PRICING AND HEDGING OF CONTINGENT CLAIMS IN INCOMPLETE MARKETS BY MODELING LOSSES AS CONDITIONAL VALUE AT RISK IN λ-GAIN LOSS OPPORTUNITIES

A THESIS

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By

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July, 2009

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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July, 2009

We combine the principles of risk aversion and no-arbitrage pricing and propose an alternative way for pricing and hedging contingent claims in incomplete markets. We re-consider the pricing problem under the condition that losses are modeled by the measure of CVaR in the concept of λ gain-loss opportunities. The proposed model enables investors to specify their preferences by putting restrictions on the parameter λ that stands for risk aversion. Using CVaR as a measure of risk enables us to account for extreme losses and yield a conservative result. The pricing problem is studied in discrete time, multi-period, stochastic linear optimization environment with a finite probability space. We extend our model to include the perspectives of writers and buyers of the contingent claims. We use duality to establish a pricing interval of the contingent claims excluding CVaR-λ gain-loss opportunities in the market. Duality results also provide a way for passing to appropriate martingale measures and we express the pricing interval also in terms of martingale measures. This pricing interval is shown to be tighter than the no-arbitrage bounds. We also present a numerical study of our work with respect to the risk aversion parameter λ and in various levels of confidence. We compute prices of the the writers and buyers of 48 European call and put options on the $S\&P500$ index on September 10, 2002 using the remaining options as market traded assets. It is possible to say that our proposed model yields good bounds as most of the bounds we obtained are very close to the true bid and ask values.

Keywords: stochastic programming, conditional value at risk, arbitrage, martingales, duality, contingent claims .

ÖZET

EKSİK PİYASALARDA KOŞULLU TALEPLERİN λ-KAZANÇ KAYIP FIRSATLARINDA KAYIPLARIN KOŞULLU RİSKE MARUZ DEĞER KULLANILARAK FİYATLANDIRILMASI

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Temmuz, 2009

Bu tez çalışmasında, riskten kaçınma ve arbitraj fiyatlama teorisi ilkeleri bir araya getirilerek eksik piyasalarda koşullu taleplerin değerlemesi için yeni bir yol önerilmektedir. λ-kazanç kayıp fırsatları konseptindeki fiyatlama problemi, kayıpların koşullu riske maruz değer (CVaR) kullanılarak modellenmesi koşulu altında tekrar değerlendirilmektedir. Önerilen model, yatırımcıların λ parametresi üzerine kısıtlama getirerek tercihlerini belirleyebilmelerine imkan sağlamaktadır. Risk ölçütü olarak CVaR kullanılması, oluşabilecek aşırı kayıpların hesaba katılabilmesini sağlamakta ve daha ihtiyatlı bir sonuç vermektedir. Fiyatlama problemi, kesikli zaman, çoklu periyot bir stokastik lineer optimizasyon ortamında çalışılmaktadır. Model, koşullu taleplerin satıcılarının ve alıcılarının bakış açılarını da içerecek şekilde genişletilmiştir. Dualite kullanılarak, piyasada koşullu talepler icin CVAR-λ kazanç kayıp fırsatı içermeyen bir fiyat aralığı tespit edilmiştir. Dualite sonuçları uygun martingale ölçütlerine geçiş imkanı sağlamış; bu sayede fiyat aralığı martingale ölçütleri cinsinden de ifade edilmiştir. Bu fiyat aralığının arbitraj fiyatlama teorisi ile tespit edilen aralıktan daha dar olduğu gösterilmiştir. Buna ek olarak, farklı güvenilirlik seviyeleri kullanılarak, riskten kaçınma parametresine göre numerik bir çalışma yapılmıştır. 10 Eylül 2002 S&P 500 indeksinde yer alan 48 Avrupa tipi alım ve satım opsiyonu için fiyatlar hesaplanmış; fiyatı hesaplanan opsiyon dışındaki opsiyonlar piyasa varlıkları olarak kabul edilmiştir. Elde ettiğimiz fiyat sınırlarının gerçek alım-satım değerlerine oldukça yakın olması, önerdiğimiz modelin iyi bir fiyat aralığı belirlediğini göstermektedir.

Anahtar sözcükler: Stokastik programlama, koşullu riske maruz değer, arbitraj, martingale, dualite, koşullu talepler.

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Chapter 1

Introduction

The question of pricing uncertain pay-offs has been studied extensively in financial economics starting after Louis Bachelier's work on option pricing in 1900. The renowned papers of Black and Scholes and Merton in 1970s paved the way for pricing uncertain payoffs in a complete and unconstrained market. Black-Scholes-Merton approach replicates uncertain payoffs using existing financial instruments and finds a unique price relative to these instruments avoiding an arbitrage opportunity. This price coincides with the expectation of claim's discounted value under the unique, risk-neutral equivalent probability measure.

The foregoing argument fails, however, unless the financial market is complete and unconstrained. In the case of incomplete markets, there ceases to exist a unique price for a contingent claim based on the absence of arbitrage opportunities. Actually, this means that on the portfolios side there is no replicating portfolio and the hedging strategy could involve a risky position; on the payoffs side, there is an infinite number of martingale measures and each of them provides a different price for the contingent claim.

In incomplete markets, there are two fundamental approaches for pricing contingent claims. The first one is usually known as "model based pricing" and is based on expected utility maximization concept. This approach equates the price of a claim to the expectation of the product of the future payoff and the marginal rate of substitution of the investor. This approach yields precise pricing of the asset due to explicit assumptions about investors preferences; however is prone to misspecification error. Since specifying investors preferences in all states is a challenging task, practical use of this approach is limited.

When investors' preferences cannot be specified, a second approach called " no arbitrage pricing" is employed. In this approach, an interval of prices consistent with no arbitrage is calculated rather than setting a unique price level. Absence of a unique martingale measure leads to a pricing interval where the minimum is called "buyer's price" and maximum is called "writer's price". If the buyers are risk averse, no one would buy a claim offered at the writer's price and similarly a risk-averse writer would not sell the claim at the buyer's price.

A writer may for various reasons settle for a price less than the writer's price. In such a case, the writer will not be able to find a super-replicating portfolio (a portfolio dominating claim's future pay-offs). Therefore, the writer runs the risk of falling short and will need to set-up his/her hedge portfolio (and equivalently determine writer's price) according to some optimality criteria. An analogous problem can be defined for the buyer as well. In order to develop an optimality criterion, Cochrane and Saa-Requejo [1] introduce "good-deal concept" which they define as an investment with a high Sharpe ratio[1].

Similarly, Bernardo and Ledoit [2] introduce the "gain-loss ratio", which is the expectation of an investment's positive excess payoffs divided by expectation of its negative excess payoffs. Building on Bernardo and Ledoit's concept of the gain-loss ratio, Pinar et al. [4] have recently developed the concept of "λ gain-loss opportunities" and investigated the derivations and computations within the framework of stochastic linear programming.

Another principle that the modern financial theory is based on is risk aversion. It is well known that the single major source of profit is risk. The expected return depends heavily on the level of risk of an investment. Although the idea of risk

[1]The Sharpe ratio or reward-to-variability ratio is a measure of the excess return (or Risk Premium) per unit of risk in an investment. It is calculated by dividing return of asset minus a benchmark rate by standard deviation of the return.

seems to be intuitively clear, it is difficult to formalize it. Different attempts have been conducted with various degrees of success. There appears an efficient way to formalize and quantify risk in most of the markets. However, each method is deeply associated with its specific market and this association limits their usefulness in other markets. Value at Risk (VaR) has been an integrated way to deal with different markets and different risks and to combine all factors into a single number which is a good indicator of the overall risk level since it was introduced by JP Morgan in 1994. It calculates maximum expected losses over a given time period at a given tolerance level. However, VaR suffers from the following drawbacks as Rockafellar and Uryasev [10] states: i) it under or over-estimates the risk when losses are not normally distributed; ii) it does not give an information on the distribution of losses exceeding VaR and iii) it does not satisfy the properties of a coherent risk measure such as sub-additivity. Conditional Value at Risk (CVaR), also called mean excess loss, mean shortfall, or tail VaR, is closely related to VaR. It has been developed as an extension of VaR and is superior to VaR for being coherent and having strong mathematical characteristics such as convexity and sub-additivity. CVaR is defined as the conditional expected loss under the condition the loss exceeds VaR. Therefore, CVaR is equal to or greater than VaR.

In this thesis, we will combine the principles of risk aversion and no-arbitrage pricing and propose an alternative way for pricing and hedging contingent claims. Investors will be able to specify their preferences by putting restrictions on the parameter λ that stands for risk aversion. Our study is mainly inspired by the work of Pinar et al. [4] and we re-consider the pricing problem under the condition that the losses are modeled by the measure of CVaR in the concept of λ gain-loss opportunities. We name this criterion as a CVaR-λ gain-loss opportunity. Using CVaR as a measure of risk will enable us to account for extreme losses and yield a conservative result. The pricing problem will be studied in discrete time, multi-period, stochastic linear optimization environment with a finite probability space. We will introduce a function that minimizes CVaR and model losses by this function. Then, we will incorporate this loss function into the stochastic program that determines the maximum expected gains of an investor that is interested in a

λ gain-loss opportunity. The λ gain-loss opportunity can be defined as a portfolio that begins with a zero initial value, makes self-financing portfolio transactions and attains a non-negative value in each future state, while in the terminal state the probability that it yields a positive value for the difference between the gains and λ times the losses is positive. We state the relationship between the existence of the CVaR-λ gain-loss opportunities and martingales. Then, we determine the pricing interval of our model excluding CVaR-λ gain-loss opportunities in the market. This pricing interval will be tighter than the no-arbitrage bounds. This is the main motivation of our study since our model enables us to obtain tighter bounds on the prices. We also note that these bounds converge to the no arbitrage bounds in the limit when the parameter λ goes to infinity in each of the specified confidence levels.

The organization of the thesis is as follows:

The next chapter starts with the review of the literature that is related to the problem under consideration. Our study is mainly about incorporating CVaR measure as losses into the λ gain-loss opportunities. The concept of λ gain-loss opportunity is in close relationship with the concepts of Sharpe Ratio, Gain-Loss ratio and Good Deals. Therefore, important works about these concepts will be examined in the literature review part. Then, the work of Rockefellar and Uryasev will be examined to give an in-depth understanding of the concept of CVaR.

In Chapter 3, the general setting and the stochastic process governing the security prices are summarized. The concept of arbitrage is defined within the framework of stochastic programming and the links between arbitrage and martingales are stated. Finally, the hedging and pricing problem of contingent claims is discussed and extended to include the perspectives of writers and buyers of the contingent claims.

Chapter 4 starts with the definition of a λ gain-loss opportunity. A stochastic linear program to determine whether a λ gain-loss opportunity exists in the system is given. Later, we elaborate on the concept of CVaR and our motivations to model losses by CVaR in the model seeking a λ gain-loss opportunity. After the notations are listed, the formulation of the model that incorporates CVAR as losses to the pricing problem of contingent claims is given. Finally, the model is developed from the perspectives of the writers and the buyers of the contingent claims.

In Chapter 5, the problem discussed in Chapter 4 is analyzed through duality. It is shown that duality results provide the means for passing to the martingale measures. We prove in Theorem 3 that the absence of a λ-gain loss opportunity is equivalent to the existence of equivalent (α, λ) compatible martingale measures. Then, the dual problems to the problems of the buyer and the writer are stated. We use the dual problems to establish a CVaR-λ pricing interval. We also express the pricing interval in terms of martingale measures.

In Chapter 6, we present a numerical study of our work with respect to the risk aversion parameter λ and in various levels of confidence (α) to give a better understanding of the model. This study enables us to compare the resulting values to the actual market prices and interpret the data numerically. We compute prices of the the writer and buyer of 48 European call and put options on the $S\&P500$ index on September 10, 2002 according to the model proposed in Chapter 4 using the remaining options as market traded assets. We illustrate a representative sample of the graphs of these options and comment on the results. It is possible to say that our proposed model yields good bounds as most of the bounds we obtained are very close to the true bid and ask values. Consequently, by giving a simple example, we show that the range of the loss aversion parameter λ decreases compared to the λ gain-loss model.

In Chapter 7, we conclude the thesis by giving an overall summary and stating some possible future research related to the model that we developed.

Chapter 2

Literature Review

This chapter consists of the review of the literature related to the model that we will constitute by using the concept of Conditional Value at Risk for measuring losses when studying with the concept of λ gain-loss opportunities.

We will begin with Bernardo and Ledoit [1] where they introduce the expected gain to loss ratio which forms the basis of the pricing methodology that we use throughout this thesis. Authors study the asset pricing in incomplete markets by developing a new approach that unifies model-based pricing and pricing by no arbitrage. Model-based pricing makes strong assumptions about a benchmark investor's preferences using utility maximization concept. These assumptions enable the calculation of a specific discount factor; thus yield exact pricing implications. Despite its preciseness, calculated prices are prone to misspecification error; therefore practical use of this approach can be limited. On the other hand, pricing by no arbitrage makes weak assumptions about only the existence of a set of basis assets and the absence of arbitrage opportunities. Thus, when the market is incomplete, this approach yields pricing implications that are robust but often too imprecise to be economically interesting. The new approach developed by the authors incorporates information from both of the approaches by making a combination of these assumptions. With this new approach, they apply the expected gain to loss ratio and obtain a duality theorem for maximizing this ratio. Another duality theorem is later used for establishing bounds on option prices. Gain-loss ratio, which is the ratio of the expectation of the investment's positive excess payoffs to the expectation of its negative excess payoffs, is introduced for measuring the attractiveness of an investment opportunity. When the expectations are taken under appropriate risk-adjusted probabilities, high gain-loss ratio constitutes desirable investments for the benchmark investor and an arbitrage opportunity in the limit. Applying duality in this new approach results in connecting the high gain-loss ratio to stage-contingent discount factors with extreme deviations from the benchmark discount factor. A finite limit $\bar{L}$ is introduced on the maximum gain-loss ratio so that the admissible set of discount factors is restricted to the ones that do not exhibit extreme deviations. Assuming that excess payoffs have a gain-loss ratio below $\bar{L}$, the bounds of the price of a non-basic asset become wider as $\bar{L}$ increases and vice versa. If $\bar{L}$ goes to infinity, the admissible set converges to the no-arbitrage case. If $\bar{L}$ goes to one which is its lower bound, the admissible set shrinks to contain only the benchmark discount factor. Therefore, $\bar{L}$ can be interpreted as the trade-off between the precision of specific benchmark pricing model and the robustness of the no-arbitrage bounds. The choice of $\bar{L}$ provides a considerable flexibility to the modeler along with the choice of a benchmark discount factor and an appropriate set of basis assets.

