SOME RESULTS ON MONOTONICITY

A Master's Thesis

by
HAYRULLAH DİNDAR

Department of
Economics
Bilkent University
Ankara
September 2010

SOME RESULTS ON MONOTONICITY

The Institute of Economics and Social Sciences
of
Bilkent University

by

HAYRULLAH DİNDAR

In Partial Fulfillment of the Requirements For the Degree
of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
BILKENT UNIVERSITY
ANKARA

Semptember 2010

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Prof. Dr. Semih Koray

Supervisor

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assist. Prof. Dr. Tarık Kara

Examining Committee Member

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

Assoc. Prof. Dr. Azer Kerimov

Examining Committee Member

Approval of the Institute of Economics and Social Sciences

Prof. Dr. Erdal Erel

Director

ABSTRACT

SOME RESULTS ON MONOTONICITY

DİNDAR, Hayrullah

M.A., Department of Economics

Supervisor: Prof. Semih Koray

September 2010

In this thesis, we investigate several issues concerning social choice rules which satisfy different degrees of Maskin type monotonicities. Firstly, we introduce $g-monotonicity$ and *monotonicity region* notions which enable one to compare monotonicity properties of non Maskin monotonic social choice rules. We compare self-monotonicities of standard scoring rules and study monotonicity of Majoritarian compromise. Secondly we determine domains of impossibility and possibility when the individual preferences are clustered around two opposing norms and the degree of clustering is measured via the $Manhattan$ *metric*. In the last chapter we investigate the relation between monotonicity and dictatoriality when agents are allowed to have thick indifference classes.

*Keywords:*Monotonicity, Self monotonicity, Manhattan metric, Impossibility, Majoritarian compromise, Standard scoring rules.

ÖZET

TEKDÜZELİK ÜZERİNE BAZI SONUÇLAR

DİNDAR, Hayrullah

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

Eylül 2010

Bu tez çalışmamızda, çeşitli Maskin tarzı tekdüzelikleri sağlayan sosyal seçme kurallarının özelliklerini inceliyoruz. İlk olarak, Maskin tekdüze olmayan sosyal seçme kurallarının tekdüzeliklerini kıyaslamamıza imkan sağlayan g-tekdüzelik ve tekdüzelik bölgesi kavramlarını tanımlıyoruz. Standart puanlamalı kuralların öz tekdüzeliklerini karşılaştırıp, Çoğunlukçu uzlaşı'nın tekdüzeliğini inceliyoruz. İkinci olarak, kişisel tercihlerin birbirine zıt iki normun etrafnda Manhattan metriğine göre yığıştığı tanım bölgelerinin imkansızlık bölgeleri olup olmadığını belirliyoruz. Tezin son kısmında, tek elemanlı olmayan eşdeğerlik sınıflarına izin verildiği durumda, tekdüzelik ve diktatörlük arasındaki ilişkiyi inceliyoruz.

Anahtar Kelimeler: Tekdüzelik, Öz tekdüzelik, Manhattan metriği, İmkansızlık, Çoğunlukçu uzlaşı, Standart puanlamalı kurallar.

ACKNOWLEDGMENTS

I feel overwhelmed with gratitude for the help of Prof. Semih Koray, not only because of his invaluable guidance throughout my study, but also because of being an exceptional role model for me. It was a great honor for me to study under his supervision. I am proud that I have had the privilege of being among his students.

I would like to express my gratitude to;

Prof. Tarık Kara, for his invaluable guidance, unlimited support and time he spared throughout my study.

Serhat Doğan, for accepting to review this material and for his valuable suggestions, moral support and close friendship. Without his help I would never be able to complete this study.

All T.A.'s and R.A.'s in Bilkent University, Department of Economics for their sincere friendship and moral support.

Finally, my family, for their endless support.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

In this thesis, we deal with several issues concerning social choice rules which satisfy different degrees of monotonicities . In particular, we extend "Maskin-type" monotonicities to all social choice rules, whether Maskin-monotonic or not. The conepts of $g-monotonicity$ and *monotonicity region* enable us to compare the monotonicity properties of non-Maskin-monotonic social choice rules. These notions are motivated by their counterparts for Maskin monotonic SCRs, namely $h - monotonicity$ and *center*, introduced by Koray(2002).

To further classify Maskin-monotonic SCRs according to their degrees of monotonicity. The notion of strongest monotonicities that a SCR satisfies, called self-monotonicities, was used by Koray and Pasin (2005) to investigate the role that self-monotonicities of a solution concept σ plays in a σ-implementabilty.

The notion of *center* extensively used in (Koray and Doğan(2008)) arises naturally from the idea of a critical profile first introduced in Koray,Adali,Erol and Ordulu (2001). The center of a Maskin-monotonic SCR F is roughly a minimal subregion of its domain such that what F does on this subdomain uniquely determines what it does on entire on the entire domain. Erol (2009) further investigated properties of *center*.

Dually, given a non-Maskin-monotonic SCR F, its *monotonicity region* refers to a maximal subregion of its domain which is "closed under monotonic transformations" and on which F is Maskin-monotonic.

Koray and Doğan (2007) classify Maskin-monotonic SCRs according to their degrees of monotonicty, employing the notion of $self-monotonicity$ that specifies, the strongest $h-monotonicities$ of a given SCR. They also employ the notion of $self-monotonicity$ to establish new $Nash-implementability$ for the two-agent case.

In the same spirit, we define $self-g-monotonicity$ for non-Maskin-monotonic SCRs and use it to compare the degrees of monotonicities of a subset of scoring rules. For non-Maskin-monotonic SCRs, we establish the relation between $self-g-monotonicity$ and *monotonicity region* . We also generalize some of the results from Koray and Dogan(2007) for scoring rules. Monotonicity properties of the Majoritarian Compromise , introduced by Murat Sertel, are also examined.

Kaya and Koray (2000) provide the first paper that relates the monotonicites of a game-theoretic solution concept σ to the set of σ-implementable SCRs, and characterize the solution concepts which implement only Maskin-monotonic SCRs. The notion of monotonicity they introduce for solution concepts is a natural modification of Maskin-monotonicty for SCRs.

Koray and Pasin (2005) introduce $H-monotonicity$ as the counterpart of $h-monotonicity$ for solution concepts. They find the unique self-monotonicity of the Nash equilibrium concept and show that it is inherited via the mechanism employed by the implemented SCRs. If one empleys Maskin-Vind type mechanisms, the inherited monotonicity is naturally nothing but "essential monotonicity". Pasin(2009) also give a new characterization of strong Nash implementability via critical profiles.

Maskin monotonicity is a necessary condition for Nash implementability (and strong Nash implementability) due to Maskin(1977). There are several game-theoretic solution concepts which themselves are not "Maskin-monotonic" and thus also implement some non-Maskin-monotonic SCRs. The notion of $G - monotonicity$ introduced here shares the same spirit as $H - monotonicity$ for solution concepts. We conjecture that in the context of non-Maskin-monotonic solution concepts, G-monotonicity will play a similar role as $H - monotonicty$ does with implementabilty according to Maskin-monotonic solution concepts.

The well-known Mueller-Satterthwaite Theorem states that a social choice function defined on the set of all linear order profiles is *onto* and $Maskin$ *monotonic* if and only if it is *dictatorial* under the presence of at least three alternatives.

A common way of escaping the impossibility results in social choice theory is to relax the full domain assumption and allow the society to choose from only a subset of preference profiles.

For a finite set A of alternatives, and a finite set N of agents, letting $L(A)$ denote the set of linear orders on A. We refer to a subset D of $L(A)$, as a domain of impossibility if a social choice function $F : D^N \to A$ is *onto* and $Maskin$ *monotonic* if and only if it is *dictatorial* under the presence of at least three alternatives. On the other hand, a domain of *possibility* is a subset D' of $L(A)$, such that there exists a non-dictatorial SCF $F : D'^N \to A$ which is *onto* and $Maskin$ *monotonic*.

Among the domains of linear orders which are r-balls with respect to the Manhattan metric about a center -a linear order representing a "social norm"- Koray,Kavlakolu and Gurer(2008) prove that a domain is one of impossibility if and only if its radius larger than $|A|$.

Erol (2009) extended the $Manhattan$ *metric* which counts the minimal number of transpositions to obtain a linear order from a given one, assigning equal weight to each transposition; by allowing different weights for transpositions at different levels. His result is interesting because he thereby also finds nested domains of impossibility and possibility.

A natural question that had been asked during this analysis of a unipolar society was how the results would be influenced by a bipolar one, whose examples are not difficult to find either in history or in the present time.

We consider domains of preferences in $L(A)$ consisting of two sections clustered around two opposing norms, respectively. Our main result here is that a domain equal to the union of two balls around the opposing norms is a domain of impossibility if and only if sum of the radii of the balls is greater or equal to $|A| - 1$.

In the last chapter we investigate how the Mueller-Satterthwaite Theorem is affected when one replaces linear orders by complee preorders in representing individual preferences. This is not meant just as a techincal exercise, as indifferences in individual preferences can hardly be denied in real life, Having gone through this exercise, we also agree with Salvador Barbera, who noted that "indifferences require attention and careful treatment and the translation of results from a world without indifferences to another where agents may be indefferent among some alternatives is not allways a straightforward exercise". (Barbera 2007)

We first note that there exists no SCF defined on the set $C(A)^N$ of complete pre-order profiles, which satisfies both on the *Maskin monotonicity* and *unanimity*. Intuitively the main reason that leads to this result is that *Maskin monotonicity* requires that an alternative a chosen at R, continues to get chosen at R', even if R' is obtained from R by moving all the strictly worse alternatives to the same indifference class with a. To deal with this problem we weaken *Maskin monotonicity* and work with this modified monotonicty. We define three kinds of dictatoriality for SCFs defined on $C(A)^N$ and investigate the relation between these dictatorialities and *monotonicity*.

CHAPTER 2

PRELIMINARIES

Let $N = \{1, 2, \cdots, n\}$ be a finite set of *individuals* and let $A = \{a_1, a_2, \cdots, a_m\}$ be a finite set of *alternatives*. We will assume $m \geq 3$ throughout the paper.

The opinion of agent i, over the set A of alternatives is described by a $preference$ relation. $L(A)$ denotes the set of linear orders over A (i.e. complete, transitive, anti-symmetric binary relations. $C(A)$ denotes the set of complete pre-orders over A (i.e. dropping the anti-symmetry assumption thus allow for indifference).

A preference profile $P \in L(A)^N$ ($R \in C(A)^N$) is the data for each agent of a linear order (complete pre-order) on A. Given $R_i \in C(A)$, strict and indifference parts of $R_i \in C(A)$ will be denoted by P_i and I_i respectively. $L(A)^N$ ($C(A)^N$) is the set of all possible preference profiles for given A and N.

Given a subset, D, of $L(A)$ or $C(A)$ a social choice rule F is a nonempty correspondence from the set D^N of preference profiles into the set A of alternatives $F : D^N \to A$. A social choice function is a single-valued social choice rule.

