IMPLEMENTATION VIA CODE OF RIGHTS

A Master's Thesis

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JULY 2008

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The Institute of Economics and Social Sciences
of
Bilkent University

by

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In Partial Fulfillment of the Requirements For the Degree
of
MASTER OF ARTS

in

THE DEPARTMENT OF
ECONOMICS
BILKENT UNIVERSITY
ANKARA

JULY 2008

I certify that I have read this thesis and have found that it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Arts in Economics.

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ABSTRACT

IMPLEMENTATION VIA CODE OF RIGHTS

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Implementation of a social choice rule can be thought of as a design of power (re)distribution in the society whose "equilibrium outcomes" coincide with the alternatives chosen by the social choice rule at any preference profile of the society. In this paper, we introduce a new societal framework for implementation which takes the power distribution in the society, represented by a code of rights, as its point of departure. We examine and identify how implementation via code of rights (referred to as gamma implementation) is related to classical Nash implementation via mechanism. We characterize gamma implementability when the state space on which the rights structure is to be specified consists of the alternatives from which a social choice is to be made. We show that any social choice rule is gamma implementable if it satisfies pivotal oligarchic monotonicity condition that we introduce. Moreover, pivotal oligarchic monotonicity condition combined with Pareto optimality is sufficient for a non-empty valued social choice rule to be gamma implementable. Finally we revisit liberal's paradox of A.K. Sen, which turns out to fit very well into the gamma implementation framework.

Keywords: Implementation, code of rights, Nash equilibrium, monotonicity,

social choice rule

ÖZET

HAKLAR YAPISI ARACILIĞIYLA UYGULANABİLİRLİK

YILDIZ, Kemal

Yüksek Lisans, Ekonomi Bölümü

Tez Yöneticisi: Prof. Semih Koray

Temmuz 2008

Bir sosyal seçim kuralının uygulanması, her terçih profilinde denge sonuçları ile sosyal seçim kuralı tarfından seçilen seçeneklerin örtüşmesini sağlayacak kuvvet dağılımının tasarımı olarak düşünülebilir. Bu çalışmamızda, toplumda haklar yapısı aracılığıyla temsil edilen kuvvet dağılımını hareket noktası alarak, uygulama kuramı içn yeni bir toplumsal çerçeve sunuyoruz. Haklar yapısı aracılığıyla uygulanabilirliğin(gama uygulanabilirlik olarak da isimlendirilmiştir), mekanizma aracılığıyla klasik Nash uygulanabilirlikle olan ilişkisi incelenmiştir. Üzerinde haklar yapısının belirleneceği durum uzayını yalnızca aralarından toplumsal seçimin yapılacağı seçeneklerin oluşturduğu durumda haklar yapısı aracılığıyla uygulanabilirliğin karakterizasyonu sunulmuştur. Herhangi bir sosyal seçim kuralının haklar yapısı aracılığıyla uygulanabilir olması için, tarafımızca ortaya atılan belirleyici oligarşik tekdüzelik koşulunu sağlaması gerektiği gösterilmiştir. Bunun yanısıra, belirleyici oligarşik tekdüzelik koşulu Pareto verimlilikle birlikte boş değerli olmayan bir sosyal seçim kuralının gama uygulanabilir olması için yeterli olmaktadır. Son olarak da A.K. Sen'e ait olan liberal ikilemi, incelenmesinin uygun düştüğü gözlemlenen haklar yapısı çerçevesinde yeniden ele alınmıştır.

Anahtar Kelimeler: Uygulama, Haklar Yapısı, Nash Dengesi, Tekdüzelik, Sosyal Seçim Kuralı.

ACKNOWLEDGMENTS

I would like to express my gratitudes to;

Prof. Semih Koray, to whom the significance of this thesis is due, for his invaluable guidance. It were a great honour for me to study under his supervision.

Prof. Tarık Kara, for his endless support throughout my study at Bilkent University.

My friends for their amity.

TÜBİTAK, for financial support.

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CHAPTER 1

INTRODUCTION

In classical implementation a rights structure among the members of the society can be induced from the mechanism, designed to implement a social choice rule under the given solution concept. In other words, in classical implementation we have an implicit specification of a power distribution among the members of the society. In this thesis, we introduce a new institutional design approach to implementation which explicitly specifies the rights structure in the society over the set of alternatives.

A constitution or a *code of rights* is used for the assignment of rights to the members of the society.First in Arrow (1967) a notion of constitution, along the same lines, is defined, where a "well-behaved" social welfare function is considered as a constitution. This notion leads to the conclusion of well known Arrow's Impossibility Theorem. In this study we define a code of rights as a set valued function, which associates each ordered pair of alternatives with a family of coalitions, indicating that each coalition in the specified family is given the right to lead a switch from the first alternative to the second one. In our framework code of rights is common knowledge, and is specified as being invariant of preferences.

The definition for *code of rights* that we use in this thesis was introduced in Sertel (2002), where it is used as a design notion in the specification of a *Rechstaat*. Parelelling the first and second welfare theorems of economics, Sertel imparted to *code of rights* an *invisible hand property* and a property of the *preservation of the best public interest*.

In a similar framework used in Sertel(2002), Peleg (1998) proposed a new definition of constitution. This constitution specifies a rights structure among the members of the society. Furthermore, Peleg (1998) investigated game forms that represent the distribution of power which is dictated by this prevailing rights structure in the society.

In classical implementation there are various examples indicating the connection between monotonicity and implementability. Maskin (1977) showed that any Nash implementable social choice rule is monotonic, and monotonicity combined with some further assumptions as *no veto power* condition is sufficient for Nash implementability. Danilov (1992), proposed an *essential monotonicity* condition which turned out to be both necessary and sufficient for Nash implementability with some further assumptions.

Kaya and Koray (2000) introduced the notion of *oligarchy* and *oligarchic monotonicity*, where they show that; any *oligarchic social choice rule* satisfies oligarchic monotonicity and oligarchic monotonicity combined with unanimity condition is sufficient for a social choice rule to be oligarchic.

The organization of the thesis is as follows: In chapter 2 we introduce the basic definitions and notation. Moreover, the relation between Nash implementation and (A, γ)-implementation is examined in this chapter by the given examples. In chapter 3, we introduce the pivotal oligarchic monotonicity condition and related definitions and in sections 3.2 and 3.3 , (A, γ)-implementation is characterized in terms of pivotal oligarchic monotonicity, and Pareto optimality. In section 3.2, we show that any (A, γ)-implementable social choice rule satisfies pivotal oligarchic monotonicity. The implementation theorem is set in section 3.3, indicating that any non-empty valued, Pareto optimal social choice rule, endowed with pivotal oligarchic monotonicity is (A, γ)-implementable. We conclude chapter 3 with the presentation of a structure for social decision making with a continuum of agents which turns out to be completely compatible with (A, γ)-implementation framework. In chapter 4, liberal's paradox of Amartya K. Sen is revisited, and investigated from (A, γ)-implementation perspective. Finally chapter 5 is devoted for the presentation of some results related with the characteristics of manhattan metric and examination of liberal's paradox in this restriced framework.

CHAPTER 2

(A, γ)- IMPLEMENTATION

2.1 Notation and Definitions

We use A to denote a non-empty, finite alternative set, while N, as usual, denotes the set of agents which is also assumed to be non-empty and finite. We will use $\mathcal{N}$ to denote the collection of all subsets of N. A coalition in N, denoted by generic element K, is a member of $\mathcal{N}$; i.e $K \in 2^N = \mathcal{N}$. A preference profile for N is an n-tuble where each component denotes the preference of the associated agent over A; i.e for any $i \in N$ and any a, b, we represent, agent i prefers b to a under preference profile R, by bR_ia. Collection of linear orders on A is denoted by $\mathcal{L}(A)$, where a linear order on A is a complete, transitive, and antisymmetric binary relation on A. The set of all linear order profiles on A is denoted by $\mathcal{L}(A)^N$. In this study, we will restrict preference profiles to the set of linear order profiles. A social choice rule F maps every preference profile on A into a subset of A; i.e. $F : \mathcal{L}(A)^N \rightarrow 2^A$. Let $R \in \mathcal{L}(A)^N$ and $a \in A$, the lower contour set of R_i with respect to alternative $a \in A$, is the set consisting of alternatives to which a is preferred by agent i under preference profile R, which is denoted by $\mathrm{L}(R_i, a)$.

A *mechanism* (or a game form) is a function g which maps every joint strategy to an outcome in the alternative set; i.e. $g : S \to A$, where $S = \Pi_{i \in N} S_i$, S_i stands for agent i's strategy set. A mechanism g, combined with a preference profile $R \in \mathcal{L}(A)^N$ forms a normal form game and the pure strategy Nash equilibria of the game is denoted by NE(g, R). We say that a social choice rule F is *Nash implementable* via a mechanism g if at each preference profile R, alternatives chosen by F coincide with the Nash equilibrium outcomes of the game for given R; i.e for any $R \in \mathcal{L}(A)^N$, we have $\{g(s) \mid s \in NE(g, R)\} = F(R)$.