Similarly, Cochrane and Saa-Requejo [2] replace the no-arbitrage conditions by the concept of a "good deal" which is defined as an investment with a high Sharpe ratio. The aim of authors in this paper is to develop a model for restricting the range of values of risky payoffs when one may not be able to trade continuously or in cases when there are state variables such as stochastic stock volatility and interest rate. Suppose that we want to learn the value of a focus payoff (x^c_{t+1}) given the prices (p_t) of a set of basis payoffs or hedging assets (x_{t+1}), then a discount factor or marginal utility growth rate (m_{t+1}) generates the value (p_t) of any payoff (x_{t+1}) by $p = E(mx)$. Therefore, if the focus payoff can be perfectly replicated from the set of basis assets, its value can be determined. However when the replication is not perfect, more restriction on discount factors is needed. For this purpose, authors add an upper limit bound on discount factor volatility (or equivalently a restriction on Sharpe ratio) in addition to the classic no-arbitrage restriction; and thereby obtain useful bounds on option prices in an incomplete market setting. Hence, the lower good-deal bound solves, $\underline{C} = \min E(mx^c)$ subject to the constraints $p = E(mx)$ which enforces that the prices of the basis assets are used to learn about the discount factor, $m \geq 0$ which is a classic characterization of marginal utility and $\sigma(m) \leq h/R^f$ which is the main innovation explained as a similar weak restriction on marginal utility and also a way of imposing weak predictions of economic models instead of imposing a full structure. The paper follows with the solution of the above model by considering different cases with different constraints binding. Firstly, the good-deal bounds are calculated in single-period and then it is shown that a recursive solution to the multi period problem exists such that the lower bound today solves the one-period problem with the lower bound tomorrow as payoff. The figure below is useful as it compares the good-deals bounds obtained by the authors with Black-Scholes and no arbitrage bounds.

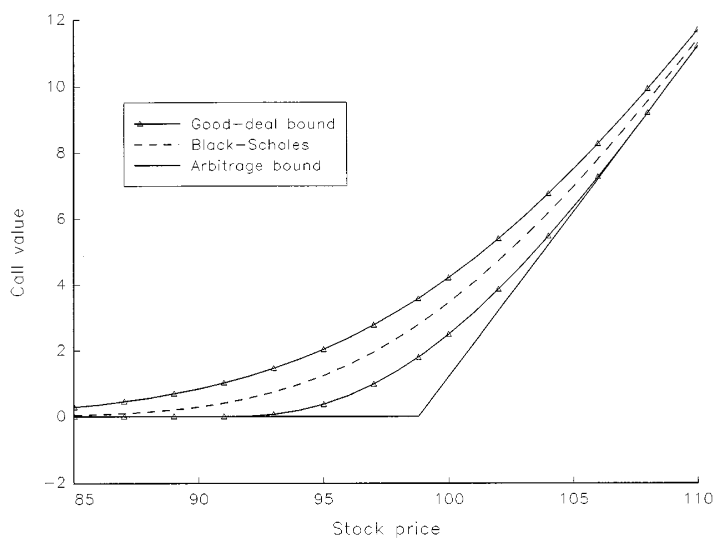

Figure 2.1: Option Price Bounds as a Function of Stock Price

King [3] presents a modeling approach for the hedging problem of contingent claims in the discrete time, discrete state case as a stochastic program. Duality is applied, leading to the arbitrage pricing theorems. The link between arbitrage and martingales is stated as the absence of arbitrage is equivalent to the existence of a probability measure that makes the price process a martingale. The relationship between the boundedness and feasibility of the problem and the requirements of the margins of the contingent claims are studied in the latter sections, stating the conditions under which a buyer should buy a claim that is offered by the writer. The model is then extended to analyze the effects of the differences in risk aversions and transaction costs. Then, the pre-existing liability positions or endowments are introduced and analyzed to see their impact on the model and it is seen that pre-existing liability structure or endowments of the market players are the reasons to trade in options. The probabilistic setting of [3] will be considered throughout the thesis.

Pinar et al. [4] study the problem of pricing and hedging contingent claims in a multi-period, linear programming setting. A concept called a λ gain-loss opportunity that is built on the Expected Gain to Loss ratio of Bernardo and Ledoit is introduced. Investors can seek a λ gain-loss opportunity in the market in absence of an arbitrage opportunity where λ stands for the loss aversion. The concept of a λ gain-loss opportunity is similar to the notion of a good-deal but the definitions are not based on ratios. Hence, resulting optimization problems are easier to analyze. The discrete time, discrete state stochastic programming that is developed by King [3] is used in the paper. The stochastic linear programming framework allows adding variables and constraints to the model and conducting numerical analysis. Firstly, the general probabilistic setting and the relationship between arbitrage and martingales are stated. Then, a stochastic linear program to seek a λ gain-loss opportunity in the market is formed. The necessary conditions for a λ gain-loss opportunity to exist in the market are stated. The cut-off value of the risk-aversion parameter is searched, and it is observed that, as the risk aversion parameter goes to infinity, the bounds of the prices not allowing a λ gain-loss opportunity converge to the no-arbitrage bounds. On the other hand, they converge to a unique value when the risk aversion parameter goes to the smallest value not allowing a λ gain-loss opportunity. Then, the financing problems are taken from the buyer's and the writer's perspectives. The problem is also considered under the assumption of proportional transaction costs. It is shown that the pricing bounds obtained are tighter than the no-arbitrage pricing bounds. The stochastic programming framework used to seek λ gain-loss opportunities forms the basis of our study in this thesis. Our main point of departure is modeling losses by CVaR instead of expected terminal wealth positions.

The key article about the optimization of CVaR by Rockefellar and Uryasev [5] will be summarized to give an in-depth understanding of the concept of CVaR. The authors introduce a new approach to minimize the CVaR of a portfolio using linear programming and non-smooth optimization techniques. It is well-known that risk management has been a concern of financial world for a long time and that the risk management techniques have been developing rapidly in recent years. VaR has been a popular risk measure, however it lacks some important mathematical characteristics such as convexity and sub-additivity which are among necessary characteristics of a coherent risk measure. That means the VaR of a combined portfolio can be larger than the sum of the VaRs of its components due to lack of sub additivity which constitutes a problem when it is required to aggregate risks of individual VaR values, and bring them together to get statistical predictability. CVaR, on the other hand, has been developed as an alternative measure of risk and is shown to be a coherent measure with strong mathematical characteristics. CVaR can be defined as the conditional expectation of the losses associated with a portfolio given that the loss at a given percentile is VaR or greater. The new approach developed by the authors that minimizes the CVaR is closely related to minimizing the VaR of the portfolio as the definitions ensure that portfolios with a small VaR necessarily have small CVaR. The important feature of the new approach is the characterization of CVaR and VaR in terms of an auxiliary function and showing that minimizing this convex and continuously differentiable function is equivalent to minimizing CVaR. Then, applications to portfolio optimization and hedging are presented to show the validity of the new approach through numerical examples. We will use the discrete-time version of this function to model losses by CVaR in the concept of λ gain-loss opportunities.

Chapter 3

Preliminaries

In this chapter, the general probability setting and the concepts of arbitrage and martingales are introduced. The connection between the arbitrage and martingales will be given through Theorem 1. Then, the financing of contingent claims, the positions of the writer and the buyer and the no-arbitrage interval will be discussed. We will start with the general probabilistic setting below.

3.1 Probabilistic Setting

Throughout the thesis, we will follow the general probabilistic setting of [3]. The behavior of the stock market is approximated by assuming that all asset values are random variables that are supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms ω are sequences of real valued vectors (security prices and payments) over the discrete time periods $t = 0, 1, \ldots, T$. In addition, we assume that the market evolves as a discrete scenario tree. In the scenario tree, the partition of probability atoms $\omega \in \Omega$ which are generated by matching path histories up to time t corresponds one-to-one with nodes $n \in N_t$ at level t in the tree. The root node $n = 0$ corresponds to trivial partition $N_0 = \Omega$, and the leaf nodes $n \in N_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$.

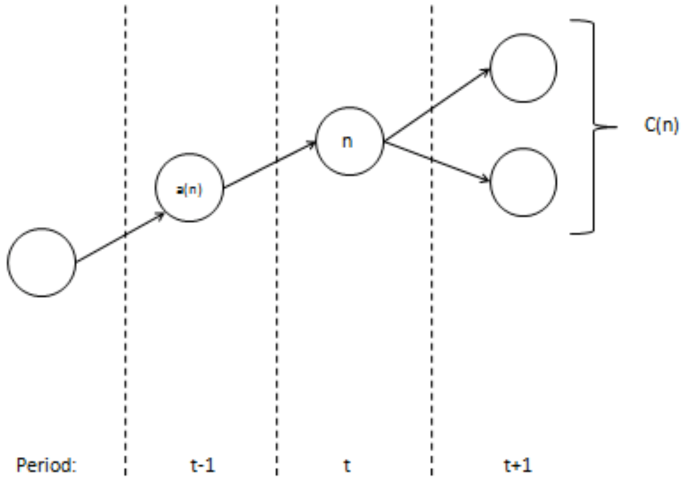

Figure 3.1: Scenario Tree

As represented in the figure above in the scenario tree, every node $n \in N_t$ for $t = 1, \ldots, T$ has a unique parent node denoted by $a(n) \in N_{t-1}$, and every node $n \in N_t$, $t = 0, 1, \ldots, T-1$ has a nonempty set of child nodes denoted by $C(n) \subset N_{t+1}$.

The probability distribution P is modeled by assigning positive weights p_n to each leaf node $n \in N_T$. The weights p_n are assigned to each leaf node $n \in N_T$ in such a way that $\sum_{n \in N_T} p_n = 1$. Each intermediate level node in the tree receives a probability mass equal to the combined mass of the paths passing through it.

$$p_n = \sum_{m \in C(n)} p_m \quad \forall n \in N_t, \quad t = T-1, \ldots, 0.$$

The ratios p_m/p_n, $m \in C_n$, are the conditional probabilities that the child node m occurs given that the parent node $n = a(m)$ has occurred.

The function $X : \Omega \to R$ is a real-valued random variable if $\{\omega : X(\omega) \leq r\} \in \mathcal{F}\ \forall r \in R$. Let X be a real-valued random variable. X can be lifted to N_t if it can be assigned a value on each node of N_t that is consistent with its definition on Ω, [3]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of N_t. A stochastic process $\{X_t\}$ is a time indexed collection of random variables such that each X_t is measurable with respect to N_t. The expected value of X_t is uniquely defined by

$$E^P[X_t] := \sum_{n \in N_t} p_n X_n.$$

The conditional expectation of X_{t+1} on N_t

$$E^P[X_{t+1}|N_t] := \sum_{m \in C(n)} \frac{p_m}{p_n} X_m$$

is a random variable taking values over the nodes $n \in N_t$.

3.2 Arbitrage and Equivalent Martingale Measures

The market consists of $J+1$ tradable securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = \left(S_n^0, \ldots, S_n^J\right)$. Suppose as in [8] that one of the securities always has strictly positive values at each node of the scenario tree. Let security 0 be such security. This security which corresponds to the risk-free asset in the classical valuation framework is chosen to be numéraire. Introducing the discount factors $\beta_n = 1/S_n^0$ we define the discounted security prices relative to the numéraire and denote it by $Z_n = \left(Z_n^0, \ldots, Z_n^J\right)$ where $Z_n^J = \beta_n S_n^J$ for $j = 0, 1, \ldots, J$. Note that, $Z_n^0 = 1$ in any state n.

The amount of security j held by the investor in state $n \in N_t$ is denoted by θ_n^j. Therefore, the value of the portfolio discounted with respect to the numéraire in state n is

$$Z_n \cdot \theta_n := \sum_{j=0}^{J} Z_n^j \theta_n^j.$$

Arbitrage can be defined as a sequence of portfolio holdings that begins with a zero initial value, makes self-financing portfolio transactions and attains a non-negative value in each future state, while in at least one terminal state it attains a strictly positive value with positive probability. It can be interpreted as making something out of nothing.

The condition of self-financing portfolio transactions

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{a(n)}, \qquad n > 0$$

states that the funds available for investment at state n are restricted to the funds generated by the price changes in the portfolio held at state $a(n)$.

The following optimization problem, called a stochastic program, is used to find an arbitrage.

$$\begin{aligned}
\max \qquad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\
& s.t. \\
Z_0 \cdot \theta_0 &= 0 \\
Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= 0, \qquad \forall n \in N_t, t \geq 1 \\
Z_n \cdot \theta_n &\geq 0, \qquad \forall n \in N_T
\end{aligned}$$

A positive optimal value for this stochastic program corresponds to an arbitrage. The program begins with a 0 valued portfolio, makes self-financing trades a each step, has a positive expected value at time T. Moreover, the problem is unbounded if the opportunity of arbitrage exists. The solution that yields a positive optimal value can be turned into an arbitrage as shown by Harrison and Pliska [8]. On the other hand, if no arbitrage is possible, the price process is called an arbitrage-free market price process.

A martingale is a stochastic process such that the expected value of the next observation, given all the past observations, is equal to the last observation.

In other words, the value of each coordinate of Z_n is equal to its conditional expectation one step ahead. The following definition is a mathematical expression of this definition.

Definition 1 *If there exists a probability measure* $Q = \{q_n\}_{n \in N_t}$ *such that*

$$Z_t = E^Q\left[Z_{t+1}|N_t\right] \quad (t \leq T-1) \tag{3.1}$$

then the vector process $\{Z_t\}$ *is called a vector-valued martingale under* Q*, and* Q *is called a martingale probability measure for the process.*

Two martingale measures are equivalent as defined in [9] whenever their null sets coincide. The definition below states this relationship.

Definition 2 *A discrete probability measure* $Q = q_{n n \in N_t}$ *is said to be equivalent to a discrete probability measure* $P = p_{n n \in N_t}$ *if* $q_n > 0$ *exactly when* $p_n > 0$*.*

The following Theorem proved by King [3] establishes the relationship between arbitrage and martingales which is of great importance to our study.

Theorem 1 *The discrete state stochastic vector process* $\{Z_t\}$ *is an arbitrage-free market price process if and only if there is at least one probability measure* Q *equivalent to* P *under which* $\{Z_t\}$ *is a martingale.*

3.3 Financing of Contingent Claims and Positions of the Writer and the Buyer

Any asset or security whose value depends upon other assets is called a contingent claim. Suppose that F is such a security, then it has payouts F_n, $n > 0$ depending on the states n of the market. Currency futures and equity options are examples of traded contingent claims. Now suppose that we would like to determine the minimum initial investment that is needed to generate payouts F_n through self-financing transactions using a riskless asset and the underlying security without the risk that the terminal positions can be negative at any state. The following stochastic program determines the minimum amount F_0 required to hedge the claim F that produces payouts F_n with no risk.

$$\begin{aligned} \min \quad & Z_0 \cdot \theta_0 \\ & s.t. \\ Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] = -\beta_n F_n & \quad \forall n \in N_t, t \geq 1 \\ Z_n \cdot \Theta_n \geq 0 \qquad & \quad \forall n \in N_T \end{aligned} \tag{3.2}$$

The dual of this problem equals to the maximum expected value of the discounted payouts over all martingale measures which is,

$$\max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right].$$

Then, we can write the proposition below which is proved by King [3].

