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SUMS WITH CONVOLUTIONS OF DIRICHLET CHARACTERS TO CUBE-FREE MODULUS

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Abstract. We find estimates for short sums of the form $\sum_{nm\leqslant X}\chi_1(n)\chi_2(m)$, where χ_1 and χ_2 are non-principal Dirichlet characters to modulus q, a cube-free integer, and X can be taken as small as $q^{\frac{1}{2}+\epsilon}$.

1. Introduction

1.1. **Notation.** Let χ_1, χ_2 be non-principal Dirichlet characters to moduli $q_1>1$ and $q_2\geqslant q_1$, respectively. The convolution of χ_1 and χ_2, denoted $\chi_1*\chi_2$, is defined formally by the relation

$$L(s,\chi_1)L(s,\chi_2)=\sum_{n=1}^{\infty}\chi_1(n)n^{-s}\sum_{n=1}^{\infty}\chi_2(n)n^{-s}=\sum_{n=1}^{\infty}(\chi_1*\chi_2)(n)n^{-s};$$

thus,

$$(\chi_1*\chi_2)(n)=\sum_{ab=n}\chi_1(a)\chi_2(b).$$

Using the truncated version of Perron's formula together with available estimates for Dirichlet L-functions one can show that the summatory function

$$\mathcal{S}_{\chi_1*\chi_2}(X):=\sum_{n\leqslant X}(\chi_1*\chi_2)(n) \tag{1}$$

satisfies the bound (see, e.g., the remark following [4, Theorem 4.16])

$$\mathcal{S}_{\chi_1*\chi_2}(X)\ll(q_1q_2)^{\frac{1}{3}}X^{\frac{1}{3}+\epsilon},$$

where the implied constant depends only on ϵ. (See [3] for recent results on related estimates as well as estimates of more general arithmetic functions.)

Note that the above estimate is worse than the trivial estimate

$$|\mathcal{S}_{\chi_1*\chi_2}(X)|\leqslant X\log X$$

unless $X\geqslant(q_1q_2)^{\frac{1}{2}+\epsilon}$.

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1.2. **Statement of results.** In this paper we estimate $\mathcal{S}_{\chi_1*\chi_2}(X)$ for small values of X in the case of two non-principal Dirichlet characters χ_1, χ_2 with a cube-free integer modulus $q>1$. Our main result in this direction is the following:

Theorem 1. *Let $q>1$ be a cube-free integer and χ_1, χ_2 non-principal Dirichlet characters to modulus q. Fix $\epsilon>0$. Then, for any integer $d>1$ and $X \geqslant q^{\frac{1}{2}+\frac{1}{2d}}$,*

$$\mathcal{S}_{\chi_1*\chi_2}(X) \ll_{\epsilon,d} \min\big\{\mathcal{S}_1(d,X), \mathcal{S}_2(d,X)\big\} \log X,$$

where

$$\mathcal{S}_1(d,X) = q^{\frac{2d^2+4d+1}{4d(d+1)^2}+\epsilon} X^{\frac{d}{d+1}}, \qquad \mathcal{S}_2(d,X) = q^{\frac{2d^2+d-1}{4d^3}+\epsilon} X^{\frac{2d^2-2d+1}{2d^2}}.$$

Theorem 1 provides a non-trivial bound if $X \gg_{\epsilon,d} q^{\frac{1}{2}+\frac{1}{2d}+\epsilon}$. For the next result, we introduce two numbers:

$$E(d) = \frac{1}{2} + \frac{1}{2d} + \frac{1}{2(d-1)} \quad \text{and} \quad A(d) = \frac{E(d+1)+E(d)}{2} \quad (d>1).$$

Proposition 2. *Let q, χ_1, χ_2, ϵ be as in Theorem 1. Then, for any integer $d>1$,*

$$\min\{\mathcal{S}_i(d',X) : d'>1, i=1,2\} = \begin{cases} \mathcal{S}_1(d,X) & \textit{if} \quad q^{E(d+1)} \leqslant X < q^{A(d)}, \\ \mathcal{S}_2(d,X) & \textit{if} \quad q^{A(d)} \leqslant X < q^{E(d)}. \end{cases}$$

Note that since $E(d+1) > \frac{1}{2} + \frac{1}{2d}$, the bound in Theorem 1 still holds when $q^{E(d+1)} \leqslant X < q^{E(d)}$ for any $d>1$.