Any social choice rule F is said to be monotonic if and only if for any $R, R' \in \mathcal{L}(A)^N$, and any $a \in F(A)$ where for any $i \in N$, $\mathrm{L}(R_i, a) \subset \mathrm{L}(R'_i, a)$ implies $a \in F(R')$. We say F is Pareto optimal if and only if there is no alternative in A which Pareto dominates a with respect to given R; i.e for any $R \in \mathcal{L}(A)^N$ and $a \in F(R)$, there is no $b \in A$ such that for any $i \in N$, bR_ia.

For any given preference profile $R \in \mathcal{L}(A)^N$, the benefit function $\beta_R : A \times A \to 2^{\mathcal{N}}$, maps any pair of alternatives $(a, b) \in A \times A$, to a member of $2^{\mathcal{N}}$; i.e. the class of all coalition families. For any $(a, b) \in A \times A$, any $K \in \mathcal{N}$, $K \in \beta_R(a, b)$ implies that; all the members of the coalition K prefers b to a under preference profile R; i.e. for any $i \in K, bR_ia$.

A code of rights is defined to be a function γ which maps any pair of alternatives $(a, b) \in A \times A$, to a coalition family; i.e $\gamma : A \times A \to 2^{\mathcal{N}}$, where for any $(a, b) \in A \times A$, and any $K \in \mathcal{N}$, $K \in \gamma(a, b)$ represents that coalition K is given the right to lead a switch from a to b, by the code of rights γ. We assume that if any coalition is given the right to lead a switch from a to b, then any coalition which contains this coalition preserves the same right; i.e for any $(a, b) \in A \times A$ and for any $K \in \mathcal{N}, K \in \gamma(a, b)$ implies for any $K' \in \mathcal{N}$ where $K \subset K'$, we have $K' \in \gamma(a, b)$. The collection of all code of rights defined on $A \times A$ for given N is denoted by $\Gamma(A, N)$.

We assume that every coalition is able to make any switch, so we do not specify an ability function $\alpha : A \times A \rightarrow 2^{\mathcal{N}}$, which specifies the able coalitions for leading a switch from an alternative to another one.

Before introducing (A, γ)-implementability notion, we need to specify an equilibrium condition which plays the role of solution concepts in classical implementation.

Definition. For any $R \in \mathcal{L}(A)^N$, and any $a \in A$, we say a is an (A, γ)-equilibrium and denote it by $a \in \epsilon(A, \gamma, \beta_R)$ if and only if for any $b \in A \setminus \{a\}, \gamma(a, b) \cap \beta_R(a, b) = \emptyset$.

If for any alternative a, there is no benefiting coalition which is given the right to lead a switch from a to any other alternative, then alternative a is referred as an (A, γ)-equilibrium.[1]

Definition. Any social choice rule F is said to be (A, γ)-implementable if there is a $\gamma \in \Gamma(A, N)$ such that for any $R \in \mathcal{L}(A)^N$, $F(R) = \epsilon(A, \gamma, \beta_R)$.

For any social choice rule F, if we can find a code of rights $\gamma : A \times A \rightarrow 2^{\mathcal{N}}$ such that; at each preference profile R, alternatives chosen by F coincide with the alternatives in the (A, γ)-equilibria for given R, then F is said to be (A, γ)-implementable.

2.2 Examples

Example 1. Let $N = \{1, 2\}, A = \{a, b, c\}$, R and R' be as specified below, and the social choice rule F be such that; $F(R) = \{a), F(R') = \{b\}$

[1]Notion of (A, γ)-equilibria as well as (A, γ) implementation can be extended to (S, γ) implementation, where S stands for any arbitrary strategy set.

R	
1	2
a	c
c	b
b	a

.

R′	
1	2
c	b
a	c
b	a

Firstly it is worth to note that F is not Nash implementable. Suppose not; i.e. there is a mechanism, $g : S \to A$ which implements F under Nash equilibrium. Now, $F(R) = g(NE(g, R)) = \{a\}$ implies there exists $(\widetilde{m_1}, \underline{m_2}) \in NE(g, R)$ such that $g(\widetilde{m_1}, \underline{m_2}) = \{a\}$ combined with for any $y \in A, yR_2a$ implies $\widetilde{m_1}$ such that for any $m_2 \in M_2, g(\widetilde{m_1}, m_2) = \{a\}$. On the other hand, $F(R') = g(NE(g, R')) = \{b\}$ indicating that there exists $(\underline{m_1}, \widetilde{m_2}) \in NE(g, R)$ such that $g(\underline{m_1}, \widetilde{m_2}\ ) = \{b\}$ with for any $y \in A$, yR_1b implies $\widetilde{m_2}$ is such that for any $m_1 \in M_1, g(m_1, \widetilde{m_2}) = \{b\}$ hence $g(\widetilde{m_1}, \widetilde{m_2}) = \{b\}$ contradicting for any $m_2 \in\ M_2,\ g(\widetilde{m_1},\ m_2) = \{a\}$.

Secondly, let us construct a code of rights γ which would implement the given social choice rule F. Let γ be such that;

$$\forall x \in \{b, c\} \quad \gamma(a, x) = \ \{\{1\}, \{1, 2\}\}$$

$$\forall x \in \{a, c\} \quad \gamma(b, x) = \ \{\{2\}, \{1, 2\}\}$$

$$\forall x \in \{a, b\} \quad \gamma(c, x) = \ \{\{1\}, \{2\}, \{1, 2\}\}$$

Now, for any $x \in \{b, c\}, \beta_R(a, x) = \{\{2\}\}$ but $\gamma(a, x) = \{\{1\}, \{1, 2\}\}$ implies $\beta_R(a, x) \cap \gamma(a, x) = \emptyset$ implies $a \in \epsilon(A, \gamma, \beta_R)$.

$\{2\} \in \beta_R(b, c) \cap\ \gamma(b, c)$ implies $b \notin \epsilon(A, \gamma, \beta_R)$.

$\{1\} \in \beta_R(c,a) \cap \ \gamma(c,a)$ implies $c \notin \epsilon(A,\gamma,\beta_R)$ implies $a = \epsilon(A,\gamma,\beta_R) = F(R)$ and for any $x \in \{a,c\}, \beta_{R'}(b,x) = \{\{1\}\}$ but $\gamma(b,x) = \{\{2\},\{1,2\}\}$ implies $\beta_{R'}(b,x) \cap \gamma(b,x) = \emptyset$ implies $b \in \epsilon(A,\gamma,\beta_{R'})$.

$\{1\} \in \beta_{R'}(a,c) \cap \ \gamma(a,c)$ implies $a \notin \epsilon(A,\gamma,\beta_{R'})$.

$\{2\} \in \beta_{R'}(c,b) \cap \ \gamma(c,b)$ implies $c \notin \epsilon(A,\gamma,\beta_{R'})$ implies $b = \epsilon(A,\gamma,\beta_{R'}) = F(R')$. Hence we can conclude that F defined on R and R', [2] is (A,γ)-implementable.

From Example 1, we can conclude that there are social choice rules which are not Nash implementable, but $(A,\gamma$)-implementable. However, converse of this holds as well; i.e there are social choice rules which are Nash implementable but not $(A,\gamma$)-implementable [3]. Following example establishes this fact.

Example 2. Let $N = \{1,2\}$, $A = \{a,b,c\}$, R, R' and R'' be as specified below, and the social choice rule F be such that; $F(R) = \{b\}$, $F(R') = F(R'') = \{a\}$.

R		R′		R″	
1	2	1	2	1	2
a	c	c	b	b	c
b	b	a	a	a	a
c	a	b	c	c	b

First let us show that F is Nash implementable. Consider the following mechanism; let $S_1 = S_2 = \{\{a,b\},\{a,c\},\{b,c\}\}, g : S \to A$, where for any $s \in S = S_1 \times S_2$, $g(s) = s_1 \cap s_2$, if there is only one $x \in \ A$ such that $x \in s_1 \cap s_2$, otherwise ties are broken with respect to the first component of first agent's strategy. Note that, for any $s \in S$, there is only one $x \in A$ such that $x \in g(s)$. Now for given R, let $\bar{s} = \ (\{a,b\},\{b,c\})$, $g(\bar{s}) = \{b\}$. For given

[2] We can extend F to the full domain by inducing $F(\tilde{R})$ from the $(A,\gamma$)-equilibria for any given $\tilde{R}$; i.e for any $\tilde{R} \in \mathcal{L}(A)^N$, $F(\tilde{R}) = \epsilon(A,\gamma,\beta_R)$.

[3] In the $(S,\gamma$)-implementation framework one can show that any Nash implementable social choice rule F is (S,γ)-implementable.

$\bar{s}_1 = \{a, b\}$, player 2 should choose either a or b, where bR_2a implies $\forall s_2 \in S_2$, $g(\bar{s})R_2g(\bar{s}_1, s_2)$ implies $\bar{s} \in NE(g, R)$. Moreover it is easy to check $\bar{s}$ is the unique Nash equilibrium of the defined game under R. If one of R' or R'' is given, then we can similarly conclude that $\{a\}$ is the unique Nash equilibrium outcome. Moreover, one can extend F to the full domain by inducing F from the Nash equilibria outcomes of the defined mechanism.

