being formed.

4.1 Extension of Symmetric Connections Model without Weights and Nonconstant Total Costs

First, consider the case where there are only 2 players i.e. $N = \{1, 2\}$. The link between player 1 and player 2 will be formed if $\delta > c$, where $0 < \delta < 1$ and $c > 0$, since $u_1(g_t), u_2(g_t) > 0$.

Now, consider the case where there are 3 players i.e. $N = \{1, 2, 3\}$. There are 7 distinct graphs which can be formed between three players where there exists at least one link in the graph. We focus on 4 of these graphs, where there are at least two links in the graph. First, consider the situation, where there is a direct link between player 1 and player 2 then player 2 and player 3 meet.



We calculate the benefits of player 2 and player 3, when they do form the link 23. If they form this link, player 2 obtains δ, since this link only provides him a direct connection with player 3. However, player 3 obtains $\delta + \delta^2$ by forming the link 23, since this link provides him an indirect connection with player 1, besides a direct link with player 2. We always assume that the total cost of cultivating a direct link is $2c$. These two players share this total cost proportional to their benefits. Let ${c_{23}}^2$ denotes the cost of link 23 for player 2. Similarly, let ${c_{23}}^3$ denotes the cost of link 23 for player 3. We have assumed that, the total cost of cultivating the link 23 is $2c$, so:

$$
\begin{aligned}
&{c_{23}}^2 + {c_{23}}^3 = 2c \\
\Rightarrow &{c_{23}}^2 + (1+\delta){c_{23}}^2 = 2c \\
\Rightarrow &2{c_{23}}^2 + \delta {c_{23}}^2 = 2c \\
\Rightarrow &{c_{23}}^2 = \frac{2}{2+\delta} c \\
\Rightarrow &{c_{23}}^3 = 2\frac{1+\delta}{2+\delta} c.
\end{aligned}
$$

Therefore, for player 2 to form this link the necessary and sufficient condition is:

$$\frac{2}{2+\delta}c < \delta.$$

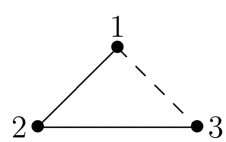
Since $c < \delta$, this condition is obviously satisfied. For player 3 to form this link the necessary and sufficient condition is:

$$2\frac{1+\delta}{2+\delta}c < \delta + \delta^2.$$

This condition holds, since both sides of this inequality are the same multiples of the corresponding parts of the first inequality, needed for player 2 to form the link 23. So, from now on we show one of the conditions needed for two players to form a particular link and the other will directly follow because of the previous argument.

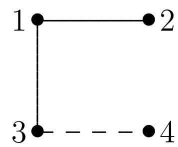
Therefore, we concluded that the condition $\delta > c$ is both necessary and sufficient for the link 23 to be formed.

Now, the links 12 and 23 are formed. Consider the case, where these two links exist then player 1 and player 3 meet.



Player 1 gains δ from the link 13 and similarly player 3's gain is δ. Here, we should remind once more that, we take into account the links in the shortest path when we calculate the payoffs of the players. Because of this assumption, when player 1 and player 3 form this link they both give up from their indirect connections to each other, so they both lose δ^2. Thus, player 1's benefit from link 13 is $\delta - \delta^2$, also player 3's benefit from this link is $\delta - \delta^2$. Since their benefits are the same, they share the total cost $2c$ equally. Therefore, complete graph is formed if and only if $\delta - \delta^2 > c$. The other possible meetings of 3 players exactly require the condition $\delta - \delta^2 > c$ for complete graph to be formed. At this point, a condition which is different than Jackson and Wolinsky's is not required. So, we continue to analyze this game with the player set $N = \{1, 2, 3, 4\}$.

Now, let N={1,2,3,4}. Assume that, there exist links between players 1 and 2, players 1 and 3 then players 3 and 4 meet.



We decide whether link 34 is formed or not. This link is formed if both players' payoffs are greater than 0. Obviously, player 3 gains δ by forming this link and player 4 gains $\delta + \delta^2 + \delta^3$, since, he benefits from the indirect connections which are previously formed by player 3 with player 1 (δ^2) and player 2 (δ^3). Now, we need to decide how total cost $2c$ is divided between these two players. Since, we proportion total cost $2c$ according to benefits, we write

$$\delta k + (\delta + \delta^2 + \delta^3)k = 2c.$$

Thus

$$k = \frac{2c}{2\delta + \delta^2 + \delta^3}.$$

We conclude that, cost of forming link 34 to player 3 is

$$\frac{2c}{2 + \delta + \delta^2}$$

and cost of forming link 34 to player 4 is

$$\frac{2c}{\delta(2 + \delta + \delta^2)}(\delta(1 + \delta + \delta^2)) = \frac{2c(1 + \delta + \delta^2)}{2 + \delta + \delta^2}.$$

Now, we can set necessary and sufficient conditions for formation of link 34, using previous calculations which informs us about benefits and costs of these two players. Since

$$\delta > c \quad \text{and} \quad \frac{2}{2 + \delta + \delta^2} < 1$$

then

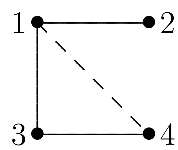
$$\frac{2c}{2 + \delta + \delta^2} < \delta.$$

So, player 3 desires to form this link. This last condition is also necessary and sufficient for player 4 to form link 34. We can see this by multiplying the both sides of this inequality by $(1 + \delta + \delta^2)$. This gives

$$\delta + \delta^2 + \delta^3 > \frac{2c(1 + \delta + \delta^2)}{2 + \delta + \delta^2},$$

which states that player 4's net benefit is greater than 0, so he will desire to form link 34.