Definition. For any given alternative a, any agent i and any preference profile $R \in C(A)^N$ the lower contour set of a w.r.t. R for agent i is :

$$L_i(a, R) = \{b \in A | aR_ib)\}$$

and the strict lower contour set of a w.r.t. R for agent i is :

$$L_i^*(a, R) = \{b \in A | aP_ib\}$$

Given an alternative $a \in A$ and a preference profile $R \in L(A)^N$, $MT(a, R)$ denotes the set of preference profiles such that for any agent the lower contour set of a does not shrink, i.e. $MT(a, R) = \{R' \in L(A)^N | \forall i \in N; L_i(a, R) \subset L_i(a, R')\}$.

Definition. Let $D \subset C(A)$, an SCR $F : D^N \to A$ is *Maskin monotonic* if and only if $\forall R, R^{'} \in D^N$ and $\forall a \in A$

$$[a \in F(R) \quad and \quad R' \in MT(a, R)] \Rightarrow a \in F(R').$$

Given an alternative $a \in A$, an agent $i \in N$ and a preference profile $R \in L(A)$; $r_i(a, R)$ denotes the rank of a for i at R,i.e. $r_i(a, R) = |\{b \in A | bR_ia\}|$. We will write $r(a, R_i)$ instead of $r_i(a, R)$ when we are working with a preference rather than a preference profile. Given an agent $i \in N$,$k \in \{1, \cdots, m\}$ and a preference profile $R \in L(A)$; $\sigma(i, k, R)$ denotes the k'th best alternative according to i,i.e. $a = \sigma(i, k, R) \Rightarrow r_i(a, R) = k$.

CHAPTER 3

MONOTONICITY PROPERTIES OF NON MASKIN MONOTONIC SOCIAL CHOICE RULES

In this chapter we study the monotonicity properties of non $Maskin\ monotonic$ social choice rules using $g-monotonicity$ and $monotonicity\ region$ notions; counterparts of $h-monotonicity$ and $center$ concepts . To make it easier for the reader to see the similarities between the two concepts, this chapter contains some of the old results, as well as some minor new results about $h-monotonicity$.[1]

Let $F : L(A)^N \to A$ be an SCR , F satisfies *unanimity* if and only if for any alternative $a \in A$ and any preference profile $R \in L(A)^N$ s.t. a is top ranked by all agents, a is among the chosen alternatives by F at R, i.e. $\forall a \in A$ $\forall R \in L(A)^N$ $[\forall i \in N \quad r_i(a, R) = 1] \Rightarrow a \in F(R)$. Let Γ denote the set of all *unanimous* SCRs and $\mathbb{M}$ denote the set of all *unanimous* and $Maskin-monotonic$ SCRs. Let $F \in \Gamma$, define $GrF = \{(a, R) \in A \times L(A)^N | a \in F(R)\}$.

Throughout this chapter *unanimity* will be assumed and the reason will be clear once we define $g-monotonicity$.

Definition. Given $F \in \Gamma$ let $h : GrF \to (2^A)^N$ and $g : GrF \to (2^A)^N$

[1] Koray(2002), Koray, Pasin(2005) Dogan (2007) contains a more detailed treatment of $h-monotonicity$ and *center*.

be two functions. We say that F is $h-monotonic$ if and only if for any $R, R' \in L(A)^N$ and any $a \in F(R)$,

$$[\forall i \in N \quad L_i(a,R) \cap h_i(a,R) \subset L_i(a,R)] \Rightarrow a \in F(R')$$

Similarly we say that F is $g-monotonic$ if and only if for any $R, R' \in L(A)^N$ and any $a \in F(R)$,

$$[\forall i \in N \quad L_i(a,R) \cup g_i(a,R) \subset L_i(a,R)] \Rightarrow a \in F(R')$$

$h : GrF \to (2^A)^N$ is a $self-h-monotonicity$ of F if and only if F is $h-monotonic$ and there is no $h' : GrF \to (2^A)^N$ with $h' \subsetneq h$ such that F is $h'-monotonic$.

Similarly $g : GrF \to (2^A)^N$ is a $self-g-monotonicity$ of F if and only if F is $g-monotonic$ and there is no $g' : GrF \to (2^A)^N$ with $g' \subsetneq g$ such that F is $g'-monotonic$.

Proposition 1. *Let F be an SCR such that for any alternative $a \in A$ there exists a preference profile $R \in L(A)^N$ with $a \in F(R)$,i.e. F is onto.*
F satisfies unanimity if and only if there exists a function $g : GrF \to (2^A)^N$ such that F is $g-monotonic$.

Proof. Assume F satisfies *unanimity*. Since for any $a \in A$ and for any $R \in L(A)^N$, $[\forall i \in N \quad r(a,R_i) = 1]$ implies $a \in F(R)$; $g : GrF \to (2^A)^N$ defined as $g_i(R,a) = (A, A, ..., A)$ $\forall i \in N$ is a $g-monotonicity$ of F.

Assume there exists a function $g : GrF \to (2^A)^N$ such that F is $g-monotonic$. Now $g : GrF \to (2^A)^N$ defined as $g_i(R,a) = (A, A, ..., A)$ $\forall i \in N$ is also a $g-monotonicity$ of F. Now for any $a \in A$ and for any $R \in L(A)^N$, $[\forall i \in N \quad L_i(a,R) = A]$ implies $a \in F(R)$ since F is onto. Thus F satisfies *unanimity*.

$\square$

Remark 1. **1)** If $L(a,R) \subset h(a,R)$ for each $(R,a) \in GrF$, $h-monotonicity$ is nothing but $Maskin-monotonicity$. Thus $\forall F \in \Gamma \setminus \mathbb{M}$; F does not have any $h-monotonicity$.

2) If $g(R,a) \subset L(a,R)$ for each $(R,a) \in GrF$, $g-monotonicity$ is nothing but $Maskin-monotonicity$. Thus $\forall F \in \mathbb{M}$; $\forall (a,R) \in GrF$ and for any agent $i \in N$ $g_i(a,R) = (\emptyset, \emptyset, ..., \emptyset)$ is the unique $self-g-monotonicity$ of F.

3) $h-monotonicity \Rightarrow Maskin-monotonicity \Rightarrow g-monotonicity$

4) Note that, if h is a $self-h-monotonicity$ of F, then for any $(a,R) \in GrF$ and any $i \in N$ $h_i(a,R) \subset L_i(a,R)$.

Definition. Let $F, G \in \Gamma$. We say that F satisfies a stronger $h-monotonicity$ condition than G if and only if

$GrG \subset GrF$ and there exist $self-h-monotonicities$ h^F and h^G of F and G , respectively s.t. for any $(a,R) \in GrG$, we have $h^F(a,R) \subset h^G(a,R)$.

Let $\underline{GrG} = \{(a,R) \in GrG \quad | \ \forall R' \in L(A)^N \ [\forall i \in N L_i(a,R) \subset L_i(a,R')]$ $implies \quad a \in G(R')\}$. We say that F satisfies a stronger $g-monotonicity$ condition than G if and only if

$\underline{GrG} \subset GrF$ and there exist $self-g-monotonicities$ g^F and g^G of F and G, respectively ,s.t. for any $(a,R) \in GrG \cap GrF$ we have $g^F(a,R) \subset g^G(a,R)$.

Remark 2. For the case $F, G \in \mathbb{M}$, $g-monotonicity$ is not very telling , since it boils down to $GrG \subset GrF$.
Similarly if $F, G \in \Gamma \setminus \mathbb{M}$, $h-monotonicity$ is not very telling since neither F nor G has any $h-monotonicity$.

Given $a \in A$, $\rho(a)$ denotes the following partition of $L(A)^N$ induced by a

$$\rho(a) = \{\{R' \in L(A)^N | \forall i \in N : L_i(a,R) = L_i(a,R') | R \in L(A)^N\}\}$$

Given $R, R' \in L(A)^N$ and an alternative a, we say that R' is a $refinement$ of R w.r.t. a if $R \in MT(R')$. We say that R' is a *strict* $refinement$ if the inclusion (of lower contour sets) is strict for at least one agent.

Definition. Let $R \in L(A)^N$ and $F \in \Gamma$. We say that R is an a-monotonicity profile for an alternative $a \in A$ relative to F if and only if
$a \in F(R)$ and $\forall R' \in\in MT(a, R)$ we have $a \in F(R')$. In which case we say that F is locally monotonic at (a, R).
Given an alternative a, we will denote the set of a-monotonicity profiles for a relative to F by $M_a(F)$, i.e.
$M_a(F) = \{R \in L(A)^N | a \in F(R) \quad and \quad \forall R' \in MT(a, R) : a \in F(R')\}$

Let $F \in \Gamma$ and $S_1, ..., S_k$ be distinct members of $\rho(a)$ s.t. $\bigcup_{i\in\{1,...,k\}} S_i = M_a(F)$. We will refer to a set $R_1, ..., R_k$ s.t. $R_i \in S_i$ for each $i \in \{1, ..., k\}$ as an $a - monotonicity\ region$ of F. Let for each $a \in A$, $MR_a(F)$ be an $a - monotonicity\ region$ of F.

Remark 3. **1)**Let $a \in A$ and $F \in \Gamma$. Note that $MR_a(F) \neq \emptyset$ since
$\emptyset \neq \{R \in L(A)^N \quad | \quad \forall i \in N \quad L_i(a, R) = A\} \subset MR_a(F)$.
2) If $F \in \mathbb{M}$, $MR_a(F) = \{R \in L(A)^N \quad | \quad a \in F(R)\}$.

Definition. A profile $R \in L(A)^N$ is an $a - critical$ profile for some $a \in A$ relative to an SCR $F \in \mathbb{M}$ if $a \in F(R)$ and for any strict refinement R' of R w.r.t. a, we have $a \notin F(R')$. We will denote the set of $a - critical\ profiles$ relative to F by $C_a(F)$.

Let $F \in \mathbb{M}$ and $S_1, ..., S_k$ be distinct members of $\rho(a)$ s.t. $\bigcup_{i\in\{1,...,k\}} S_i = C_a(F)$. We will refer to a set $R_1, ..., R_k$ s.t. $R_i \in S_i$ for each $i \in \{1, ..., k\}$ as an $a - center$ of F. Let for each $a \in A$, $CE_a(F)$ be an a-center of F. We will refer to a set $\bigcup_{a\in A} CE_a(F)$ as a *center* of F.

Proposition 2. *Let $a \in A$, $R \in L(A)^N$ and $F \in \Gamma$. $R \in M_a(F)$ if and only if for any $self - g - monotonicity$ of F; $g_i^F(a, R) = \emptyset$ $\forall i \in N$.*

Proof. Obvious □

Proposition 3. *Let $F, G \in \Gamma$. F satisfies a stronger $g - monotonicity$ condition than G if and only if $\forall a \in A$ $M_a(G) \subset M_a(F)$.*

Proof. Assume F satisfies a stronger $g-monotonicity$ condition than G. Let $a \in A$ and $R \in M_a(G)$.

Note that $\underline{GrG} = \bigcup_{a \in A} M_a(G)$. Now $\underline{GrG} \subset Gr(F)$ implies $(a, R) \in GrF$. Since $(a, R) \in GrG$ we have $(a, R) \in GrG \cap GrF$.