Now let us show that F is not (A, γ)-implementable. Suppose not; i.e. there exists a $\gamma \in \Gamma(A, N)$ such that for any $R \in \mathcal{L}(A)^N$, $F(R) = \epsilon(A, \gamma, \beta_R)$ implies $F(R') = \epsilon(A, \gamma, \beta_{R'}) = \{a\}$ and $\{2\} \in \beta_{R'}(a, b)$ implies $\{2\} \notin \gamma(a, b)$, similarly from $F(R'') = \{a\}$, we get $\{2\} \notin \gamma(a, c)$, with $\{\{2\}\} = \beta_R(a, b) = \beta_R(a, c)$ implies for any $x \in A \setminus \{a\}, \gamma(a, x) \cap \beta_R(a, x) = \emptyset$ implies $\{a\} \in \epsilon(A, \gamma, \beta_R) = F(R)$, contradicting $F(R) = \{b\}$. Hence we can conclude that F is not (A, γ)-implementable.

CHAPTER 3

CHARACTERIZATION OF (A, γ)-IMPLEMENTABILITY

3.1 Pivotal Oligarchic Monotonicity

In order to state our monotonicity condition, first we need to introduce some auxiliary notions.

Definition. For any $R \in \mathcal{L}(A)^N$, and any $(a, b) \in A \times A$, $\mathcal{M}_R(a, b)$ stands for the maximal coalition in the coalition family $\beta_R(a, b)$; i.e $\mathcal{M}_R(a, b) \in \beta_R(a, b)$ and for any $K \in \beta_R(a, b), K \subset \mathcal{M}_R(a, b)$.

Since N is finite we know that; there always exists a unique maximal coalition, possibly empty set, in the coalition family $\beta_R(a, b)$.

Definition. A social choice rule F is said to be monotonic if and only if for any $R, R' \in \mathcal{L}(A)^N$, any $a \in F(R)$ satisfying condition

$$\forall b \in A, \mathcal{M}_{R'}(a, b) \subset \mathcal{M}_R(a, b) \tag{3.1}$$

implies $a \in F(R')$.

Maskin introduced the monotonicity condition in terms of sets consisting alternatives, specified for each agent; here we restate the monotonicity condition by specifying coalitions for each alternative associated with the ones chosen by F.

Definition. For any $(a, b) \in A \times A$, any $K \in 2^N$, K is said to be an (a, b)-oligarchy if and only if for any $R \in \mathcal{L}(A)^N$, $bR_K a$ implies $a \notin F(R)$.

If there is a coalition K such that; b is preferred to a by all the members of K implies a is not chosen by F, then we call K; an a-oligarchy via b or simply an (a, b)-oligarchy.

Definition. For any $R \in \mathcal{L}(A)^N$, any $a \in F(R)$, any $b \in A$, and any $K \in 2^N$, K is said to be a *pivotal (a, b, R) oligarchy* if and only if $\mathcal{M}_R(a, b) \cup K$ is an (a, b)-oligarchy.

Any coalition K is considered as a pivotal coalition for having an (a, b)-oligarchy, if the coalition formed by unification of the largest coalition which prefers b to a under R and K forms an (a, b)-oligarchy.

Definition. For any $R \in \mathcal{L}(A)^N$, any $a \in F(R)$, any $b \in A$, and any $K \in 2^N$, K is said to be a non-pivotal (a, b, R)-oligarchy denoted by $K \in C^{NPO}(a, b, R)$ [$C^{NPO}(a, b, R)$ stands for family of non-pivotal (a, b, R)-oligarchies] if and only if K is not a pivotal (a, b, R) oligarchy. Moreover, K is said to be a maximal non-pivotal (a, b, R)-oligarchy denoted by $K \in C^{MNPO}(a, b, R)$ if and only if $K \in C^{NPO}(a, b, R)$ and there is no $K' \in C^{NPO}(a, b, R)$ such that $K \subset K'$.

Remark 1. Any alternative a, being chosen by F under R indicates that; $\mathcal{M}_R(a, b)$ is not an (a, b) oligarchy, if not clearly a should not be chosen by F, hence we know that $\mathcal{M}_R(a, b)$ is in the family of non-pivotal (a, b, R)-oligarchies, $C^{NPO}(a, b, R)$, and clearly any member of $\mathrm{C}^{MNPO}(a, b, R)$ contains $\mathcal{M}_R(a, b)$.

Definition. (Pivotal oligarchic monotonicity, POM) Any social choice rule F satisfies POM if and only if for any $R, R' \in \mathcal{L}(A)^N$ and any $a \in F(R)$

satisfying condition

$$\forall b \in A, \exists K \in C^{MNPO}(a,b,R) : \mathcal{M}_{R'}(a,b) \subset \mathcal{M}_R(a,b) \cup K \tag{3.2}$$

implies $a \in F(R')$.

Intuitively, POM means that alternative a continues to be chosen by F, unless there is an (a,b)-oligarchy which prefers b to a under R'.

Lemma 1. *Any social choice rule F endowed with POM is monotone.*

Proof. Take any $R, R' \in \mathcal{L}(A)^N$, and $a \in F(R)$, where condition (1) is satisfied. Now for any $b \in A$, $\mathcal{M}_{R'}(a,b) \subset \mathcal{M}_R(a,b)$ implies (3.2) holds, hence $a \in F(R')$. □

3.2 Necessity of POM for (A, γ)-implementability

Lemma 2. *For any (A,γ)-implementable social choice rule F, let γ be a code of rights which implements F, for any $(a,b) \in A \times\ A$, and any $K \in 2^N$ such that $K \neq \emptyset$, we have $K \in \gamma(a,b)$ if and only if K is an (a,b)-oligarchy.*

Proof. ($\Rightarrow$) For any$(a,b) \in A \times\ A$, assume that $\emptyset \neq K \in \gamma(a,b)$. Now $K \in \gamma(a,b)$ implies for any $R \in \mathcal{L}(A)^N$ such that $K \in \beta_R(a,b)$, $K \in \gamma(a,b) \cap \beta_R(a,b)$, and $K \neq \emptyset$ implies $\gamma(a,b) \cap \beta_R(a,b) \neq \emptyset$ hence we get $a \notin \epsilon(A, \gamma, \beta_R)$, now since F is (A,γ)-implementable we get $a \notin F(R)$.

($\Leftarrow$) Assume not; i.e. K is an (a,b)-oligarchy but $K \notin \gamma(a,b)$. Take any R such that for any $i \in N \setminus K$, aR_ib, and bR_Ka; [i.e. $K = \mathcal{M}_R(a,b)$] . Now K is an (a,b)-oligarchy implies $a \notin F(R)$, and F $is (A,\gamma)$-implementable indicates that $a \notin \epsilon(A,\gamma,\beta_R)$ thus, we can conclude that $\exists K' \subset K$ such that $K' \in \gamma(a,b)$ implies $K \in \gamma(a,b)$ contradicting $K \notin \gamma(a,b)$. □

Theorem 1. *Any (A,γ)-implementable social choice rule F satisfies POM.*

Proof. Take any (A, γ)-implementable social choice rule F, any $a \in F(R)$, and any $R, R' \in \mathcal{L}(A)^N$ such that condition (2) holds.

Now condition (2) implies for any $b \in A$, there exists $K \in C^{MNPO}(a, b, R)$ such that $\mathcal{M}_{R'}(a, b) \subset \mathcal{M}_R(a, b) \cup K$ where $\mathcal{M}_R(a, b) \cup K$ is not an (a, b)-oligarchy, hence $\mathcal{M}_{R'}(a, b)$ is not an (a, b)-oligarchy, by the lemma above we get; $\mathcal{M}_{R'}(a, b) \notin \gamma(a, b)$ combined with $\mathcal{M}_{R'}(a, b)$ being maximal implies $\gamma(a, b) \cap \beta_{R'}(a, b) = \emptyset$, so $a \in \epsilon(A, \gamma, \beta_{R'})$ now, F being (A, γ)-implementable implies $a \in F(R')$ hence F satisfies POM. □

3.3 The Implementation Theorem

In this section we state a converse result to Theorem 1. We construct a code of rights to implement a social choice rule F, which is non-empty valued Pareto optimal, and which satisfies pivotal oligarchic monotonicity.

Theorem 2. *Any non-empty valued, Pareto optimal social choice rule F, endowed with POM, is* (A, γ)*-implementable.*

Proof. First let us construct the code of rights, γ such that; for any $(a, b) \in A \times A$, and any $K \in 2^N$, we have $K \in \gamma(a, b)$ if and only if K is an (a, b)-oligarchy. Now, for any $R \in \mathcal{L}(A)^N$, $a \in F(R)$, and $b \in A$; $a \in F(R)$ implies $\mathcal{M}_R(a, b)$ is not an (a, b)-oligarchy indicating that $\mathcal{M}_R(a, b) \notin \gamma(a, b)$, $\mathcal{M}_R(a, b)$ being maximal implies $\gamma(a, b) \cap \beta_R(a, b) = \emptyset$, so $a \in \epsilon(A, \gamma, \beta_R)$. This implies $F(R) \subset \epsilon(A, \gamma, \beta_R)$.