Now, we proceed by assuming that link 34 is added to the graph and players 1 and 4 meet.



Obviously, player 1 gains δ from this direct connection with player 4, but nothing else, since we take into account the shortest path between two players. Thus player 4's direct connection with player 3 is worthless for player 1. However, as it can be seen from the graph, player 1 has an indirect connection with player 4 and if he forms link 14, he looses the benefit which is provided by that indirect connection. Thus, player 1's benefit from link 14, should be noted as $\delta - \delta^2$. Player 4 gains δ from the direct connection with player 1. Moreover, if link 14 is formed, number of links between player 4 and player 2 decreases from 3 to 2. Thus, player 4 gains δ^2, when he looses δ^3. Also, player 4 looses his indirect connection with player 1, i.e. $-\delta^2$. So, player 4's benefit is

$$-\delta^2 - \delta^3 + \delta + \delta^2 = \delta - \delta^3.$$

As we did previously, we need to find the costs to players of forming link 14. We know that

$$(\delta - \delta^2)k + (\delta - \delta^3)k = 2c$$

then

$$k = \frac{2c}{2\delta - \delta^2 - \delta^3}.$$

We conclude that, cost of forming link 14 to player 1 is

$$\frac{2c(1-\delta)}{2-\delta-\delta^2}$$

and cost of forming link 14 to player 4 is

$$\frac{2c(1-\delta^2)}{2-\delta-\delta^2}.$$

The necessary and sufficient condition for player 1 to form this link is

$$\frac{2c(1-\delta)}{2-\delta-\delta^2} < \delta-\delta^2.$$

This inequality can be written as:

$$\frac{2c(1-\delta)}{2-\delta-\delta^2} < \delta(1-\delta) \Rightarrow \frac{2c}{2-\delta-\delta^2} < \delta.$$

Since, $\frac{2}{2-\delta-\delta^2} < 1$, $0 < \delta < 1$ and $\delta > c$[1] we get

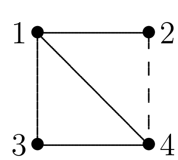
$$\frac{2c(1-\delta)}{2-\delta-\delta^2} < \delta(1-\delta).$$

So, player 1 desires to form link 14. The necessary and sufficient condition for player 4 to form link 14,

$$\frac{2c(1-\delta^2)}{2-\delta-\delta^2} < \delta-\delta^3$$

holds, since both sides of this inequality are the same multiples of the corresponding parts of the first inequality, needed for player 1 to form the link 14. So, player 4 also desires to form this link.

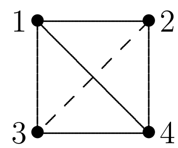
Now, links 12, 13, 34 and 14 are formed. Assume that, players 2 and 4 meet.



If these two players form this link then player 2's and player 4's payoffs will be $\delta-\delta^2$, so they share the total cost equally. The necessary and sufficient condition

[1]we assumed this condition, since it is required for complete graph to be formed.

for link 24 to be formed is $\delta - \delta^2 > c$. This condition is the same with the condition which is sufficient for complete graph to be formed in the symmetric connections model. Suppose that, this last condition is satisfied and link 24 is formed. For complete graph to be formed, we should consider the case where players 2 and 3 meet.

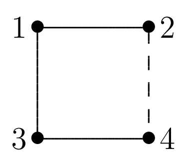


Each of these two players gains $\delta - 2\delta^2$ from link 23, since both loose their indirect connections with each other by forming this direct connection. Since their benefits are the same, they share total cost $2c$ equally. Thus, $\delta - 2\delta^2 > c$ is both necessary and sufficient for link 23 to be formed. Note that, this last condition is a different condition than those stated by Jackson and Wolinsky in symmetric connections model, which are sufficient for complete graph to be reached.

We have reached this new condition as a result of a particular order in which four players meet. If we change this ordering, we may end up with other new conditions. So, we should do a similar analysis by changing the order in which four players meet. I will briefly summarize our findings.

Let $N = \{1, 2, 3, 4\}$. Suppose that, there exist links between players 1 and 2, players 1 and 3, players 3 and 4 then players 2 and 4 meet.

Step 1:



Links which are already formed: link 12, link 13, link 34

Link which can be formed: link 24

Benefit of player 2 form link 24: $\delta - \delta^3$

Benefit of player 4 from link 24: $\delta - \delta^3$

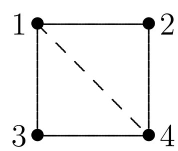
Cost of the link 24 to player 2: c

Cost of the link 24 to player 4: c

Necessary and sufficient condition for link 24 to be formed by player 2: $\delta - \delta^3 > c$. This condition holds, since $\delta - \delta^3 > \delta - 2\delta^2$.[2]

Necessary and sufficient condition for link 24 to be formed by player 4: $\delta - \delta^3 > c$. This condition holds because of the same reason for player 2.

Step 2:



Links which are already formed: 12, 13, 24, 34

Link which can be formed: link 14

Benefit of player 1 form link 14: $\delta - 2\delta^2$

Benefit of player 4 from link 14: $\delta - 2\delta^2$

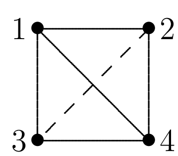
Cost of the link 14 to player 1: c

Cost of the link 14 to player 4: c

Necessary and sufficient condition for link 14 to be formed by player 1: $\delta - 2\delta^2 > c$.

