

The Best Gain-Loss Ratio is a Poor Performance Measure*

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Abstract. The gain-loss ratio is known to enjoy very good properties from a normative point of view. As a confirmation, we show that the best market gain-loss ratio in the presence of a random endowment is an acceptability index, and we provide its dual representation for general probability spaces. However, the gain-loss ratio was designed for finite Ω and works best in that case. For general Ω and in most continuous time models, the best gain-loss is either infinite or fails to be attained. In addition, it displays an odd behavior due to the scale invariance property, which does not seem desirable in this context. Such weaknesses definitely prove that the (best) gain-loss is a *poor* performance measure.

Key words. gain-loss ratio, acceptability indexes, incomplete markets, martingales, quasi-concave optimization, duality methods, market modified risk measures

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1. Introduction. The gain-loss ratio was introduced by Bernardo and Ledoit [2] to provide an alternative to the classic Sharpe ratio (SR) in portfolio performance evaluation. Cochrane and Saa-Requejo [9] call portfolios with high SR “good deals.” These opportunities should, informally speaking, be regarded as quasi arbitrages and therefore should be ruled out. Ruling out good deals, or equivalently restricting SR, produces in turn restrictions on pricing kernels. Restricted pricing kernels are desirable since they provide narrower lower and upper price intervals for contingent claims in comparison to arbitrage free price intervals. This criterion is based on the assumption that a high SR is attractive and a low SR is not. The SR criterion works well in a Gaussian returns context, but in general it does not since it is incompatible with no-arbitrage. In fact a positive gain with finite first moment but infinite variance has zero SR, but it is very attractive as it is an arbitrage. The SR has another drawback: it is not monotone and thus violates a basic axiom in theory of choice. To remedy the aforementioned shortcomings of the SR, Bernardo and Ledoit proposed as performance measure the gain-loss ratio

$$\alpha(X) = \frac{E[X^+]}{E[X^-]},$$

where the expectation is taken under the historical probability measure P . The gain-loss ratio α is well defined on nonnull payoffs X as soon as X^+ or X^- are integrable; it has an intuitive significance and is easy to compute. It also enjoys many properties: monotonicity across X s;

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scale invariance, that is, $\alpha(cX) = \alpha(X)$ for all $c > 0$; law invariance, as two payoffs with the same distribution have the same α ; and a classic continuity property (Fatou property). Restricted to portfolios with positive expectation, it becomes a quasi-concave map, consistent with second order stochastic dominance, as shown by Cherny and Madan in [8], and is thus an *acceptability index* in their terminology.

Let α^* denote the best gain-loss ratio from the market, i.e., from the set \mathcal{X} of nontrivial, *discounted* portfolio gains with finite first moment

$$\alpha^* := \sup_{X \in \mathcal{X}, X \neq 0} \alpha(X).$$

In the case P is already a pricing kernel, $\alpha^* = 1$ as $E[X] = E[X^+ - X^-] = 0$ for all gains. This gives a flavor of the main result by Bernardo and Ledito, which is the equivalence between

- (i) $\alpha^* < +\infty$,
- (ii) existence of pricing kernels with state price density Z satisfying $c \leq Z \leq C$ for some constants $C, c > 0$.

That is, restrictions on the best gain-loss ratio are equivalent to the existence of special, restricted pricing kernels bounded and bounded away from 0. Bernardo and Ledito also prove a duality formula for α^* ,

$$\alpha^* = \min_Z \frac{\text{ess sup } Z}{\text{ess inf } Z},$$

where Z varies over all the pricing kernels as in item (ii) above. Though stated for a general probability space and in a biperiodal market model, Bernardo and Ledito's derivation is correct only if Ω is finite. In fact, what they actually show is

$$\alpha^* = \max_{X \in \mathcal{X}, X \neq 0} \alpha(X) = \min_Z \frac{\text{ess sup } Z}{\text{ess inf } Z},$$

i.e., that the best ratio is always attained. This is true only if Ω is finite.

Against this background, the present paper develops an analysis of the gain-loss ratio for general probability spaces. The rest of the paper is organized as follows. In section 2 we show the above equivalence (i) \iff (ii) in the presence of a continuous time market for general Ω . The duality technique employed here extends also the treatment by Pinar, Altay-Salih, and Camci [14] and Pinar [15]. The assumptions made on the market model are quite general, as we do not require the underlyings process S to be either a continuous diffusion or locally bounded.

The duality formula for α^* is correctly reformulated as $\sup \dots = \min \dots$ in Theorem 2.6, and a simple counterexample where the supremum α^* , though finite, is not attained is provided in section 2.4.

In section 2.3 pros and cons of the best gain-loss ratio are discussed. While in discrete time models there is a full characterization of models with finite best gain-loss ratio, in continuous time the situation is hopeless. In most commonly used models, $\alpha^* = +\infty$ as any pricing kernel is unbounded as shown in details for the Black-Scholes model in Example 2.9. Finally, in section 3 we analyze the best gain-loss ratio $\alpha^*(B)$ in the presence of a random endowment B . In section 3.1 $\alpha^*(B)$ is shown to be an acceptability index on integrable payoffs, according to the definition given by Biagini and Bion-Nadal [4]. There we briefly highlight the difference

between the notions of acceptability index as given in [8] and [4], and we motivate the reason why the choice made by [4] is preferable here. Then, in section 3.2 we prove an extension of Theorem 2.6 in the presence of B and we provide a dual representation for $\alpha^*(B)$. Section 3.3 concludes by pointing out other gain-loss drawbacks when an endowment is present, which prove that the (best) gain-loss is a poor performance measure.

2. The market best gain-loss α^* and its dual representation.

2.1. The market model. Let $(\Omega, (\mathcal{F}_t)_{t \in [0, T]}, P)$ be a continuous time stochastic basis satisfying the usual assumptions. S is an \mathbb{R}^d -valued semimartingale on this basis and models the (discounted) time evolution of d underlyings up to the finite horizon T . A strategy ξ is a predictable, S -integrable process and the stochastic integral $\xi \cdot S$ is the corresponding gain process. Now, some integrability condition must be imposed on S in order to ensure the presence of strategies ξ with well-defined gain-loss ratio. In some cases in fact it may happen that every nonnull terminal gain $K = \xi \cdot S_T$ verifies $E[K^+] = E[K^-] = +\infty$; see section 2.4 for a simple one period model of such an extreme situation.

