VALUING RISKY PROJECTS IN INCOMPLETE MARKETS

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By

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June, 2009

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ABSTRACT

VALUING RISKY PROJECTS IN INCOMPLETE MARKETS

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We study the problem of valuing risky projects in incomplete markets. We develop a new method to value risky projects by restricting the so-called gain-loss ratio. We calculate the project value bounds on a numerical example and compare the results of our method with the option pricing analysis method. The proposed method yields tighter price bounds to the projects than option pricing analysis method. Moreover, for a specific value of gain-loss preference parameter, λ^*, our new method may yield a unique project value. Interestingly, replicating portfolios are different in the upper and lower bound problems for λ^*. The results are obtained in a discrete time, discrete space framework. We also extend our method to markets with transaction costs and situations with uncertain state probabilities.

Keywords: Option Pricing Theory, Valuation, Transaction Costs, Arbitrage .

ÖZET

EKSİK PİYASALARDA RİSKLİ PROJELERİN FİYATLANDIRILMASI

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Haziran, 2009

Bu tez çalışmasında riskli projelerin eksik piyasalarda fiyatlandırılması üzerinde çalışılmıştır. Projenin kazanç kayıp oranı kısıtlanarak, riskli projeleri fiyatlamak için yeni bir yöntem geliştirilmiştir. Sayısal bir örnek üzerinde proje değerinin sınırları hesaplanmış ve opsiyon fiyatlama yönteminin sonuçlarıyla karşılaştırılmıştır. Geliştirilen yöntem opsiyon fiyatlama yöntemine göre daha dar sınırlar vermektedir. Hatta, belirli bir λ^* değeri için tek bir proje değeri vermektedir, fakat çoğaltma portföyleri üst ve alt sınır problemleri için de farklıdır. Sonuçlar ayrık zaman, ayrık uzay çerçevesinde elde edilmiştir. Bu method ayrıca işlem maliyeti olan piyasalar ve durum olasılıkları tam olarak belli olmayan piyasalar için de geliştirilmiştir.

Anahtar sözcükler: Opsiyon Fiyatlama Teorisi, Fiyatlama, İşlem Maliyeti, Arbitraj.

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Chapter 1

Introduction

Investors have a desire to predict the future value of projects in which they plan to invest. If they can come up with accurate estimates, they may invest in profitable projects or they may decline projects where they will lose money. Under complete markets, standard valuation methods are sufficient to get a unique project value. However, the complete market assumption is far from being realistic. In incomplete markets, which can mimic the behavior of the real markets, available securities are not sufficient to replicate a project's cash flow, so it is not possible to get a unique value for the project. Therefore, valuing risky projects in incomplete markets has been a popular subject in academic literature. Due to high interest in this topic, many scientists worked on this subject and proposed different methods. With the increase in the number of methods proposed, it has been a debate which method is superior to others.

The main goal of this study is to propose a new method for valuing risky project in incomplete markets and compare it with the option pricing analysis method. We will use the gain-loss approach as in Bernardo and Ledoit [1] to develop our new method by restricting the gain-loss ratio of the projects. This new method provides us with a means to find tighter price bounds for the risky projects. Moreover, in most cases we can compute a unique project value in incomplete markets.

Firstly, we will introduce the basic framework and notation that will be used throughout this thesis. In Chapter 2, we will review the literature that is related to the problem under consideration.

The organization of the rest of the thesis is as follows:

In Chapter 3, we develop a new method to value risk projects in incomplete markets. In this chapter, we also assume that market is frictionless. By working on a capital budgeting example of [10], we compute the value of the project by option pricing analysis method and by our new method. Then, we compare these results. Moreover, we state and prove the Consistency Theorem in an incomplete market without transaction costs.

In Chapter 4, we work in an incomplete market with transaction costs. We state the problems that gives the project value bounds with these assumptions. Similar to Chapter 3, we compute the project value of the example of [10] and compare the result of the option pricing analysis method and our new method. We also state and prove Consistency Theorem with these assumptions.

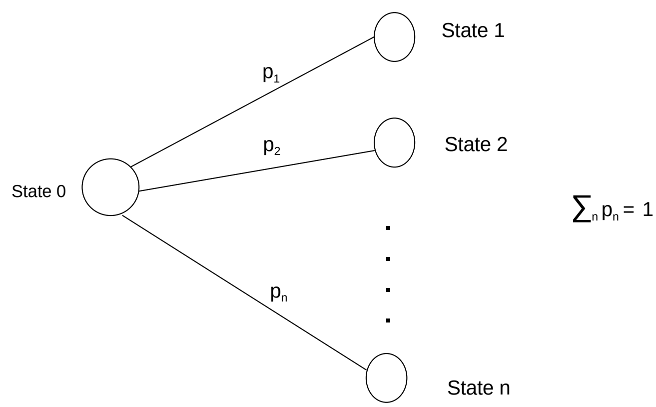
In Chapter 5, we assume that state probabilities are uncertain. According to this assumption we modify our new method and compute the project value bounds in an incomplete market with transaction costs and without transaction costs. We compare these results with option pricing bounds. As in previous chapters, we state and prove Consistency Theorem in an incomplete market with unknown state probabilities.

In Chapter 6, we conclude the thesis and review our contributions to the literature.

1.1 Basic Framework and Notation

Throughout this thesis we model the behavior of stock market by assuming that securities prices and other payments are discrete random variables. We model the beliefs and preferences of a single market participant, referred to as the 'firm'. This firm's belief and preferences are attributed as if it were privately owned and operated by a single owner or, equivalently, its owners were of one mind. This is consistent with the decision tree analysis method where analysts work with the firm's top officers to assess

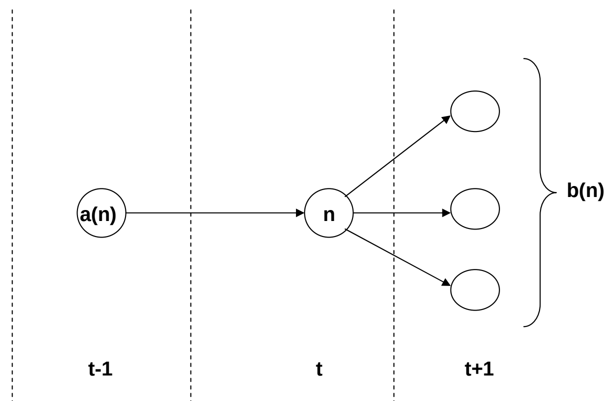
Figure 1.1: Decision Tree-1



the probabilities, p_n, for relevant uncertainties, and we assume that these uncertainties have been resolved and trading takes places at time $t = 0, 1, \ldots, T$. As shown in the Figure 1.1, we assume that each node of the decision tree represents the state of the market at a given time. N_t is the set of nodes at time t, and the set of all nodes is denoted by N.

In the decision tree N_0 represents the root node. As shown in Figure 1.2, every node $n \in N_t$ for $t = 1, \ldots, T$ has a unique parent node denoted by $a(n) \in N_{t-1}$, and every node $n \in N_t$ for $t = 1, \ldots, T-1$ has a set of child nodes denoted by $b(n) \subset N_{t+1}$. A positive probability p_n is attached to each leaf node $n \in N_T$, so $\sum_{n \in N_T} p_n = 1$. For each intermediate node of the tree,

Figure 1.2: Decision Tree-2



$$p_n = \sum_{m \in b(n)} p_m, \quad \forall\, n \in N_t, \quad t = 0, \ldots, T-1.$$

A project is a risky cash flow stream c_n that specifies the project's payoff at every possible state n. There are $M+1$ traded securities. The prices of the securities at state n are given by a vector:

$$s(n)=(\theta_n^0, \theta_n^1, \ldots, \theta_n^M)$$

where θ_n^i denotes the price of the ith security at state n. We also assume that there is a risk free security (the 0th security) whose time-t price is $(1+r_f)^t$ in all states n, r_f is referred as risk free rate. We let

$$\beta(n) = (\alpha_n, \xi_n^1, \ldots, \xi_n^M)$$

denote a trading strategy that specifies a portfolio of securities held from node n to its child nodes during $(t, t+1]$. The 0th component, α_n, of this vector is specified for the trading strategy of the risk free security which will used in Chapter 4 frequently.

Chapter 2

Literature Review

In this chapter we study the literature that is related to the problem under consideration. We will discuss different valuation methods and their relations. We begin with the paper of Smith and Nau [10] which intends to fill the gap between decision analysis and finance disciplines. In the literature, there are many alternative and competing methods for valuing risky projects. This paper compares and contrasts three different approaches: risk-adjusted discount rate analysis, decision analysis and option pricing analysis, and focuses on the last two approaches. The first goal of this work is to show that, if market opportunities to borrow and trade that are considered in option pricing analysis are included in the decision tree analysis and, time and risk preferences are captured by a utility function, then these two methods give consistent values to the risky projects. This is contrary to the work of Copeland, Koller and Murrin [4] that states option pricing analysis method superior over decision analysis methods. When option pricing analysis gives a unique value and optimal strategy, decision tree analysis also gives the same value and optimal strategy. If option pricing analysis gives bounds to the value of risky projects and a set of optimal strategies, decision tree analysis gives a value that lies within the same bounds and an optimal strategy that is in this set of optimal strategies.

The authors give a capital budgeting example and compare the values obtained from the naive decision tree analysis, option pricing analysis and full decision tree analysis. These three methods give the same project value for all states in complete markets. To obtain this result a firm's time and risk preferences captured by a utility function , and market opportunities are included by defining value of project in terms of breakeven buying price and breakeven selling price. As a result, when we neglect rounding errors, full decision tree analysis method also gives the same project value in complete markets. In incomplete markets, they expand the same example and show that naive decision tree analysis, option pricing analysis and full decision tree analysis give consistent results. Here, the firm is uncertain about the efficiency of plant, in addition to being uncertain about the level of demand. However, since plant efficiency should not affect the risk-adjusted discount rate, discount rates that are found in complete market case are used in naive decision tree analysis, and it gives an identical project value for the complete markets case. To use the option pricing methods in this expanded problem, dominating and dominated trading strategies were introduced as replicating trading strategies, we cannot construct a perfect replicating trading strategy since there is no market equivalent for the efficiency uncertainty . By using dominating and dominated strategies and including arbitrage conditions, upper and lower bounds of project value can be found. These values are consistent with bounds that are computed by considering the set of risk-neutral distributions consistent with market information. When the market is complete, there is a replicating strategy, and these bounds collapse to a unique value.

The trading strategy β dominates the project if future cash flows generated by β are always greater than or equal to those of the project. Conversely, a trading strategy β is dominated by the project if future cash flows generated by β are always less than or equal to those of the project. By these definitions upper and lower bounds of the project value can be computed.

The second goal of this work is to show how option pricing analysis and decision tree analysis techniques can be profitably integrated. In complete markets, option pricing analysis provide a way to decompose the decision analysis problem into two subproblems: The financing problem and the investment problem. The investment problem can be solved by option pricing methods using only market information, and the financing problem can be solved by decision analysis methods using subjective beliefs and preferences. Separation theorem states the method of finding solution of the grand problem by using these subproblems. In complete and incomplete markets, consistency and separation theorem holds, however there are some modifications in incomplete markets.

The Consistency Theorem, which will be also proved for different conditions in this thesis, states that if the securities market is complete, then the firm's breakeven buying price and breakeven selling prices for any project are both equal to option pricing value; if the securities market is incomplete, then the firm's breakeven buying price and breakeven selling prices for any project may differ but both lie between the bounds given by the option pricing analysis. Furthermore, when the firm chooses a project management strategy to maximize the project's value, option pricing and decision analysis approach should give the same optimal project management strategy. However their inputs and outputs are different. Both methods require the firm to specify state-contingent cash flows for the project and state-contingent values of the securities for all possible states of the world and time. However, option pricing approach also requires the firm to specify probabilities and a utility function describing its preferences. In return for this additional input we get an additional output: the optimal strategy for investing in securities. If the firm is not interested in this additional output, then option pricing provides a simpler way to compute project value.

Bernardo and Ledoit [1] developed an approach for asset pricing in incomplete markets that bridges the gap between two fundamental approaches, model-based pricing and pricing by no-arbitrage. Model-based pricing makes explicit assumptions about an investor's preferences and get a specific pricing kernel which shows investor's willingness to pay for consumption across states. This approach yields pricing implications that are exact but sensitive to misspecification errors. The second approach, no-arbitrage pricing, assumes only the existence of a set of basis assets and the absence of arbitrage opportunities to restrict the admissible set of pricing kernels that correctly price assets. This approach yields pricing implications in incomplete markets that are robust but often too imprecise to be economically interesting. As a result, investors have to choose between robustness and precision. The goal of this paper is to incorporate information of both method and strengthen the no-arbitrage to preclude investment opportunities whose attractiveness exceeds a specified threshold. The combination of these assumptions yields a restricted set of admissible pricing kernels to restrict asset prices. The developed approach measures the attractiveness of an investment by the gain-loss ratio. The gain-loss ratio is defined as the expectation of the investment's positive excess payoffs divided by the expectation of its negative excess payoffs. In general, investments with high gain-loss ratio are very attractive to the benchmark investor and, in the limit, investments with infinite gain-loss ratios constitute arbitrage opportunities.

