FINANCIAL VALUATION OF FLEXIBLE SUPPLY CHAIN CONTRACTS

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By
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November, 2008

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

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We consider a single buyer - single supplier multiple period quantity flexibility contract in which the buyer has options to buy in case of a higher than expected demand in addition to the committed purchases at the beginning of each period of the contract. We take the buyer's point of view and find the maximum value of the contract for the buyer by analyzing the financial and real markets simultaneously. We assume both markets evolve as discrete scenario trees. Furthermore, under the assumption that the demand of the item correlates perfectly with the price of the risky security we present a model to find the buyer's maximum acceptable price of the contract. Applying duality, we develop sufficient conditions on some parameters to decrease the value of the contract. Then, an experimental study is presented to illustrate the impacts of all the parameters on the value of the contract and the option. We show that the model can also be extended to the case of partially correlated demand and the risky asset price under the assumption that the markets evolve as binomial trees. Finally, we apply duality and perform numerical analysis for the latter assumption.

Keywords: Flexible supply chain contract, options, arbitrage, martingales, duality, binomial trees.

ÖZET

ESNEK TEDARİK ZİNCİRİ SÖZLEŞMELERİNİN FİNANSAL DEĞERLERİ

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Perakendecinin her periyot başında alınacak olan önceden belirlenmiş siparişlere ek olarak, beklenenden fazla talep olması durumunda kullanılmak üzere opsiyonlara sahip olduğu, çok periyotlu, tek perakendeci - tek tedarikçi esnek miktarlı sözleşmeler ele alınmaktadır. Problem, perakendecinin bakış açısından değerlendirilip, sözleşme için ödenmesi kabul edilmesi gereken maksimum deger, finansal ve reel marketler, her iki marketin de kesikli senaryo ağacı olarak hareket ettiği varsayımı altında ilişkilendirilerek analiz edilmektedir. Bunun yanında, talebin riskli menkul kıymet ile tam korelasyon gösterdiği varsayımı altında, perakendecinin sözleşme için kabul edeceği maksimum fiyat için bir model geliştirilmektedir. Dualite uygulanarak, sözleşmenin fiyatını düşürebilmek amacıyla bazı model parametreleri için yeterli koşullar geliştirilmektedir. Modeldeki parametrelerin, sözleşme ve opsiyon değerleri üzerindeki etkilerini görebilmek açısından deneysel bir çalışma uygulanmaktadır. Modelin, marketlerin ikili kesikli senaryo ağacı olarak hareket ettiği ve talebin riskli menkul kıymet ile kısmen korelasyon gösterdiği varsayımı altındaki durumda da uygulanabilir olduğu gösterilmektedir. Son olarak, bu varsayım için de dualite uygulanıp numerik çalışmalar yapılmaktadır.

Anahtar sözcükler: Esnek tedarik zinciri sözleşmeleri, opsiyonlar, martingale, dualite, ikili kesikli senaryo ağaçları.

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Chapter 1

Introduction

In recent years, economic globalization lead to an increased importance of a firms' ability to adapt to the changing market needs quickly to compete effectively in the market. For example, in the case of a single buyer - single supplier system where the demand is highly unpredictable, flexibility for the buyer to be able to get additional products in response to demand changes is essential. While this flexibility is in benefit for the buyer, it incurs extra costs for the supplier.

These costs are due to some inflexibilities that the supplier faces; additional inventories of the finished goods or raw materials, long production or procurement lead times and accelerating or out-sourcing production. This leads the supplier to provide a flexibility less than what is requested by the buyer.

Consider a single buyer - single supplier system where the buyer receives the finished products from the supplier, stores and sells them to customers in the end market at a fixed market price that is exogenously specified. The buyer and the supplier agree on a multiple quantity flexibility contract in which the buyer has options for the case of higher than expected demand in addition to the committed purchases at the beginning of each period. The buyer is faced with two decisions before the horizon. First of these decisions is to place orders for goods to be delivered at the beginning of each period. The other is to purchase options from the supplier which enable the buyer to order additional units of goods before the beginning of the next period after observing the actual demand of the current period.

The total procurement costs for the buyer associated with this contract consist of the cost of the committed quantity of goods, the cost of the options purchased and the cost of using the option.

When we consider the supplier's point of view, there may be raw materials/components that he procures from his upstream suppliers with long lead times. Therefore, the supplier may have to order raw materials at the beginning, covering both the committed quantity of goods and the number of options purchased as there is no opportunity of reordering during the periods due to the long lead times. This brings the supplier some uncertainty with regard to the number of orders of additional units that the buyer may place. Furthermore, he has to carry the additional units of raw materials which is costly. Therefore, to provide flexibility to the buyer, the supplier makes a commitment to produce additional goods up to a number and offers options at a price to share the associated risks. That is, to get the flexibility to purchase additional units besides the committed quantity of goods, there is a certain price that the buyer has to pay for the contract to the supplier. Intuitively, the supplier wishes to gain as much as he can from the contract. However, the buyer will be willing to buy the contract up to a certain price.

In this thesis, we assume that financial and real markets evolve as discrete scenario trees. We further assume that there is a perfect correlation between the demand and the price of some risky asset traded in a financial market which implies that the scenario trees of the markets coincide. We then find the maximum price that the buyer should accept to pay for the contract by studying the financial and real markets simultaneously. We analyze the problem of the buyer who has zero initial portfolio, and hence, makes short sales of risky assets to buy the contract and then repay his debts by self-financing transactions in the financial market and the cash flows generated in the real market by operations. Therefore, at each node the portfolio value of the buyer is composed of the portfolio value of parent node and the cash flow generated in the real market at that node. We then relax the assumption of perfect correlation of demand of the item with the price of some risk asset and analyze the value of the contract to the buyer in presence of the partially correlated demand and the risky asset.

The organization of the thesis is as follows:

In Chapter 2, we provide a review of the literature that is closely related to the problem under consideration.

In Chapter 3, we review the stochastic process governing the security prices. Furthermore, we introduce financial markets and the basic concepts (arbitrage and martingales) of our analysis. Finally, we review the pricing problems of writers and buyers of contingent claims.

In Chapter 4, the real market is described and the relation between the financial and real markets are analyzed. Also, the assumptions and notations are listed. Then, the model is developed under the assumption that the demand of the item correlates perfectly with the price of the risky asset.

In Chapter 5, the problem discussed in Chapter 4 is analyzed through the duality. Then, by analyzing the dual, the effects of some parameters on the value of the contract are stated through observations.

In Chapter 6, we present an experimental study to illustrate the effects of the parameters on the values of the contract and the cost of the flexibility available to the buyer. This study will enable us to derive managerial insights and interpret the data numerically.

In Chapter 7, the model developed in Chapter 4 is extended to the case of partially correlated demand and the risky asset price under the assumption that both markets evolve as binomial trees. Then, by analyzing the extended model through the dual, the validity of observations made in Chapter 5 is discussed. The chapter ends with the analysis of the effects of the parameters on the values of the contract and the cost of the flexibility available to the buyer by an experimental study.

In Chapter 8, we conclude the thesis by giving an overall summary and listing some possible future research directions.

Chapter 2

Literature Survey

In this chapter, we review the literature that is related to the problem under consideration. We begin with the paper of King [1] who models the hedging of contingent claims in the discrete time, discrete state case as a stochastic program. In his paper, claims are treated as a liability of the writer and a mathematical structure based on duality is developed to analyze them. The conditions under which the buyer buys the claim offered by the writer are discussed and it is observed that differences in liability/endowment structures must be introduced to buy/sell options. The model is extended to incorporate differences in risk aversion and transaction costs. It is shown that arbitrage pricing in incomplete markets does not lead to trade of options. The author also considers another extension of the model in which pre-existing liabilities or endowments are introduced. He observes that pre-existing liability structure or endowments of the market players are the reasons to trade in options.

Delft and Vial [3] propose a practical approach to construct stochastic programming models to solve sequential decision-making problems under uncertainty. In their paper, it is shown that complex problems can be formulated even by non-professional users by means of algebraic modeling languages and solved by commercial solvers. To point out their approach, they provide an example of an option contract in the area of supply chain management. In the example, they consider a general single buyer-single supplier contract with periodical commitment of options as introduced by Barnes-Schuster et al. [4]. They assume stochastic demand. They consider the buyer's point of view and maximize the profit of the buyer. Furthermore, they make the assumption that the decision variables do not have any effect on the underlying stochastic process and the stochastic process is discrete with discrete or a discretized state space. They use event tree representation to formalize the process. In addition, they perform numerical studies to show the reliability of their approach.

Chen and Parlar [8] extend the standard newsvendor problem by introducing a put option into it, so that the risk-averse newsvendor protects himself against lower than expected demand. The objective of their work is to analyze the value of a put option to a risk-averse newsvendor. They discuss the cases where the newsvendor decides on the strike price and/or the strike quantity besides the order quantity to enhance the risk return profile. They assume that the option is priced using historical demand data Hull [5], and information is symmetric. They showed that the optimal order quantity is independent of the use of option. That is, the option parameters do not impact the newsvendor's expected profit, whereas they impact the variance of the profit. Furthermore, it is shown that as long as the utility function of the newsvendor is quadratic and the order quantity that maximizes the expected profit is used, the buyer is indifferent between maximizing the expected utility and minimizing the variance of the profit.

The problem of hedging in the newsvendor setting when perfect or partial correlation of demand of the item with the price of a tradable financial asset is assumed is considered by Gaur and Seshadri [6]. They handle this problem for discretionary purchase items based on a forecasting model combining the personal opinion of the retailer with the price information of the underlying asset. The objective of their work is to derive an optimal hedging strategy that minimizes the variance of the profit and increases the expected utility for a risk-averse decision maker. Unlike previous research, they analyze the effect of hedging on decision making. They show that hedging has an effect on both risk-neutral and risk-averse decision makers in a way that it reduces the variance of the profit and the investment in inventory, whereas it increases the expected utility of a risk-averse decision maker and optimal inventory level for a wide range of utility functions. Furthermore, they present a numerical example to point out the results of their model.

Barnes-Schuster et al. [7] study the role of options in a buyer-supplier system by considering a two-period correlated demand model. They assume that before the beginning of the horizon, the retailer places firm orders to be delivered at the beginning of periods 1 and 2 at the same unit price and purchases options from the supplier that allow him to order additional units in period 2 after observing the actual demand of period 1 by paying an exercise price. The flexibility of the buyer to adjust order quantity of period 2 to the observed demand of period 1 provided by options is highlighted. Furthermore, in analyzing the model various channel structures are considered and performance of them are numerically compared.

The problem of valuing a supply contract that requires the manufacturer to deliver fixed quantities of a finished good according to a deterministic delivery schedule at a predetermined unit price is considered by Kamrad and Ritchken [9]. They formulate the problem using a contingent claims approach since it requires less data and more fully exploits market information. In addition, they assume that the input price processes are correlated Ito processes with general drift components, and are constrained only in their volatility structures. The goal of their paper is to formulate a model valuing a fixed price supply contract characterized by multiple input price uncertainty and significant operating flexibility. Even though the formulation of their model follows arbitrage pricing procedures, it does not yield analytical solutions. Therefore, they establish a multinomial lattice approximation procedure that allows optimal solutions to be obtained. In addition, they present an example that illustrates the valuation procedure and highlights how the value of supply contracts with flexibility can be determined.

Chapter 3

Review of Financial Markets

3.1 Arbitrage and Martingales

In this section, financial markets and the concepts of arbitrage and martingales are introduced. The link between arbitrage and martingales is analyzed.

A financial market is a mechanism that allows people to buy and sell financial securities. (It is also the coming together of buyers and sellers to trade financial securities.)

Throughout the thesis we consider the general probabilistic setting of [1]. We assume that all random quantities are supported on a finite probability space $(\Omega, \mathcal{F}, P)$ whose atoms ω are sequences of real valued vectors (security prices and payments) over the discrete time periods $t = 0, 1, \ldots, T$. In addition, we assume that the market evolves as a discrete scenario tree. In the scenario tree, the partition of probability atoms $\omega \in \Omega$ which are generated by matching path histories up to time t corresponds one-to-one with nodes $n \in N_t$ at level t in the tree. The root node $n = 0$ corresponds to trivial partition $N_0 = \Omega$, and the leaf nodes $n \in N_T$ correspond one-to-one with the probability atoms $\omega \in \Omega$.

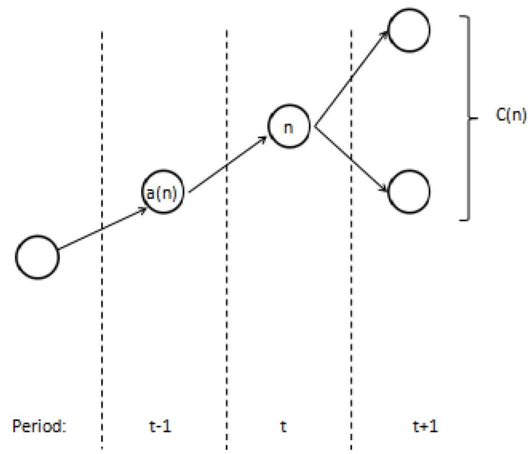

Figure 3.1: Financial Market Scenario Tree

As represented in the figure above in the scenario tree, every node $n \in N_t$ for $t = 1, \ldots, T$ has a unique parent node denoted by $a(n) \in N_{t-1}$, and every node $n \in N_t$, $t = 0, 1, \ldots, T-1$ has a nonempty set of child nodes denoted by $C(n) \subset N_{t+1}$.

The probability distribution P is modeled by assigning positive weights p_n to each leaf node $n \in N_T$. The weights p_n are assigned to each leaf node $n \in N_T$ in such a way that $\sum_{n \in N_T} p_n = 1$. Each intermediate level node in the tree receives a probability mass equal to the combined mass of the paths passing through it.

$$p_n = \sum_{m \in C(n)} p_m \quad \forall n \in N_t, \quad t = T-1, \ldots, 0.$$

The ratios p_m/p_n, $m \in C_n$, are the conditional probabilities that the child node m occurs given that the parent node $n = a(m)$ has occurred.

The function $X : \Omega \to R$ is a real-valued random variable if $\{\omega : X(\omega) \leq r\} \in \mathcal{F}\ \forall r \in R$. Let X be a real-valued random variable. X can be lifted to N_t if it can be assigned a value on each node of N_t that is consistent with its definition on Ω, [1]. This kind of random variable is said to be measurable with respect to the information contained in the nodes of N_t. A stochastic process $\{X_t\}$ is a time indexed collection of random variables such that each X_t is measurable with respect to N_t. The expected value of X_t is uniquely defined by

$$E^P[X_t] := \sum_{n \in N_t} p_n X_n.$$

The conditional expectation of X_{t+1} on N_t

$$E^P[X_{t+1}|N_t] := \sum_{m \in C(n)} \frac{p_m}{p_n} X_m$$

is a random variable taking values over the nodes $n \in N_t$.

The market consists of $J+1$ tradable securities indexed by $j = 0, 1, \ldots, J$ with prices at node n given by the vector $S_n = \left(S_n^0, \ldots, S_n^J\right)$. Suppose one of the securities always has strictly positive values at each node of the scenario tree. Let security 0 be such security. This security which corresponds to the risk-free asset in the classical valuation framework is chosen to be numéraire. Introducing the discount factors $\beta_n = 1/S_n^0$ we define the discounted security prices relative to the numéraire and denote it by $Z_n = \left(Z_n^0, \ldots, Z_n^J\right)$ where $Z_n^J = \beta_n S_n^J$ for $j = 0, 1, \ldots, J$. Note that, $Z_n^0 = 1$ in any state n.

The amount of security j held by the investor in state $n \in N_t$ is denoted by θ_n^j. The value of the portfolio discounted with respect to the numéraire in state n is

$$Z_n \cdot \theta_n := \sum_{j=0}^{J} Z_n^j \theta_n^j.$$

Throughout the thesis, we will use the following definition of arbitrage: An arbitrage is a sequence of portfolio holdings that begins with a zero initial value, makes self-financing portfolio transactions and attains a non-negative value in each future state, while in at least one terminal state it attains a strictly positive value with positive probability.

The condition of self-financing portfolio transactions

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{a(n)} \quad n > 0$$

states that the funds available for investment at state n are restricted to the funds generated by the price changes in the portfolio held at state $a(n)$.

The following optimization problem, called a stochastic program, is used to find an arbitrage.

$$\begin{aligned} \max \quad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\ & \text{s.t.} \\ Z_0 \cdot \theta_0 &= 0 \\ Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= 0, \quad \forall n \in N_t, t \geq 1 \\ Z_n \cdot \theta_n &\geq 0, \quad \forall n \in N_T \end{aligned}$$

A positive optimal value for this stochastic program corresponds to an arbitrage. The solution that yields a positive optimal value can be turned into an arbitrage as shown by Harrison and Pliska [10]. On the other hand if no arbitrage is possible, the price process is called an arbitrage-free market price process.

A martingale is a stochastic process such that the expected value of the next observation, given all the past observations, is equal to the last observation. Martingale properties needed for our study are formalized in the following definition.

Definition 1 *If there exists a probability measure $Q = \{q_n\}_{n \in N_t}$ such that*

$$Z_t = E^Q \left[Z_{t+1} \,|N_t\right] \quad (t \leq T - 1) \tag{3.1}$$

then the vector process $\{Z_t\}$ *is called a vector-valued martingale under* Q*, and* Q *is called a martingale probability measure* (MPM) *for the process.*

We further need the following definition.

Definition 2 *A discrete probability measure* $Q = \{q_n\}_{n \in N_t}$ *is said to be equivalent to a discrete probability measure* $P = \{p_n\}_{n \in N_t}$ *if* $q_n > 0$ *exactly when* $p_n > 0$.

The key link between arbitrage and martingales is proved by King in the following theorem (c.f. Theorem 1 of [1]).

Theorem 1 *The discrete state stochastic vector process* $\{Z_t\}$ *is an arbitrage-free market price process if and only if there is at least one probability measure* Q *equivalent to* P *under which* $\{Z_t\}$ *is a martingale.*

3.2 Financing of Contingent Claims and Positions of the Writer and the Buyer

A contingent claim F is a security that has payouts F_n, $n > 0$ depending on the states n of the market. Currency futures and equity options are examples of traded contingent claims. The minimum initial investment needed to generate payouts F_n through self-financing transactions using a riskless asset and the underlying security with no risk of terminal positions being negative at any state can be captured in a stochastic program. The following stochastic program determines the minimum amount F_0 required to hedge the claim F that produces payouts F_n with no risk.

$$\begin{aligned}
\min \quad & F_0 \\
& \text{s.t.} \\
Z_0 \cdot \theta_0 &= F_0 \\
Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= -\beta_n F_n \quad && \forall n \in N_t, t \geq 1 \\
Z_n \cdot \Theta_n &\geq 0 && \forall n \in N_T
\end{aligned}$$

King proved the following (c.f. Proposition 2 of [1]):

Proposition 1 *Let F_n be a contingent claim on an arbitrage-free market price process $\{Z_t\}$. The claim is attainable if and only if its price F_0 satisfies*

$$\beta_0 F_0 \geq \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right] \tag{3.2}$$

where $\mathcal{M} = \left\{Q : Z_t = E^Q \left[Z_{t+1} \left| N_t \right.\right] (t \leq T-1)\right\}$, and $\max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t\right]$ is the maximum expected value of the discounted payouts over all possible martingale measures.