The following is thus the integrability assumption on S which holds throughout the paper.

Assumption 2.1. Let $S_T^* = \sup_{t \leq T} |S_t|$ denote the maximal functional at T . Then $S_T^* \in L^1(P)$.

Note that S_T^* coincides with the running maximum at the terminal date T if S is nonnegative. This assumption is verified in many models used in practice:

- If time is discrete with finite horizon, or equivalently, S is a pure jump process with jumps occurring only at fixed dates t_1, \dots, t_n , the assumption is equivalent to $S_{t_i} \in L^1(P)$ for all t_i ;
- If S is a Lévy process, the assumption is equivalent to the integrability of S_T only (or of S_t at any fixed $0 < t \leq T$). This is a particular case of a more general result on moments of Lévy process; see [19, section 5.25, Theorem 5.25.18].

Therefore, at least in normal market conditions Assumption 2.1 is quite reasonable. From a strict mathematical perspective it ensures that the gains processes are true (and not local) martingales under bounded pricing kernels. The admissible strategies we consider are the linear space $\Xi = \{\xi \mid \xi \text{ is simple, predictable, and bounded}\}$, i.e., those ξ which may be written as $\sum_{i=1}^{n-1} H_i \mathbb{1}_{] \tau_i, \tau_{i+1}]}$ for some stopping times $0 \leq \tau_1 < \dots < \tau_n \leq T$ with H_i bounded and \mathcal{F}_{τ_i} -measurable. These strategies represent the set of buy-and-hold strategies on S over finitely many trading dates. The set of terminal admissible gains, which are replicable at zero cost via a simple strategy, is thus the linear space

$$\mathcal{K} = \{K \mid K = \xi \cdot S_T \text{ for some } \xi \in \Xi\}.$$

Thanks to Assumption 2.1, $\mathcal{K} \subseteq L^1(P)$. Note that $\xi = \mathbb{1}_A \mathbb{1}_{]s, t]}$ and its opposite $-\xi$ are in Ξ for all $A \in \mathcal{F}_s$ and for all $0 \leq s < t \leq T$, so that $K = \mathbb{1}_A(S_t - S_s)$ and $-K$ are in \mathcal{K} .

The best gain-loss in the above market is then

$$\alpha^* := \sup_{K \in \mathcal{K}, K \neq 0} \alpha(K).$$

The best gain-loss α^* is always greater or equal to 1, and it is equal to 1 if and only if (iff) P is already a martingale measure for S . These facts can be easily proved using the linearity of \mathcal{K} and the above observation $\pm \mathbb{1}_A \mathbb{1}_{]s, t]} \in \Xi$.

2.2. No λ gain-loss, its dual characterization, and the duality formula for α^* . The market best gain-loss α^* is the value of a nonstandard optimization problem. In fact, the gain-loss ratio α is not concave, and not even quasi-concave on $L^1(P)$. However, when restricted to variables with nonnegative expectation it becomes quasi-concave, as shown in detail by [8]. Since the optimization can be restricted to gains with nonnegative expectations without loss of generality, in the end α^* can be seen as the optimal value of a quasi-concave problem.

To characterize α^* and to link it to a no-arbitrage type result, we rely on a parametric family of auxiliary utility maximization problems with piecewise linear utility U_λ :

$$U_\lambda(x) = x^+ - \lambda x^-, \quad \lambda \geq 1.$$

The convex conjugate of U_λ , $V_\lambda(y) = \sup_x (U_\lambda(x) - xy)$ is the functional indicator of the interval $[1, \lambda]$:

$$V_\lambda(y) = \begin{cases} 0 & \text{if } 1 \leq y \leq \lambda, \\ +\infty & \text{otherwise.} \end{cases}$$

By mere definition of the conjugate, the Fenchel inequality holds:

$$(1) \quad U_\lambda(x) - xy \leq V_\lambda(y) \quad \forall x, y \in \mathbb{R}.$$

Definition 2.2. Fix $\lambda \in [1, +\infty)$. Then the set of probabilities \mathcal{Q}_λ which have finite V_λ entropy is

$$\mathcal{Q}_\lambda := \{Q \text{ probab.}, Q \ll P \mid \exists y > 0, E \left[V_\lambda \left(y \frac{dQ}{dP} \right) \right] < +\infty\}.$$

Remark 2.3. The set \mathcal{Q}_λ is not empty, as $\mathcal{Q}_1 = \{P\}$ and $P \in \mathcal{Q}_\lambda$ for all $\lambda \geq 1$. It is also easy to check that \mathcal{Q}_λ is convex and the family $(\mathcal{Q}_\lambda)_{\lambda \geq 1}$ is nondecreasing in the parameter. With the usual convention $\frac{c}{0} = +\infty$ for $c > 0$, $\mathcal{Q}_\lambda = \{Q \text{ probab.}, Q \ll P \mid \frac{\text{ess sup } \frac{dQ}{dP}}{\text{ess inf } \frac{dQ}{dP}} \leq \lambda\}$.

The next definition is understood as follows. The market is gain-loss free at a certain level $\lambda > 1$ if not only there is no gain with $\alpha \geq \lambda$, but also λ cannot be approximated arbitrarily well with gains in \mathcal{K} .

Definition 2.4. For a given $\lambda \in (1, +\infty)$, the market is λ gain-loss free if $\alpha^* < \lambda$.

Theorem 2.6 below, first shown by Bernardo and Ledoit in a two periods setup, states the equivalence between absence of λ gain-losses and existence of a martingale measure whose density satisfies precise bounds.

Some notation first. Let $\mathcal{C} = \{X \in L^1 \mid X \leq K \text{ for some } K \in \mathcal{K}\}$ denote the set (convex cone) of claims which are super replicable at zero cost, and consider its polar set $\mathcal{C}^0 = \{Z \in L^\infty \mid E[ZX] \leq 0 \text{ for all } X \in \mathcal{C}\}$. As $\mathcal{C} \supseteq -L^1_+$, $\mathcal{C}^0 \subseteq L^1_+$. \mathcal{C}^0 is a convex cone and thus not empty as $0 \in \mathcal{C}^0$.