For comparison with [1, Corollary 2], we state our result explicitly for $d=2$ and $d=3$, which follows by combining Theorems 1 and 2:

Corollary 3. *Let q, χ_1, χ_2, ϵ be as in Theorem 1. Then,*

$$\mathcal{S}_{\chi_1*\chi_2}(X) \ll \log X \begin{cases} q^{\frac{31}{192}+\epsilon} X^{\frac{3}{4}} & \textit{if} \quad X \in \big[q^{\frac{19}{24}}, q^{\frac{41}{48}}\big), \\ q^{\frac{5}{27}+\epsilon} X^{\frac{13}{18}} & \textit{if} \quad X \in \big[q^{\frac{41}{48}}, q^{\frac{11}{12}}\big), \\ q^{\frac{17}{72}+\epsilon} X^{\frac{2}{3}} & \textit{if} \quad X \in \big[q^{\frac{11}{12}}, q^{\frac{13}{12}}\big), \\ q^{\frac{9}{32}+\epsilon} X^{\frac{5}{8}} & \textit{if} \quad X \in \big[q^{\frac{13}{12}}, q^{\frac{5}{4}}\big). \end{cases}$$

We remark that our method follows mainly that of [1] and consists of dissecting $\mathcal{S}_{\chi_1*\chi_2}(X)$ (see section 2.2) and then applying Burgess' bound (see Lemma 4).

1.3. **Previous work.** An analogue of the sum $\mathcal{S}_{\chi_1*\chi_2}(X)$ has been previously estimated by Moshchevitin (see the proof of [5, Theorem 5]) in the special case that $\chi_2 = \overline{\chi}_1$ and $q_1 = q_2 = p$ is a prime number and has been shown to be important for some problems on continued fractions of rational numbers. This sum also makes its appearance in a paper by Moshchevitin and Ushanov [6], where they generalize a theorem by Larcher on good lattice points and multiplicative subgroups modulo a prime.

More recently, Banks and Shparlinski [1] have estimated $\mathcal{S}_{\chi_1*\chi_2}(X)$ for primitive Dirichlet characters χ_1, χ_2 of conductors $q_1>1$, and $q_2 \geqslant q_1$, respectively. Their result improves and generalizes the bounds given in [5] and [6] and holds for $X \geqslant q_2^{\frac{2}{3}}$ with $\log X = q_2^{o(1)}$. Assuming in our case that the modulus q of the characters is cube-free allows us to make full use of Burgess' bound. In this way, we extend the range of X down to $q^{\frac{1}{2}+\epsilon}$ and achieve a slight improvement (see Corollary 3) over those in [1, Corollary 2].

1.4. **Related problems.** One can consider estimating $\mathcal{S}_{\chi_1*\chi_2}(X)$ in the case of two characters to distinct moduli, both of which are cube-free. Another direction would be to consider the convolution of a number of Dirichlet characters, namely, to estimate

$$\sum_{a_1\cdots a_k\leqslant X}\chi_1(a_1)\cdots\chi_k(a_k),$$

where the χ_i are characters to moduli $q_i > 1$. This is the summatory function associated with the product

$$L(s,\chi_1)\cdots L(s,\chi_k).$$

Note that when the characters are the same, say χ, we have

$$\mathcal{S}_{\chi*\chi}(X)=\sum_{n\leqslant X}\tau(n)\chi(n),$$

where $\tau(n)$ is the number of positive divisors of n. Related sums of the form

$$\sum_{n\leqslant X}\tau(n)\chi(n+a)\qquad(a\in\mathbb{Z};\gcd(a;q)=1)$$

have also been studied but are generally approached using different methods.