Necessary and sufficient condition for link 14 to be formed by player 4: $\delta - 2\delta^2 > c$.

Step 3:



[2] we assumed this condition, since we have seen from previous results that, it is required for complete graph to be formed.

Links which are already formed: 12, 13, 24, 34, 14.

Link which can be formed: link 23

Benefit of player 2 form link 23: $\delta - 2\delta^2$

Benefit of player 3 from link 23: $\delta - 2\delta^2$

Cost of the link 23 to player 2: c

Cost of the link 23 to player 3: c

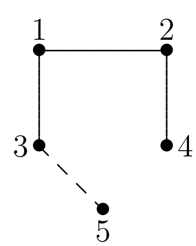
Necessary and sufficient condition for link 23 to be formed by player 2: $\delta - 2\delta^2 > c$.

Necessary and sufficient condition for link 23 to be formed by player 3: $\delta - 2\delta^2 > c$.

We continued to analyze different orders in which four players meet, then we performed this analysis for five players, also for six players. We observed that, $\delta - (n-2)\delta^2 > c$ is both necessary and sufficient condition for complete graph to be formed. We will prove this at the end of this section.

Now, let N={1,2,3,4,5}.

Step 1:



Links which are already formed: link 12, link 13, link 24.

Link which can be formed: link 35

Benefit of player 3 form link 35: δ

Benefit of player 5 from link 35: $\delta + \delta^2 + \delta^3 + \delta^4$

Cost of the link 35 to player 3:

$$\frac{2c}{2+\delta+\delta^2+\delta^3}$$

Cost of the link 35 to player 5:

$$\frac{2c(1+\delta+\delta^2+\delta^3)}{2+\delta+\delta^2+\delta^3}$$

Necessary and sufficient condition for link 35 to be formed by player 3:

$$\frac{2c}{2+\delta+\delta^2+\delta^3} < \delta$$

This condition holds, since

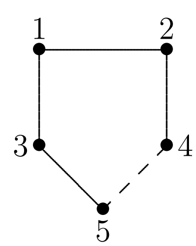
$$\frac{2c}{2+\delta+\delta^2+\delta^3} < c < \delta.$$

Necessary and sufficient condition for link 35 to be formed by player 5:

$$\frac{2c(1+\delta+\delta^2+\delta^3)}{2+\delta+\delta^2+\delta^3} < \delta+\delta^2+\delta^3+\delta^4$$

This condition follows from the previous condition which is necessary and sufficient for player 3 to form the link 35.

Step 2:



Links which are already formed: link 12, link 13, link 24, link 35.

Link which can be formed: link 45

Benefit of player 4 form link 45: $\delta - \delta^4$

Benefit of player 5 from link 45: $\delta - \delta^4$

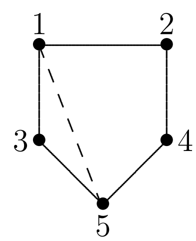
Cost of the link 45 to player 4: c

Cost of the link 45 to player 5: c

Necessary and sufficient condition for link 45 to be formed by player 4: $\delta - \delta^4 > c$. This condition holds, since $\delta - \delta^4 > \delta - \delta^2 > c$.[3]

Necessary and sufficient condition for link 45 to be formed by player 5: $\delta - \delta^4 > c$. This condition holds as we have shown at the last part.

Step 3:



Links which are already formed: link 12, link 13, link 24, link 35, link 45.

Link which can be formed: link 15

Benefit of player 1 form link 15: $\delta - \delta^2$

Benefit of player 5 from link 15: $\delta - \delta^2$

Cost of the link 15 to player 1: c

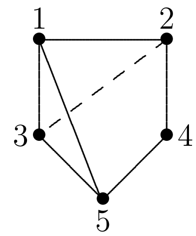
Cost of the link 15 to player 5: c

Necessary and sufficient condition for link 15 to be formed by player 1: $\delta - \delta^2 > c$. This condition holds, since we have already assumed that $\delta - \delta^2 > c$, for complete graph to be formed in previous cases.

Necessary and sufficient condition for link 15 to be formed by player 5: $\delta - \delta^2 > c$. This condition holds as we have shown at the last part.

[3]we assumed $\delta - \delta^2 > c$, since we have seen from previous results that, it is required for complete graph to be formed.

Step 4:



Links which are already formed: link 12, link 13, link 24, link 35, link 45, link 15.

Link which can be formed: link 23

Benefit of player 2 form link 23: $\delta - \delta^2$

Benefit of player 3 from link 23: $\delta - \delta^2$

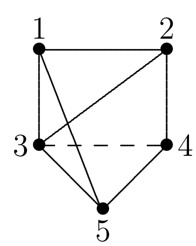
Cost of the link 23 to player 2: c

Cost of the link 23 to player 3: c

Necessary and sufficient condition for link 23 to be formed by player 2: $\delta - \delta^2 > c$. This condition holds, since we have already assumed that $\delta - \delta^2 > c$, for complete graph to be formed in previous cases.

Necessary and sufficient condition for link 23 to be formed by player 3: $\delta - \delta^2 > c$. This condition holds as we have shown at the last part.