Central result of this new approach is the duality result that connects gain-loss ratio and pricing kernels exhibiting extreme deviations from the benchmark pricing kernel. By imposing a bound on the maximum gain-loss ratio, the admissible pricing kernel can be restricted to those that do not exhibit such extreme deviations. When the gain-loss bound is equal to one, the admissible set contains only benchmark pricing kernel and we get the model-based pricing implications in this case. When the gain-loss bound goes to infinity, the admissible set contains all pricing kernels and we derive the no-arbitrage pricing implications. The main advantage of this duality result for deriving asset pricing implications is that the gain-loss ratio characterizes the set of arbitrage and approximate arbitrage opportunities.

By using the duality result, authors demonstrate how to derive pricing implications that lie between results of model-based pricing and those of no-arbitrage pricing. The first assumption that defines pricing methodology is: Excess payoffs have a gain-loss ratio below $\bar{L}$. This assumption expresses the idea that if the benchmark model is reasonable, then high gain-loss ratio investment opportunities are inconsistent with well-functioning capital markets: if high gain-loss ratio investments existed, they would be arbitraged away. The bounds on the price of a nonbasic asset found by using the above assumption get wider (narrower) as $\bar{L}$ increases (decreases). In the limit as $\overline{L}$ goes to infinity, they converge to the no-arbitrage bounds. The authors show the implication of a gain-loss ratio restriction by computing bounds on the price of options on a stock when there is no intermediate trading, and they conclude that their method offers a general way to chart the middle ground between a specific asset pricing model and no-arbitrage. Moreover, it is demonstrated that model-based pricing and no-arbitrage pricing techniques represent extreme cases of this new approach.

This new method involves several choices that the modeler must make ex ante in order to obtain implications. These are Ceiling on the Maximum Gain-Loss Ratio, Basis assets, and Benchmark Pricing Kernel.

Maximum Gain-Loss Ratio: Parameter $\bar{L}$ controls the trade off between the precision of model-based pricing method and the robustness of no-arbitrage pricing method. Choosing the value $\bar{L}$ is difficult, some people may choose a specific model or some others insist on nothing stronger than no-arbitrage assumption. However, neither one of these commonly made choices is optimal for deriving useful pricing implications in practice. Since the no-arbitrage principle is weak, it is always better to use a large but finite value of $\bar{L}$.

Basis assets: It is recommended to include basis assets with known prices and payoffs that nearly mimic the assets to be priced. Including them to the basis assets result in tighter pricing bounds. Moreover, modeler should include only basis assets that is available to the investor.

Benchmark Pricing Kernel: There are many alternative views to obtain benchmark pricing kernel, but the benchmark pricing kernel must be strictly positive to eliminate the possibility of arbitrage opportunities. Moreover, the choice of the pricing kernel should account for the characteristics of the investor in question.

Bernardo and Ledoit [1] developed a novel way to compute pricing bounds based on gain-loss ratio. This new approach provides us to find tighter price bounds; however there is an important disadvantage: numerical computations of the pricing bounds are complex. Longarela [8] provides a simple procedure that allows us to solve this problem by linear programming approach. The main idea of this approach is finding an equivalent linear constraint to the nonlinear constraint.

In this paper the notation in [1] is followed and the same set of assumptions is accepted. A two-period economy with S future states of nature which occur with strictly positive probabilities p_j is considered. Let Z be the space of portfolio payoffs which is spanned by a set of N payoffs $\tilde{z}^1,...,\tilde{z}^N$. Every $\tilde{z} \in Z$ is a random variable $\tilde{z} = [z_1,...,z_S]$.

Asset prices are given by a linear function π defined on Z, that is, the portfolio with payoff $\tilde{z} \in Z$ has price $\pi(\tilde{z})$. There is no-arbitrage and hence, there exists at least one random variable $\tilde{m} > 0$ such that $E(\tilde{m}\tilde{z}) = \pi(z)\ \forall\ \tilde{z} \in Z$ and M is the set of admissible stochastic discount factors. Each $\tilde{m} \in M$ has an associated vector of state prices given by $\mu_j = p_j m_j, = 1, ..., S$, where m_j represents the value of $\tilde{m}$ at state of nature j.

For each $\tilde{m} \in M$ define the value $L_{\tilde{m}} \equiv \frac{max_{j=1,...,S}(\frac{m_j}{m_j^*})}{min_{j=1,...,S}(\frac{m_j}{m_j^*})}$

$L_{\tilde{m}}$ gives the maximum gain-loss ratio and Bernardo and Ledoit [1] define pricing bounds on $\tilde{z^*}$ as the solution to the programs

$$min_{\tilde{m}\in M, L_{\tilde{m}} \leq \bar{L}} E(\tilde{m}\tilde{z^*}) \tag{2.1}$$

and

$$max_{\tilde{m}\in M, L_{\tilde{m}} \leq \bar{L}} E(\tilde{m}\tilde{z^*}) \tag{2.2}$$

where $\bar{L}$ is a ceiling to be set by the user which must satisfy $\bar{L} \geq min_{\tilde{m}\in M} L_{\tilde{m}}$

The following proposition is the fundamental result of this approach.

Proposition: 1. *$L_{\tilde{m}} \leq \bar{L}$ if and only if there exist two constants θ_1^* and θ_2^* such that $\frac{\theta_2^*}{\theta_1^*} = \bar{L}$ and $\theta_1^* \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2^*$, $j = 1, ..., S$*

The above proposition allows us to transform nonlinear constraint $L_{\tilde{m}} \leq \bar{L}$ into a linear one, and computation of the bounds can be done by solving

$$\begin{aligned}
&\min_{\mu_1,...,\mu_S,\theta_1,\theta_2} \sum_{j=1}^{S} \mu_j z_j^* \\
&s.t \\
&\sum_{j=1}^{S} \mu_j z_j^* = \pi(\tilde{z^i}), i = 1, ..., N \\
&\theta_1 \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2, j = 1, ..., S \\
&\theta_2 = \theta_1 \bar{L} \\
&\theta_1 \geq 0.
\end{aligned}$$

and

$$\max_{\mu_1,\ldots,\mu_S,\theta_1,\theta_2} \sum_{j=1}^{S} \mu_j z_j^*$$
$$s.t$$
$$\sum_{j=1}^{S} \mu_j z_j^* = \pi(\tilde{z^i}), i = 1, \ldots, N$$
$$\theta_1 \leq \frac{\mu_j}{\mu_j^*} \leq \theta_2, j = 1, \ldots, S$$
$$\theta_2 = \theta_1 \bar{L}$$
$$\theta_1 \geq 0.$$

Moreover, from the dual of above two linear programs, Bernardo and Ledoit's dual expression of the bounds in 2.1 and 2.2 are obtained.

Now, we will examine another approach that values uncertain payoffs by restricting discount factors. Bernardo and Ledoit [1] do this by restricting the gain-loss ratio and Cochrane and Saa-Requejo [3] restrict pricing kernels by putting a bound to the sharpe ratio and they define assets with high sharp ratios as good-deals.

The basic idea of this work is most simply explained in one-period environment. The value of a focus payoff x^c_{t+1} is calculated by taking as given the prices p_t of a set of basis payoffs or hedging assets x_{t+1} . The discount factor m_{t+1} generates the value p_t of any payoff x_{t+1} by p=$E(mx)$. When the focus payoff x^c_{t+1} can be perfectly replicated by basis asset payoffs x, there is enough information to determine its exact value. However, when the replication is not perfect, the existence of a discount factor or law of one price says nothing about the value of focus payoff. Therefore, more restriction on discount factor is required.

The authors state that the more restriction on the discount factor, the more information about asset values. It is required that discount factor price is a set of basis assets, that it is nonnegative and an upper bound on its volatility is imposed. Therefore, the lower good-deal bound solves

$$\underline{C} = \min_m E(mx^c)$$
$$s.t$$
$$p = E(mx^c)$$
$$m \geq 0$$
$$\sigma(m) \leq \frac{h}{R^f}.$$

where $\underline{C}$ is the lower good-deal bound; m is the discount factor; x^c is the focus payoff to be valued; p and x are the price and payoffs of basis assets, h is the prespecified volatility bound, R^f is the risk-free interest rate; E and σ are the conditional mean and variance; and the upper good-deal bound $\overline{C}$ solves the corresponding maximum.

The first constraint, $p = E(mx^c)$, enforces the relative pricing idea. The second constraint, $m \geq 0$, is a classic and weak characterization of weak utility. The portfolio interpretation of this assumption is equivalent to the absence of arbitrage opportunities, which means that if a payoff is nonnegative in every state of nature; its value must also be nonnegative. The above problem with the first two constraints leads to well-known arbitrage bounds on the value.

The third constraint, $\sigma(m) \leq \frac{h}{R^f}$, is the innovation of this paper. The authors intend it as a similar weak restriction on marginal utility, a natural next step when absence of arbitrage alone does not give precise enough answers. It has also a portfolio interpretation. The discount factor volatility restriction is equivalent to an upper limit on the sharp ratio of mean excess return to standard deviation. The discount factor volatility constraint is also a way of imposing weak or robust predictions of economic models. Furthermore, the volatility constraint is an easy way to reduce unreasonable discount factors within the arbitrage bounds and it weeds out some of the arbitrage-free but still "‘unreasonable"’ discount factors and their corresponding option prices. This new approach is illustrated with a simple example. In this example, arbitrage bounds and Good-deal bounds are found and they are compared.

Not all values outside the good-deal bounds imply high Sharpe ratios or arbitrage opportunities. Such values might be generated by a positive but highly volatile discount factor, and generated by another less volatile but sometimes negative discount factor, but discount factor that generates these values cannot be nonnegative and cannot respect the volatility constraint simultaneously.

Good-deal bounds should be useful in many situations in which a relative pricing approach is appropriate but perfect replication is not possible. For example, a trader can use the bounds as buy and sell points in the search for "‘good-deals"’ in asset markets.

In this paper, how to calculate good-deal bounds in single period, multi period, and continuous-time contexts is shown. Now, we will examine single period case in more detail.

One Period:

There is one period and no intermediate trading until the payoff x^c is realized and one of the basis payoffs is riskless, so $E(m) = \frac{1}{R^f}$. Then, the problem to obtain good-deal bounds becomes:

$$
\begin{aligned}
&\underline{C} = \min_m E(mx^c) \\
&s.t \\
&p = E(mx^c) \\
&m \geq 0 \\
&\sigma(m^2) \leq A^2.
\end{aligned}
$$

where $A^2 = \frac{(1+h^2)}{(R^f)^2}$

For any solution to exist, one must pick a sufficiently large bound A on price the basis assets:

$$
\begin{aligned}
& A^2 \leq \min_m E(m^2) \\
& s.t \\
& p = E(mx^c), m \geq 0.
\end{aligned}
$$

The problem has two inequality constraints, therefore the solution can be found by trying all the combinations of binding and non binding constraints. Firstly, it is assumed that the volatility constraint is binding and the positivity constraint is slack. If the resulting discount factor is nonnegative, this is the solution. If not, it is assumed that the volatility constraint is slack and the positivity constraint is binding. This configuration delivers the arbitrage bound on value and the minimum variance discount factor that generates the arbitrage bound is found. If this discount factor satisfies the volatility constraint, this is the solution. If not, the problem is solved with both constraints binding.

Now, focusing on the works Bernardo and Ledoit [1] and Smith and Nau [10], we will propose a new method to value risky projects.