3.2.1 Position of the Writer

This section analyzes the position of the writer of the contingent claim. The writer of the claim receives F_0 from the buyer of the claim at state $n = 0$ and pays F_n in each state $n > 0$ in the future. In the meantime, the writer invests this money to generate a profit in such a way as to maximize expected value at the end of the horizon while hedging the claim. The problem of the writer can be modeled as the stochastic program

$$
\begin{aligned}
\max \quad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\
\text{s.t.} & \\
Z_0 \cdot \theta_0 &= \beta_0 F_0 \\
Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= -\beta_n F_n \quad \forall n \in N_t, t \geq 1 \\
Z_n \cdot \theta_n &\geq 0 \quad\quad\quad \forall n \in N_T.
\end{aligned}
$$

The necessary and the sufficient condition needed for the writer's problem to have an optimal solution and the condition on the price F_0 charged by the writer are derived in the following theorem. (c.f. Theorem 2 of [1]).

Theorem 2 *The writer's problem has an optimum if and only if*

1. *The collection of equivalent martingale probability measures on the market price process $\{Z_t\}$ is nonempty, and*
2. *The price F_0 charged by the writer to generate payouts F_n satisfies*

$$\beta_0 F_0 \geq \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.3}$$

Furthermore, this price is invariant under the changes of the original probability measure P.

Therefore, the writer's minimum acceptable price to sell the claim is

$$F_0^{writer} = \beta_0^{-1} \max_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.4}$$

3.2.2 Position of the Buyer

This section analyzes the position of the buyer of the contingent claim. The buyer of the claim pays F_0 to the writer at state $n = 0$ and receives payments F_n in each state $n > 0$ in the future. Like the writer, the buyer wishes to maximize expected value at the end of the horizon by trading. The problem of the buyer can be modeled as the following stochastic program

$$\begin{aligned} \max \quad & \sum_{n \in N_T} p_n Z_n \cdot \theta_n \\ & \text{s.t.} \\ Z_0 \cdot \theta_0 &= -\beta_0 F_0 \\ Z_n \cdot \left[\theta_n - \theta_{a(n)}\right] &= \beta_n F_n \qquad \forall n \in N_t, t \geq 1 \\ Z_n \cdot \theta_n &\geq 0 \qquad \forall n \in N_T. \end{aligned}$$

The results derived for the writer's problem are independent of the sign of F. Therefore, the buyer's acceptable price to buy the claim can be computed by reversing the signs in the equation derived in the writer's problem. Hence, the buyer's acceptable price F_0 satisfies

$$\beta_0 F_0 \leq \min_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.5}$$

Therefore, the buyer's maximum acceptable price to buy the claim is

$$F_0^{buyer} = \beta_0^{-1} \min_{Q \in \mathcal{M}} E^Q \left[\sum_{t=1}^{T} \beta_t F_t \right]. \tag{3.6}$$

Chapter 4

Model

In this part of the thesis, a model for the financial valuation of supply chain contract is introduced. We consider a general single buyer-single supplier contract having periodical commitment with options. We consider the case where the demand forecast for the item is perfectly correlated with the price of an underlying security traded in the financial markets.

The general setting of the contract is as follows. The buyer is an intermediary between the market and the supplier. He buys the finished products from the supplier and sells them to customers at the end market at a fixed market price that is exogenously specified. The demand of the customers at the end market is assumed to be uncertain.

The buyer and the supplier sign a multiple period quantity flexibility contract, in which the buyer has options to buy in case of a higher than expected demand in addition to the committed purchases at the beginning of each period of the contract.

In our study, we assume that the demand of the customers for the finished products is uncertain, i.e., demand follows a stochastic process. We further assume that this stochastic process evolves as a discrete scenario tree. We now describe the scenario tree in more detail.

The nodes of the scenario tree represent the state of the discrete state stochastic process at a given period. The arcs correspond to the probabilistic transitions from one node at a given period to another node at the next period. As represented in the figure below there exists exactly one arc leading to a node, while there may be many arcs emanating from a node. As in the financial market scenario tree we denote the nodes obtained by the arcs emanating from node n, $n \in N_t$ for $t = 0, \ldots, T-1$ by $C(n) \subset N_{t+1}$, and the unique node that gives rise to node n, $n \in N_t$ for $t = 1, \ldots, T$ by $a(n) \in N_{t-1}$.

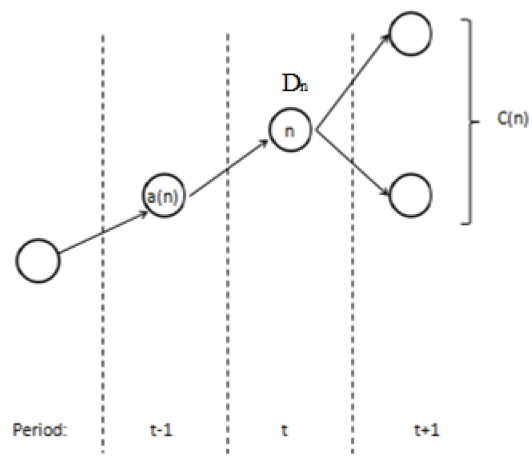

Figure 4.1: Demand Market Scenario Tree

Now, consider a periodic review inventory problem with horizon T. The decisions made by the buyer at the beginning of the horizon are as follows. The buyer orders Q_t units to be delivered in period t for $t = 1, \ldots, T$ at a unit purchase price of p_t. We refer to Q_t as firm orders. In addition, the buyer purchases options from the supplier which give him an opportunity to purchase additional units later by paying an exercise price. We assume that one option gives the buyer a right to purchase one additional unit of product, and this additional unit is delivered at the beginning of the next period that the option is exercised. We further assume that the number of options exercised by the buyer at each node n, $n \in N_t$ for $t = 1, \ldots, T-1$ is bounded above by a constant M. In each state n, $n \in N_t$ for $t = 1, \ldots, T-1$, after observing the actual demand of node n, the buyer decides to exercise options or not. If at state n for $n \in N_t$, $t = 1, \ldots, T-1$ the buyer decides to exercise m_n options at a unit price of e_t where $m_n \leq M$, the additional units are delivered at the beginning of period $t+1$.

In each period t for $t = 1, \ldots, T-1$ excess demand is assumed to be backlogged to the next period at the unit shortage cost s_t. However, at the end of the horizon shortage is not allowed. In addition, in each period t, $t = 1, \ldots, T$, excess inventory is carried to the next period at the unit holding cost of h_t.

One of our most important initial assumptions is that demand forecast for the item is perfectly correlated with the price of a risky security traded in the financial market. This actually implies that the scenario tree of the financial market and the demand market coincide as shown in the figure below.

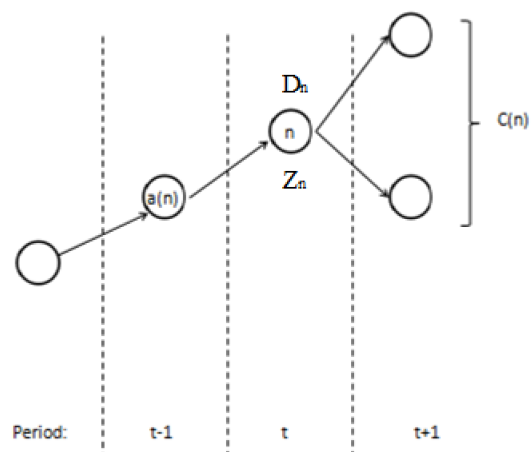

Figure 4.2: Financial and Demand Market Scenario Tree

Before moving on to the mathematical formulation of the model, we now summarize the notations that will be used throughout the thesis.

4.1 Notations

Decision Variables

Q_t : Firm order to be delivered in period t

θ_n : The vector amount of securities held at node n

m_n : Number of options exercised at node n

V : Value of the contract

I_n^+ : Positive inventory at the end of node n

I_n^- : Negative inventory at the end of node n

I_n : Net inventory at the end of node n

V_M : Contract value when the buyer is allowed to exercise at most M options

Parameters

M : Maximum number of options that can be exercised at node n

r_t : Sales price of finished product at the end market in period t

p_t : Purchase price of unit firm order Q_t in period t

h_t : Unit holding cost for finished products in period t

s_t : Unit stock-out cost for finished products in period t

Z_n: The vector of security prices at node n

D_n : Demand at node n

e_t : Unit price for an option exercised in period t

4.2 Assumptions

- The demand forecast for the item is perfectly correlated with the price of an underlying security traded in the financial markets.

- In the financial market, the price process $\{Z_t\}$ is an arbitrage-free market price process. This is equivalent to the existence of a martingale probability measure Q for the price process $\{Z_t\}$.

- At each state n, $n \in N_t$ for $t = 1, \ldots, T-1$, the buyer is allowed to exercise at most M options and the options exercised are delivered at the beginning of period $t+1$.

- In the real market, in period t for $t = 1, \ldots, T-1$ excess demand is backlogged and excess inventory is carried to the next period. However, at the end of the horizon, shortage is not allowed.
- The backorders are met at the present price.
- To avoid the trivial cases it is assumed that the sales price r_t is greater than the purchase price p_t and the stock-out cost s_t is greater than the holding cost h_t in period t for $t = 1, \ldots, T$.

Before moving on to the mathematical formulation of the model, we now explain the main motivation of our model and discuss the financial constraints.

In Section 3.2.2, the position of the buyer of the contingent claim who pays F_0 at state $n = 0$ and receives payments F_n in each future state $n > 0$ is discussed. In the thesis, we extend this analysis and study the financial and real markets simultaneously. We analyze the problem of the buyer who borrows money at the beginning of the horizon by making short sales of stocks to acquire the contract and buy bonds with the rest of the money. The buyer then repays his debts by making self-financing transactions in the financial market and cash flows generated in the real market.

The goal of our study is to find the maximum value that the buyer will accept to pay for the contract. Hence the objective function is formulated in such a way to maximize the value of the contract that the buyer will accept to pay. Since the portfolio of the buyer was *zero* before borrowing money, and the money borrowed at the beginning of the horizon is used to acquire the contract and buy bonds to later trade in the financial market, the portfolio of stocks, bonds and the value of the contract must add up to *zero*.

The portfolio value at each node n, $Z_n \cdot \theta_n$, is composed of the portfolio value of parent node $a(n)$, $Z_n \cdot \theta_{a(n)}$, and the cash flow generated in the real market at node n denoted by F_n. Therefore, the following equation describes the self-financing nature of portfolio transactions:

$$Z_n \cdot \theta_n = Z_n \cdot \theta_{a(n)} + F_n,$$

or,

$$Z_n \cdot \left(\theta_n - \theta_{a(n)}\right) = F_n.$$

Denote $\theta_n - \theta_{a(n)}$ by $\Delta\theta_n$ then we have

$$Z_n \cdot \Delta\theta_n = F_n.$$

With the above specifications, our model that we refer to as $(P1)$ can be formulated as follows.

$\max V$

s.t.

$$Z_0 \cdot \theta_0 + V = 0 \tag{4.1}$$

$$Z_n \cdot \Delta\theta_n = r_1 \left(D_n - I_n^-\right) - \left(p_1 Q_1 + e_1 m_n + h_1 I_n^+ + s_1 I_n^-\right) \qquad \forall n \in N_1 \tag{4.2}$$

$$Z_n \cdot \Delta\theta_n = r_t \left(D_n - I_n^- + I_{a(n)}^-\right) - \left(p_t Q_t + e_t m_n + h_t I_n^+ + s_t I_n^-\right) \qquad \forall n \in N_t, t = 2, \dots, T-1 \tag{4.3}$$

$$Z_n \cdot \Delta\theta_n = r_T \left(D_n + I_{a(n)}^-\right) - \left(p_T Q_T + h_T I_n\right) \qquad \forall n \in N_T \tag{4.4}$$

$$Z_n \cdot \theta_n \geq 0 \qquad \forall n \in N_T \tag{4.5}$$

$$I_n = Q_1 - D_n \qquad \forall n \in N_1 \tag{4.6}$$

$$I_n = I_{a(n)} + Q_t + m_{a(n)} - D_n \qquad \forall n \in N_t, t = 2, \dots, T \tag{4.7}$$

$$I_n = I_n^+ - I_n^- \qquad \forall n \in N_t, t = 1, \dots, T-1 \tag{4.8}$$

$$I_n \geq 0 \qquad \forall n \in N_T \tag{4.9}$$

$$m_n \leq M \qquad \forall n \in N_t, t = 1, \dots, T-1 \tag{4.10}$$

$$Q_t \geq 0 \qquad t = 1, \dots, T \tag{4.11}$$

$$I_n^+ \geq 0 \qquad \forall n \in N_t, t = 1, \dots, T \tag{4.12}$$

$$I_n^- \geq 0 \qquad \forall n \in N_t, t = 1, \dots, T-1 \tag{4.13}$$

Constraint 4.2 implies that F_n for $n \in N_1$ is the revenue in period 1, which is the amount of the product sold at a unit sales price of r_1, minus the expenditure in period 1, which is the firm order at a unit purchase price of p_1, the amount of options exercised to be used in the second period at a unit exercise price of e_1, the positive inventory at a unit cost of h_1 and the backorder amount at a unit cost of s_1.

$$F_n = r_1 \left(D_n - I_n^-\right) - \left(p_1 Q_1 + e_1 m_n + h_1 I_n^+ + s_1 I_n^-\right) \quad \forall n \in N_1$$

Constraint 4.3 states that F_n for $n \in N_t$, $t = 2, \ldots, T-1$ is the revenue in period t, $t = 2, \ldots, T-1$, that is the demand at node n plus the backorder amount at node $a(n)$ minus the shortage at node n at a unit sales price of r_t, minus the expenditure in period t, $t = 2, \ldots, T-1$, that is, the firm order, the number of options exercised in period t to be used in period $t+1$, the positive inventory and the backorder amount at unit prices of p_t, e_t, h_t and s_t.

$$F_n = r_t \left(D_n - I_n^- + I_{a(n)}^-\right) - \left(p_t Q_t + e_t m_n + h_t I_n^+ + s_t I_n^-\right) \quad \forall n \in N_t, t = 2, \ldots, T-1$$

Constraint 4.4 ensures that F_n for $n \in N_T$ is the revenue in the last period, which is the demand at node n plus the backorder amount coming from parent node $a(n)$ at a unit sales price of r_T since shortage is not allowed in the last period , minus the expenditure, which is the firm order at a unit purchase price p_T plus the positive inventory held at node n at a unit cost of h_T since in the last period options cannot be exercised and shortage is not allowed.

$$F_n = r_T \left(D_n + I_{a(n)}^-\right) - \left(p_T Q_T + h_T I_n\right) \quad \forall n \in N_T$$

Constraint 4.5 guarantees that the value of the portfolio in the terminal states are non-negative. This is needed to assure that the buyer has repaid all the debts.

Constraints 4.6, 4.7, 4.8 and 4.9 are the inventory balance constraints. Constraint 4.6 implies that in the first period, the net inventory at each state n,

$n \in N_1$ is equal to the firm order for period 1 minus the demand at that node since there is no backorder to cover or positive inventory carried from the previous period.

Constraint 4.7 states that in period t, $t = 2, \ldots, T$ the net inventory at each state n, $n \in N_t$ is equal to the sum of the net inventory of the parent node $a(n)$, the firm order of period t and the number of options exercised in period $t-1$ to be delivered in period t minus the demand at state n. The reason is that except the first period, the buyer is allowed to have inventory either positive or negative coming from the previous periods. Furthermore, the buyer has an opportunity to use options that bring him as many additional units as the number of options exercised.

Constraint 4.8 implies that in period t, $t = 1, \ldots, T-1$, the net inventory at any node is equal to positive inventory minus the negative inventory at that node. However the net inventory in the last period is simply the positive inventory. This is due to the fact that shortage is not allowed at the end of the horizon. This is guaranteed in constraint 4.9.

Constraint 4.10 shows the flexibility of the buyer. It states that at any node that the buyer is allowed to exercise options which is all the periods except the last period, he is permitted to exercise at most M options.

Chapter 5

Analysis of Optimal Solutions via Duality

This section analyzes the problem discussed in Chapter 4 through an equivalent problem called the dual. We first examine the financial constraints in the dual corresponding to the decision variables θ_n for $n \in N_t$, $t = 0, \ldots, T$. The first step in calculating the dual is to assign dual variables to each constraint in the model. We assign q_n as dual variables for all the nodes of the financial constraints (4.1)-(4.4), and w_n for the non-negativity constraint of the portfolio in the terminal nodes, that is constraint (4.5), $\forall n \in N_T$.

Firstly, the dual constraint corresponding to the decision variable V, that is the value of the contract, is

$$q_0 = 1. \tag{5.1}$$

Next, the dual constraint corresponding to θ_n, $n \in N_t$ for $t = 0, \ldots, T-1$ is the martingale condition

$$q_n Z_n = \sum_{m \in C(n)} q_m Z_m \quad n \in N_t, t = 0, \ldots, T-1. \tag{5.2}$$

The dual constraint corresponding to the decision variables θ_n for $n \in N_T$ is

$$(q_n + w_n)\, Z_n = 0 \quad n \in N_T,$$

and since the first component $Z_n^0 = 1$ for all states n we have

$$q_n + w_n = 0 \quad n \in N_T.$$

In addition, by the non-negativity of the portfolio in the terminal positions

$$w_n \le 0 \quad n \in N_T.$$

Finally, combining the above two constraints, one has the following constraint in the dual.

$$q_n \ge 0 \quad n \in N_T. \tag{5.3}$$

Next, we analyze the constraints in the dual arising from the constraints of the real market. We assign y_n as dual variables for the inventory balance constraints (4.6) and (4.7), $\forall n \in N_t, t = 1, \ldots, T$, k_n for the constraint (4.8), $\forall n \in N_t, t = 1, \ldots, T-1$, and f_n for the flexibility constraint (4.10), $\forall n \in N_t, t = 1, \ldots, T-1$. The dual constraint corresponding to the firm orders Q_t is

$$\sum_{n \in N_t} p_t q_n + y_n \ge 0 \quad t = 1, \ldots, T. \tag{5.4}$$

The constraint in the dual arising from the number of options exercised, i.e. m_n, $n \in N_t, t = 1, \ldots, T-1$ is

$$e_t q_n + f_n + \sum_{m \in C(n)} y_m \ge 0 \quad n \in N_t, t = 1, \ldots, T-1. \tag{5.5}$$

The dual constraint corresponding to the net inventory at state n, $n \in N_t, t = 1, \ldots, T-1$ is

$$-y_n + \sum_{m \in C(n)} y_m + k_n = 0 \quad n \in N_t, t = 1, \ldots, T-1.$$

Reformulating the above constraint, one obtains

$$k_n = y_n - \sum_{m \in C(n)} y_m \quad n \in N_t, t = 1, \ldots, T-1.$$

The constraint in the dual arising from the positive inventory at state n, $n \in N_t, t = 1, \ldots, T-1$ is

$$h_t q_n - k_n \geq 0 \quad n \in N_t, t = 1, \ldots, T-1,$$

and the constraint in the dual arising from the negative inventory at state n, $n \in N_t, t = 1, \ldots, T-1$ is

$$(r_t + s_t)\, q_n - r_{t+1} \sum_{m \in C(n)} q_m + k_n \geq 0 \quad n \in N_t, t = 1, \ldots, T-1.$$

Replacing k_n by $y_n - \sum_{m \in C(n)} y_m$ one has the following constraints in the dual corresponding to, respectively, positive and negative inventory at state n, $n \in N_t, t = 1, \ldots, T-1$

$$h_t q_n - y_n + \sum_{m \in C(n)} y_m \geq 0 \quad n \in N_t, t = 1, \ldots, T-1, \tag{5.6}$$

$$(r_t + s_t)\, q_n - r_{t+1} \sum_{m \in C(n)} q_m + y_n - \sum_{m \in C(n)} y_m \geq 0 \quad n \in N_t, t = 1, \ldots, T-1. \tag{5.7}$$

Finally, the dual constraint corresponding to the net inventory at the terminal positions which is also the positive inventory since shortages are not allowed in the last period is

$$h_T q_n - y_n \geq 0 \quad n \in N_T. \tag{5.8}$$

To obtain the objective function of the dual program we leave the parameters of the model at the right hand side and multiply them by respective dual variables.