However, \mathcal{C}^0 may be trivially $\{0\}$, i.e., its basis $\mathcal{C}^0_1 = \{Z \in \mathcal{C}^0 \mid E[Z] = 1\}$ may be empty. This may happen in common models such as the Black–Scholes model; see Remark 2.3 and Example 2.9 for a discussion and more details. The basis \mathcal{C}^0_1 however is important for gain-loss analysis. The following Lemma in fact proves that \mathcal{C}^0_1 is the set of bounded martingale probability densities, which in turn appear in the characterization of the market best gain-loss in Theorem 2.6.

Lemma 2.5. $Z \in \mathcal{C}_1^0$ iff it is a bounded martingale density.

Proof. If $Z \in \mathcal{C}_1^0$, it is bounded nonnegative and integrates to 1, so it is a probability density of a $Q \ll P$. Moreover, $\pm \mathbb{1}_A(S_t - S_s) \in \mathcal{C}$ for all $A \in \mathcal{F}_s, s < t$, so that $E[Z \mathbb{1}_A(S_t - S_s)] = 0$, which precisely means $E_Q[S_t | \mathcal{F}_s] = S_s$. Conversely, if Q is a martingale probability for S with bounded density Z , then

$$S_T^* \in L^1(P) \subseteq L^1(Q).$$

As S_T^* is Q -integrable and ξ is bounded, the integral $\xi \cdot S$ has maximal functional $(\xi \cdot S)_T^* \in L^1(Q)$ and is thus a martingale of class $\mathcal{H}^1(Q)$; see [16, Chapter IV, section 4]. Now, if $K \in \mathcal{C}$ by definition it can be super replicated at zero cost: $K \leq \xi \cdot S_T$ for some ξ , whence

$$E[ZK] = E_Q[K] \leq E_Q[\xi \cdot S_T] = 0.$$

The above inequality implies $Z \in \mathcal{C}^0$. ■

Theorem 2.6. *The following conditions are equivalent:*

- (a) *The market is λ gain-loss free.*
- (b) *There exists an (equivalent) martingale probability Q such that*

$$(2) \quad \frac{\text{ess sup } \frac{dQ}{dP}}{\text{ess inf } \frac{dQ}{dP}} < \lambda.$$

In case any of the two conditions above holds, the market best gain-loss α^ admits a dual representation as*

$$(3) \quad \alpha^* = \min_{Q \in \mathcal{M}_\infty} \frac{\text{ess sup } \frac{dQ}{dP}}{\text{ess inf } \frac{dQ}{dP}}$$

in which \mathcal{M}_∞ is the set of equivalent martingale probabilities Q with densities $Z \in \mathcal{C}_1^0$ which are (bounded and) bounded away from 0, i.e., $\{Z \in \mathcal{C}_1^0 \mid Z > c \text{ for some } c > 0\}$.

The equivalence will be proved by duality methods via the auxiliary utility maximization problem

$$u_\mu := \sup_{K \in \mathcal{K}} E[U_\mu(K)].$$

The reason is that $u_\mu < +\infty$ is equivalent to $\alpha^* \leq \mu$. In fact, the functional $E[U_\mu(K)] = E[K^+ - \mu K^-]$ is positively homogeneous so that

$$u_\mu < +\infty \Leftrightarrow u_\mu = 0,$$

and the latter condition in turn is equivalent to $\alpha^* \leq \mu$ because $0 \in \mathcal{K}$.

Before starting the proof, recall also that the Fenchel pointwise inequality (1) gives for any random variable Y

$$U_\mu(K) - KY \leq V_\mu(Y).$$

Proof of Theorem 2.6. (b) \Rightarrow (a). If there exists a Q with the stated properties, its density Z belongs to \mathcal{C}_1^0 by Lemma 2.5. Set $Y = \frac{Z}{\text{ess inf } Z} \in \mathcal{C}^0$. As $1 \leq Y \leq \frac{\text{ess sup } Z}{\text{ess inf } Z} := \mu < \lambda$, $V_\mu(Y) = 0$ and thus for all K the Fenchel inequality simply reads as $U_\mu(K) - KY \leq 0$.

Taking expectations, $E[U_\mu(K)] \leq 0$ for all $K \in \mathcal{K}$, which is in turn equivalent to $u_\mu = 0$ and to $\alpha^* \leq \mu < \lambda$.

(a) \Rightarrow (b). Set $\mu = \alpha^*$. Then $u_\mu = 0$. The existence of a Q is now a standard duality instance. Note that U_μ is monotone, so $u_\mu = \sup_{K \in \mathcal{C}} E[U_\mu(K)]$. Also, the monotone concave functional $E[U_\mu(\cdot)]$ is finite and thus continuous on L^1 by the extended Namioka theorem (see [5], [13]). Therefore the Fenchel duality theorem applies (see, e.g., [6, Theorem I.11] or [3] for a survey of duality techniques in the utility maximization problem) and gives the formula

$$u_\mu = \min_{Y \in \mathcal{C}^0} E[V_\mu(Y)].$$

In particular the infimum in the dual is obtained by a $Y^* \in \mathcal{C}^0$. Therefore $1 \leq Y^* \leq \mu = \alpha^* < \lambda$, and its scaling $Z^* = Y^*/E[Y^*]$ is a martingale density with the property required in (2).

Suppose now any of the two conditions above holds true. Then, the proof of the arrow (b) \Rightarrow (a) actually shows

$$(4) \quad \alpha^* = \sup_{K \in \mathcal{K}, K \neq 0} \frac{E[K^+]}{E[K^-]} \leq \inf_{Q \in \mathcal{M}_\infty} \frac{\text{ess sup } Z}{\text{ess inf } Z},$$

and the proof of the arrow (a) \Rightarrow (b) shows that the infimum is obtained by Z^* and there is no duality gap. ■

The next corollary is essentially a slight rephrasing of the theorem just proved. It gives an alternative expression for the dual representation of α^* , which will be generalized in Corollary 3.5, section 3.