$R \in M_a(G)$ implies $g^G(a, R) = (\emptyset, \emptyset, ..., \emptyset)$ for any $self - g - monotonicity$ of G. Since F satisfies a stronger $g - monotonicity$ condition than G , there exist a $self - g - monotonicity$ of F, say g^F, s.t. $g^F(a, R) = (\emptyset, \emptyset, ..., \emptyset)$. Thus $R \in M_a(F)$.

Assume $\forall a \in A \; M_a(G) \subset M_a(F)$.

Note that $\underline{GrG} = \bigcup_{a \in A} M_a(G) \subset \bigcup_{a \in A} M_a(F) \subset GrF$.

Let $a \in A$ and $R \in L(A)^N$ s.t. $a \in F(R) \cap G(R)$, and let g^G be a $self - g - monotonicity$ of G.

Note that $\forall R' \in L(A)^N$

$$[\forall i \in N L_i(a, R) \cup g_i^G(a, R) \subset L_i(a, R')] \Rightarrow a \in G(R')$$

Now $R' \in M_a(G)$ since $\forall R'' \in L(A)^N$

$$\begin{aligned} &[\forall i \in N L_i(a, R') \subset L_i(a, R'')] \\ \Leftrightarrow &[\forall i \in N L_i(a, R) \cup g_i^G(a, R) \subset L_i(a, R'')] \\ &\Rightarrow a \in G(R') \end{aligned}$$

Now $R' \in M_a(G)$ which implies $\forall R'' \in L(A)^N$

$$[\forall i \in N L_i(a, R) \cup g_i^G(a, R) \subset L_i(a, R'')] \Rightarrow a \in G(R')$$

Define $g : GrF \to (2^A)^N$ as follows:

$\forall (a, R) \in GrG \cap GrF \qquad g(a, R) = g^G(R, a)$

and $\forall (a, R) \in GrF \setminus GrG \qquad g(a, R) = A$

Note that g defined as above is a $g - monotonicity$ of F and there exists

$g^F \subset g$ s.t. g^F is a $self - g - monotonicity$ of F, now

$\forall (a,R) \in GrG \cap GrF \qquad g^F(a,R) \subset g(a,R) \subset g^G(a,R)$ $\square$

The next proposition , which we borrow from Koray,Dogan(2008) will be used in the next section.

Proposition 4. *Let $F, G \in \mathbb{M}$. F satisfies a stronger $h - monotonicity$ condition than G if and only if for any $a \in A$, $R \in C_a(G)$ there exists some $R' \in C_a(F)$ s.t. R' is a refinement of R w.r.t. a.*

3.1 Standard Scoring Rules

Throughout this section $m, n \geq 3$ will be assumed.
A score vector is an m-tuple $v = (v_1, v_2, ..., v_m) \in \mathbb{R}^m$ with $v_i \geq v_{i+1}$ for all $i \in \{1, 2, ..., m-1\}$ and $v_1 > v_m$.
We say that a social choice rule F is a scoring rule induced by a score vector $v \in \mathbb{R}^m$ if and only if for any $R \in L(A)^N$ we have,

$$F(R) = \{a \in A \quad | \sum_{i \in N} v_{r_i(a,R)} \geq \sum_{i \in N} v_{r_i(b,R)} \quad \forall b \in A\}$$

Remark 4. 1) Any scoring rule satisfies *unanimity* thus any scoring rule satisfies a $g - monotonicity$.

For sake of simplifying the notation we will assume $v \in [0,1]^m$, $v_1 = 1$ and $v_m = 0$ and the following proposition shows that this does not affect the generality of the results.

Proposition 5. *Let F be a scoring rule induced by a scoring vector $v \in \mathbb{R}^m$. If G is a scoring rule induced by a scoring vector $w \in \mathbb{R}^m$ where $w = (\frac{v_1 - v_m}{v_1 - v_m}, \frac{v_2 - v_m}{v_1 - v_m}, ..., \frac{v_m - v_m}{v_1 - v_m})$, $G = F$.*

Proof. Note that $v_1 - v_m > 0$ so w is a well defined vector in $\mathbb{R}^m$ and $w_i \geq w_{i+1}$ for all $i \in \{1, 2, ..., m-1\}$ and $w_1 > w_m$ follows directly from v being a scoring vector. Finally $\forall R \in L(A)^N$ $\forall a, b \in A$

$$\sum_{i\in N} v_{r_i(a,R)} \geq \sum_{i\in N} v_{r_i(b,R)} \quad \Leftrightarrow \quad \sum_{i\in N} w_{r_i(a,R)} \geq \sum_{i\in N} w_{r_i(b,R)}$$

Thus $G(R) = F(R)$. $\square$

Proposition 6. *Let F be a scoring rule with score vector v and let $g : GrF \to (2^A)^N$ be a function.*
F is $g-monotonic$ if and only if
$\forall (a, R) \in GrF$ and $\forall b \in A \setminus \{a\}$

$$\sum_{i\in N_a^b} (1 - v_{m-c_i}) \leq \sum_{i\in N_b^a} min_{j\in\{1,\ldots,m-c_i\}}(v_j - v_{j+1}) \tag{3.1}$$

where $N_b^a = \{i \in N \ \ |b \in L_i(a, R) \cup g_i(a, R)\}$, $N_a^b = N \setminus N_b^a$ and $c_i = |L_i(a, R) \cup g_i(a, R)| - 1$.

Proof. Assume F is $g-monotonic$, let $(a, R) \in GrF$ and $b \in A \setminus \{a\}$. Consider the preference profile R' with $\forall i \in N$ $L_i(a, R') = L_i(a, R) \cup g_i(a, R)$. Note that $\forall i \in N$ $r_i(a, R) = c_i$. Now consider the profile R'' obtained R' as follows:
For all agents i that ranks b above a at R', interchange b with the top ranked alternative, leaving everything else the same,i.e. $\forall i \in N$ s.t. $bR_i'a$: $r_i(b, R'') = 1$,$r_i(\sigma(i, 1, R'), R'') = r_i(b, R'$, $\forall c \in A \setminus \{b, \sigma(i, 1, R')\}$ $r_i(c, R'') = r_i(c, R'')$
For all agents i that ranks a above b at R',move b just belove a and move a, b as a block $\begin{matrix} a \\ b \end{matrix}$ to an upper position so that the difference of score gain between a and b is minimal,i.e.
$\forall i \in N$ s.t. $aR_i'b$: $R'' \in S$ and $\forall R_x \in S$ $\quad v_{r(a,R'')} - v_{r(a,R'')+1} \leq v_{r(a,R_x)} - v_{r(a,R_x)+1}$ where $S = \{R_x \in L(A) \ \ | \ \ [L(a, R_i') \subset L(a, R_x)] \ \ r(b, R_x) = r(a, R_x) + 1\}$

Note that

$$[\forall i \in N L_i(a,R) \cup g_i(a,R) \subset L_i(a,R'')]$$

thus $g-monotonicity$ of F implies $a \in F(R'')$, i.e.

$$\sum_{\{i\in N|bR''_i a\}} 1 + \sum_{\{i\in N|aR''_i b\}} v_{r(b,R''_i)} \leq \sum_{\{i\in N|bR''_i a\}} v_{m-c_i} + \sum_{\{i\in N|aR''_i b\}} v_{r(a,R''_i)}$$

now , it easy to see that equation 1 is satisfied by construction of R''. Since for any $b \in A \setminus \{a\}$ there exists such R'' we are done.

For the converse , let $g : GrF \to (2^A)^N$ be a function s.t. $\forall (a,R) \in GrF$ and $\forall b \in A \setminus \{a\}$ equation 1 holds. Let R' be a preference profile s.t. $[\forall i \in N L_i(a,R) \cup g_i(a,R) \subset L_i(a,R')]$, and $b \in A \setminus \{a\}$. Note that equation 1 implies total score of b is less than total score of a at R''

$$\sum_{\{i\in N|bR''_i a\}} 1 + \sum_{\{i\in N|aR''_i b\}} v_{r(b,R''_i)} \leq \sum_{\{i\in N|bR''_i a\}} v_{m-c_i} + \sum_{\{i\in N|aR''_i b\}} v_{r(a,R''_i)}$$

where R'' is defined in the first part of the proof. By construction of we have total score of b is less than total score of a at R''. Now using the same procedure for every alternative other than a, we get total score of b is less than total score of a at R' for any $b \in A \setminus a$,i.e. $a \in F(R')$. Since R' is an arbitrary preference profile s.t. $[\forall i \in N L_i(a,R) \cup g_i(a,R) \subset L_i(a,R')]$ F is $g-monotonic.$

□

Corollary. *Let F be a scoring rule with score vector v.*
F is $Maskin-monotonic$ if and only if

$\forall (a,R) \in GrF$ *and* $\forall b \in A \setminus \{a\}$

$$\sum_{i \in N_a^b} (1 - v_{m-c_i}) \leq \sum_{i \in N_b^a} min_{j \in \{1,...,m-c_i\}} (v_j - v_{j+1})$$

where $N_b^a = \{i \in N \quad | b \in L_i(a,R)\}$, $N_a^b = N \setminus N_b^a$ *and* $c_i = |L_i(a,R)| - 1$.

Proof. Note that a SCR is Maskin monotonic if and only if it is $g-monotonic$ for $g(a,R) = (\emptyset, ..., \emptyset)$ $\forall (a,R) \in GrF$. Applying the previous proposition we get the desired result. □

Proposition 7. *Let F be a scoring rule with score vector v and let $h : GrF \to (2^A)^N$ be a function.*
F is $h - monotonic$ if and only if
$\forall (a,R) \in GrF$ *and* $\forall b \in A \setminus \{a\}$

$$\sum_{i \in N_a^b} (1 - v_{m-c_i}) \leq \sum_{i \in N_b^a} min_{j \in \{1,...,m-c_i\}} (v_j - v_{j+1}) \tag{3.2}$$

where $N_b^a = \{i \in N \quad | b \in L_i(a,R) \cap h_i(a,R)\}$, $N_a^b = N \setminus N_b^a$ *and* $c_i = |L_i(a,R) \cap h_i(a,R)| - 1$.

Proof. Almost the same proof as the previous proposition. □

The following theorem by Doǧan (2008) characterizes the $Maskin-monotonic$ scoring rules.

Theorem 1. *Let F be a scoring rule with score vector v and let $k = min\{i \in \{1,2,...,m-1\} | v_{i+1} \neq v_1\}$.*
F is Maskin-monotonic if and only if i or ii holds;
i) $m = 3$, $n = 4$ *and* $v_1 > v_2 = v_3$
ii) $k > \frac{m(n-1)}{n}$.

Proposition 8. *Let F and G be scoring rules with score vectors v and w respectively. Let $k^F = min\{i \in \{1,2,...,m-1\} | v_{i+1} \neq v_1\}$ and $k^G = min\{i \in$*

$\{1, 2, ..., m-1\} | w_{i+1} \neq w_1\}$.