Using our method one can also estimate the summatory function associated with the more general product

$$\sum_n f(n)n^{-s}\sum_n g(n)n^{-s}=\sum_n (f*g)(n)n^{-s},$$

where f and g are two arithmetic functions, as long as one can estimate, for small values of X, the sums

$$\sum_{n\leqslant X}f(n)\qquad\text{and}\qquad\sum_{n\leqslant X}g(n).$$

2. Proof of Theorem 1 and Proposition 2

2.1. **Preliminaries.** The following result, due to D. A. Burgess [2], plays a central role in our work:

Lemma 4. *Let $q>1$ be a cube-free integer, $\rho\geqslant 1$ a fixed integer, and $\epsilon>0$ a fixed real number. If χ is a non-principal Dirichlet character to modulus q, then for any pair of integers M and $N>0$,*

$$\sum_{M\leqslant n\leqslant M+N}\chi(n)\ll N^{1-\frac{1}{\rho}}q^{\frac{\rho+1}{4\rho^2}+\epsilon},\tag{2}$$

where the implied constant depends on ϵ and ρ.

Burgess' bound holds in general for any integer $q>1$, in which case $1\leqslant\rho\leqslant 3$. This bound is useful, for a fixed $\rho\geqslant 1$, when $N\gg q^{\frac{1}{4}+\frac{1}{4\rho}+\epsilon}$. In case q is prime, one can prove (see, e.g., what follows [4, Theorem 12.6]) a slightly stronger bound, namely, that

$$\left|\sum_{M\leqslant n\leqslant M+N}\chi(n)\right|\leqslant 30N^{1-\frac{1}{\rho}}q^{\frac{\rho+1}{4\rho^2}}(\log q)^{\frac{1}{\rho}}.$$

2.2. **Hyperbola method.** We write $\mathcal{S}_{\chi_1*\chi_2}(X)$ as $S_1+S_2-S_3$, where

$$S_1=\sum_{\substack{nm\leqslant\sqrt{X}\\ n\leqslant\sqrt{X}}}\chi_1(n)\chi_2(m),\qquad S_2=\sum_{\substack{nm\leqslant\sqrt{X}\\ m\leqslant\sqrt{X}}}\chi_1(n)\chi_2(m),\tag{3}$$

and

$$S_3=\sum_{n\leqslant\sqrt{X}}\chi_1(n)\sum_{m\leqslant\sqrt{X}}\chi_2(m).\tag{4}$$

From now on we shall assume that $\epsilon>0$ is fixed. Using (2) with $\rho=R>1$ we see that

$$S_3\ll E_R(\epsilon):=X^{1-\frac{1}{R}}q^{\frac{R+1}{2R^2}+\epsilon}.\tag{5}$$

2.3. **Bounding the sums S_1 and S_2.** Due to the symmetry and the fact that the bound in (2) depends only on the conductor of the character and not on the character itself, it is enough to estimate S_1.

Following [1] we introduce two parameters $\theta\in(0,1/2)$ and $\gamma\in(1/2,1]$, and write S_1 as $S_{11}+S_{12}$, where

$$S_{11}=\sum_{\substack{nm\leqslant X\\ n\leqslant\gamma X^\theta}}\chi_1(n)\chi_2(m),\qquad S_{12}=\sum_{\substack{nm\leqslant X\\ \gamma X^\theta<n\leqslant\sqrt{X}}}\chi_1(n)\chi_2(m).\tag{6}$$

Applying (2) to S_{11} with $\rho=R-1$ we obtain

$$|S_{11}|\ll X^{1+\frac{\theta-1}{R-1}}\,q^{\frac{R}{4(R-1)^2}+\epsilon}.$$

For $R>2$, we choose $\theta=\frac{1}{R}$ and deduce that

$$|S_{11}|\ll X^{1-\frac{1}{R}}q^{\frac{R}{4(R-1)^2}+\epsilon}\leqslant E_R(\epsilon),\tag{7}$$

since

$$\frac{R}{4(R-1)^2}<\frac{R+1}{2R^2}.$$

For $R=2$, we choose $\theta=\frac{1}{3}$, obtaining

$$|S_{11}|\ll X^{\frac{1}{3}}q^{\frac{1}{2}+\epsilon}\leqslant E_2(\epsilon),$$

whenever $X\geqslant q^{\frac{3}{4}}$.