Chapter 3

Markets without Transaction Costs

In this chapter, we study markets without transaction costs. There are many alternative and competitive methods for valuing risky projects. When markets are complete in that every project risk can be hedged by securities, these methods yield a unique project value without making any assumption about investor preferences. However, when markets are incomplete, a unique project value can not be found. Smith and Nau [10] compute bounds to the project value by option pricing analysis method in incomplete markets. To compute these bounds, dominating and dominated trading strategies introduced as an extension of replicating trading strategy. If the future project cash flows generated by a trading strategy β are always greater then or equal to those of the project, then β is the dominating trading strategy. If the future project cash flows generated by a trading strategy β are always less than or equal to those of the project then β is the dominated trading strategy. They also assume that the securities market is frictionless and the project is traded in a market that does not allow arbitrage. If the project traded in a arbitrage free market, the project's current value must be less than or equal to the current market value of every dominating trading strategy and greater than or equal to the current market value of every dominated trading strategy. By these definitions and assumptions the option pricing approach gives the upper and lower bounds:

Upper bound is computed by finding:

$$\bar{v} = c_0 + \min\{\beta(0)s(0) : [\beta(a(n)) - \beta(n)]s(n) \geq c_n \text{ for all } n > 0\} \quad (3.1)$$

Lower bound is computed by finding:

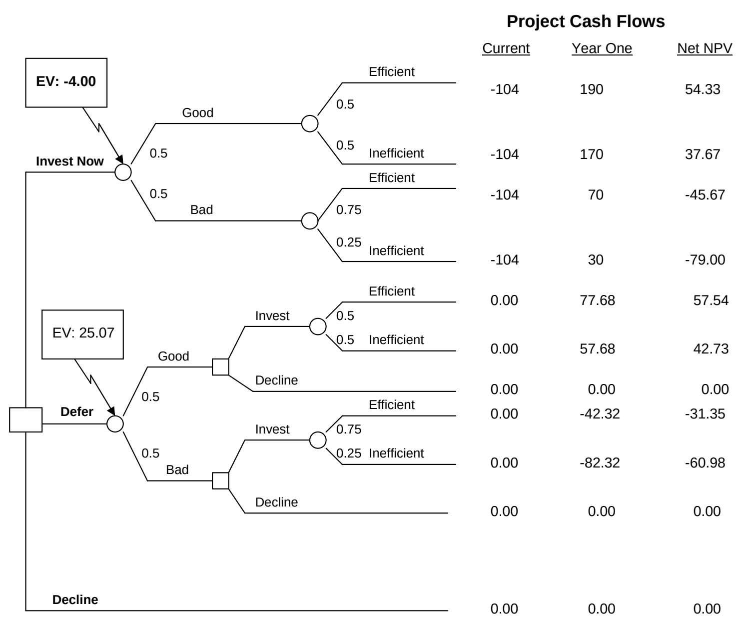
$$\underline{v} = c_0 + \max\{\beta(0)s(0) : [\beta(a(n)) - \beta(n)]s(n) \leq c_n \text{ for all } n > 0\} \quad (3.2)$$

3.1 A Capital Budgeting Example

We compute the project value bounds for the example in Smith and Nau [10]: This is a two period capital budgeting example. There are two securities in the market, a risk-free security that allows the firm to borrow and lend at 8% percent and a 'twin security' whose values depend on the uncertain level of demand. The current price of the twin security is \$20, in the good state it will be worth \$36 and in the bad state it will be worth \$12. The firm is presented with the opportunity to invest \$104 now to built a plant that a year later will have a payoff that depends on the 'level of demand' and 'uncertain efficiency'. As shown in Figure 3.1 In the 'good' state if plant is 'efficient', it pays \$190 and if plant is 'inefficient', it pays \$170. In the 'Bad' state if plant is 'efficient', its payoff is \$70 and if plant is 'inefficient', it pays \$30. Alternatively, for a fee to be negotiated, the firm may obtain a one-year license to allow them to defer the construction of the plant until after the state is known. If they choose this option, they may invest \$112.32 one year from now and get either \$190 or \$170 in the good state, get either \$70 or \$30 in the bad state, or decline to invest and let the option expire. The firm may also decline to invest without paying or receiving any money. The probabilities of these uncertainties are known and can be seen from the Figure 3.1.

Now, we will compute the smallest value of the portfolio that dominates the project cash flow as upper bound of the project value, and we will compute the largest value of the portfolio that is dominated by the project cash flow as lower bound of the project value.

Figure 3.1: Decision Tree for the Capital Budgeting Example



Invest Now Alternative upper bound:

$$
\begin{aligned}
&\min_{\alpha,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \geq 190 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \geq 170 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \geq 70 \\
&1.08(\alpha_0 - \alpha_4) + 12(\xi_0 - \xi_4) \geq 30 \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0 \\
&1.08\alpha_4 + 12\xi_4 \geq 0.
\end{aligned}
$$

Invest Now Alternative lower bound:

$$
\begin{aligned}
&\max_{\beta,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \leq 190 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \leq 170 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \leq 70 \\
&1.08(\alpha_0 - \alpha_4) + 12(\xi_0 - \xi_4) \leq 30 \\
&1.08\alpha_1 + 36\xi_1 \leq 0 \\
&1.08\alpha_2 + 36\xi_2 \leq 0 \\
&1.08\alpha_3 + 12\xi_3 \leq 0 \\
&1.08\alpha_4 + 12\xi_4 \leq 0.
\end{aligned}
$$

Defer alternative upper bound:

$$
\begin{aligned}
&\min_{\alpha,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \geq 77.68 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \geq 57.68 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \geq 0 \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0.
\end{aligned}
$$

Defer alternative lower bound:

$$
\begin{aligned}
&\max_{\beta,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \leq 77.68 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \leq 57.68 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \leq 0 \\
&1.08\alpha_1 + 36\xi_1 \leq 0 \\
&1.08\alpha_2 + 36\xi_2 \leq 0 \\
&1.08\alpha_3 + 12\xi_3 \leq 0.
\end{aligned}
$$

By solving these linear programs we can find upper and lower bounds of project value for Invest Now and Defer Alternative, that are consistent with the value computed by considering the set of risk neutral distributions consistent with market information. As a result of these methods we obtain upper and lower bounds 5.26 and −24.37 for the Invest now alternative and 28.77 and 21.36 for the Defer alternative, respectively .

3.2 Modeling Gain-Loss Bounds for the Project Value

In this section, we will modify the linear programs of the previous section inspired by the contributions of Bernardo and Ledoit [1]. They introduce gain-loss criterion which suggests to choose the portfolio which gives the best value of the difference of expected positive final positions and a parameter λ (greater than one) times expected negative final positions, $E^P[X^+] - \lambda E^P[X^-]$ where $X^+ = \{x_n^+\}$ and $X^- = \{x_n^-\}$ and we define the project cash flow at state n by $\beta(n)c(n) = x_n^+ - x_n^-$ where x_n^+ and x_n^- are non-negative numbers, i.e, the final portfolio at terminal state n is expressed as the sum of positive and negative positions. This criterion gives rise to a new concept 'λ gain-loss opportunity'. This new concept is defined in [9] as a portfolio which can be set up at no cost but yields a positive value for the difference between gains and 'λ-losses'. Our new method provides us to find maximum and minimum prices which do not introduce λ gain-loss opportunities in the market. This price interval is contained in the no-arbitrage price interval which is named as consistency theorem throughout this thesis. As λ gets larger, investors become more averse to loss and they begin to prefer near-arbitrage positions so, price bounds approach the no-arbitrage bounds. As λ gets closer to one, gain and loss are equally weighted, and if it exists, we can find a unique value for the project. In fact, in most cases, project value may become unique at a value of λ larger than one which will be denoted as λ^* from now on. As it is stated in Pınar, Salih, and Camci [9], λ^* is the maximum gain-loss ratio that λ gain-loss opportunity continue to exist. When $\lambda \leq \lambda^*$, there is λ gain-loss opportunity in the market and the problems that gives us the project value become unbounded.

Now, we will state our new model. Since computations and derivations are carried out using linear programming models, we can easily add gain-loss constraint, $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0$, to the model of the previous section, that we used in option pricing approach. So, by this method upper bound is computed by finding:

$$
\begin{aligned}
GL(U) \qquad & \min \beta(0)s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n))s(n) \geq c_n, \ \forall \ n \in N_t, t > 0 \\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
& \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Lower Bound is computed by finding:

$$
\begin{aligned}
GL(L) \qquad & \max -\beta(0)s(0) \\
& s.t \\
& (\beta(n)-\beta(a(n)))s(n) \leq c_n, \ \forall \ n \in N_t \\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
& \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

To compare the result of the option pricing method and our new method, we compute project value bounds of the example in [10]. For Invest Now opportunity upper and lower bounds can be computed by solving the following problems:

$$
\begin{aligned}
&\min_{\alpha,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \geq 190 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \geq 170 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \geq 70 \\
&1.08(\alpha_0 - \alpha_4) + 12(\xi_0 - \xi_4) \geq 30 \\
&1.08\alpha_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\alpha_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\alpha_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&1.08\alpha_4 + 12\xi_4 - x_4^+ + x_4^- = 0 \\
&x_n^+ \geq 0 \text{ for all } n > 0 \\
&x_n^- \geq 0 \text{ for all } n > 0 \\
&(0.25x_1^+ + 0.25x_2^+ + 0.375x_3^+ + 0.125x_4^+) \\
&-\lambda(0.25x_1^- + 0.25x_2^- + 0.375x_3^- + 0.125x_4^-) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&\max_{\alpha,\xi} -\alpha_0 - 20\xi_0 \\
&s.t \\
&1.08(\alpha_1 - \alpha_0) + 36(\xi_1 - \xi_0) \leq 190 \\
&1.08(\alpha_2 - \alpha_0) + 36(\xi_2 - \xi_0) \leq 170 \\
&1.08(\alpha_3 - \alpha_0) + 12(\xi_3 - \xi_0) \leq 70 \\
&1.08(\alpha_4 - \alpha_0) + 12(\xi_4 - \xi_0) \leq 30 \\
&1.08\alpha_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\alpha_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\alpha_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&1.08\alpha_4 + 12\xi_4 - x_4^+ + x_4^- = 0 \\
&x_n^+ \geq 0 \text{ for all } n > 0 \\
&x_n^- \geq 0 \text{ for all } n > 0 \\
&(0.25x_1^+ + 0.25x_2^+ + 0.375x_3^+ + 0.125x_4^+) \\
&-\lambda(0.25x_1^- + 0.25x_2^- + 0.375x_3^- + 0.125x_4^-) \geq 0.
\end{aligned}
$$

For Defer Alternative upper and lower bounds can be computed by solving the following problems:

$$
\begin{aligned}
&\min_{\alpha,\xi} \alpha_0 + 20\xi_0 \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36(\xi_0 - \xi_1) \geq 77.68 \\
&1.08(\alpha_0 - \alpha_2) + 36(\xi_0 - \xi_2) \geq 57.68 \\
&1.08(\alpha_0 - \alpha_3) + 12(\xi_0 - \xi_3) \geq 0 \\
&1.08\alpha_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\alpha_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\alpha_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&x_n^+ \geq 0 \text{ for all } n > 0 \\
&x_n^- \geq 0 \text{ for all } n > 0 \\
&(0.25x_1^+ + 0.25x_2^+ + 0.5x_3^+) - \lambda(0.25x_1^- + 0.25x_2^- + 0.5x_3^-) \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&\max_{\alpha,\xi} -\alpha_0 - 20\xi_0 \\
&s.t \\
&1.08(\alpha_1 - \alpha_0) + 36(\xi_1 - \xi_0) \leq 77.68 \\
&1.08(\alpha_2 - \alpha_0) + 36(\xi_2 - \xi_0) \leq 57.68 \\
&1.08(\alpha_3 - \alpha_0) + 12(\xi_3 - \xi_0) \leq 0 \\
&1.08\alpha_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\alpha_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\alpha_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&x_n^+ \geq 0 \text{ for all } n > 0 \\
&x_n^- \geq 0 \text{ for all } n > 0 \\
&(0.25x_1^+ + 0.25x_2^+ + 0.5x_3^+) - \lambda(0.25x_1^- + 0.25x_2^- + 0.5x_3^-) \geq 0.
\end{aligned}
$$

As it can be seen from the Table 3.3 and Table 3.4 , as λ gets larger project value bounds that we compute with this new method goes to the no-arbitrage bounds that we compute with option pricing analysis. Moreover, as λ gets smaller price bounds of project become tighter and for λ^*, we can find a unique value to the risky projects in incomplete markets.

Interestingly, although at the level $\lambda = 1.5$ upper and lower bound problems give us a unique project value, replicating portfolios need not be identical. For the Invest Now upper bound, at $n = 0$ replicating portfolio consist of borrowing 62.5 unit risk free security and buying 8.125 twin security. If node 1 were to be reached, twin security is liquidated, position in risk free security is zeroed out and position of 32.497 units risk free security is taken. In case of node 2, similarly twin security is liquidated, position in risk free security is zeroed out and position of 50.926 units risk free security is taken. Finally in node 3, twin security is liquidated, but a short position of 37.037 units remains in the risk free security. For the Invest Now lower bound at $n = 0$ replicating portfolio consist of borrowing 20.833 unit of risk free security and 3.958 units of twin security. If node 1 were to be reached, twin security is liquidated, position in risk free security is zeroed out and a position of 23.148 units risk free security is taken. In case of node 2, twin security is liquidated, the position in risk free security is zeroed out and position of 4.630 units risk free security is taken. Finally in node 4, twin security is liquidated, but a short position of 37.037 units remains in the risk free security.

For the Defer upper bound at $n = 0$ replicating portfolio consists of borrowing 45.222 unit of risk free security and buying 3.514 twin security. If node 2 were to be reached twin security is liquidated, position in risk free security is zeroed out and the position of 18.519 unit of the risk free security is taken. In the case of node 3, twin security is liquidated, but a short position of 6.173 units remains in the riskless asset. For the Defer lower bound at $n = 0$ replicating portfolio consist of borrowing 2.143 twin security and buying 8.023 units of risk free security. If node 1 were to be reached risk free security is liquidated, position in twin security is zeroed out, and position of 18.519 units risk free security is taken. In case of node 3, risk free security is liquidated, position in twin security is zeroed out and but a short position of 5.787 units remains in the riskless asset.

Table 3.1 and Table 3.2 show how the replicating portfolios differ for upper and lower bounds for the $\lambda = 1.5$, for which value upper and lower bound coincide.