Conversely to show that; $\epsilon(A, \gamma, \beta_R) \subset F(R)$, for any $R \in \mathcal{L}(A)^N$, take any $a \in \epsilon(A, \gamma, \beta_R)$, and assume that $a \notin F(R)$. Now F is non-empty valued implies there exists $b \in A \setminus \{a\}$ such that $b \in F(R)$. Since F is Pareto optimal, there exists $K \in 2^N$ such that $K \neq \emptyset$, and $K \in \beta_R(a, b)$. Assume without loss of generality that $K = \mathcal{M}_R(a, b)$.

Now construct a new preference profile R' such that for any $j \in N \setminus K$,

$\mathrm{L}(R'_j, a) = A$, and for any $c \neq a$, $\mathrm{L}(R'_j, c) \setminus \{a\} = \mathrm{L}(R_j, c) \setminus \{a\}$, moreover let for any $i \in K$, $R'_i = R_i$. We claim that $a \notin F(R')$, suppose not; i.e. $a \in F(R')$. Take any $c \in A$, and consider $\mathcal{M}_{R'}(a, c)$, clearly we have $\mathcal{M}_{R'}(a, c) \subset K$, and$\mathcal{M}_{R'}(a, c) = \mathcal{M}_R(a, c) \cap K$, as $R'_K = R_K$. Let $\bar{K} \in 2^N$ such that $\bar{K} = \mathcal{M}_R(a, c) \cap (N \setminus K)$; i.e. $\bar{K}$ is the maximal subcoalition in $N \setminus K$ which prefers c to a under R, it is clear that $\bar{K} \cup \mathcal{M}_{R'}(a, c) \in \beta_R(a, c)$. Now, $a \in \epsilon(A, \gamma, \beta_R)$ implies $\gamma(a, c) \cap \beta_R(a, c) = \emptyset$ hence $\bar{K} \cup \mathcal{M}_{R'}(a, c) \notin \gamma(a, c)$ implies $\bar{K} \cup \mathcal{M}_{R'}(a, c)$ is not an (a, c)-oligarchy, thus we get $\bar{K}$ is an non-pivotal (a, c, R')-oligarchy. This implies that, there exists $\tilde{K} \in C^{MNPO}(a, c, R')$ such that $\bar{K} \subset \tilde{K}$. Now we have shown that; for any $c \in A$, there exists $\tilde{K} \in C^{MNPO}(a, c, R')$ such that $\mathcal{M}_R(a, c) \subset \mathcal{M}_{R'}(a, c) \cup \tilde{K}$. Thus by POM we can say that $a \in F(R)$, contradicting that $a \notin F(R)$. Hence we can conclude that $a \notin F(R')$.

Let preference profile, R'' be such that for any $j \in N \setminus K$, $R''_j = R'_j$, and for any $i \in K$, $\mathrm{L}(R''_i, a) = A \setminus \{b\}$, and for any $c \in A \setminus \{a, b\}$, $\mathrm{L}(R''_i, c) \setminus \{a, b\} = \mathrm{L}(R'_i, c) \setminus \{a, b\}$. We claim that; $a \notin F(R'')$, assume contrary; i.e. $a \in F(R'')$. Now, take any $c \in A \setminus \{a, b\}$, we have $\mathcal{M}_{R''}(a, c) = \emptyset$. Let $\underline{K}$ be such that $\underline{K} = \mathcal{M}_R(a, c) \cap K$, note that by construction of R' we have; $\mathcal{M}_{R'}(a, c) = \mathcal{M}_R(a, c) \cap K$, and clearly $\underline{K} \in \beta_R(a, c)$. Now $a \in \epsilon(A, \gamma, \beta_R)$ implies $\gamma(a, c) \cap \beta_R(a, c) = \emptyset$ implies $\underline{K} \cup \mathcal{M}_{R''}(a, c) = \underline{K} \cup \emptyset = \underline{K} \notin \gamma(a, c)$ indicating $\underline{K}$ is not an (a, c)-oligarchy, so $\underline{K}$ is an non-pivotal $(a, c, R'')-$oligarchy. This implies there exists $\tilde{K} \in C^{MNPO}(a, c, R'')$ such that $\underline{K} \subset \tilde{K}$. Moreover if $c = b$, we have $\mathcal{M}_{R'}(a, b) = K = \mathcal{M}_{R''}(a, b)$ implies there exists $\tilde{K} \in C^{MNPO}(a, b, R'')$ such that $\emptyset \subset \tilde{K}$. Thus for any $c \in A$, there exists $\tilde{K} \in C^{MNPO}(a, c, R'')$ such that $\mathcal{M}_{R'}(a, c) \subset \mathcal{M}_{R''}(a, c) \cup \tilde{K}$ by POM, implies $a \in F(R')$, contradicting that $a \notin F(R')$. Hence we can conclude $a \notin F(R'')$.

Now we know that; $a \notin F(R'')$ where $K = \mathcal{M}_{R''}(a, b)$; i.e. K is the largest coalition which prefers b to a under R'', moreover for any $\tilde{R} \in \mathcal{L}(A)^N$ such that $b\tilde{R}_K a$, we clearly have; for any $i \in N$, $\mathrm{L}(\tilde{R}_i, a) \subset \mathrm{L}(R_i'', a)$ combined with monotonicity which is known to be implied by POM from Lemma 1 shows that $a \notin F(\tilde{R})$ indicating that K is an (a, b)-oligarchy, thus $K \in \gamma(a, b)$ implies $K \in \gamma(a, b) \cap \beta_R(a, b)$, with $K \neq \emptyset$ we can say that $\gamma(a, b) \cap \beta_R(a, b) \neq \emptyset$, contradicting $a \in \epsilon(A, \gamma, \beta_R)$. Hence we can conclude that; $a \in F(R)$, indicating; $\epsilon(A, \gamma, \beta_R) \subset F(R)$. □

3.4 Implementation of SDRs with a Continuum of Agents

One of the famous impossibility theorems in economics, Muller-Satterthwaite (1977) Theorem tells that in Arrowian framework the unique onto social choice function which satisfies monotonicity condition is dictatoriality. This conclusion leads pessimisim for implementabilty, since monotonicity is a necessary condition for a social choice rule to be Nash-implementable. Although under implementation via code of rights framework the class of implementable social choice rules is extended, our results from the previous chapter indicates that monotonicity preserves its necessity for implementation in this framework. This yields a rather pessimistic conclusion for implementability. On the other hand, for many social choice situations as general elections for assemblies, any particular voter is quite negligible for changing the result of the election. This observation motives to form a notion of social decision rule with a continuum of agents associated with a non-atomic measure space. Following definitions serves for establishing counter part of a social choice rule in such a setting with its most generality. It is worth to observe that, in this framework it can be directly shown that any social decision rule is (A, γ)-implementable.

Suppose $X = [0, 1]$ is the set of agents, and A denotes the non-empty, finite set of alternatives. We will use Σ to denote the Borel σ-algebra on X. Lebesgue measure defined on Σ will be denoted by λ. $\mathcal{F}^+(X,\Sigma,\lambda)$ will stand for the collection of all non-negative, Lebesgue measurable functions defined on Σ. For any $K \in \Sigma$ and any $a, b \in A$, we represent, coalition K prefers b to a under R, by $bR_K a$.

The set of all preference profiles on A is denoted by $\mathcal{L}(A)^\Sigma$.

A complete preorder on A is a complete binary relation defined on A. We will use $l(A)$ to denote the set of all complete orderings on A with generic element $\succeq \in l(A)$.

We define a code of rights similar to the one defined for finite set of agents, as a function γ which maps any pair of alternatives $(a,b) \in A \times A$, to a coalition family in 2^Σ ; i.e $\gamma : A \times A \rightarrow\ 2^\Sigma$, where for any $(a,b) \in A \times A$, and any $K \in \Sigma$, $K \in \gamma(a,b)$ implies that coalition K is given the right to lead a switch from a to b, by the code of rights γ. The only difference in the definition is that we require any coalition to be a member of Borel σ-algebra on $[0,1]$.

For any given preference profile $R \in \mathcal{L}(A)^N$, the benefit function β_R : A $\times$ A$\rightarrow 2^\Sigma$, notions of (A,γ)-equilibrium and $(A,\gamma) - \textit{implementation}$ are defined accordingly.

Definition. A social aggregation rule on X, $\bar{F}$ is described by the collection $\{(f^{a,b}, \alpha^{a,b})\}_{a,b \in A: a \neq b}$ where for any $a, b \in A$ such that $a \neq b$,

$$f^{a,b} \in \mathcal{F}^+(X,\Sigma,\lambda), \int_X f^{a,b} d\lambda \leq 1, \alpha^{a,b} \in [0, \int_X f^{a,b}]. \tag{3.3}$$

$$f^{a,b} = f^{b,a}, \alpha^{b,a} = \int_X f^{a,b} d\lambda - \alpha^{a,b} \tag{3.4}$$

Now, we can define $\bar{F} : \mathcal{L}(A)^\Sigma \rightarrow l(A)$ such that for any $R \in \mathcal{L}(A)^\Sigma$, and any

$a, b \in A$ where $a \neq b$, $b \succ_{\bar{F}(R)} a$ if and only if $\int_{\mathcal{M}_R(a,b)} f^{a,b} d\lambda > \alpha^{a,b}$.