Proposition 1 *Let F_n be a contingent claim on an arbitrage-free market price process $\{Z_t\}$. The claim is attainable if and only if its price F_0 satisfies*

$$\beta_0 F_0 \geq \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right] \tag{3.3}$$

where $\mathcal{M} = \left\{Q : Z_t = E^Q\left[Z_{t+1}\left|N_t\right.\right](t \leq T-1)\right\}.$

3.3.1 Position of the Writer

This section will discuss the position of the writer of the contingent claim. The writer of the claim receives F_0 from the buyer of the claim at state $n = 0$ and pays F_n in each state $n > 0$ in the future. In the meantime, the writer invests this money to generate a profit to maximize expected value at the end of the horizon while hedging the claim. The problem of the writer can be modeled as the stochastic program

$$\begin{aligned} \max \quad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\ & s.t. \\ Z_0 \cdot \theta_0 &= \beta_0 F_0 \\ Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= -\beta_n F_n \qquad \forall n \in N_t, t \geq 1 \\ Z_n \cdot \theta_n &\geq 0 \qquad \forall n \in N_T. \end{aligned}$$

The necessary and the sufficient condition needed for the writer's problem to have an optimal solution and the condition on the price F_0 charged by the writer are derived in the following theorem proved by King [3].

Theorem 2 *The writer's problem has an optimum if and only if*

1. *The collection of equivalent martingale probability measures on the market price process* $\{Z_t\}$ *is nonempty, and*

2. *The price* F_0 *charged by the writer to generate payouts* F_n *satisfies*

$$\beta_0 F_0 \geq \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t\right]. \tag{3.4}$$

Furthermore, this price is invariant under the changes of the original probability measure P.

Therefore, the writer's minimum acceptable price to sell the claim is

$$F_0^{writer} = \beta_0^{-1} \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.5}$$

3.3.2 Position of the Buyer

This section analyzes the position of the buyer of the contingent claim. The buyer of the claim pays F_0 to the writer at state $n = 0$ and receives payments F_n in each state $n > 0$ in the future. Like the writer, the buyer wishes to maximize expected value at the end of the horizon by trading. The problem of the buyer can be modeled as the following stochastic program

$$\begin{aligned} \max \quad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\ & s.t. \\ Z_0 \cdot \theta_0 &= -\beta_0 F_0 \\ Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= \beta_n F_n \qquad \forall n \in N_t, t \geq 1 \\ Z_n \cdot \theta_n &\geq 0 \qquad\qquad \forall n \in N_T. \end{aligned}$$

The results derived for the writer's problem are independent of the sign of F. Therefore, the buyer's acceptable price to buy the claim can be computed by reversing the signs in the equation derived in the writer's problem. Hence, the buyer's acceptable price F_0 satisfies

$$\beta_0 F_0 \leq \min_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.6}$$

Therefore, the buyer's maximum acceptable price to buy the claim is

$$F_0^{buyer} = \beta_0^{-1} \min_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.7}$$

In the previous section, we have stated that the writer's minimum offering price was

$$F_0^{writer} = \beta_0^{-1} \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right].$$

Then we have, $F_0^{buyer} \leq F_0^{writer}$ and the interval $[F_0^{buyer}, F_0^{writer}]$ is called the no-arbitrage interval.

Chapter 4

Modeling Losses as 'CVaR'

In this part of the thesis, we introduce our model which develops the concept of λ gain-loss opportunities using CVaR when measuring losses. In our study, we assume that the scenario tree of the financial market evolves as described in Chapter 3. Before moving on to formulation of the model, we shall elaborate on the concepts of λ gain-loss opportunities and CVaR.

4.1 λ Gain- Loss Opportunities

Firstly, a λ gain-loss opportunity occurs when it is possible to form a portfolio such that the difference between the gains and λ times the losses is positive with a positive probability at the terminal state where we start with a zero valued initial portfolio. When an arbitrage opportunity does not exist in the market, this kind of criteria enable investors to determine the attractiveness of an investment. As we have stated earlier, gain-loss ratio of Bernardo and Ledoit [2] and good-deals of Cochrane and Saa-Requejo[1] are other examples of such a criterion. We can formulate this as follows:

Let $Z_n \cdot \theta_n = x_n^+ - x_n^-$ for $n \in N_T$ where x_n^+ and x_n^- are non-negative numbers. This means that the final portfolio value at terminal state n can be written as the difference of two non-negative numbers. Suppose that there exists a set of vectors θ_n, $\forall n \in N$ such that:

$$
\begin{aligned}
& Z_0 \cdot \theta_0 && = 0 \\
& Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] && = 0, \quad \forall n \in N_t, t \geq 1 \\
& E^P\left[X^+\right] - \lambda E^P\left[X^-\right] && > 0,
\end{aligned}
$$

Where $\lambda > 1$ and $X^+ = x^+_{n\, n \in N_T}$, $X^- = x^-_{n\, n \in N_T}$.

Such portfolio holdings are said to allow a "λ gain-loss opportunity at level λ". Formulating the problem as a linear program provides us computational ease as well as the benefit of the ability of adding extra constraints to the model when needed. Therefore, we can capture the problem of an investor seeking a λ gain-loss opportunity even if an arbitrage opportunity does not exist in the stochastic linear program below:

$$
\begin{aligned}
& \max \sum_{n \in N_T} p_n x_n^+ - \lambda \sum_{n \in N_T} p_n x_n^- && && (4.1) \\
& s.t. && && (4.2) \\
& Z_0 \cdot \theta_0 && = 0 && (4.3) \\
& Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] && = 0, \quad \forall n \in N_t, t \geq 1 && (4.4) \\
& Z_n \cdot \theta_n - x_n^+ + x_n^- && = 0, \quad \forall n \in N_T && (4.5) \\
& x_n^+ && \geq 0, \quad \forall n \in N_T && (4.6) \\
& x_n^- && \geq 0, \quad \forall n \in N_T && (4.7) \\
& && && (4.8)
\end{aligned}
$$

The solution is said to allow a λ gain-loss opportunity at level λ if there is an optimal solution to the above problem with a positive optimal value. Conversely, the discrete state stochastic vector process $\{Z_t\}$ does not admit a λ gain-loss opportunity at level λ if the value of the stochastic program is zero. Moreover, Pinar et al. [4] proves that if the market price process does not admit a λ gain-loss opportunity at level λ, then there exists an equivalent measure that makes the price process a martingale.

4.2 Losses as CVaR

In our model, the risk component of the objective function is modeled by the CVaR measure instead of the expected value of negative terminal wealth positions. The main motivation to express the risk component of our model using CVaR instead of the expected value of negative terminal wealth positions is that CVaR is a conservative measure of risk with strong mathematical characteristics. Moreover, unlike VaR, CVaR accounts for potential losses beyond itself and measures extreme risk. VaR can be defined as the maximum tolerable loss that could occur with a given probability within a given period of time, i.e, losses larger than VaR occur with probability not exceeding α, where α is the specified confidence level. A mathematical definition for VaR can be given as follows, let $H_C(c) = Pr(C \leq c)$ be the cdf of the random variable c and $\alpha \in (0, 1)$. Then, the VaR can be defined as:

$$H_C^{-1}(1-\alpha) = \inf\left\{t : Pr(C \leq t) \geq 1-\alpha\right\} = \inf\left\{t : Pr(C > t) \leq \alpha\right\}.$$

Although, VaR has been a popular risk measure, it lacks some important mathematical characteristics such as convexity and sub-additivity which are among necessary characteristics of a coherent risk measure. For instance, due to lack of sub additivity, VaR of the combination of two portfolios may be greater than sum of their individual VaRs or non-convexity may cause some computational difficulties. The problem underlying the VaR models is that risk assessed is limited, since the tail end of the distribution of loss is not typically assessed and VaR is criticized for not considering losses beyond itself. CVaR, on the other hand, has been developed as an alternative measure of risk and is shown to be a coherent measure with strong mathematical characteristics. CVaR can be defined as the conditional expectation of the losses associated with a portfolio given that the loss at a given percentile is VaR or greater.

Minimization of the CVaR of the portfolios can be modeled by linear programming as shown by Rockafellar and Ursayev [5]. The new approach developed by the authors to minimize the CVaR is closely related to minimizing the VaR of the portfolio as the definitions ensure that portfolios with a small VaR necessarily have small CVaR. The function $f_\alpha(X^-, \gamma)$ developed by Rockafellar and Uryasev [5] will be used to model the negative terminal wealth positions in developing the concept of λ gain-loss opportunities in our model, which is defined as:

$$f_\alpha(X^-, \gamma) := \gamma + (1/1-\alpha) \sum_{n \in N_T} p_n \max(0, x_n^- - \gamma).$$

Now, we will discuss the development of this function according to [5]. Let $f(x, y)$ be the loss associated with the decision vector x, that is chosen from a set $X \in R^n$ and the random vector $y \in R^m$. We can interpret the vector x to represent a portfolio where X represents the set of available portfolios. The vector y stands for uncertainties in the market that could have an affect on the loss. For each x the loss $f(x, y)$ is a random variable having a distribution in R determined by the distribution of y. For simplicity, the underlying probability distribution of y is assumed to have a density, denoted by $p(y)$.

The probability of $f(x, y)$ not exceeding a threshold γ is given by:

$$\Psi(x, \gamma) = \int_{f(x,y) \leq \gamma} p(y) dy.$$

As a function of γ for fixed x, Ψ is the cumulative distribution function for the loss associated with x which is of fundamental importance when determining VaR and CVaR. Here, again for simplicity we can make one more assumption that $\Psi(x, \gamma)$ is everywhere continuous with respect to γ.

Let $\gamma_\alpha(x)$ and $\phi_\alpha(x)$ represent α-VaR and α-CVaR respectively for the loss random variable associated with x and a specified probability level $\alpha \in (0, 1)$. Then,

$$\gamma_\alpha(x) = \min\{\gamma \in R : \Psi(x,\gamma) \geq \alpha\}$$

and

$$\phi_\alpha(x) = (1-\alpha)^{-1} \int_{f(x,y)\geq\gamma_\alpha(x)} f(x,y)p(y)dy.$$

The logic behind these formulations is as follows: The first formula gives us the left endpoint of the nonempty interval consisting of the γ values satisfying $\Psi(x,\gamma) = \alpha$, as $\Psi(x,\gamma)$ is continuous and nondecreasing with respect to γ. In the second formula the probability that $f(x,y) \geq \gamma_\alpha(x)$ is equal to $1-\alpha$. Hence $\phi_\alpha(x)$ gives us the conditional expectation of the loss associated with x relative to the loss being $\gamma_\alpha(x)$ or greater.

The next step to get to the function that we use is the definition of the function F_α on $X \times R$ which is a characterization of $\phi_\alpha(x)$ and $\gamma_\alpha(x)$.

$$F_\alpha(x,\gamma) = \gamma + (1-\alpha)^{-1} \int_{y\in R^m} [f(x,y)-\gamma]^+ p(y)dy,$$

where $[t]^+ = t$, when $t > 0$ and $[t]^+ = 0$, when $t \leq 0$. The following theorems are proved by Rockafellar and Uryasev [5]:

Theorem 3 *As a function of α, $F_\alpha(x,\gamma)$ is convex and continuously differentiable. The α-CVaR of the loss associated with any $x \in X$ can be determined from the formula*

$$\phi_\alpha(x) = \min_{\gamma\in R} F_\alpha(x,\gamma).$$

Theorem 4 *Minimizing the α-CVaR of the loss associated with x over all $x \in X$ is equivalent to minimizing $F_\alpha(x,\gamma)$ over all $(x,\gamma) \in X \times R$ in the sense that*

$$\min_{x\in X} \phi_\alpha(x) = \min_{(x,\gamma)\in(X\times R)} F_\alpha(x,\gamma).$$

Shapiro et al. [6] explain the idea behind the development of this function in a similar way with a similar notation. Suppose that we want to satisfy

$$VaR[C_x] \leq 0$$

at a specified confidence level α. Since VaR was already defined as

$$VaR_\alpha[c] = \inf \{t : Pr(C \leq t) \geq 1 - \alpha\}.$$

we can rewrite the constraint we want to satisfy as

$$Pr(C_x > 0) = E[1_{(0,\infty)}(C_x)] \leq \alpha$$

where the function $1_{(0,\infty)}(.)$ is the indicator function such that $1_{(0,\infty)}(c) = 0$ if $c \leq 0$ and $1_{(0,\infty)}(c) = 1$ if $c > 0$.

The difficulty with this constraint is that the step function is non-convex and discontinuous at zero. Therefore, a convex approximation of the expected value can be constructed as follows: Let φ $R \to R$ be a nonnegative valued, nondecreasing, convex function such that

$$\varphi(c) \geq 1_{(0,\infty)}(c) \forall c \in R.$$

Since $1_{(0,\infty)}(tc) = 1_{(0,\infty)}(c), \forall t > 0$ and $c \in R$, we have

$$\varphi(tc) \geq 1_{(0,\infty)}(c).$$

Therefore, the inequality $\inf_{t>0} E[\varphi(tC)] \geq E[1_{(0,\infty)}(C)]$ holds. We can then approximate the constraint

$$VaR[C_x] \leq 0$$

as

$$\inf_{t>0} E[\varphi(tC_x)] \leq \alpha.$$

It is obvious that the approximation is better if the function $\varphi(.)$ is smaller. Therefore, the best choice of $\varphi(.)$ would be to take the piecewise linear function:

$$\varphi(c) := [1 + \epsilon c]_+ \text{ for some } \epsilon > 0.$$

But, the constraint

$$\inf_{t>0} E[\varphi(tC_x)] \leq \alpha$$

is invariant to the scale changes and hence the best choice of the function becomes

$$\varphi(c) = [1 + c]_+.$$

With this function, the initial constraint becomes,

$$\inf_{t>0}(tE[t^{-1} + C]_+ - \alpha) \leq 0,$$

or dividing both sides of the inequality by α we get,

$$\inf_{t>0}(\alpha^{-1}E[C + t^{-1}]_+ - t^{-1}) \leq 0.$$

Replacing t with $-t^{-1}$ we obtain,

$$\inf_{t<0}(t + \alpha^{-1}E[C - t]_+) \leq 0.$$

The quantity,

$$CVaR_\alpha(c) := \inf_{t \in R}(t + \alpha^{-1}E[C - t]_+)$$

is called the conditional value (or average value) at risk of C at level α which in turn corresponds to our function.

We will introduce auxiliary variables u_n in order to incorporate the function

$$f_\alpha(X^-, \gamma) = \gamma + (1/1 - \alpha) \sum_{n \in N_T} p_n \max(0, x_n^- - \gamma).$$

into our model.

The notation that will be used throughout the thesis are summarized in the following section.

4.3 Notation

Decision Variables

θ_n^j : The amount of security j held by the investor in state $n \in N_t$

x_n^+ : Gains of the investor in the final portfolio value at terminal state n

x_n^- : Losses of the investor in the final portfolio value at terminal state n

u_n : Auxiliary variables introduced for the function $\max(0, x_n^- - \gamma)$ $\forall n \in N_T$

γ : The threshold value that the loss function does not exceed, namely value at risk.