Table 3.1: Invest Now Alternative Replicating Portfolios for $\lambda = 1.5$

	Upper	Lower
β_0	-62.5	-20.833
β_1	32.407	-23.148
β_2	50.926	4.630
β_3	-37.037	0.00
β_4	0.00	-37.037
ξ_0	8.125	-3.958
ξ_1	0.00	0.00
ξ_2	0.00	0.00
ξ_3	0.00	0.00
ξ_4	0.00	0.00

Table 3.2: Defer Alternative Replicating Portfolios for $\lambda = 1.5$

	Upper	Lower
β_0	-45.222	8.023
β_1	0.00	18.519
β_2	-6.173	0.00
β_3	-37.037	-5.787
ξ_0	3.514	-2.143
ξ_1	0.00	0.00
ξ_2	0.00	0.00
ξ_3	0.00	0.00

Now, we will state and prove the Consistency Theorem for incomplete markets without transaction costs.

Theorem 1. ***Consistency Theorem(Incomplete Markets without Transaction Costs)*** *In an incomplete, frictionless market, the firm's gain-loss upper bound and gain-loss lower bound for any project may differ, but both lie between the bounds given by the option pricing analysis method.*

Proof. Let us begin with forming the linear programming dual of problem:

Table 3.3: Invest Now Alternative Gain-Loss Bounds

λ	Upper	Lower
100000000	5.26	-24.37
1000	5.24	-24.28
500	5.22	-24.19
100	5.11	-23.50
50	4.96	-22.69
20	4.53	-20.51
10	3.83	-17.53
5	2.48	-13.26
4	1.84	-11.67
3	0.81	-9.56
2	-1.09	-5.85
1.8	-1.69	-5.12
1.6	-2.89	-4.37
1.5	-4	-4

$$
\begin{aligned}
GL(U) \qquad & \min \beta(0)s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n))s(n) \geq c_n, \ \ \forall \ \ n \in N_t, t > 0 \\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \ \forall \ \ n \in N_t \\
& \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \ \forall \ \ n \in N_t \\
& x_n^+ \geq 0, \ \ \forall \ \ n \in N_t \\
& x_n^- \geq 0, \ \ \forall \ \ n \in N_t.
\end{aligned}
$$

Forming the Lagrangian function after attaching multipliers $v_n \geq 0$, w_n, $V \geq 0$, we obtain:

$L(\beta, X^+, X^-, v, w, V) = \beta(0)s(0) + V(\lambda \sum_n p_n x_n^- - \sum_n p_n x_n^+) + \sum_n v_n(c_n + [\beta(n) - \beta(a(n))]s(n)) - \sum_n w_n(\beta(n)s(n) - x_n^+ + x_n^-)$ then we maximize over the variables β , X^+ and X^- separately again. This results in the dual problem:

Table 3.4: Defer Alternative Gain-Loss Bounds

λ	Upper	Lower
100000000	28.77	21.36
1000	28.76	21.37
500	28.75	21.38
100	28.70	21.44
50	28.63	21.51
20	28.42	21.72
10	28.10	22.04
5	27.54	22.60
4	27.29	22.84
3	26.92	23.22
2	25.99	24.14
1.8	25.68	24.45
1.6	25.30	24.84
1.5	25.07	25.07

$$GL(D1) \qquad \max \sum_n v_n c(n) \tag{3.3}$$

$$s.t$$

$$\sum_n v_n s(n) \leq s(0) \tag{3.4}$$

$$w_n s(n) - v_n s(n) \leq 0 \tag{3.5}$$

$$V p_n \leq w_n \tag{3.6}$$

$$w_n \leq \lambda V p_n \tag{3.7}$$

$$v_n \geq 0 \tag{3.8}$$

$$V \geq 0. \tag{3.9}$$

Then form the linear programming dual of problem:

$$
\begin{aligned}
OP(U) \qquad & \min \beta(0)s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n))s(n) \geq c_n, \ \forall \ n \in N_t \\
& \beta(n)s(n) \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Similarly, forming the Lagrangian function after attaching multipliers $v_n \geq 0$, $w_n \geq 0$, we obtain:

$L(\beta, v, w) = \beta(0)s(0) + \sum_n v_n(c_n + [\beta(n) - \beta(a(n))]s(n)) - \sum_n w_n(\beta(n)s(n))$

Then we maximize over the variable β. This results in the dual problem:

$$
\begin{aligned}
OP(D1) \qquad & \max \sum_n v_n c_n && (3.10) \\
& s.t \\
& \sum_n v_n s(n) \leq s(0) && (3.11) \\
& w_n s(n) - v_n s(n) \leq 0 && (3.12) \\
& v_n \geq 0 && (3.13) \\
& w_n \geq 0. && (3.14)
\end{aligned}
$$

Now we have to show that optimal solution of the problem GL(D1) cannot be larger than OP(D1)'s optimal solution.

The problems GL(D1) and OP(D1) have identical objective functions. Moreover, constraints 3.4, 3.5, 3.8 are identical to constraints 3.11, 3.12, 3.13, respectively and constraints 3.6, 3.7, 3.9 imply constraint 3.14 with the fact $p_n \geq 0$ and $\lambda > 1$. Therefore, the feasible set of GL(D1) is a subset of feasible set of OP(D1).

So we get the desired result. Then by the Strong Duality Theorem if the primal problem has an optimal solution than the dual also has the optimal solution and these problems' optimal values are same. Therefore optimal value of GL(U) is less than or equal to optimal value of OP(U).

The other part of the proof is similar. Firstly, form the linear programming dual of problem:

$$
\begin{aligned}
GL(L) \qquad & \max -\beta(0)s(0) \\
& s.t \\
& (\beta(n)-\beta(a(n)))s(n) \leq c_n, \ \forall \ n \in N_t \\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
& \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Again by attaching multipliers $v_n \geq 0$, w_n, $V \geq 0$, we obtain dual problem as:

$$GL(D2) \qquad \min \sum_n v_n c(n) \tag{3.15}$$

$$s.t$$

$$\sum_n v_n s(n) \geq s(0) \tag{3.16}$$

$$w_n s(n) - v_n s(n) \geq 0 \tag{3.17}$$

$$w_n \geq V p_n \tag{3.18}$$

$$\lambda V p_n \geq w_n \tag{3.19}$$

$$v_n \geq 0 \tag{3.20}$$

$$V \geq 0. \tag{3.21}$$

Now form the dual of problem:

$$
\begin{aligned}
OP(L) \qquad & \max \beta(0)s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n))s(n) \le c_n, \ \forall \ n \in N_t \\
& \beta(n)s(n) \le 0, \ \forall \ n \in N_t.
\end{aligned}
$$

After attaching multipliers $v_n \ge 0$, $w_n \ge 0$, we obtain:

$$
\begin{aligned}
OP(D2) \qquad & \min \sum_n v_n c(n) && (3.22) \\
& s.t \\
& \sum_n v_n s(n) \ge s(0) && (3.23) \\
& w_n s(n) - v_n s(n) \ge 0 && (3.24) \\
& v_n \ge 0 && (3.25) \\
& w_n \ge 0. && (3.26)
\end{aligned}
$$

The objective functions of both dual problem are identical and we can easily show that feasible set of GL(D2) is a subset of feasible set of OP(D2).

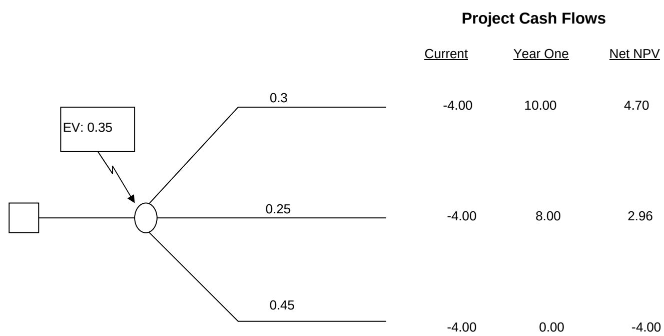
Constraints 3.16, 3.17,3.20 of GL(D2) are identical to constraints 3.23, 3.24, 3.25 of OP(D2), respectively and constraints 3.18, 3.19, 3.21 imply constraint 3.26 with the facts $\lambda > 1$ and $p_n \ge 0$.

So, optimal value of GL(L) is greater than or equal to optimal value of OP(L).

□

Let us return to the behavior of project value bounds when λ decreases. In the

Figure 3.2: Payoffs of the Stock

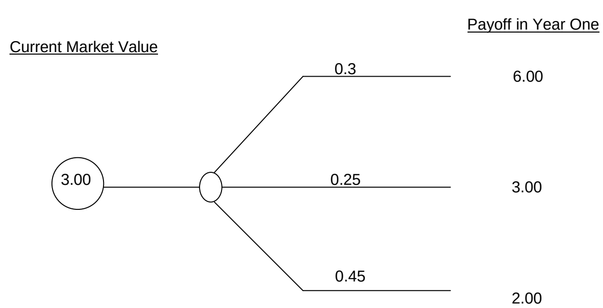


example of Smith and Nau [10] that is stated above, we can find a unique project value for λ^*, however this is not always the case. The following example shows that upper and lower bounds of the project value need not to coincide for some value of λ.

Example: Let us assume that market consists of a riskless asset with zero growth rate and 1 stock. At $n = 0$ stock price is 3. At node 1, 2, 3 stock's price can take the values 6, 3, 2 with probabilities 0.3, 0.25, 0.45, respectively. Figure 3.3 shows the behavior of the stock. As shown in Figure 3.2, the project payoff at node 1, 2, 3 can take the values 10, 8, 0, respectively. We find that λ^* is 2. However, for $\lambda = 2$ the price interval for the project value is [3.45; 4.12]. When $\lambda \leq 2$ GL(U) and GL(L) become unbounded, so [3.45; 4.12] is the tightest interval that we can compute for the project value.

Although these types of examples are nongeneric, this example shows that the bounds of the project value do not necessarily reduce to a single point for the smallest λ.

Figure 3.3: Decision Tree for the Simple Capital Budgeting Example



Chapter 4

Proportional Transaction Costs

So far we have assumed a frictionless market, and developed our results based on the no-arbitrage assumption, ignoring transaction costs. In the literature there are many works on the problem of pricing and hedging the contingent claims in presence of transaction costs. [6, 7, 11] are examples of the works in that area. Leland [7] develops an option replicating strategy which depends on the size of transaction costs, Edirisinghe, Naik and Uppal [11] provide a method to solve the cost minimization problem when there are fixed and variable transaction costs and finds the least cost strategies that yield payoffs at least as large as the desired one. Hodges and Neuberger [6] work on the problem of best replication of a contingent claim under transactions costs and considered the effect of the transaction costs on pricing and hedging. It is assumed that the cost of trading a stock is proportional to the price.

This chapter is devoted to investigate valuing risky project in incomplete markets in presence of transaction costs. Similar to [6], throughout this chapter we assume that cost of trading a security (excluding risk free security) is proportional to the price, also transaction costs for buying and selling a security are different and there is no transaction cost for risk free security. An investor who buys one share of security j when the security price is ξ_j pays $(1+\eta)\xi_j$ and who sells one share of security j gets $(1-\zeta)\xi_j$, where η and $\zeta \in [0,1)$. With these assumptions, we extend the option pricing model that is introduced in the previous chapter and we compute the project value bounds by solving the following optimization problems. It is also important to note that when $\eta = \zeta = 0$, problems OPT(U) and OPT(L) reduce to the problems 3.1 and 3.2, respectively.

We propose to solve the following problem for upper bounds

$$
\begin{aligned}
OPT(U) \qquad & \min \alpha_0 + \theta_0\xi_0^+ - \theta_0\xi_0^- + \theta_0\eta\xi_0^+ + \theta_0\zeta\xi_0^- \\
& s.t \\
& (1+r_f)(\alpha_{a(n)} - \alpha_n) + \theta_n\xi_n^- - \theta_n\xi_n^+ - \theta_n\eta\xi_n^+ - \theta_n\zeta\xi_n^- \geq c_n, \ \ \forall \ n \in N_t, t > 0 \\
& \xi_0 = \xi_0^+ - \xi_0^- \\
& \xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^- \\
& (1+r_f)\alpha_n + \theta_n\xi_n \geq 0, \ \ \forall \ n \in N_t \\
& \xi_0^+ \geq 0 \\
& \xi_0^- \geq 0 \\
& \xi_n^- \geq 0, \ \ \forall \ n \in N_t \\
& \xi_n^+ \geq 0, \ \ \forall \ n \in N_t.
\end{aligned}
$$

and the following problem for lower bounds

$$
\begin{aligned}
OPT(L) \quad & \max -\alpha_0 - \theta_0\xi_0^+ + \theta_0\xi_0^- - \theta_0\eta\xi_0^+ - \theta_0\zeta\xi_0^- \\
& s.t \\
& (1+r_f)(\alpha_n - \alpha_{a(n)}) + \theta_n\xi_n^+ - \theta_n\xi_n^- + \theta_n\eta\xi_n^+ + \theta_n\zeta\xi_n^- \leq c_n,, \ \forall \ n \in N_t, t > 0 \\
& \xi_0 = \xi_0^+ - \xi_0^- \\
& \xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^- \\
& (1+r_f)\alpha_n + \theta_n\xi_n \geq 0, \ \forall \ n \in N_t \\
& \xi_0^+ \geq 0 \\
& \xi_0^- \geq 0 \\
& \xi_n^- \geq 0, \ \forall \ n \in N_t \\
& \xi_n^+ \geq 0, \ \forall \ n \in N_t
\end{aligned}
$$

We notice that problems OPT(U) and OPT(L) are different from problems of Chapter 3. Since transaction cost is not applied to risk free security and it is applied to other securities, we write constraints $(\beta(n) - \beta(a(n)))s(n) \leq c_n$ and $(\beta(a(n)) - \beta(n))s(n) \geq c_n$ explicitly.

