RAMANUJAN'S CONGRUENCES FOR THE PARTITION FUNCTION

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By

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ABSTRACT

RAMANUJAN'S CONGRUENCES FOR THE PARTITION FUNCTION

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In this thesis, we study Ramanujan's congruences for the partition function and some of their combinatorial interpretations. Our main tools are from the theory of theta function.

Keywords: Ramanujan's congruences, partition, rank, crank, theta functions .

ÖZET

RAMANUJAN'IN BÖLÜŞÜM FONKSİYONU İÇİN VERMİS OLDUĞU DENKLEMLER

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Bu tez'de Ramanujan'ın bölüşüm fonksiyonu için vermis olduğu denklemleri ve bunların kombinatoriksel izahlarını çalışacağız. Yöntemlerimiz ağırlıklı olarak teta fonksiyonları teorisininden olacak.

Anahtar sözcükler: Ramanujan'ın denklemleri, bölüşüm, rank, crank, teta fonksiyonları .

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Ramanujan's Congruences for the Partition Function

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Chapter 1

Introduction

A partition π of a natural number n is a finite sequence of non increasing positive integers $\pi = (\lambda_1, \lambda_2, ..., \lambda_k)$ that sums to n. The number of all partitions of n is denoted by $p(n)$. For convenience, we assume that $p(0) = 1$ with the empty partition as being the only partition of 0. There are 5 partition of 4, namely,

$$\begin{aligned}
&4\\
&3+1\\
&2+2\\
&2+1+1\\
&1+1+1+1.
\end{aligned}$$

S. Ramanujan's celebrated congruences for the partition function are

$$p(5n+4) \equiv 0 \pmod{5}, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

In their most general form they can be stated as follows

$$p(5^n m + l_n) \equiv 0 \pmod{5^n}, \tag{1.4}$$

$$p(7^n m + k_n) \equiv 0 \pmod{7^{[\frac{n+2}{2}]}}, \tag{1.5}$$

$$p(11^n m + t_n) \equiv 0 \pmod{11^n}, \tag{1.6}$$

where for $p = 5, 7$ and 11, the numbers l_n, k_n and t_n are the least positive solutions of $24x \equiv 1 \pmod{p^n}$ respectively.

Ramanujan [16] proved (1.1) and (1.2) in 1919. Later in 1921 [17], he gave a proof of (1.3) by employing different methods. Ramanujan also sketched proofs of (1.4) and (1.5) with $n = 2$. In 1938, G. N. Watson [18] proved (1.4) and (1.5). Ramanujan's original formulation of (1.5) was in fact incorrect. The congruence (1.6) has remained unproven until A.O.L. Atkin [4] gave a proof in 1967. The works of Newman [14], and of Atkin and J. N. O'Brien [5], and of Atkin and H. P. F. Swinnerton-Dyer [7] have shown that there are many other congruences for the partition function. For example, Atkin and O'Brien [5] found that

$$p(594 \cdot 13n + 111247) \equiv 0 \pmod{13}.$$

In 2000, K. Ono [15] proved that if $m \geq 5$ is a prime, then there are infinitely many integers a and b such that $p(an + b) \equiv 0 \pmod{m}$ for all n. Ramanujan stated that other than those he found there seemed to be no other congruence in the form $p(an + b) \equiv 0 \pmod{n}$ with n prime. His guess was proven to be correct by M. Boylan and Ahlgren [1] in 2003.

In 1944, F. J. Dyson [9] gave the first combinatorial interpretation of Ramanujan's partition congruences for the modulus 5 and 7. He defined the rank of a partition π to be the biggest part of π minus the number of parts in π and conjectured that this rank divides partitions of $5n + 4$ and $7n + 5$ into 5 and 7 equinumerous classes. Following his notation let $N(m, n)$ be the number of partitions of n with rank m and $N(m, t, n)$ be the number of partitions of n with rank m modulo t. Dyson conjectured that

$$N(i,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leq i \leq 4,$$

and

$$N(i,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \leq i \leq 6.$$

He also conjectured another partition statistic which he called "crank" that would divide partitions of $11n+6$ into 11 equinumerous classes. His conjectures about the rank was proven by Atkin and Swinnerton-Dyer in 1958 [6]. The existence of a "crank" was first proved by F. G. Garvan[10] in terms of vector partitions. In his notation a vector partition is a triplet (π_1, π_2, π_3) where π_1 is a partition with distinct parts while π_2 and π_3 are arbitrary partitions. The set of all vector partitions is denoted by V. For the vector partition $\pi = (\pi_1, \pi_2, \pi_3)$, we define the number of parts of π_1 to be $\#(\pi_1)$, sum of parts of π to be $s(\pi)$, the weight of π to be $\omega(\pi) = (-1)^{\#(\pi_1)}$, and the crank of π to be $r(\pi) = \#(\pi_2) - \#(\pi_3)$. If $s(\pi) = n$, then we say that π is a vector partition of n. Let $N_v(m,n)$ be the number of vector partitions of n with crank m counted according to their weight that is

$$N_v(m,n) = \sum_{\pi \in V,\, s(\pi)=n,\, r(\pi)=m} \omega(\pi).$$

Finally, let $N_v(m,t,n)$ be the number of vector partitions of n with crank m modulo t counted according to their weight that is

$$N_v(m,t,n) = \sum_{l=-\infty}^{\infty} N_v(lt+m,n) = \sum_{\pi \in V,\, s(\pi)=n,\, r(\pi)\equiv m \pmod t} \omega(\pi).$$