Step 5:



Links which are already formed: link 12, link 13, link 24, link 35, link 45, link 15, link 23.

Link which can be formed: link 34

Benefit of player 3 form link 34: $\delta - 2\delta^2$

Benefit of player 4 from link 34: $\delta - 2\delta^2$

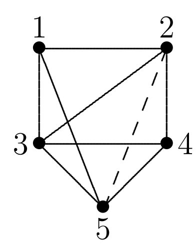
Cost of the link 34 to player 3: c

Cost of the link 34 to player 4: c

Necessary and sufficient condition for link 34 to be formed by player 3: $\delta - 2\delta^2 > c$. This condition holds, since we have already assumed that $\delta - 2\delta^2 > c$, for complete graph to be formed in previous cases.

Necessary and sufficient condition for link 34 to be formed by player 4: $\delta - 2\delta^2 > c$. This condition holds as we have shown at the last part.

Step 6:



Links which are already formed: link 12, link 13, link 24, link 35, link 45, link 15, link 23, link 34.

Link which can be formed: link 25

Benefit of player 2 form link 25: $\delta - 3\delta^2$

Benefit of player 5 from link 25: $\delta - 3\delta^2$

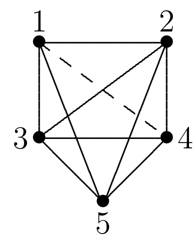
Cost of the link 25 to player 2: c

Cost of the link 25 to player 5: c

Necessary and Sufficient condition for link 25 to be formed by player 2: $\delta - 3\delta^2 > c$.

Necessary and sufficient condition for link 25 to be formed by player 5: $\delta - 3\delta^2 > c$.

Step 7:



Links which are already formed: link 12, link 13, link 24, link 35, link 45, link 15, link 23, link 34, link 25.

Link which can be formed: link 14

Benefit of player 1 form link 14: $\delta - 3\delta^2$

Benefit of player 4 from link 14: $\delta - 3\delta^2$

Cost of the link 14 to player 1: c

Cost of the link 14 to player 4: c

Necessary and Sufficient condition for link 14 to be formed by player 1: $\delta - 3\delta^2 > c$.

Necessary and sufficient condition for link 14 to be formed by player 4: $\delta - 3\delta^2 > c$.

These two last conditions are different conditions than those we have found which are necessary and sufficient for complete graph to be reached, so we need to impose $\delta - 3\delta^2$ as a new condition for complete graph to be formed. As we have mentioned we observed that, $\delta - (n-2)\delta^2 > c$ is both necessary and sufficient condition for complete graph to be formed for n players. Now, we will prove this observation.

Theorem 4.1.1 $\forall n \in \mathbb{N}$, *complete graph forms if and only if* $\forall i, j \in \{1, \ldots, n\}$ *the following condition is satisfied*

$$\delta - (n-2)\delta^2 > c. \qquad (4.1)$$

Proof: First of all let us prove that 4.1 is a sufficient condition for complete graph to be formed. Let t_0 be any time and fixed. Let at time t_0, player i and player j meet. Let there are $k \geq 0$ players who have formed paths with the shortest length between players i and j. Let there are $m \geq 0$ players who are connected to player i and connected to player j via player i and those k players. Let there are $p \geq 0$ players who are connected to player j and connected to player i via player j and those k players. Note that, $k + m + p \leq n - 2$. Let B_i denotes the benefit of player i from link ij, B_j denotes the benefit of player j from link ij, C_i denotes the cost of player i from link ij, C_j denotes the cost of player j from link ij.

Note that, when $m = p = 0$, $k = n - 2$, where player i and player j form links via k players, $\{k_1, k_2, \ldots, k_k\}$, with length 2, we have

$$B_i = B_j = \delta - \delta^2(n-2),$$
$$C_i = C_j = c.$$

Note that, the condition

$$\delta - \delta^2(n-2) > c \tag{4.2}$$

satisfies $B_i > C_i$ and $B_j > C_j$.

Hence, in this case the link between player i and player j forms.

Moreover, when $k = 0$, $m \geq 0$, $p \geq 0$, where player i has links with m players with length 1 and player j has links with p players with length 1, we have

$$B_i = \delta + \delta^2 p$$
$$B_j = \delta + \delta^2 m$$
$$C_i = \frac{2c(\delta + \delta^2 p)}{2\delta + \delta^2(m+p)}$$
$$C_j = \frac{2c(\delta + \delta^2 m)}{2\delta + \delta^2(m+p)}$$

Note that, the condition

$$\delta > c \tag{4.3}$$

satisfies $B_i > C_i$ and $B_j > C_j$. Hence, in this case the link between player i and player j forms.

Furthermore, note that, for any combination of m, k, p, with arbitrary path lengths, the sufficient condition for forming link ij is restricted by condition 4.2 and condition 4.3. Therefore, the condition 4.1 is sufficient for complete graph to be formed by n players.