If $\quad 2 \leq k^G \leq k^F \leq \frac{m(n-1)}{n}$ *F satisfies a stronger* $g-monotonicity$ *condition than* G*.*

Proof. Let $a \in A$ and $R \in M_a(G)$. Let $N_a^b = \{i \in |bR_i\}$ and $N_b^a = N \setminus N_a^b$ for any $b \in A \setminus \{a\}$. We have

$$\sum_{i \in N_a^b} (1 - w_{r_i(a,R)}) \leq \sum_{i \in N_b^a} min_{j \in \{1,...,r_i(a,R)\}} (w_j - w_{j+1})$$

.Note that $0 = w_1 - w_2 = min_{j \in \{1,...,r_i(a,R)\}} (w_j - w_{j+1})$so $(1 - w_{r_i(a,R)}) = 0 \Rightarrow w_{r_i(a,R)} = 1 \Rightarrow v_{r_i(a,R)} = 1 \quad \forall i \in N_a^b$. Since v is a scoring vector $v_j - v_{j+1} \geq 0$ $\forall j \in 1, 2, ..., m-1$ and

$$\sum_{i \in N_a^b} (1 - v_{r_i(a,R)}) \leq 0 \leq \sum_{i \in N_b^a} min_{j \in \{1,...,r_i(a,R)\}} (v_j - v_{j+1})$$

thus $R \in M_a(F)$. Since a and R are arbitrary , $M_a(G) \subset M_a(F)$. By proposition 4 , F satisfies a stronger monotonicity condition than G. □

Example 1. Let $N = \{1, 2, 3\}$ and $A = \{a, b, c\}$. Let F and G be scoring rules with score vectors $(1, 0, 0)$ and $(1, 1, 0)$ respectively,i.e. 1-plurality and 2-plurality. Consider the preference profile

$$R = \begin{array}{ccc} a & b & b \\ c & c & c \\ b & a & a \end{array}$$

note that $R \in M_b(F)$ and $R \in M_c(G)$ but $F(R) = \{b\}$ and $G(R) = \{c\}$ so F and G are not comparable in terms of their $g-monotonicities.$

Proposition 9. *Let F and G be scoring rules with score vectors v and w respectively. Let $k^F = min\{i \in \{1, 2, ..., m-1\} | v_{i+1} \neq v_1\}$ and $k^G = min\{i \in \{1, 2, ..., m-1\} | w_{i+1} \neq w_1\}$.*

If $k^F \geq k^G > \frac{m(n-1)}{n}$ F satisfies a stronger $h-monotonicity$ condition than G.

Proof. Assume $k^F \geq k^G > \frac{m(n-1)}{n}$. Note that $k > \frac{m(n-1)}{n} \Rightarrow kn > mn - m \Rightarrow m > n(m-k)$ since $n > 0$. Note that for any profile $R \in L(A)^N$, we must have some $a \in A$ with $|\{i \in N| \quad r_i(a, R) \leq k\}| = n$, which is the total number of participants.

So for any alternative a and any preference profile R , $a \in F(R) \Leftrightarrow |\{i \in N| \quad r_i(a, R) \leq k^F\}| = n$ and similarly $a \in G(R) \Leftrightarrow |\{i \in N| \quad r_i(a, R) \leq k^G\}| = n$. Note that $k^F \geq k^G \Rightarrow GrG \subset GrF$.

Let $a \in A$ and $R \in C_a(G)$. Note that we have $a \in F(R)$, if R is an a critical profile we are done. If not by definition of an a-critical profile $\exists R^{(1)} \in$ s.t. $a \in F(R^{(1)})$ and $R^{(1)}$ is a strict refinement of R w.r.t. a. If $R^{(1)}$ is also not an a-critical profile there exists $R^{(2)}$ s.t. $R^{(2)}$ is a strict refinement of $R^{(1)}$ and $a \in F(R^{(2)})$. After we continue this way after at most finitely many steps, we will reach $R^{(t)}$, $t \in \mathbb{N}$, s.t. $R^{(t)} \in C_a(F)$. Note that $R^{(t)}$ is a refinement of R thus by Proposition 5, F satisfies a stronger monotonicity condition than G. □

3.2 Majoritarian Compromise

Majoritarian Compromise SCR is introduced by Murat Sertel. It satisfies desired properties such as Majoritarian-optimality while it is not Maskin monotonic and violates Condorcet consistency and Condorcet Loser criterion. It is subgame perfect implementable but not nash implementable.[2] We will borrow Sertel and Yılmaz's formulation with slight modifications.

Every $R_i \in L(A)$ determines a (ordinal) utility $\Pi : A \to \{1, 2, ..., m\}$ representing R_i through $\Pi(a) = |L(a, R_i)|$ at each $a \in A$

For each coalition $K \subset N$, at each $R_K = \{R_i\}_{i \in K} \in L(A)^K$ we also define a (ordinal) "welfare" $\Pi_K : A \to \{1, 2, ..., m\}$ representing R_K through $\Pi_K(a) = Min\{\Pi_i(a)|i \in K\}|$ at each $a \in A$, where Π_i is the utility represent-

[2]For proofs and further analysis of Majoritarian Compromise, interested reader is referred to Sertel, Yılmaz (1998).

ing the preference $R_i \quad (i \in K)$.

We say that an alternative 'gains k^{th} degree approval or support' from a coalition $K \subset N$ with (coalitional) preference profile $R_K \in L(A)^K$ iff $\Pi_K(a) \geq m - k + 1$.

A coalition $K \subset N$ is called a majority (in N) iff $|K| \geq |N \setminus K|$. For a given integer n, $\lceil \frac{n}{2} \rceil$ denotes the smallest integer which is no less than $\frac{n}{2}$,i.e.

$$\lceil \frac{n}{2} \rceil = \begin{cases} \frac{n}{2}, & \text{if n is even;} \\ \frac{n+1}{2}, & \text{if n is odd.} \end{cases}$$

At any profile $R \in L(A)^N$, we write $\overline{\Pi} = Max\{\Pi_K(a)|K \in \mu \quad, a \in A\}$ for the highest majority welfare achievable (by suitable choice of a) at R, and we define $\overline{M}(R) = \{a \in A|\Pi_i(a) = \overline{\Pi}\}$ as the set of alternatives giving this majority welfare.

At any $R \in L(A)^N$ and $a \in A$ define $K(a, R) = \{i \in N|\Pi_i(a) \geq \overline{\Pi}\}$ as the set of agents enjoying at least $\overline{\Pi}$ utility at a.

Definition. The Majoritarian Compromise is the SCR $M : L(A)^N \to A$ defined by $M(R) = \{a \in \overline{M}(R)|b \in \overline{M}(R) \Rightarrow |K(b, R)| \leq |K(a, R)|\}$.

Specifically, given any set A of alternatives, for any profile R of strict preferences (linear orders) on A, M picks that subset $M(R)$ of the alternatives in A which gain the largest number of agents's $k^*(R)^{th}$ degree approvals, where $k^*(R)$, the critical degree of majority approval at R, is the smallest integer k for which some alternative is commonly regarded as k^{th} best or better by at least half of the n agents whose preferences are recorded by the profile R.

Lemma 1. *Let $R \in L(A)^N$. There exists $a \in A$ s.t. M is locally monotonic at $(a, R) \in GrM$ only if $k^*(R) \leq 2$.*

Proof. Assume to the contrary that $k^*(R) \geq 3$.

Let $a \in M(R)$, since $k^*(R) \geq 3$ number of agents who rank a at second rank or better is strictly less than $\lceil \frac{n}{2} \rceil$,i.e. $|\{i \in N|r(a, R_i) \leq k^*(R) - 1\}| < \lceil \frac{n}{2} \rceil$ and there exists an agent who ranks a $k^*(R)^{th}$. Pick one such agent, say agent j, and the alternative, say b, which is top ranked by agent j.

Consider R' obtained from R via the following algorithm

Move a to the top position and b to the second position for all agents who rank a in $\{1, 2, .., k^*(R) - 1\}$and b to the second position.

Move a to the top position and b to the second position for some agents other than agent j who rank a $k^*(R)^{th}$ so that $|\{i \in N|r(a, R'_i) = 1\}| = \lceil \frac{n}{2} \rceil - 1$. Keeping everything else fixed.

Now note that $|\{i \in N|r(b, R'_i) \leq 2\}| = \lceil \frac{n}{2} \rceil$ and $|\{i \in N|r(a, R'_i) \leq 2\}| < \lceil \frac{n}{2} \rceil$. Thus a is not chosen at R' even though R' is a monotonic transformation of R w.r.t. a, the desired contradiction. □

Lemma 2. *Let $(a, R) \in GrM$. M is locally monotonic at (a, R) only if one of the following conditions hold*

i) $k^*(R) = 1$.

ii)$k^*(R) = 2$ and a is the Condorcet winner at R.

Proof. $k^*(R) = 1$ case is obvious.

Assume $k^*(R) = 2$ and a is not the Condorcet winner at R. There exists $b \in A \setminus \{a\}$ s.t. $|\{i \in N|bR_ia\}| \geq \lceil \frac{n}{2} \rceil$. Consider R' obtained from R by moving b to the top position in every such agents ranking (i.e. the agents who strictly prefer b to a), leaving everything else the same. Note that $k^*(R') = 1$ and $a \notin M(R')$ but R' is a monotonic transformation of R w.r.t. a. Completing the proof. □

Remark 5. The preceding lemma also implies that for any preference profile R and two distinct alternatives $a, b \in A$ M cannot be locally monotonic at both (a, R) and (b, R) except the case that [n is even, $k^*(R) = 1$ and $|\{i \in N|r(a, R_i) = 1\}| = |\{i \in N|r(b, R_i) = 1\}| = \frac{n}{2}$]. It is a direct implication of the fact that for all other cases we must have a and b to be the Condorcet winner at R which is impossible.

Let $R \in L(A)^N$, $a \in A$ and $b \in A \setminus \{a\}$ .We will use the following notations in the following proposition which characterizes local monotonicities of M.

$x = |\{i \in N | r(a, R_i) = 1\}|$ (number of agents that top rank a)

$y = |\{i \in N | r(a, R_i) = 2\}|$ (number of agents that second rank a)

$z = |\{i \in N | bR_i a \quad and \quad r(a, R_i) = 2\}|$ (number of agents who top rank b and second rank a)

$t = |\{i \in N | bR_i a \quad and \quad r(a, R_i) < 2\}|$ (number of agents who prefer b to a and does not second rank a)

Proposition 10. *Let $(a, R) \in GrM$. M is locally monotonic at (a, R) if and only if one of the following conditions hold*

i) $k^*(R) = 1$

ii) $k^*(R) = 2$ *and for all* $b \in A \setminus \{a\}$

$z + min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\} + t \leq y$.[3]

Proof. Assume M is locally monotonic at (a, R). By preceding lemmas $k^*(R) \leq 2$ and $[k^*(R) = 1$ and $|\{i \in N | r(a, R_i) = 1\}| \geq < \frac{n}{2} >]$ or $[k^*(R) = 2$ and a is the Condorcet winner at $R]$.

If $k^*(R) = 1$ $|\{i \in N | r(a, R_i) = 1\}| \geq < \frac{n}{2} >$ by preceding lemma.