4.1 Capital Budgeting Example with Transaction Costs

Now, we can compute the upper and lower bounds of 'Invest Now' and 'Defer' Alternative for the Capital Budgeting example of Smith and Nau [10].

Invest Now Alternative upper and lower bounds can be computed by solving the following problems respectively:

$$
\begin{aligned}
&\min \alpha_0 + 20\xi_0^+ - 20\xi_0^- + 20\eta\xi_0^+ + 20\zeta\xi_0^- \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36\xi_1^- - 36\xi_1^+ - 36\eta\xi_1^+ - 36\zeta\xi_1^- \geq 190 \\
&1.08(\alpha_0 - \alpha_2) + 36\xi_2^- - 36\xi_2^+ - 36\eta\xi_2^+ - 36\zeta\xi_2^- \geq 170 \\
&1.08(\alpha_0 - \alpha_3) + 12\xi_3^- - 12\xi_3^+ - 12\eta\xi_3^+ - 12\zeta\xi_3^- \geq 70 \\
&1.08(\alpha_0 - \alpha_4) + 12\xi_4^- - 12\xi_4^+ - 12\eta\xi_4^+ - 12\zeta\xi_4^- \geq 30 \\
&\xi_0 = \xi_0^+ - \xi_0^- \\
&\xi_1 - \xi_0 = \xi_1^+ - \xi_1^- \\
&\xi_2 - \xi_0 = \xi_2^+ - \xi_2^- \\
&\xi_3 - \xi_0 = \xi_3^+ - \xi_3^- \\
&\xi_4 - \xi_0 = \xi_4^+ - \xi_4^- \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0 \\
&1.08\alpha_4 + 12\xi_4 \geq 0 \\
&\xi_0^+ \geq 0 \\
&\xi_0^- \geq 0 \\
&\xi_n^- \geq 0, \ \forall \ n \in N_t \\
&\xi_n^+ \geq 0, \ \forall \ n \in N_t
\end{aligned}
$$

$$
\begin{aligned}
&\max -\alpha_0 - 20\xi_0^+ + 20\xi_0^- - 20\eta\xi_0^+ - 20\zeta\xi_0^- \\
&s.t \\
&1.08(\alpha_1 - \alpha_0) + 36\xi_1^+ - 36\xi_1^- + 36\eta\xi_1^+ + 36\zeta\xi_1^- \leq 190 \\
&1.08(\alpha_2 - \alpha_0) + 36\xi_2^+ - 36\xi_2^- + 36\eta\xi_2^+ + 36\zeta\xi_2^- \leq 170 \\
&1.08(\alpha_3 - \alpha_0) + 12\xi_3^+ - 12\xi_3^- + 12\eta\xi_3^+ + 12\zeta\xi_3^- \leq 70 \\
&1.08(\alpha_4 - \alpha_0) + 12\xi_4^+ - 12\xi_4^- + 12\eta\xi_4^+ + 12\zeta\xi_4^- \leq 30 \\
&\xi_0 = \xi_0^+ - \xi_0^- \\
&\xi_1 - \xi_0 = \xi_1^+ - \xi_1^- \\
&\xi_2 - \xi_0 = \xi_2^+ - \xi_2^- \\
&\xi_3 - \xi_0 = \xi_3^+ - \xi_3^- \\
&\xi_4 - \xi_0 = \xi_4^+ - \xi_4^- \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0 \\
&1.08\alpha_4 + 12\xi_4 \geq 0 \\
&\xi_0^+ \geq 0 \\
&\xi_0^- \geq 0 \\
&\xi_n^- \geq 0, \ \forall \ n \in N_t \\
&\xi_n^+ \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Defer alternative upper bound can be computed by solving the problem:

$$
\begin{aligned}
&\min \alpha_0 + 20\xi_0^+ - 20\xi_0^- + 20\eta\xi_0^+ + 20\zeta\xi_0^- \\
&s.t \\
&1.08(\alpha_0 - \alpha_1) + 36\xi_1^- - 36\xi_1^+ - 36\eta\xi_1^+ - 36\zeta\xi_1^- \geq 77.68 \\
&1.08(\alpha_0 - \alpha_2) + 36\xi_2^- - 36\xi_2^+ - 36\eta\xi_2^+ - 36\zeta\xi_2^- \geq 57.68 \\
&1.08(\alpha_0 - \alpha_3) + 12\xi_3^- - 12\xi_3^+ - 12\eta\xi_3^+ - 12\zeta\xi_3^- \geq 0 \\
&\xi_0 = \xi_0^+ - \xi_0^- \\
&\xi_1 - \xi_0 = \xi_1^+ - \xi_1^- \\
&\xi_2 - \xi_0 = \xi_2^+ - \xi_2^- \\
&\xi_3 - \xi_0 = \xi_3^+ - \xi_3^- \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0 \\
&\xi_0^+ \geq 0 \\
&\xi_0^- \geq 0 \\
&\xi_n^- \geq 0, \ \forall \ n \in N_t \\
&\xi_n^+ \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Defer alternative lower bound can be computed by solving the problem:

$$
\begin{aligned}
&\max -\alpha_0 - 20\xi_0^+ + 20\xi_0^- - 20\eta\xi_0^+ - 20\zeta\xi_0^- \\
&s.t \\
&1.08(\alpha_1 - \alpha_0) + 36\xi_1^+ - 36\xi_1^- + 36\eta\xi_1^+ + 36\zeta\xi_1^- \leq 77.68 \\
&1.08(\alpha_2 - \alpha_0) + 36\xi_2^+ - 36\xi_2^- + 36\eta\xi_2^+ + 36\zeta\xi_2^- \leq 57.68 \\
&1.08(\alpha_3 - \alpha_0) + 12\xi_3^+ - 12\xi_3^- + 12\eta\xi_3^+ + 12\zeta\xi_3^- \leq 0 \\
&\xi_0 = \xi_0^+ - \xi_0^- \\
&\xi_1 - \xi_0 = \xi_1^+ - \xi_1^- \\
&\xi_2 - \xi_0 = \xi_2^+ - \xi_2^- \\
&\xi_3 - \xi_0 = \xi_3^+ - \xi_3^- \\
&1.08\alpha_1 + 36\xi_1 \geq 0 \\
&1.08\alpha_2 + 36\xi_2 \geq 0 \\
&1.08\alpha_3 + 12\xi_3 \geq 0 \\
&\xi_0^+ \geq 0 \\
&\xi_0^- \geq 0 \\
&\xi_n^- \geq 0, \ \forall \ n \in N_t \\
&\xi_n^+ \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

As a result of solving these problems we get different project values for different η and ζ value. From the Table 4.1 and Table 4.2 we can see how project values change for some values of η and ζ.

4.2 Model with Transaction Costs

Now, with the same approach as in the previous chapter we will restrict the gain-loss ratio and we will obtain tighter bounds to the project value in incomplete markets with transaction costs. For computing upper bounds we will solve the problem :

Table 4.1: No-Arbitrage Invest Now Alternative Upper and Lower Bounds with Transaction Cost

	Upper	Lower
$\eta = 0.01, \zeta = 0.01$	6.26	-25.54
$\eta = 0.05, \zeta = 0.05$	10.26	-30.2
$\eta = 0.1, \zeta = 0.05$	15.26	-30.2
$\eta = 0.1, \zeta = 0.1$	15.26	-36.04
$\eta = 0.15, \zeta = 0.1$	20.26	-36.04
$\eta = 0.15, \zeta = 0.15$	20.26	-41.87
$\eta = 0.1, \zeta = 0.15$	25.26	-41.87
$\eta = 0.15, \zeta = 0.2$	20.26	-47.7
$\eta = 0.2, \zeta = 0.2$	25.26	-47.7

Table 4.2: No-Arbitrage Defer Alternative Upper and Lower Bounds with Transaction Cost

	Upper	Lower
$\eta = 0.01, \zeta = 0.01$	29.42	20.88
$\eta = 0.05, \zeta = 0.05$	32.01	18.96
$\eta = 0.1, \zeta = 0.05$	35.24	18.96
$\eta = 0.1, \zeta = 0.1$	35.24	16.56
$\eta = 0.15, \zeta = 0.1$	38.48	16.56
$\eta = 0.15, \zeta = 0.15$	38.48	14.15
$\eta = 0.1, \zeta = 0.15$	41.72	14.15
$\eta = 0.15, \zeta = 0.2$	38.48	11.75
$\eta = 0.2, \zeta = 0.2$	41.72	11.75

$$
\begin{aligned}
GLT(U) \qquad & \min \alpha_0 + \theta_0\xi_0^+ - \theta_0\xi_0^- + \theta_0\eta\xi_0^+ + \theta_0\zeta\xi_0^- \\
& s.t \\
& (1+r_f)(\alpha_{a(n)} - \alpha_n) + \theta_n\xi_n^- - \theta_n\xi_n^+ - \theta_n\eta\xi_n^+ - \theta_n\zeta\xi_n^- \geq c_n, \ \forall \ n \in N_t, t > 0 \\
& \xi_0 = \xi_0^+ - \xi_0^- \\
& \xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^- \\
& (1+r_f)\alpha_n + \theta_n\xi_n - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
& \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \forall \ n \in N_t \\
& \xi_0^+ \geq 0 \\
& \xi_0^- \geq 0 \\
& \xi_n^- \geq 0, \ \forall \ n \in N_t \\
& \xi_n^+ \geq 0, \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

For computing lower bounds we will solve the problem :

$$
\begin{aligned}
GLT(L) \quad & \max -\alpha_0 - \theta_0\xi_0^+ + \theta_0\xi_0^- - \theta_0\eta\xi_0^+ - \theta_0\zeta\xi_0^- \\
& s.t \\
& (1+r_f)(\alpha_n - \alpha_{a(n)}) + \theta_n\xi_n^+ - \theta_n\xi_n^- + \theta_n\eta\xi_n^+ + \theta_n\zeta\xi_n^- \leq c_n, \ \forall \ n \in N_t, t > 0 \\
& \xi_0 = \xi_0^+ - \xi_0^- \\
& \xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^- \\
& (1+r_f)\alpha_n + \theta_n\xi_n - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
& \sum_n p_n t x_n^+ - \lambda \sum_n p_n x_n^- \geq 0, \ \forall \ n \in N_t \\
& \xi_0^+ \geq 0 \\
& \xi_0^- \geq 0 \\
& \xi_n^- \geq 0, \ \forall \ n \in N_t \\
& \xi_n^+ \geq 0, \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

Problems GLT(U) and GLT(L) reduce to the problems GL(U) and GL(L) of the Chapter 3, respectively, when we choose transaction costs 0, i.e., $\eta = \zeta = 0$

Continuing the example in [10] with $\eta = \zeta = 0.01$, we compute 'Invest Now' and 'Defer' Alternative upper and lower bounds for different λ values. From Table 4.5 and Table 4.6 as λ gets smaller upper and lower bounds get closer and for $\lambda = 1.44499$ 'Invest Now' alternative upper and lower bounds become equal to -3 and 'Defer' Alternative upper and lower bounds become equal to 25.63. As λ gets larger upper and lower bounds computed by this new method approach the no-arbitrage bounds.

Although we can compute a unique project value when $\lambda = 1.44499$, hedging policies are different for dominated and dominating strategies. Table 4.3 and Table 4.4 show how replicating portfolios differ from each other.

Now we can generalize the consistency theorem of the previous chapter to the

Table 4.3: Invest Now Alternative Replicating Portfolios for $\lambda = 1.44499$ and $\eta = \zeta = 0.01$

	Upper	Lower
β_0	-101.0	-61.736
β_1	74.926	-237.662
β_2	56.408	-219.143
β_3	-36.185	-126.551
β_4	-73.222	-89.514
ξ_0	0.00	8.056
ξ_1	0.00	8.056
ξ_2	0.00	8.056
ξ_3	0.00	8.056
ξ_4	0.00	8.056

Table 4.4: Defer Alternative Replicating Portfolios for $\lambda = 1.44499$ and $\eta = \zeta = 0.01$

	Upper	Lower
β_0	-45.575	-25.631
β_1	-117.501	46.295
β_2	-98.982	27.777
β_3	-45.575	-25.631
ξ_0	3.525	0.00
ξ_1	3.525	0.00
ξ_2	3.525	0.00
ξ_3	3.525	0.00

incomplete markets with transaction costs.

