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Closely embedded Kreĭn spaces and applications to Dirac operators

Petru Cojuhari [a], Aurelian Gheondea [b,c,*]

[a] Department of Applied Mathematics, AGH University of Science and Technology, Al. Mickievicza 30, 30-059 Cracow, Poland
[b] Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey
[c] Institutul de Matematică al Academiei Române, C.P. 1-764, 014700 București, Romania

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ABSTRACT

Motivated by energy space representation of Dirac operators, in the sense of K. Friedrichs, we recently introduced the notion of closely embedded Kreĭn spaces. These spaces are associated to unbounded selfadjoint operators that play the role of kernel operators, in the sense of L. Schwartz, and they are special representations of induced Kreĭn spaces. In this article we present a canonical representation of closely embedded Kreĭn spaces in terms of a generalization of the notion of operator range and obtain a characterization of uniqueness. When applied to Dirac operators, the results differ according to a mass or a massless particle in a dramatic way: in the case of a particle with a nontrivial mass we obtain a dual of a Sobolev type space and we have uniqueness, while in the case of a massless particle we obtain a dual of a homogenous Sobolev type space and we lose uniqueness.

1. Introduction

Among other important ideas and results, the celebrated article of K. Friedrichs [21] indicated that Hilbert spaces associated to the quadratic form $\langle H\cdot,\cdot\rangle$, where H is the Hamiltonian of a system, can be viewed as energy spaces of the system. This theory triggered the theory of nonnegative quadratic forms, e.g. see T. Kato [23], M. Reed and B. Simon [27,28] and the vast bibliography cited there, and has deep connections with the theory of operator ranges, as initiated by the pioneering works of G. Mackey [25], J. Dixmier [15,16], and L. Schwartz [29], and surveyed in a more modern presentation by P.A. Fillmore and J.P. Williams [19]. In [6] we extended some of these ideas to Hilbert spaces induced by unbounded operators. However, the notion of a Hilbert space induced by a positive selfadjoint operator is rather abstract and its uniqueness is determined only up to unitary equivalence. For more practical reasons, when the ambient Hilbert space of the Hamiltonian is a function space, it is desirable to get an energy space that is also a function space on the same base set and, in addition, have appropriate uniqueness properties. Put in this way, this question leads to the notion of replacing a continuous embedding of Hilbert spaces by a closed embedding, and we did this in [8] by showing that what we get is a special type of induced Hilbert space, more or less the equivalent of an operator range, and applied this theory to certain homogenous Sobolev spaces. So, roughly speaking, we cannot always get the energy space as a function space but if we accept that “a few” elements are allowed to live in distributions spaces on the same base set instead, then the theory provides sufficiently useful answers.

On the other hand, when the Hamiltonian is no longer a positive operator, but only a selfadjoint one, then the first modification that one has to perform is to replace the Hilbert space by a Kreĭn space, paying the price of all the geometric-topological difficulties that usually show up in the underlying operator theory on Kreĭn spaces. We made this step in [7]

* Corresponding author at: Department of Mathematics, Bilkent University, 06800 Bilkent, Ankara, Turkey.
E-mail addresses: cojuhari@uci.agh.edu.pl (P. Cojuhari), aurelian@fen.bilkent.edu.tr, A.Gheondea@imar.ro (A. Gheondea).

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when we introduced the notion of a Kreĭn space induced by a selfadjoint operator and studied mainly the uniqueness question. The answer to the uniqueness question is that the "lateral spectral gap" condition of T. Hara [22] (see also [11,17,14] for equivalent results) governs the uniqueness of the induced Kreĭn space in pretty much the same way as in the bounded case. Our main application and motivation for pursuing the indefinite generalizations are strongly connected with the Dirac operators (see [20,31] and, for a recent approach of Hilbert–Kreĭn structures in similar problems see [10]).

Having in mind these same Dirac operators, the next step in this enterprise was to investigate the possibility of getting a notion of Kreĭn spaces closely embedded, and we recently did this in [9] by means of a generalization of the de Branges spaces of Kreĭn type and applied this to the free Dirac operator corresponding to a particle with nontrivial mass. In this article we continue this direction of investigation by providing a more general model in the spirit of operator ranges and by getting a characterization of uniqueness in terms of the lateral spectral gap. Our main results refer to applications to Dirac operators and we show that the pictures differ according to a mass or a massless particle in a dramatic way: in the case of a particle with a nontrivial mass we obtain a dual of a Sobolev type space and we can prove uniqueness, while in the case of a massless particle we obtain the dual of a homogenous Sobolev type space and we lose uniqueness.