Definition. A social decision rule, $F : \mathcal{L}(A)^{\Sigma} \rightarrow 2^A \setminus \{\emptyset\}$, such that for any $R \in \mathcal{L}(A)^{\Sigma}$, $a \in Aa$ $a \in F(R)$ iff $\forall b \neq a, b \not\succ_{\bar{F}(R)} a$

Note that our definition of a social decision rule is quite general that it neither necessarily needs to be neutral nor anonymous.

Claim. *Any social decision rule F is (A, γ)-implementable.*

Proof. Simply define the code of rights γ such that, for any $a, b \in A$ such that $a \neq b$, any coalition $K \in \Sigma$, $K \in \gamma(a, b)$ if and only if $\int_{K)} f^{a,b} d\lambda > \alpha^{a,b}$. Now, for any $R \in \mathcal{L}(A)^{\Sigma}$, any $a \in A$, $a \in F(R)$ implies that for any $b \neq a$, $\int_{\mathcal{M}_R(a,b)} f^{a,b} d\lambda \leq \alpha^{a,b}$ indicating that for any $K \in \beta_R(a, b), \int_K d\lambda \leq \alpha^{a,b}$, hence $K \notin \gamma(a, b)$ implies that $a \in \epsilon(A, \gamma, \beta_R)$.
Conversely for any $a \in A$, $a \in \epsilon(A, \gamma, \beta_R)$ implies that for any $b \neq a$, $\beta_R(a, b) \cap \gamma(a, b) = \emptyset$. Hence $\mathcal{M}_R(a, b) \notin \gamma(a, b)$ indicating that $\int_{\mathcal{M}_R(a,b)} f^{a,b} d\lambda \leq \alpha^{a,b}$. Hence we can conclude that there does not exist $b \in A$ such that $b \succ_{\bar{F}(R)} a$ indicating that $a \in F(R)$, thus we have $F(R) = \epsilon(A, \gamma, \beta_R)$.

$\square$

CHAPTER 4

SEN'S LIBERAL PARADOX

4.1 (A, γ)-IMPLEMENTATION and SEN'S LIBERAL PARADOX

In this section, we consider Sen's paradox of the Paretian liberal from the (A, γ)-implementation perspective that we have introduced in section 3. We show that; we can design codes of rights that are consistent with Sen's minimal liberalism, and Pareto optimality. Finally we revisit Sen's conclusion of impossibility of a Paretian liberal in terms of (A, γ)-implementability. To establish the desired result we first introduce the familiar definitions used by Sen, under the general framework that is described in section 2.

Definition. Any social choice rule F satisfies *minimal liberalism* if there exist $\{i, j\} \subset N$ such that $i \neq j$, and for any $l \in \{i, j\}$, there exist $x^l, y^l \in A$ such that for any $R \in \mathcal{L}(A)^N$, $x^l R_l y^l$ implies $y^l \notin F(R)$, and respectively $y^l R_l x^l$ implies $x^l \notin F(R)$.

Minimal liberalism implies that there are at least two individuals such that for each of them there are at least a pair of alternatives (x,y) over which he is decisive, that is whenever he prefers x to y, y is not chosen, and respectively whenever he prefers y to x, x is not chosen. In other words any social choice rule F satisfies minimal liberalism if there are at least two individuals $\{i,j\} \subset N$ such that $i \neq j$, where for each of them there are at least a pair of alternatives (x^i,y^i), (x^j,y^j) such that i is an (x^i,y^i)-oligarchy, and j is an (x^j,y^j)-oligarchy. Moreover, let us characterize minimal liberalism in terms of codes of rights.

Definition. Any code of rights γ is said to satisfy minimal liberalism, and denoted by γ^L, if

(3) There exist $\{i,j\} \subset N$ such that $i \neq j$, and for any $l \in \{i,j\}$ there exists $x^l, y^l \in A$ such that for any $K \in 2^N$, $K \in \gamma(x^l, y^l)$ or $K \in \gamma(y^l, x^l)$ if and only if $l \in K$ holds.

Now, let us show that for any social choice rule F, being (A, γ^L)-implementable that is; having code of rights which satisfies minimal liberalism and which implements F, implies F satisfies minimal liberalism.

Lemma 3. *Any (A, γ^L)-implementable social choice rule F satisfies minimal liberalism.*

Proof. Let F be an (A, γ^L)-implementable social choice rule then there is a code of rights,γ, which implements F and satisfies (3) implies there exist $\{i,j\} \subset N$ such that $i \neq j$,and for any $l \in \{i,j\}$ there exist $x^l, y^l \in A$ such that for any $R \in \mathcal{L}(A)^N$ such that $x^l R_l y^l$, $[\{l\} \in \gamma^L(x^l, y^l) \cap \beta_R(x^l, y^l)]$ implies $y^l \notin \epsilon(A, \gamma^L, \beta_R)$, thus $y^l \notin F(R)$ as F is (A, γ^L)-implementable. Similarly for any $R \in \mathcal{L}(A)^N$ such that $y^l R_k x^l, \{l\} \in \gamma^L(y^l, x^l) \cap \beta_R(y^l, x^l)$ implies $x^l \notin \epsilon(A, \gamma, \beta_R)$, so $x^l \notin F(R)$ indicating that F satisfies minimal liberalism. $\square$

Moreover, via Lemma 2 it can easily be shown that; any social choice rule F which is $(A, \gamma\ )$-implementable, and which satisfies minimal liberalism is indeed (A, γ^L)-implementable.

Definition. Any code of rights γ is said to satisfy Pareto optimality, and denoted by γ^P, if for any $a, b \in A$ such that $a \neq b$, $N \in \gamma(a, b)$.

Lemma 4. *Any (A, γ^P)-implementable social choice rule F satisfies Pareto optimality.*

Proof. Assume not; i.e. F is (A, γ^P)-implementable, but F is not Pareto optimal implies there exists $R \in \mathcal{L}(A)^N$, and there exist $a, b \in A$ such that $a \in F(R)$, for any $i \in N$ $\mathrm{bR}_i a$ implies $N \in \beta_R(a, b)$ thus $N \in \beta_R(a, b) \cap \gamma(a, b)$ indicating $a \notin \epsilon(A, \gamma, \beta_R)$ this implies that $a \notin F(R)$ as F is $(A, \gamma$)-implementable, contradicting $a \in F(R)$. □

Now we can state the theorem indicating impossibility of a Paretian liberal, in terms of (A, γ)-implementability.

Theorem 3. *There is no non-empty valued social choice rule F which is (A, γ^{PL})-implementable [i.e implementable by a γ, which satisfies minimal liberalism, and Pareto optimality].*

Proof. Assume not; i.e. there is a non-empty valued social choice rule F such that for any $R \in \mathcal{L}(A)^N$, and F is (A, γ^{PL})-implementable for $N = \{1, 2\}$ implies (3) that is; there exist $x, y, z, w \in A$ such that for any $K \in 2^N$, $K \in \gamma(x, y)$ or $K \in \gamma(y, x)$ if and only if $1 \in K$ and $K \in \gamma(z, w)$ or $K \in \gamma(w, z)$ if and only if $2 \in K$ holds. Now, if $(x, y) = (z, w)$, then let $A = \{x, y\}$, and consider R such that xR_1y, yR_2x, implies $\{1\} \in \beta_R(x, y) \cap \gamma(x, y)$, and $\{2\} \in \beta_R(y, x) \cap \gamma(y, x)$ implies $\epsilon(A, \gamma, \beta_R) = \emptyset$, hence we get$F(R) = \emptyset$, contradicting F being non-empty valued. Assume without loss of generality, $x = z$, and $y \neq w$. Now for $A = \{x, y, w\}$ consider R given below, note that only Pareto optimal outcomes are x, y, this implies $\epsilon(A, \gamma, \beta_R) \subset \{x, y\}$.

R

1	2
x	y
y	w
w	x

However, $\{2\} \in \beta_R(x, w) \cap \gamma(x, w)$ implies $x \notin \epsilon(A, \gamma, \beta_R)$; $\{1\} \in \beta_R(y, x) \cap \gamma(y, x)$ implies $y \notin \epsilon(A, \gamma, \beta_R)$, so $\epsilon(A, \gamma, \beta_R) = \emptyset$, but F is non-empty valued, contradicting F is (A, γ^{PL})-implementable.