Theorem 2. ***Consistency Theorem (Incomplete Markets with Transaction Costs)*** *In an incomplete market with transaction costs, the firm's gain-loss upper bound and gain-loss lower bound for any project may differ, but both lie between the bounds given by the option pricing analysis method.*

Proof. Similar to the proof of the Consistency Theorem in incomplete markets, take the dual of the problems OPT(U) and GLT(U).

Dual of OPT(U):

$$OPT(D1) \qquad \max \sum_n v_n c_n \tag{4.1}$$
$$s.t$$
$$-(1+r_f)v_n + (1+r_f)w_n \leq 0 \tag{4.2}$$
$$(1+r_f)v_n \leq 0 \tag{4.3}$$
$$v_n\theta_n + v_n\theta_n\eta \leq 0 \tag{4.4}$$
$$-v_n\theta_n - v_n\theta_n\zeta \leq 0 \tag{4.5}$$
$$w_n\theta_n \leq 0 \tag{4.6}$$
$$v_n \geq 0 \tag{4.7}$$
$$w_n \geq 0. \tag{4.8}$$
$$\tag{4.9}$$

Dual of GLT(U):

$$GLT(D1) \qquad \max \sum_t v_n c_n) \tag{4.10}$$
$$s.t$$
$$-(1+r_f)v_n + (1+r_f)w_n \leq 0 \tag{4.11}$$
$$(1+r_f)v_n \leq 0 \tag{4.12}$$
$$v_n\theta_n + v_n\theta_n\eta \leq 0 \tag{4.13}$$
$$-v_n\theta_n - v_n\theta_n\zeta \leq 0 \tag{4.14}$$
$$w_n\theta_n \leq 0 \tag{4.15}$$
$$Vp_n \leq w_n \tag{4.16}$$
$$w_n \leq \lambda V p_n \tag{4.17}$$
$$v_n \geq 0 \tag{4.18}$$
$$V \geq 0. \tag{4.19}$$
$$\tag{4.20}$$

The constraints 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 are equivalent to constraints 4.11, 4.12, 4.13, 4.14, 4.15, 4.18, respectively. Furthermore constraints 4.16, 4.17, 4.18 and $p_n \geq 0$, $\lambda > 1$ imply constraint 4.8. Since the objective functions of the problems OPT(D1) and GLT(D1) are identical, but the problem GLT(D1)'s feasible set is more restricted. Therefore optimal value of the GLT(D1) should be less than or equal to optimal value of OPT(D1).

Similarly, for the other side of the proof we take the dual of the problems OPT(L) and GLT(L).

Dual of OPT(L):

$$OPT(D2) \qquad \min \sum_n v_n c_n \tag{4.21}$$

$$s.t$$

$$(1+r_f)v_n + (1+r_f)w_n \geq 0 \tag{4.22}$$

$$-(1+r_f)v_n \geq 0 \tag{4.23}$$

$$v_n\theta_n + v_n\theta_n\eta \geq 0 \tag{4.24}$$

$$-v_n\theta_n + v_n\theta_n\zeta \geq 0 \tag{4.25}$$

$$w_n\theta_n \geq 0 \tag{4.26}$$

$$v_n \geq 0 \tag{4.27}$$

$$w_n \geq 0. \tag{4.28}$$

Dual of GLT(L):

$$GLT(D2) \qquad \min \sum_n v_n c_n \tag{4.29}$$

$$s.t$$

$$(1+r_f)v_n + (1+r_f)w_n \geq 0 \tag{4.30}$$

$$-(1+r_f)v_n \geq 0 \tag{4.31}$$

$$v_n\theta_n + v_n\theta_n\eta \geq 0 \tag{4.32}$$

$$-v_n\theta_n + v_n\theta_n\zeta \geq 0 \tag{4.33}$$

$$w_n\theta_n \geq 0 \tag{4.34}$$

$$Vp_n \geq w_n \tag{4.35}$$

$$w_n \geq \lambda V p_n \tag{4.36}$$

$$v_n \geq 0 \tag{4.37}$$

$$V \geq 0. \tag{4.38}$$

The objective function of both problem is identical and the constraints 4.22, 4.23, 4.24, 4.25, 4.26, 4.27 of the problem OPT(L) is identical to the constraints 4.30, 4.31, 4.32, 4.33, 4.34, 4.37 of the problem GLT(D2) and the constraint 4.28 is implied by the constraints 4.35, 4.36, 4.38 and the facts $p_n \geq 0$ and $\lambda > 0$.

So, the feasible set of GLT(D2) is subset the feasible set of OPT(D2), therefore optimal value of the GLT(D2) should be greater than or equal to optimal value of OPT(D2). By the strong duality theorem of linear programming this gives us the desired result.

□

Comparing Table 4.5 with Table 3.3 and Table 4.6 with Table 3.4, we explain the impact of the transaction costs to the value of project. When there is transaction costs in the market, option pricing analysis method gives wider bounds. As λ decreases these bounds approximate to each other and for $\lambda = 1.44499$, Invest Now Alternative project value bounds coincide in -3 and Defer Alternative project value bounds coincide in

Table 4.5: Invest Now Alternative Gain-Loss Bounds with Transaction for $\eta = 0.01$ and $\zeta = 0.01$

λ	Upper	Lower
100000000	6.26	-25.54
1000	6.24	-25.45
500	6.23	-25.36
100	6.11	-24.66
50	5.96	-23.53
20	5.54	-21.61
10	4.85	-18.6
5	3.52	-16.22
4	2.89	-14.26
3	1.88	-4
2	0	-12.65
1.8	-0.58	-5.88
1.6	-1.28	-5.15
1.5	-2.37	-4
1.44499	-3	-3

25.63 when $\eta = \zeta = 0.01$. When $\eta = \zeta = 0.2$, for $\lambda = 1.00000001$ Invest Now Alternative project value bounds approximate to 7.11 and Defer Alternative project value bounds approximate to 31.33. So, we can conclude that when there is transaction costs in the market, as ratio of the transaction costs increase λ^* decrease, the project value bounds get wider, and the value that these bounds coincide increase.

4.3 Counter Example

Let us now look at the behavior of the bounds when λ decreases. Consider the dual problems GLT(D1) and GLT(D2),which give us the gain loss upper and lower bounds respectively. If both problems have an unique optimal feasible solution, the upper and lower bounds coincide. However, the following example shows that the bounds do not have to coincide for the smallest λ value, λ^* .

Example: Let us assume that market consists of a riskless asset with zero growth rate and 2 stocks. Also assume that %10 transaction cost is applied when selling and

Table 4.6: Defer Alternative Gain-Loss Bounds with Transaction for $\eta = 0.01$ and $\zeta = 0.01$

λ	Upper	Lower
100000000	29.42	20.88
1000	29.41	20.89
500	29.40	20.90
100	29.34	20.95
50	29.27	21.02
20	29.06	21.23
10	28.73	21.54
5	28.16	22.09
4	27.90	22.33
3	27.52	22.76
2	26.68	23.70
1.8	26.38	24.02
1.6	26	24.41
1.5	25.77	25.07
1.44499	25.63	25.63

buying these stocks. At $n = 0$ stock price is 10 for both of the stocks. As shown in Figure 4.1 , at state 1, 2, 3, 4 first stock's price can take the values 11, 13, 15, 9 and as shown in Figure 4.2, the second stock's price can take values 12, 11, 18, 6 with probabilities 0.3, 0.3, 0.3, 0.1, respectively. Therefore, at node 1, $\theta(1) = (11 \quad 12)^T$ with $p_1 = 0.3$; at node 2, $\theta(2) = (13 \quad 11)^T$ with $p_2 = 0.3$; at node 3, $\theta(3) = (15 \quad 18)^T$ with $p_3 = 0.3$; at node 4, $\theta(4) = (9 \quad 6)^T$ with $p_4 = 0.1$. The project payoff at $t = 1$ can take the values $c(1) = (40, 10, 5, 0)^T$. We find that λ^* is 9. However, for $\lambda = 9$ the price interval for the project value is [9.17; 25.77].

This example shows that the bounds of the project value do not necessarily reduce to a single point for the smallest λ.

Figure 4.1: Payoffs of the Stock-1

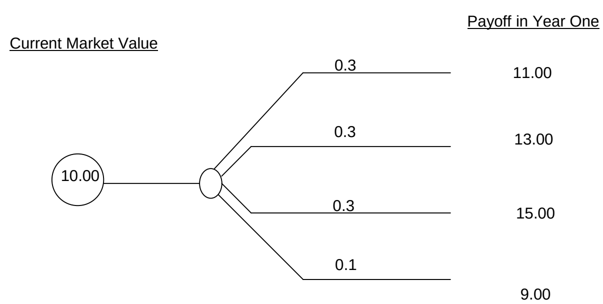
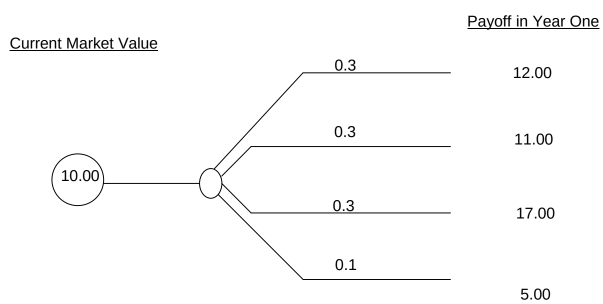


Figure 4.2: Payoffs of the Stock-2



Chapter 5

Uncertain Probabilities

We have assumed thus far that the state probabilities are exactly known, however this requires a perfect knowledge about the market. In practice, these probabilities prone to errors. Valuing risky projects based on the inaccurate state probabilities may be highly misleading. A similar problem is also discussed in [2, 5]. Ghaoui, Oks, and Oustry [5] worked on the problem of the computing and optimizing the worst-case Value-at-Risk, which can be solved by solving a semi-definite programming problem. They assume that the true distribution of returns is only partially known. Carr, Geman and Madan [2] considered the problem of hedging, pricing and positioning in incomplete market and developed a new approach which bridges between the standard arbitrage pricing and expected utility maximization. This approach involves specifying a set of probability measure and associated floors. So, probability measures are not exactly known and it is defined that the investment opportunity will be acceptable, if the expected payoffs under these measurements exceed associated floors.

In this chapter we will apply our new method and find project value bounds when the state probabilities are partially known as in [5]. Let us assume that $p_n \in P = \{\mu_n \leq p_n \leq \kappa_n, \ \ \sum_{n \in N} p_n = 1, \ \ p_n \geq 0\}$ where μ_n and κ_n are known positive numbers. In option pricing method, uncertain state probabilities do not affect the project value, since these probabilities are not used in this method. However, in our new method state probabilities have significant roles. The constraint $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0$ should be satisfied for all $p_n \in P$. In fact this expression is equivalent to:

$$\min_{p_n \in P} \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0. \tag{5.1}$$

Since this problem is semi-infinite it is not practical to use. Therefore by taking its dual we get the following linear problem:

$$\begin{aligned}
&\max \sum_v \mu_n y_n - \sum_n \kappa_n z_n + K \\
&s.t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \ \forall \ \ n \in N_t, t > 0 \\
&y_n \geq 0, \ \ \forall \ \ n \in N_t \\
&z_n \geq 0, \ \ \forall \ \ n \in N_t.
\end{aligned}$$

The above linear problem can be used instead of the gain-loss constraint. So upper and lower bounds of the project value can be found by solving the expanded version of the problems in Chapter 3 and 4. We will divide this chapter into two section. In the first and second section we will examine how the project value bounds will change when state probabilities are not known exactly in a market without transaction costs and in a market with transaction costs, respectively.