Garvan proved that [10]

$$N_v(i,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le i \le 4, \tag{1.7}$$

$$N_v(i,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le i \le 6, \tag{1.8}$$

and

$$N_v(i,11,11n+6) = \frac{p(11n+6)}{11}, \quad 0 \le i \le 10. \tag{1.9}$$

Later in the same year G. E. Andrews and Garvan [3] discovered another "crank" in terms of regular partitions. The methods of Garvan and of Atkin and Swinnerton-Dyer were purely analytical but in 2003 Garvan, D. Kim and D. Stanton [12] found yet another crank along with explicit bijections between equinumerous classes. Several identities stated in Ramanujan's "Lost Notebook" were very influential in Garvan's discovery of crank. These identities and their relation to the works of Atkin and Swinnerton-Dyer on Dyson's rank together with further contributions of Ramanujan to partition congruences with numerous references can be found in [11].

The rest of this theses is organized as follows. In the next chapter, we collect the necessary theta function identities which we employ in our proofs. Then, we present Ramanujan's own proofs for his partition congruences for modulus 5 and 7 along with Winquist's [19] proof of Ramanujan's partition congruence for modulus 11. In the last chapter, we give Garvan's proof of (1.7)—(1.9).

Chapter 2

Ramanujan's Partition Congruences

2.1 Preliminary Results

We start this chapter by Euler's generating function identity for the partition function. We employ the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$

here and throughout the manuscript we assume that q is a complex number with $|q|<1$.

Theorem 2.1. *(Euler) [2, pp. 4–5] The generating function of $p(n)$ is*

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} p(n)q^n. \tag{2.1}$$

Proof. Let us first prove that

$$\prod_{i=1}^{m}(1-q^i)^{-1} = \sum_{n=0}^{\infty} p_m(n)q^n,$$

where $p_m(n)$ is the number of partitions of n with largest part at most m. we expand each term in the product by using geometric series and conclude that

$$\begin{aligned}\prod_{i=1}^{m}(1-q^i)^{-1} =& \prod_{i=1}^{m}(1+q^i+q^{2i}+q^{3i}+...)\\ =&(1+q^1+q^{2.1}+q^{3.1}+...)\times\\ &(1+q^2+q^{2.2}+q^{3.2}+...)\times ....\\ &(1+q^m+q^{2.m}+q^{3.m}+...)\\ =& \sum_{k_1\geq 0, k_2\geq 0,...} q^{k_1\cdot 1+k_2\cdot 2+k_3\cdot 3+\cdots+k_m\cdot m}.\end{aligned}$$

Observe that the exponent of q is just the partition where the part j, $1 \leq j \leq m$, repeated k_j times unless $k_j = 0$ in which case j does not appear as a part. Given a fixed integer n, q^n will be present in the above sum once for each such partition of n. Hence, the coefficient of q^n is exactly the number of partitions of n with largest part at most m. This argument is justified since we are multiplying finitely many absolutely convergent series. Clearly,

$$\sum_{j=0}^{m} p(j)q^j \leq \sum_{j=0}^{\infty} p_m(j)q^j = \prod_{i=1}^{m}(1-q^i)^{-1} \leq \prod_{i=1}^{\infty}(1-q^i)^{-1} < \infty.$$

Therefore, $\sum_{j=0}^{m} p(j)q^j$ is a bounded increasing sequence hence it converges. On the other hand, we also have

$$\sum_{j=0}^{\infty} p(j)q^j \geq \sum_{j=0}^{\infty} p_m(j)q^j = \prod_{i=1}^{m}(1-q^i)^{-1} \to \prod_{i=1}^{\infty}(1-q^i)^{-1} \quad \text{as} \quad m \to \infty.$$

Hence,

$$\sum_{j=0}^{\infty} p(j)q^j = \prod_{i=1}^{\infty}(1-q^i)^{-1}.$$

□

Next, we derive appropriate representations for the infinite products $(q;q)_\infty$, $(q;q)_\infty^3$, and $(q;q)_\infty^{10}$ that we will employ in our proofs of (1.1)—(1.3). We start with Jacobi's Triple Product Identity. The proof we gave dates back to Gauss.

Theorem 2.2. *For all z with $z \neq 0$, we have*

$$\sum_{n=-\infty}^{\infty} q^{n(n-1)/2} z^n = (-z;q)_\infty(-q/z;q)_\infty(q;q)_\infty. \tag{2.2}$$

Proof. Let

$$F(z) := (-z;q)_\infty(-q/z;q)_\infty(q;q)_\infty. \tag{2.3}$$

By replacing z by zq, we find that

$$\begin{aligned} F(zq) &= (-zq;q)_\infty(-1/z;q)_\infty(q;q)_\infty \\ &= \frac{(-z;q)_\infty}{1+z}(1+1/z)(-q/z;q)_\infty(q;q)_\infty \\ &= z^{-1}F(z). \end{aligned}$$

Therefore, we have $F(z) = zF(qz)$.
Now let us consider Laurent series of $F(z)$

$$F(z) = \sum_{n=-\infty}^{\infty} a_n(q) z^n.$$

Since $F(z) = zF(qz)$, we find that

$$\begin{aligned} F(z) = \sum_{n=-\infty}^{\infty} a_n(q) z^n &= z \sum_{n=-\infty}^{\infty} a_n(q)(qz)^n \\ &= \sum_{n=-\infty}^{\infty} a_n(q) q^n z^{n+1} = \sum_{n=-\infty}^{\infty} a_{n-1}(q) q^{n-1} z^n. \end{aligned}$$