Now if x,y,z,w are all distinct then consider R given below, again note that only Pareto optimal outcomes are w, y implies $\epsilon(A, \gamma, \beta_R) \subset \{w, y\}$

R

1	2
w	y
x	z
y	w
z	x

However, $\{2\} \in \beta_R(w, z) \cap \gamma(w, z)$ implies $w \notin \epsilon(A, \gamma, \beta_R)$; $\{1\} \in \beta_R(y, x) \cap \gamma(y, x)$ implies $y \notin \epsilon(A, \gamma, \beta_R)$, thus $\epsilon(A, \gamma, \beta_R) = \emptyset$, but F is non-empty valued, contradicting F is (A, γ^{PL})-implementable. $\square$

4.2 The Manhattan Metric

4.2.1 Preliminaries

For any given alternative set A, social ordering $s \in \mathcal{L}(A)$ and $k \in \{0, 1, ...\frac{n*(n-1)}{2}\}$, $D_k(A, s)$ stands for the set of linear orderings such that any linear ordering, p is contained in this set if and only if the distance between p and s is less than or equal to k, with respect to Manhattan metric; i.e. $D_k(A, s) = \{p \in \mathcal{L}(A) : \delta(p, s) \leq k\}$. Moreover $\mathrm{C}_k(A, s)$ consists of linear orderings over A such that any linear ordering, p is contained in this set if and only if the distance between p and s is equal to k, with respect to Manhattan metric; i.e. $C_k(A, s) = \{p \in \mathcal{L}(A) : \delta(p, s) = k\}$. We can also represent $\mathrm{D}_k(A, s)$ as the union of $\{\mathrm{C}_h(A, s)\}_{h \leq k}$; i.e $\mathrm{D}_k(A, s) = \cup_{h \leq k} \mathrm{C}_h(A, s)$. Moreover for any given set of agents N, $\mathrm{D}_k^N(A, s)$ stands for the collection of linear order profiles where each agent's ordering, R_i belongs to $\mathrm{D}_k(A, s)$. In section 4.3, $\mathrm{D}_k(A, s) is going to be used instead of \mathrm{D}_k^N(A, s)$, where it leads no confusion.

Definition. Let p_1 and p_2 be two linear orderings. Distance between these two orderings with respect to Manhattan metric denoted by, $\delta(p_1, p_2)$, is the minimum number of binary alterations needed to obtain p_2 from p_1.

Remark 2. Note that $\delta(p_1, p_2)$ is equal to the number of pairs $\{a_1, a_2\}$ such that their relative rankings in p_1 and p_2 are different. This observation leads us to conclude that indeed Manhattan metric is equivalent to well known Kemeny metric.

4.2.2 Uniqueness of Center in Socially Centered Domains

Definition. For any $p \in \mathcal{L}(A)$ and $B \subset A$, B is said to be a *block* in p if for any $b \in A$, there exist $a, c \in B$ such that $a \succ_p b \succ_p c$ implies $b \in B$.

Definition. Let p_1 and p_2 be two linear orderings and B be a block in p_1. $T(p_1, p_2, B)$ is defined as a member of $\mathcal{L}(A)$ such that;

$$T(p_1, p_2, B) \mid_{A \setminus B} = p_1 \mid_{A \setminus B}$$

$$T(p_1, p_2, B) \mid_B = p_2 \mid_B$$

$$\forall a \in A \setminus B \ \forall b \in B, \ T(p_1, p_2, B) \mid_{\{a,b\}} = p_1 \mid_{\{a,b\}}$$

holds.

Remark 3. For any p_1 and $p_2 \in \mathcal{L}(A)$ and $B \subset A$ such that B is a block in p_1, by Remark 1 we have;

$$\delta(p_1, p_2) = \delta(p_1, T(p_1, p_2, B)) + \delta(T(p_1, p_2, B), p_2) \tag{4.1}$$

Lemma 5. *For any $p_1, p_2 \in \mathcal{L}(A)$ such that $\delta(p_1, p_2) < \frac{n*(n-1)}{2}$, there is a $p_3 \in \mathcal{L}(A)$ such that;*

$$\delta(p_2, p_3) = 1 \tag{4.2}$$

$$\delta(p_1, p_3) = \delta(p_1, p_2) + 1 \tag{4.3}$$

holds.

Proof. Since $\delta(p_1, p_2) < \frac{n*(n-1)}{2}$ we know that $p_2 \neq p_1^{-1}$ where $\delta(p_1, p_1^{-1}) =$

$\frac{n*(n-1)}{2}$. Hence we can conclude that there exist $a, b \in A$ such that $\{a, b\}$ forms a block in p_2 and if $a \succ_{p_2} b$ then $b \succ_{p_1^{-1}} a$; similarly if $b \succ_{p_2} a$ then $a \succ_{p_1^{-1}} b$. Now, set $B = \{a, b\}$. By our choice of $\{a, b\}$ it is clear that $\delta(p_2, T(p_2, p_1^{-1}, B)) = 1$ as we obtain $T(p_2, p_1^{-1}, B)$ from p_2 only by replacing the positions of a and b. Moreover, by Remark 2,

$$\delta(p_2, p_1^{-1}) = \delta(p_2, T(p_2, p_1^{-1}, B)) + \delta(T(p_2, p_1^{-1}, B), p_1^{-1}) \Rightarrow \delta(T(p_2, p_1^{-1}, B), p_1^{-1})$$
$$= \delta(p_2, p_1^{-1}) - \delta(p_2, T(p_2, p_1^{-1}, B)) = \delta(p_2, p_1^{-1}) - 1 = (\frac{n*(n-1)}{2} - \delta(p_1, p_2)) - 1 \ .$$

Since, $\delta(T(p_2, p_1^{-1}, B), p_1^{-1}) = \frac{n*(n-1)}{2} - \delta(T(p_2, p_1^{-1}, B), p_1)$ we get $\delta(T(p_2, p_1^{-1}, B), p_1) = \frac{n*(n-1)}{2} - ((\frac{n*(n-1)}{2} - \delta(p_1, p_2)) - 1) = \delta(p_1, p_2) + 1$. Now, finally by setting $p_3 = T(p_2, p_1^{-1}, B)$ we can conclude that both 5.2 and 5.3 holds.

□

Theorem 4. *For any $s \in \mathcal{L}(A)$ and $\bar{d} < \frac{n*(n-1)}{2}$, there does not exist $s^{'} \in \mathcal{L}(A)$ such that $s^{'} \neq s$ and $\mathrm{D}_{\bar{d}}(A, s^{'}) = \mathrm{D}_{\bar{d}}(A, s)$*

Proof. For any $s^{'} \in \mathcal{L}(A)$ such that $s^{'} \neq s$, if $\delta(s, s^{'}) = \frac{n*(n-1)}{2}$ then since $\bar{d} < \frac{n*(n-1)}{2}$ we have $s^{'} \notin \mathrm{D}_{\bar{d}}(A, s)$, hence $\mathrm{D}_{\bar{d}}(A, s) \neq \mathrm{D}_{\bar{d}}(A, s^{'})$. Suppose $1 \leq \delta(s, s^{'}) < \frac{n*(n-1)}{2}$. Now, by Lemma 1, we know that there is a $p_1 \in \mathcal{L}(A)$ such that $\delta(s^{'}, p_1) = 1$ and $\delta(s, p_1) = \delta(s, s^{'}) + 1$. If $\delta(s, s^{'}) = \bar{d}$ or $\delta(s, s^{'}) + 1 = \frac{n*(n-1)}{2}$, then we will be done; if not then we have $\delta(s, p_1) < \frac{n*(n-1)}{2}$ hence again by using Lemma 2 we get that there exist $p_2 \in \mathcal{L}(A)$ such that $\delta(p_1, p_2) = 1$ and $\delta(s, p_2) = \delta(s, p_1) + 1 = \delta(s, s^{'}) + 2$. Note that $\delta(s^{'}, p_2) \leq \delta(s^{'}, p_1) + \delta(p_1, p_2) = 2$ by triangle inequality. Thus, if $\delta(s, s^{'}) = \bar{d} - 1$ or $\delta(s, s^{'}) + 2 = \frac{n*(n-1)}{2}$, then we will be done; if not then we have $\delta(s, p_2) < \frac{n*(n-1)}{2}$ again by proceeding similarly after finitely many steps we will reach the conclusion that there exist $\bar{p} \in \mathcal{L}(A)$ such that $\delta(s^{'}, \bar{p}) \leq \bar{d}$ and $\delta(s, \bar{p}) = min\{(\delta(s, \bar{p}) + \bar{d}), \frac{n*(n-1)}{2}\}$. Indicating that $\bar{p} \in \mathrm{D}_{\bar{d}}(A, s^{'})$ but $\bar{p} \notin \mathrm{D}_{\bar{d}}(A, s)$, hence we can conclude that $\mathrm{D}_{\bar{d}}(A, s) \neq \mathrm{D}_{\bar{d}}(A, s^{'})$.

$\square$

4.2.3 On the Density of Socially Centered Domains

Proposition 1. *For any alternative set A where $|A| = n > 1$, any social ordering $s \in \mathcal{L}(A)$, any distance $d \in \{0, 1, ...\frac{n*(n-1)}{2}\}$, and any $x \in A$, the relation;*

$$|\underset{d}{\mathrm{C}}(A, s)| = \sum_{k=0}^{min\{n-1,d\}} |\underset{d-k}{\mathrm{C}}(A \setminus \{x\}, s)| \tag{4.4}$$

holds.