We briefly present now the organization of this article. In order to combine the positive definiteness of closely embedded Hilbert spaces with indefiniteness of the induced Kreĭn spaces, after briefly recalling the definition and the basic results on induced Kreĭn spaces in Section 2, we provide in Section 3 a canonical representation in the spirit of operator ranges, which actually is a generalization of de Branges spaces of Kreĭn type as in [9]. Then, in Section 4 we recall the definition of closely embedded Kreĭn spaces and their properties as in [9]. The main result concerns the uniqueness properties obtained in Theorem 4.7. Finally, in Section 5 we apply our results to general free Dirac operator by calculating energy spaces and establishing their properties.

In order to keep this article to a reasonable size, we assume that the reader is familiar with the basic notions of indefinite inner product spaces and their linear operators, e.g. see [3]. In this respect, our notation follows the one we used in [7,8]. Since [9] is not yet published, we made our article independent by briefly recalling, in Subsection 4.1, all the definitions and the results from that article. Also, we will freely use the main concepts and results in the operator theory of unbounded selfadjoint operators, especially their spectral theory, borelian functional calculus, and polar decompositions. All these can be found in the classical textbooks of M.S. Birman and M.Z. Solomyak [2], T. Kato [23], M. Reed and B. Simon [27,28]. In the application section, we will also use the basic notions on Sobolev spaces, e.g. see R.A. Adams [1], and V.G. Maz'ja [26], as well the theory of Dirac operators, e.g. see B. Thaller [31] and L.D. Landau [24].

2. Preliminaries on Kreĭn spaces induced by symmetric operators

If A is a symmetric densely defined linear operator in the Hilbert space $\mathcal{H}$ we can define a new inner product $[\cdot,\cdot]_A$ on $\mathrm{Dom}(A)$, the domain of A, by

$$[x, y]_A = \langle Ax, y\rangle_{\mathcal{H}}, \quad x, y \in \mathrm{Dom}(A). \tag{2.1}$$

In this subsection we recall the existence and the properties of some Kreĭn spaces associated to this kind of inner product space, cf. [7].

A pair $(\mathcal{K}, \Pi)$ is called a *Kreĭn space induced by A* if:

(iks1) $\mathcal{K}$ is a Kreĭn space;
(iks2) Π is a linear operator from $\mathcal{H}$ into $\mathcal{K}$ such that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi)$;
(iks3) $\Pi\,\mathrm{Dom}(A)$ is dense in $\mathcal{K}$;
(iks4) $[\Pi x, \Pi y]_{\mathcal{K}} = \langle Ax, y\rangle_{\mathcal{H}}$ for all $x \in \mathrm{Dom}(A)$ and $y \in \mathrm{Dom}(\Pi)$.

The operator Π is called *the canonical operator*. In case the operator A is bounded, this is a definition first considered in [12,13].

Remark 2.1.

(1) $(\mathcal{K}, \Pi)$ is a Kreĭn space induced by A if and only if it satisfies the axioms (iks1)–(iks3) and
(iks4′) $\Pi^\sharp\Pi \supseteq A$, in the sense that $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(\Pi^\sharp\Pi)$ and $Ax = \Pi^\sharp\Pi x$ for all $x \in \mathrm{Dom}(A)$.
(2) If A is selfadjoint, hence maximal symmetric, the axiom (iks4′) is equivalent with
(iks4″) $\Pi^\sharp\Pi = A$, in the sense that $\mathrm{Dom}(\Pi^\sharp\Pi) = \mathrm{Dom}(A)$ and $Ax = \Pi^\sharp\Pi x$ for all $x \in \mathrm{Dom}(A)$.
(3) Without loss of generality we can assume that Π is closed.
(4) If the symmetric densely defined operator A admits an induced Kreĭn space $(\mathcal{K}, \Pi)$ such that Π is bounded, then A is bounded. The converse is not true, in general, that is, if A is bounded and selfadjoint operator then it may happen that Π is unbounded. However, if A is not only bounded but also everywhere defined (in particular, if A is bounded selfadjoint), then the operator Π is bounded as well.