By comparing the coefficients of z^n, we obtain the recurrence relation

$$a_n(q) = q^{n-1}a_{n-1}(q),$$

from which we deduce that $a_n(q) = a_0(q)q^{n(n-1)/2}$. By using this in the Laurent series representation of $F(z)$, we find that

$$F(z) = a_0(q) \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} z^n. \tag{2.4}$$

Therefore, it remains to prove that $a_0(q) = 1$. Let

$$G(z) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} z^n. \tag{2.5}$$

Therefore,

$$a_0(q) = \frac{F(z)}{G(z)}. \tag{2.6}$$

Since $F(-1) = 0$, we observe from (2.4) (or by direct calculation) that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n-1)/2} = q^{\frac{1}{8}} \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n-1)^2/8} = 0. \tag{2.7}$$

By definition (2.5) and by (2.7) with q replaced by q^4, we find that

$$\begin{aligned}
G(iq^{1/2}) &= \sum_{n=-\infty}^{\infty} i^n q^{n^2/2} && (2.8)\\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2} - i \sum_{n=-\infty}^{\infty} (-1)^n q^{(2n-1)^2/2} && (2.9)\\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2}. && (2.10)
\end{aligned}$$

From (2.3), we have

$$\begin{aligned}
F(iq^{1/2}) &= (-iq^{1/2}; q)_\infty (iq; q)_\infty (q; q)_\infty && (2.11)\\
&= (-q, q^2)_\infty (q; q)_\infty && (2.12)\\
&= (-q; q^2)_\infty (q; q^2)_\infty (q^2; q^2)_\infty && (2.13)\\
&= (q^2; q^4)_\infty (q^2; q^2)_\infty && (2.14)\\
&= (q^2; q^4)_\infty^2 (q^4; q^4)_\infty. && (2.15)
\end{aligned}$$

From definitions (2.3) and (2.5), we easily find that

$$F(-q^{1/2}) = (q^{1/2};q)_\infty^2 (q;q)_\infty \tag{2.16}$$

and

$$G(-q^{1/2}) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/2}. \tag{2.17}$$

By comparing these last two equations with (2.10) and (2.15), we conclude from (2.6) that

$$a_0(q) = a_0(q^4). \tag{2.18}$$

Therefore, by iteration, we find

$$a_0(q) = a_0(q^4) = a_0(q^{16}) = \cdots \tag{2.19}$$

From this we conclude that $a_0(q) = 1$ since $q^n \to 0$ as $n \to 0$. □

Theorem 2.3. *(Euler's Pentagonal Number Theorem) [8, Theorem 1.3.5]*

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{2.20}$$

Proof. By employing Jacobi Triple Product Identity with q and z replaced by q^3 and $-q$ respectively, we conclude that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} = (q;q^3)_\infty (q^2;q^3)_\infty (q^3;q^3)_\infty = (q;q)_\infty.$$

□

Theorem 2.4. *(Jacobi) [8, Theorem 1.3.9]*

$$\sum_{n=0}^{\infty} (-1)^n (2n+1) q^{n(n+1)/2} = (q;q)_\infty^3. \tag{2.21}$$

Proof. In (2.2), we replace z by $-zq$ and divide both sides of the resulting equation by $(1-1/z)$, we find that

$$\frac{z}{z-1}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}z^n = (zq;q)_\infty(q/z;q)_\infty(q;q)_\infty.$$

For the left hand side, we have that

$$\begin{aligned}
&\frac{z}{z-1}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}z^n \\
&= \frac{z}{z-1}\{\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}z^n + \sum_{n=-\infty}^{-1}(-1)^n q^{n(n+1)/2}z^n\} \\
&= \frac{z}{z-1}\{\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}z^n - \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}z^{-n-1}\} \\
&= \frac{z}{z-1}\{\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}(\frac{z^{2n+1}-1}{z^{n+1}})\} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}z^{-n}(\frac{z^{2n+1}-1}{z-1}).
\end{aligned}$$

Therefore, we find that

$$(zq;q)_\infty(q/z;q)_\infty(q;q)_\infty = \sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}z^{-n}(\frac{z^{2n+1}-1}{z-1}). \tag{2.22}$$

Finally, by letting z approach to 1, we conclude that

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2} = (q;q)_\infty^3.$$

□

Throughout this manuscript by $\sum_{n=0}^{\infty}\alpha_n q^n \equiv \sum_{n=0}^{\infty}\beta_n q^n \pmod{r}$ we mean that $\alpha_n \equiv \beta_n \pmod{r}$ for all n.

Lemma 2.5. *If* $\sum_{n=0}^{\infty}a_n q^n \sum_{n=0}^{\infty}b_n q^n \equiv 0 \pmod{r}$ *and* $a_0 = 1$*, then* $b_n \equiv 0 \pmod{r}$ *for all* n*.*

Proof.

$$\sum_{n=0}^{\infty} a_n q^n \sum_{n=0}^{\infty} b_n q^n = \sum_{n=0}^{\infty} \sum_{s=0}^{n} a_s b_{n-s} q^n.$$

Thus by our assumption

$$\sum_{s=0}^{n} a_s b_{n-s} \equiv 0 \pmod{r} \text{ for all } n.$$

For $n = 0$, we have $a_0 b_0 \equiv 0 \pmod{r}$. Since $a_0 = 1$ we have $b_0 \equiv 0 \pmod{r}$. For $n \geq 1$,

$$\sum_{s=0}^{n} a_s b_{n-s} = a_0 b_n + a_1 b_{n-1} + ... + a_n b_0 \equiv 0 \pmod{r}.$$

But since $a_0 = 1$ we have by induction that $b_n \equiv 0 \pmod{r}$. □

Now we are ready to prove Ramanujan's partition congruences (1.1)—(1.3).