Proof. Clearly for $n = 1$ we have $\frac{n*(n-1)}{2} = 0$, hence $|\mathrm{C}_0(A, s)| = 1$, where simply $s = a$. For any $n > 1$, consider any social ordering $s \in \mathcal{L}(A)$. Assume w.l.o.g. that x is the bottom ranked alternative in s; i.e $\sigma(x, s) = n$. Now, for any $d \in \{0, 1, ...\frac{n*(n-1)}{2}\}$ and any $m \in \{1, ...n\}$, let $P_m(x, d) = \{p \in \mathrm{C}_d(A, s) | \sigma(x, p) = m\}$. It is evident that, $\mathrm{C}_d(A, s) = \cup_{m\in\{1,.....n\}} P_m(x, d)$ where for some $m \in \{1, ...n\}$ it is possible that $P_m(x, d) = \emptyset$.

Claim. *For any $p \in \mathrm{C}_d(A, s)$ and $m \in \{1, ...n\}$, we have $p \in P_m(x, d)$ if and only if $p|_{A\setminus x} \in \mathrm{C}_{d-(n-m)}(A \setminus x, s|_{A\setminus x})$*

Proof. For any $p \in \mathrm{C}_d(A, s)$, firstly suppose $p \in P_m(x, d)$ for some $m \in \{1, ...n\}$. Now, $L(p, x) = \{y \in A | x \succ_p y\}$ clearly forms a block in p, hence by Remark 2 we know that;

$$\delta(p, s) = \delta(p, T(p, s, L(p, x))) + \delta(T(p, s, L(p, x)), s) \tag{4.5}$$

Moreover, $L(p, x) \cup \{x\}$ forms a block in $T(p, s, L(p, x))$, hence we get;

$$\delta(T(p, s, L(p, x)), s) = \delta(T(p, s, L(p, x)), T(T(p, s, L(p, x)), s, L(p, x) \cup \{x\}))$$

$$+\,\delta(T(T(p,s,L(p,x)),s,L(p,x)\cup\{x\}),s)\ \ .$$

Note that by construction, for any $z \in L(p,x)$, $\sigma(z, T(p,s,L(p,x))) = \sigma(z, s|_{L(p,x)})$, thus combined with $\sigma(x,s) = n$, we can conclude that $\delta(T(p,s,L(p,x)), T(T(p,s,L(p,x)),s,$ … $\{x\})) = \sigma(x,s) - \sigma(x,p) = n - m$. Moreover substituting this in the above equation implies;

$$\delta(T(T(p,s,L(p,x)),s,L(p,x)\cup\{x\}),s) = \delta(T(p,s,L(p,x)),s) - (n-m)$$

and by using 4.5,

$$\delta(T(p,s,L(p,x)),s) - (n-m) = \delta(p,s) - \delta(p,T(p,s,L(p,x))) - (n-m)$$

hence by rearranging we get;

$$\delta(T(T(p,s,L(p,x)),s,L(p,x)\cup\{x\}),s) + \delta(p,T(p,s,L(p,x)))$$

$$= \delta(p,s) - (n-m) = d - (n-m) \quad as \quad p \in \underset{d}{\mathrm{C}}(A,s)\ \ . \tag{4.6}$$

Now, consider $\delta(p|_{A\backslash x}, s|_{A\backslash x})$, by construction we know $x \notin L(p,x)$, so $L(p,x)$ continues to form a block in $p|_{A\backslash x}$. Now, by applying 4.5 for $p|_{A\backslash x}$ and $s|_{A\backslash x}$ we get that;

$$\delta(p|_{A\backslash x}, s|_{A\backslash x}) = \delta(p|_{A\backslash x}, T(p|_{A\backslash x}, s|_{A\backslash x}, L(p,x))) + \delta(T(p|_{A\backslash x}, s|_{A\backslash x}, L(p,x)), s|_{A\backslash x}) \tag{4.7}$$

In addition as $\sigma(x,s) = n = \sigma(x, T(T(p,s,L(p,x)),s,L(p,x)\cup\{x\}))$ we have

$$\delta(T(T(p,s,L(p,x)),s,L(p,x)\cup\{x\}),s) = \delta(T(p|_{A\backslash x}, s|_{A\backslash x}, L(p,x)), s|_{A\backslash x}) \tag{4.8}$$

Since $\sigma(x,p) = \sigma(x, T(p,s,L(p,x)))$ and clearly $L(x,p) = L(x, T(p,s,L(p,x)))$ we get;

$$\delta(p, T(p,s,L(p,x))) = \delta(p|_{A\backslash x}, T(p|_{A\backslash x}, s|_{A\backslash x}, L(p,x))) \tag{4.9}$$

Finally, by substituting 4.8 and 4.9 in 4.7 and by using the relation in 4.6 we can conclude that;

$$\delta(p|_{A\backslash x}, s|_{A\backslash x}) = d - (n-m)$$

indicating that $p|_{A\backslash x} \in \mathrm{C}_{d-(n-m)}(A, s|_{A\backslash x})$.

Conversely, suppose $p|_{A\backslash x} \in \mathrm{C}_{d-(n-m)}(A, s|_{A\backslash x})$ this combined with equations 4.7,4.8 and 4.9 implies that;

$$\begin{aligned} d-(n-m) = \delta(p|_{A\backslash x}, s|_{A\backslash x}) &= \delta(p, T(p,s,L(p,x))) \\ &+ \delta(T(T(p,s,L(p,x)), s, L(p,x)\cup\{x\}), s)\ . \end{aligned} \tag{4.10}$$

Moreover, by adding $\delta(T(p,s,L(p,x)), T(T(p,s,L(p,x)), s, L(p,x)\cup\{x\}))$ to both sides of equation, which we have shown to be equal to $n-m$, we get;

$$\begin{aligned} d = \delta(p, T(p,s,L(p,x))) &+ \delta(T(p,s,L(p,x)), T(T(p,s,L(p,x)), s, L(p,x)\cup\{x\})) \\ &+ \delta(T(T(p,s,L(p,x)), s, L(p,x)\cup\{x\}), s) = \delta(p,s) \end{aligned} \tag{4.11}$$

Thus, we can conclude that $p \in \mathrm{C}_d(A,s)$. $\square$

Now, note that for any $m \in \{1, ...n\}$, and any $p, q \in P_m(x,d)$, $\sigma(x,p) = \sigma(x,q) = m$ implies that $p \neq q$ if and only if $p|_{A\backslash x} \neq q|_{A\backslash x}$, indicating that $P_m(x,d)$ and $\mathrm{C}_{d-(n-m)}(A\setminus x, s|_{A\backslash x})$ are in one to one correspondence, combined with $\mathrm{C}_d(A,s) = \cup_{m\in\{1,.....n\}} P_m(x,d)$ indicates that;

$$|\underset{d}{\mathrm{C}}(A,s)| = \sum_{m=1}^{n} |P_m(x,d)| = \sum_{m=1}^{n} |\underset{d-(n-m)}{\mathrm{C}}(A\setminus x, s|_{A\backslash x})| = \sum_{k=0}^{n-1} |\underset{d-k}{\mathrm{C}}(A\setminus x, s|_{A\backslash x})|\ .$$

Since for any $k > d$, we have $d - k < 0$ indicates that $|\,\mathrm{C}_{d-k}(A, s)| = 0$, hence from above expression we finally get;

$$|\underset{d}{\mathrm{C}}(A, s)| = \sum_{k=0}^{min\{n-1,d\}} |\underset{d-k}{\mathrm{C}}(A \setminus x, s|_{A \setminus x})|$$

$\square$

Following table demonstrates the cardinalities of $\mathrm{C}_d(A, s)$ for $n \in \{1, 2, 3, 4, 5\}$ and for corresponding distances.

$n \setminus d$	0	1	2	3	4	5	6	7	8	9	10
1	1										
2	1	1									
3	1	2	2	1							
4	1	3	5	6	5	3	1				
5	1	4	9	15	20	22	20	15	9	4	1

4.3 Paretian Liberal in Socially Centered Domains

Following example shows that when $|A| = 4$, for any social ordering s, there is a social choice function F defined on $\mathrm{D}_3(A, s)$ which satisfies Pareto optimality and minimal liberalism.

Example 3. Let $A = \{a, b, c, d\}$, and $N = \{1, 2\}$. Assume w.l.o.g that $a \succ_s b \succ_s c \succ_s d$, and let $F : \mathrm{D}_3(A, s) \to A$ be such that for any $R \in \mathrm{D}_3(A, s)$;

$$F(R) \in \{T(R_1), T(R_2)\} \setminus \{c, d\} \text{ if } \{T(R_1), T(R_2)\} \setminus \{c, d\} \neq \emptyset$$

$$F(R) = c \text{ if } \{T(R_1), T(R_2)\} = \{c\}$$

$$F(R) = d \text{ if } \{T(R_1), T(R_2)\} = \{d\}$$

$$F(R) = a, \text{ otherwise.}$$

Where, $T(R_i$ stands for the top ranked alternative in the ordering of i^{th} individiual under preference profile R.It is clear that in any of the four cases, F is a well-defined function on $\mathrm{D}_3(A, s)$. Moreover in any of the first three cases $F(R) \in \{T(R_1), T(R_2)\}$ indicates that $F(R) \in PO(R)$ Now, we need to show that, if none of the first three cases hold, then a is Pareto optimal. Since we have none of the first three cases we can suppose that w.l.o.g that $c = T(R_1)$ and $d = T(R_2)$. Now, $R_2 \in \mathrm{D}_3(A, s)$ implies that we have $d \succ_{R_2} a \succ_{R_2} b \succ_{R_2} c$, moreover since $R_1 \in \mathrm{D}_3(A, s)$ we have $a \succ_{R_1} d$, hence a can not be Pareto dominated by any other alternative. Thus we can conclude that F satisfies Pareto optimality.
For any $x \in \{c, d\}, R \in \mathrm{D}_3(A, s), x = F(R)$ if and only if $x = T(R_1)$ and $x = T(R_2)$ implies that both agent 1 and agent 2 have liberal rights over (c, d) pair.