Therefore, the dual program that we refer to as $(D1)$ is formulated as follows.

$$\begin{aligned}
\min \quad & \sum_{t=1}^{T} \sum_{n \in N_t} D_n \left(r_t q_n + y_n\right) + M \sum_{t=1}^{T-1} \sum_{n \in N_t} f_n \\
\text{s.t.} & \\
& (5.1) - (5.8) \\
& f_n \geq 0 \qquad n \in N_t, t = 1, \ldots, T-1.
\end{aligned}$$

The basic theorem of linear programming states that problem $(P1)$ has an optimal solution if and only if the dual $(D1)$ does too, and both optimal values are equal. Furthermore, it follows again from the theory of linear programming that problem $(P1)$ has an optimal solution if and only if it is feasible and bounded. Moreover, $(P1)$ is bounded if and only if there exists at least one probability measure Q under which the price process $\{Z_t\}$ is martingale, and there exists y_n and f_n satisfying (5.4) - (5.8).

Now, assume the financial market is arbitrage-free. Then, we can summarize our findings above in the result below.

Theorem 3 *The maximum value that the buyer will accept to pay for the contract is*

$$\min_{Q \in \mathcal{M}} \left\{ \sum_{t=1}^{T} \sum_{n \in N_t} D_n \left(r_t q_n + y_n^*\right) + M \sum_{t=1}^{T-1} \sum_{n \in N_t} f_n^* \right\}$$

where y^* and f^* are the optimal solution of the following linear program that we refer to as $(D2)$.

$$\min \quad \sum_{t=1}^{T} \sum_{n \in N_t} D_n y_n + M \sum_{t=1}^{T-1} \sum_{n \in N_t} f_n$$

s.t.

$$\sum_{n \in N_t} y_n \geq -\sum_{n \in N_t} p_t q_n \qquad t = 1, \ldots, T \tag{5.9}$$

$$f_n + \sum_{m \in C(n)} y_m \geq -e_t q_n \qquad n \in N_t, t = 1, \ldots, T-1 \tag{5.10}$$

$$y_n - \sum_{m \in C(n)} y_m \leq h_t q_n \qquad n \in N_t, t = 1, \ldots, T-1 \tag{5.11}$$

$$y_n - \sum_{m \in C(n)} y_m \geq r_{t+1} \sum_{m \in C(n)} q_m - (r_t + s_t)\, q_n \qquad n \in N_t, t = 1, \ldots, T-1 \tag{5.12}$$

$$y_n \leq h_T q_n \qquad n \in N_T \tag{5.13}$$

$$f_n \geq 0 \qquad n \in N_t, t = 1, \ldots, T-1 \tag{5.14}$$

From the theorem above, we make the following observations:

Observation 1 *It is obvious that if $f_n^* = 0$, an increase in the value of M does not have any effect on the value of the contract since*

$$M \sum_{t=1}^{T-1} \sum_{n \in N_t} f_n^* = 0.$$

This actually means that the buyer is flexible enough to exercise as many options as he wants even before an increase in the value of M, that is, the primal constraints corresponding to f_n for $n \in N_t, t = 1, \ldots, T-1$ are all non-binding.

We will observe the effects of exercise price and purchase price on the value of the contract under the following assumptions.

1. $f_n = 0$ for $n \in N_t$, $t = 1, \ldots, T-1$. That is, the buyer has enough flexibility to exercise as many options as he wants.

2. Before any changes $p_1 = p_2 = \ldots = p_T$. That is, unit cost of placing a firm order is the same at all the periods.

Before moving on to the observations we will reorganize the constraints of $(D2)$.

Constraint 5.10 exists $\forall n \in N_t$, $t = 1, \ldots, T-1$. Writing the constraints 5.10 for a fixed t and summing them $\forall n \in N_t$ we have

$$\sum_{n \in N_t} f_n + \sum_{n \in N_t} \sum_{m \in C(n)} y_m \geq -\sum_{n \in N_t} e_t q_n.$$

Furthermore by the assumption $\sum_{n \in N_t} f_n = 0$. Therefore, one obtains

$$\sum_{n \in N_t} \sum_{m \in C(n)} y_m \geq -\sum_{n \in N_t} e_t q_n. \tag{5.15}$$

Similar to constraint 5.10, constraint 5.11 exists $\forall n \in N_t$, $t = 1, \ldots, T-1$. Hence, writing the constraints 5.11 for a fixed t and summing them $\forall n \in N_t$ we have

$$\sum_{n \in N_t} y_n - \sum_{n \in N_t} \sum_{m \in C(n)} y_m \leq \sum_{n \in N_t} h_t q_n.$$

Reorganizing the above constraint one can obtain

$$\sum_{n \in N_t} \sum_{m \in C(n)} y_m \geq \sum_{n \in N_t} y_n - \sum_{n \in N_t} h_t q_n. \tag{5.16}$$

In addition, constraint 5.9 can be rewritten as

$$0 \geq -\sum_{n\in N_t} y_n - \sum_{n\in N_t} p_t q_n. \tag{5.17}$$

Summing the constraints 5.16 and 5.17 we obtain

$$\sum_{n\in N_t}\sum_{m\in C(n)} y_m \geq -\sum_{n\in N_t}(p_t+h_t)\,q_n. \tag{5.18}$$

Finally, constraints 5.15 and 5.18 imply that

$$\sum_{n\in N_t}\sum_{m\in C(n)} y_m \geq max\left\{-\sum_{n\in N_t}(p_t+h_t)\,q_n, -\sum_{n\in N_t} e_t q_n\right\}. \tag{5.19}$$

Having obtained the above constraint we make the following observations.

Proposition 2 *If $p_t + h_t \leq e_t$, decreasing the purchase price of period t, $t = 1,\ldots,T-1$, while leaving the purchase prices of other periods unchanged increases the value of the contract.*

Proof: From constraint 5.19, if $p_t + h_t \leq e_t$, i.e.,

$$max\left\{-\sum_{n\in N_t}(p_t+h_t)\,q_n, -\sum_{n\in N_t} e_t q_n\right\} = -\sum_{n\in N_t}(p_t+h_t)\,q_n$$

decreasing p_t decreases $\sum_{n\in N_t}(p_t+h_t)\,q_n$ and hence increases $\left(-\sum_{n\in N_t}(p_t+h_t)\,q_n\right)$. This implies that an increase in the lower bound of our minimization program. Hence, y_n^* achieve bigger values. This, moreover, increases the value of the contract.

On the other hand, if $p_t + h_t \geq e_t$ then $-\sum_{n\in N_t} e_t q_n \geq -\sum_{n\in N_t}(p_t+h_t)\,q_n$. Therefore, decreasing p_t decreases $\sum_{n\in N_t}(p_t+h_t)\,q_n$ and hence increases $\left(-\sum_{n\in N_t}(p_t+h_t)\,q_n\right)$, but $max\left\{-\sum_{n\imath N_t}(p_t+h_t)\,q_n, -\sum_{n\in N_t} e_t q_n\right\} = -\sum_{n\in N_t} e_t q_n$ and y_n^*, $n \in N_{t+1}$ does not change unless p_t is decreased to a value such that

$p_t + h_t \leq e_t$. Hence change in the value of the contract depends on the constraint 5.9. If it is binding decreasing p_t decreases $\sum_{n \in N_t} p_t q_n$ and hence increases $-\left(\sum_{n \in N_t} p_t q_n\right)$. This means that an increase in the lower bound of our minimization program. Hence, y_n^* achieve bigger values. This, moreover, increases the value of the contract. $\square$

Proposition 3 *If $e_t \leq p_t + h_t$ decreasing the exercise price of period t, i.e., e_t, while the exercise prices of other periods are unchanged increases the value of the contract.*

Proof: From constraint 5.19 a change in e_t, changes y_n^*, $n \in N_{t+1}$ if the maximum in constraint 5.19 is obtained by $-\left(\sum_{n \in N_t} e_t q_n\right)$. This happens when $e_t \leq p_t + h_t$. Therefore, if $e_t \leq p_t + h_t$ decreasing e_t decreases $\sum_{n \in N_t} e_t q_n$ and hence increases $-\left(\sum_{n \in N_t} e_t q_n\right)$. This means that that an increase in the lower bound of our minimization program. Hence, y_n^* achieve bigger values. This, moreover, increases the value of the contract. $\square$

The impacts of the rest of the parameters are not independent of the other parameters. Therefore, the analysis of these parameters are studied in the next chapter by taking the relative positions of the parameters into consideration.

Chapter 6

Experimental Study

In the previous chapter, the effects of buyer flexibility, exercise price and purchase price on the value of the contract were analyzed analytically. In this chapter the analysis of these parameters are done numerically and are extended to all parameters to give a better understanding of the previous chapter. In addition, the effect of the parameters on the cost of the flexibility provided by the use of options is studied. We refer to this cost as the option value. In order to observe how a change in the value of the parameter affects the value of the option, both the value of the contract with $M = 0$ and $M > 0$ is examined. Then the difference is taken to find the value of the option as the model we study has an operating profit even when the use of option is not allowed. For simplicity we first make all the analysis in a two-period model and consider the binomial tree shown in Figure 6.1. A three-period model is considered when need arises. For all the analysis, we assume that there is only one risky security and one riskless asset.

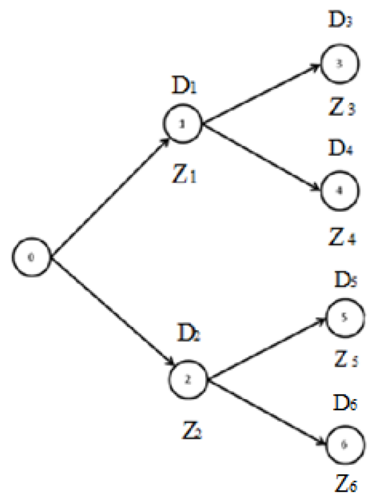

Figure 6.1: Two-Period Binomial Tree

As can be seen from Figure 6.1

$$N_0 = \{0\}, \qquad N_1 = \{1,2\}, \qquad N_2 = \{3,4,5,6\}$$

$$a\,(1) = 0, \qquad a\,(2) = 0$$

$$a\,(3) = 1, \qquad a\,(4) = 1$$

$$a\,(5) = 2, \qquad a\,(6) = 2$$

$$Z_n = (Z_n^0, Z_n^1) \qquad n = 0, \ldots, 6$$

where Z_n^0 denotes the price of the riskless asset, and Z_n^1 denotes the price of the risky security.

Before we move on to the analysis, in order to make our model more clear we write our model for the two-period case explicitly.

$$\max V$$

s.t.

$$Z_0 \cdot \theta_0 + V = 0 \tag{6.1}$$

$$Z_1 \cdot (\theta_1 - \theta_0) = r_1 \left(D_1 - I_1^-\right) - \left(p_1 Q_1 + e_1 m_1 + h_1 I_1^+ + s_1 I_1^-\right) \tag{6.2}$$

$$Z_2 \cdot (\theta_2 - \theta_0) = r_1 \left(D_2 - I_2^-\right) - \left(p_1 Q_1 + e_1 m_2 + h_1 I_2^+ + s_1 I_2^-\right) \tag{6.3}$$

$$Z_3 \cdot (\theta_3 - \theta_1) = r_2 \left(D_3 + I_1^-\right) - (p_2 Q_2 + h_2 I_3) \tag{6.4}$$

$$Z_4 \cdot (\theta_4 - \theta_1) = r_2 \left(D_4 + I_1^-\right) - (p_2 Q_2 + h_2 I_4) \tag{6.5}$$

$$Z_5 \cdot (\theta_5 - \theta_2) = r_2 \left(D_5 + I_2^-\right) - (p_2 Q_2 + h_2 I_5) \tag{6.6}$$

$$Z_6 \cdot (\theta_6 - \theta_2) = r_2 \left(D_6 + I_2^-\right) - (p_2 Q_2 + h_2 I_6) \tag{6.7}$$

$$Z_n \cdot \theta_n \geq 0 \qquad \forall n \in N_2 \tag{6.8}$$

$$I_1 = Q_1 - D_1 \tag{6.9}$$

$$I_2 = Q_1 - D_2 \tag{6.10}$$

$$I_3 = I_1 + Q_2 + m_1 - D_3 \tag{6.11}$$

$$I_4 = I_1 + Q_2 + m_1 - D_4 \tag{6.12}$$

$$I_5 = I_2 + Q_2 + m_2 - D_5 \tag{6.13}$$

$$I_6 = I_2 + Q_2 + m_2 - D_6 \tag{6.14}$$

$$I_1 = I_1^+ - I_1^- \tag{6.15}$$

$$I_2 = I_2^+ - I_2^- \tag{6.16}$$

$$I_n \geq 0 \qquad \forall n \in N_2 \tag{6.17}$$

$$m_1 \leq M \tag{6.18}$$

$$m_2 \leq M \tag{6.19}$$

$$Q_1 \geq 0 \tag{6.20}$$

$$Q_2 \geq 0 \tag{6.21}$$

$$I_1^+ \geq 0 \tag{6.22}$$

$$I_2^+ \geq 0 \tag{6.23}$$

$$I_1^- \geq 0 \tag{6.24}$$

$$I_2^- \geq 0 \tag{6.25}$$

During the analysis, stock is chosen as an underlying security. In order to observe the effect of volatility of stock prices, the stock prices are chosen in such a way that the average price remains constant at all periods:

$$Z_0 = (Z_1 + Z_2)/2 = (Z_3 + Z_4 + Z_5 + Z_6)/4.$$

Similar to stock prices, demand values are chosen in a way that the average values are constant in all the periods so that effect of demand volatility can be observed.

$$(D_1 + D_2)/2 = (D_3 + D_4 + D_5 + D_6)/4.$$

Taking the above specifications into consideration, the values of the parameters and the corresponding decision variables in a base case are represented in Table 6.1. The value of the contract denoted by V_M is 146.7857 and the value of the option is $\mathrm{V}_M - V_0 = 146.7857 - 81.1607 = 65.625$.

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n	θ_n^0	θ_n^1	F_n
	$\mathrm{Q}_1 = 45$	0	10	15		45.902	-62.754	
$\mathrm{r}_t = 20$	$\mathrm{Q}_2 = 20$	1	12	20	45	-13.064	-26.875	10
$\mathrm{p}_t = 12$	$\mathrm{V}_0 = 416.68$	2	12	10	25	19.767	-38.393	-70
$\mathrm{h}_t = 1.5$	$\mathrm{V}_M = 482.30$	3	14.4	25	55			860
$\mathrm{s}_t = 2.5$	$\mathrm{I}_2^+ = 20$	4	14.4	5	30			322.5
$\mathrm{M} = 100$	$\mathrm{I}_4 = 25$	5	14.4	22	40			560
$\mathrm{e} = 10$	$\mathrm{I}_6 = 25$	6	14.4	8	15			22.5

Table 6.1: Parameters and the decision variables in base case

Now, we will investigate how the value of the option and the contract is affected by the changes in the value of the parameters and compare the results of this chapter with the previous chapter. Throughout the analysis, graphs are formed by taking the sample size of the parameters large enough to recognize the general pattern and in the graphs solid lines represent the value of the contract and the dashed lines represent the value of the option.

Case 1 : Effect of Buyer Flexibility

The buyer is allowed to purchase options from the supplier at the beginning of the horizon to later exercise and obtain additional units. The buyer, however, is not fully flexible to adjust order quantities to the observed demands. At each state n, $n \in N_t$, $t = 1, \ldots, T-1$, he is allowed to exercise at most M options. Thus, the flexibility available to the buyer, that is the value of M, plays an important role on determining the value of the contract and the option. The value of the contract and the option corresponding to the different values of M are presented in Table 6.2. The values of other parameters are taken as in the base case.

M	V_0	V_M	$V_M - V_0$	m_1
100	416.68	482.3	65.62	35
35	416.68	482.3	65.62	35
30	416.68	472.93	56.25	30

Table 6.2: Decision variables in case 1

The first row of Table 6.2 states that if the buyer is allowed to exercise at most 100 options while the values of the rest of the parameters are taken as in the base case, the buyer exercises 35 options in node 1. The value of the contract is $V_M = 482.3$ and the value of the option is $V_M - V_0 = 65.62$.

If the value of M decreases to 35 while the values of the other parameters are kept unchanged, the buyer still exercises 35 options. This implies that the buyer is still flexible enough to follow the base case scenario. Therefore, neither the value of the contract nor the value of the option changes.

Next, we observe that decreasing the flexibility of the buyer to 30 options decreases the value of the contract and the option, respectively, to 472.93 and 56.25. This is due to the fact that the buyer is not flexible enough to exercise as many options as he wants. Therefore, the buyer places more firm orders to meet the demand in case of higher than expected demand. Since the buyer is not flexible enough to respond to market changes and places more firm orders which may also result in an increase in positive inventory, both the value of the contract and the option decrease.

To summarize, as shown in the figure below the values of the contract and the option are unchanged as long as the buyer is flexible enough to exercise the amount used in the base case. However, decreasing the value of M to an amount lower than the amount of options exercised in the base case decreases the values of the contract and the option.

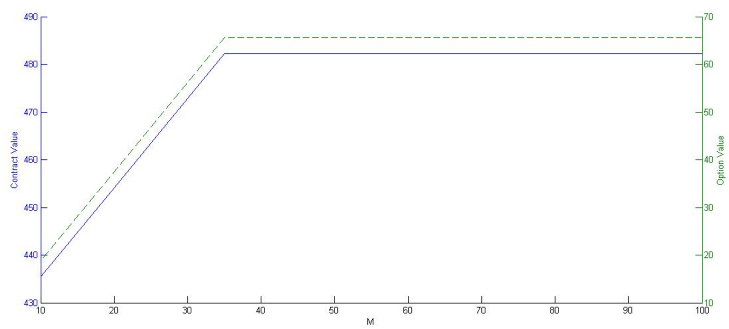

Figure 6.2: Contract and Option Values vs M

Case 2 : Effect of Exercise Price

The buyer has a limited flexibility to purchase options from the supplier at the beginning of the horizon. The buyer then use these options to obtain additional units by paying an exercise price. Thus, the price that the buyer pays to exercise options affects the values of the option and the contract. The values of the contract and the option corresponding to the different values of exercise prices are summarized in Table 6.3. The values of other parameters are taken as in the base case.

e	V_0	V_{100}	$V_{100} - V_0$	m_1	m_2
10	416.68	482.3	65.62	35	
11	416.68	458.97	42.29	35	
9	416.68	522.3	105.62	55	20

Table 6.3: Decision variables in case 2

The first row in Table 6.3 states that if in the contract it is agreed that the buyer pays exercise price 10 to obtain one additional unit of product while the values of the rest of the parameters are as in the base case, the buyer exercises 35 options. In addition, the values of the contract and the option, respectively, are 482.3 and 65.62.

If the exercise price is increased to 11, the value of the contract and the option, respectively, decreases to 458.97 and 42.29. This is due to the fact that it becomes more expensive to adjust order quantities to the observed demands by exercising options. This implies that the contract becomes less valuable for the buyer. Thus, the buyer is willing to pay less for the contract.

Next we observe that a decrease in the exercise price results in an increase in both the value of the contract and the option. As summarized in Table 6.3 when exercise price is decreased to 9, the value of the contract and the option, respectively, increases to 522.3 and 105.62. This is so because it becomes cheaper to adjust order quantities by exercising options. Therefore, the contract becomes more profitable for the buyer and he accepts to pay more for the contract.

To summarize, as shown in the figure below, as long as options are used to meet the demand, the values of the contract and the option decrease with an increase in the exercise price, whereas they both increase with a decrease in the exercise price.