Corollary 2.7. *Let $\lambda \in [1, +\infty)$ and let $Q_\lambda \cap \mathcal{M}$ be the (convex) set of martingale measures with finite V_λ -entropy. The conditions $\alpha^* < +\infty$ and $Q_\lambda \cap \mathcal{M} \neq \emptyset$ for some $\lambda \geq 1$ are equivalent; and in case α^* is finite, it admits the representation*

$$\alpha^* = \min\{\lambda \geq 1 \mid Q_\lambda \cap \mathcal{M} \neq \emptyset\}.$$

In particular, $\alpha^ = 1$ iff P is already a martingale measure.*

Proof. Note that $\mathcal{M}_\infty = \cup_{\lambda \geq 1} Q_\lambda \cap \mathcal{M}$ and $(Q_\lambda \cap \mathcal{M})_{\lambda \geq 1}$ is a parametric family nondecreasing in λ with $Q_1 \cap \mathcal{M} = \{P\} \cap \mathcal{M}$ either empty or equal to $\{P\}$. The rest of the proof is then a straightforward consequence of (the proof of) Theorem 2.6. ■

2.3. Pros and cons of gain-loss ratio. The requirement of gain-loss free market can thus be seen as a result à la fundamental theorem of asset pricing (FTAP) also in general probability spaces. A comprehensive survey of no-arbitrage concepts and results is the reference book by Delbaen and Schachermayer [10]. Compared to those theorems, the above proof looks surprisingly easy. Of course, there is a (twofold) reason. First, there is an integrability condition on S ; second, and most importantly, the assumption of λ gain-loss free market is much stronger than absence of arbitrage (or absence of free lunch with vanishing risk).

The stronger requirement of absence of λ gain-loss arbitrage allows a straightforward reformulation in terms of a standard utility maximization problem. This reformulation as such is not possible for the general FTAP case. The reader is however referred to [17] for a proof of the FTAP in discrete time based on a technique which relies in part on the ideas of utility maximization.

In discrete time trading there is a full characterization of the models which have finite best gain-loss ratio. On one side, the Dalang–Morton–Willinger theorem ensures that under no-arbitrage condition there always exists a bounded pricing kernel. Such a kernel is not necessarily bounded away from 0. On the other side, the characterization of arbitrage free markets which admit pricing kernels satisfying prescribed lower bounds is provided by [18].

In continuous time there is no such characterization, and α^* is very likely to be infinite in common models; see Example 2.9 for an illustration in the Black–Scholes model. And even if it is finite, the supremum may not be obtained. This is not due to our specific assumptions, i.e., restriction to simple strategies in Ξ . In general the market best gain-loss is intrinsically not obtained, due to the nature of the functional considered. As it is scale invariant, maximizing sequences can be selected without loss of generality of unitary L^1 -norm. But the unit sphere in L^1 is not (weakly) compact, unless L^1 is finite dimensional or, equivalently, unless Ω is finite. So, when Ω is infinite maximizing sequences may fail to converge, as shown in Example 2.10 in a one period market.

Of course, an enlargement of strategies would certainly help in capturing optimizers in some specific model. But given the intrinsic problems of gain-loss optimization, in the end we choose to work with simple, bounded strategies, as they have a clear financial meaning and allow for a plain mathematical treatment.

2.4. Examples.

Example 2.8 (a model where no gain has well-defined gain-loss ratio). When Assumption 2.1 does not hold, gain-loss ratio criterion may lose significance. Suppose S consists of only one jump which occurs at time T . So, $S_t = 0$ up to time $T-$, while S_T has the distribution of the jump size. If the filtration is the natural one, then a strategy is simply a real constant $\xi = c$ and terminal wealths K are of the form $K = cS_T$. Suppose the jump has a symmetric distribution with infinite first moment. Although this is an arbitrage free model, if $c \neq 0$ both $E[K^+]$ and $E[K^-]$ are infinite.

Example 2.9 (gain-loss ratio is infinite in a Black–Scholes world). In the Black–Scholes market model, the density of the unique pricing kernel is

$$Z = (Z_T =) \exp\left(-\pi W_T - \frac{\pi^2 T}{2}\right)$$

in which W_T stands for the Brownian motion at terminal date T and $\pi = \frac{\mu-r}{\sigma}$ is the market price of risk. This density is both unbounded and not bounded away from 0, so \mathcal{C}^0 is trivial and its basis empty. Therefore, though there is no-arbitrage when $\mu \neq r$ the Black–Scholes market is not gain-loss free for any level λ : $\alpha^* = +\infty$.

Not surprisingly, the idea behind the construction of explicitly arbitrarily large gain-loss ratios is playing with sets where the density Z is either very small or very large. The former sets have a low cost if compared to the physical probability of happening, while the latter in turn happen with small probability but have a (comparatively) high cost. We give examples of both. Without loss of generality, suppose $r = 0$ and fix $1 > \epsilon > 0$. Let $A_\epsilon := \{Z < \epsilon\}$, p_ϵ its probability, and $X_\epsilon = \mathbb{1}_{A_\epsilon}$, while $B_\epsilon := \{Z > \frac{1}{\epsilon}\}$, q_ϵ its probability, and $Y_\epsilon = \mathbb{1}_{B_\epsilon}$. Some calculations show that X_ϵ and Y_ϵ are cash-or-nothing digital options on $S_T =$

$S_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$, either of call type with very large strike or of put type with very small strike when ϵ goes to zero:

1. Let $c_\epsilon = E[ZX_\epsilon]$ be the cost of X_ϵ , which is much smaller than p_ϵ as $c_\epsilon < \epsilon p_\epsilon < 1$. Since the market is complete $K_\epsilon := X_\epsilon - c_\epsilon$ is a gain. Its gain-loss ratio is then

$$\frac{E[K_\epsilon^+]}{E[K_\epsilon^-]} = \frac{(1 - c_\epsilon)p_\epsilon}{c_\epsilon(1 - p_\epsilon)} > \frac{1 - c_\epsilon}{\epsilon} > \frac{1}{\epsilon} - p_\epsilon$$

which tends to $+\infty$ as $\epsilon \downarrow 0$.