2.2 Proofs of Ramanujan's congruences

Theorem 2.6. *[8, Theorem 2.3.1]*

$$p(5n + 4) \equiv 0 \pmod{5}.$$

Proof. By Euler's identity, (2.1), for the generating function of $p(n)$, we have

$$(q; q)_\infty^4 = (q; q)_\infty^5 \sum_{n=0}^{\infty} p(n) q^n. \tag{2.1}$$

By Binomial Theorem, we also have

$$(q; q)_\infty^5 \equiv (q^5; q^5)_\infty \pmod{5}. \tag{2.2}$$

Therefore, by (2.1) and (2.2), we find that

$$(q;q)_\infty^4 \equiv (q^5;q^5)_\infty \sum_{n=0}^{\infty} p(n)q^n \pmod 5. \tag{2.3}$$

On the other hand, by employing (2.20) and (2.21), we deduce that

$$\begin{aligned}(q;q)_\infty^4 &= (q;q)_\infty (q;q)_\infty^3 \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2} \sum_{m=-\infty}^{\infty} (-1)^m (2m+1) q^{m(m+1)/2} \\ &= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+n} (2m+1) q^{\frac{n(3n+1)}{2} + \frac{m(m+1)}{2}}. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we have

$$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(n)q^n \equiv \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} (-1)^{m+n} (2m+1) q^{\frac{n(3n+1)}{2} + \frac{m(m+1)}{2}} \pmod 5.$$

It is easy to check that

$$\frac{n(3n+1)}{2} + \frac{m(m+1)}{2} \equiv 4 \pmod 5$$

if and only if

$$\frac{n(3n+1)}{2} \equiv 1 \pmod 5 \quad \text{and} \quad \frac{m(m+1)}{2} \equiv 3 \pmod 5.$$

But then,

$$(2m+1) \equiv 0 \pmod 5,$$

which implies that

$$(q^5;q^5)_\infty \sum_{n=0}^{\infty} p(5n+4)q^{5n+4} \equiv 0 \pmod 5.$$

Hence, by Lemma 2.5, we conclude that

$$p(5n+4) \equiv 0 \pmod 5.$$

$\square$

Next, we prove Ramanujan's partition congruence modulo 7. The proof is very similar to that of Theorem 2.6.

Theorem 2.7. *[8, Theorem 2.4.1]*

$$p(7n+5) \equiv 0 \pmod{7}.$$

Proof. By (2.1), we have

$$(q;q)_\infty^6 = (q;q)_\infty^7 \sum_{n=0}^{\infty} p(n)q^n. \tag{2.5}$$

By Binomial Theorem, we also have

$$(q;q)_\infty^7 \equiv (q^7;q^7)_\infty \pmod{7}. \tag{2.6}$$

From these last two equations, we find that

$$(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n)q^n \equiv (q;q)_\infty^6 \pmod{7}. \tag{2.7}$$

On the other hand, by (2.21), we find that

$$\begin{aligned}(q;q)_\infty^6 &= \{(q;q)_\infty^3\}^2 \\ &= \{\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}\}^2 \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(2n+1)(2m+1)q^{n(n+1)/2+m(m+1)/2}.\end{aligned} \tag{2.8}$$

Therefore, by (2.7) and (2.8), we have that

$$(q^7;q^7)_\infty \sum_{n=0}^{\infty} p(n)q^n \equiv \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}(2n+1)(2m+1)q^{n(n+1)/2+m(m+1)/2} \pmod{7}.$$

Observe that

$$n(n+1)/2 + m(m+1)/2 \equiv 5 \pmod 7 \tag{2.9}$$

if and only if

$$n, m \equiv 3 \pmod 7,$$

which in turn implies that

$$(2n+1)(2m+1) \equiv 0 \pmod 7.$$

Therefore, we find that

$$(q^7; q^7)_\infty \sum_{n=0}^{\infty} p(7n+5)q^{7n+5} \equiv 0 \pmod 7.$$

Hence, by Lemma 2.5, we conclude that

$$p(7n+5) \equiv 0 \pmod 7.$$

□

Next, we state an identity given by Winquist[19]. For a short proof of this identity see [13]. Winquist used his identity to find an appropriate representation for $(q;q)_\infty^{10}$ which allowed him to give and elementary proof of Ramanujan's partition congruence modulo 11. His proof is very similar to that of Ramanujan and so we skip some details to avoid repetitions.