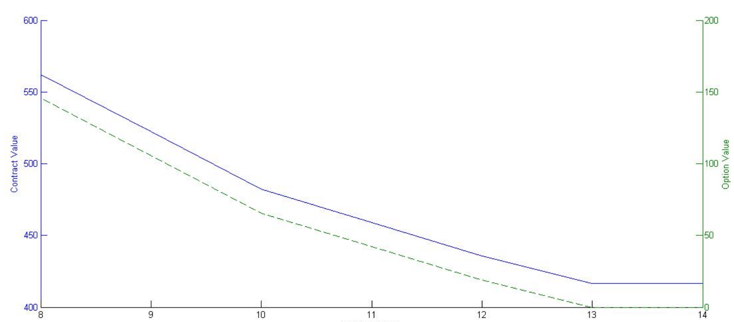

Figure 6.3: Contract and Option Values vs Exercise Price

Case 3 : Effect of Purchase Price

At the beginning of the horizon, the buyer gives firm orders Q_t to be delivered in period t, $t = 1, \ldots, T$ at a unit purchase price of p_t. Hence, the value of the purchase price has an effect on the value of the contract and the option. The values of the contract and the option corresponding to different values of purchase prices are shown in Table 6.4. We assume that other parameters take their base case values.

p_1	p_2	V_0	V_{100}	$V_{100} - V_0$	Q_1	Q_2
12	12	416.68	482.3	65.62	45	20
11	11	492.37	533.69	41.32	45	20
13	13	340.98	444.8	103.82	45	0
11	12	454.18	519.8	65.62	45	20
12	11	454.87	496.19	41.32	45	20
13	12	379.18	444.8	65.62	45	20
12	13	378.48	482.3	103.82	45	0

Table 6.4: Decision variables in case 3

The first row of Table 6.4 states the situation in the base case. It shows that if the buyer gives firm orders for both periods at unit purchase prices of 12, the buyer orders 45 units for period 1 and 20 units for period 2. In addition, the value of the option and the contract, respectively, are 65.62 and 482.3.

We first decrease the purchase price of period 1 to 11 while keeping the purchase price of period 2 constant and observe that the value of the contract is increased to 519.8. The reason behind this is that it becomes cheaper to give firm orders for period 1. On the other hand, the value of the option does not change. This is due to the fact that even though it becomes cheaper to give firm orders for period 1, it is still more expensive than exercising an option unless $p_1 + h_1 < e_1$. The reason is that there is a holding cost that needs to be paid for one unit of firm order delivered in period 1 and carried to next period. If, however, $p_1 + h_1 < e_1$ the value of the option decreases as purchase price of period 1 decreases. This is due to the fact that the buyer then neither gives firm orders for period 2 nor uses options, instead he places more firm orders in period 1 and carries to the next period. The buyer prefers to meet the demand in period 2 by firm orders of period 1 to exercise options in period 1 since it is cheaper to purchase quantities in period 1 and carry to the second period. On the other hand, if the purchase price in period 1 where the buyer cannot have any additional units by the use of options as additional units are delivered at the beginning of the next period is increased to 13 while the purchase price in period 2 is unchanged, it becomes more expensive to place firm orders for period 1. Therefore, the buyer accepts to pay less for the contract. The value of the contract decreases to 444.8. However, the value of the option does not change. This is so because as the cost of placing firm order for period 1 increases while it is unchanged for period 2 the buyer will place fewer firm order for period 1 and more firm order for period 2. The shortage of period 1 will be covered by the additional firm order of period 2. That is, the need for options does not change. Thus, the value of the option remains the same.

To summarize, the value of the contract and the option with changes in the purchase price of period 1 while the second period purchase price is constant is shown in the figure below.

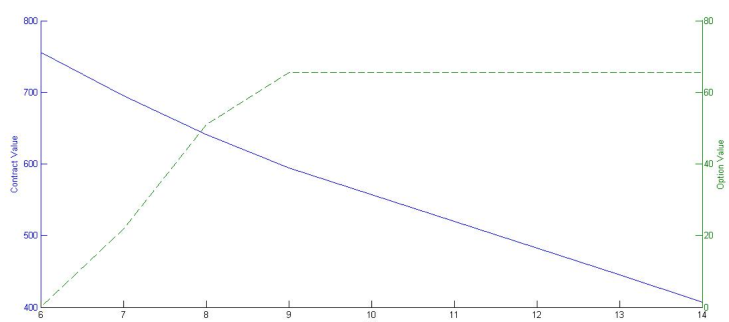

Figure 6.4: Contract and Option Values vs Purchase Price of Period 1

Next, the purchase price in period 2 is decreased to 11 while the purchase price in period 1 is unchanged. Since it becomes cheaper to purchase firm orders for period 2, the buyer accepts to pay more for the contract. The value of the contract increases to 496.19. However, the value of the option decreases. The reason behind this is that, it becomes cheaper to give firm orders for period 2.

This means that the cost of placing firm orders for period 2 becomes relatively less expensive than exercising options. Therefore, the value of the option decreases to 41.32. On the other hand, if the purchase price in period 2 is increased to 13 while the purchase price for period 1 is kept constant, the buyer does not give any firm orders for period 2. The buyer neither wants to pay the additional cost of purchasing firm order in period 2, nor wants to place more firm orders for period 1 and pay the cost of carrying them to the period 2. Instead, the buyer exercises more options to meet the demand in period 2 and this increases the value of the option to 103.82. The value of the contract, however, does not change. This happens, since flexibility of the buyer is enough to meet the demand of period 2 by using options. This allows the buyer not to place any firm order for period 2 at higher unit price.

To summarize, the value of the contract and the option with changes in the purchase price of period 2 while the second period purchase price is constant is shown in the figure below.

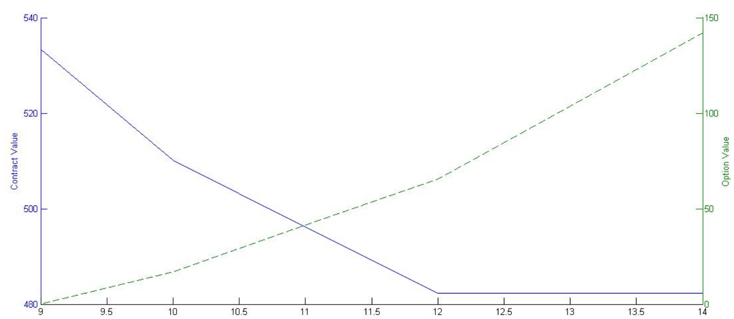

Figure 6.5: Contract and Option Values vs Purchase Price of Period 2

Case 4 : Effect of Demand Volatility

The buyer receives the finished products from the supplier and sells them to customers at the end market. Therefore, the demand of the customers for the finished products plays an important role on the value of the option. To observe its effect we diversify the demand volatility while keeping the mean of the demands same in all the periods.

$$(D_1 + D_2)/2 = (D_3 + D_4 + D_5 + D_6)/4$$

The values of options corresponding to different demand volatilities are summarized in Table 6.5. The values of the other parameters are taken as in the base case.

D_1	D_2	D_3	D_4	D_5	D_6	Q_1	Q_2	m_1	m_2	V_0	V_{100}	$V_{100} - V_0$
45	25	55	30	40	15	45	20	35		416.68	482.3	65.62
40	30	55	30	40	15	40		55	30	421.26	468.13	46.87
50	20	55	30	40	15	50	10	45		412.07	496.47	84.4
45	25	50	35	40	15	45		50	20	409.8	466.05	56.25
45	25	60	25	40	15	45	20	40		423.55	498.55	75

Table 6.5: Decision variables in case 4

The first row of Table 6.5 states the situation in the base case. It shows that if demand follows the pattern represented above, the value of the option is 65.62. Furthermore, the buyer places ,respectively, 45 and 20 units firm orders for period 1 and 2 and exercises 35 options in node 1.

We first observe that decreasing the volatility of demand decreases the value of the option. This is due to the fact that a decrease in demand volatility allows the buyer to make more correct decisions. This implies that the uncertainty in the market has decreased. Therefore, options become less valuable.

Next, we observe that increasing the volatility of demand increases the value of the option. This happens, since more volatile demand leads to more mismatch between the supply of the buyer and the demand. Therefore, options are used to correct mismatches of period 1 and to minimize the possible mismatch of period 2 by adjusting orders to the observed demands. This, therefore, makes the option more valuable.

Case 5 : Volatility of Stock Prices

The buyer borrows money to acquire the contract by making short sales of stocks. Then, he pays the debt by generating cash flows in the real market and making self-financing portfolio transactions in the financial market. Furthermore, we make the assumption that the demand forecast for the item is perfectly correlated with the price of an underlying asset. As stock is the underlying asset which is traded, the stock price has an effect on the value of the option. To analyze its effect we vary the volatility of the stock prices while keeping the mean of the stock prices constant in all the periods:

$$Z_0 = (Z_1 + Z_2)/2 = (Z_3 + Z_4 + Z_5 + Z_6)/4$$

The value of the option corresponding to different values of stock prices are summarized in Table 6.6. We assume that other parameters take their base case values.

Z_0	Z_1	Z_2	Z_3	Z_4	Z_5	Z_6	θ_0	θ_1	θ_2	V_0	V_{100}	$V_{100} - V_0$
15	20	10	25	5	22	8	-62.754	-26.875	-38.393	416.68	482.3	65.62
15	19	11	25	5	22	8	-70.285	-26.875	-38.393	464.91	505.93	41.02
15	20	10	28	2	24	6	-55.970	-20.673	-29.861	389.22	454.84	65.62
15	21	9	28	2	24	6	-50.853	-20.673	-29.861	356.19	438.22	82.03
15	19	11	28	2	24	6	-63.646	-20.673	-29.861	436.84	477.86	41.02

Table 6.6: Decision variables in case 5

The first row of Table 6.6 states the situation in the base case. It shows that if the stock prices follow the pattern presented above, at the beginning of the horizon the buyer makes 62.754 short sales of stocks. The portfolio of stocks in node 1 and node 2, respectively, are -26.875 and -38.393. This implies that the buyer has paid the part of his debt and has 26.875 and 38.393 remaining stocks to pay in node 1 and node 2 respectively. In addition, it is pointed out that the value of the option is 65.62.

Initially, we keep the stock prices in period 2 constant and analyze the effect of stock prices in period 1. We first observe that as volatility of the stock prices in period 1 decreases the value of the option also decreases. This is so because we assume that demand is perfectly correlated with the price of a risky security and in period 1 where the decision to exercise options or not is made the prices of stock are less volatile.

Next, we investigate the case where the stock prices in period 1 are unchanged. We observe that the stock prices in period 2 do not have any impact on the value of the option. This is due to the fact that in period 2, that is, at the terminal position, there is no action taken by the buyer to exercise options or not, and in period 1 where the decision of whether to exercise options or not is made the prices of stocks are unaltered. However, the stock prices in period 2 impacts the portfolio of stock in period 1. This is explained below in more detail.

Observation 2 *The stock prices in period* 2 *do not impact the value of the option, whereas they impact the portfolio of stock in period* 1*.*

This is due to the fact that to repay the debt the buyer needs to cover all his short sales, and the buyer forms his portfolio in period 1 by taking into account the cost of unit stock in the next period, that is, the pattern that stock prices follow in period 2.

Case 6 : Effect of Interest Rate on the Riskless Asset

At the beginning of the horizon, after paying for the contract the buyer buys bonds and uses them later to repay the debt. Thus, the price of the riskless asset affects the value of the option. To observe its effect we change the interest rate on the riskless asset. The value of the option corresponding to different interest rates are summarized in Table 6.7. The values of the other parameters are taken as in the base case.

The first row of Table 6.7 states the situation in the base case. It shows that if the interest rate on the riskless asset is 20%, the value of the option is 65.62. Furthermore, the buyer places 45 and 20 units firm orders for period 1 and 2

interest rate(%)	Q_1	Q_2	m_1	m_2	V_0	V_{100}	$V_{100} - V_0$
20	45	20	35		416.68	482.3	65.62
10	45		55	20	286.66	458.66	172
0	25		75	40	111.2	452.45	341.25
25	45	20	35		469.22	497.22	28

Table 6.7: Decision variables in case 6

respectively. In addition, it is pointed out that in node 1 the buyer exercises 35 options.

We observe from Table 6.7 that as the interest rate on the riskless asset decreases, the buyer places fewer firm orders and uses more options to meet the demand. The reason behind this is that as the interest rate on the riskless asset decreases, in case of higher than expected demand to meet the demand the buyer can make more short sales of bonds and exercise options with the money borrowed. This, therefore, increases the value of the option.

Thus far all the cases are analyzed in a two-period model. However the analysis of the remaining parameters requires a higher dimensional model. Thus, we will extend our model to a three-period model, and consider the binomial tree shown in Figure 6.6.

As can be seen from Figure 6.6

$$N_0 = \{0\}, \qquad N_1 = \{1, 2\}, \qquad N_2 = \{3, 4, 5, 6\}$$

$$N_3 = \{7, 8, 9, 10, 11, 12, 13, 14\}$$

$$a\,(1) = a\,(2) = 0$$

$$a\,(3) = a\,(4) = 1, \quad a\,(5) = a\,(6) = 2$$

$$a\,(7) = a\,(8) = 3, \quad a\,(9) = a\,(10) = 4$$

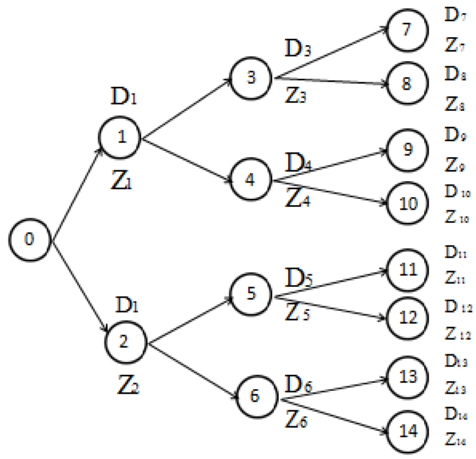

Figure 6.6: Three-Period Binomial Tree

$$a\,(11) = a\,(12) = 5, \quad a\,(13) = a\,(14) = 6$$

$$Z_n = (Z_n^0, Z_n^1) \quad n = 0, \ldots, 14$$

where Z_n^0 denotes the price of the riskless asset, and Z_n^1 denotes the price of the risky security.

Before moving on to the illustrations, as in the two-period case we need to choose a base case to start analysis.

As in the two-period model, in the base case the stock prices are chosen in such a way that the average values of them are the same in all the periods.

$$Z_0 = \sum_{i=1}^{2} Z_i/2 = \sum_{i=3}^{6} Z_i/4 = \sum_{i=7}^{14} Z_i/8$$

Similar to stock prices, demand values are chosen in a way that the average values of them are the same in all the periods.

$$\sum_{i=1}^{2} D_i/2 = \sum_{i=3}^{6} D_i/4 = \sum_{i=7}^{14} D_i/8$$

Taking the above specifications into account, the values of the parameters and the corresponding decision variables in the base case are shown in Table 6.8.

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n	θ_n^0	θ_n^1	F_n
		0	10	15		10.44	-60.223	
$r_t = 20$	$Q_1 = 40$	1	11	20	45	-71.663	-29.702	-292.5
$p_t = 12$	$V_0 = 353.70$	2	11	10	25	-23.363	-23.300	-2.5
$h_t = 1.5$	$V_M = 799.10$	3	12.1	25	55	23.479	-53.750	550
$s_t = 2.5$	$m_1 = 60$	4	12.1	5	30	-4.132	-80.625	562.5
$M = 100$	$m_3 = 65$	5	12.1	22	40	31.528	-76.786	-512.5
$e_t = 10$	$m_4 = 10$	6	12.1	8	15	28.174	-107.500	-50
	$m_5 = 75$	7	13.31	30	65			1300
	$m_6 = 35$	8	13.31	20	40			762.5
		9	13.31	8	35			700
		10	13.31	4	20			377.5
		11	13.31	25	50			1500
		12	13.31	18	25			962.5
		13	13.31	10	35			700
		14	13.31	5	10			162.5

Table 6.8: Parameters and the decision variables in base case

Now, we will investigate the value of the option and the contract with changes in the value of the parameters.

Case 7 : Effect of Sales Price

The buyer sells the finished products at the end market to the customers at unit sales prices of r_1, r_2 and r_3 in period 1, 2 and 3 respectively. Therefore, the values of the sales prices impact the value of the contract and the option. The values of the contract and the option corresponding to the different values of sales prices are summarized in Table 6.8. The values of the other parameters are taken as in the base case.

r_1	r_2	r_3	V_0	V_{100}	$V_{100} - V_0$	Q_1
20	20	20	353.7	799.1	445.4	40
20	19	18	250.45	694.31	443.86	45
18	20	20	286.59	749.8	463.21	25
20	21	22	456.94	910.51	453.57	25
22	20	20	422.79	867.66	444.87	45

Table 6.9: Decision variables in case 7

The first row of Table 6.9 states the situation in the base case. It shows that if sales price is 20 in all periods, the buyer purchases 40 units of firm orders for period 1. In addition, the value of the contract and the option, respectively, are 799.1 and 445.4.

If the revenue of selling unit product in period 2 and 3 is decreased, respectively, to 19 and 18, the value of the contract decreases to 694.31. This is due to the fact that the profit of selling unit product has decreased. This implies that the contract becomes less profitable for the buyer. In addition, the value of the option decreases to 443.86. The reason behind this is that options that are used to meet demands generate fewer cash flows.

When the sales price of period 1 is decreased to 18 while sales prices of other periods are unchanged, the value of the contract decreases to 749.8. The reason behind this is that the buyer will gain less for the unit he sells in period 1. Furthermore, since the sales price in period 1 is less than sales prices of other periods, the buyer will prefer to sell the finished products in periods 2 and 3.

Thus, he will give less firm orders for period 1 and meet the demand of period 1 in later periods. Therefore, the buyer will exercise more options to cover the shortage of period 1. Intuitively, this will result in an increase in the value of the option.

To observe changes in the value of the contract and the option in case of increases in sales prices, we first increase sales prices in period 2 and 3, respectively, to 21 and 22 while keeping the sales price in period 1 constant. Then since the revenue of selling unit product in periods 2 and 3 is increased, the value of the contract is also increased. This is due to the fact that increase in sales price of periods 2 and 3 makes the contract more profitable. This, however, decreases the firm orders for period 1. The buyer prefers to meet the demand of period 1 in later periods so that he can generate more profit. Therefore, he will exercise more options to cover the shortage of period 1. This implies that the need for options is increased. So, the value of the option increases.

Finally, the sales price of period 1 is increased to 22 while the sales prices of other periods are constant. As sale of unit product in period 1 generates more profit, the buyer accepts to pay more for the contract. The value of the contract increases to 867.66. However, the value of the option decreases to 444.87. The reason behind this is that, sale of unit product in period 1 becomes more profitable than other periods and the buyer will prefer to sell whole demand of period 1 in period 1. Therefore he will purchase more firm orders for period 1. This implies that, there will not be any shortage but possibly some positive inventory carried to the next periods. Therefore, the buyer will need fewer options to exercise.

The value of the contract and the option with changes in the sales prices are summarized in the figures below.