If $k^*(R) = 2$ consider R'_b obtained from R by the following changes:

Moving b to just below a for every agent who top ranks a,

Moving a to the top position and b to the second position in $min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\}$ agents preferences who initially second rank a and rank b further below,

Moving b to the top position for any agent who prefers b to a and ranks a below third position,

Keeping everything else fixed.

Clearly each R'_b obtained is a monotonic transformation of R w.r.t. a and $k^*(R'_b) = 2$. Now M being locally monotonic at (a, R) implies $a \in M(R'_b)$ which in turn gives $z + min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\} + t \leq y$.

To see the conserve, note that

[3] Note that $z \leq y$ by definition and $x \leq \lceil \frac{n}{2} \rceil - 1$ when $k^*(R) = 2$.

If $k^*(R) = 1$ and a is the Condorcet winner, $|\{i \in N | r(a, R'_i) = 1\}| \geq l+1$ for any monotonic transformation of R' of R w.r.t. a, thus $M(R') = \{a\}$.

If $k^*(R) = 2$ and for all $b \in A \setminus \{a\}$ $z + min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\} + t \leq y$ we have $x + z + min\{y - z, l - x\} + t \leq x + y$ which in turn implies $a \in M(R')$ for any monotonic transformation of R' of R w.r.t. a by construction of R'_b in the preceding part. Because R'_b is the most advantageous profile for b in the sense that if b cannot prevent a from being chosen at R'_b it cannot prevent a from being chosen at any other monotonic transformation R' of R w.r.t. a. $\square$

Remark 6. The preceding proposition can be interpreted as follows:

If $min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\} = \lceil \frac{n}{2} \rceil - x - 1$: $z + y - z + t \leq y \Rightarrow t = 0$. Intuitively, if $x + (y - z)$ is less than the threshold $\lceil \frac{n}{2} \rceil - 1$ for some alternative b, we must have $t = 0$,i.e. b cannot be ranked above a for any agents that rank a strictly below second row.

If $min\{y - z, \lceil \frac{n}{2} \rceil - x - 1\} = \lceil \frac{n}{2} \rceil - x - 1$: $z + \lceil \frac{n}{2} \rceil - x - 1 + t \leq y \Rightarrow \lceil \frac{n}{2} \rceil - 1 \geq (x + (y - z)) - t$.Intuitively, if $x + (y - z)$ is greater than the threshold $\lceil \frac{n}{2} \rceil - 1$ for some alternative b, $(x + (y - z)) - t$ is also greater than the threshold.

Example 2. The following example shows that M being locally monotonic at (a, R) need not imply $|M(R)| = 1$ even if n is odd.

$$\begin{pmatrix} & R^1 & \\ 1 & 2 & 3 \\ a & b & c \\ b & a & d \\ c & c & a \\ d & d & b \end{pmatrix}$$

Note that M is locally monotonic at (a, R^1) but $M(R^1) = \{a, b\}$.

Let $(a, R) \in GrM$ and assume n is odd. The next example shows that if $k^*(R) = 2$ even if a is the Condorcet winner at R , M may not be locally monotonic at (a, R).

$$\begin{array}{c} R^2 \\ \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} a & b & c & d & a & a & b \\ e & e & e & e & c & d & a \\ b & a & a & a & b & c & c \\ c & c & b & b & d & b & d \\ d & d & d & c & e & e & e \end{pmatrix} \end{array}$$

Note that $M(R^2) = \{a, e\}$ and a is the Condorcet winner at R^2 but M is not locally monotonic at (a, R^2).

3.3 G-monotonicity

We will start by reminding some well-known definitions about implementation theory.[4]

Let $M = \prod_{i \in N} M_i$ denote a joint message space and $C(M)^N$ denote the set of complete preorders on M.

Given a joint message space M, $o : M \to A$ denotes an onto outcome function and O denotes the set of all such functions. $\succeq \in C(M)^N$ is said to be admissible if and only if, for any $m, m' \in M$, $[m \sim_i m'$ for some $i \in N]$ implies $[m \sim_i m'$for all $i \in N]$, i.e. indifference classes of each agent coincides.

Let $\mathbb{A}$ denote the set of all admissible profiles in $C(M)^N$. Given an admissible profile $\succeq \in \mathbb{A}$ $\rho(\succeq)$ denotes the partition of M into indifference classes induced by $\succeq$.

For a given outcome function $o \in O$, $p(o) = \{g^{-1}(a) | a \in A\}$,i.e. the partition of M induced by o. For any $o \in O$ let $\mathbb{A}(o) = \{\succeq \in \mathbb{A} | p(o) = \rho(\succeq)\}$. For any $R \in L(A)^N$ let $\succeq_R$ denote the complete pre-order on M induced by R. And let $U_i^*(m, \succeq)$ denote the strict upper contour set of m w.r.t. $\succeq$ for

[4]Koray, Pasin (2005) introduce an elegant formulation which enables one to treat a solution concept for normal form games as an SCR, we will borrow their formulation with some minor changes.

agent i; for any $m \in M$, $\succeq \in \mathbb{A}$ and $i \in N$.

Remark 7. Each $\succeq \in \mathbb{A}$ leads to a unique linear order profile on A. Conversely , $R \in L(A)^N$ leads to a unique complete pre-order profile $\succeq \in \mathbb{A}$.

When we fix the player set N and the joint message space $M = \prod_{i \in N} M_i$ a solution concept for normal form games can be viewed as an SCR so that the notions of g-monotonicity and self-g-monotonicity will apply to solution concepts as well. A solution concept for normal form games with a joint message space M, now becomes a mapping $\sigma : C(M)^N \to 2^M$. In this setting the joint strategy space is considered as the alternative set and the agent's rankings over the joint strategies as the preferences over the set of alternative set. A solution concept assigns a subset of the joint message space to each preference profile in the same way an SCR does. Let S denote the set of all solution concepts and let $Gr\sigma = \{(m, \succeq) \in M \times C(M)^N | m \in \sigma(\succeq)\}$,i.e. graph of σ. The notions of g-monotonicity and self-g-monotonicity now become applicable to solution concepts for normal form games. g-monotonicity of a solution concept will be denoted by G for convenience.

Now we are ready to give the definition of $self - G - monotonicity$ of a solution concept relative to a mechanism.

Definition. Let $\mu = (M, o)$ be a mechanism with joint message space M and outcome function o. Let σ be a solution concept and let $G : Gr\sigma \to (2^M)^N$. G is said to be a G-monotonicity for σ relative to μ if and only if

For all $\succeq, \succeq' \in \mathbb{A}(o)$ and for all $m \in \sigma(\succeq)$

$$[\forall i \in N \quad L_i(m, \succeq) \cup G_i(m, \succeq) \subset L_i(m, \succeq')] \Rightarrow \exists m' \in \sigma(\succeq') : \quad m' \sim m.$$

Minimal G-monotonicities for σ relative to μ are called self-G-monotonicities for σ relative to μ.

The following proposition summarizes the inheritance of monotonicity properties from solution concepts to SCRs they implement via a pre-specified mechanism.

Proposition 11. *Let σ be a solution concept. Given $F : L(A)^N \to A$ an SCR which is σ-implementable and $\mu = (M, o)$ a mechanism that σ implements F. Let $G : Gr\sigma \to (2^M)^N$ be a G-monotonicity of σ relative to μ. Now $g_i^\mu : GrF \to (2^A)^N$ defined as follows is a g-monotonicity of F:*

for any preference profile $R \in L(A)^N$, any $a \in F(R)$, any joint message $m \in \sigma(\succeq_R)$ s.t. $a \in o(m)$ and any agent $i \in N$

$g_i^\mu(a, R) = \{o(m')|m' \in G_i(m, \succeq_R)\}$. [5]

Proof. Let $(a, R) \in GrF$ and $R' \in L(A)^N$ s.t.

$$L_i(a, R) \cup g_i^\mu(a, R) \subset L_i(a, R') \quad \forall i \in N$$

$(a, R) \in GrF$ implies there exists $m \in M$ s.t. $a \in o(m)$. Now by the above inclusions and definition of g^μ

$$L_i(m, \succeq_R) \cup G_i(m, \succeq_R) \subset L_i(m, \succeq'_R)$$

and since σ is $G-monotonic$ via μ this implies there exists $m' \in \sigma(\succeq'_R)$ s.t. $m' \sim_{R'} m$. Note that $R \in L(A)^N$ thus $m' \sim_{R'} m$ implies $a \in o(m')$, i.e. $a \in F(R')$. $\square$

Kaya and Koray (2000) introduce the notion of universal monotonicity for solution concepts. We will mention some of their results before we proceed.[6]

First set $\mathbb{A}(O') = \bigcup_{o \in O'} \mathbb{A}(o)$ for any nonempty subset O' of O. Now, given a solution concept σ and a nonempty subset O' of O, we say that $O' - monotonic$ if and only if, for any $\succeq, \succeq' \in \mathbb{A}(O')$ with $\rho(\succeq) = \rho(\succeq')$ and $m \in \sigma(\succeq)$ there exists some $m' \in \sigma(\succeq')$ with $m' \sim m$ whenever $L_i(m, \succeq$

[5]Existence of m is guaranteed by F being σ-implementable via $\mu = (M, o)$.

[6]They also fix the joint message space so each outcome function defines a mechanism. For a more detailed discussion of universal monotonicity interested reader is referred to Kaya and Koray (2000).

$) \subset L_i(m, \succeq')$ for any $i \in N$. We refer to $O-monotonicity$ as universal monotonicity.

We now associate a class O_σ of outcome functions with each solution concept σ through $O_\sigma = \{o \in O| \quad |o(\sigma(\succeq))| = 1$ for each $\succeq \in \mathbb{A}(o)\}$. Given σ, this is the class of all outcome functions via which σ only implements singleton valued SCRs.[7]

Remark 8. A solution concept σ is universally monotonic [$O_\sigma - monotonic$] if and only if it is $G^\emptyset$ monotonic , where $G^\emptyset$ is a G-monotonicity function assigning emptyset for any agent, message, preference relation triple and for any outcome function $o \in O$ [$o \in O_\sigma$].

Remark 9. "Maskin monotonicity" is nothing but strongest form of G - monotonicity, thus G-monotonicity notion does not give any information when working with universally monotonic solution concepts. This relation is direct consequence of Maskin monotonicity , g-monotonicity relation.

Kaya and Koray note that Nash and strong Nash solution concepts are universally monotonic thus they implement only Maskin monotonic SCRs. The following lemma summarizes their results on undominated strategies solution concept and undominated Nash solution concept respectively.

Lemma 3. *1) Let σ stand for the undominated strategies solution concept. Let $O^*_\sigma = \{o \in O_\sigma | (M, o)$ is a bounded mechanism $\}$.[8] Then, σ is O^*_σ monotonic. σ is not $O_\sigma - monotonic$ (thus not universally monotonic).*

2) Undominated Nash equilibrium solution concept is not universally monotonic, nor O_σ-monotonic where σ stand for the undominated Nash equilibrium solution concept.

[7]Kaya and Koray characterize solution concepts which only implement maskin monotonic SCRs as universally monotonic solution concepts. They also characterize solution concepts which only implement dictatorial social choice functions as $O_\sigma - monotonic$ solution concepts when $|A| \geq 3$.