Parameters

λ: Loss Aversion parameter

α: Parameter specifying the level of confidence

p_n: Probability weights assigned to each leaf node n

Z_n: The vector of security prices at node n

4.4 Formulation and Constraints

With above specifications, the mathematical formulation of the model that we refer to as $(P1)$ can be formulated as follows:

$$\max \sum_{n \in N_T} p_n x_n^+ - \lambda(\gamma + \frac{1}{1-\alpha} \sum_{n \in N_T} p_n u_n)$$

$s.t.$

$$Z_0 \cdot \theta_0 = 0 \tag{4.1}$$

$$Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] = 0, \qquad \forall n \in N_t, t \geq 1 \tag{4.2}$$

$$Z_n \cdot \theta_n - x_n^+ + x_n^- = 0, \qquad \forall n \in N_T \tag{4.3}$$

$$x_n^+ \geq 0, \qquad \forall n \in N_T \tag{4.4}$$

$$x_n^- \geq 0, \qquad \forall n \in N_T \tag{4.5}$$

$$u_n \geq 0, \qquad \forall n \in N_T \tag{4.6}$$

$$u_n \geq x_n^- - \gamma, \qquad \forall n \in N_T \tag{4.7}$$

Here, constraint (4.1) guarantees that the funds available at initial state is zero. Constraint (4.2), known as the condition of self-financing portfolio transactions, states that the funds available for investment at state n are restricted to the funds generated by the price changes in the portfolio held at state $a(n)$. Constraint (4.3) states that the final portfolio value at terminal state n can be expressed in terms of the non-negative variables x_n^+ and x_n^-. Constraints (4.4) and (4.5) are the non-negativity constraints of the variables. Constraint (4.6) and (4.7) assure that the auxiliary variables u_n are equal to zero, when $x_n^- - \gamma \leq 0$ and to $x_n^- - \gamma$, when $x_n^- - \gamma > 0$.

The solution to $P1$ gives rise to a CVaR-λ gain-loss opportunity at level 'λ' and confidence level 'α' whenever there exists an optimal solution to the above stochastic problem with a positive optimal value. In fact, the problem is unbounded if a λ gain-loss opportunity exists. Because when the problem is solvable, by fundamental theorem of linear programming, it always has a basic optimal solution such that x_n^+, x_n^- can not both be positive. Therefore, the discrete state stochastic vector process $\{Z_t\}$ does not admit a CVaR-λ gain-loss opportunity at level λ and confidence level α if the value of the stochastic program is zero.

4.5 Exploring the effects of the parameters λ and α

Firstly, we will start with the effect of the parameter λ on the objective function. λ can be interpreted as the loss aversion parameter as the gains of the investor at the terminal state will be λ times the losses. Investors can decide on the level of loss that they are willing to undertake by specifying the parameter λ. As λ increases, the investor chooses less-risky positions, whereas when λ decreases the risk that the investor undertakes increases. When we think of the case that λ tends to infinity in the limit, we observe that we obtain the no-arbitrage problem of King defined in Chapter 3. In this case, the investor chooses near arbitrage positions. On the other hand, in the case that λ is 1, the gains and the losses of the individual will be equally shared.

Secondly, we will discuss the effect of the parameter α which is the confidence level. As we have stated above, it is useful to notice that the optimal value of the variable γ corresponds to the VaR at the specified confidence level α and the expression $\gamma + \frac{1}{1-\alpha}\sum_{n\in N_T} p_n \max(0, x_n^- - \gamma)$ corresponds to the CVaR at level α. The figure below is useful to illustrate the relationship between VaR, CVaR and α.

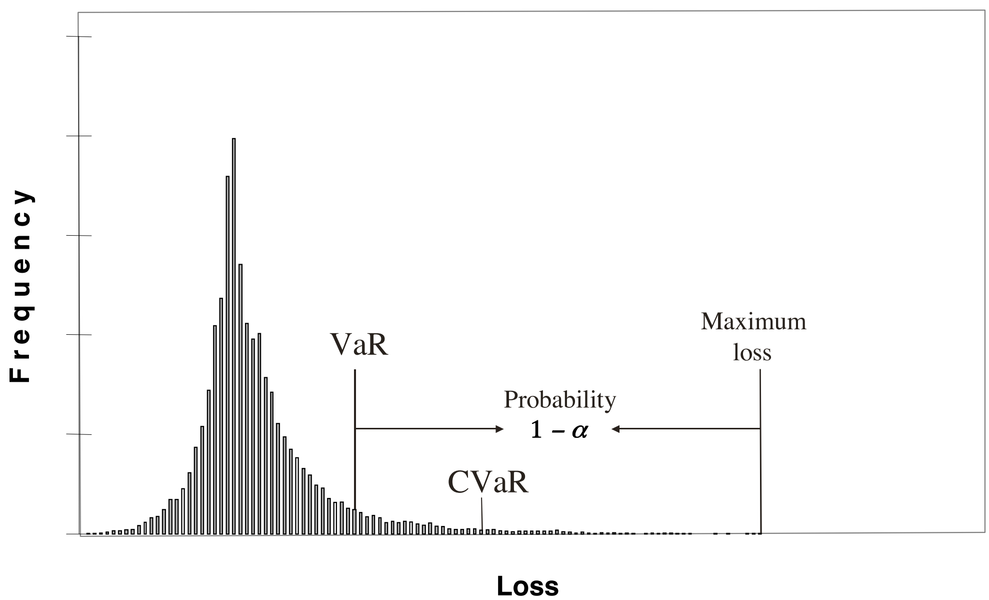

Figure 4.1: VaR, CVaR and α

Now, let us suppose that α is zero. This implies that γ is zero because the value at risk at level $\alpha = 0$ is zero. When we insert $\alpha = 0$ and $\gamma = 0$ to our objective function, we obtain the problem of a λ gain-loss opportunity that was given in section 4.1. This is expected because the effect of CVaR also decreases as the effect of α decreases. CVaR is defined as expected loss in the worst $q\%$ of the cases where $q = (1-\alpha) \times 100$. When we set $\alpha = 0$, we cover 100% of the cases and CVaR equals the expected value of losses. This results in measuring losses by expected values of negative terminal wealth positions instead of CVaR in our model. On the contrary, let us suppose that α is increased to 1, this means that $(1-\alpha)$ is zero. Then, VaR becomes the maximum loss. As CVaR will be the average of a single point in this special case, we will have CVaR equals maximum loss as well. We also observe that VaR and CVaR are increasing functions of α.

This is reasonable since we expect to incur more losses as the confidence level increases. This relationship will be reflected in our λ values in the following way: The maximum value of the parameter λ allowing a CVaR-λ-gain loss opportunity decreases as the value of α increases. This follows from the fact that increasing α values imply higher level of expected loss as explained above; thereby leads to a smaller gain-loss opportunity.

4.6 Positions of the Writer and the Buyer

Now, we will extend our model by considering the perspectives of potential writers and buyers. First, consider the position of the writer of the contingent claim F who has received F_0 in return for a promise to pay F_n in the future, depending on the states of the market. The writer would seek an answer to the following question: What is the minimum initial investment to replicate the pay-outs F_n using securities available in the market so that the positive expected wealth at the terminal state would be greater than λ times the expected negative terminal wealth? Therefore, the writer would be interested in the solution of the following stochastic linear programming problem:

$$
\begin{array}{lll}
\min Z_0 \cdot \theta_0 & & \\
s.t. & & \\
Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] & = -\beta_n F_n, & \forall n \in N_t, t \geq 1 \\
Z_n \cdot \theta_n - x_n^+ + x_n^- & = 0 & \forall n \in N_T \\
\displaystyle\sum_{n \in N_T} p_n x_n^+ - \lambda(\gamma + 1/1 - \alpha \sum_{n \in N_T} p_n u_n) & \geq 0 & \\
x_n^+ & \geq 0, & \forall n \in N_T, \\
x_n^- & \geq 0, & \forall n \in N_T, \\
u_n & \geq 0, & \forall n \in N_T, \\
u_n & \geq x_n^- - \gamma, & \forall n \in N_T
\end{array}
$$

On the other hand, when we consider the point of view of a buyer, it is reasonable that a buyer who pays F_0 in return for a promise of payments F_n in each state $n > 0$ would be interested in the answer of the following question: What is the maximum price that I should pay for the claim such that the expected positive terminal wealth positions do not fall short of λ times the expected negative terminal wealth positions? Then, the problem of the buyer could be expressed as below:

$$
\begin{aligned}
&\max -Z_0 \cdot \theta_0 && && \\
&s.t. && && \\
&Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] && = \beta_n F_n, && \forall n \in N_t, t \geq 1 \\
&Z_n \cdot \theta_n - x_n^+ + x_n^- && = 0 && \forall n \in N_T \\
&\sum_{n \in N_T} p_n x_n^+ - \lambda(\gamma + \frac{1}{1-\alpha} \sum_{n \in N_T} p_n u_n) && \geq 0 && \\
&x_n^+ && \geq 0, && \forall n \in N_T, \\
&x_n^- && \geq 0, && \forall n \in N_T, \\
&u_n && \geq 0, && \forall n \in N_T, \\
&u_n && \geq x_n^- - \gamma, && \forall n \in N_T
\end{aligned}
$$

In this chapter, we have introduced our model with the formulations and explanations. Then, we extended the model to include the problems of writer and the buyer. In the next chapter, we will obtain the duals to the problems that we have stated. We will construct equivalent martingale measures similar to defined on Chapter 3 and obtain a price interval for prices of buyers and writers of contingent claims not allowing a CVaR-λ-gain loss opportunity in the system.

Chapter 5

Duality and Martingales

This chapter analyzes the problem discussed in Chapter 4 through an equivalent problem called the dual. We establish the connection between CVaR-λ gain-loss opportunities and martingales which is similar to the connection between arbitrage and martingales as discussed in Chapter 3.

5.1 Forming the dual problem of the model

We first examine the financial constraints in the dual corresponding to the decision variables θ_n for $n \in N_t$, $t = 0, \ldots, T$, x_n^+ for $n \in N_T$, x_n^- for $n \in N_T$, γ for $n \in N_T$ and u_n for $n \in N_T$. The first step in calculating the dual is to assign dual variables to each constraint in the model. We assign y_n as dual variables for all nodes of the financial constraints (4.1) and (4.2), w_n for constraint (4.3) $\forall n \in N_t$ and k_n for the last constraint which is constraint (4.7), $\forall n \in N_T$.

Firstly, the dual constraint corresponding to the decision variable θ_n, for $n \in N_t$, $t = 0, \ldots, T-1$ is

$$y_n Z_n = \sum_{m \in c(n)} y_m Z_m \quad n \in N_t, t = 0, \ldots, T-1. \tag{5.1}$$

Next, the dual constraint corresponding to the decision variables θ_n for $n \in N_T$ is

$$(y_n + w_n)\, Z_n = 0 \quad n \in N_T,$$

and since the first component $Z_n^0 = 1$ for all states n we have

$$y_n + w_n = 0 \quad n \in N_T \Rightarrow y_n = -w_n n \in N_T.$$

Another dual constraint corresponding to the variable x_n^+ is

$$-w_n \geq p_n \Rightarrow y_n \geq p_n \quad n \in N_T.$$

The dual constraint corresponding to the variable x_n^- is that

$$w_n \geq k_n \Rightarrow y_n \leq -k_n \qquad n \in N_T.$$

The dual constraint corresponding to the variable γ is that

$$\sum_{n \in N_t} k_n = -\lambda$$

The dual constraint corresponding to the last set of variables u_n is

$$k_n \geq -(\lambda/1 - \alpha)p_n$$

Finally, combining the above constraints, one has the following constraint in the dual.

$$p_n \leq y_n \leq -k_n \leq (\lambda/1 - \alpha)p_n, \forall n \in N_T.$$

Furthermore, we can impose the condition that

$$y_0 = \lambda$$

The reason behind this condition is as follows: Suppose that we have another problem P' with a corresponding dual problem D'. Problem P' is the same problem as P except that the variables x_n^+ are now free, which means that we have the additional constraint $w_n \leq k_n$, $\forall n \in N_T$ in D'. This means that D is a relaxation to D'. The constraints $w_n \leq k_n$, $\forall n \in N_T$ and $w_n \geq k_n$, $\forall n \in N_T$ in D' together imply that $w_n = k_n$, $\forall n \in N_T$. Now, let us suppose that there is a solution $[y_n\ w_n\ k_n]^T$ to D such that $w_n > k_n$, $\forall n \in N_T$. We will try to form a corresponding alternative solution of the form $w_n = k_n$, $\forall n \in N_T$ for every possible solution of the form $w_n > k_n$, $\forall n \in N_T$. The equality of the variables $w_n = k_n$, $\forall n \in N_T$ will imply that $\sum_{n \in N_t} w_n = -\lambda$ and hence $\sum_{n \in N_t} y_n = \lambda$. We know together from constraint (5.1) and from $Z_n^0 = 1$ that $y_n = \sum_{m \in s(n)} y_m, \forall n \in N_t, t = 0, \ldots, T-1$. This means that the sum of y_n over all states $n \in N_t$ in each time period t sums to y_0. Therefore, $\sum_{n \in N_t} y_n = \lambda$ will imply that $y_0 = \lambda$.

To get to the case when $w_n = k_n$,$\forall n \in N_T$, we can either increase k_n or decrease w_n. Firstly, let us try to increase k_n, we should check if the constraints including k_n can still be satisfied. Checking the constraints alone would be sufficient as the objective function is 0. But, this is not possible because the constraint $\sum_{n \in N_t} k_n = -\lambda$ would be violated as we can not increase λ accordingly.

Therefore,we need to decrease w_n. Now, we should check if the constraints containing w_n can still be satisfied. From $y_n = -w_n$, $\forall n \in N_T$ we see that we need to increase y_n with an increment of $(w_n - k_n)$; this equality can still be satisfied as the upper bound for y_n is $-k_n$ which we do not exceed in this case and the constraint (5.1) can be satisfied by increasing y_n's in the final period.

This shows that we can form an alternative solution of the form $w_n = k_n$,$\forall n \in N_T$ for every possible solution of the form $w_n > k_n$,$\forall n \in N_T$ and that $y_0 = \lambda$ and $w_n = k_n$,$\forall n \in N_T$ is always feasible to D. But, we also know that D is a feasibility problem and hence this feasible solution will in fact be the optimal solution.

For the signs of the dual variables we have,

$$y_n \geq 0, w_n \leq 0, k_n \leq 0.$$

The non-negativity of the variables y_n follows from the non-negativity of p_n which implies the negativity of w_n since $y_n = -w_n$ and the negativity of the last set of the variables k_n follows from the inequality that $w_n \geq k_n$. To obtain the objective function of the dual program we leave the parameters of the model at the right hand side and multiply them by respective dual variables. Hence, the objective function of the dual problem will be zero as the right hand side of the constraints in the primal problem is zero. Moreover, we can get rid of the variables w_n and k_n since $y_n = -w_n$, $\forall n \in N_T$ and $w_n = k_n$, $\forall n \in N_T$ as explained above. Therefore, the dual problem becomes a feasibility problem in the variables $y_n \geq 0$, $\forall n$.