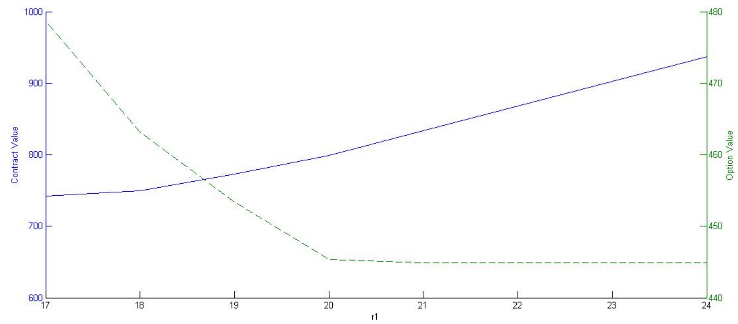

Figure 6.7: Contract and Option Values vs Sales Price of Period 1

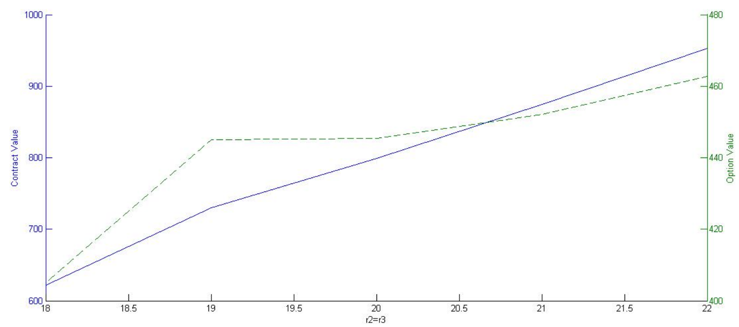

Figure 6.8: Contract and Option Values vs Sales Price of Period $2-3$

Case 8 : Effect of Holding Cost

The buyer purchases firm orders Q_t to be delivered in period t, $t = 1, \ldots, T$. In some periods the buyer may give firm orders more than the demand forecast as purchase price of that period is cheaper or the demand realized may be less than what is expected. In such situations, the buyer carries inventory to the next period and pays the cost of holding it. Therefore, the value of the holding cost has an effect on the value of the contract and the option. The values of the contract and the option corresponding to different values of holding costs are shown in Table 6.10. The values of the other parameters are taken as in the base case.

The first row in Table 6.10 states the situation in the base case. It shows that if the cost of carrying positive inventory for all the periods is 1.5, the buyer purchases 40 units of firm orders for period 1. In addition, the value of the contract and the option, respectively, are 799.1 and 445.4.

h_1	h_2	h_3	V_0	V_{100}	$V_{100} - V_0$	Q_1
1.5	1.5	1.5	353.7	799.1	445.4	40
2	2	2	326.54	793.36	466.82	25
1	1	1	380.85	805.66	424.81	45
1.5	2	2	329.73	795.75	466.02	40
2	1.5	1.5	350.52	796.71	446.19	25
1.5	1	1	377.67	802.48	424.81	40
0.5	1.5	1.5	360.06	804.93	444.87	45

Table 6.10: Decision variables in case 8

We first observe that increasing the holding cost of periods 2 and 3 to 2 while keeping the first period holding cost constant decreases the value of the contract to 795.75. This is so because the loss of the buyer caused by carrying inventory in periods 2 and 3 are increased. This implies that the contract becomes less profitable. The value of the option, however, increases to 466.02. This is due to the fact that the cost of carrying unit inventory becomes more expensive and the buyer prefers to exercise options when needed instead of purchasing more firm orders and paying higher inventory costs by carrying inventories. Whereas, if holding cost of periods 2 and 3 are decreased to 1 while the holding cost of period 1 is constant, the value of the contract increases to 802.48. The reason is that the cost of carrying unit inventory has decreased. Hence, the contract becomes more profitable. However, the value of the option decreases to 424.81. The reason behind this is that as cost of carrying inventory becomes cheaper the buyer will prefer to give more firm orders and carry inventory to exercise more options.

Next, we keep holding costs of periods 2 and 3 constant and observe the impact of first period holding cost. We first increase the holding cost of period 1 to 2. Since the cost of carrying inventory becomes more expensive, the expenses of the buyer in case of having positive inventory are increased. Hence, the contract becomes less profitable for the buyer. Therefore he accepts to pay less for the contract. The value of the contract decreases to 796.71. However, the value of the option increases to 446.19. This is due to the fact that the cost of carrying inventory in period 1 has increased and therefore the buyer purchases fewer firm orders for period 1. This implies that the need of options in later periods to cover shortage of period 1 is increased. Finally, decreasing the first period holding cost to 0.5 increases the value of the contract to 804.93. This is due to the fact that the expense of the buyer caused by carrying unit inventory is decreased. Therefore, the contract becomes more profitable for the buyer. However, the value of the option decreases to 444.87. The reason is that a decrease in the value of the holding cost of period 1 triggers the buyer to purchase more firm orders in case of larger demands. Hence, instead of using more options the buyer prefers to purchase more firm orders and carry them to the later periods.

The value of the contract and the option with changes in the holding costs are summarized in the figures below.

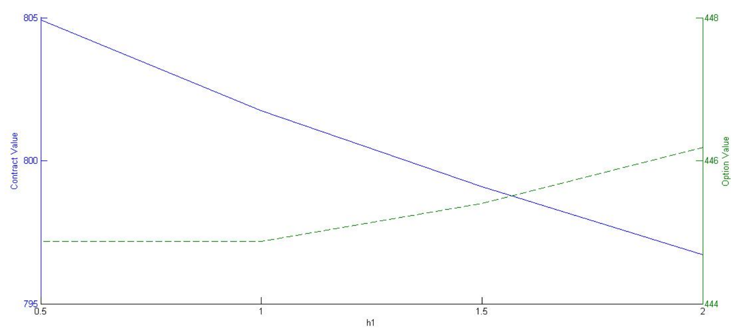

Figure 6.9: Contract and Option Values vs Holding Cost of Period 1

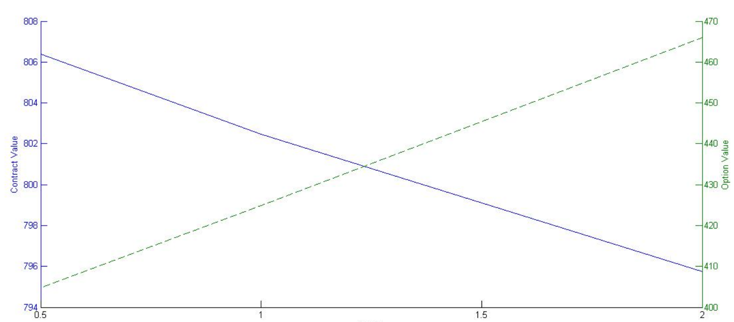

Figure 6.10: Contract and Option Values vs Holding Cost of Period $2-3$

Case 9 : Effect of Stock-out Cost

In the model it is assumed that the demand of the customers are uncertain. In addition, stock-out in all the periods except the last period is allowed. Hence, the value of the contract and the option are independent of the value of the stock-out cost of the last period, whereas they are dependent of the stock-out costs of the other periods. The values of the contract and the option corresponding to different values of stock-out costs are summarized in Table 6.11. The values of the other parameters are taken as in the base case.

s_1	s_2	s_3	V_0	V_{100}	$V_{100} - V_0$	Q_1
2.5	2.5	2.5	353.7	799.1	445.4	40
2	2.5	2.5	353.7	801.16	447.46	25
3	2.5	2.5	353.7	798.56	444.86	45
2.5	2	2.5	353.7	799.87	446.17	40
2.5	5	2.5	353.7	795.47	441.77	45

Table 6.11: Decision variables in case 9

The first row of Table 6.11 states the situation in the base case. It shows that if the holding cost in all the periods are 2.5, the buyer accepts to pay 799.1 for the contract. In addition, the value of the option is 445.4.

We observe that an increase in the stock-out cost of period 1 and/or 2 results in a decrease in the value of the contract. This is due to the fact that having stock-out becomes more expensive. This implies that the contract becomes less profitable in case of shortages in period 1 and/or 2. In addition, since the buyer does not want to have shortages he gives more firm orders for period 1. This implies that in period 2 the buyer needs fewer options since he will not have any backorder to cover. Furthermore, he may have some positive inventory. Therefore, the need of options is decreased. Thus, the value of the option decreases. Whereas, decreasing the stock-out cost in period 1 and/or 2 increases the value of the contract. The reason behind this is that the loss of the buyer in case of having shortages is decreased. This means that the contract becomes more profitable. Therefore, the buyer accepts to pay more for the contract. However, the change in the value of the option is opposite. Since unit cost of not meeting the demand is decreased, the buyer undertakes the risk of having shortages and purchases less firm orders for period 1. This implies that in later periods the buyer needs more options to cover the shortages of period 1. Thus, the value of the option increases.

The value of the contract and the option with changes in stock-out costs are summarized in the figures below.

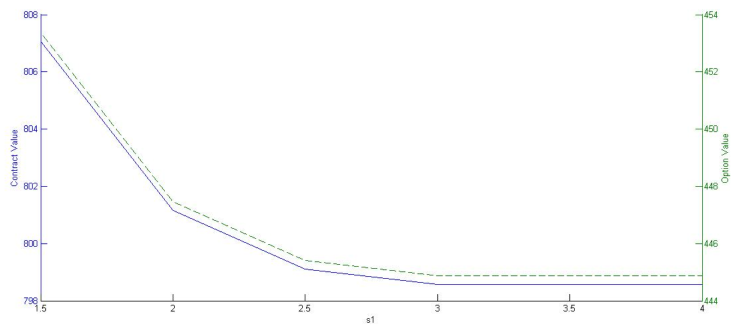

Figure 6.11: Contract and Option Values vs Stock-out Cost of Period 1

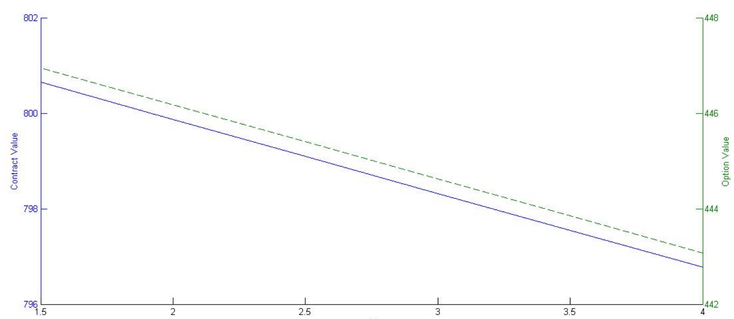

Figure 6.12: Contract and Option Values vs Stock-out Cost of Period 2

Chapter 7

Demand Partially Correlated with the Price of a Risky Security

Until now we made all the analysis under the assumption that the demand forecast for the item is perfectly correlated with the price of an underlying security. In this chapter, however, we relax this assumption and analyze our model in presence of partial correlation of demand with the price of a risky security traded in the financial markets. Furthermore, we assume that demand and risky asset price processes evolve as binomial trees.

7.1 Scenario Tree

In Chapter 4, we showed that financial and demand market trees coincide under the assumption of perfect correlation of demand of the item with the price of a risky security. However, once we relax this assumption we will come up with a different tree structure. We explain the new tree structure below in more details.

As represented in Figure 4.2 perfect correlation assumption implies one-to-one correspondence between demand of node n, D_n, and price of a risky security at node n, Z_n. On the other hand, partial correlation assumption may result in two different scenarios. First one is that two different demands may correspond to a unique risky security price at node n as can be seen from Figure 7.1. This means that there exists two different demand processes corresponding to a unique risky asset price process. In Figure 7.1 D_{1n} and D_{2n}, respectively, represent the values of the first and the second demand processes at node n. The other probability as represented in Figure 7.2 is that two different risky security prices may correspond to a unique demand at node n. This implies that there exists two different risky asset price processes corresponding to a unique demand process. In Figure 7.2 Z_{1n} and Z_{2n}, respectively, represent the values of the first and the second risky asset price processes at node n.

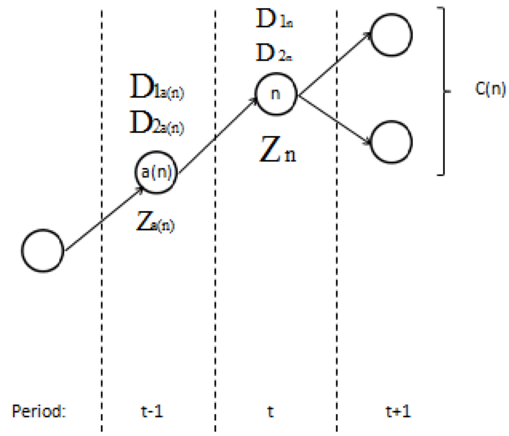

Figure 7.1: Scenario Tree in Presence of Two Demand Processes

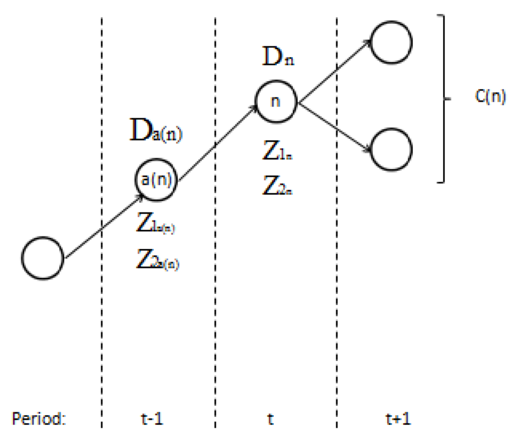

Figure 7.2: Scenario Tree in Presence of Two Risky Asset Price Processes

One can reorganize the above figures and obtain a simpler form to navigate through the arcs. Without loss of generality we reorganize Figure 7.1 by a two-period example. In the example we assume there is one risky security and one riskless asset whose prices are denoted below the nodes. In addition, values of the two different demand processes corresponding to each node are denoted above the nodes, where the uppermost ones represent the values of the first demand process and the other ones represent the values of the second demand process. This implies that, for example in node 1 when the price of the risky security is 20, demand value will be either 40 or 45. Moreover, if from node 1 node 3 is reached, the price of the risky security will be 25, and the demand value will be 50 or 55. In the previous chapters, however, this was not possible. That is, if the price of the risky security changes from 20 to 25 it was certain that the demand value will change from, say, 45 to 55 as there is a perfect correlation.

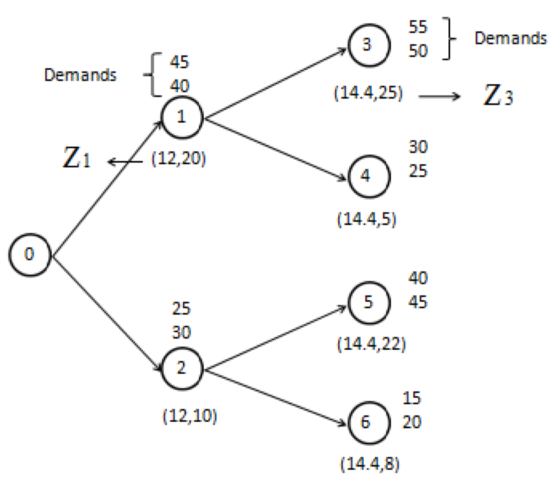

Figure 7.3: Example of a Scenario Tree in Presence of Two Demand Processes

In order to simplify the above figure, from each node we will obtain two different nodes whose prices of the risky securities are the same but the demand values are different. That is, there will be two different numbered nodes with the same risky security prices but different demands. Therefore, each node n for $n \in N_2$ will be reached by a node following the same demand process with itself and by another node following the other demand process. In Figure 7.4

the numbers at the right of each node denotes the demand and the price of the riskless and risky assets. The nodes which are obtained by those following the same demand process are denoted by solid arrows, and those following the other demand process are denoted by dashed arrows. This procedure can also be applied to the case where there is a unique demand value corresponding to a node but different risky security prices.

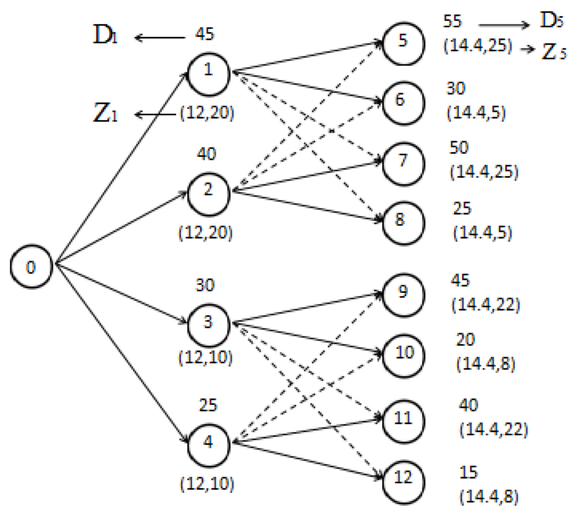

Figure 7.4: Two-Period Financial and Demand Market Scenario Tree

We make the following observations from the figure above.

Observation 3 *Each node n for $n \in N_t$, $t = 2, \dots, T$ has two parent nodes, while there exists a unique parent node for each node $n \in N_1$.*

The parent node of each node n for $n \in N_1$ is unique since at the initial period there exists a unique node which gives rise to all nodes of the first period. On the other hand, there exists two parent nodes for each node n, $n \in N_t$, $t = 2, \dots, T$ since each node can be achieved by a node following the same demand process with itself and by another node following the other demand process.

Observation 4 *Each node n for $n \in N_t$, $t = 0, \dots, T-1$ has four child nodes. Except the initial period, at all the periods each node gives rise to two child nodes* *that follows the same demand process with itself and two child nodes that follows the other demand process.*

We will denote the set of child nodes of node n for $n \in N_t$, $t = 0, \ldots, T-1$ that follows the same demand process with itself by $C_1(n) \subset N_{t+1}$. On the other hand, we will represent the set of child nodes of node n for $n \in N_t$, $t = 1, \ldots, T-1$ that follows the other demand process by $C_2(n) \subset N_{t+1}$.

Before moving on to the mathematical formulation, we explain the notation numerically by using Figure 7.4.

As explained in Observation 4, in the first period each node has a unique parent node.

$$a(1) = a(2) = a(3) = a(4) = \{0\}.$$

However, in other periods each node has two parent nodes; one with the same demand process with itself and one with the other demand process.

$$a(5) = a(6) = a(7) = a(8) = \{1, 2\},$$

$$a(9) = a(10) = a(11) = a(12) = \{3, 4\}.$$

As explained in Observation 5, four nodes emanate from the initial period all following the same demand process with node 0 since at the beginning of the horizon both process are the same. Therefore,

$$C_1(0) = \{1, 2, 3, 4\} \qquad C_2(0) = \emptyset$$

However, in other periods each node gives rise to two nodes with the same demand process with itself and two nodes with the other demand process.

$$C_1(1) = \{5, 6\} \qquad C_2(1) = \{7, 8\}$$

$$C_1(2) = \{7, 8\} \qquad C_2(2) = \{5, 6\}$$

$$C_1(3) = \{9, 10\} \qquad C_2(3) = \{11, 12\}$$

$$C_1(4) = \{11, 12\} \qquad C_2(4) = \{9, 10\}$$

7.2 Mathematical Formulation

In this section, we will reformulate our model according to the new tree structure. However, since the tree structure has changed, the adjustments explained below need to be done.

Each node n in period 1 can be achieved by a single path. However, as can be seen from Figure 7.4 each node n in period 2 can be obtained by two different paths. For example, node 5 can be achieved by the following paths.

$$0 \to 1 \to 5$$

$$0 \to 2 \to 5$$

Furthermore, each node n in period 3 can be obtained by four different paths. The reason behind this is that there exists two different paths for each node $n \in N_2$ and each node in period 3 is achieved by two different nodes in period 2. This can also be observed from Figure 7.5.