2.4. **Estimating the sum S_{12}.** Fix a real number λ such that

$$3X^{-\theta}<\lambda\leqslant 1.\tag{8}$$

Let I be the positive integer determined by the relation

$$(1+\lambda)^I\geqslant X^{1/2-\theta}>(1+\lambda)^{I-1}.$$

It immediately follows from this definition that for $X>e^{\frac{\lambda}{2\theta}}$,

$$I<1+\frac{(\frac{1}{2}-\theta)\log X}{\log(1+\lambda)}<\frac{2(\frac{1}{2}-\theta)\log X+\lambda}{\lambda}<\lambda^{-1}\log X.\tag{9}$$

If we choose $\gamma=X^{\frac{1}{2}-\theta}(1+\lambda)^{-I}$ we see that

$$\frac{1}{2}\leqslant(1+\lambda)^{-1}<\gamma\leqslant 1,$$

that is, $\gamma\in\left(\frac{1}{2},1\right]$, as needed.

Finally, we put $Z_0 = \gamma X^{\theta}$ and $Z_i = Z_{i-1}(1+\lambda)$ for $i = 1, \ldots, I$. Notice that $Z_I = \sqrt{X}$.

We now rewrite S_{12} as $S'_{12} + S''_{12}$, where

$$S'_{12} = \sum_{i=1}^{I} \sum_{Z_{i-1} < n \leqslant Z_i} \chi_1(n) \sum_{m \leqslant \frac{X}{Z_i}} \chi_2(m),$$

$$S''_{12} = \sum_{i=1}^{I} \sum_{Z_{i-1} < n \leqslant Z_i} \chi_1(n) \sum_{\frac{X}{Z_i} < m \leqslant \frac{X}{n}} \chi_2(m).$$

Note that $Z_i - Z_{i-1} = \lambda Z_{i-1} \geqslant \lambda Z_0 > 1$. Since

$$\frac{X}{n} - \frac{X}{Z_i} < \frac{X}{Z_{i-1}} - \frac{X}{Z_i} < \frac{X\lambda}{Z_{i-1}} \qquad (i = 1, \ldots, I),$$

and $X\lambda / Z_{i-1} > X^{\frac{1}{2}-\theta}$, it follows by (9) and (2) applied with $\rho = s$ that

$$\begin{aligned} |S''_{12}| &\ll \sum_{i=1}^{I} \lambda Z_{i-1} \left(\frac{X\lambda}{Z_{i-1}} \right)^{1-\frac{1}{s}} q^{\frac{s+1}{4s^2}+\epsilon} \\ &\leqslant q^{\frac{s+1}{4s^2}+\epsilon} \lambda^{1-\frac{1}{s}} X^{1-\frac{1}{2s}} \log X. \end{aligned} \tag{10}$$

As for S'_{12}, using (2) twice with $\rho = t$ and $\rho = r$ we deduce that

$$\begin{aligned} |S'_{12}| &\ll \sum_{i=1}^{I} (\lambda Z_{i-1})^{1-\frac{1}{r}} q^{\frac{r+1}{4r^2}+\frac{\epsilon}{2}} \left(\frac{X}{Z_{i-1}} \right)^{1-\frac{1}{t}} q^{\frac{t+1}{4t^2}+\frac{\epsilon}{2}} \\ &\leqslant q^{\frac{r+1}{4r^2}+\frac{t+1}{4t^2}+\epsilon} \lambda^{-\frac{1}{r}} X^{1-\frac{r+t}{2rt}} \log X. \end{aligned} \tag{11}$$

We now choose $\lambda = \lambda(s,r,t)$ in order to balance (10) and (11); that is, we set

$$\lambda(s,r,t) = X^{f(s,r,t)} q^{g(s,r,t)}, \tag{12}$$

where

$$\begin{aligned} f(s,r,t) &= \left(\frac{1}{2s} - \frac{1}{2r} - \frac{1}{2t} \right) \left(1 + \frac{1}{r} - \frac{1}{s} \right)^{-1}, \\ g(s,r,t) &= \left(\frac{r+1}{4r^2} + \frac{t+1}{4t^2} - \frac{s+1}{4s^2} \right) \left(1 + \frac{1}{r} - \frac{1}{s} \right)^{-1}. \end{aligned}$$