Eventually, the dual program that we refer to as $(D1)$ is formulated as follows.

$$\min 0$$

$$s.t.$$

$$y_0 = \lambda \tag{5.4}$$

$$y_n Z_n = \sum_{m \in c(n)} y_m Z_m, \qquad \forall n \in N_t, t = 0, \ldots, T-1. \tag{5.5}$$

$$p_n \leq y_n \leq \frac{\lambda}{1-\alpha} p_n, \qquad \forall n \in N_T. \tag{5.6}$$

$$y_n \geq 0, \qquad \forall n \in N_t, t = 0, \ldots, T-1. \tag{5.7}$$

The basic theorem of linear programming states that problem $(P1)$ has an optimal solution if and only if the dual $(D1)$ does too, and both optimal values are equal. Furthermore, it follows again from the theory of linear programming that problem $(P1)$ has an optimal solution if and only if it is feasible and bounded. Moreover, $(P1)$ is bounded if and only if there exists y_n satisfying the above feasibility problem. We will connect this dual feasibility to appropriate martingale measures.

Definition 3 *Given $\lambda > 1$ and $\alpha \in [0,1]$ and a discrete probability measure $Q = \{q_n\}_{n\in N_t}$ is said to be (α,λ)-compatible to a discrete probability measure $P = \{p_n\}_{n\in N_t}$ if it is equivalent to P (as defined in Chapter 3) and satisfies*

$$(1/\lambda)\max_{n\in N_T} p_n/q_n \leq 1 \leq \frac{1}{1-\alpha}\min_{n\in N_T} p_n/q_n.$$

In Chapter 3, we have stated Theorem 1 that provides a way to interpret the absence of an arbitrage opportunity in terms of martingales. Similarly, we will prove Theorem 5 below which is essential as it relates CVaR-λ gain-loss opportunities to martingales.

Theorem 5 *The discrete state stochastic vector process Z_t does not admit a CVaR-λ gain-loss opportunity at a fixed level λ and confidence level α if and only if there is at least one probability measure $Q - (\alpha,\lambda)$ compatible to P under which Z_t is a martingale.*

Proof: Let us start with proving the necessity part first. Consider D1, for passing to the martingales, we define the process $q_n = y_n/\lambda$ for each $n \in N_T$. This defines a probability measure Q over the leaf nodes $n \in N_T$. We can rewrite D1 as the feasibility of the following system with the newly defined weights:

$$q_0 = 1 \tag{5.8}$$

$$q_n Z_n = \sum_{m\in c(n)} q_m Z_m, \qquad \forall n \in N_t, t = 0,\ldots,T-1. \tag{5.9}$$

$$(1/\lambda)p_n \leq q_n \leq \frac{1}{1-\alpha}p_n, \qquad \forall n \in N_T. \tag{5.10}$$

$$q_n \geq 0, \qquad \forall n \in N_t, t = 0,\ldots,T-1. \tag{5.11}$$

The inequality (5.10) can be rearranged to be in the following form:

$$(1/\lambda)\max_{n\in N_T} p_n/q_n \leq 1 \leq \frac{1}{1-\alpha}\min_{n\in N_T} p_n/q_n.$$

Therefore, we constructed an equivalent measure $Q - (\alpha, \lambda)$ compatible to P by Definition 3 under which Z_t is a martingale. This proves the necessity part.

To prove the reverse direction, suppose that Q is a (α, λ) compatible measure for the price process Z_t. Then, we must have;

$$q_0 = 1 \tag{5.8}$$

$$q_n Z_n = \sum_{m \in c(n)} q_m Z_m, \qquad \forall n \in N_t, t = 0, \ldots, T-1. \tag{5.9}$$

$$(1/\lambda) p_n \leq q_n \leq \frac{1}{1-\alpha} p_n, \qquad \forall n \in N_T. \tag{5.10}$$

$$q_n \geq 0, \qquad \forall n \in N_t, t = 0, \ldots, T-1. \tag{5.11}$$

and

$$(1/\lambda) \max_{n \in N_T} p_n / q_n \leq 1 \leq \frac{1}{1-\alpha} \min_{n \in N_T} p_n / q_n$$

If the above inequality is obtained as an equality, the right or left hand side of the inequality can be set as y_0 and $y_n = q_n y_0$. Otherwise, we can choose a factor y_0 in the interval $\left[(1/\lambda) \max_{n \in N_T} p_n / q_n, (\frac{1}{1-\alpha} \min_{n \in N_T} p_n / q_n\right]$ and set $y_n = q_n y_0$, $\forall n$. These values satisfy D1. Since the dual is feasible, the primal is feasible and bounded from the theory of linear programming and the system does not admit a CVaR-λ gain-loss opportunity. This concludes the proof of Theorem 3. □

We observe that Theorem 1 in Chapter 3 and Theorem 2 of [4] relating λ-gain-loss opportunities to martingales are special cases of Theorem 5 for values of $\alpha = 0$, $\lambda = \infty$ and for $\alpha = 0$ respectively.

5.2 Establishing bounds on the prices of the buyer and the writer via duality

In this section, we will construct the dual programs to the writer's and buyer's hedging problems in a similar way. We will pass to the martingale measures from these duality results. It is known that when we are pricing the assets, we adjust the calculated expected values for the risk involved by an appropriate discount factor. But, under the assumption that there is no CVaR-λ gain-loss opportunity in the market, constructing the equivalent martingale measures provides us an alternative way to do this calculation. Instead of first taking the expectation and then adjusting for risk, we can first adjust the probabilities of future outcomes such that they incorporate the effects of risk, and then take the expectation under these different probabilities. Eventually, these adjusted probabilities are called risk-neutral probabilities and they constitute the risk-neutral measure. Therefore,we will establish the price interval of the buyer and the writer both in terms of the dual problems and martingale measures.

Definition 4 *A contingent claim F with price F_0 is said to be (α, λ)-attainable if there exist vectors θ_n, $\forall n \in N$ satisfying:*

$$Z_0\theta_0 \leq \beta_0 F_0$$
$$Z_n(\theta_n - \theta_{a_n}) = -\beta_n F_n, \forall n \in N_t, t \geq 1$$
$$E^P\left[X^+\right] - \lambda f_\alpha(X^-, \gamma) = 0$$

Proposition 2 *At a fixed level $\lambda > 1$ and $\alpha \in (0, 1)$, assume that the discrete price process Z_t does not allow a CVaR λ-gain loss opportunity. Then, the minimum initial investment W_0 required to hedge the claim such that the gains at the terminal state are at least λ times the losses is;*

$$W_0 = \frac{1}{\beta_0\lambda} \max_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n.$$

where $Y(\alpha, \lambda)$ denotes the feasible set of $D1$ defined in Chapter 4.

Proof: We start by forming the dual to the writer's problem. This time, attaching the multipliers v_n, w_n and V we have;

$$
\begin{aligned}
& \max \sum_{n>0} v_n \beta_n F_n \\
& s.t. \\
& v_n Z_n = \sum_{m \in c(n)} v_m Z_m, && \forall n \in N_t, t = 0, \ldots, T-1. \\
& V p_n \leq v_n \leq V \frac{\lambda}{1-\alpha} p_n, && \forall n \in N_T. \\
& v_n \geq 0, && \forall n \in N_t, t = 0, \ldots, T-1. \\
& V \geq 0
\end{aligned}
$$

Observing that V can not be 0 as this would lead to infeasibility of the variable y_n, we insert $V = 1/y_0$ and $v_n = y_n/y_0$ to obtain

$$
\begin{aligned}
& \max_{y \in Y(\alpha,\lambda)} (1/y_0) \sum_{n>0} y_n \beta_n F_n \\
& s.t. \\
& y_0 = \lambda \\
& y_n Z_n = \sum_{m \in c(n)} y_m Z_m, && \forall n \in N_t, t = 0, \ldots, T-1. \\
& p_n \leq y_n \leq \frac{\lambda}{1-\alpha} p_n, && \forall n \in N_T.
\end{aligned}
$$

We note that the feasible set of this problem is the same with the feasible set of $D1$, namely $Y(\lambda)$. Furthermore, under the assumption that the system does not admit a CVaR λ gain loss opportunity we can say that there exist a feasible solution to D1, but this means that there is a feasible solution to the above problem. Since the dual of the writer problem is feasible, the writer's problem is solvable by linear programming duality.□

Therefore, we can furthermore impose the following:

Corollary 1 *At a fixed level $\lambda > 1$ and $\alpha \in (0,1)$, assume that the discrete price process Z_t does not allow a λ-gain loss opportunity. Then, the contingent claim is (α, λ) attainable if and only if ;*

$$\beta_0 F_0 \geq \frac{1}{\lambda} \max_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n.$$

The minimum acceptable price for the writer of the contingent claim becomes:

$$F_0^{writer} = \frac{1}{\beta_0 \lambda} \max_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n.$$

Moreover, we can pass to martingale measures through this dual problem. Below is the martingale representation of the problem:

$$\max_{Q \in Q(\lambda)} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right].$$

where $Q(\lambda)$ is the set of q_n's satisfying:

$$\begin{aligned}
&Z_0 = \sum_{m \in c(0)} q_m Z_m \\
&q_n Z_n = \sum_{m \in c(n)} q_m Z_m, && \forall n \in N_t, t = 1, \ldots, T-1. \\
&p_n/\lambda \leq q_n \leq p_n/(1-\alpha), && \forall n \in N_T.
\end{aligned}$$

Now, we construct the dual to the buyer's problem in a similar way:

$$
\begin{aligned}
&\min_{y \in Y(\alpha,\lambda)} (1/y_0) \sum_{n>0} y_n \beta_n F_n \\
&s.t. \\
&y_0 = \lambda \\
&y_n Z_n = \sum_{m \in s(n)} y_m Z_m, && \forall n \in N_t, t = 0, \ldots, T-1. \\
&p_n \leq y_n \leq \frac{\lambda}{1-\alpha} p_n, && \forall n \in N_T. \\
&y_n \geq 0, && \forall n \in N_t, t = 0, \ldots, T-1.
\end{aligned}
$$

And the maximum acceptable price for the buyer is:

$$
F_0^{buyer} = \frac{1}{\beta_0 \lambda} \min_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n.
$$

Passing to martingale measures, we can equivalently pose the problem of the buyer as:

$$
\min_{Q \in Q(\lambda)} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right].
$$

Now, we can state the price bounds of the writer and buyer in the following equivalent forms:

$$
\left[\frac{1}{\beta_0 \lambda} \min_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n, \frac{1}{\beta_0 \lambda} \max_{y \in Y(\alpha,\lambda)} \sum_{n>0} y_n \beta_n F_n \right]
$$

or equivalently,

$$
\left[\beta_0^{-1} \min_{Q \in Q(\lambda)} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right], \beta_0^{-1} \max_{Q \in Q(\lambda)} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right] \right]
$$

It is worth reminding that unless the two values are equal, no trading occurs.

Recall that the no-arbitrage bounds of [3] stated in Chapter 3 was:

$$\left[\beta_0^{-1} \min_{Q\in\mathcal{M}} E^Q\left[\sum_{t=1}^{T}\beta_t F_t\right], \beta_0^{-1} \max_{Q\in\mathcal{M}} E^Q\left[\sum_{t=1}^{T}\beta_t F_t\right]\right]$$

Therefore, by noting that $Q(\lambda)$ is included in $\mathcal{M}$ for fixed λ and α we say that the price interval that we obtained is tighter than the no-arbitrage interval defined on Chapter 3.

In this chapter, we have stated the dual problems to our model and to the problems of writer and buyer. We have shown that the existence of a (α, λ) compatible probability measure Q that makes the price process a martingale ensures that the price process does not allow a CVaR-λ gain-loss opportunity and vice versa. We used the dual problems of the writer's and the buyer's problems for establishing the CVaR-λ bounds. We presented these bounds also in terms of equivalent martingale measures and showed that these bounds are tighter than the no-arbitrage bounds. Next chapter will present a numerical study of our work.

Chapter 6

Experimental Study

In the preceding two chapters, we incorporated CVaR measure into the concept of λ-gain-loss opportunities and stated our model for pricing contingent claims. We then stated the dual problems to our model that allowed us to obtain price intervals for the buyer and the writer. In this chapter, we will present a numerical analysis of the model with respect to the risk aversion parameter λ and in various levels of confidence (α) to give a better understanding of the previous chapters in a discrete market model.

6.1 Calibrated Option Bounds

The model of the writer and the buyer that we defined in section 4.6 will be modified in this chapter when we are conducting the experimental analysis. Let the index set $k = 1, ..., K$ denote available contingent claims, and H^k, $k = 1, ..., K$ be contingent claims with bid and ask prices $C_b^k \leq C_a^k$ and pay-offs H_n^k. The basic modification in the model is to allow the writer and the buyer to apply buy and hold strategies on these available options. This modification results in the following problem:

$$
\begin{aligned}
&\min Z_0 \cdot \theta_0 + C_a \cdot \xi_+ - C_b \cdot \xi_- && & \\
&s.t. && & \\
&Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &&= G_n \cdot [\xi_+ - \xi_-] - \beta_n F_n, & \forall n \in N_t, t \geq 1, \\
&Z_n \cdot \theta_n - x_n^+ + x_n^- &&= 0 & \forall n \in N_T, \\
&\sum_{n \in N_T} p_n x_n^+ - \lambda(\gamma + \frac{1}{1-\alpha} \sum_{n \in N_T} p_n u_n) &&\geq 0 & \\
&x_n^+ &&\geq 0, & \forall n \in N_T, \\
&x_n^- &&\geq 0, & \forall n \in N_T, \\
&u_n &&\geq 0, & \forall n \in N_T, \\
&u_n &&\geq x_n^- - \gamma, & \forall n \in N_T, \\
&\xi_+ &&\geq 0, & \\
&\xi_- &&\geq 0. &
\end{aligned}
$$

where ξ_+^i and ξ_+^i are the amounts bought and shorted of H^k at time $t = 0$.