Now, let us prove that 4.1 is a necessary condition for complete graph to be formed. We use induction method to prove this part. Note that, we have shown that for three players case, if complete graph is formed then $\forall i, j \in \{1, 2, 3\}$, we have

$$\delta - \delta^2 > c.$$

Now assume that for $n-1$ players, if complete graph is formed, then we have $\forall i, j \in \{1, 2, \ldots, n-1\}$,

$$\delta - \delta^2(n-1) > c.$$

Now, let us add player n to the game. Let without loss of generality, first of all, player 1 and player n form a link, then

$$\begin{aligned}
B_1 &= \delta, \\
B_n &= \delta + \delta^2(n-2), \\
C_1 &= \frac{2c}{2 + \delta(n-2)}, \\
C_n &= \frac{2c(1 + \delta(n-2))}{2 + \delta(n-2)}.
\end{aligned}$$

Note that, $B_1 > C_1$, $B_n > C_n$ holds since the condition $\delta > c$ is satisfied, due to induction assumption. Now let player 2 and player n form a link, then

$$\begin{aligned}
B_2 &= B_n = \delta - \delta^2, \\
C_2 &= C_n = c.
\end{aligned}$$

Hence, $B_2 = B_n > C_2 = C_n$ holds since the condition $\delta - \delta^2 > c$ is satisfied, as a result of induction assumption.

Similarly, let player n forms links with player 1,. . ., player $k-1$, then the condition $\delta - \delta^2(k-1) > c$ is necessary for player k and player n to form a link. Thus, if complete graph forms, then

$$\delta - \delta^2(n-2) > c.$$

holds.

Corollary 4.1.2 *If $\delta - (n-2)\delta^2 > c$ then complete graph is stable.*

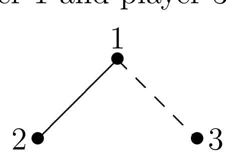
4.2 Extension of Symmetric Connections Model with Weights and Nonconstant Cost

In section 4.1, we have shown that, for n players, complete graph forms if and only if $\delta - (n-2)\delta^2 > c$. Here, one should notice that for this condition to hold $\frac{1}{n-2} > \delta$ is required. So, $\forall n \in \mathbb{N}$, we can not have complete graph.

We desire to analyze the complete graph formation for all $n \in \mathbb{N}$. In this section, we assign a weight to each link that can be formed between any two players. We denote the weight which is assigned to the link between players x and y with $f(x,y)$ and we assume that $f(x,y) = f(y,x)$ and $f(x,y) \in [0,1]$. Moreover, we denote the total cost of the link between players x and y with $2c(x,y)$ and we assume that $c(x,y) = c(y,x)$ with $\delta > c(x,y) > 0$. Our other assumptions are the same as the ones in section 4.1. However, from now on, the order in which players meet is important for us, since they are not identical anymore. We perform the same analysis in the previous section.

We first analyze the two players case i.e. $N = \{1,2\}$. We will assume that $\delta > c(1,2)$. Suppose that, player 1 and player 2 meet. Obviously, they form a link if $f(1,2)\delta > c(1,2)$, i.e. $f(1,2) > \frac{c(1,2)}{\delta}$.

Now, we will consider the case where there are 3 players, $N = \{1,2,3\}$. Suppose that, link 12 is formed and player 1 and player 3 meet.



Since link 12 has already been formed, we have $f(1,2) > \frac{c(1,2)}{\delta}$. We again start by calculating the benefits of players and then costs to players to observe the conditions which are necessary and sufficient for link 13 to be formed. The benefit of player 1 from this link is

$$\delta f(1,3).$$

The benefit of player 3 from this link is

$$\delta f(1,3) + \delta^2 f(1,2) f(1,3) = \delta f(1,3)[1 + \delta f(1,2)]$$

since player 3 gains an indirect connection with player 2 by forming a direct connection with player 1. The intuition behind multiplying the second term by $f(1,2)$ is that player 3 benefits from player 2 through player 1. The cost to players of forming link 13 is again proportional to their benefits and total cost of cultivating this link is $2c(1,3)$. Thus

$$[\delta f(1,3)]k + \delta f(1,3)[1 + \delta f(1,2)]k = 2c(1,3)$$

then

$$k = \frac{2c(1,3)}{\delta f(1,3)[2 + \delta f(1,2)]}.$$

So, cost to player 1 of forming link 13 is

$$\frac{2c(1,3)}{2 + \delta f(1,2)}$$

and cost to player 3 of link 13 is

$$\frac{2c(1,3)[1 + \delta f(1,2)]}{2 + \delta f(1,2)}.$$

We conclude that, player 1 desires to form link 13 if and only if

$$\delta f(1,3) > \frac{2c(1,3)}{2 + \delta f(1,2)}$$

and player 3 desires to form link 13 if and only if

$$\delta f(1,3)[1 + \delta f(1,2)] > \frac{2c(1,3)[1 + \delta f(1,2)]}{2 + \delta f(1,2)}.$$

Actually, if we divide both sides of the last inequality by $[1 + \delta f(1,2)]$, we end up with the same condition for player 1 to form link 13, that is

$$\delta f(1,3) > \frac{2c(1,3)}{2 + \delta f(1,2)}.$$

Therefore,

$$f(1,3) > \frac{2c(1,3)}{\delta(2 + \delta f(1,2))}$$

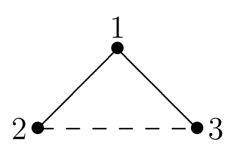
is both necessary and sufficient condition for this link to be formed. This condition holds if and only if

$$f(1,3) > \frac{c(1,3)}{\delta},$$

since

$$\frac{2}{2 + \delta f(1,2)} < 1.$$

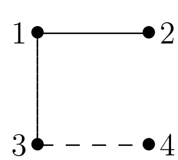
Now, suppose that links 12 and 13 are formed then player 2 and player 3 meet.