2. Let $b_\epsilon = E[Z Y_\epsilon]$ be the cost of Y_ϵ . Then, $1 > b_\epsilon > \frac{q_\epsilon}{\epsilon}$. As before, $C_\epsilon := Y_\epsilon - b_\epsilon$ and its opposite K_ϵ are gains. The gain-loss ratio of K_ϵ is then

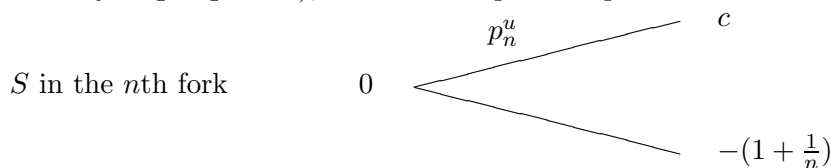
$$\frac{E[K_\epsilon^+]}{E[K_\epsilon^-]} = \frac{b_\epsilon(1 - q_\epsilon)}{(1 - b_\epsilon)q_\epsilon} > \frac{1 - q_\epsilon}{\epsilon}$$

which also tends to $+\infty$ as $\epsilon \downarrow 0$.

The two items together show better why in a gain-loss free market there must be a pricing kernel bounded above *and* bounded away from 0. As a final remark, the strategies that lead to the digital terminal gains $X_\epsilon - c_\epsilon$ and $Y_\epsilon - b_\epsilon$ are *not* bounded. However stochastic integration theory, see, e.g., the book by Karatzas and Shreve [11, Chapter 3], ensures they can be approximated arbitrarily well by simple bounded strategies with L^2 convergence of the terminal gains, so the approximating strategies are in Ξ and their gain-loss ratio blows up.

Example 2.10 (the market best gain-loss ratio may not be attained). Let us consider a one period model consisting of a countable collection of one-step binomial trees with initial uncertainty on the particular binomial fork we are in. The idea is to set the odds and the (single) risky underlying so that the best gain-loss ratio in the n th binomial fork is less than the best gain-loss in the subsequent $(n + 1)$ th binomial fork. This prevents the existence of an optimal solution.

Suppose then $S_0 = 0$, the interest rate $r = 0$, and the probability of being in the n th fork is $\pi_n > 0$. If we are in the n th fork, S_1 can either go up to a constant $c > 0$, independent of n , or go down to $-(1 + \frac{1}{n})$ with conditional probability of going up p_n^u (and $p_n^d = 1 - p_n^u$ is the conditional probability of going down), as summed up in the picture below:



Since S is bounded, Assumption 2.1 is satisfied; there is no-arbitrage and $\mathcal{M}_\infty \neq 0$. In fact, the probability Q which gives to each fork the same probability as P and gives to S a conditional probability of going up in the n th fork equal to $q_n^u = \frac{1 + 1/n}{c + 1 + 1/n}$ is a martingale probability which has density bounded and bounded away from 0. Note that a strategy ξ can be identified with the sequence $(\xi_n)_n$ of its values, chosen at the beginning of each fork. Now,

the scale invariance property implies that the best gain-loss ratio α_n^* in each fork is given by the best between a long position in the underlying and a short one:

$$\alpha_n^* = \max \left(\frac{cp_n^u}{(1+1/n)p_n^d}, \frac{(1+1/n)p_n^d}{cp_n^u} \right).$$

If in addition the parameters $(p_n^u)_{n \geq 1}, c$ satisfy $\alpha_n^* < \alpha_{n+1}^*$, then actively trading in the $n+1$ th fork only and doing nothing in the other forks is always better than trading in the first n forks. To fix the ideas, suppose that in each fork being long in S is better than being short, i.e., $\alpha_n^* = \frac{cp_n^u}{(1+1/n)p_n^d}$. This is satisfied iff $c \geq (1+1/n)\frac{p_n^d}{p_n^u}$ for all $n \geq 1$. Then, the condition $\alpha_n^* < \alpha_{n+1}^*$ for all n becomes

$$1 - \frac{1}{(n+1)^2} < \frac{p_n^d p_{n+1}^u}{p_n^u p_{n+1}^d}.$$

A simple case when this is verified is when the conditional historical probabilities do not depend on n . So, suppose from now on that $p_n^u = p^u$ for all n and that $c \geq 2\frac{p^d}{p^u}$. Then,

$$(5) \quad \alpha^* = \lim_{n \rightarrow +\infty} \alpha_n^* = c \frac{p^u}{p^d}$$

and for any strategy ξ such that $K = \xi \cdot S_1 \in L^1$

$$\alpha(K) < \alpha^*.$$

This is intuitive from the construction but can be verified by (a bit tedious and thus omitted) explicit computations with series.

As the strategies with integrable terminal gain form the largest conceivable domain in gain-loss ratio maximization, this example also proves that the best gain-loss ratio is intrinsically not attained. Namely, it is not a matter of strategy restrictions (boundedness or other).

From an analytic point of view, let us see what goes wrong. Define the sequence of strategies ξ^n :

$$\xi^n = \begin{cases} 1 & \text{if we are initially in the } n\text{th fork,} \\ 0 & \text{otherwise.} \end{cases}$$

ξ^n is the optimizer in the n th fork, and (5) implies it is a maximizing sequence for α^* . The maximizing gains $k^n = \xi^n \cdot S_1$ converge in L^1 to 0, but in 0 α is not defined. By scale invariance, the normalized version

$$K^n = \frac{k_n}{E[|k_n|]}$$

is still maximizing but is not uniformly integrable and thus has no limit.

We finally remark that a $Q \in \mathcal{M}_\infty$ in our model exists because the ratio of the upper value to the lower value of S_1 in each fork, $(S_1)_n^u / (S_1)_n^d$, remains bounded and bounded away from zero when n tends to infinity. A simple modification, with, e.g., $(S_1)_n^u = 1$ and $(S_1)_n^d = -2^{-n}$ as in [10, Remark 6.5.2], leads to an arbitrage free market model with no Q bounded away from zero.