Proposition 2. *For $|A| = n \geq 5$, for any $s \in \mathcal{L}(A)$ and $k \in \{0, 1, ...\frac{n*(n-1)}{2}\}$ if $k \geq 3n - 9$, then there is no social choice function $F : D_k(A, s) \rightarrow A$ which satisfies minimal liberalism and Pareto optimality.*

Proof.

Case 1. Suppose both agents 1 and 2 have minimal rights over the pair of alternatives (a, b). Assume w.l.o.g that $a \succ_s b$ and consider the following contradictory profile where $R_1|_{A\setminus\{a,b\}} = s|_{A\setminus\{a,b\}}$ and similarly $R_2|_{A\setminus\{b\}} = s|_{A\setminus\{b\}}$.

R

1	2
a	b
b	.
.	.

Now, $\delta(s, R_1) = [\sigma(a, s) - 1] + [\sigma(b, s) - 2] \leq 2n - 4 \leq 3n - 9$ for $n \geq 5$ and $\delta(s, R_2) = \sigma(b, s) - 1 \leq n - 1 \leq 3n - 9$ for $n \geq 5$. Hence we get contradictory profile $R \in \mathrm{D}_{3n-9}(A, s)$.

Case 2. Suppose agent 1 has minimal rights over (a, b)-pair and agent 2 has minimal rights over the (b, c)-pair. Consider the following two contradictory profiles where for any $i \in \{1, 2\}$, $R_i|_{A\setminus\{a,b,c\}} = s|_{A\setminus\{a,b,c\}}$ and similarly $R_i^{'}|_{A\setminus\{a,b,c\}} = s|_{A\setminus\{a,b,c\}}$.

R

1	2
c	b
.	c
.	.
a	.
b	a
.	.

R'

1	2
a	b
.	a
.	.
c	.
b	c
.	.

Now, let $[s]$ stands for the class of linear orderings such that for any $x \in A \setminus \{a, c\}$ we have $x \succ_s a \succ_s c$ for any $s \in [s]$. Similarly let $[s^{'}]$ stands for the class of linear orderings such that for any $x \in A \setminus \{c, b\}$ we have $x \succ_{s'} c \succ_{s'} b$ for any $s^{'} \in [s^{'}]$. Form of linear orderings belonging to $[s]$ and $[s^{'}]$ are depicted below.

s	$s^{'}$
.	.
.	.
.	.
.	.
a	c
c	b

It is easy to see that for any $p \in \mathcal{L}(A) \setminus [s], \delta(R_1, p) < \delta(R_1, s)$ for any $s \in [s]$ and similarly for any $p \in \mathcal{L}(A) \setminus [s^{'}], \delta(R_2, p) < \delta(R_2, s^{'})$ for any $s^{'} \in [s^{'}]$. Note that, $\delta(R_1, s) = (n-1) + (n-2) = 2n-3 > 3n-9$ for $n = 5$ and $\delta(R_2, s^{'}) = (n-1) + (n-2) = 2n-3 > 3n-9$ for $n = 5$. However, for any $p \in \mathcal{L}(A) \setminus [s], \delta(R_1, p) < \delta(R_1, s) = 2n-3$ implies that $\delta(R_1, p) \leq 2n-4 \leq 3n-9$ for $n \geq 5$ and similarly for any $p \in \mathcal{L}(A) \setminus [s^{'}], \delta(R_2, p) < \delta(R_2, s^{'}) = 2n-3$ implies that $\delta(R_2, p) \leq 2n-4 \leq 3n-9$ for $n \geq 5$. Thus we can conclude that for any $p \in \mathcal{L}(A) \setminus ([s] \cup [s^{'}]), R \in \mathrm{D}_{3n-9}(A, p)$. Moreover it can easily be checked that, for any $s \in [s]$ and $i \in \{1, 2\}$, $\delta(R_i^{'}, s) \leq 3n-9$ for $n \geq 5$ and $\delta(R_i, s^{'}) \leq 3n-9$ for $n \geq 5$. This indicates that, for any $p \in [s] \cup [s^{'}], R^{'} \in \mathrm{D}_{3n-9}(A, p)$. Now, we can conclude that for any $p \in \mathcal{L}(A)$ we have $R \in \mathrm{D}_{3n-9}(A, p)$ or $R^{'} \in \mathrm{D}_{3n-9}(A, p)$ where R and $R^{'}$ are contradictory profiles.

Case 3. Suppose agent 1 has minimal rights over (a, b)-pair and agent 2 has minimal rights over the (c, d)-pair where a, b, c, d are all different than each other. Now consider the following three $\{a, b, c, d\}$ restricted social orderings;

p_1	p_2	p_3
a	a	a
d	c	b
c	b	c
b	d	d

Observe that from these 3 orderings by replacing (a, b) we can get 6 different orderings, further by replacing (c, d) we can get 12 different orderings and finally by replacing the positions of (a, b)-pair and (c, d)-pair we can get all 24 possible different $\{a, b, c, d\}$ restricted orderings. Now, consider the following two contradictory profiles;

R

1	2
d	a
b	.
a	c
.	d
c	b
.	.

R'

1	2
c	a
b	.
a	d
.	c
d	b
.	.

Let for any $k \in \{1, 2, 3\}$, $[s_k] = \{s \in \mathcal{L}(A) | s|_{a,b,c,d} = p_1\}$. Now, for any $s \in [s_1]$, $\delta(s, R_1) = [\sigma(d, s) - 1] + [\sigma(b, s^{'}) - 2] + [\sigma(a, s^{''} - 3)] \leq (n - 3) + (n - 2) + (n - 4) = 3n - 9$ where $s^{'}$ stands for the linear ordering such that $\sigma(d, s^{'}) = \sigma(d, R_1)$ and $s^{'}|_{A \setminus \{d\}} = s|_{A \setminus \{d\}}$, similarly $s^{''}$ is such that $\sigma(b, s^{''}) = \sigma(b, R_1)$ and $s^{''}|_{A \setminus \{b\}} = s^{'}|_{A \setminus \{b\}}$. By calculating the distances similarly we get $\delta(s, R_2) \leq 2n - 7 < 3n - 9$ for $n \geq 5$. Hence we can conclude that contradictory profile $R \in \mathrm{D}_{3n-9}(A, s)$.

Moreover, for any $s \in [s_2]$, $\delta(s, R_1^{'}) \leq 3n - 10 < 3n - 9$ and $\delta(s, R_2^{'}) \leq 3n - 11 < 3n - 9$, thus contradictory profile $R^{'} \in \mathrm{D}_{3n-9}(A, s)$. Finally for any $s \in [s_3]$, $\delta(s, R_1^{'}) \leq 3n - 9$ and $\delta(s, R_2^{'}) \leq 3n - 9$ indicating that contradictory profile $R^{'} \in \mathrm{D}_{3n-9}(A, s)$. Now, for any $s \in \mathcal{L}(A)$ such that s belongs neither of $[s_1], [s_2], [s_3]$ we know that s can be obtained from these classes by making the necessary replacements among $\{a, b, c, d\}$ and also by making the same replacements over R or $R^{'}$ one gets the associated contradictory profile which belongs to $\mathrm{D}_{3n-9}(A, s)$. Thus we can conclude that for any $s \in \mathcal{L}(A)$ a contradictory profile in the form of R or R' belongs to $\mathrm{D}_{3n-9}(A, s)$.

These three cases show that regardless of how we define the minimal rights of agents 1 and 2, we can reach to a contradictory profile R such that $R \in \mathrm{D}_{3n-9}(A, s)$. Hence, we can conclude that there is no social choice function $F : \mathrm{D}_{3n-9}(A, s) \rightarrow 2^A$ which satisfies minimal liberalism and Pareto optimality.

$\square$

CHAPTER 5

CONCLUSION

In this thesis, we introduced the notion of (A, γ)-implementation, and provided a characterization in terms of Pareto optimality, and pivotal oligarchic monotonicity.(A, γ)-implementation differs from classical implementation mainly in two respects: (i) In (A, γ)-implementation, we explicitly specify a rights structure among the members of the society, which is independent of their preferences, where outcomes are determined as a result of this rights structure and preferences. (ii) In classical implementation we deal with general strategy sets whereas in (A, γ)-implementation we choose the strategy set being equivalent to the alternative set, which leads to a rather simple framework.

Our work in this thesis also paves the way for the analysis of (S, γ)-implementation, and its characterization. Moreover, identifying the relation between implementation under other solution concepts, and (A, γ)-implementation are other subjects for further research.

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