Here the parameters s, r and t must be chosen so that (8) is satisfied. Assuming this holds for some triple (s,r,t), we conclude upon combining (10) and (11) that

$$|S_{12}| \ll B^{\epsilon}_{X,q}(s,r,t) := X^{F(s,r,t)} q^{G(s,r,t)+\epsilon} \log X, \tag{13}$$

where

$$\begin{aligned} F(s,r,t) &= \frac{s-1}{s} f(s,r,t) + 1 - \frac{1}{2s}, \\ G(s,r,t) &= \frac{s-1}{s} g(s,r,t) + \frac{s+1}{4s^2}. \end{aligned}$$

From now on we shall omit the subscripts X and q, and write $B^{\epsilon}(s,r,t)$ instead of $B^{\epsilon}_{X,q}(s,r,t)$.

2.5. **Proof of Theorem 1.** Combining (5), (7) and (13) we conclude that

$$\begin{aligned}|\mathcal{S}_{\chi_1*\chi_2}(X)| &\leqslant |S_1| + |S_2| + |S_3| \leqslant 2\big(|S_{11}| + |S_{12}|\big) + |S_3| \\ &\ll B^{\epsilon}(s,r,t) + E_R(\epsilon) \leqslant \max\big\{B^{\epsilon}(s,r,t),\, E_R(\epsilon)\big\},\end{aligned}$$

where (s, r, t) is a triple for which (8) holds with $\theta = \frac{1}{R}$ if $R > 2$ and with $\theta = \frac{1}{3}$ if $R = 2$, in which case we assume that $X \geqslant q^{\frac{3}{4}}$.

Fix an integer $d > 1$, and take all parameters R, s, r, t equal to d. One can then easily check that for $X \geqslant q^{\frac{1}{2}+\frac{1}{2d}}$,

$$E_d(\epsilon) \leqslant B^{\epsilon}(d,d,d) = \mathcal{S}_2(d,X)\log X,$$

and that (8) is satisfied with these parameters.

Similarly, if we choose $R, s, r = d + 1$ and $t = d$, then for $X \geqslant 1$,

$$E_{d+1}(\epsilon) \leqslant B^{\epsilon}(d+1,d+1,d) = \mathcal{S}_1(d,X)\log X,$$

and one can easily verify that (8) holds with these parameters as well. This establishes the proof of Theorem 1.

2.6. **Proof of Proposition 2.** We first note that the inequality

$$\mathcal{S}_2(d,X) \leqslant \mathcal{S}_1(d,X)$$

holds if and only if $X \geqslant q^{A(d)}$, while

$$\mathcal{S}_1(d,X) \leqslant \mathcal{S}_2(d+1,X)$$

holds if and only if $X \geqslant q^{E(d+1)}$. This implies that the minimal choice among $S_i(d', X)$ with $i = 1, 2$ and $d' > 1$ is $\mathcal{S}_1(d,X)$ for $q^{E(d+1)} \leqslant X < q^{A(d)}$, and $\mathcal{S}_2(d,X)$ for $q^{A(d)} \leqslant X < q^{E(d)}$. This concludes the proof of Proposition 2.

3. Determining the optimal bound

The following result justifies our choice of the triples (d, d, d) and $(d+1, d+1, d)$ in the proof of Theorem 1 among other possible triples (s, r, t) for which $s, r, t \geqslant d$:

Lemma 5. *For any integer $d > 1$ and any choice of triples (s, r, t) with $s, r, t \geqslant d$, we have*

$$\begin{aligned} B^{\epsilon}(d+1,d+1,d) &\leqslant B^{\epsilon}(s,r,t), \qquad \textit{if} \quad q^{E(d+1)} \leqslant X < q^{A(d)}, \\ B^{\epsilon}(d,d,d) &\leqslant B^{\epsilon}(s,r,t), \qquad \textit{if} \quad q^{A(d)} \leqslant X < q^{E(d)}. \end{aligned}$$

Proof. Note that for triples (s, r, t) and (s', r', t'), the inequality

$$B^{\epsilon}(s,r,t) \leqslant B^{\epsilon}(s',r',t')$$

holds whenever $X \leqslant q^{\mathcal{P}(s,r,t;s',r',t')}$, where

$$\mathcal{P}(s,r,t;s',r',t') := \frac{G(s',r',t') - G(s,r,t)}{F(s,r,t) - F(s',r',t')}, \tag{14}$$

provided that both the numerator and denominator of (14) are positive. In case the denominator vanishes, we only require that the numerator be non-negative.