5.1 Market Without Transaction Costs

In Chapter 3 we have computed project value bounds with the assumption that there is no transaction cost applied when buying and selling the securities. Now with the same assumption we will compute project value bounds when state probabilities p_n are uncertain. To obtain the desired results we replace the constraint $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0$

with the constraints

$$
\begin{aligned}
&\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0 \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0,\ \ \forall\ \ n \in N_t, t > 0 \\
&y_n \geq 0,\ \ \forall\ \ n \in N_t \\
&z_n \geq 0,\ \ \forall\ \ n \in N_t.
\end{aligned}
$$

Upper Bound is computed by finding:

$$
\begin{aligned}
&\min \beta(0)s(0) \\
&s.t \\
&(\beta(a(n)) - \beta(n))s(n) \geq c_n,\ \ \forall\ \ n \in N_t, t > 0 \\
&\beta(n)s(n) - x_n^+ + x_n^- = 0,\ \ \forall\ \ n \in N_t \\
&-y_n + z_n - K + x_n^+ - \lambda x_n^- \geq 0,\ \ \forall\ \ n \in N_t \\
&y_n \geq 0,\ \ \forall\ \ n \in N_t \\
&z_n \geq 0,\ \ \forall\ \ n \in N_t \\
&x_n^+ \geq,\ \ \forall\ \ n \in N_t \\
&x_n^- \geq 0,\ \ \forall\ \ n \in N_t \\
&\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0.
\end{aligned}
$$

Lower Bound is computed by finding:

$$
\begin{aligned}
&\max -\beta(0)s(0) \\
&s.t \\
&(\beta(n)-\beta(a(n)))s(n) \leq c_n, \ \forall \ n \in N_t, t>0 \\
&\beta(n)s(n) - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t \\
&y_n \geq 0, \ \forall \ n \in N_t \\
&z_n \geq 0, \ \forall \ n \in N_t \\
&x_n^+ \geq 0, \ \forall \ n \in N_t \\
&x_n^- \geq 0, \ \forall \ n \in N_t \\
&\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0.
\end{aligned}
$$

When the state probability bounds collapse to a p_n, i.e., $\kappa_n = \mu_n = p_n$, these problems reduce to the problems GL(U) and GL(L), respectively. Let us show this:

Let $\kappa_n = \mu_n = p_n$ and the constraint $y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0$ imply that

$$
y_n \leq z_n + x_n^+ - \lambda x_n^- - K. \tag{5.2}
$$

Then the constraint $\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0$ become:

$$
\begin{aligned}
0 &\leq \sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \\
&= \sum_n p_n y_n - \sum_n p_n z_n + K \\
&\leq \sum_n p_n z_n + \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- - \sum_n p_n K - \sum_n p_n z_n + K \\
&= \sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^-
\end{aligned}
$$

So we get the result $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0$, which is exactly the gain-loss constraint of the problem GL(U).

5.1.1 The Consistency Theorem

In Chapter 3 and 4, we proved the Consistency Theorem in a market without transaction costs and with transaction costs, respectively. Now, we will state and prove the Consistency Theorem in a market with uncertain state probabilities.

Theorem 3. ***Consistency Theorem*** *In an incomplete market, the firm's gain-loss upper bound and gain-loss lower bound for any project when the state probabilities uncertain, ie.* $p_n \in P = \{\mu_n \leq p_n \leq \kappa_n, \ \ \sum_n p_n = 1, \ \ p_n \geq 0\}$*, may differ, but both lie between the bounds given by the option pricing analysis method.*

Proof. First we have to show that upper bound given by the option pricing analysis is greater than or equal to gain-loss upper bound when the state probabilities are uncertain. To reach the desired result, we form dual of the problem:

$$
\begin{array}{ll}
UP(U) & \min \beta(0)s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n))s(n) \geq c_n, \ \ \forall \ n \in N_t, t > 0 \\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \ \forall \ n \in N_t \\
& -y_n + z_n - K + x_n^+ - \lambda x_n^+ \geq 0, \ \ \forall \ n \in N_t \\
& y_n \geq 0, \ \ \forall \ n \in N_t \\
& z_n \geq 0, \ \ \forall \ n \in N_t \\
& x_n^+ \geq 0, \ \ \forall \ n \in N_t \\
& x_n^- \geq 0, \ \ \forall \ n \in N_t 0 \\
& \sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0.
\end{array}
$$

and we get

$$
\begin{aligned}
UP(D1) \qquad & \max \sum_n v_n c_n && (5.3)\\
& s.t \\
& \sum_n v_n s(n) \leq s(0) && (5.4)\\
& w_n s(n) - v_n s(n) \leq 0 && (5.5)\\
& \gamma_n \leq w_n && (5.6)\\
& w_n \leq \lambda \gamma_n && (5.7)\\
& \mu_n \varepsilon_n \leq \gamma_n && (5.8)\\
& \gamma_n \leq \kappa_n \varepsilon_n && (5.9)\\
& \gamma_n \geq 0, \;\; \forall \;\; n \in N_t, t > 0 && (5.10)\\
& v_n \geq 0, \;\; \forall \;\; n \in N_t && (5.11)\\
& \varepsilon_n \geq 0, \;\; \forall \;\; n \in N_t. && (5.12)
\end{aligned}
$$

Now, form the dual of the problem:

$$
\begin{aligned}
OP(U) \qquad & \min \beta(0) s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n)) s(n) \geq c_n, \;\; \forall \;\; n \in N_t, t > 0 \\
& \beta(n) s(n) \geq 0, \;\; \forall \;\; n \in N_t.
\end{aligned}
$$

We get:

$$
\begin{aligned}
OP(D1) \qquad & \max \sum_{t} v_n c_n && (5.13)\\
& s.t \\
& \sum_{n} v_n s(n) \leq s(0) && (5.14)\\
& w_t s(n) - v_n s(n) \leq 0 && (5.15)\\
& v_n \geq 0 && (5.16)\\
& w_n \geq 0. && (5.17)
\end{aligned}
$$

Both dual problems have the identical objective function and the constraints 5.4, 5.5, 5.11 and 5.14, 5.15, 5.16 are equivalent each other, respectively. Moreover, 5.6, 5.7, 5.10 imply 5.17 with the fact that $\lambda \geq 0$. Therefore optimal value of UP(D1) is less than or equal to OP(D1).

For the other part of the proof, form dual of the problem:

$$
\begin{aligned}
UP(L) \qquad & \max -\beta(0)s(0)\\
& s.t\\
& (\beta(n) - \beta(a(n)))s(n) \leq c_n, \ \forall \ n \in N_t, t > 0\\
& \beta(n)s(n) - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t\\
& y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t\\
& y_n \geq 0, \ \forall \ n \in N_t\\
& z_n \geq 0, \ \forall \ n \in N_t 0\\
& x_n^+ \geq 0, \ \forall \ n \in N_t\\
& x_n^- \geq 0, \ \forall \ n \in N_t\\
& \sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
UP(D2) \qquad & \min \sum_n v_n c_n && (5.18)\\
& s.t \\
& \sum_n v_n s(n) \geq s(0) && (5.19)\\
& w_n s(n) - v_n s(n) \geq 0 && (5.20)\\
& w_n \geq \gamma_n && (5.21)\\
& \lambda\gamma_n \geq w_n && (5.22)\\
& \mu_n \varepsilon_n + \gamma_n \geq && (5.23)\\
& -\gamma_n - b_n \varepsilon_n \geq && (5.24)\\
& \gamma_n \geq 0, \ \forall \ n \in N_t, t > 0 && (5.25)\\
& v_n \geq 0, \ \forall \ n \in N_t && (5.26)\\
& \varepsilon_n \geq 0, \ \forall \ n \in N_t. && (5.27)
\end{aligned}
$$

and form the dual of the problem:

$$
\begin{aligned}
OP(L) \qquad & \max \beta(0) s(0) \\
& s.t \\
& (\beta(a(n)) - \beta(n)) s(n) \leq c_n, \ \forall \ n \in N_t, t > 0 \\
& \beta(n) s(n) \leq 0, \ \forall \ n \in N_t.
\end{aligned}
$$

$$OP(D2) \qquad \min \sum_n v_n c_n \tag{5.28}$$

$$s.t \tag{5.29}$$

$$\sum_n v_n s(n) \geq s(0) \tag{5.30}$$

$$w_n s(n) - v_n s(n) \geq 0 \tag{5.31}$$

$$v_n \geq 0 \tag{5.32}$$

$$w_n \geq 0. \tag{5.33}$$

Similar to the first part of the proof, both dual problem has the identical objective function and the constraints 5.19, 5.20, 5.26 is equivalent to 5.30, 5.31, 5.32, respectively. Furthermore, constraints 5.21 and 5.25 imply constraint 5.33. So optimal value of UP(D2) is greater than or equal to optimal value of OP(D2). Then, by strong duality theorem we are done.

□

5.1.2 An Example

In this chapter we will also continue to work on the Capital Budgeting Example of [10] and we assume that there is $\mp 10\%$ error in the probabilities of this example. For Invest Now opportunity upper and lower bounds can be computed by solving the following problems respectively :

$$
\begin{aligned}
&\min_{\beta} \beta_0 + 20\xi_0 \\
&s.t \\
&1.08(\beta_0 - \beta_1) + 36(\xi_0 - \xi_1) \geq 190 \\
&1.08(\beta_0 - \beta_2) + 36(\xi_0 - \xi_2) \geq 170 \\
&1.08(\beta_0 - \beta_3) + 12(\xi_0 - \xi_3) \geq 70 \\
&1.08(\beta_0 - \beta_4) + 12(\xi_0 - \xi_4) \geq 30 \\
&1.08\beta_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\beta_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\beta_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&1.08\beta_4 + 12\xi_4 - x_4^+ + x_4^- = 0 \\
&x_n^+ \geq 0, \ \forall \ n \in N_t, t > 0 \\
&x_n^- \geq 0, \ \forall \ n \in N_t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t \\
&y_n^+ \geq 0, \ \forall \ n \in N_t \\
&z_n^- \geq 0, \ \forall \ n \in N_t \\
&0.225y_1 + 0.225y_2 + 0.3375y_3 + 0.1125y_4 \\
&-0.275z_1 - 0.275z_2 - 0.4125z_3 - 0.1375z_4 + K \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
& \max_{\beta} -\beta_0 - 20\xi_0 \\
& s.t \\
& 1.08(\beta_1 - \beta_0) + 36(\xi_1 - \xi_0) \leq 190 \\
& 1.08(\beta_2 - \beta_0) + 36(\xi_2 - \xi_0) \leq 170 \\
& 1.08(\beta_3 - \beta_0) + 12(\xi_3 - \xi_0) \leq 70 \\
& 1.08(\beta_4 - \beta_0) + 12(\xi_4 - \xi_0) \leq 30 \\
& 1.08\beta_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
& 1.08\beta_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
& 1.08\beta_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
& 1.08\beta_4 + 12\xi_4 - x_4^+ + x_4^- = 0 \\
& x_n^+ \geq 0, \ \forall \ n \in N_t, t > 0 \\
& x_n^- \geq 0, \ \forall \ n \in N_t \\
& y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t \\
& y_n^+ \geq 0, \ \forall \ n \in N_t \\
& z_n^- \geq 0, \ \forall \ n \in N_t \\
& 0.225y_1 + 0.225y_2 + 0.3375y_3 + 0.1125y_4 \\
& -0.275z_1 - 0.275z_2 - 0.4125z_3 - 0.1375z_4 + K \geq 0.
\end{aligned}
$$

For Defer alternative upper and lower bounds can be computed by solving the following problems respectively :

$$
\begin{aligned}
&\min_{\beta} \beta_0 + 20\xi_0 \\
&s.t \\
&1.08(\beta_0 - \beta_1) + 36(\xi_0 - \xi_1) \geq 77.68 \\
&1.08(\beta_0 - \beta_2) + 36(\xi_0 - \xi_2) \geq 57.68 \\
&1.08(\beta_0 - \beta_3) + 12(\xi_0 - \xi_3) \geq 0 \\
&1.08\beta_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\beta_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\beta_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&x_n^+ \geq 0, \ \forall \ n \in N_t, t > 0 \\
&x_n^- \geq 0, \ \forall \ n \in N_t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t \\
&y_n^+ \geq 0, \ \forall \ n \in N_t \\
&z_n^- \geq 0, \ \forall \ n \in N_t \\
&0.225y_1 + 0.225y_2 + 0.45y_3 - 0.275z_1 - 0.275z_2 - 0.55z_3 + K \geq 0.
\end{aligned}
$$

$$
\begin{aligned}
&\max_{\beta} -\beta_0 - 20\xi_0 \\
&s.t \\
&1.08(\beta_1 - \beta_0) + 36(\xi_1 - \xi_0) \leq 77.68 \\
&1.08(\beta_2 - \beta_0) + 36(\xi_2 - \xi_0) \leq 57.68 \\
&1.08(\beta_3 - \beta_0) + 12(\xi_3 - \xi_0) \leq 0 \\
&1.08\beta_1 + 36\xi_1 - x_1^+ + x_1^- = 0 \\
&1.08\beta_2 + 36\xi_2 - x_2^+ + x_2^- = 0 \\
&1.08\beta_3 + 12\xi_3 - x_3^+ + x_3^- = 0 \\
&x_n^+ \geq 0, \ \forall \ n \in N_t, t > 0 \\
&x_n^- \geq 0, \ \forall \ n \in N_t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t \\
&y_n^+ \geq 0, \ \forall \ n \in N_t \\
&z_n^- \geq 0, \ \forall \ n \in N_t \\
&0.225y_1 + 0.225y_2 + 0.45y_3 - 0.275z_1 - 0.275z_2 - 0.55z_3 + K \geq 0.
\end{aligned}
$$

As it can be seen from the Table 5.1 and Table 5.2 project value bounds coincide for some value of λ. For $\lambda = 1.22727052$, we get -4 for Invest Now Alternative and 25.07 for Defer Alternative as project value. Interestingly, in Chapter 1 we computed the same project values for $\lambda = 1.5$. Now, we assume that there is $\mp 5\%$ error in state probabilities. In this case, we get the same project values for $\lambda = 1.30737$. So, it can be observed that when the ratio of the error become smaller, λ approximate to the λ^* that is computed with the exact state probabilites. However, it is not possible in all cases to find a λ value that gives us unique project value. Example 3.2 of Chapter 3 shows that for $\lambda = 1.636359$ project value interval is [3.48; 3.94].