Theorem 2.8. *(Winquist Identity)[19]*

$$(q;q)_\infty^2 (y;q)_\infty (q/y;q)_\infty (z;q)_\infty (q/z;q)_\infty (y/z;q)_\infty (zq/y;q)_\infty (yz;q)_\infty (q/yz;q)_\infty$$

$$= \sum_{i=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{i+j} \{(y^{-3i} - y^{3i+3})(z^{-3j} - z^{3j+1}) + (y^{-3j+1} - y^{3j+2})(z^{3i+2} - z^{-3i-1})\} q^{3i(i+1)/2 + j(3j+1)/2}. \tag{2.10}$$

Corollary 2.9.

$$(q;q)_\infty^{10} = \frac{1}{6}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(6i+3)(6j+1)(9j^2+3j-9i^2-9i-2)\}q^{3i(i+1)/2+j(3j+1)/2}. \tag{2.11}$$

Proof. First we divide both sides of (2.10) by $(1-y)$ and take the limit as $y \to 1$, for the left hand side we have

$$\lim_{y\to 1}\frac{(q;q)_\infty^2(y;q)_\infty(q/y;q)_\infty(z;q)_\infty(q/z;q)_\infty(y/z;q)_\infty(zq/y;q)_\infty(yz;q)_\infty(q/yz;q)_\infty}{(1-y)}$$
$$= (q;q)_\infty^4(z;q)_\infty^2(q/z;q)_\infty^2(1/z;q)_\infty(zq;q)_\infty.$$

For the right hand side with the help of L'Hopital's rule, we find that

$$\lim_{y\to 1}\frac{1}{(1-y)}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(y^{-3i}-y^{3i+3})(z^{-3j}-z^{3j+1})$$
$$+(y^{-3j+1}-y^{3j+2})(z^{3i+2}-z^{-3i-1})\}q^{3i(i+1)/2+j(3j+1)/2}$$
$$=\lim_{y\to 1}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j+1}\{((-3i)y^{-3i-1}-(3i+3)y^{3i+2})(z^{-3j}-z^{3j+1})$$
$$+((-3j+1)y^{-3j}-(3j+2)y^{3j+1})(z^{3i+2}-z^{-3i-1})\}q^{3i(i+1)/2+j(3j+1)/2}$$
$$=\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j+1}\{(-6i-3)(z^{-3j}-z^{3j+1})+(-6j-1)(z^{3i+2}-z^{-3i-1})\}q^{3i(i+1)/2+j(3j+1)/2}.$$

Therefore, we have

$$(q;q)_\infty^4(z;q)_\infty^2(q/z;q)_\infty^2(1/z;q)_\infty(zq;q)_\infty$$
$$=\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j+1}\{(-6i-3)(z^{-3j}-z^{3j+1})+(-6j-1)(z^{3i+2}-z^{-3i-1})\}q^{3i(i+1)/2+j(3j+1)/2}. \tag{2.12}$$

Next we divide both sides of (2.12) by $\frac{-(1-z)^3}{z}$, and let $z \to 1$. For the left hand side, we find that

$$\lim_{z\to 1}\frac{(q;q)_\infty^4(z;q)_\infty^2(q/z;q)_\infty^2(1/z;q)_\infty(zq;q)_\infty}{\frac{-(1-z)^3}{z}}$$
$$=(q;q)_\infty^{10}.$$

For the right hand side, by imposing L'Hopital rule three times, we find that

$$\lim_{z\to 1}\frac{1}{(1-z)^3}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(-6i-3)(z^{-3j+1}-z^{3j+2})$$
$$+(-6j-1)(z^{3i+3}-z^{-3i})\}q^{3i(i+1)/2+j(3j+1)/2}$$

$$=\frac{1}{6}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(6i+3)(6j+1)(9j^2+3j-9i^2-9i-2)\}q^{3i(i+1)/2+j(3j+1)/2}.$$

By equating these last two equations, we arrive at (2.11).

□

Theorem 2.10.

$$p(11n+6)\equiv 0 \pmod{11}.$$

Proof. By arguing as in (2.5)–(2.7), we find that

$$(q^{11};q^{11})_\infty\sum_{n=0}^{\infty}p(n)q^n\equiv(q;q)_\infty^{10} \pmod{11}.$$

Next we employ (2.11) to deduce that

$$(q^{11};q^{11})_\infty\sum_{n=0}^{\infty}p(n)q^n$$
$$\equiv\frac{1}{6}\sum_{i=0}^{\infty}\sum_{j=-\infty}^{\infty}(-1)^{i+j}\{(6i+3)(6j+1)(9j^2+3j-9i^2-9i-2)\}q^{3i(i+1)/2+j(3j+1)/2} \pmod{11}.$$

Observe that

$$3i(i+1)/2+j(3j+1)/2\equiv 6 \pmod{11}$$

if and only if

$$i \equiv 5 \pmod{11} \quad \text{and} \quad j \equiv 9 \pmod{11}$$

which in turn implies that

$$(6i+3)(6j+1)(9j^2+3j-9i^2-9i-2) \equiv 0 \pmod{11}.$$

Since $\gcd(6,11) = 1$ and all the coefficients are integers, we find that

$$(q^{11};q^{11})_\infty \sum_{n=0}^{\infty} p(11n+6)q^{11n+6} \equiv 0 \pmod{11}.$$

By employing Lemma 2.5, we conclude that

$$p(11n+6) \equiv 0 \pmod{11}$$

. $\square$

Chapter 3

Combinatorial Interpretations

Recall that a vector partition is a triplet (π_1, π_2, π_3) where π_1 is a partition with distinct parts while π_2 and π_3 are arbitrary partitions. The set of all vector partitions is denoted by V. For the vector partition $\pi = (\pi_1, \pi_2, \pi_3)$, we define the number of parts of π_1 to be $\#(\pi_1)$, sum of parts of π to be $s(\pi)$, the weight of π to be $\omega(\pi) = (-1)^{\#(\pi_1)}$, and the crank of π to be $r(\pi) = \#(\pi_2) - \#(\pi_3)$. If $s(\pi) = n$, then we say that π is a vector partition of n. Let $N_v(m, n)$ be the number of vector partitions of n with crank m counted according to their weight, namely,

$$N_v(m,n) = \sum_{\pi \in V,\, s(\pi)=n,\, r(\pi)=m} \omega(\pi).$$

Finally, let $N_v(m, t, n)$ be the number of vector partitions of n with crank m modulo t counted according to their weight, namely,

$$N_v(m,t,n) = \sum_{l=-\infty}^{\infty} N_v(lt+m, n) = \sum_{\pi \in V,\, s(\pi)=n,\, r(\pi)\equiv m \pmod t} \omega(\pi).$$