For example, node 13 can be achieved by the following paths.

$$0 \to 1 \to 5 \to 13$$

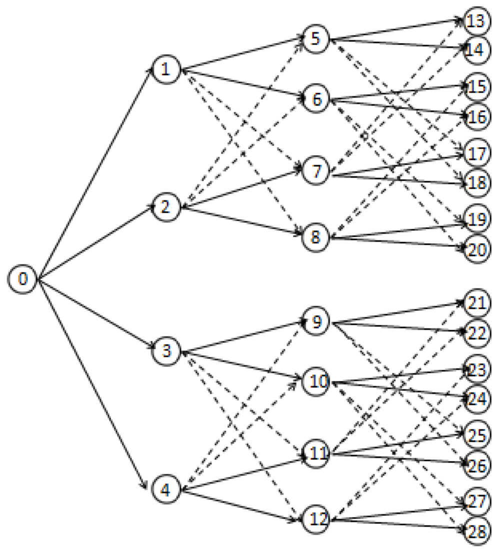

Figure 7.5: Three-Period Financial and Demand Market Scenario Tree

$$0 \to 2 \to 5 \to 13$$

$$0 \to 1 \to 7 \to 13$$

$$0 \to 2 \to 7 \to 13$$

We can generalize the number of paths that the node $n \in N_t$ for $t = 1, \ldots, T$ can be obtained as follows.

Claim 1 *There exists* 2^{t-1} *different paths for each node* $n \in N_t$ *for* $t = 1, \ldots, T$.

Proof: By induction on t:

$t = 1$: Each node $n \in N_1$ can only be achieved by node 0. Therefore, there exists unique path $\forall n \in N_1$ as satisfied by $2^{t-1} = 2^{1-1} = 2^0 = 1$.

$t = 2$: Each node $n \in N_2$ can be achieved by two different nodes in period 1 as illustrated in Figures 7.4 and 7.5. Therefore, there exists two paths $\forall n \in N_2$ as satisfied by $2^{t-1} = 2^{2-1} = 2^1 = 2$.

Now assume there exists $2^{(T-1)-1} = 2^{T-2}$ paths for each node $n \in N_{T-1}$.

Since each node $n \in N_{T-1}$ can be obtained by 2^{T-2} paths, and each node $n \in N_T$ can be obtained by two different nodes in period $T-1$ there exists $2^{T-2} * 2 = 2^{T-1}$ paths $\forall n \in N_T$. $\square$

All the decisions at node n are made according to the path the node n is obtained. Hence, for each path the node n is achieved, there must exist different decision variables for each decision. We handle this situation by giving indices to decision variables of node n. The decision variable at node n will be indexed by n_i if node n is obtained by path i. The paths will be numbered according to the following algorithm:

Algorithm 1 The path numbering algorithm

Step 0: Write all the paths that the node n can be obtained.

Step 1: Starting from the last node of the path, which is node n, look at the previous nodes of each path, pick the one that follows the same process with node n and start indexing with this path. If $\exists$ more than one node following the same process with node n go to step 2.

Step 2: Starting from the last nodes of the paths, look at the previous nodes of the paths until there is no tie. Then start indexing with the paths including the nodes that follow the same process with the node that the last tie occurs.

Step 3: Starting with the number that has not been given as an index yet, apply step 2 to the remaining paths.

In order to make it more clear, I will show how the paths are numbered so that the decision variables corresponding to that node are indexed by examples based on the Figures 7.4 and 7.5.

Without loss of generality, I will consider the paths ending with node 5 and node 7 in Figure 7.4.

$$0 \rightarrow 1 \rightarrow 5 \quad (1)$$

$$0 \rightarrow 2 \rightarrow 5 \quad (2)$$

$$0 \to 2 \to 7 \qquad (1)$$

$$0 \to 1 \to 7 \qquad (2)$$

As can be seen above, there are two paths ending with node 5. Therefore, starting from the last node of the paths, which is node 5, we will look at the previous nodes of each path. We see that node 1 and node 2 are the only nodes giving rise to node 5. Hence, we will number the path including node 1 as the first path and the other path as the second path, since nodes 1 and 5 follow the same process. Furthermore, the decision variables at node 5 will be indexed by 5_i if node 5 is achieved by path i where $i = 1, 2$. The numbering of the paths ending with node 7 and the indexing of the decision variables at node 7 are done similarly.

Now, without loss of generality, I will consider the paths ending with node 13 in Figure 7.5.

$$0 \to 1 \to 5 \to 13 \qquad (1)$$

$$0 \to 2 \to 5 \to 13 \qquad (2)$$

$$0 \to 2 \to 7 \to 13 \qquad (3)$$

$$0 \to 1 \to 7 \to 13 \qquad (4)$$

As can be seen above there are four paths ending with node 13. Therefore, starting from the last node of the paths, which is node 13, we will look at the previous nodes of each path. We see that node 5 and node 7 are the nodes giving rise to node 13. Hence, we will start indexing with the paths including node 5 as nodes 5 and 13 follow the same process. Since there are two paths including node 5, we will look at the nodes preceding the node 5 in the paths. We observe that node 5 is obtained by nodes 1 and 2 in the paths. Therefore, we will index the path including node 1 as the first path and the path including node 2 as the second path. Next, we will apply the same algorithm for the paths including node 7. Similar to node 5, node 7 is obtained by nodes 1 and 2 in the paths. Therefore, starting from the smallest number that has not been given as index yet, we will index the paths including node 2 first and the path including node 1 next. Furthermore, the decision variables at node 13 will be indexed by 13_i if node 13 is achieved by path i where $i = 1, 2, 3, 4$.

Note that we number the path $0 \to 2 \to 7$ as path 1 and index the decision variables at node 7 as 7_1. Moreover, we number the path $0 \to 2 \to 7 \to 13$ as path 3 and the decision variables at node 13 as 13_3. We will index the decision variables at the parent node of node 13 in path 3 as 7_3. However, in two period when the same path is followed, the decision variables at node 7 are indexed by 7_1. The reason behind this is that there are $2^{3-1} = 2^2 = 4$ paths in period 3 and $2^{2-1} = 2^1 = 2$ paths in period 2. Hence, whenever node n, $n \in N_t$ is achieved by path i where $i > 2^{(t-1)-1} = 2^{t-2}$ the indices of the decision variables of the parent node of node n must be equated to those of $t-1$ period case. For example, $\theta_{a(13)_3} = \theta_{7_3} = \theta_{7_1}$. Therefore, since there are 2^{t-2} paths in period $t-1$, assuming parent node of node n, $a(n)$, for $n \in N_t$, is node α where $\alpha \in N_{t-1}$, the relationship between the decision variables of the parent node of node n are given below.

$$\theta_{a(n)_i} = \theta_{\alpha_{i-2^{t-2}}} \quad m_{a(n)_i} = m_{\alpha_{i-2^{t-2}}} \quad I_{a(n)_i} = I_{\alpha_{i-2^{t-2}}}$$

$$I^-_{a(n)_i} = I^-_{\alpha\, i-2^{t-2}} \quad I^+_{a(n)_i} = I^+_{\alpha_{i-2^{t-2}}}$$

Finally, we will denote the changes in the portfolio, i.e., $\theta_{n_i} - \theta_{a(n)_i}$ by $\Delta\theta_{n_i}$.

With the above adjustments our model that we refer to as $(P2)$ can be formulated as follows.

$\max V$

s.t.

$$Z_0 \cdot \theta_0 + V = 0 \tag{7.1}$$

$$Z_n \cdot \Delta\theta_{n_i} = r_1 \left(D_n - I^-_{n_i}\right) - \left(p_1 Q_1 + e_1 m_{n_i} + h_1 I^+_{n_i} + s_1 I^-_{n_i}\right) \quad \forall n \in N_1, i = 1 \tag{7.2}$$

$$Z_n \cdot \Delta\theta_{n_i} = r_t \left(D_n - I^-_{n_i} + I^-_{a(n)_i}\right) - \left(p_t Q_t + e_t m_{n_i} + h_t I^+_{n_i} + s_t I^-_{n_i}\right) \quad \forall n \in N_t, i = 1, \ldots, 2^{t-1},\ t = 2, \ldots, T-1 \tag{7.3}$$

$$Z_n \cdot \Delta\theta_{n_i} = r_T \left(D_n + I^-_{a(n)_i}\right) - \left(p_T Q_T + h_T I_{n_i}\right) \quad \forall n \in N_T, i = 1, \ldots, 2^{T-1} \tag{7.4}$$

$$Z_n \cdot \theta_{n_i} \geq 0 \quad \forall n \in N_T, i = 1, \ldots, 2^{T-1} \tag{7.5}$$

$$I_{n_i} = Q_1 - D_n \quad \forall n \in N_1, i = 1 \tag{7.6}$$

$$I_{n_i} = I_{a(n)_i} + Q_t + m_{a(n)_i} - D_n \quad \forall n \in N_t, i = 1, \ldots, 2^{t-1},\ t = 2, \ldots, T \tag{7.7}$$

$$I_{n_i} = I^+_{n_i} - I^-_{n_i} \quad \forall n \in N_t, i = 1, \ldots, 2^{t-1},\ t = 1, \ldots, T-1 \tag{7.8}$$

$$I_{n_i} \geq 0 \quad \forall n \in N_T, i = 1, \ldots, 2^{t-1} \tag{7.9}$$

$$m_{n_i} \leq M \quad \forall n \in N_t, i = 1, \ldots, 2^{t-1},\ t = 1, \ldots, T-1 \tag{7.10}$$

$$Q_t \geq 0 \quad t = 1, \ldots, T \tag{7.11}$$

$$I^+_{n_i} \geq 0 \quad \forall n \in N_t, i = 1, \ldots, 2^{t-1},\ t = 1, \ldots, T-1 \tag{7.12}$$

$$I^{-}_{n_i} \geq 0 \qquad \forall n \in N_t, i = 1, \ldots, 2^{t-1}$$
$$t = 1, \ldots, T-1 \tag{7.13}$$

The difference of the above model from the former (given in Chapter 4) one is that, we do not have unique paths for nodes. Therefore, we introduce indices to the decision variables of each node to denote by which path it is achieved. However, the corresponding constraints are the same as in Chapter 4, but this time repeated for each path.

7.3 Duality

This section considers the problem discussed in section 7.1 and 7.2 through its dual. As it is not possible to write the constraints of the dual in a compact form for higher dimensional periods, this section will only be considered for two period. Hence, before moving on to the duality analysis I will rewrite the primal problem for two period and refer to it as $(P3)$.

$$\max V$$

s.t.

$$Z_0 \cdot \theta_0 + V = 0 \tag{7.14}$$

$$Z_n \cdot \Delta\theta_{n_i} = r_1 \left(D_n - I^-_{n_i}\right) - \left(p_1 Q_1 + e_1 m_{n_i} + h_1 I^+_{n_i} + s_1 I^-_{n_i}\right) \quad \forall n \in N_1, i = 1 \tag{7.15}$$

$$Z_n \cdot \Delta\theta_{n_i} = r_2 \left(D_n + I^-_{a(n)_i}\right) - \left(p_2 Q_2 + h_2 I_{n_i}\right) \quad \forall n \in N_2, i = 1, 2 \tag{7.16}$$

$$Z_n \cdot \theta_{n_i} \geq 0 \quad \forall n \in N_2, i = 1, 2 \tag{7.17}$$

$$I_{n_i} = Q_1 - D_n \quad \forall n \in N_1, i = 1 \tag{7.18}$$

$$I_{n_i} = I_{a(n)_i} + Q_2 + m_{a(n)_i} - D_n \quad \forall n \in N_2, i = 1, 2 \tag{7.19}$$

$$I_{n_i} = I^+_{n_i} - I^-_{n_i} \quad \forall n \in N_1, i = 1 \tag{7.20}$$

$$I_{n_i} \geq 0 \quad \forall n \in N_2, i = 1, 2 \tag{7.21}$$

$$m_{n_i} \leq M \quad \forall n \in N_1, i = 1 \tag{7.22}$$

$$Q_t \geq 0 \quad t = 1, 2 \tag{7.23}$$

$$I^+_{n_i} \geq 0 \quad \forall n \in N_1, i = 1 \tag{7.24}$$

$$I^-_{n_i} \geq 0 \quad \forall n \in N_1, i = 1 \tag{7.25}$$

As in Chapter 5, we begin forming the dual problem to $(P3)$ by attaching dual variables q_n for all the nodes of the financial constraints (7.14) and (7.15), variables q_{n_i} for all the nodes of the constraints (7.16), and w_{n_i} with the set of constraints (7.17).

Then we obtain the following financial constraints in the dual corresponding to decision variables V, θ_0, θ_{n_i} for $n \in N_1$, $i = 1$ and $n \in N_2$, $i = 1, 2$ respectively.

$$q_0 = 1 \tag{7.26}$$

$$q_0 Z_0 = \sum_{m \in C_1(0)} q_m Z_m \tag{7.27}$$

$$q_n Z_n = \sum_{i=1}^{2} \sum_{m \in C_i(n)} q_{m_i} Z_m \quad n \in N_1 \tag{7.28}$$

$$q_{n_i} \geq 0 \quad n \in N_2, i = 1, 2 \tag{7.29}$$

Next, we analyze the constraints in the dual arising from the constraints of the real market. We assign dual variables y_n and y_{n_i} for all the nodes of the inventory balance constraints (7.18) and (7.19) respectively. In addition, variables k_n and f_n are assigned with the set of constraints (7.20) and (7.22) respectively. Then the following dual constraints are obtained.

$$\sum_{n \in N_1} p_1 q_n + y_n \geq 0 \tag{7.30}$$

$$\sum_{i=1}^{2} \sum_{n \in N_2} p_2 q_{n_i} + y_{n_i} \geq 0 \tag{7.31}$$

$$e_1 q_n + f_n + \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0 \tag{7.32}$$

$$h_1 q_n - y_n + \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0 \quad n \in N_1 \tag{7.33}$$

$$\left(r_1 + s_1\right) q_n - r_2 \left(\sum_{i=1}^{2} \sum_{m \in C_i(n)} q_{m_i} \right) + y_n - \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0 \quad n \in N_1 \tag{7.34}$$

$$h_2 q_{n_i} - y_{n_i} \geq 0 \quad n \in N_2, i = 1, 2 \tag{7.35}$$

Finally, the objective function of the dual problem is obtained by multiplication of the terms that contains only the parameters of the problem $(P3)$ with the respective dual variables.

Therefore, the dual program that we refer to as $(D2)$ is formulated as follows.

$$\begin{aligned} \min \quad & \sum_{n \in N_1} D_n \left(r_1 q_n + y_n\right) + \sum_{n \in N_2} \sum_{i=1}^{2} D_n \left(r_2 q_{n_i} + y_{n_i}\right) + M \sum_{n \in N_1} f_n \\ & \text{s.t.} \\ & (7.26) - (7.35) \\ & f_n \geq 0 \quad n \in N_1 \end{aligned}$$

Again by the basic theorem of linear programming an optimal solution for $(P3)$ exists if and only if it exists for the dual $(D2)$ too, and that their optimal values are equal. Furthermore, an optimal solution for $(P3)$ exists if and only if it is feasible and bounded. Moreover, $(P3)$ is bounded if and only if there exists at least one probability measure Q under which the price process $\{Z_t\}$ is martingale, and there exists y_n and f_n satisfying (7.30) - (7.35).

Now, assume that the financial market is arbitrage-free.

Theorem 4 *The maximum value that the buyer will accept to pay for the contract is*

$$\min_{Q\in\mathcal{M}}\left\{\sum_{n\in N_1} D_n\left(r_1 q_n + y_n^*\right) + \sum_{n\in N_2}\sum_{i=1}^{2} D_n\left(r_2 q_{n_i} + y_{n_i}^*\right) + M\sum_{n\in N_1} f_n^*\right\}$$

where y^* and f^* are the optimal solution of the linear program

$$\min \quad \sum_{n\in N_1} D_n y_n + \sum_{n\in N_2}\sum_{i=1}^{2} D_n y_{n_i} + M\sum_{n\in N_1} f_n$$

s.t.

$$\sum_{n\in N_1} y_n \geq -\sum_{n\in N_1} p_1 q_n \tag{7.36}$$

$$\sum_{i=1}^{2}\sum_{n\in N_2} y_{n_i} \geq -\sum_{i=1}^{2}\sum_{n\in N_2} p_2 q_{n_i} \tag{7.37}$$

$$f_n + \sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq -e_1 q_n \qquad n\in N_1 \tag{7.38}$$

$$y_n - \sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \leq h_1 q_n \qquad n\in N_1 \tag{7.39}$$

$$y_n - \sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq r_2\left(\sum_{i=1}^{2}\sum_{m\in C_i(n)} q_{m_i}\right) - \left(r_1 + s_1\right) q_n \qquad n\in N_1 \tag{7.40}$$

$$y_{n_i} \leq h_2 q_{n_i} \qquad n\in N_2 \tag{7.41}$$

$$f_n \geq 0 \qquad n\in N_1 \tag{7.42}$$

By taking the value of $T = 2$ in Observation 1 of Chapter 5, we can make the same observation from the above theorem. That is, if $f_n^* = 0$, an increase in the value of M does not have any effect on the value of the contract. Furthermore, by reorganizing the constraints of the above program under the assumptions of Chapter 5 , we obtain an equation similar to one obtained in Chapter 5.

$$\sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq max\left\{-\sum_{n\in N_1}(p_1+h_1)\,q_n, -\sum_{n\in N_1} e_1 q_n\right\}. \tag{7.43}$$

Hence, by taking the value of $t = 1$ in Observation 2 and 3 of Chapter 5, we can also make the same observations from the above theorem. That is, if $p_1 + h_1 \leq e_1$ decreasing the purchase price of period 1, while the purchase price of period 2 is unchanged increases the value of the contract and if $e_1 \leq p_1 + h_1$ decreasing the exercise price of period 1 increases the value of the contract.

Finally, we make the following observation about the contract value in presence of the partial correlation of the demand of the item with the risky security price.

Proposition 4 *The value of the contract in the case of partially correlated demands and risky security prices is less than or equal to that of perfectly correlated demands and risky security prices.*

Proof: By taking the value of the partially correlated demands and risky security prices suitably, it is possible to make them correlate perfectly. This implies that all the feasible solutions of $(D1)$ is also satisfied by $(D2)$. This moreover implies that optimal solution of $(D1)$ is a feasible solution for $(D2)$. Hence, the optimal value of $(D2)$ is less than or equal to that of $(D1)$. $\square$

The effects of the other parameters on the value of the contract and the option cannot be derived from the theorem above since the relative positions of the parameters become important. Hence, we will illustrate how the other parameters affect the value of the contract and the option in the next section by taking the relative positions of the parameters into account.

7.4 Experimental Study

In this section, in order to observe how relaxing the perfect correlation assumption affects the role of the parameters on the value of the contract and the option, we will perform numerical studies. As in Chapter 6, we will begin the analysis with a two-period model and consider the binomial tree shown in Figure 7.6. A higher dimensional model will be considered when need arises. For all the analysis it is assumed that there is one risky security and one riskless asset.

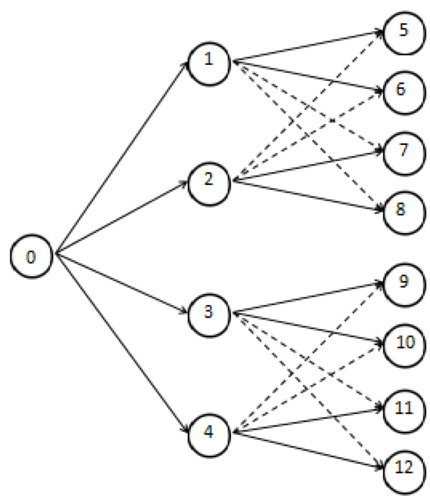

Figure 7.6: Two-Period Financial and Demand Market Scenario Tree

As can be seen from Figure 7.6

$$N_0 = \{0\}, \qquad N_1 = \{1, 2, 3, 4\}$$

$$N_2 = \{5, 6, 7, 8, 9, 10, 11, 12\}$$

$$a(1) = a(2) = a(3) = a(4) = \{0\}$$

$$a(5) = a(6) = a(7) = a(8) = \{1, 2\}$$

$$a\,(9) = a\,(10) = a\,(11) = a\,(12) = \{3, 4\}$$

$$Z_n = (Z_n^0, Z_n^1) \qquad n = 0, \ldots, 14$$

where Z_n^0 denotes the price of the riskless asset, and Z_n^1 denotes the price of the risky security.