Similarly, the hedging strategy of the buyer when there are other options available for trading becomes:

$$
\begin{aligned}
&\max -Z_0 \cdot \theta_0 - C_a \cdot \xi_+ + C_b \cdot \xi_- && & \\
&s.t. && & \\
&Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &&= G_n \cdot [\xi_+ - \xi_-] + \beta_n F_n, & \forall n \in N_t, t \geq 1, \\
&Z_n \cdot \theta_n - x_n^+ + x_n^- &&= 0 & \forall n \in N_T, \\
&\sum_{n \in N_T} p_n x_n^+ - \lambda(\gamma + \frac{1}{1-\alpha} \sum_{n \in N_T} p_n u_n) &&\geq 0 & \\
&x_n^+ &&\geq 0, & \forall n \in N_T, \\
&x_n^- &&\geq 0, & \forall n \in N_T, \\
&u_n &&\geq 0, & \forall n \in N_T, \\
&u_n &&\geq x_n^- - \gamma, & \forall n \in N_T, \\
&\xi_+ &&\geq 0, & \\
&\xi_- &&\geq 0. &
\end{aligned}
$$

We will use the bid and ask closing prices of 48 European call and put options on the $S\&P500$ index on September 10, 2002 [7]. We will present a numerical approach for computing bounds for price bounds of an option not allowing a λ gain-loss opportunity when some other options are available for trading. Numerical tests on $S\&P500$ index show the accuracy of our proposed method. Table 1 below displays the bid and ask closing prices of these 48 European call and put options and the columns named ‘STR’ and ‘MAT’ represent the strike prices and maturities of the options respectively. The first 21 options are call options whereas the remaining 27 options are put options. We will compute CVaR bounds for each of the 48 options for various values of λ at two most common different confidence levels with $\alpha = 0.95$ and $\alpha = 0.99$ by using the remaining 47 options as market-traded claims. This means that for each option and for each value of λ at each confidence level, we will solve problems of the buyer and the writer separately using the remaining 47 options. The resulting values will then be compared with the actual market prices that are given below in Table 1

Table 6.1: Data for Call Options

Option	STR	MAT	C_b	C_a
1	890	17	31.5	33.5
2	900	17	24.4	26.4
3	905	17	21.2	23.2
4	910	17	18.5	20.1
5	915	17	15.8	17.4
6	925	17	11.2	12.6
7	935	17	7.6	8.6
8	950	17	3.8	4.6
9	955	17	3	3.7
10	975	17	0.95	1.45
11	980	17	0.65	1.15
12	900	37	42.3	44.3
13	925	37	28.2	29.6
14	950	37	17.5	19
15	875	100	77.1	79.1
16	900	100	61.6	63.6
17	950	100	35.8	37.8
18	975	100	26	28
19	995	100	19.9	21.5
20	1025	100	12.6	14.2
21	1100	100	3.4	3.8

Table 6.2: Data for Put Options

Option	STR	MAT	C_b	C_a
22	750	17	0.4	0.6
23	790	17	1	1.3
24	800	17	1.3	1.65
25	825	17	2.5	2.85
26	830	17	2.6	3.1
27	840	17	3.4	3.8
28	850	17	3.9	4.7
29	860	17	5.5	5.8
30	875	17	7.2	7.8
31	885	17	9.4	10.4
32	750	37	5.5	5.9
33	775	37	6.9	7.7
34	800	37	9.3	10
35	850	37	16.7	8.3
36	875	37	23	24.3
37	900	37	31	33
38	925	37	41.8	43.8
39	975	37	73	75
40	995	37	88.9	90.9
41	650	100	5.7	6.7
42	700	100	9.2	10.2
43	750	100	14.7	15.8
44	775	100	17.6	19.2
45	800	100	21.7	23.7
46	850	100	33.3	35.3
47	875	100	40.9	42.9
48	900	100	50.3	52.3

Generating the scenario trees and establishing a relationship between them is the main issue when conducting the experimental analysis. We have written a GAMS program that produces scenario trees according to the preferences of the user and incorporates it into the optimization model. We solve the primal problems in our analysis as they are more natural from the view of hedging and easier to set up. We will solve a three period model and we choose the branching structure as (50, 10, 10). This means that $v_1 = 50$, $v_2 = 10$ and $v_3 = 10$ which means that there are $5,000$ scenarios. We insert the Gauss-Hermite table as an input and produce a tree with given periods and the given branching structure. We will use $S = (1, S^1)$ as the dynamically traded securities where S^1 is the $S\&P500$ index. The structure of the periods is chosen according to the maturity of the options. This means that we allow the period changes to occur on days $0, 17, 37$ and 100 when the trading occurs. The scenario tree is built by approximating S^1 by the Gauss-Hermite process. We assign Gauss-Hermite quadrature values to each node for probability and volatility calculations. Therefore, we will give a more detailed information on the process of Gauss-Hermite in the following section.

6.2 Gauss-Hermite Processes

This section will summarize the Gauss-Hermite processes as in [7]. Suppose that we have an asset whose price S_t follows a continuous geometric Brownian motion with daily drift d and volatility σ. For $t = 1, ..., T$ the logarithm of the price S_t, $\xi_t = \ln S_t$ satisfies

$$\xi_t = \xi_{t-1} + d_t + e_t$$

where e_t are normally distributed with zero mean and standard deviation σ_t, and

$$\begin{aligned} d_t &= l_t d, \\ \sigma_t &= \sqrt{l_t}\sigma \end{aligned}$$

Here, l_t is the length of period t in days. Given the parameters in the above equation and the initial value ξ_0, we generate a scenario tree by using the Gauss-Hermite quadrature and obtain a sample $(e_1^{i_1})_{i_1=1}^{v_1}$ of size v_1 of e_1 with the associated probabilities $(\pi_1^{i_1})_{i_1=1}^{v_1} \subset (0, \infty)$. This enables us to approximate the possible values of the logarithmic index at time $t = 1$:

$$\xi_1^{i_1} = \xi_0 + d_1 + e_1^{i_1}, i_1 = 1, ..., v_1.$$

Similarly, we then generate a sample $(e_2^i)_{i=1}^{v_2}$ of e_2 in the second period and the possible values of the logarithmic index at time $t = 2$ are,

$$\xi_1^{i_1,i_2} = \xi_1^{i_1} + d_1 + e_2^{i_2}, i_1 = 1, ..., v_1, i_2 = 1, ..., v_2.$$

We obtain a scenario tree whose nodes N_t at time t are labeled by $(i_1, ..., i_t)$ for $t = 2, ..., T$ Defining,

$$\begin{gathered}
N = N_1 \cup ... \cup N_T, \\
\alpha(i_1, ..., i_t) = (i_1, ..., i_{t-1}), \\
C(i_1, ..., i_t) = (i_1, ..., i_{t+1}) \in N_{T+1} | i_{t+1} \in 1, ..., v_{t+1}, \\
S_n = \exp(\xi_n), \forall n \in N, \\
p(i_1, ..., i_t) = \pi_1^{i_1} ... \pi_t^{i_t}.
\end{gathered}$$

This brings us to the probabilistic setting defined in Chapter 3. Such discrete processes are called Gauss-Hermite processes [8], which converge weakly to the discrete time geometric Brownian motion [9], as the number of branches increases. The probabilities in the Gauss- Hermite processes do not depend on the parameters μ, σ or the step length l_t. For Gauss-Hermite processes, the discretized one-step conditional probabilities of the logarithmic index match a maximum number of moments of the normal distribution.

6.3 Plots

In this section, we will present the plots that we draw with respect to the risk aversion parameter λ and in various levels of confidence (α) according to the results of the stochastic program that we described in Chapter 4. The relationship between various λ values and the prices are shown on each plot. The bid-price, ask price, the price of the writer and buyer are all displayed in figures in order to make an easier comparison between the actual and predicted values. The figures for $\alpha = 0.95$ and $\alpha = 0.99$ are presented respectively for each option. It is observed that for each of the options the bounds we obtained are very close to the true values. In most cases, either one or both of the values fall between the true values. In general, better bounds are obtained when there are many benchmark options having strike prices similar to the strike price of the option that we price. This is expected since the bounds are obtained by hedging the cash-flows of the given option using market-traded options. Therefore, better hedges can be obtained when the remaining options are similar to the option that is being hedged. Since including all 96 graphs of the 48 options in here would be too much to look into we have selected a sample of the options representing relatively different patterns. We will start with Option 1.

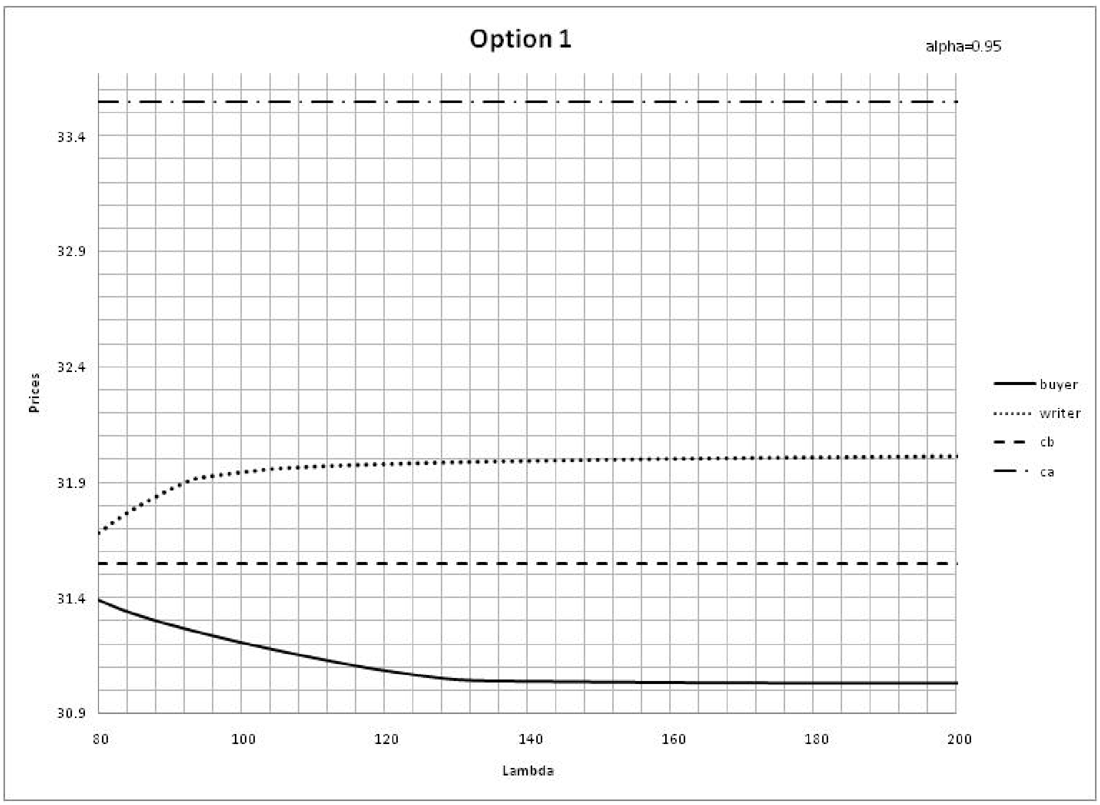

Figure 6.1: Option1, alpha=0.95

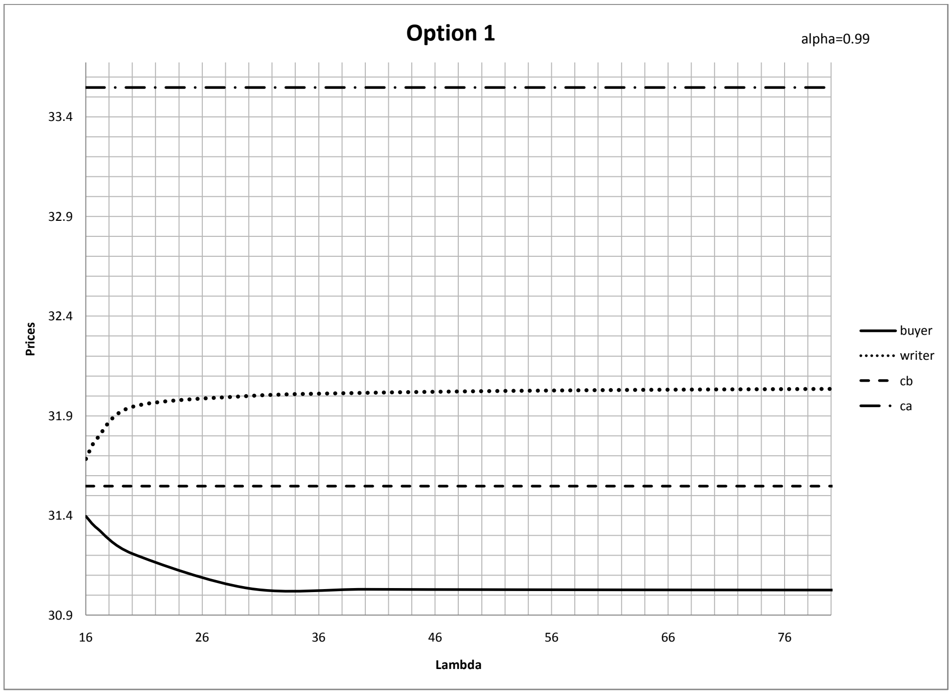

Figure 6.2: Option1, alpha=0.99

Figures 6.1 and 6.2 show the relationship between various λ values and the price of the first option. It is observed that the same values for the prices of writer and buyer are obtained for different values of λ in different confidence levels. We see that the limiting value of the loss-aversion parameter is 80 at $\alpha = 0.95$ and 16 at $\alpha = 0.99$ which is the same for all options. This means that the minimum value of λ not allowing a CVaR-gain-loss opportunity in the system is 80 at $\alpha = 0.95$ and 16 at $\alpha = 0.99$. This is expected because increasing the confidence level would decrease the supremum of the λ values allowing a CVaR-λ-gain-loss opportunity. It can be seen from the figure that the price of the buyer starts with the value of 31.348 for the limiting value of $\lambda = 80$ and converges to the value of 30.97 for $\lambda = 500$. For the price of the writer, these values are 31.637 and 32.987 respectively. These values are obtained for the limiting values of $\lambda = 16$ at $\alpha = 0.99$ and $\lambda = 80$ at $\alpha = 0.95$. We also observe that when λ tends to ∞, the bounds obtained from our model converge to no-arbitrage bounds as expected.

Since the peak is slower for bigger values of λ and the convergence of the curve can be noticed, the biggest value of λ that we include in the figures will be 200 for $\alpha = 0.95$ and 80 for $\alpha = 0.95$. We see for Option 1 that the price of the writer is within the bounds of the true price interval and converges to the actual price of the writer. The price of the buyer however starts with a very close value to the actual price of the buyer in the limiting case of λ and converges to a lower value as λ is increased gradually.