3. Best gain loss with a random endowment.

3.1. The best gain-loss $\alpha^*(B)$ is an acceptability index on L^1 . Suppose the investor at time T has a nonreplicable random endowment $B \in L^1, B \notin \mathcal{K}$. If she optimizes over the market in order to reduce her exposure, the best gain-loss in the presence of B will be

$$\sup_{K \in \mathcal{K}} \alpha(B + K),$$

which is well defined as $B + K$ never vanishes on \mathcal{K} . This expression can be rewritten as $\sup_{K \in \mathcal{K}, K+B \neq 0} \alpha(B + K)$, which makes sense also if $B = 0$ or, more generally, if $B \in \mathcal{K}$, and in that case it coincides with α^* . From now on, the value α^* defined in section 2.1 is denoted by $\alpha^*(0)$. So, let us define on L^1 the map

$$\alpha^*(B) := \sup_{K \in \mathcal{K}, B+K \neq 0} \alpha(B + K).$$

Lemma 3.1. *The map α^* satisfies*

1. $\alpha^* : L^1 \rightarrow [\alpha^*(0), +\infty]$;
2. *nondecreasing monotonicity*;
3. *quasi-concavity, i.e., for any $B_1, B_2 \in L^1$ and for any $c \in [0, 1]$*

$$(6) \quad \alpha^*(cB_1 + (1-c)B_2) \geq \min(\alpha^*(B_1), \alpha^*(B_2));$$

4. *scale invariance: $\alpha^*(B) = \alpha^*(cB)$ for all $c > 0$;*
5. *continuity from below, i.e.,*

$$B_n \uparrow B \Rightarrow \alpha^*(B_n) \uparrow \alpha^*(B).$$

Proof.

1. Without loss of generality, assume $B \notin \mathcal{K}$ and fix $K \neq 0$. For any $t > 0$, $tK \in \mathcal{K}$ and by the scale invariance property of α

$$\alpha(B + tK) = \alpha\left(\frac{B}{t} + K\right).$$

An application of dominated convergence gives $\lim_{t \uparrow +\infty} \alpha\left(\frac{B}{t} + K\right) \rightarrow \alpha(K)$ and consequently $\sup_{t > 0} \alpha\left(\frac{B}{t} + K\right) \geq \alpha(K)$. So,

$$\alpha^*(B) = \sup_{K \in \mathcal{K}} \alpha(B + K) = \sup_{K, t > 0} \alpha(B + tK) = \sup_K \left(\sup_{t > 0} \alpha\left(\frac{B}{t} + K\right) \right) \geq \sup_{K \neq 0} \alpha(K) = \alpha^*(0).$$

2. Nondecreasing monotonicity is a consequence of the monotonicity of α .
3. Quasi-concavity is equivalent to convexity of the upper level sets $A_b := \{B \in L^1 \mid \alpha^*(B) > b\}$ for any fixed $b > \alpha^*(0) = \min_B \alpha^*(B)$. Pick $B_1, B_2 \in A_b$. By Corollary 2.7, $\alpha^*(0) \geq 1$, and since $b > \alpha^*(0) \geq 1$ we can assume that any maximizing sequence K_n^i for $\alpha^*(B_i), i = 1, 2$ satisfies $\alpha(B_i + K_n^i) > 1$, or, equivalently, $B_i + K_n^i$ has positive expectation for all $n \geq 0$ and $i = 1, 2$. It can be easily checked that α is quasi-concave

when restricted to variables with positive expectation (we refer to [8] for a proof). Therefore, for any fixed $c \in [0, 1]$, if $W_n := cB_1 + (1 - c)B_2 + cK_n^1 + (1 - c)K_n^2$ we have

$$\alpha(W_n) \geq \min(\alpha(B_1 + K_n^1), \alpha(B_2 + K_n^2))$$

and $\alpha^*(cB_1 + (1 - c)B_2) \geq \alpha(W_n)$ for all n . Letting $n \rightarrow +\infty$, $\alpha^*(cB_1 + (1 - c)B_2) \geq \min(\alpha^*(B_1), \alpha^*(B_2)) > b$ and thus $cB_1 + (1 - c)B_2 \in A_b$.

4. The scale invariance property easily follows from the scale invariance of α and the cone property of \mathcal{K} .
5. Suppose $B_n \uparrow B$. Select a maximizing sequence $(K_m)_m \in \mathcal{K}$ for $\alpha^*(B)$:

$$\alpha(B + K_m) \uparrow \alpha^*(B).$$

For any fixed m , $B_n + K_m \uparrow B + K_m$ and continuity from below of the expectation of positive and negative part implies the existence of n_m such that $\alpha(B_{n_m} + K_m) \geq \alpha(B + K_m) - \frac{1}{m}$. By the monotonicity property of α^*

$$\alpha^*(B) \geq \lim_n \alpha^*(B_n) \geq \alpha^*(B_{n_m}) \geq \alpha(B_{n_m} + K_m) \geq \alpha(B + K_m) - \frac{1}{m}$$

and, passing to the limit on m , we get $\alpha^*(B) = \lim_n \alpha^*(B_n)$. ■

The above lemma shows that α^* is an acceptability index continuous from below in the sense of Biagini and Bion-Nadal [4]. Acceptability indexes were axiomatically introduced by Cherny and Madan [8], as maps β defined on bounded variables with the following properties:

1. nonnegativity,
2. nondecreasing monotonicity,
3. quasi-concavity,
4. scale invariance,
5. continuity from above: $B_n \downarrow B \Rightarrow \beta(B_n) \downarrow \beta(B)$.

Biagini and Bion-Nadal extend the analysis of performance measures beyond bounded variables and in a dynamic context. In particular, here the continuity from below property replaces continuity from above. This nontrivial point is the key to the extension of the concept of acceptability indexes beyond bounded variables and solves the value-at-0 puzzle for indexes. In fact, continuity from above for an index, which is $+\infty$ -valued on positive random variables (as the gain-loss ratio α and the optimized α^*), implies the index should be $+\infty$ -valued also at 0. This is awkward for any index, but in particular the best gain-loss index α^* loses meaning if we redefine it to be $+\infty$ at 0 only for the sake of the (wrong) continuity requirement.