If link 23 is formed then both players gain $\delta f(2,3)$, but they both loose their indirect connections with each other, that is $\delta^2 f(1,2)f(1,3)$. Thus, link 23 is formed if and only if $\delta f(2,3) - \delta^2 f(1,2)f(1,3) > c(2,3)$. Hence, for 3 player case, complete graph is reached if and only if $\forall i, j \in \{1,2,3\}$ with $i \neq j$, we have

$$\delta f(i,j) - \delta^2 \sum_{\substack{l=1 \\ l \neq i \\ l \neq j}}^{3} f(l,i)f(l,j) > c(i,j).$$

Now, let $N = \{1,2,3,4\}$. Assume that, links 12 and 13 are formed then player 3 and player 4 meet.



If player 3 forms link 34 then he will gain

$$\delta f(3,4)$$

where as player 4 will gain

$$\delta f(3,4) + \delta^2 f(3,4)f(3,1) + \delta^3 f(3,4)f(3,1)f(1,2).$$

When player 4 forms this direct link with player 3, he also benefits from player 3's connections from player 1 and player 2. Player 4 will benefit from player 1 through the link he will form with player 3 and then player 3's direct connection with player 1. We denote this situation by writing

$$\delta^2 f(3,4)f(3,1).$$

Similarly, player 4 will benefit from player 2 through the links 34, 13, 12 that is

$$\delta^3 f(3,4)f(3,1)f(1,2).$$

Thus

$$[\delta f(3,4)]k + [\delta f(3,4) + \delta^2 f(3,4)f(3,1) + \delta^3 f(3,4)f(3,1)f(1,2)]k = 2c(3,4)$$

then

$$k = \frac{2c(3,4)}{\delta f(3,4)[2 + \delta f(1,3) + \delta^2 f(1,3)f(1,2)]}.$$

This result yields that, cost of player 3 from forming link 34 is

$$\frac{2c(3,4)}{2 + \delta f(1,3) + \delta^2 f(1,3)f(1,2)}$$

and cost of player 4 from forming link 34 is

$$\frac{2c(3,4)[2 + \delta f(1,3) + \delta^2 f(1,3)f(1,2)]}{2 + \delta f(1,3) + \delta^2 f(1,3)f(1,2)}.$$

We conclude that, link 34 is formed if and only if

$$\delta f(3,4) > \frac{2c(3,4)}{2 + \delta f(1,3) + \delta^2 f(1,3)f(1,2)}.$$

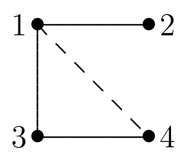
This condition holds if and only if

$$f(3,4) > \frac{c(3,4)}{\delta},$$

since

$$\frac{2}{2+\delta f(1,3)+\delta^2 f(1,3)f(1,2)} < 1.$$

Step 1: Now, suppose that links 12, 13 and 34 are formed then players 1 and 4 meet.



We summarize our findings:

Benefit of player 1 form link 14:

$$\delta f(1,4) - \delta^2 f(1,3)f(3,4) = \delta[f(1,4) - \delta f(1,3)f(3,4)].$$

Benefit of player 4 from link 14:

$$\begin{aligned}-\delta^2 f(3,4)f(1,3) - \delta^3 f(3,4)f(1,3)f(1,2) + \delta f(1,4) + \delta^2 f(1,4)f(1,2) &= \\ \delta f(1,4)[1+\delta f(1,2)] - \delta^2 f(1,3)f(3,4)[1+\delta f(1,2)] &= \\ \delta[1+\delta f(1,2)][f(1,4) - \delta f(1,3)f(3,4)].\end{aligned}$$

Cost of the link 14 to player 1:

$$\frac{2c(1,4)}{2+\delta f(1,2)}$$

Cost of the link 14 to player 4:

$$\frac{2c(1,4)[1+\delta f(1,2)]}{2+\delta f(1,2)}$$

Necessary and sufficient condition for link 14 to be formed by player 1:

$$\delta[f(1,4) - \delta f(1,3)f(3,4)] > \frac{2c(1,4)}{2+\delta f(1,2)}.$$

Necessary and sufficient condition for link 14 to be formed by player 4:

$$\delta[1+\delta f(1,2)][f(1,4)-\delta f(1,3)f(3,4)] > \frac{2c(1,4)[1+\delta f(1,2)]}{2+\delta f(1,2)}.$$

Obviously, these two necessary and sufficient conditions are equivalent, so it is enough for us to focus on one of them to decide whether or not link 14 is formed. I will consider the necessary and sufficient condition for player 1:

$$\begin{aligned}\delta[f(1,4)-\delta f(1,3)f(3,4)] &> \frac{2c(1,4)}{2+\delta f(1,2)}\\ &\Rightarrow [2+\delta f(1,2)][\delta f(1,4)-\delta^2 f(1,3)f(3,4)]\\ &> (2+c(1,4))[\delta f(1,4)-\delta^2 f(1,3)f(3,4)].\end{aligned}$$

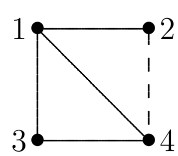
If

$$\begin{aligned}&[\delta f(1,4)-\delta^2 f(1,3)f(3,4)] > c(1,4)\\ \Rightarrow&(2+c(1,4))[\delta f(1,4)-\delta^2 f(1,3)f(3,4)] > 2c(1,4)+(c(1,4))^2 > 2c(1,4).\end{aligned}$$

We conclude that, link 14 is formed if and only if

$$[\delta f(1,4)-\delta^2 f(1,3)f(3,4)] > c(1,4).$$

Step 2: Assume that link 14 is formed.