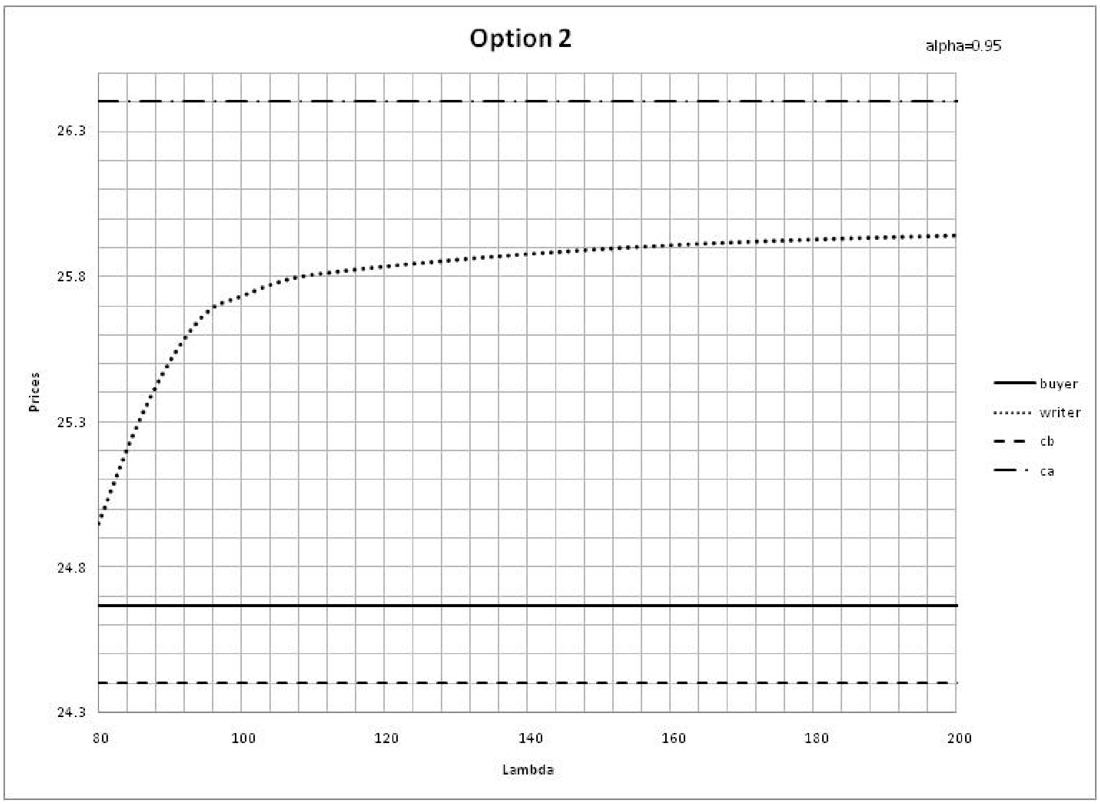

Figure 6.3: Option2,alpha=0.95

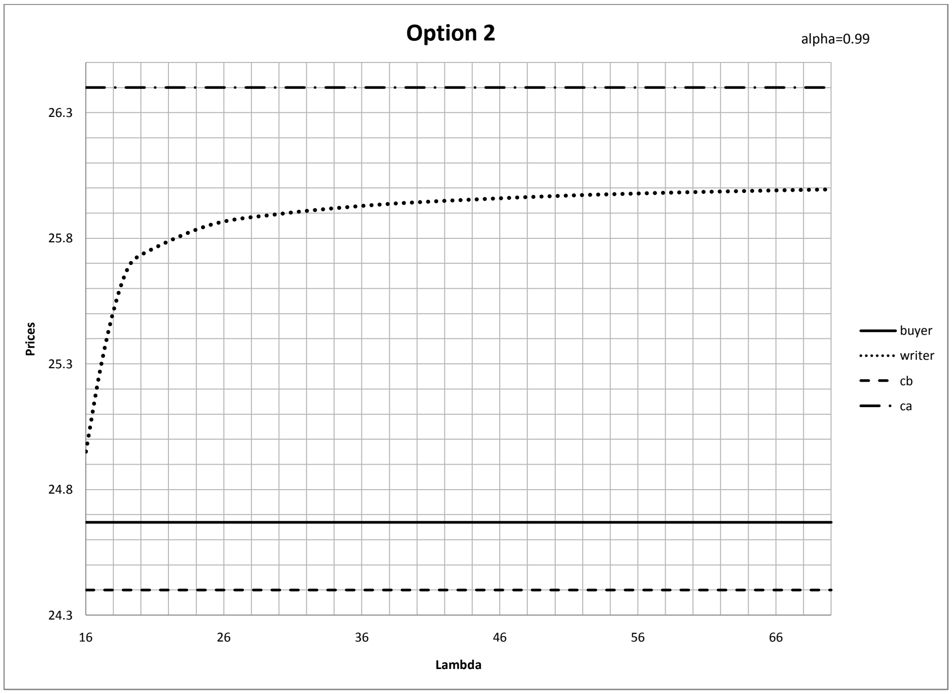

Figure 6.4: Option2, alpha=0.99

Figures 6.3 and 6.4 show the values we obtained for Option 2 along with the true bid-ask prices. Option 2 is an example of the options where both of the predicted values fall within the bounds of true prices. Here, whereas the price of the buyer is constant at the value 24.67, the price of the writer starts with the value of 24.95 for the limiting value of $\lambda = 80$ and converges to the value of 26 for $\lambda = 500$ at $\alpha = 0.95$. Same prices are obtained both for the writer and the buyer for $\lambda = 16$ and $\lambda = 80$ at $\alpha = 0.99$ respectively. Unlike Option 1, we see that the CVaR-λ bounds are contained in the interval $[c_b, c_a]$.

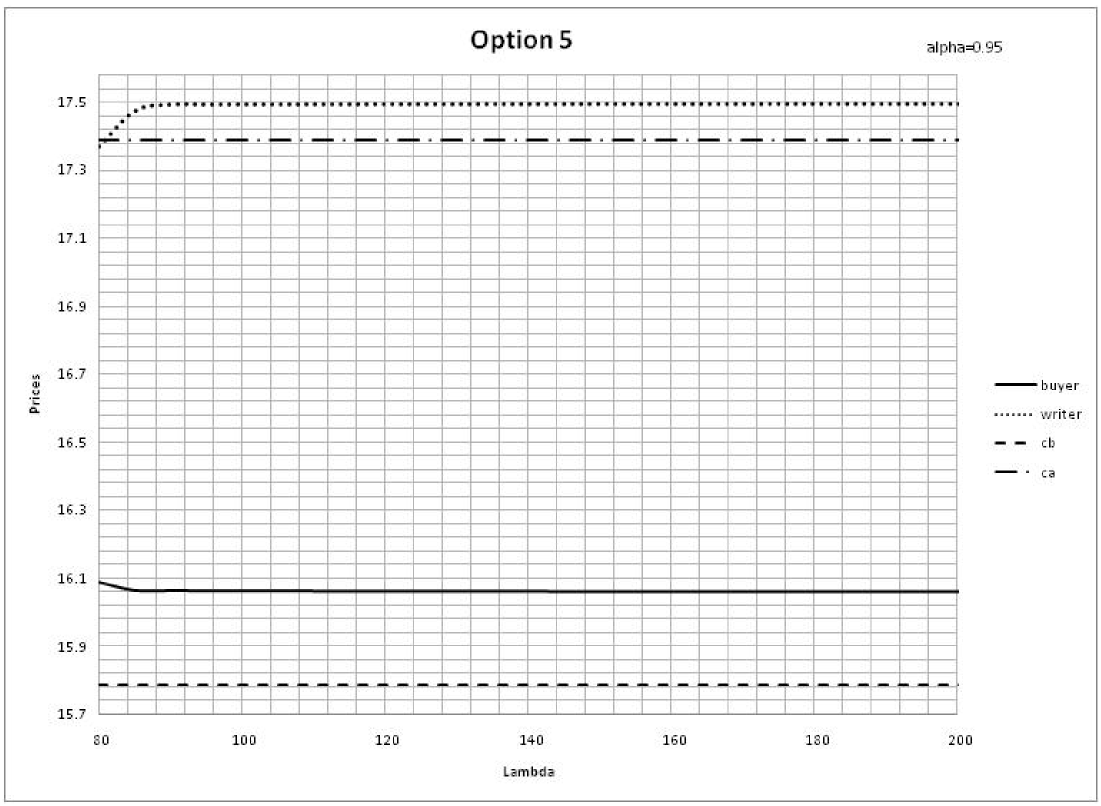

Figure 6.5: Option5, alpha=0.95

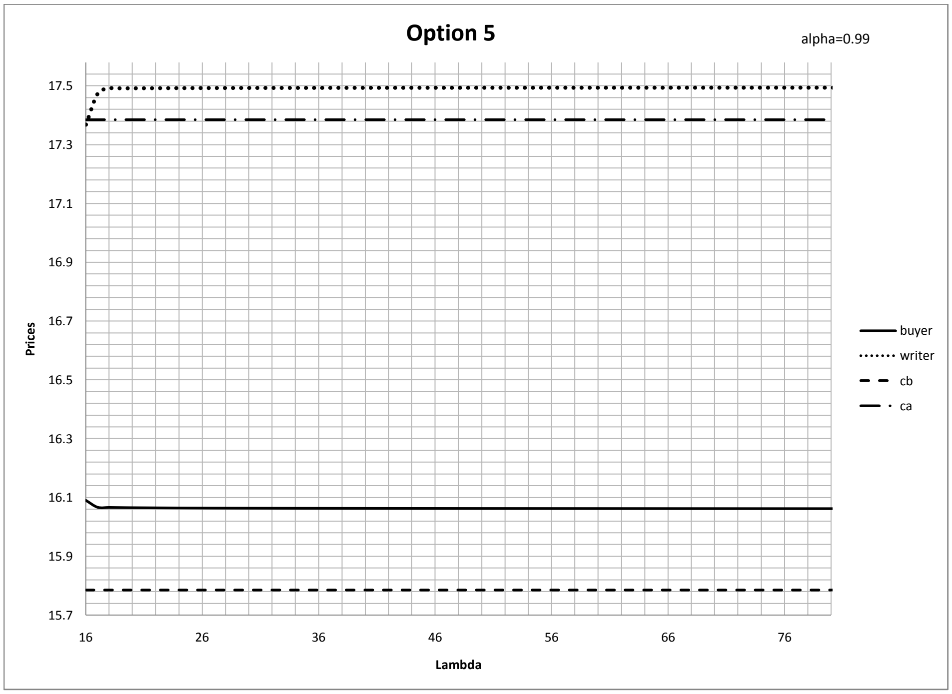

Figure 6.6: Option5, alpha=0.99

The next option we include is Option 5 that is shown in figures 6.5 and 6.6. Starting with the value of 16.105 for the buyer and 17.383 for the writer the values converge to 16.076.786 and 17.509 respectively. Option 5 is an example where only the price of the buyer is within the interval of true prices. Here, the price of the buyer was within the limits in the limiting case of λ. However, as λ increases, the price of the writer also increases, slightly passing the actual price of the writer.

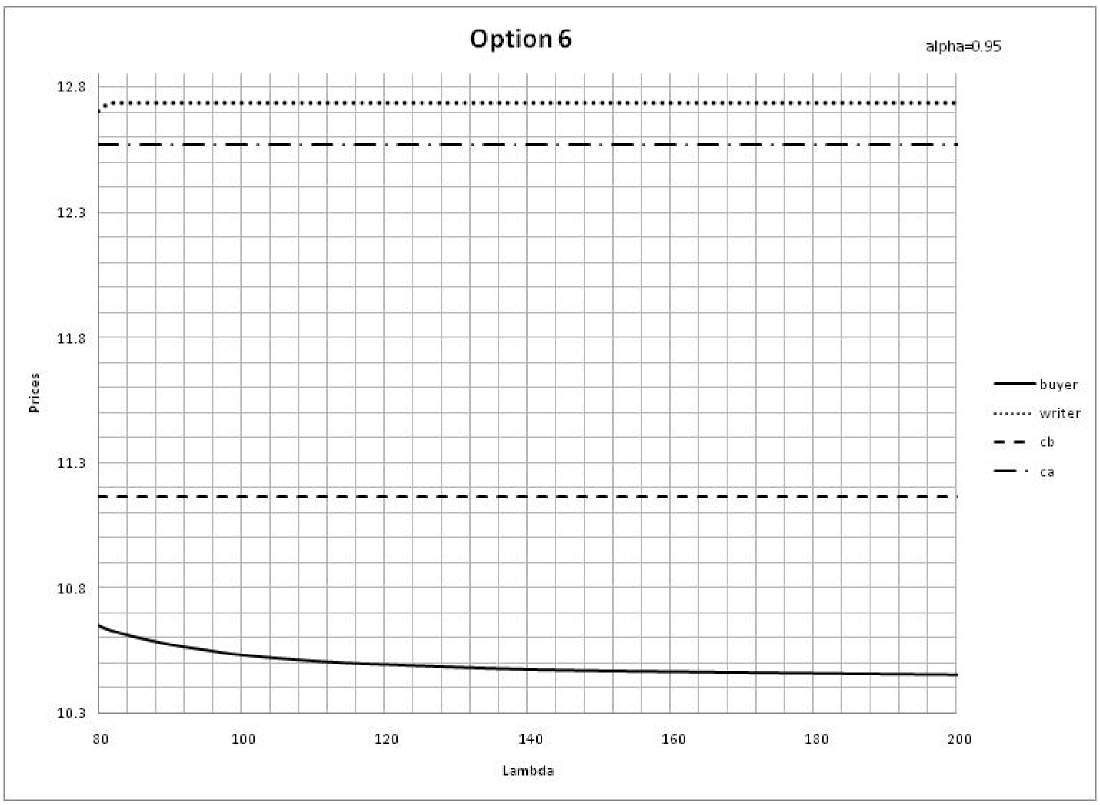

Figure 6.7: Option6, alpha=0.95

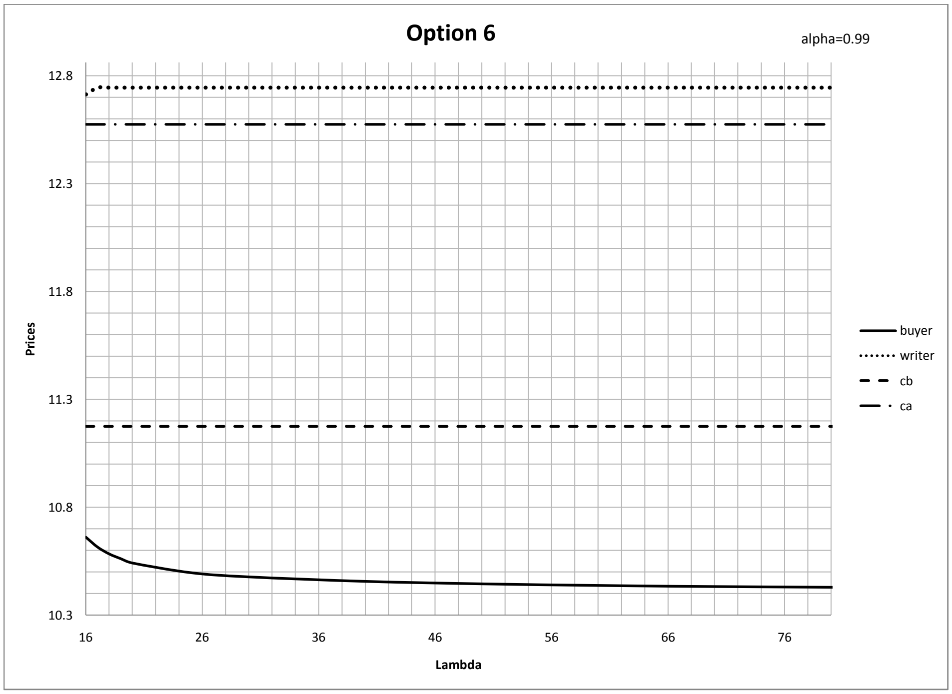

Figure 6.8: Option6, alpha=0.99

Figures 6.7 and 6.8 show the values we obtained for option 6. For all of the options, the increase and decrease of the predicted values are most apparent within the range of $\lambda \in [80, 100]$ for $\alpha = 0.95$ and $\lambda \in [16, 20]$ for $\alpha = 0.99$. For the remaining λ values, the prices are almost convergent to the final predicted values. Option 6 is an example for options with both of the predicted values are out of the interval of the actual prices. It is observed that the bounds we obtain using the our model are remarkably tighter than the no-arbitrage bounds.