For a densely defined symmetric operator A in a Hilbert space, various necessary and sufficient conditions of existence of Hilbert spaces induced by A are available (see [7]). In this paper we are interested mainly in the case of selfadjoint operators, when the existence is guaranteed by the spectral theorem. The first example starts with a selfadjoint operator A and describes a construction of a Kreĭn space induced by A, more or less the equivalent of the quotient completion method.

Example 2.2. Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. We consider the polar decomposition of A

$$A = S_A|A|, \tag{2.2}$$

where, by borelian functional calculus, there is defined $|A| = (A^*A)^{1/2} = (A^2)^{1/2}$, the *modulus* (or the *absolute value*) of the operator A, and $S_A = \operatorname{sgn}(A)$ is a selfadjoint partial isometry on $\mathcal{H}$. Recall that $\operatorname{Dom}(A) = \operatorname{Dom}(|A|)$ and that $|A|$ is a nonnegative selfadjoint operator. We now consider the quotient completion of $\operatorname{Dom}(A)$ with respect to the nonnegative selfadjoint operator $|A|$ as follows. Since $|A|$ is a nonnegative selfadjoint operator in the Hilbert space $\mathcal{H}$, then $|A|^{1/2}$ exists as a nonnegative selfadjoint operator in $\mathcal{H}$, $\operatorname{Dom}(|A|^{1/2}) \supseteq \operatorname{Dom}(|A|) = \operatorname{Dom}(A)$ and $\operatorname{Dom}(A)$ is a core of $|A|^{1/2}$ (e.g. see [2]). In particular we have

$$\langle |A|x, y\rangle_{\mathcal{H}} = \langle |A|^{1/2}x, |A|^{1/2}y\rangle_{\mathcal{H}}, \quad x \in \operatorname{Dom}(A),\ y \in \operatorname{Dom}(|A|^{1/2}),$$

which shows that we can consider the seminorm $\||A|^{1/2}\cdot\|$ on $\operatorname{Dom}(A)$ and make the quotient completion with respect to this seminorm in order to get a Hilbert space $\mathcal{K}_A$. We denote by Π_A the corresponding canonical operator. It is seen that $(\mathcal{K}_A, \Pi_A)$ is a Hilbert space induced by $|A|$, cf. [6].

Further on, $\operatorname{Ker}(S_A) = \operatorname{Ker}(A)$ and S_A leaves invariant $\operatorname{Dom}(A)$. Since S_A is a selfadjoint partial isometry, its spectrum coincides with its point spectrum and is contained in $\{-1, 0, +1\}$ and $\operatorname{Dom}(A) = \mathcal{D}_+ \oplus \operatorname{Ker}(A) \oplus \mathcal{D}_-$, where

$$\mathcal{D}_\pm = \operatorname{Dom}(A) \cap \operatorname{Ker}(S_A \mp I). \tag{2.3}$$

This implies that we can identify naturally $\operatorname{Dom}(A)/\operatorname{Ker}(A)$ with $\mathcal{D}_+ \oplus \mathcal{D}_-$. Now observe that we can complete $\mathcal{D}_\pm$ with respect to the norm $\||A|^{1/2}\cdot\|$ and let these completions be denoted by $\mathcal{K}_A^\pm$ and that $\mathcal{K}_A$ can be naturally identified with $\mathcal{K}_A^+ \oplus \mathcal{K}_A^-$, and considering this as a fundamental decomposition,

$$\mathcal{K}_A = \mathcal{K}_A^+\,[+]\,\mathcal{K}_A^- \tag{2.4}$$

it yields an indefinite inner product $[\cdot,\cdot]$ with respect to which $\mathcal{K}_A$ becomes a Kreĭn space.