Links which are already formed: link 12, link 13, link 34 and link 14

Link which can be formed: link 24

Benefit of player 2 form link 24:

$$\delta f(2,4)-\delta^2 f(1,2)f(1,4).$$

Benefit of player 4 from link 24:

$$\delta f(2,4)-\delta^2 f(1,2)f(1,4).$$

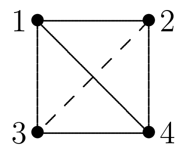
Cost of the link 24 to player 2: $c(2,4)$

Cost of the link 24 to player 4: $c(2,4)$

Necessary and sufficient condition for link 24:

$$\delta f(2,4) - \delta^2 f(1,2)f(1,4) > c(2,4)$$

Step 3: Assume that link 24 is formed.



Links which are already formed: link 12, link 13, link 34, link 14 and link 24

Link which can be formed: link 23

Benefit of player 2 form link 23:

$$\delta f(2,3) - \delta^2[f(1,2)f(1,3) + f(2,4)f(3,4)]$$

Benefit of player 3 from link 23:

$$\delta f(2,3) - \delta^2[f(1,2)f(1,3) + f(2,4)f(3,4)]$$

Cost of the link 23 to player 2: $c(2,3)$

Cost of the link 23 to player 3: $c(2,3)$

Necessary and sufficient condition for link 23 to be formed is:

$$\delta f(2,3) - \delta^2[f(1,2)f(1,3) + f(2,4)f(3,4)] > c(2,3)$$

If we perform this analysis for all possible orders in which players meet, then we obtain the following result:

For four players case, the complete graph forms if and only if $\forall i,j \in \{1,2,3,4\}$ with $i \neq j$, we have

$$\delta f(i,j) - \delta^2 \sum_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{4} f(l,i)f(l,j) > c(i,j).$$

Here, we observed that necessity and sufficiency condition for complete graph formation can be generalized in the following way:

Theorem 4.2.1 $\forall n \in \mathbb{N}$, *complete graph forms if and only if* $\forall i, j \in \{1, \ldots, n\}$ *with* $i \neq j$, *the following condition is satisfied*

$$\delta f(i,j) - \delta^2 \sum_{\substack{l=1 \\ l \neq i \\ l \neq j}}^{n} f(l,i) f(l,j) > c(i,j). \tag{4.4}$$

Proof: First of all, let us prove that 4.7 is a sufficient condition for complete graph to be formed. Let t_0 be any time and fixed. Let at time t_0, player i and player j meet. Let there are $k \geq 0$ players who have formed paths with the shortest length between players i and j. Let there are $m \geq 0$ players who are connected to player i and connected to player j via player i and those k players. Let there are $p \geq 0$ players who are connected to player j and connected to player i via player j and those k players. Note that, $k + m + p \leq n - 2$. Let B_i denotes the benefit of player i from link ij, B_j denotes the benefit of player j from link ij, C_i denotes the cost of player i from link ij, C_j denotes the cost of player j from link ij.
Note that, when $m = p = 0$, $k = n - 2$, where player i and player j form links via k players, $\{k_1, k_2, \ldots, k_k\}$, with length 2, we have

$$B_i = B_j = \delta f(i,j) - \delta^2 \sum_{l=1}^{n-2} f(i, k_l) f(j, k_l),$$

$$C_i = C_j = c(i,j).$$

Note that, the condition

$$\delta f(i,j) - \delta^2 \sum_{l=1}^{n-2} f(i, k_l) f(j, k_l) > c(i,j) \tag{4.5}$$

satisfies $B_i > C_i$ and $B_j > C_j$. Hence, in this case the link between player i and player j forms.
Moreover, when $k = 0$, $m \geq 0$, $p \geq 0$, where player i has links with m players,

$\{m_1, \ldots, m_m\}$, with length 1 and player j has links with p players, $\{p_1, \ldots, p_p\}$, with length 1, we have

$$
\begin{aligned}
B_i &= \delta f(i,j) + \delta^2 f(i,j) \sum_{k=1}^{p} f(j, p_k) \\
B_j &= \delta f(i,j) + \delta^2 f(i,j) \sum_{k=1}^{m} f(i, m_k) \\
C_i &= \frac{2c(i,j)(\delta f(i,j) + \delta^2 f(i,j) \sum_{k=1}^{p} f(j,p_k))}{2\delta f(i,j) + \delta^2 f(i,j)(\sum_{k=1}^{p} f(j,p_k) + \sum_{k=1}^{m} f(i,m_k))} \\
C_j &= \frac{2c(i,j)(\delta f(i,j) + \delta^2 f(i,j) \sum_{k=1}^{m} f(i,m_k))}{2\delta f(i,j) + \delta^2 f(i,j)(\sum_{k=1}^{p} f(j,p_k) + \sum_{k=1}^{m} f(i,m_k))}
\end{aligned}
$$

Note that, the condition

$$
\delta f(i,j) > c(i,j) \tag{4.6}
$$

satisfies $B_i > C_i$ and $B_j > C_j$. Hence, in this case the link between player i and player j forms.

Furthermore, note that, for any combination of m, k, p, with arbitrary path lengths, the sufficient condition for forming link ij is restricted by condition 4.5 and condition 4.6. Therefore, the condition 4.7 is sufficient for complete graph to be formed by n players.