[8]A mechanism (M, o) is bounded if and only if $\forall i \in N$, $\succeq inC(M)^N$, $m_i \in M_i$: m_i is either undominated or $\exists \overline{m_i} \in M_i$ which dominates m_i and is undominated. Note that (M, o) is whenever M is finite.

In light of the preceding remark $G - monotonicity$ notion is only useful for further classifying monotonicity properties of non universally monotonic SCRs. By previous lemma both undominated strategies solution concept and undominated Nash equilibrium solution concept are non universally monotonic. We also characterized $G - monotonicity$ of Majoritarian Compromise, a non Maskin monotonic subgame perfect implementable SCR, so it is worthwhile to search for a way to define $G - monotonicity$ for extensive form mechanisms.

CHAPTER 4

SOCIAL CHOICE PROBLEMS WITH BIPOLAR PREFERENCE PROFILES

Given a subset, D, of $L(A)$, an SCF $F : D^N \to A$ is *dictatorial* if there exists a unique $i \in N$ such that $\forall R \in D^N$

$$F(R) = \{a \in A | \forall b \in A; \quad aR_ib\}$$

Given two preferences $R_i, R'_i \in L(A)$ the Manhattan distance between R_i and R'_i is defined as follows [1];

$$m(R_i, R'_i) = \sum_{a \in A} \frac{|r(a, R_i) - r(a, R'_i)|}{2}$$

Let $\begin{array}{c} \underline{P^1} \\ a_1 \\ \cdots \\ a_m \end{array}$ and $\begin{array}{c} \underline{P^2} \\ a_m \\ \cdots \\ a_1 \end{array}$; $r^1, r^2 \in \{0, 1, \ldots, \frac{m(m-1)}{2}\}$, define

[1] Koray,Kavlakoglu,Gurer(2008) contains a more detailed treatment of Manhattan metric.

$$D_{r^1,r^2}(P^1,P^2) = D_{r^1}(P^1) \bigcup D_{r^2}(P^2) \quad \text{where}$$

$$D_{r^1}(P^1) = \{P_i \in L(A) | m(P^1, P_i) \le r^1\}$$
$$D_{r^2}(P^2) = \{P_i \in L(A) | m(P^2, P_i) \le r^2\}$$

without loss of generality we will assume $r^1 \ge r^2$ through this section.

Definition. A domain $D \in L(A)$ satisfies unique seconds property if there exists $a, b \in A$ s.t. for all $R_i \in D$.

$$r(a, R_i) = 1 \Rightarrow r(a, R_i) = 0$$

The following simple lemma will be helpful in construction of the nondictatorial SCF in the following theorem.

Lemma 4. *For* $0 \le r^1 + r^2 \le m - 2 \quad D_{r^1,r^2}(P^1,P^2)$ *satisfies unique seconds property.*

Proof. Let $a = a_{r^1+1}$ and $b = a_1$ in the definition of unique seconds property. Note that $r^1 < m - r^2$ so there does not exist P_i in $D_{r^2}(P^2)$ with $r_1(P_i) = a_{r^1+1}$. Thus $P_i \in D_{r^1,r^2}(P^1,P^2)$ and $r_1(P_i) = a_{r^1+1}$ implies $P_i \in D_{r^1}(P^1)$ and it takes r^1 elementary transformations to take a_{r^1+1} to the top position starting from P^1, so for any such $P_i \quad r_2(P_i) = a_1$ □

Theorem 2. *For* $0 \le r^1 + r^2 \le m - 2 \quad D_{r^1,r^2}(P^1,P^2)$ *is a possibility domain.*

Proof. **Case 1:** $r^1 + r^2 = 0$ In this trivial case $D_{r^1,r^2}(P^1,P^2) = \{P^1, P^2\}$. Define $F : D_{r^1,r^2}(P^1,P^2)^N \to A$ as follows: $\forall R \in D_{r^1,r^2}(P^1,P^2)^N$

$$F(R) = \begin{cases} a_m, & \text{if } P_1 = P^2 \text{ and } \exists j \neq 1 \text{ with } P_j = P^2; \\ a_1, & \text{otherwise.} \end{cases}$$

Nondictoriality, *unanimity* and also *Maskin monotonicity* of F defined as above are easy to check.

Case 2: $0 < r^1 + r^2 \leq m - 2$ Define $F : D_{r^1,r^2}(P^1, P^2)^N \to A$ as follows: $\forall R \in D_{r^1,r^2}(P^1, P^2)^N$

$$F(R) = \begin{cases} \sigma(1,1,R), & \text{if } \sigma(1,1,R) \neq a_{r^1+1}; \\ a_{r^1+1}, & \text{if } \sigma(1,1,R) = a_{r^1+1} \text{ and } a_{r^1+1} R_2 a_1; \\ a_1, & \sigma(1,1,R) = a_{r^1+1} \text{ and } a_1 R_2 a_{r^1+1}. \end{cases}$$

Clearly F is unanimous.

F is non-dictatorial since : $R^* = \begin{pmatrix} a_{r^1+1} & a_1 & \dots & a_1 \\ a_1 & . & . & . \\ . & . & . & . \\ a_m & a_m & . & a_m \end{pmatrix} \in D_{r^1,r^2}(P^1, P^2)^N$

and $F(R^*) = a_1 \neq a_{r^1+1}$. (the fact that other agents cannot be dictator is easy to see)

F is $Maskin\ monotonic$:

If $F(R) = \sigma(1,1,R) \neq a_{r^1+1}$ $\quad \forall R' \in D_{r^1,r^2}(P^1, P^2)^N$ s.t. $R' \in MT(\sigma(1,1,R), R)$, $\sigma(1,1,P') \neq \sigma(1,1,R)$, thus $F(R') = \sigma(1,1,R)$

If $F(R) = a_{r^1+1}$ then $\sigma(1,1,R) = a_{r^1+1}$ and $a_{r^1+1} P_2 a_1$, clearly for any $R' \in D_{r^1,r^2}(P^1, P^2)^N, r_1(P_1)$ s.t $R' \in MT(a_{r^1+1}, R)$, $\sigma(1,1,R') = \sigma(1,1,R)$ and $a_{r^1+1} R'_2 a_1$, thus $F(R') = a_{r^1+1}$.

If $F(R) = a_1$ where $\sigma(1,1,R) = a_{r^1+1}$ and $a_1 R_2 a_{r^1+1}$, by preceding *Lemma* $\sigma(1,2,R) = a_1$. For any $P' \in D_{r^1,r^2}(P^1, P^2)^N, a_1)$ s.t. $R' \in MT(a_1, R)$, $[a_1$ is at top of $R'_1]$ or $[a_{r^1+1}$ is at the top of R'_1 and a_1 is in the second place$]$. (thinking of preferences as column vectors).

If $r_1(R'_1) = a_1$ $F(R') = a_1$ and if $[a_{r^1+1}$ is at the top of R'_1 and a_1 is in the second place$]$ $a_1 R_2 a_{r^1+1}$ implies $a_1 R'_2 a_{r^1+1}$ $F(R') = a_1$ by definition of F .

□

Theorem 3. *For $r^1 + r^2 = m - 1$, $D_{r^1,r^2}(P^1, P^2)$ is an impossibility domain.*

Proof. To simplify notation we will write D instead of $D_{r^1,r^2}(P^1, P^2)$ throughout this proof ; let $F : D^N \to A$ be a M.M. and *unanimous* SCF , $k \in \{1, \ldots, r^1\}$ and define P^k and B^k as follows:

$$T^k = \begin{array}{c} a_k \\ a_1 \\ a_2 \\ \ldots \\ a_m \end{array} \qquad B^k = \begin{array}{c} a_1 \\ a_2 \\ \ldots \\ a_{k+r^1-1} \\ a_k \\ \ldots \\ a_m \end{array}$$

Consider $P^k = [T^k \ldots T^k] \in D^N$; by *unanimity*, $F(P^k) = a_k$.

Now starting from P^k, column by column, take a_k as down as possible so that a_k is still chosen and the new profile is in the domain, keeping all the other orderings the same. Let the final ordering be P'^k; at least one column of P'^k must be T^k, otherwise a_1 would be chosen by *unanimity*. Now take any $i \in N$ s.t. $P_i'^k \neq T^k$ and $P_i'^k \neq B^k$, if such i exists. Let a_m be just below a_k in $P_i'^k$. Since $P_i'^k \neq B^k$, by switching a_m and a_k in $P_i'^k$, the new profile, say $P_i''^k \in D$. It is also clear that $T'^k = \begin{array}{c} a_1 \\ a_k \\ a_2 \\ \ldots \\ a_m \end{array} \in D$ then consider

$P''^k = [\ldots P_{j-1}{}'^k \quad P'^k \quad P_{j+1}{}'^k \ldots P_{i-1}{}'^k \quad P_i''^k \quad P_{i+1}{}'^k \ldots]$ in D^N .

By construction of P^k, $F(P''^k) \neq a_k$; by M.M. $F(P''^k)$ should be both a_1 and a_m contradiction.

Therefore, for any $i \in N$; $P_i'^k = T^k$ or B^k; w.l.o.g. let $P_1'^k, \ldots, P_l'^k = T^k$ and $P_{l+1}'^k, \ldots, P_N'^k = B^k$.

$l \geq 1$ is allready proven, assume to the contrary that $l \geq 2$:

Let $T''^k = \begin{matrix} a_2 \\ a_k \\ a_1 \\ a_3 \\ ... \\ a_m \end{matrix}$ and for $j \in \{1, .., l\}$ $P'^{k,j} = [P_1'^k \quad P_2'^k ... P_{j-1}'^k \quad T''^k \quad P_{l+1}'^k ... P_N'^k]$

We will preove by induction $F(P'^{k,2}) = a_k$:

Initial step is $F(P'^{k,l}) = a_k$. By M.M. $F(P'^{k,l}) = a_2$ or a_k.

Note that $\begin{matrix} a_k \\ a_2 \\ a_1 \\ a_3 \\ ... \\ a_m \end{matrix} \in D$ and consider first three rows of $l-1$ and l^{th} columns of $P'^{k,l}$ (considered as a $m \times n$ matrix); keeping other rankings fixed:

$$\begin{matrix} a_k & a_k \\ a_1 & a_1 \\ a_2 & a_2 \end{matrix} \longrightarrow^F a_k \Rightarrow \begin{matrix} a_k & a_k \\ a_1 & a_2 \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_k \Rightarrow \begin{matrix} a_1 & a_k \\ a_k & a_2 \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_k \text{ or } a_1$$

If $\begin{matrix} a_1 & a_k \\ a_k & a_2 \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_k$ quad then $\begin{matrix} a_1 & a_k \\ a_k & a_1 \\ a_2 & a_2 \end{matrix} \longrightarrow^F a_k$ which is a contradiction with definition of P'^k.