Garvan proved that [10]

$$N_v(i,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \le i \le 4, \tag{3.1}$$

$$N_v(i,7,7n+5) = \frac{p(7n+5)}{7}, \quad 0 \le i \le 6, \tag{3.2}$$

and

$$N_v(i,11,11n+6) = \frac{p(11n+6)}{11}, \quad 0 \le i \le 10. \tag{3.3}$$

Let us verify (3.1) for $n = 4$.

$$
\begin{array}{ll}
\pi = (3+1, \varnothing, \varnothing) & (r(\pi), \omega(\pi)) = (0,1) \\
\pi = (\varnothing, 3, 1) & (r(\pi), \omega(\pi)) = (0,1) \\
\pi = (\varnothing, 1+1, 1+1) & (r(\pi), \omega(\pi)) = (0,1) \\
\pi = (\varnothing, 1, 3) & (r(\pi), \omega(\pi)) = (0,1) \\
\pi = (\varnothing, 2, 2) & (r(\pi), \omega(\pi)) = (0,1) \\
\pi = (4, \varnothing, \varnothing) & (r(\pi), \omega(\pi)) = (0,-1) \\
\pi = (1, 2, 1) & (r(\pi), \omega(\pi)) = (0,-1) \\
\pi = (2, 1, 1) & (r(\pi), \omega(\pi)) = (0,-1) \\
\pi = (1, 1, 2) & (r(\pi), \omega(\pi)) = (0,-1) \\
\pi = (3, 1, \varnothing) & (r(\pi), \omega(\pi)) = (1,-1)
\end{array}
$$

$\pi = (2, 2, \varnothing)$ $(r(\pi), \omega(\pi)) = (1, -1)$

$\pi = (1, 3, \varnothing)$ $(r(\pi), \omega(\pi)) = (1, -1)$

$\pi = (1, 1 + 1, 1)$ $(r(\pi), \omega(\pi)) = (1, -1)$

$\pi = (\varnothing, 4, \varnothing)$ $(r(\pi), \omega(\pi)) = (1, 1)$

$\pi = (\varnothing, \varnothing, 1 + 1 + 1 + 1)$ $(r(\pi), \omega(\pi)) = (1, 1)$

$\pi = (\varnothing, 1 + 1, 2)$ $(r(\pi), \omega(\pi)) = (1, 1)$

$\pi = (\varnothing, 2 + 1, 1)$ $(r(\pi), \omega(\pi)) = (1, 1)$

$\pi = (2 + 1, 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (1, 1)$

$\pi = (2, 1 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (2, -1)$

$\pi = (1, 2 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (2, -1)$

$\pi = (1, \varnothing, 1 + 1 + 1)$ $(r(\pi), \omega(\pi)) = (2, -1)$

$\pi = (\varnothing, 3 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (2, 1)$

$\pi = (\varnothing, 2 + 2, \varnothing)$ $(r(\pi), \omega(\pi)) = (2, 1)$

$\pi = (\varnothing, 1 + 1 + 1, 1)$ $(r(\pi), \omega(\pi)) = (2, 1)$

$\pi = (\varnothing, \varnothing, 2 + 1 + 1)$ $(r(\pi), \omega(\pi)) = (2, 1)$

$\pi = (2, \varnothing, 1 + 1)$ $(r(\pi), \omega(\pi)) = (3, -1)$

$\pi = (1, 1 + 1 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (3, -1)$

$\pi = (1, \varnothing, 2 + 1)$ $(r(\pi), \omega(\pi)) = (3, -1)$

$\pi = (\varnothing, 2 + 1 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (3, 1)$

$\pi = (\varnothing, 1, 1 + 1 + 1)$ $(r(\pi), \omega(\pi)) = (3, 1)$

$\pi = (\varnothing, \varnothing, 2 + 2)$ $(r(\pi), \omega(\pi)) = (3, 1)$

$\pi = (\varnothing, \varnothing, 3 + 1)$ $(r(\pi), \omega(\pi)) = (3, 1)$

$\pi = (3, \varnothing, 1)$ $(r(\pi), \omega(\pi)) = (4, -1)$

$\pi = (2, \varnothing, 2)$ $(r(\pi), \omega(\pi)) = (4, -1)$

$\pi = (1, \varnothing, 3)$ $(r(\pi), \omega(\pi)) = (4, -1)$

$\pi = (1, 1, 1 + 1)$ $(r(\pi), \omega(\pi)) = (4, -1)$

$\pi = (2 + 1, \varnothing, 1)$ $(r(\pi), \omega(\pi)) = (4, 1)$

$\pi = (\varnothing, 1 + 1 + 1 + 1, \varnothing)$ $(r(\pi), \omega(\pi)) = (4, 1)$

$\pi = (\varnothing, \varnothing, 4)$ $(r(\pi), \omega(\pi)) = (4, 1)$

$\pi = (\varnothing, 2, 1 + 1)$ $(r(\pi), \omega(\pi)) = (4, 1)$

$\pi = (\varnothing, 1, 2 + 1)$ $(r(\pi), \omega(\pi)) = (4, 1)$

So $N_v(i, 5, 4) = 1$ for $i = 0, 1, 2, 3, 4$.