Now, let us prove that 4.7 is a necessary condition for complete graph to be formed. We use induction method to prove this part. Note that, we have shown that for three players case, if complete graph is formed then $\forall i, j \in \{1, 2, 3\}$, we have

$$
\delta f(i,j) - \delta^2 \sum_{\substack{l=1 \\ l \neq i \\ l \neq j}}^{3} f(l,i) f(l,j) > c(i,j).
$$

Now assume that for $n-1$ players, if complete graph is formed, then we have $\forall i, j \in \{1, 2, \ldots, n-1\}$,

$$
\delta f(i,j) - \delta^2 \sum_{\substack{l=1 \\ l \neq i \\ l \neq j}}^{n-1} f(l,i) f(l,j) > c(i,j).
$$

Now, let us add player n to the game. Let without loss of generality, first of all, player 1 and player n form a link, then

$$\begin{aligned}
B_1 &= \delta f(1,n),\\
B_n &= \delta f(1,n) + \delta^2 f(1,n)\sum_{k=2}^{n-1} f(1,k),\\
C_1 &= \frac{2c(1,n)}{2+\delta\sum_{k=2}^{n-1} f(1,k)},\\
C_n &= \frac{2c(1,n)(1+\delta\sum_{k=2}^{n-1} f(1,k))}{2+\delta\sum_{k=2}^{n-1} f(1,k)}.
\end{aligned}$$

Note that, $B_1 > C_1$, $B_n > C_n$ holds if the condition $\delta f(1,n) > c(1,n)$ is satisfied. Now let player 2 and player n form a link, then

$$\begin{aligned}
B_2 &= B_n = \delta f(2,n) - \delta^2 f(1,2) f(1,n),\\
C_2 &= C_n = c(2,n).
\end{aligned}$$

Hence, $B_2 = B_n > C_2 = C_n$ holds if the condition $\delta f(2,n) - \delta^2 f(1,2)f(1,n) > c(2,n)$ is satisfied.

Similarly, let player n forms links with player 1,. . ., player $k-1$, then the condition $\delta f(k,n) - \delta^2 \sum_{l=1}^{k-1} f(l,n)f(k,l) > c(k,n)$ is necessary for player k to form a link with player n. Thus, if complete graph forms, then $\forall i,j \in \{1,2,\ldots,n\}$ with $i \neq j$,

$$\delta f(i,j) - \delta^2 \sum_{\substack{l=1\\ l\neq i\\ l\neq j}}^{n} f(l,i)f(l,j) > c(i,j).$$

holds.

Corollary 4.2.2 *If $\forall i,j \in \{1,\ldots,n\}$ with $i \neq j$, the following condition is satisfied*

$$\delta f(i,j) - \delta^2 \sum_{\substack{l=1\\ l\neq i\\ l\neq j}}^{n} f(l,i)f(l,j) > c(i,j).$$

then complete graph is stable.

Example 1 *Let* $f(i,j) = \left(\frac{1}{ij}\right)^{\alpha}$, $c(i,j) = \left(\frac{1}{ij}\right)^{\alpha+k}$ *such that* $\alpha \geq 1$ *and* δ *be fixed reel number such that* $\frac{6}{\pi^2} > \delta > 0$. *Then, we need to find* k *such that*

$$\left(\frac{1}{ij}\right)^{\alpha} - \delta \sum_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{n} \left(\frac{1}{ijl^2}\right)^{\alpha} > \left(\frac{1}{ij}\right)^{\alpha+k} \frac{1}{\delta}.$$

holds $\forall i, j \in \{1, 2, \ldots, n\}$ *with* $i \neq j$. *That is*

$$(ij)^k > \frac{1}{\delta} \left(\frac{1}{1 - \delta \sum_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{n} \frac{1}{l^{2\alpha}}} \right).$$

Note that

$$(ij)^k \geq 2^k.$$

Moreover

$$\left(\frac{1}{1 - \delta \sum_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{n} \frac{1}{l^{2\alpha}}} \right) \leq \left(\frac{1}{1 - \delta \frac{\pi^2}{6}} \right)$$

since

$$\sum_{l=1}^{n} \frac{1}{l^2} = \frac{\pi^2}{6}.$$

Thus, if we choose k *such that*

$$k > \frac{\log\left(\frac{1}{1-\delta\frac{\pi^2}{6}}\right) - \log \delta}{\log 2}, \tag{4.7}$$

we have

$$\left(\frac{1}{ij}\right)^{\alpha} - \delta \sum_{\substack{l=1 \\ l\neq i \\ l\neq j}}^{n} \left(\frac{1}{ijl^2}\right)^{\alpha} > \left(\frac{1}{ij}\right)^{\alpha+k} \frac{1}{\delta}.$$

$\forall n \in \mathbb{N}$. *Thus, complete graph forms for any number of players with condition* (4.7) *with assumptions which are specified at the beginning of the example.*

CHAPTER 5

CONCLUSION

In this thesis, We have concluded that, it is possible to reach to the complete graph $\forall n \in \mathbb{N}$ by assigning weights to direct links between any to players in the game which we have extended. These weights can be thought as if they represent the performance of the players in the game. As illustrated by the example, for any two players i and j, we can find $f(i,j)$ and $c(i,j)$ in neighbourhood of zero such that the complete graph is reached.

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