Two Kreĭn spaces $(\mathcal{K}_i, \Pi_i)$, $i = 1, 2$, induced by the same symmetric operator A, are called *unitary equivalent* if there exists a bounded unitary operator $U : \mathcal{K}_1 \to \mathcal{K}_2$ such that

$$U\Pi_1 x = \Pi_2 x, \quad x \in \operatorname{Dom}(A). \tag{2.5}$$

In order to exploit the full power of induced Kreĭn spaces we need to know which linear operators can be lifted to induced Kreĭn spaces. We answered affirmatively this question in [7] for the Kreĭn spaces in the unitary orbit of $(\mathcal{K}_A, \Pi_A)$, that is, for any other Kreĭn space $(\mathcal{K}, \Pi)$ that is unitary equivalent with $(\mathcal{K}_A, \Pi_A)$. Throughout, $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ denotes the collection of all bounded linear operator $T : \mathcal{H}_1 \to \mathcal{H}_2$.

Theorem 2.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let A and B be selfadjoint operators in $\mathcal{H}_1$ and respectively $\mathcal{H}_2$. We consider the induced Kreĭn spaces $(\mathcal{K}_A, \Pi_A)$ and $(\mathcal{K}_B, \Pi_B)$. Then for any operators $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, and $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \operatorname{Dom}(B),\ y \in \operatorname{Dom}(A), \tag{2.6}$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{L}(\mathcal{K}_A, \mathcal{K}_B)$ and $\widetilde{S} \in \mathcal{L}(\mathcal{K}_B, \mathcal{K}_A)$ such that $\widetilde{T}\Pi_A x = \Pi_B T x$ for all $x \in \operatorname{Dom}(A)$ and $\widetilde{S}\Pi_B y = \Pi_A S y$, for all $y \in \operatorname{Dom}(B)$ and

$$\langle \widetilde{S}h, k\rangle_{\mathcal{K}} = \langle h, \widetilde{T}k\rangle_{\mathcal{K}}, \quad h \in \mathcal{K}_B,\ k \in \mathcal{K}_A.$$

In the special case of a selfadjoint operator, we recall the characterization of uniqueness induced Kreĭn spaces in spectral terms, cf. [6] (a generalization of results in [22,11,17,14]). In the following, $\rho(A)$ denotes the resolvent set of the operator A.

Theorem 2.4. *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$. The following statements are equivalent*:

(i) *The Kreĭn space induced by A is unique, modulo unitary equivalence.*
(ii) *A has a lateral spectral gap, that is, there exists an $\epsilon > 0$ such that either $(0, \epsilon) \subset \rho(A)$ or $(-\epsilon, 0) \subset \rho(A)$.*

3. The Kreĭn space $\mathcal{R}(T)$

In this section we investigate a generalization of the space $\mathcal{R}(T)$, as introduced and studied in [8] for the Hilbert space case, to Kreĭn spaces. The Kreĭn space $\mathcal{R}(T)$ is associated to a closed operator on a Kreĭn space. This construction will shed more light on the definition of a closely embedded Kreĭn space, already introduced in [9]. The generalized de Branges space $\mathcal{B}_A$ presented in [9] is actually of type $\mathcal{R}(T)$.

Let T be a linear operator acting between two Kreĭn spaces $\mathcal{G}$ and $\mathcal{H}$. We assume that *the domain of T is dense and that* $\operatorname{Ker}(T)$ *is a regular subspace of $\mathcal{G}$.*

On the linear submanifold $\operatorname{Ran}(T)$ of $\mathcal{H}$ we can define an indefinite inner product

$$[Tx, Ty]_T := [x, y]_{\mathcal{G}}, \quad x, y \in \operatorname{Dom}(T),\ x, y\,[\perp]\operatorname{Ker}(T), \tag{3.1}$$

as well as a quadratic norm

$$\|Tx\|_T := \|x\|_G, \quad x \in \operatorname{Dom}(T),\ x\,[\perp]\operatorname{Ker}(T), \tag{3.2}$$

where $\|\cdot\|_G$ is a fixed fundamental norm on the Kreĭn space $\mathcal{G}$, that is, induced by a fixed fundamental symmetry G on $\mathcal{G}$, subject to the property that it leaves the regular subspace $\operatorname{Ker}(T)$ invariant. To be more precise, for such a fundamental symmetry we note that $\operatorname{Ker}(T)^\perp$ is also invariant under G, note that $x \perp \operatorname{Ker}(T)$ if and only if $x\,[\perp]\operatorname{Ker}(T)$. Note that $(\operatorname{Ran}(T); [\cdot,\cdot]_T)$ has a (abstract) completion, with respect to the quadratic norm $\|\cdot\|_T$, to a Kreĭn space that we denote by $\mathcal{R}(T)$. Further, consider the *embedding operator* j_T with domain $\operatorname{Dom}(j_T) = \operatorname{Ran}(T) \subseteq \mathcal{R}(T)$ and valued in $\mathcal{H}$, defined by