Hence $\begin{matrix} a_1 & a_k \\ a_k & a_2 \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_1 \Rightarrow \begin{matrix} a_1 & a_2 \\ a_k & a_k \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_1 \Rightarrow \begin{matrix} a_k & a_2 \\ a_1 & a_k \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_1$ or a_k

But we also have that $\begin{matrix} a_k & a_k \\ a_1 & a_1 \\ a_2 & a_2 \end{matrix} \longrightarrow^F a_k \Rightarrow \begin{matrix} a_k & a_2 \\ a_1 & a_k \\ a_2 & a_1 \end{matrix} \longrightarrow^F a_2$ or a_k

Therefore $\begin{array}{cc} a_k & a_2 \\ a_1 & a_k \\ a_2 & a_1 \end{array} \longrightarrow^F a_k$, i.e. $F(P'^{k,l}) = a_k$.

To show $F(P'^{k,l-1}) = a_k$ we use $F(P'^{k,l}) = a_k$ and the same methodology, applying this method $l-j+1$ times, we get $F(P'^{k,2}) = a_k$ as claimed.

To see $F(P'^{k,1}) = a_2$ note that $F(P'^k) = a_k$ implies $F(P'^k \quad ...P'^k \quad P''^k ... P''^k) = a_k \Rightarrow F(P'^k) = a_k$ or a_2 on the other hand

let $P''' = \begin{array}{c} a_2 \\ a_1 \\ ... \end{array} \in D$, $F(P'''^N) = a_2 \Rightarrow^{M.M.} F(P'^{k,1}) = a_2$ or a_1 thus

$F(P'^{k,1}) = a_2$

Now consider the first two columns and three rows of $P'^{k,2}$

$\begin{array}{cc} a_k & a_2 \\ a_1 & a_k \\ a_2 & a_1 \end{array} \longrightarrow^F a_k$ and $\begin{array}{cc} a_2 & a_2 \\ a_k & a_k \\ a_1 & a_1 \end{array} \longrightarrow^F a_2$.

$\begin{array}{cc} a_k & a_2 \\ a_1 & a_k \\ a_2 & a_1 \end{array} \longrightarrow^F a_k \Rightarrow \begin{array}{cc} a_k & a_2 \\ a_2 & a_k \\ a_1 & a_1 \end{array} \longrightarrow^F a_k \Rightarrow \begin{array}{cc} a_k & a_2 \\ a_2 & a_1 \\ a_1 & a_k \end{array} \longrightarrow^F a_k$ or a_1; but it cannot be a_k by definition of P'^k.

Now $\begin{array}{cc} a_k & a_2 \\ a_2 & a_1 \\ a_1 & a_k \end{array} \longrightarrow^F a_1 \Rightarrow \begin{array}{cc} a_2 & a_2 \\ a_k & a_1 \\ a_1 & a_k \end{array} \longrightarrow^F a_1$

however $\begin{array}{cc} a_2 & a_2 \\ a_k & a_k \\ a_1 & a_1 \end{array} \longrightarrow^F a_2 \Rightarrow \begin{array}{cc} a_2 & a_2 \\ a_k & a_1 \\ a_1 & a_k \end{array} \longrightarrow^F a_2$

the desired contradiction, thus $l = 1$,i.e. $\forall k \in \{1, 2, ..., r^1\} \quad \exists i_k \in N$ with $F(\underline{P}^k) = a_k$ where

$\underline{P}^k_{i_k} = T^k$ and $\underline{P}^k_j = B^k \; \forall j \neq i_k$.

For $k \in \{m - r^2 + 1, ..., m\}$ same proof applies replacing a_1 with a_m; a_2

with a_{m-1} . T^k , B^k in that case respectively are: $\begin{array}{c} a_k \\ a_m \\ a_{m-1} \\ ... \\ a_1 \end{array}$ and $\begin{array}{c} a_m \\ a_{m-1} \\ ... \\ a_k \\ ... \\ a_1 \end{array}$ a_k is in $m-k+1^{th}$ place in B^k.

For $k = r^1 + 1 = n - r^2$; first note that $P^1 = \begin{array}{c} a_k \\ a_1 \\ ... \\ a_m \end{array}$, $P^2 = \begin{array}{c} a_1 \\ ... \\ a_m \\ a_k \end{array}$, $P^3 = \begin{array}{c} a_1 \\ ... \\ a_k \\ a_m \end{array}$,

$P^4 = \begin{array}{c} a_k \\ a_m \\ ... \\ a_1 \end{array}$, $P^5 = \begin{array}{c} a_m \\ ... \\ a_1 \\ a_k \end{array}$, $P^6 = \begin{array}{c} a_m \\ ... \\ ... \\ a_1 \end{array}$ $\in D$.

We have shown that $\exists i_1 \in N$ s. t. whenever i_1 top ranks a_1, a_1 gets chosen under F; w.l.o.g. let $i_1 = 1$.

Consider $P = \begin{array}{ccccc} a_1 & a_m & ... & ... & a_m \\ a_k & ... & ... & ... & ... \\ ... & a_1 & ... & ... & a_1 \\ a_m & a_k & ... & ... & a_k \end{array}$; $F(P) = a_1$ by agent 1 being the dictator for alternative a_1, furthermore it is easy to see that dictator for alternative a_m (cause all the other agents top rank a_m and still a_m is not chosen); these two together gives agent 1 is dictator for a_j for all $j \in \{1, ..., r^1, m - r^2 + 1, ..., m\}$.

Only thing remaining to show is agent 1 is dictator for a_{r^1+1},let $k = r^1 + 1$ this time and assume to the contrary that $\exists P^* \in D$ s.t. $r_1(P_1^*) = a_k$ but $F(P^*) \neq a_k$

Consider $P' = \begin{array}{ccc} a_k & a_m & ... \\ a_1 & ... & ... \\ a_2 & ... & ... \\ ... & a_1 & ... \\ a_m & a_k & ... \end{array} \in D \quad F(P') \neq a_k$ otherwise it would imply $F(P^*) \neq a_k$.

Passing from P to P' by M.M. $F(P') = a_k$ or a_1, so $F(P') = a_1$. Obtain P'' form P' by moving a_k just above a_1 in all the agents preferences except agent 1 and keep agent 1's preference the same; $F(P'') = a_k$ or a_1.

Assume to the contrary that $F(P'') = a_k$ and define P''' as follows $P_1''' = \begin{array}{c} a_k \\ a_m \\ ... \\ a_1 \end{array}$

and $P_i''' = P_i''$ for $i \neq 1$

$F(P'') = a_k \Rightarrow F(P''') = a_k$

Let $P'''' = \begin{array}{cccc} a_k & a_m & ... & a_m \\ a_m & ... & ... & ... \\ ... & a_1 & ... & a_1 \\ a_1 & a_k & ... & a_k \end{array}$

$F(P'''') = a_1$ or a_k by M.M.; neither is possible since: $F(P'''') = a_1 \Rightarrow F(P^5....P^5) = a_1$ contradicting *unanimity*

$F(P'''') = a_k \Rightarrow F(P') = a_k \neq a_1$ contradiction.

Thus agent 1 is dictator for a_{r^1+1} as well, implying F is dictatorial , i.e. D is an impossibility domain. $\square$

Definition. A domain D is *top − bottom rich* iff $\quad \forall a \in A \quad \exists P, P' \in D$ with

$$r(a, P) = 1 \quad and \quad r(a, P') = m$$

Lemma 5. *For* $r^1 + r^2 = m - 1 \quad D_{r^1,r^2}(P^1, P^2)$ *is top − bottom rich.*

Proof. Pick $a_i \in A$

If $i \leq r^1 + 1 \quad \exists P^i \in D_{r^1}(P^1)$ with $r_1(P^i) = a_i$ and if $i \geq m - r^2 \quad \exists P^i \in D_{r^2}(P^2) \quad r_2(P^i) = a_i$, noting $r^1 + r^2 = m - 1$ implies $r^1 = m - r^2$, we get $\exists P^i \in D_{r^1,r^2}(P^1, P^2)$ with $r_1(P^i) = a_i \ \forall a_i \in A$.

Similarly if $i \geq m - r^2 \quad \exists P^i \in D_{r^2}(P^2)$ with $r_m(P^i) = a_i$ and if $i \leq r^1 + 1$ $\exists P^i \in D_{r^1}(P^1)$ with $r_m(P^i) = a_i$, we get $\forall a_i \in A \quad \exists P^i \in D_{r^1,r^2}(P^1, P^2)$ with $r_m(P^i) = a_i \ \forall a_i \in A$. □

Lemma 6. *Let $D \subset L(A)$ be a top$-$bottom rich domain and $D \subset K \subset L(A)$. If D^N is a domain of impossibility then so also is K^N.*

Proof. Let $F : K^N \to A$ be a *Maskin monotonic* and *unanimous* SCF. Then restriction of F to D^N, $F|_{D^N}$, is still M.M. and *unanimous*; hence $F|_{D^N}$ is *dictatorial*, w.l.o.g. let 1^{st} agent be the dictator. Given $a \in A$, let $P, P' \in D$ s.t. $r_1(P) = r_m(P') = a$. $F(P, P', ..., P') = a$ and when we pass to any profile $S \in K^N$ with $r_1(S_1) = a$, clearly S is a improvement for alternative a, thus $F(S) = a$. □

The following corollary summarizes our results on "bipolar" societies :

Corollary. $D_{r^1,r^2}(P^1, P^2) \in L(A)$ *is an impossibility domain if and only if* $r^1 + r^2 \geq m - 1$.

CHAPTER 5

INDIFFERENCES AND DICTATORIALITY

The following simple proposition points out that *Maskin monotonicity* is too demanding on $C(A)^N$ because it turns out that only *Maskin monotonic* SCFs defined on $C(A)^N$ are the family of constant SCFs.

Proposition 12. *Let $F : C(A)^N \rightarrow A$ be an SCF. F satisfies Maskin monotonicity only if F is constant.*

Proof. Assume F is not constant , i.e. there exists $a, b \in A$ $a \neq b$ and $R, R' \in C(A)^N$ s.t. $F(R) = a$ and $F(R') = b$. Consider the preference profile $R'' \in C(A)^N$ s.t. a and b are both top ranked in any agents preference,i.e. $\forall i \in N$ $\forall c \in A$ aR_ic and aR_ic. Maskin monotonicity implies $a, b \in F(R'')$ which contradicts with F being an SCF since $a \neq b$. □

The following definition of monotonicity requires preservation both lower counter sets and strictly lower counter sets for all agents in order to guarantee that the chosen alternative does not change.

Given an alternative $a \in A$ and a preference profile $R \in C(A)^N$, let $MT^*(a, R) = \{R' \in C(A)^N | \quad L_i(a, R) \subseteq L(a, R'_i) \quad and \quad L^*_i(a, R) \subseteq L^*_i(a, R')\}$.

Definition. An SCF $F : C(A)^N \rightarrow A$ satisfies *monotonicity* if and only if

$\forall\ R, R' \in C(A)$, $\forall i \in N$

$F(R) = a$, $R' \in MT^*(a, R)$ implies $F(R') = a$.

Note that *monotonicity* is equivalent to M.M. when indifferences are not allowed.

Definition. An SCF $F : C(A)^N \to A$ satisfies *weak dictatoriality* if and only if

$\exists i \in N$ s.t. $\forall R \in C(A)^N$ $F(R) \in I_1(R_i)$

where $I_1(R_i)$ denotes the top indifference class of agent i.