3.2. The dual representation of $\alpha^*(B)$. There is a natural generalization of the results in Theorem 2.6 in the presence of a claim. First, we need an auxiliary result.

Lemma 3.2. *Fix $B \in L^1$ and suppose $\alpha^*(B) > \alpha^*(0)$. Then, any maximizing sequence $(K_n)_n$ for $\alpha^*(B)$ is bounded in L^1 .*

Proof. Select a maximizing sequence for $\alpha^*(B)$, $K_n \in \mathcal{K}$, $\alpha(B + K_n) \uparrow \alpha^*(B)$. Let $(c_n)_n$ denote the corresponding sequence of L^1 - norms, i.e., $c_n = E[|K_n|]$. If $(c_n)_n$ were unbounded, by passing to a subsequence, still denoted in the same way, we could assume $c_n \uparrow +\infty$. Let $k_n = \frac{K_n}{c_n}$. The scale invariance property of α would imply

$$\alpha(B + K_n) = \frac{E[(B + K_n)^+]}{E[(B + K_n)^-]} = \frac{E[(\frac{B}{c_n} + k_n)^+]}{E[(\frac{B}{c_n} + k_n)^-]}.$$

Since $\frac{B}{c_n} \rightarrow 0$ in L^1 , $\alpha^*(B) = \lim_n \alpha(B + K_n) = \lim_n \frac{E[k_n^+]}{E[k_n^-]}$, whence we would get the contradiction $\alpha^*(B) \leq \alpha^*(0)$. ■

Theorem 3.3. *The following conditions are equivalent:*

- (i) $\alpha^*(B) < +\infty$,
- (ii) $E_Q[B] \leq 0$ for some $Q \in \mathcal{M}_\infty$.

If any of the two conditions (i), (ii) is satisfied, α^* admits the dual representation

$$(7) \quad \alpha^*(B) = \min_{Q \in \mathcal{M}_\infty, E_Q[B] \leq 0} \frac{\text{ess sup } Z}{\text{ess inf } Z},$$

which becomes

$$(8) \quad \alpha^*(B) = \min_{Q \in \mathcal{M}_\infty, E_Q[B]=0} \frac{\text{ess sup } Z}{\text{ess inf } Z}$$

when $+\infty > \alpha^*(B) > \alpha^*(0)$.

Proof.

- (i) \Rightarrow (ii). Set $b = \alpha^*(B)$. Then $b \geq \alpha^*(0) \geq 1$. So,

$$0 = \alpha^*(B) - b = \sup_{K \in \mathcal{K}} \frac{E[U_b(B + K)]}{E[(B + K)^-]}.$$

The denominator is positive, whence the above relation implies $E[U_b(B + K)] \leq 0$ for all K . Therefore $\sup_K E[U_b(B + K)] \leq 0$ with possibly strict inequality. Since this supremum is finite, the Fenchel duality theorem applies, similarly to Theorem 2.6, and gives

$$\sup_K E[U_b(B + K)] = \min_{Q \in \mathcal{C}_1^0, y \geq 0} \left\{ yE \left[\frac{dQ}{dP} B \right] + E \left[V_b \left(y \frac{dQ}{dP} \right) \right] \right\} \leq 0.$$

Given the structure of V_b , any couple of minimizers y^*, Q^* satisfies $y^* > 0$ and $dQ^* = Z^* dP \in \mathcal{Q}_b \cap \mathcal{C}_1^0 = \mathcal{Q}_b \cap \mathcal{M} \subseteq \mathcal{M}_\infty$, which is then not empty. So, $E[V_b(y^* \frac{dQ^*}{dP})] + y^* E_{Q^*}[B] \leq 0$ implies $E_{Q^*}[B] \leq 0$ and (ii) follows.

(ii) \Rightarrow (i). Fix a martingale measure $dQ = Z dP$ with the stated properties, and let $y = \frac{1}{\text{ess inf } Z}$, $\mu = \frac{\text{ess sup } Z}{\text{ess inf } Z}$ so that $1 \leq yZ \leq \mu$. The Fenchel inequality applied to the couple U_μ, V_μ , on $B + K$ and yZ , respectively, gives

$$U_\mu(B + K) - (K + B)yZ \leq V_\mu(yZ) = 0 \quad \forall K \in \mathcal{K}.$$

Taking expectations, $E[U_\mu(B + K)] \leq yE_Q[B] \leq 0$ for all K , which implies $\alpha^*(B) \leq \mu$.

The duality formula (7) has been implicitly proved in the above lines. In fact, with the same notation as in the implications (i) \rightarrow (ii), we have the relation

$$\alpha^*(B) \leq \frac{\text{ess sup } Z^*}{\text{ess inf } Z^*} \leq b,$$

where the first inequality follows from the arrow (ii) \rightarrow (i) and the second from $Q^* \in \mathcal{Q}_b$. But since $\alpha^*(B) = b$, the inequalities are in fact equalities.

To show the representation (8), suppose by contradiction that there exists a B such that $+\infty > \alpha^*(B) > \alpha^*(0)$ and the minimum in (7) is obtained at a Q^* with $E_{Q^*}[B] < 0$. Pick a maximizing sequence $(K_n)_n$ for $\alpha^*(B)$, which by Lemma 3.2 is bounded in L^1 -norm. With the same notation as of the implication (i) \Rightarrow (ii) above, we have the inequality

$$E[U_b(B + K_n)] \leq y^* E_{Q^*}[B] < 0.$$

From this, dividing by $E[(B + K_n)^-]$ and adding b to both members we derive

$$\alpha(B + K_n) = \frac{E[(B + K_n)^+]}{E[(B + K_n)^-]} \leq b + y^* \frac{E_{Q^*}[B]}{E[(B + K_n)^-]} \leq b + y^* \frac{E_{Q^*}[B]}{L} < b = \alpha^*(B),$$

where L is a uniform upper bound for $E[(B + K_n)^-]$. Letting $n \uparrow +\infty$, we get the contradiction $\alpha^*(B) = \lim_n \alpha(B + K_n) < \alpha^*(B)$. ■

Remark 3.4. The representations (7) and (8) are interesting per se. In fact, the abstract dual representation of a quasi-concave map is known (Volle [20, Theorem 3.4]), but there are few examples in which such a dual representation can be explicitly computed.