$$j_T u = u, \quad u \in \operatorname{Ran}(T). \tag{3.3}$$

Lemma 3.1. *Let T be a linear operator with domain dense in the Kreĭn space $\mathcal{G}$ and valued in the Kreĭn space $\mathcal{H}$, such that* $\operatorname{Ker}(T)$ *is a regular subspace of $\mathcal{G}$. Consider the Kreĭn space $\mathcal{R}(T)$ and the embedding j_T defined as in* (3.1)–(3.3)*. Then, there exists a unique* (*Kreĭn space*) *coisometry $U_T \in \mathcal{L}(\mathcal{G}, \mathcal{R}(T))$, such that* $\operatorname{Ker}(U_T) = \operatorname{Ker}(T)$ *and $T \subseteq j_T U_T$. Since* $\operatorname{Ker}(T)$ *is closed we get $T = j_T U_T$.*

Proof. The proof is very similar to the one of Lemma 2.5 in [8], so we skip repetitive arguments. Letting $U_T : \operatorname{Dom}(T)(\subseteq \mathcal{G}) \to \operatorname{Ran}(T)(\subseteq \mathcal{H})$ be defined by

$$U_T x := Tx, \quad x \in \operatorname{Dom}(T), \tag{3.4}$$

it follows that U_T is isometric both with respect to the indefinite inner products $[\cdot,\cdot]_{\mathcal{G}}$ and $[\cdot,\cdot]_T$, as well as with respect to the quadratic norms $\|\cdot\|_{\mathcal{G}}$ and $\|\cdot\|_T$. Thus, U_T extends uniquely to a (bounded) coisometry of Kreĭn spaces $U_T \in \mathcal{L}(\mathcal{G}, \mathcal{R}(T))$ such that $\operatorname{Ker}(U_T) = \operatorname{Ker}(T)$ and $T \subseteq j_T U_T$. $\square$

Remark 3.2. The assumption in Lemma 3.1 that T is densely defined can be replaced by the more general one that $\overline{\operatorname{Dom}(T)}$ is a regular subspace of $\mathcal{G}$. In this case, in order to define the Kreĭn space $\mathcal{R}(T)$ we have to use a fundamental symmetry G subject to the condition that it leaves invariant both $\operatorname{Ker}(T)$ and $\overline{\operatorname{Dom}(T)}$. Then, the coisometry U_T is obtained with the properties $\operatorname{Ker}(U_T) = \operatorname{Ker}(T)\,[+]\,(\mathcal{G}\,[-]\operatorname{Dom}(T))$ and $TP_{\overline{\operatorname{Dom}(T)}} = j_T U_T$.

Similar statements (some of them identical) with the ones in Propositions 2.7 and 2.8 in [8] can be stated and proved immediately. Additional information is produced by the following

Proposition 3.3. *If $T \in \mathcal{C}(\mathcal{G}, \mathcal{H})$ for some Kreĭn spaces $\mathcal{G}$ and $\mathcal{H}$ such that* $\operatorname{Ker}(T)$ *is a regular subspace in $\mathcal{G}$, then the Kreĭn space $\mathcal{R}(T)$ defined at* (3.1) *and* (3.2) *has the following properties*:

(i) *The embedding operator j_T defined at* (3.3) *is densely defined and closed.*
(ii) $\operatorname{Dom}(j_T) = \operatorname{Ran}(T) = \mathcal{D}_-\,[+]\,\mathcal{D}_+$ *for some uniformly negative/positive definite linear manifolds $\mathcal{D}_\pm$ in the Kreĭn space $\mathcal{R}(T)$.*
(iii) $(\mathcal{R}(T), j_T^\sharp)$ *is a Kreĭn space induced by the selfadjoint operator $A = j_T j_T^\sharp$.*

Proof. (i) This follows by Lemma 3.1.