Weak dictatoriality only requires that at each preference profile the chosen alternative must be from the top indifference class of the dictator. If we also add the requirement that the choice at any profile only depends on the preference of the dictator we get the following definition.

Definition. An SCF $F : C(A)^N \to A$ satisfies *dictatoriality* iff F satisfies *weak dictatoriality* (say agent i is the dictator) and $\forall R, R' \in C(A)^N$

$$R_i = R'_i \quad \Rightarrow \quad F(R) = F(R')$$

Given an alternative $a \in A$, a preference profile $R \in C(A)^N$ we will denote the top indefferece class of agent i at preference profile R by $I_1(R_i)$,i.e. $I_1(R_i) = \{a \in A | aR_i b \forall b \in A\}$

On top of requirements of *dictatoriality* adding the intuitive condition that when the dictator has a finer top indifference class ,containing the previously chosen alternative , say a, a is still chosen, we get the strongest dictatoriality that we will consider.

Definition. An SCF $F : C(A)^N \to A$ satisfies *strict dictatoriality* iff F satisfies *dictatoriality* (say agent i is the dictator) and $\forall R, R' \in C(A)^N$

$$F(R) = a \quad and \quad a \in I_1(R'_i) \subseteq I_1(R_i) \quad \Rightarrow \quad F(R) = F(R')$$

Theorem 4. *Let $F : C(A)^N \to A$ be a monotonic and unanmious SCF, then F satisfies weak dictatoriality.*

Proof. Let F be a *monotonic* and *unanimous* SCF. Note that F restricted to $L(A)^N$ is $Maskin$ *monotonic* and *unanimous* thus there exists an unique agent $i \in N$ s.t. $\forall R \in L(A)^N$, $F(R)$ is the top ranked alternative in R_i by the well known Mueller-Satthertwaite Theorem. Without loss of generality assume agent 1 is the dictator.

Assume to the contrary that F is not $weakly - dictatorial$, i.e. $\exists R \in C(A)^N$ s.t. $F(R) \notin I_1(R_1)$: Let $R = \begin{matrix} \{x, y, z\} & ... & ... & ... \\ \cdots & ... & ... & ... \\ \{w, ...\} & ... & ... & ... \\ \cdots & ... & ... & ... \end{matrix}$ and $F(R) = w$. Obtain the preference profile R' from R by breaking indifference classes which does not contain w, for all agents. Note that $F(R') = w$ by monotonicity.

Now consider $R'' \in L(A)^N$ s.t. x is top ranked in agent 1's preference and for all other agents x is bottom ranked . Note that $F(R'') = x$ and passing from R'' to R' monotonicity implies $F(R') = x \neq w$, the desired contradiction.

□

The following example shows that *monotonicity* and *unanimity* does not imply *dictatoriality* (thus *str. dictatoriality*)

Example 3. Let $A = \{a, b, c\}$ and $N = \{1, 2\}$. Define $F : C(A)^N \to A$ as follows: F is a weak-dictatorial function of agent 1 and for $\forall R \in C(A)^N$ s.t. $I_1(R_1) = \{a, b\}$ if bP_2a $F(R) = b$ and $F(R) = a$ if aP_2b. For any other preference profile define F suitably. Note that F satisfies *monotonicity* and *unanimity* but it is not *dictatorial*.

Following example shows that *dictatoriality* (thus *weak dictatoriality*) does not imply *monotonicity*.

Example 4. Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$ .Define $F : C(A)^N \to A$ as follows: F is a dictatorial function of agent 1 and $\forall R \in C(A)^N$; $F(R) = a$ if $I_1(R_1) = \{a, b\}$ or $I_1(R_1) = \{a, c\}$, $F(R) = c$ if $I_1(R_1) = \{b, c\}$ and $F(R) = b$

if $I_1(R_1) = \{a, b, c\}$. Note that passing from a preference profile where agent 1 ranks b and c as the most desirable alternatives to a preference profile where agent 1 is indifferent between a,b and c monotonicity implies c continues to get chosen , thus F does not satisfy monotonicity.

Definition. Let $F : C(A)^N \to A$ be an SCF. We say that F is $tie-breaker$ iff $\exists P \in L(A)$ s.t. $\forall R \in C(A)^N$

$$F(R) = F(R^P)$$

where $R^P \in L(A)^N$ is the preference profile obtained from R by breaking all the indifferences according to P.

The following simple lemma will be useful in characterization of *str. dictatorial* SCFs.

Lemma 7. *Any str. dictatorial SCF is a* $tie-breaker$.

Proof. Let $F : C(A)^N \to A$ be a *str. dictatorial* SCF. Without loss of generality assume agent 1 is the dictator. Let $R^1, R^2, ..., R^m \in C(A)^N$ be preference profiles s.t. $I_1(R_1^1) = A$, $I_1(R_1^2) = I_1(R_1^1) \setminus F(R^1)$, ... ,$I_1(R_1^m) = I_1(R_1^{m-1}) \setminus F(R^{m-1})$.
<u>Claim:</u> F is a $tie-breaker$ with $P = (F(R^1), F(R^2), ..., F(R^m) \in L(A)$.
<u>Proof of The Claim:</u>Assume it is not the case and there exists $R \in C(A)^N$ s.t. $a = F(R) \neq F(R^P) = b$. Let $b = F(R^k)$ for some $k \in \{1, 2, ..., m\}$, note that $F(R^P) = F(R^k)$ implies $I_1(R_1) \subset I_1(R_1^k)$. Now passing from R^k to R , we get $F(R^k) = F(R)$ the desired contradiction. □

Theorem 5. *An SCF* $F : C(A)^N \to A$ *is monotonic, unanimous and* $tie-breaker$ *iff* F *is strictly dictatorial.*

Proof. Assume F is *strictly dictatorial* and w.l.o.g. assume agent 1 is the dictator. Let $R, R' \in C(A)^N$ s.t. $F(R) = a$, $R' \in MT^*(a, R)$. $F(R) = a$

implies $a \in I_1(R_1)$ and $R' \in MT^*(a, R)$ implies $I_1(R_1) \subset I_1(R'_1)$ and $a \in I_1(R'_1)$. Now $F(R') = a$ since F is a *strictly dictatorial* SCF with agent 1 as the dictator. Thus F satisfies *monotonicity*.

By the previous lemma F is $tie-breaker$ and *unanimity* of F is obvious.

Assume F is *monotonic*, *unanimous* and $tie - breaker$.

Unanimity and *monotonicity* together imply F is a $weak - dictatorial$. W.l.o.g. assume agent 1 is the dictator and let $P \in L(A)^N$ be the preference according to which F is a $tie - breaker$.

Let $R, R' \in C(A)^N$ s.t. $R_1 = R_2$, now $R_1^P = R_1'^P$, $F(R) = R^P$ and $F(R') = R'^P$ implies $F(R) = F(R')$ by $weak - dictatoriality$ of F. Thus F is dictatorial.

Let $R, R' \in C(A)^N$ s.t. $F(R) = a$ and $a \in I_1(R'_i) \subseteq I_1(R_i)$. Note that $F(R) = a$ implies $F(R^p) = a$. Now $F(R^p) = a$ and $a \in I_1(R'_i) \subseteq I_1(R_i)$ gives $F(R'^p) = a$, thus F is $strictly - dictatorial$. $\square$

CHAPTER 6

CONCLUSION

Concerning monotonicity properties of non Maskin monotonic SCRs, we defined notions of $g-monotonicity$ and *monotonicity region*. Although we did not have enough time to reach the results we aimed for in this chapter concerning implementability, the results are still promising. We investigated monotonicity properties of standard scoring rules and gave a comparison of scoring rules in terms of monotonicities they satisfy. We also characterized local monotonicity properties of Majoritarian Compromise SCR. We showed that Majoritarian Compromise SCR's local monotonicity properties are closely related to generalized Condorcet type conditions. In the final section of this chapter we show that $g-monotonicity$ of an SCR is inherited from solution concept which implements it ,via the mechanisms implementation takes place. We did not have enough time to reach characterization results in implementation that we aimed for, but we strongly believe that $G-monotonicity$ (coupled with some other monotonicity notion) is promising in characterization of self-monotonicities of non universally monotonic solution concepts. So, we are planning to focus our future research on that subject.

In chapter 4, we show that a "bipolar" domain D, is a domain of impossibility if and only if sum of the two radiuses is greater or equal to $|A|-1$. This result was conjectured by Koray,S., Kavlakoglu,S., Gurer,E.,(2008) in their conclusion. The proof given, although a bit long, mostly depends on the careful use of critical profiles (w.r.t. a restricted domain) and ones again shows usefulness of critical profiles notion while working with Maskin monotonic SCRs. The results in this part can be generalized in two ways. Firstly considering polarized societies instead of bipolar ones may result in a better understanding of historically standard preference domains in a more generalized fashion. Secondly following the idea of Erol (2009), instead of using Manhattan metric; defining some other metrics and working with domains clustered according to them will be useful generalizations of our results.

Finally, in the last chapter concerning the impossibility on $C(A)^N$; we show that Maskin monotonicity is too demanding when we restrict our attention to SCFs. We give a natural extension of Maskin monotonicity, define three kinds of dictatorialities and investigate the relation between our monotonicity definition and these dictatorilaties. We give a characterization of *strictly dictatorial* SCFs as the SCFs satisfying *unanimity*, *monotonicity* and $tie-breaking$. Our results in this section strengthen the warning for the researchers planning to work on complete pre-orders that indifferences may change the results dramatically.

BIBLIOGRAPHY

Maskin,E.(1977): Nash Equilibrium and Welfare Optimality,mimeo, M.I.T.

Koray,S.,Adali,A.,Erol,S.,Ordulu,N.(2001): A Simple Proof of Muller Satterthwaite Theorem, mimeo, Bilkent University.

Jackson, Matthew O.,:(2001) A crash course in implementation theory, Social Choice and Welfare.

Koray,Semih,(2002): A Classification of Maskin-Monotonic Social Choice Rules via the Notion of Self Monotonicity, mimeo, Bilkent University.

Koray,S.,Pasin,P.(2005): Self Monotonicity For Nash Equilibrium Concept, mimemo, Bilkent University.

Dogan, Battal(2007):Explorations On Monotonicity In Social Choice Theory,mimeo, Bilkent University.

Dogan, Battal(2008):Maskin Monotonic Scoring Rules, mimeo, Bilkent University.

Koray,S., Kavlakoglu,S., Gurer,E.,(2008): Do Impossibility Results Survive in Historically Standard Domains?, mimeo, Bilkent University.

Erol, Selman,(2009): Essays In Social Choice Theory,mimeo, Bilkent University.

Barbera, Salvador,(29/2007): Indifferences and Domain Restrictions, Analyse & Kritik.

Kaya, A. and Koray S. ,(2000): Characterization of Solution Concepts which only Implement Maskin-monotonic Social Choice Rules, mimeo, Bilkent University.

Sertel, M. R. and Yilmaz, B. (1998): The Majoritarian Compromise is Majoritarian-Optimal and Subgame-Perfect Implementable.