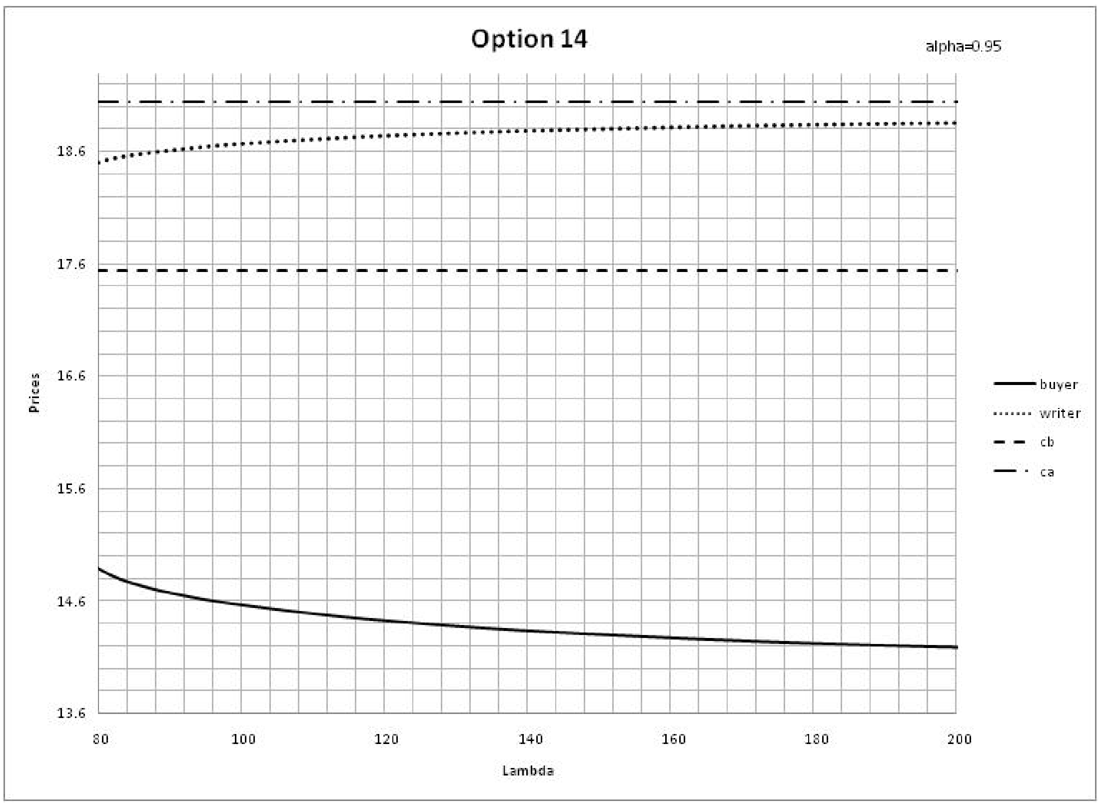

Figure 6.9: Option14, alpha=0.95

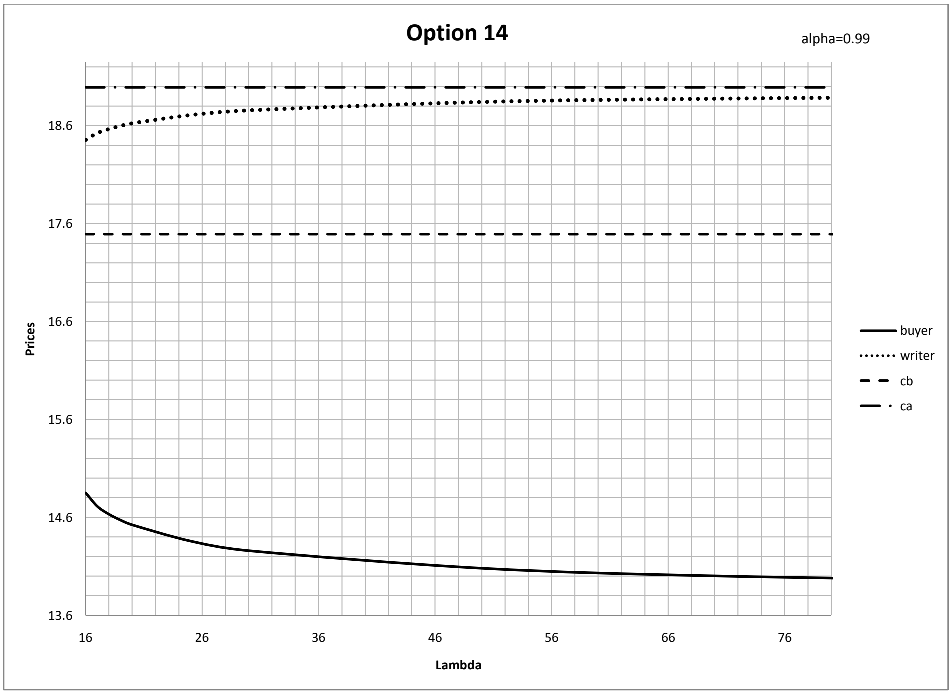

Figure 6.10: Option14, alpha=0.99

We continue with Option 14 that is shown in figures 6.9 and 6.10. Starting with the value of 14.86 for the buyer and 18.46 for the writer the values converge to 13.83 and 18.97 respectively. We observe in Option 14 that the price of the writer is within the interval formed by the actual prices and converges to the actual price of the writer as λ increases. The price of the buyer, on the other hand, is below the price of the buyer.

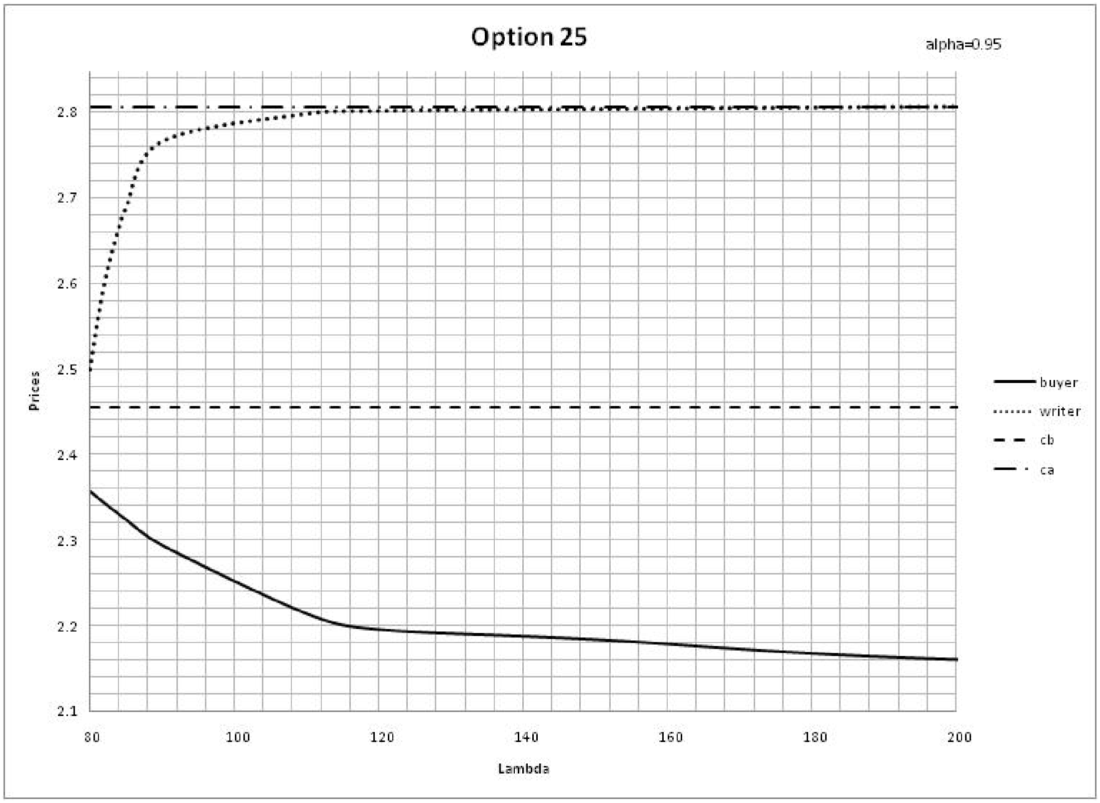

Figure 6.11: Option25, alpha=0.95

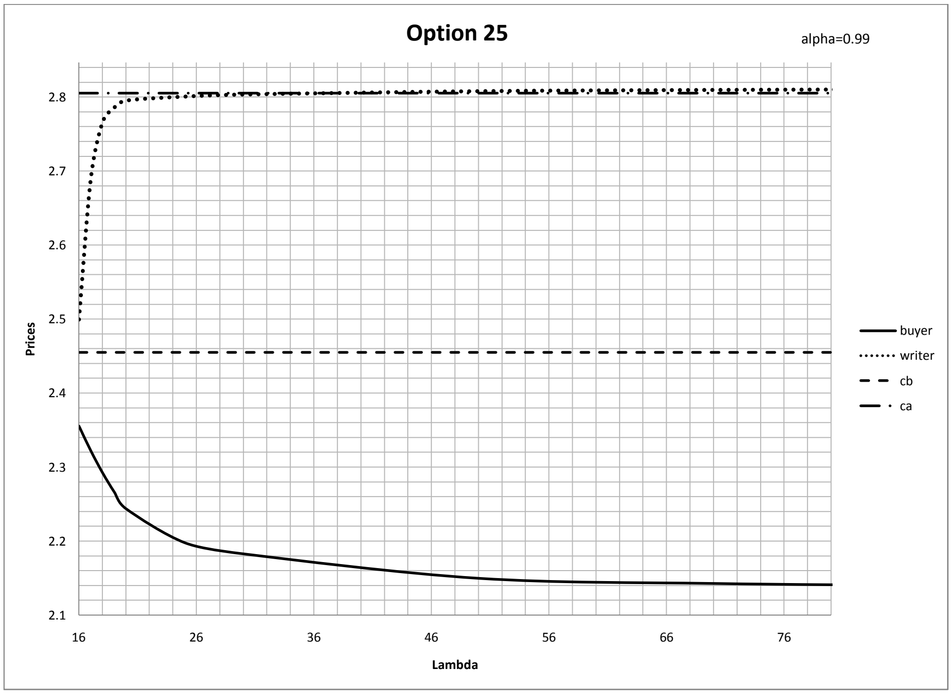

Figure 6.12: Option25, alpha=0.99

Figures 6.11 and 6.12 show the values we obtained for Option 25 along with the true bid-ask prices. Option 25 is an interesting example of the options where the price of the writer converges exactly to the actual price of the writer for values of $\lambda \geq 120$ for $\alpha = 0.95$ and of $\lambda \geq 30$ for $\alpha = 0.99$ Here, the price of the buyer is closest to the actual value for the limiting cases of $\lambda = 80$ for $\alpha = 0.95$ and $\lambda = 16$ for $\alpha = 0.99$.

Another important thing to note is that, the values of the parameter λ allowing no CVaR-λ-gain loss opportunity in the system have significantly decreased compared to the model capturing λ-gain-loss opportunities. Although we observe this while pricing these options, we now present a simple single model example to clearly illustrate this relationship:

Example : Consider a single-period numerical example for simplicity. Let us assume that the market consists of a riskless asset with zero growth rate, and of a stock and that the riskless asset has price equal to 1 throughout. The stock price evolves according a trinomial tree as follows. At time $t = 0$, the stock price is 10. Hence $Z_0 = (1\ 10)^T$. At time $t = 1$, the stock price can take the values $20, 15$ and 7.5 with equal probability. Hence, $Z_1 = (1\ 20)^T$; $Z_2 = (1\ 15)^T$ and $Z_3 = (1\ 7.5)^T$. We can construct an equivalent martingale measure with $q_1 = q_2 = 1/8$ and $q_3 = 3/4$, therefore the system is arbitrage-free. Solving the problems $P1$ and $D1$, we see that the limiting value of λ not allowing a CVaR-λ gain-loss opportunity in the market is 2.67 which is smaller than the limiting value of $\lambda = 6$ allowing no λ gain- loss opportunity in the system. This example shows that the limiting value of the loss aversion parameter can be decreased in our model compared to the λ gain-loss model.

In this chapter, we have computed prices of the the writer and buyer of 48 European call and put options on the $S\&P500$ index on September 10, 2002 according to the model proposed in Chapter 4 using the remaining options as market traded assets. We have illustrated a representative sample of the graphs of these options and commented on the results. It is possible to say that our proposed model yields good bounds as most of the bounds we obtained are very close to the true bid and ask values.

Chapter 7

Conclusion

In this thesis, we study the pricing problem of contingent claims by incorporating CVaR measure as losses to the concept of λ gain-loss opportunities in a discrete time, multi-period, stochastic linear optimization environment with a finite probability space. This enables us to combine the principles of risk aversion and no-arbitrage pricing. Investors can decide on the risk level that they are willing to take by putting restrictions on the parameter λ without having to deal with complex utility functions. Although VaR has been a popular measure in recent years for its ability to measure aggregate risk, it suffers from not being a coherent risk measure. CVaR, on the other hand, has been proposed as an alternative risk measure to VaR which is a coherent measure. Using CVaR instead with nice mathematical properties as a measure of risk enables us to account for extreme losses and yield a conservative result. We introduce a function that is proved to minimize CVaR by Rockefellar and Uryasev [5] and incorporate this function to represent losses into the model seeking for a λ gain-loss opportunity. We state the relationship between the existence of the CVaR-λ gain-loss opportunities and the martingales via Theorem 5. Then, the model is extended to include the perspectives of the writers and the buyers of the contingent claims. Then, the dual problems to the problems of the buyer and the writer are stated. By doing so, we were able to determine the pricing interval of the model not including CVaR-λ gain-loss opportunities in the market. Analyzing the problem through duality also provided us the means for passing to martingale measures. We express the pricing interval both in terms of duality and in terms of martingale measures. This pricing interval is shown to be tighter than the no-arbitrage interval in width theoretically in Chapter 5. We also note that these bounds converge to the no arbitrage bounds in the limit when the parameter λ goes to infinity in each of the specified confidence levels. Moreover, the values of the parameter λ allowing no CVaR-λ-gain loss opportunity in the system have significantly decreased compared to the model seeking λ-gain-loss opportunities. Therefore, the main contrubition of the thesis is to obtain tighter bounds on the prices of the contingent claim taking into account the investors preferences without complex utility functions. Moreover, modeling losses by CVaR provided us to obtain systems not allowing CVaR-λ gain-loss opportunities with smaller values of λ compared to the λ gain-loss opportunities. We also present a numerical study of our work. We compute prices of the the writers and the buyers of 48 European call and put options on the $S\&P500$ index on September 10, 2002 according to the model proposed in Chapter 4 using the remaining options as market traded assets. This study enables us to compare the resulting values to the actual market prices and interpret the data numerically. Based on our results, it is possible to say that our proposed model yields good bounds as most of the bounds we obtained are very close to the true bid and ask values.

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