The behaviour of the Defer Alternative gain-loss bounds with $\mp 5\%$ error in p_n can be seen from the Figure 5.1. As λ gets larger upper bound approximate to 28.77 and lower bound approximates to 21.36. As λ gets smaller, upper and lower bounds get

Table 5.1: Invest Now Alternative Gain-Loss Bounds with $\mp 10\%$ error in p_n

λ	Upper	Lower
100000000	5.26	-24.37
1000	5.25	-24.3
500	5.23	-24.22
100	5.13	-23.64
50	5.01	-22.94
20	4.65	-21.04
10	4.06	-18.37
5	2.96	-14.34
4	2.42	-12.8
3	1.56	-10.75
2	-0.07	-7.44
1.8	-0.59	-6.59
1.6	-1.21	-5.69
1.5	-1.58	-5.23
1.4	-2.18	-4.78
1.3	-3.01	-4.33
1.25	-3.69	-4.10
1.23	-3.96	-4.01
1.22727052	-4	-4

closer and for $\lambda = 1.35715$ they become equal to 25.07.

5.2 Market With Transaction Costs

In this section, we assume that our state probabilities are uncertain in a market with transaction costs. So, we extend our model in Chapter 4 by replacing the constraint $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \geq 0$ with the constraints

Figure 5.1: Defer Alternative Gain-Loss Bounds with $\mp 5\%$ error in p_n

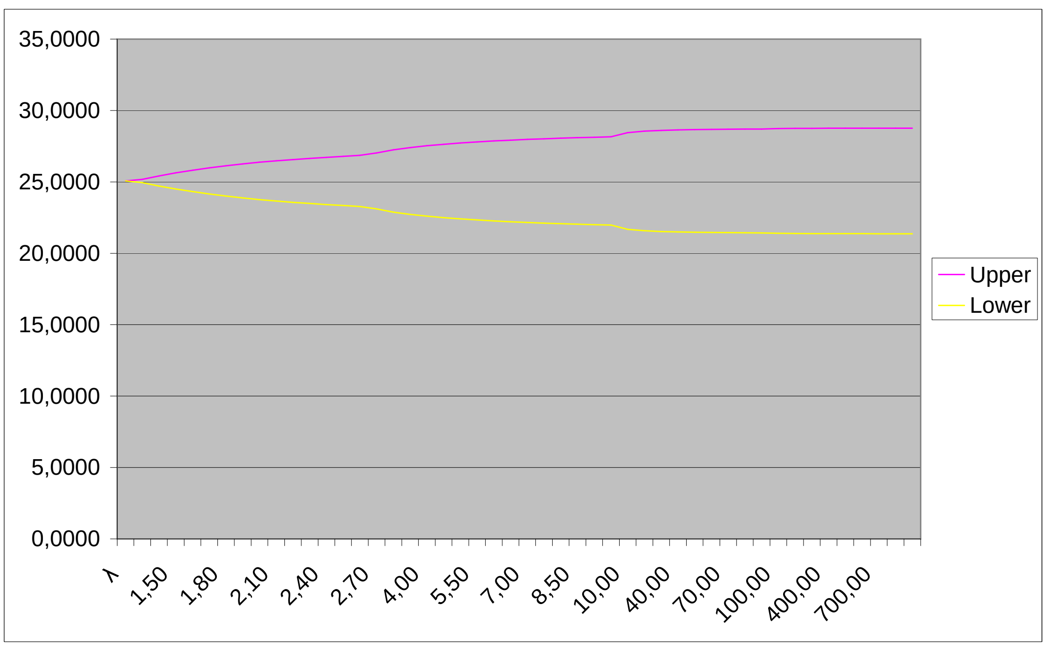


Table 5.2: Defer Alternative Gain-Loss Bounds with $\mp$10% error in p_n

λ	Upper	Lower
100000000	28.77	21.36
1000	28.76	21.37
500	28.76	21.38
100	28.71	21.42
50	28.65	21.48
20	28.48	21.65
10	28.21	21.92
5	27.73	22.41
4	27.51	22.62
3	27.14	23.00
2	26.43	23.71
1.8	26.20	23.94
1.6	25.93	24.21
1.5	25.74	24.40
1.4	25.52	24.61
1.3	25.27	24.86
1.25	25.13	25.00
1.22727052	25.07	25.07

$$\begin{aligned}
&\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0 \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \ \forall \ \ n \in N_t, t > 0 \\
&y_n \geq 0, \ \ \forall \ \ n \in N_t \\
&z_n \geq 0, \ \ \forall \ \ n \in N_t.
\end{aligned}$$

For computing upper bound we will solve the problem:

$$\min \alpha_0 + \theta_0\xi_0^+ - \theta_0\xi_0^- + \theta_0\eta\xi_0^+ + \theta_0\zeta\xi_0^-$$
$$s.t$$
$$(1+r_f)(\alpha_{a(n)} - \alpha_n) + \theta_n\xi_n^- - \theta_n\xi_n^+ - \theta_n\eta\xi_n^+ - \theta_n\zeta\xi_n^- \geq c_n, \ \forall \ n \in N_t, t > 0$$
$$\xi_0 = \xi_0^+ - \xi_0^-$$
$$\xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^-$$
$$(1+r_f)\alpha_n + \theta_n\xi_n - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t$$
$$y_n - z_n + K - x_n^+ + \lambda x_n^- \leq 0, \ \forall \ n \in N_t, t > 0$$
$$y_n \geq 0, \ \forall \ n \in N_t$$
$$z_n \geq 0, \ \forall \ n \in N_t.$$
$$\xi_0^+ \geq 0$$
$$\xi_0^- \geq 0$$
$$\xi_n^- \geq 0, \ \forall \ n \in N_t$$
$$\xi_n^+ \geq 0, \ \forall \ n \in N_t$$
$$x_n^- \geq 0, \ \forall \ n \in N_t$$
$$x_n^+ \geq 0, \ \forall \ n \in N_t.$$
$$\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \geq 0$$

For computing lower bounds we will solve the problem :

$$
\begin{aligned}
&\max -\alpha_0 - \theta_0\xi_0^+ + \theta_0\xi_0^- - \theta_0\eta\xi_0^+ - \theta_0\zeta\xi_0^- \\
&s.t \\
&(1+r_f)(\alpha_n - \alpha_{a(n)}) + \theta_n\xi_n^+ - \theta_n\xi_n^- + \theta_n\eta\xi_n^+ + \theta_n\zeta\xi_n^- \le c_n, \ \forall \ n \in N_t, t > 0 \\
&\xi_0 = \xi_0^+ - \xi_0^- \\
&\xi_n - \xi_{a(n)} = \xi_n^+ - \xi_n^- \\
&(1+r_f)\alpha_n + \theta_n\xi_n - x_n^+ + x_n^- = 0, \ \forall \ n \in N_t \\
&y_n - z_n + K - x_n^+ + \lambda x_n^- \le 0, \ \forall \ n \in N_t, t > 0 \\
&y_n \ge 0, \ \forall \ n \in N_t \\
&z_n \ge 0, \ \forall \ n \in N_t. \\
&\xi_0^+ \ge 0 \\
&\xi_0^- \ge 0 \\
&\xi_n^- \ge 0, \ \forall \ n \in N_t \\
&\xi_n^+ \ge 0, \ \forall \ n \in N_t \\
&x_n^- \ge 0, \ \forall \ n \in N_t \\
&x_n^+ \ge 0, \ \forall \ n \in N_t. \\
&\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \ge 0
\end{aligned}
$$

When $\kappa_n = \mu_n = p_n$, these problems reduce to problems GLT(U) and GLT(L), respectively. As we have showed before the constraints $y_n - z_n + K - x_n^+ + \lambda x_n^- \le 0$ and $\sum_n \mu_n y_n - \sum_n \kappa_n z_n + K \ge 0$ imply the constraint $\sum_n p_n x_n^+ - \lambda \sum_n p_n x_n^- \ge 0$. This is an expected result, because when $\kappa_n = \mu_n = p_n$ there is no uncertainty on the state probabilities. So we get back to the case of Chapter 4.

5.2.1 An Example

Continuing to work on the Capital Budgeting example of Smith and Nau [10], we compute upper and lower bounds for the 'Invest Now' and 'Defer' alternatives. As it can be seen from the Table 5.3 and Table 5.4, as λ goes to infinity, project's price bounds approach the no-arbitrage price bounds. As λ gets smaller these bounds get closer and for $\lambda = 1.18227$, they collapse to -3. When we compare this result with the results of Chapter 4, it can be seen that project value is -3 in both cases, however in Chapter 4 this value is computed for $\lambda = 1.44499$. So, it can be deduced that when the uncertainties increase, the critical λ values approach 1.

Table 5.3: Invest Now Alternative Gain-Loss Bounds for $\eta = 0.01$ and $\zeta = 0.01$ and $\mp 10\%$ error in p_n

λ	Upper	Lower
100000000	6.26	-25.54
1000	6.25	-25.46
500	6.23	-25.39
100	6.13	-24.80
50	6.01	-24.09
20	5.65	-22.16
10	5.06	-19.45
5	3.98	-15.36
4	3.46	-13.80
3	2.61	-11.70
2	1.01	-8.21
1.8	0.50	-7.33
1.6	-0.12	-6.44
1.5	-0.48	-6.00
1.4	-0.88	-5.56
1.3	-1.66	-5.11
1.2	-2.75	-3.40
1.18227	-3	-3

Figure 5.2 shows the behavior of the Invest Now Alternative Gain-Loss Bounds with $\mp 10\%$ error in p_n and $\eta = 0.01$ and $\zeta = 0.01$. For large values of λ, upper and lower bounds approach option pricing bounds, and for $\lambda = 1.18227$ upper and lower bounds of the project value coincide in -3.

Figure 5.2: Invest Now Alternative Gain-Loss Bounds for $\eta = 0.01$ and $\zeta = 0.01$ and $\mp 10\%$ error in p_n

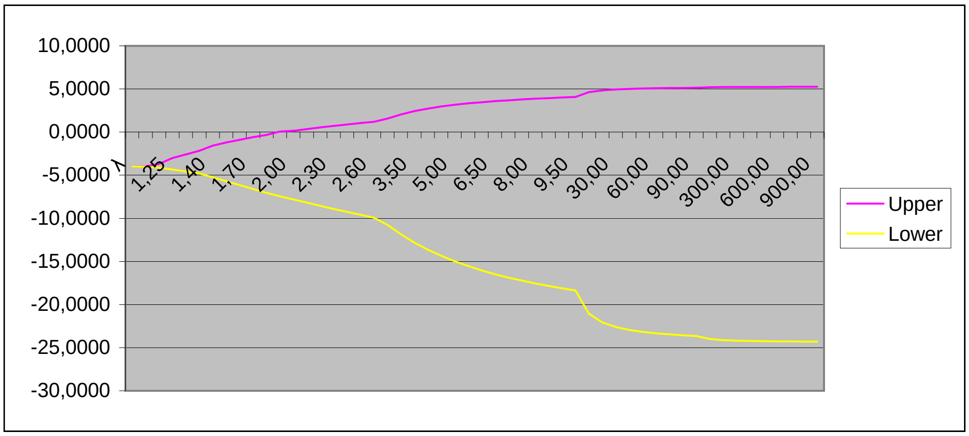


Table 5.4: Defer Alternative Gain-Loss Bounds for $\eta = 0.01$ and $\zeta = 0.01$ and $\mp 10\%$ error in p_n

λ	Upper	Lower
100000000	29.42	20.88
1000	29.41	20.89
500	29.41	20.89
100	29.36	20.94
50	29.30	21.00
20	29.12	21.17
10	28.85	21.43
5	28.35	21.90
4	28.13	22.14
3	27.78	22.52
2	27.07	23.23
1.8	26.85	23.45
1.6	26.57	23.77
1.5	26.41	23.96
1.4	26.22	24.18
1.3	25.97	24.43
1.2	25.69	25.41
1.18227	25.63	25.63

Chapter 6

Conclusion

In this study, we studied on valuing risky projects in incomplete markets. We have developed a new method to value risky projects in incomplete markets. This method is an extension of the option pricing analysis approach. By restricting the gain-loss ratio, we have found tighter bounds than the option pricing analysis method. Furthermore, we were able to find unique values for some cases. In this case of unique project value, we have shown that replicating trading strategies for upper and lower part of the problem differ from one other. We managed to include proportional transaction costs to our problem. We not only considered the cases where the state probabilities are certain, but also the cases where we have uncertain state probabilities. We demonstrated the strength of our new method by computing project value of a capital budgeting example, and compared the results with the option pricing bounds of the projects. When gain-loss preference parameter, λ, goes to infinity, project value bounds computed by this method approximate the option pricing bounds and as λ become smaller bounds collapse to each other. We were able to conclude that in all cases bounds that we computed with this new method are tighter than option pricing bounds, what we label as Consistency Theorem.

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