A generating function for $N_v(m, n)$ is given by [10, eq. (1.25)]

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_v(m, n) z^m q^n = \frac{(q;q)_\infty}{(zq;q)_\infty (z^{-1}q;q)_\infty}. \tag{3.4}$$

We will only sketch a proof of (3.4) by identifying each piece on the right of (3.4). By arguing as in the proof of Theorem 2.1, we find that

$$\begin{aligned}\frac{1}{(zq;q)_\infty} = \prod_{n=0}^{\infty}(1 - zq^n)^{-1} &= \prod_{n=0}^{\infty}(1 + z^1q^n + z^2q^{2n} + z^3q^{3n} + ...) \\ &= (1 + z^1q^1 + z^2q^{2\cdot 1} + z^3q^{3\cdot 1} + ...) \times \\ &\quad \cdot (1 + z^1q^{1\cdot 2} + z^2q^{2\cdot 2} + z^3q^{3\cdot 2} + ...) \times \\ &\quad ... \\ &= \sum_{k_i \in N} z^{k_1+k_2+k_3..} q^{1.k_1+2.k_2+3.k_3...} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} p(m, n) z^m q^n,\end{aligned}$$

where $p(m, n)$ is the number of partitions of n with m parts. Similarly,

$$(q;q)_\infty = (1-q)(1-q^2)(1-q^3).... \tag{3.5}$$

$$= 1 - q^1 - q^2 + q^{1+2} - q^3 + ... \tag{3.6}$$

$$= \sum_{n=0}^{\infty} p_e(n) q^n - \sum_{n=0}^{\infty} p_o(n) q^n, \tag{3.7}$$

where $p_e(n)$ $(p_o(n))$is the number of partitions of n with even (odd) number of distinct parts.

If we let $z = 1$ in the above equation we find that

$$\sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_v(m, n) q^n = \frac{1}{(q;q)_\infty}.$$

Therefore, by (2.1), we deduce that

$$p(n) = \sum_{m=-\infty}^{\infty} N_v(m,n) = \sum_{k=0}^{t-1} N_v(k,t,n).$$

Lemma 3.1. *[10, Lemma (2.2)] Let t be a prime and r_t be the reciprocal of t modulo 24. Then*

$$N_v(i,t,tn+r_t) = \frac{p(tn+r_t)}{t}, \quad 0 \leq i \leq t. \tag{3.8}$$

if and only if $a_{tn+r_t} = 0$ where

$$\sum_{n=0}^{\infty} a_n q^n = \frac{(q;q)_\infty}{(\omega q;q)_\infty (\omega^{-1}q;q)_\infty}$$

and ω is the t^{th} root of unity.

Proof. By (3.8), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n q^n &= \frac{(q;q)_\infty}{(\omega q;q)_\infty (\omega^{-1}q;q)_\infty} \\
&= \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_v(m,n)\omega^m q^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{t-1} \sum_{m \equiv k \pmod t} N_v(m,n)\omega^m q^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{t-1} \omega^m \sum_{m \equiv k \pmod t} N_v(m,n) q^n \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{t-1} \omega^m N_v(k,t,n) q^n.
\end{aligned}$$

If we equate the coefficient of q^{tn+r_t}, we find that

$$a_{tn+r} = \sum_{k=0}^{t-1} \omega^m N_v(k,t,tn+r_t). \tag{3.9}$$

Since t is prime and ω is the t^{th} root of unity, we have

$$\sum_{k=0}^{t-1} \omega^m = 0.$$

Now assume that (3.8) is true, then, by (3.9), we find that

$$a_{tn+r_t} = N_v(0, t, tn + r_t) \sum_{k=0}^{t-1} \omega^m = 0.$$

Conversely, suppose that $a_{tn+r_t} = 0$, then, by (3.9), we have that

$$\sum_{k=0}^{t-1} \omega^m N_v(k, t, tn + r_t) = 0. \tag{3.10}$$

By using the fact that the minimal polynomial of ω over Q is $f(x) = 1 + x + x^2 + x^3 + ... + x^{t-1}$, we conclude from (3.10) that

$$N_v(i, t, tn + r_t) = N_v(1, t, tn + r_t) = ... = N_v(t - 1, t, tn + r_t).$$

Moreover,

$$p(tn + r_t) = \sum_{k=0}^{t-1} N_v(k, t, tn + r_t) = tN_v(0, t, tn + r_t),$$

which is (3.8). □

We are now ready to prove (3.1)–(3.3). We start with (3.1). By Lemma 3.1, it suffices to prove that $a_{5n+4} = 0$. Let α be the 5th root of unity. By (2.20) and (2.22), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n q^n =& \frac{(q;q)_\infty}{(\alpha q;q)_\infty(\alpha^{-1}q;q)_\infty} \\
=& \frac{(q;q)_\infty\{(q;q)_\infty(\alpha^2q;q)_\infty(\alpha^{-2}q;q)_\infty\}}{(q;q)_\infty(\alpha q;q)_\infty(\alpha^2q;q)_\infty(\alpha^{-2}q;q)_\infty(\alpha^{-1}q;q)_\infty} \\
=& \frac{(q;q)_\infty\{(q;q)_\infty(\alpha^2q;q)_\infty(\alpha^{-2}q;q)_\infty\}}{(q^5;q^5)_\infty} \\
=& \frac{\sum_{m=-\infty}^{\infty}(-1)^m q^{m(3m-1)/2}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}(\alpha^{-2})^n((\alpha^2)^{2n+1}-1)/(\alpha^2-1)}{(q^5;q^5)_\infty} \\
=& \frac{\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{n+m} q^{m(3m-1)/2+n(n+1)/2}(\alpha^{-2})^n((\alpha^2)^{2n+1}-1)/(\alpha^2-1)}{(q^5;q^5)_\infty}.
\end{aligned}$$

By a similar justification as in the proof of (1.1), we see that when the exponent of q is congruent to 4 modulo 5, we have $2n+1 \equiv 0 \pmod 5$. Therefore,

$$((\alpha^2)^{2n+1}-1) = 0.$$

Hence, $a_{5n+4} = 0$.