As mentioned at the beginning of the chapter, there are two possible scenarios in presence of the partial correlation assumption. We will call the one that two different demands may correspond to a unique risky security price at node n as scenario 1 and the other scenario that two different risky security prices may correspond to a unique demand at node n as scenario 2. For each scenario, we need to choose a base case to perform the experiments.

In both scenarios, the stock prices and demand values will be chosen in such a way that the average price and average demand value remain constant at all periods in order to observe the effect of volatility of stock price and demand.

$$Z_0 = \sum_{i=1}^{4} Z_i/4 = \sum_{i=5}^{12} Z_i/8$$

$$\sum_{i=1}^{4} D_i/4 = \sum_{i=5}^{12} D_i/8$$

With the above specifications, the values of the parameters and the corresponding decision variables in the base case in scenario 1 and 2 are represented respectively in Table 7.1 and Table 7.2.

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n	θ_n^0	θ_n^1
$r_t = 20$	$Q_1 = 40$	0	10	15		41.561	-53.671
$p_t = 12$	$Q_2 = 30$	1	12	20	45	-4.667	-25.559
$h_t = 1.5$	$V_0 = 295.7$	2	12	20	40	2.734	-26.875
$s_t = 2.5$	$V_M = 389.45$	3	14.4	10	30	39.163	-45.293
M = 100		4	14.4	10	25	28.621	-38.393
e = 10		5	14.4	25	55		5.353
		6	14.4	5	30		21.5
		7	14.4	25	50		
		8	14.4	5	25		
		9	14.4	22	45		
		10	14.4	8	20		
		11	14.4	22	40		
		12	14.4	8	15		

Table 7.1: Parameters and decision variables in base case in scenario 1

Now, we analyze the value of the option and the contract by varying the value of the parameters and list the main results without repeating all the details that are similar to the previous chapter.

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n	θ_n^0	θ_n^1
$r_t = 20$	$Q_1 = 45$	0	10	15		37.128	-50.726
$p_t = 12$	$Q_2 = 0$	1	12	20	45	-32.741	-18.205
$h_t = 1.5$	$V_0 = 261.92$	2	12	23	45	-30.948	-23.370
$s_t = 2.5$	$V_M = 388.12$	3	14.4	7	25	6.985	-37.524
$M = 100$		4	14.4	10	25	0.434	-33.594
$e = 10$		5	14.4	25	55		6.936
		6	14.4	5	30		
		7	14.4	28	55		
		8	14.4	2	30		
		9	14.4	24	40		
		10	14.4	6	15		
		11	14.4	22	40		
		12	14.4	8	15		

Table 7.2: Parameters and decision variables in base case in scenario 2

Case 1 : Effect of Buyer Flexibility

To study the effect of flexibility available to a buyer we vary the value of M. We observe that in both scenarios the values of the contract and the option are unaffected as long as the buyer is flexible enough to exercise the amount used in the base case. On the other hand, it is observed that decreasing the value of M to an amount lower than the amount of options exercised in the base case results in a decrease in both the values of the contract and the option.

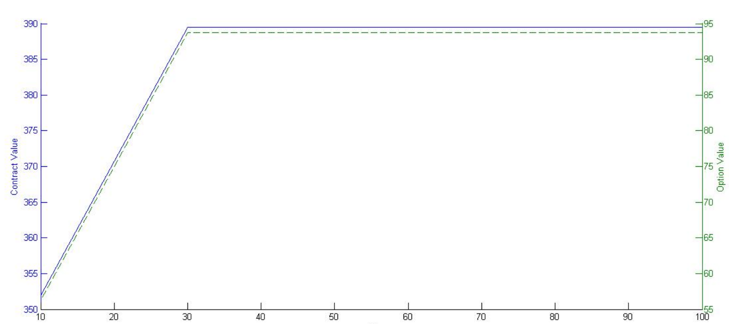

Figure 7.7: Contract and Option Values vs M in Scenario 1

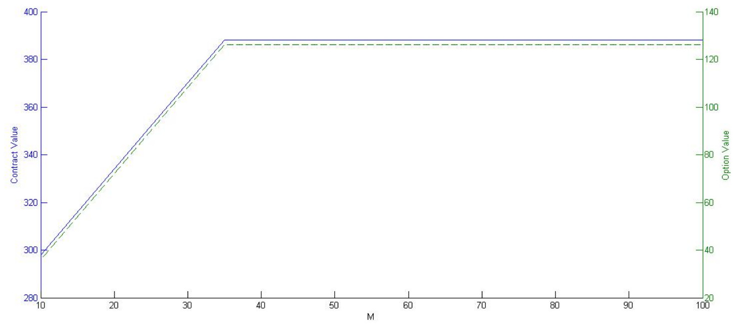

Figure 7.8: Contract and Option Values vs M in Scenario 2

Case 2 : Effect of Exercise Price

We test the effect of exercise price by changing the unit price paid to exercise an option. In both scenarios, we observe that as long as options are used to satisfy the demand, as exercise price decreases the values of the contract and the option increases.

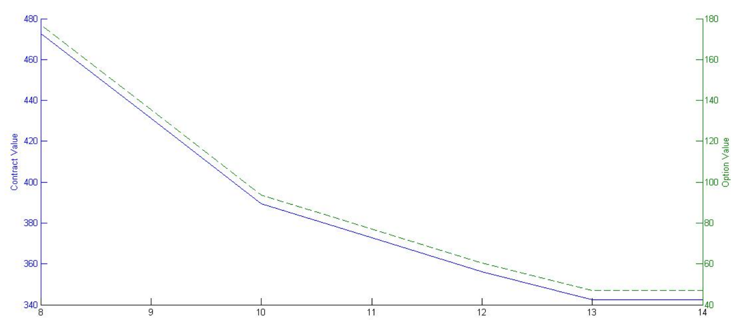

Figure 7.9: Contract and Option Values vs Exercise Price in Scenario 1

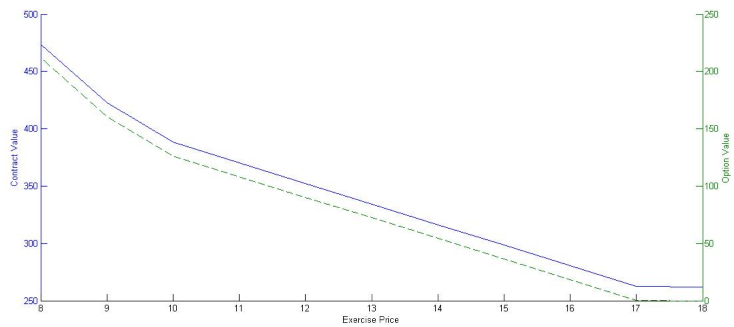

Figure 7.10: Contract and Option Values vs Exercise Price in Scenario 2

Case 3 : Effect of Purchase Price

We observe that in both scenarios as the unit cost of placing a firm order for period 1 is decreased, while keeping it constant for period 2, the value of the contract increases, whereas the value of the option decreases if and only if $p_1 + h_1 < e_1$.

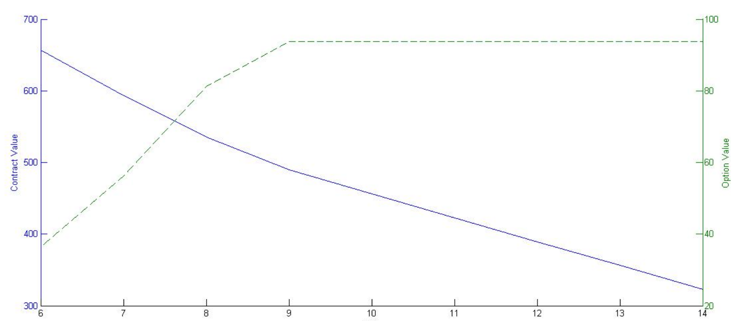

Figure 7.11: Contract and Option Values vs p_1 in Scenario 1

Again, in both scenarios decreasing the purchase price of period 2 while keeping it constant for period 1, increases the value of the contract as long as $p_2 < p_1$, whereas decreases the value of the option at all times.

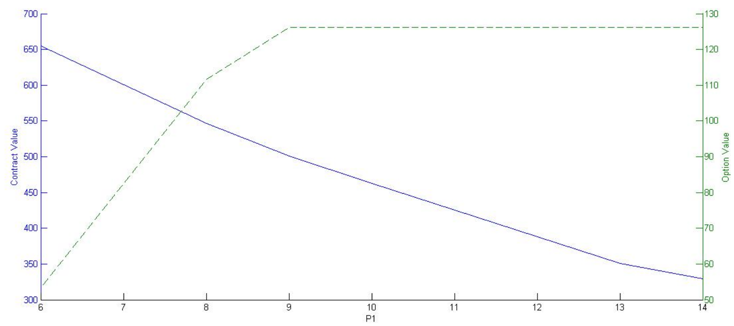

Figure 7.12: Contract and Option Values vs p_1 in Scenario 2

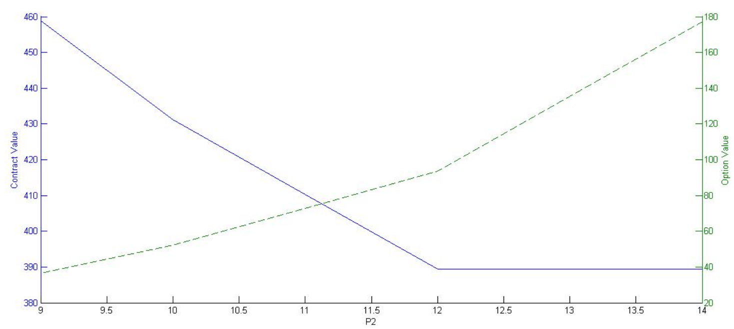

Figure 7.13: Contract and Option Values vs p_2 in Scenario 1

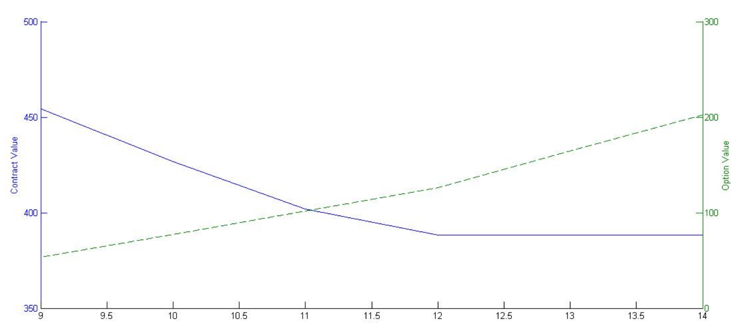

Figure 7.14: Contract and Option Values vs p_2 in Scenario 2

Case 4 : Effect of Demand Volatility

The effect of demand volatility is studied by varying the values of the demand processes and the results are summarized in Table 7.3 and 7.4 for scenario 1 and 2 respectively. The values of the other parameters are taken as in the base case. In both scenarios, we observe that increasing the volatility of demand results in an increase in the value of the option.

n	D_n	D_n	D_n	D_n	D_n
1	45	40	50	45	45
2	40	40	40	40	40
3	30	30	30	30	30
4	25	30	20	25	25
5	55	55	55	50	60
6	30	30	30	35	25
7	50	50	50	50	50
8	25	25	25	25	25
9	45	45	45	45	45
10	20	20	20	20	20
11	40	40	40	40	40
12	15	15	15	15	15
$V_{100} - V_0$	93.75	37.5	150	84.37	103.12

Table 7.3: Parameters and decision variables in case 4 in scenario 1

n	D_n	D_n	D_n	D_n	D_n
1	45	40	50	45	45
2	45	40	50	45	45
3	25	30	20	25	25
4	25	30	20	25	25
5	55	55	55	50	60
6	30	30	30	35	25
7	55	55	55	50	60
8	30	30	30	35	25
9	40	40	40	40	40
10	15	15	15	15	15
11	40	40	40	40	40
12	15	15	15	15	15
$V_{100} - V_0$	126.2	90.14	162.26	108.17	144.23

Table 7.4: Parameters and decision variables in case 4 in scenario 2

Case 5 : Volatility of Stock Prices

The effect of volatility of stock prices is tested by changing the values of the risky security price processes and the results are summarized in Tables 7.5 and 7.6 for scenarios 1 and 2 respectively. The values of the other parameters are taken as in the base case.

In both scenarios we first keep the stock prices in period 2 constant and observe that the value of the option decreases as volatility of the stock prices in period 1 decreases. The reason behind this is that even it is not perfect we assume that there is a correlation between demand and the price of a risky security and in period 1 where the decision to exercise options or not is given the prices of stock are less volatile.

Next, we keep the stock prices in period 1 constant and observe that the value of the option is unaffected from the stock prices of period 2. This is so because in period 2 there is no action taken by the buyer to exercise options or not, and in period 1 where the decision of whether to exercise options or not is given the prices of stocks are unaltered. However, the stock prices in period 2 impact the

n	Z_n	Z_n	Z_n	Z_n	Z_n
0	15	15	15	15	15
1	20	19	20	21	19
2	20	19	20	21	19
3	10	11	10	9	11
4	10	11	10	9	11
5	25	25	28	28	28
6	5	5	2	2	2
7	25	25	28	28	28
8	5	5	2	2	2
9	22	22	24	24	24
10	8	8	6	6	6
11	22	22	24	24	24
12	8	8	6	6	6
$V_{100} - V_0$	93.75	76.17	93.75	105.47	76.17

Table 7.5: Parameters and decision variables in case 5 in scenario 1

portfolio of stocks in period 1 since the buyer needs to cover all his short sales to repay the debt, and forms his portfolio in period 1 by taking into account the price of the stock in the next period.

n	Z_n	Z_n	Z_n	Z_n	Z_n
0	15	15	15	15	15
1	20	19	20	21	19
2	23	23	23	23	23
3	7	7	7	7	7
4	10	11	10	9	11
5	25	25	28	28	28
6	5	5	2	2	2
7	28	28	28	28	28
8	2	2	2	2	2
9	24	24	24	24	24
10	6	6	6	6	6
11	22	22	24	24	24
12	8	8	6	6	6
$V_{100} - V_0$	126.2	136.72	126.2	117.19	136.72

Table 7.6: Parameters and decision variables in case 5 in scenario 2

Case 6 : Effect of Interest Rate on the Riskless Asset

The effect of interest rate on the riskless asset is analyzed simply by observing the value of the option for different interest rates. The results are summarized in Tables 7.7 and 7.8 for scenario 1 and 2 respectively. The values of the other parameters are taken as in the base case.

interest rate(%)	$V_{100} - V_0$
0	343.75
10	191.64
20	93.75
25	63.2

Table 7.7: Parameters and decision variables in case 6 in scenario 1

In both scenarios, we observe that as the interest rate on the riskless asset decreases the value of the option increases. This is due to the fact that lower interest rate allows the buyer to make more short sales of bonds and exercise options with the money borrowed to meet larger than expected demand.

interest rate(%)	$V_{100} - V_0$
0	396.92
10	236.98
20	126.2
25	28

Table 7.8: Parameters and decision variables in case 6 in scenario 2

For the analysis of the remaining parameters we will extend our model to a three-period model and consider the binomial tree shown in the figure below.

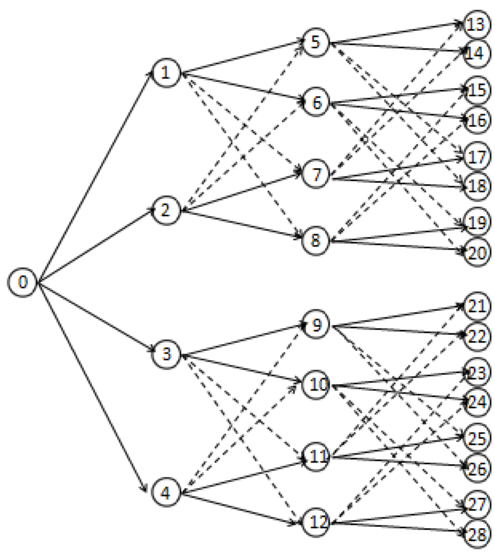

Figure 7.15: Three-Period Financial and Demand Market Scenario Tree

As can be seen from Figure 7.11

$$N_0 = \{0\}, \qquad N_1 = \{1, 2, 3, 4\}$$

$$N_2 = \{5, 6, 7, 8, 9, 10, 11, 12\}, \qquad N_3 = \{13, 14, \ldots, 27, 28\}$$

$$a(1) = a(2) = a(3) = a(4) = \{0\}$$

$$a(5) = a(6) = a(7) = a(8) = \{1, 2\}$$

$$a\,(9) = a\,(10) = a\,(11) = a\,(12) = \{3, 4\}$$

$$a\,(13) = a\,(14) = a\,(17) = a\,(18) = \{5, 7\}$$

$$a\,(15) = a\,(16) = a\,(19) = a\,(20) = \{6, 8\}$$

$$a\,(21) = a\,(22) = a\,(25) = a\,(26) = \{9, 11\}$$

$$a\,(23) = a\,(24) = a\,(27) = a\,(28) = \{10, 12\}$$

$$Z_n = (Z_n^0, Z_n^1) \qquad n = 0, \ldots, 28$$

where Z_n^0 denotes the price of the riskless asset, and Z_n^1 denotes the price of the risky security.

Before moving on to the illustrations, as in the two-period model we need to choose a base case for each scenario to start analysis. By the similar reasoning of two-period case in both scenarios the stock prices and the demand values will be chosen in such a way that the average of them are constant at all periods. The values of the parameters and the corresponding decision variables in the base case for scenario 1 and 2 are given in appendix. We will now illustrate the roles of the remaining parameters on the values of the contract and the option.

Case 7 : Effect of Sales Price

By varying the sales prices and keeping the values of the other parameters constant, we observe that in both scenarios the value of the contract increases as the unit revenue of selling finished products increases. Furthermore, we observe that the value of the option decreases by an increase in sales price of period 1 unless the buyer places as many firm order as in period 1 to meet the demand of period 1 in order not to miss the opportunity of generating higher profits. On the other hand, we observe that the value of the option increases by an increase in sales price of periods 2 and 3 as long as the buyer places fewer firm orders for period 1 and exercises more options to meet the demand of period 1 in later periods to generate higher revenues.