Note also that if the market is complete and the unique martingale measure Q^* is in \mathcal{M}_∞ , then $\alpha^*(B) = +\infty$ iff $E_{Q^*}[B] > 0$, and $\alpha^*(B)$ is finite (and equal to $\alpha^*(0)$) iff $E_{Q^*}[B] \leq 0$.

Corollary 3.5. *With the convention $\sup \emptyset = \alpha^*(0)$, α^* admits the representation*

$$(9) \quad \alpha^*(B) = \sup\{\lambda \geq 1 \mid E_Q[B] > 0 \ \forall Q \in \mathcal{Q}_\lambda \cap \mathcal{M}\}.$$

Proof. With the usual convention $\inf \emptyset = +\infty$, the proof of Theorem 3.3 shows that

$$\alpha^*(B) = \inf\{\lambda \mid E_Q[B] \leq 0 \text{ for some } Q \in \mathcal{Q}_\lambda \cap \mathcal{M}\}$$

and that $\alpha^*(B)$ is finite iff the infimum is a minimum. As $\mathcal{Q}_\lambda \cap \mathcal{M}$ is a set of probabilities which is nondecreasing in the parameter, the right-hand side of the above equation is an interval I , either $[\alpha^*(B), +\infty)$ when $\alpha^*(B)$ is finite or empty when $\alpha^*(B)$ is infinite. Since

$$\{\lambda \geq 1 \mid E_Q[B] > 0 \ \forall Q \in \mathcal{Q}_\lambda \cap \mathcal{M}\}$$

corresponds to the interval $I^c \cap [1, +\infty)$, its supremum coincides with $\alpha^*(B)$ both in the finite and infinite cases. ■

Remark 3.6. A general result on acceptability indexes and performance measures is that any such map can be represented in terms of a one-parameter, nondecreasing family of risk measures (see [8, 4]). In [8, Theorem 1, Proposition 4] it is shown that the gain-loss index α admits a representation in terms of the family $(\rho_\lambda)_\lambda$:

$$\rho_\lambda(X) := \sup_{Q \in \mathcal{Q}_\lambda} E_Q[-X].$$

The formula (9) proves an intuitive fact: the market optimized gain-loss index α^* admits a representation via the risk measures $(\rho_\lambda^M)_\lambda$ induced by $(\mathcal{Q}_\lambda \cap \mathcal{M})_{\lambda \geq 1}$

$$\rho_\lambda^M(X) := \sup_{Q \in \mathcal{Q}_\lambda \cap \mathcal{M}} E_Q[-X],$$

where we adopt the convention $\rho_\lambda^M = -\infty$ if $\mathcal{Q}_\lambda \cap \mathcal{M} = \emptyset$. The family $(\rho_\lambda^M)_\lambda$ consists of the so-called market modifications of the collection of risk measures $\rho_\lambda(X) := \sup_{\mathcal{Q}_\lambda} E_Q[-X]$. For the concept of market modified risk measure and its relation to hedging, the reader is referred to [7] and [1, section 3.1.3].

3.3. Final comments. The results just found constitute the basis for a strong objection against best gain-loss ratio as a performance criterion in the presence of an endowment. To start with, Lemma 3.1 shows that possessing a claim whatsoever can never be worse than the case $B = 0$ since $\alpha^*(B) \geq \alpha^*(0)$, which does not make economic sense.

Second, by Theorem 3.3 the index α^* can be of little use in discriminating payoffs, as $\alpha^*(B)$ is finite iff the claim belongs to $\cup_{Q \in \mathcal{M}_\infty} \{B \mid E_Q[B] \leq 0\}$, and we have seen that \mathcal{M}_∞ is empty in most continuous time models.

Moreover, if there is a unique pricing kernel, say P , then $\alpha^*(B) = +\infty$ if $E[B] > 0$, or if $E[B] < 0$, it is optimal to take infinite risk to offset the negative expectation of B and end up with $\alpha^*(B) = \alpha^*(0) = 1$, along the same lines of the proof of item 1 in Lemma 3.1. This is also unreasonable.

From a strict mathematical viewpoint, there is quite a difference from what happens in standard utility maximization: For example, if P is a martingale measure and $B = m$ is constant, the optimal solution is simply to not invest in the market. This is due to risk aversion, and mathematically it is a consequence of Jensen's inequality:

$$E[U(m + K)] \leq U(m + E[K]) = U(m).$$

On the contrary, when $m < 0$, $0 = \alpha(m) < \alpha^*(m) = 1 = \alpha^*(0)$. The scale invariance property $\alpha^*(B) = \alpha^*(cB)$ for all $c > 0$ implies

$$\alpha^*(B) = \sup_{c > 0} \alpha^*(cB) = \sup_{c > 0, K \in \mathcal{K}} \alpha(K + cB).$$

As a consequence, our optimization problem better compares with the so-called static-dynamic utility maximization, see, e.g., Ilhan, Jonsson, and Sircar [12], where the optimization is made dynamically in the underlyings and statically in the claim

$$u(B) := \sup_{c > 0, K \in \mathcal{K}} E[U(K + cB)],$$

where only long positions are permitted in the claim to mirror the constraint we have for gain-loss. When P is a martingale measure and $B = m < 0$ the value of the static-dynamic utility maximization verifies

$$U(m) < u(m) = u(0) = U(0),$$

and this result is exactly in the spirit of the equality $\alpha^*(m) = \alpha^*(0)$ found before.

As a final remark, the scale invariance property may be questionable for performance measures in general. In fact, α^* can be seen as an evaluation of the whole half ray generated by B , cB , $c > 0$, rather than B itself. So, it is desirable only if the (large) investor seeks an information on the "direction of trade," as illustrated by Cherny and Madan [8], and it is not appropriate for small investors, e.g., if *quantity* matters. The cited work [4] is entirely dedicated to the definition of a good notion of performance measures in an intertemporal setting.

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