Next, we prove (3.2). Let β be the seventh root of unity.
According to Lemma 3.1, we need to show that $a_{7n+5} = 0$. By employing (2.22) twice, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} a_n q^n = \frac{(q;q)_\infty}{(\beta q;q)_\infty(\beta^{-1}q;q)_\infty} \\
=& \frac{\{(q;q)_\infty(\beta^2q;q)_\infty(\beta^{-2}q;q)_\infty\}\{(q;q)_\infty(\beta^3q;q)_\infty(\beta^{-3}q;q)_\infty\}}{\prod_{k=0}^{6}(\beta^k q;q)_\infty} \\
=& \frac{\{(q;q)_\infty(\beta^2q;q)_\infty(\beta^{-2}q;q)_\infty\}\{(q;q)_\infty(\beta^3q;q)_\infty(\beta^{-3}q;q)_\infty\}}{(q^7;q^7)_\infty} \\
=& \sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}(\beta^{-2})^n((\beta^2)^{2n+1}-1)}{(\beta^2-1)}\sum_{m=-\infty}^{\infty}(-1)^m q^{n(n+1)/2}(\beta^{-3})^m\frac{((\beta^3)^{2m+1}-1)}{(\beta^3-1)(q^7;q^7)_\infty} \\
=& \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^{n+m}q^{n(n+1)/2+m(m+1)/2}(\beta^{-2})^n(\beta^{-3})^m\frac{((\beta^2)^{2n+1}-1)}{(\beta^2-1)}\frac{((\beta^3)^{2m+1}-1)}{(\beta^3-1)(q^7;q^7)_\infty}.
\end{aligned}$$

By a similar justification as in the proof of (1.2), we see that when the exponent of q is congruent to 5 modulo 7, we have $2n+1 \equiv 0 \pmod 7$ and $2m+1 \equiv 0 \pmod 7$. Therefore, $((\beta^2)^{2n+1}-1)=0$ and $((\beta^3)^{2m+1}-1)=0$. Hence, $a_{7n+5}=0$.

Lastly, we prove (3.3). Let γ be the eleventh root of unity. In Winquist's identity, (2.10), we replace y by γ^9 and z by γ^5, we obtain

$$(q;q)_\infty^2(\gamma^9;q)_\infty(q\gamma^2;q)_\infty(\gamma^5;q)_\infty(q\gamma^6;q)_\infty(\gamma^4;q)_\infty(\gamma^7q;q)_\infty(\gamma^3;q)_\infty(q\gamma^8;q)_\infty$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\{(\gamma^{6i}-\gamma^{5i+5})(\gamma^{7j}-\gamma^{4j+5})+(\gamma^{6j+9}-\gamma^{5j+7})(\gamma^{4i+10}-\gamma^{7i-5})\}q^{3i(i+1)/2+j(3j+1)/2}. \tag{3.11}$$

According to Lemma 3.1, we have to show that $a_{11n+6}=0$. By (3.11), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}a_nq^n =&\frac{(q;q)_\infty}{(\gamma q;q)_\infty(\gamma^{-1}q;q)_\infty}\\ =&\frac{(q;q)_\infty^2\prod_{k=2}^{9}(\gamma^kq;q)_\infty}{\prod_{k=0}^{10}(\gamma^kq;q)_\infty}\\ =&\frac{1}{(1-\gamma^3)(1-\gamma^4)(1-\gamma^5)(1-\gamma^9)(q^{11};q^{11})_\infty}\\ &\cdot\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\{(\gamma^{6i}-\gamma^{5i+5})(\gamma^{7j}-\gamma^{4j+5})\\ &+(\gamma^{6j+9}-\gamma^{5j+7})(\gamma^{4i+10}-\gamma^{7i-5})\}q^{3i(i+1)/2+j(3j+1)/2}\\ =&\frac{1}{(1-\gamma^3)(1-\gamma^4)(1-\gamma^5)(1-\gamma^9)(q^{11};q^{11})_\infty}\\ &\cdot\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\{(\gamma^{6i}-\gamma^{5i+5})(\gamma^{7j}-\gamma^{4j+5})\\ &+(\gamma^{6j+9}-\gamma^{5j+7})(\gamma^{4i+10}-\gamma^{7i-5})\}q^{3i(i+1)/2+j(3j+1)/2}.\end{aligned}$$

By a similar justification as in the proof of (1.3), we see that when the exponent of q, $3i(i+1)/2+j(3j+1)/2$, is congruent to 6 modulo 11, we have $i \equiv 5 \pmod{11}$ and $j \equiv 9 \pmod{11}$. Therefore, $\gamma^{6i}-\gamma^{5i+5}=0$ and $\gamma^{6j+9}-\gamma^{5j+7}=0$. Hence, $a_{11n+6}=0$.

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