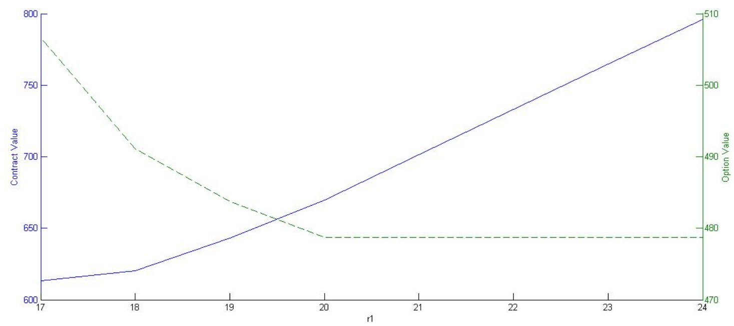

Figure 7.16: Contract and Option Values vs r_1 in Scenario 1

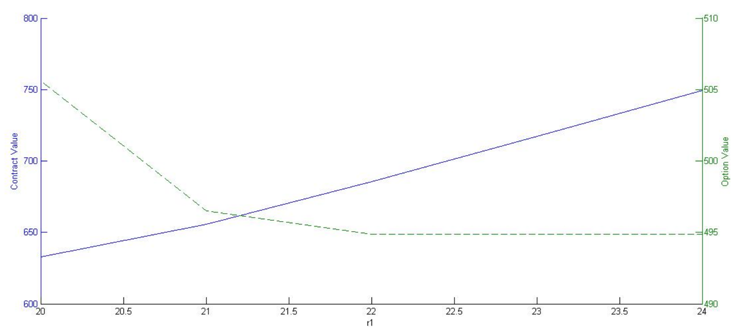

Figure 7.17: Contract and Option Values vs r_1 in Scenario 2

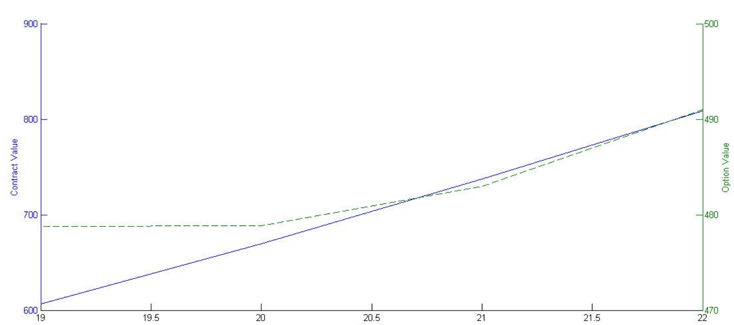

Figure 7.18: Contract and Option Values vs $r_2 - r_3$ in Scenario 1

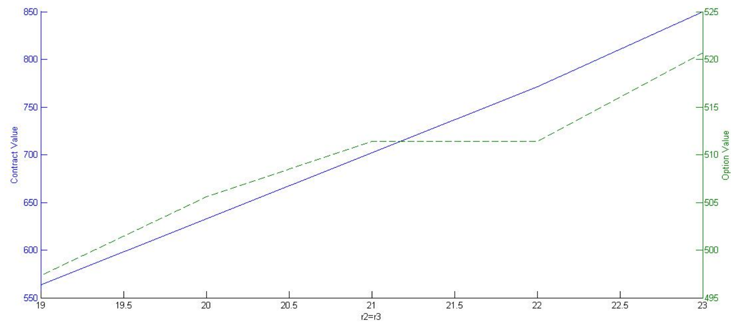

Figure 7.19: Contract and Option Values vs $r_2 - r_3$ in Scenario 2

Case 8 : Effect of Holding Cost

We test the effect of holding cost on the values of the contract and the option by changing the unit cost of holding a positive inventory. We first observe that as long as the buyer holds inventories, the contract becomes more profitable and the buyer accepts to pay more for the contract as the unit cost of holding a positive inventory decreases. On the other hand, we observe that the value of the option decreases as the unit cost of holding a positive inventory decreases.

As can be seen from Figure 7.21, in scenario 2 the value of the contract is unaffected by the changes in h_1. The reason is that, due to the choice of the values of the parameters in scenario 2, the buyer does not hold inventories in any node of period 1. Therefore, we consider another setting and change the values of Z_2^1 to 19 and Z_3^1 to 11 and call this scenario as scenario 2^1. We then observe the value of the contract by changing the cost of holding inventory in period 1. As expected, we observe that the result that is shown in Figure 7.22 is similar to that of previous chapter and that of scenario 1 in this chapter.

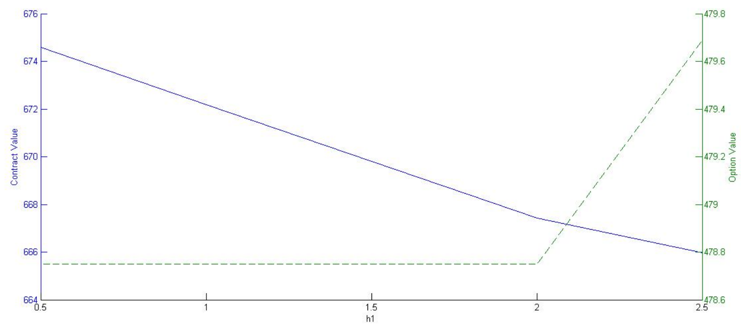

Figure 7.20: Contract and Option Values vs h_1 in Scenario 1

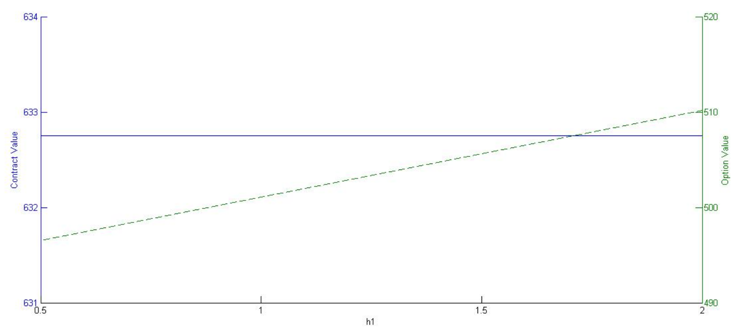

Figure 7.21: Contract and Option Values vs h_1 in Scenario 2

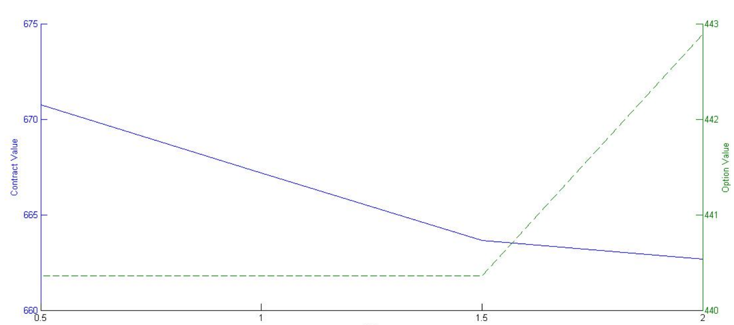

Figure 7.22: Contract and Option Values vs h_1 in Scenario 2[1]

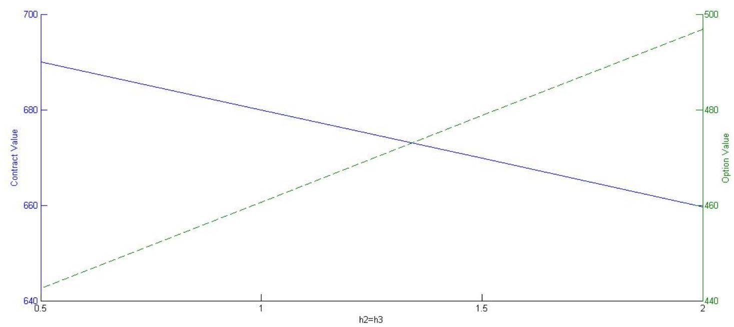

Figure 7.23: Contract and Option Values vs $h_2 - h_3$ in Scenario 1

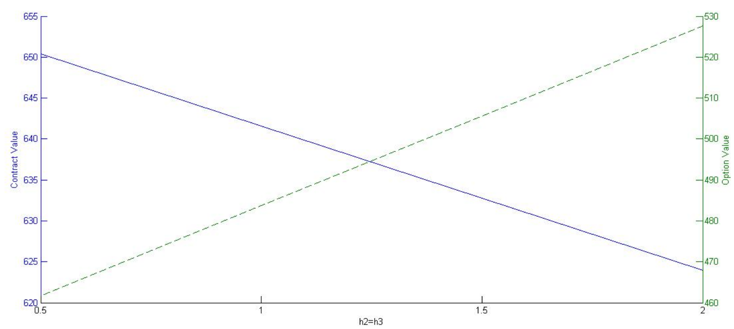

Figure 7.24: Contract and Option Values vs $h_2 - h_3$ in Scenario 2

Case 9 : Effect of Stock-out Cost

The impact of the stock-out cost is tested by varying the unit cost of having shortages. As expected, it is observed that as long as there is a backorder, the values of the contract decreases as the unit penalty cost in period 1 increases, since the buyer will generate fewer profits in case of shortages in period 1. Moreover, it is observed the value of the option also decreases by an increase in the penalty cost of period 1, since the buyer will place more firm orders for period 1 in order not to pay the higher stock-out cost and hence necessitates fewer options in later periods.

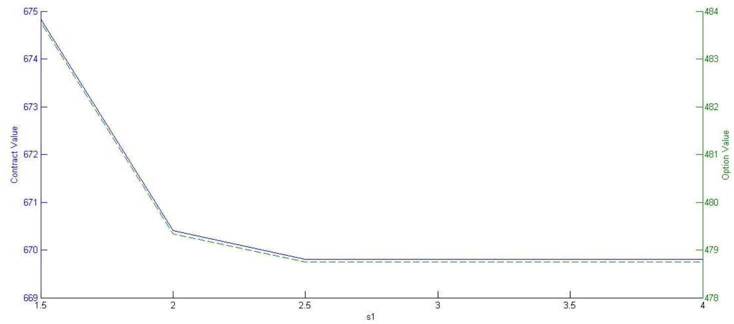

Figure 7.25: Contract and Option Values vs s_1 in Scenario 1

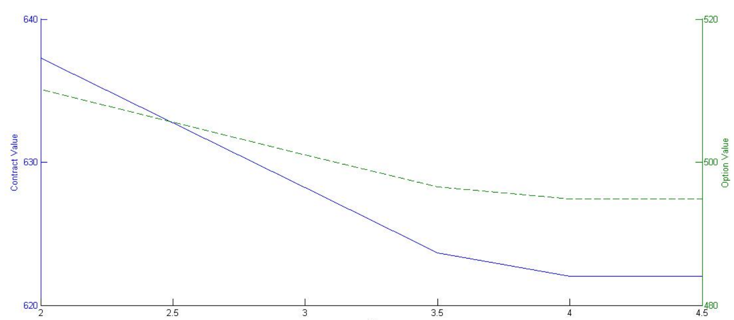

Figure 7.26: Contract and Option Values vs s_1 in Scenario 2

Chapter 8

Conclusion

In this thesis, we considered a general single buyer - single supplier quantity flexibility contract with options for multiple periods. We investigate the problem of the buyer of the contract and find the maximum acceptable price of the contract for the buyer by analyzing both financial and real markets. In our study, we assume that the markets evolve as discrete scenario trees. The essence of the thesis is that it incorporates financial and real markets in this setting. The details of this approach are discussed throughout the thesis.

Firstly, a model to find the maximum acceptable price of the contract for the buyer is presented under the assumption of the perfect correlation between the demand of the item and the price of the risky asset traded in the financial markets. Then, we make use of duality theory to analyze the problem. From duality considerations, we make some inferences through observations of the effects of the position of the parameters on the value of the contract. One of the observation that we have made is that, increasing the flexibility available to a buyer increases the value of the contract up to a level, but when the buyer has the desired flexibility, the value of the contract is unaffected from an increase in the value of M.

Next, we assume that the unit cost of placing a firm order is the same at all periods and the buyer is flexible enough to use as much as options he wants and show that; (i) decreasing the purchasing price in a period decreases the value of the contract as long as the cost of exercising an option is more expensive than the cost of placing a firm order plus the cost of holding a positive inventory (ii) decreasing the cost of exercising an option decreases the value of the contract as long as the cost of placing a firm order plus the cost of holding a positive inventory is more expensive than the cost of exercising an option.

We then relax the assumption of perfect correlation and extend our model to the case of partial correlation between the demand and the price of the risky security traded in the financial market. We further assume that both markets evolve as binomial trees and show that our model works under this assumption and check the validity of the observations derived under the perfect correlation assumption.

Under both of the assumptions, we perform experimental studies to see the effects of the parameters on the value of the contract and the cost of the flexibility available to the buyer. Under both assumptions, we observe mainly; (i) as the volatility of the demand increases the value of the option increases (ii) the value of the option increases by an increase in the volatility of the price of the risky security traded in the financial market in period 1, whereas it is unaffected by that of period 2 (iii) as the interest on the riskless asset increases the value of the option decreases. We further observe the behavior of the value of the contract and the option by changing other operational costs, which are mainly; (iv) an increase in sales price of period 1 increases the value of the contract, whereas it decreases the value of the option (v) an increase in sales prices of other periods increases both the values of the contract and the option (vi) the values of the contract and the option decrease as the cost of holding positive inventory increases (vii) as the cost of having backorder increases, both the values of the contract and the option decrease. The details and the conditions on the validities of these observations are mentioned through the thesis without repeating all the details for the extended model when similar results are obtained.

There is room for improvement and future research of the model under consideration in the case of partial correlation of the demand and the price of the risky security. For instance, the assumption that both financial and real markets evolve as binomial trees can be relaxed.

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Appendix A

Derivation of Equations

A.1 Derivation of Equation 7.43

Summing the constraints 7.38 $\forall n \in N_1$ we have

$$\sum_{n\in N_1} f_n + \sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq -\sum_{n\in N_1} e_1 q_n.$$

Furthermore by the assumption $\sum_{n\in N_1} f_n = 0$. Therefore, one obtains

$$\sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq -\sum_{n\in N_1} e_1 q_n. \tag{A.1}$$

Similar to constraint 7.38, summing the constraints 7.39 $\forall n \in N_1$ we have

$$\sum_{n\in N_1} y_n - \sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \leq \sum_{n\in N_1} h_1 q_n.$$

Reorganizing the above constraint one can obtain

$$\sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq \sum_{n\in N_1} y_n - \sum_{n\in N_1} h_1 q_n. \tag{A.2}$$

In addition, constraint 7.36 can be rewritten as

$$0 \geq -\sum_{n\in N_1} y_n - \sum_{n\in N_1} p_1 q_n. \tag{A.3}$$

Summing the constraints $A.2$ and $A.3$ we obtain

$$\sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq -\sum_{n\in N_1} (p_1+h_1)\, q_n. \tag{A.4}$$

Finally, constraints $A.1$ and $A.4$ imply that

$$\sum_{n\in N_1}\sum_{i=1}^{2}\sum_{m\in C_i(n)} y_{m_i} \geq max\left\{-\sum_{n\in N_1} (p_1+h_1)\, q_n, -\sum_{n\in N_1} e_1 q_n\right\}. \tag{A.5}$$

A.2 Derivation of Dual Constraints of Chp. 7

The dual constraint corresponding to the decision variable V, that is the value of the contract, is

$$q_0 = 1. \tag{A.6}$$

Next, the dual constraint corresponding to θ_n, $n \in N_0$ is the martingale condition

$$q_0 Z_0 = \sum_{m\in C_1(0)} q_m Z_m, \tag{A.7}$$

and the dual constraint corresponding to θ_{n_i}, $n \in N_1$ is

$$q_n Z_n = \sum_{i=1}^{2} \sum_{m \in C_i(n)} q_{m_i} Z_m \quad n \in N_1. \tag{A.8}$$

The dual constraint corresponding to the decision variables θ_{n_i} for $n \in N_2$, $i = 1, 2$ is

$$(q_{n_i} + w_{n_i})\, Z_n = 0 \quad n \in N_2, i = 1, 2,$$

and since the first component $Z_n^0 = 1$ for all states n we have

$$q_{n_i} + w_{n_i} = 0 \quad n \in N_2, i = 1, 2.$$

In addition, by the non-negativity of the portfolio in the terminal positions

$$w_{n_i} \leq 0 \quad n \in N_2, i = 1, 2.$$

Combining the above two constraints one has the following constraint in the dual.

$$q_{n_i} \geq 0 \quad n \in N_2, i = 1, 2. \tag{A.9}$$

The dual constraint corresponding to the firm orders Q_t is

$$\sum_{n \in N_1} p_1 q_n + y_n \geq 0, \tag{A.10}$$

$$\sum_{i=1}^{2} \sum_{n \in N_2} p_2 q_{n_i} + y_{n_i} \geq 0. \tag{A.11}$$

The constraint in the dual arising from the number of options exercised, i.e. m_{n_i}, $n \in N_1$ is

$$e_1 q_n + f_n + \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0. \tag{A.12}$$

The dual constraint corresponding to the net inventory at state n, $n \in N_1$ is

$$-y_n + \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} + k_n = 0 \quad n \in N_1.$$

Reformulating the above constraint one obtains

$$k_n = y_n - \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \quad n \in N_1.$$

The constraint in the dual arising from the positive inventory at state n, $n \in N_1$ is

$$h_1 q_n - k_n \geq 0 \quad n \in N_1,$$

and the constraint in the dual arising from the negative inventory at state n, $n \in N_1$ is

$$(r_1 + s_1)\, q_n - r_2 \left(\sum_{i=1}^{2} \sum_{m \in C_i(n)} q_{m_i} \right) + k_n \geq 0 \quad n \in N_1.$$

Replacing k_n by $y_n - \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \quad n \in N_1$ one has the following constraints in the dual corresponding to, respectively, positive and negative inventory at state n, $n \in N_1$.

$$h_1 q_n - y_n + \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0 \quad n \in N_1, \tag{A.13}$$

$$(r_1 + s_1) q_n - r_2 \left(\sum_{i=1}^{2} \sum_{m \in C_i(n)} q_{m_i} \right) + y_n - \sum_{i=1}^{2} \sum_{m \in C_i(n)} y_{m_i} \geq 0 \quad n \in N_1 \tag{A.14}$$

Finally, the dual constraint corresponding to the net inventory at the terminal positions which is also the positive inventory since shortages are not allowed in the last period is

$$h_2 q_{n_i} - y_{n_i} \geq 0 \quad n \in N_2, i = 1, 2. \tag{A.15}$$

A.3 Base Case Values in Scenario 1 and 2

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n
		0	10	15	
$r_t = 20$	$Q_1 = 40$	1	11	20	45
$p_t = 12$	$V_0 = 191.06$	2	11	20	40
$h_t = 1.5$	$V_M = 669.81$	3	11	10	30
$s_t = 2.5$		4	11	10	25
M = 100		5	12.1	25	55
e = 10		6	12.1	5	30
		7	12.1	25	50
		8	12.1	5	25
		9	12.1	22	45
		10	12.1	8	20
		11	12.1	22	40
		12	12.1	8	15
		13	13.31	30	65
		14	13.31	20	40
		15	13.31	8	35
		16	13.31	4	20
		17	13.31	30	60
		18	13.31	20	35
		19	13.31	8	30
		20	13.31	4	15
		21	13.31	25	55
		22	13.31	18	30
		23	13.31	10	40
		24	13.31	5	15
		25	13.31	25	50
		26	13.31	18	25
		27	13.31	10	35
		28	13.31	5	10

Table A.1: Parameters and decision variables in base case in scenario 1

Parameters	Decision Var.	n	Z_n^0	Z_n^1	D_n
		0	10	15	
$r_t = 20$	$Q_1 = 25$	1	11	20	45
$p_t = 12$	$V_0 = 127.14$	2	11	23	45
$h_t = 1.5$	$V_M = 632.75$	3	11	7	25
$s_t = 2.5$		4	11	10	25
M = 100		5	12.1	25	55
e = 10		6	12.1	5	30
		7	12.1	28	55
		8	12.1	2	30
		9	12.1	24	40
		10	12.1	6	15
		11	12.1	22	40
		12	12.1	8	15
		13	13.31	30	65
		14	13.31	20	40
		15	13.31	8	35
		16	13.31	4	20
		17	13.31	32	65
		18	13.31	24	40
		19	13.31	5	35
		20	13.31	1	20
		21	13.31	28	50
		22	13.31	19	25
		23	13.31	9	35
		24	13.31	2	10
		25	13.31	25	50
		26	13.31	18	25
		27	13.31	10	35
		28	13.31	5	10

Table A.2: Parameters and decision variables in base case in scenario 2