We now choose $s' = r' = d + 1$ and $t' = d$ and compute (14). With the aid of a computer or otherwise, one can easily check that

(1) For $s, r \geqslant 0$, not both zero,

$$F(d+1+s, d+1+r, d) - F(d+1, d+1, d) = \frac{(d-1)(rd+s+rs)}{2d(1+d)(1+2s+rs+2d+rd+sd+d^2)} > 0,$$

$$\begin{aligned} &G(d+1, d+1, d) - G(d+1+s, d+1+r, d) \\ &= \Big\{ r^2(d^3+d^2-2d-1)\left(s+s^2+2ds+d^2+d\right) \\ &\quad + s(1+d)\left(d^4+3d^3-3d-1+s(d^3+d^2-2d-1)\right) \\ &\quad + r(2s+d)(1+d)^2(d^3+2d^2-2d-1) \\ &\quad + rs^2\left(d^4+4d^3+d^2-5d-2\right)\Big\}\Big(4d^2\left(1+Q(d,s,r)\right)\Big)^{-1} > 0, \end{aligned}$$

where $Q(d, s, r)$ is a polynomial with positive coefficients, and

$$\begin{aligned} &\mathcal{P}(d+1+s, d+1+r, d; d+1, d+1, d) \\ &\quad = E(d+1) - \frac{d\left(r^2(d+d^2+s+2ds)+s^2(1+r+r^2+d)\right)}{2(d^2-1)(1+r+d)(1+d+s)(rd+s+rs)} \\ &\quad < E(d+1). \end{aligned}$$

(2) For non-negative integers s, r and t, and $d' = d+1$,

$$F(d'+s, d'+r, d'+t) - F(d', d', d) = \frac{1+P_1(d,s,r,t)}{2+Q_1(d,s,r,t)} > 0,$$

$$G(d', d', d) - G(d'+s, d'+r, d'+t) = \frac{1+P_2(d,s,r,t)}{4d+Q_2(d,s,r,t)} > 0,$$

$$\mathcal{P}(d'+s, d'+r, R+t; d', d', d) = E(d') - \frac{r^2d^2+s^2d+td+P_3(d,s,r,t)}{2d+Q_3(d,s,r,t)},$$

where $P_i(d,s,r,t)$ and $Q_i(d,s,r,t)$, $i = 1,2,3$, are polynomials in d, s, r, t with positive coefficients and $P_3(d,0,0,0) = 0$. It follows from the last equation that

$$\mathcal{P}(d'+s, d'+r, d'+t; d', d', d) < E(d')$$

unless r, s and t are all zero, in which case we have equality.

(3) For any $d > 1$,

$$\mathcal{P}(d, d+1, d+1; d+1, d+1, d) = E(d+1) - \frac{1}{2d(d^2-1)},$$

$$\mathcal{P}(d+1, d, d+1; d+1, d+1, d) = E(d+1) - \frac{1}{2(d+1)}.$$

(4) For any $X \geqslant 1$,

$$B^{\epsilon}(d+1, d+1, d) < B^{\epsilon}(d, d, d+1).$$

(5) For $X \leqslant q^{A(d)}$, and $d' = d + 1$,

$$B^\epsilon(d', d', d) \leqslant \min\{B^\epsilon(d', d, d), B^\epsilon(d, d', d), B^\epsilon(d, d, d)\},$$

and for $X \geqslant q^{A(d)}$,

$$B^\epsilon(d, d, d) \leqslant \min\{B^\epsilon(d', d, d), B^\epsilon(d, d', d), B^\epsilon(d, d, d'), B^\epsilon(d', d', d)\}.$$

The result follows upon combining all the comparisons above. □

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