(ii) Let $\mathcal{G} = \mathcal{G}^+\,[+]\,\mathcal{G}^-$ be the fundamental decomposition associated to the fundamental symmetry G that was used to define the strong topology, see (3.2). Then $\mathcal{D}_\pm := U_T(\mathcal{K}^\pm) \cap \operatorname{Ran}(T)$ are uniformly definite linear manifolds in the Kreĭn space $\mathcal{R}(T)$ and $\mathcal{R}(T) = \mathcal{D}_+\,[+]\,\mathcal{D}_-$.

(iii) $\mathcal{R}(T)$ is a Kreĭn space, $j_T^\sharp$ is a linear operator from $\mathcal{H}$ in $\mathcal{R}(T)$ and $\operatorname{Dom}(A) = \operatorname{Dom}(j_T j_T^\sharp) \subseteq \operatorname{Dom}(j_T^\sharp)$. Also, $j_T^\sharp \operatorname{Dom}(A) = j_T^\sharp \operatorname{Dom}(j_T j_T^\sharp)$. To see this, let $u \in \operatorname{Ran}(j_T^\sharp)$ be such that it is $[\cdot,\cdot]_T$-orthogonal on $j_T^\sharp \operatorname{Dom}(A)$. Then, $u = j_T^\sharp y$ for some $y \in \operatorname{Dom}(j_T^\sharp)$ and for all $x \in \operatorname{Dom}(j_T j_T^\sharp)$ we have

$$0 = \left[j_T^\sharp x, j_T^\sharp y\right]_T = \left[j_T j_T^\sharp x, y\right]_{\mathcal{H}},$$

hence y is orthogonal to $\operatorname{Ran}(A)$, which is dense in $\mathcal{H}$, and hence $y = 0$ and $u = j_T^\sharp y = 0$.

Finally, for all $x \in \operatorname{Dom}(A) = \operatorname{Dom}(j_T j_T^\sharp)$ and all $y \in \operatorname{Dom}(j_T^\sharp)$ we have

$$\left[j_T^\sharp x, j_T^\sharp y\right]_{\mathcal{R}(T)} = \left[j_T j_T^\sharp x, y\right]_{\mathcal{H}} = [Ax, y]_{\mathcal{H}},$$

and hence $(\mathcal{R}(T), j_T^\sharp)$ is a Kreĭn space induced by A. $\square$

Remark 3.4. The Kreĭn space structure of $\mathcal{H}$ (the codomain space) of the operator T does not play an essential role in the construction of the Kreĭn space $\mathcal{R}(T)$; all we need is its Hilbert space strong topology. On the contrary, the Kreĭn space structure of $\mathcal{G}$ (the domain space) and the additional constraints are essential for the construction of the Kreĭn space $\mathcal{R}(T)$. Because of this and in order to keep the notation simpler, we will mainly consider the case when the ambient space $\mathcal{H}$ is a Hilbert space.

Remark 3.5. Let A be a selfadjoint operator in a Hilbert space $\mathcal{H}$. With the notation as in Example 2.2, note that the space $\mathcal{G} = \mathcal{H} \ominus \operatorname{Ker}(A)$ has a natural structure of Kreĭn space, letting S_A (compressed to $\mathcal{H}$) be its fundamental symmetry. Then, letting $T = |A|^{1/2}$, considered as a linear operator from $\mathcal{G}$ in $\mathcal{H}$, the Kreĭn space $\mathcal{B}_A$ defined in [9] is exactly the Kreĭn space $\mathcal{R}(T)$.

4. Closely embedded Kreĭn spaces

The notion of closely embeded Kreĭn spaces makes the connection between induced Kreĭn spaces and L. de Branges and J. Rovnyak [5,4], and L. Schwartz [29] theory of Hilbert/Kreĭn spaces continuously contained. Our main concern is a characterization of uniqueness. To this end we first recall the definition and the basic properties, cf. [9].

4.1. Definition and basic properties of closely embedded Kreĭn spaces

In view of Proposition 3.3 the natural definition of a closely embedded Kreĭn space can be given. According to Remark 3.4, without loss of generality the ambient space $\mathcal{H}$ will be considered a Hilbert space. Thus, a Kreĭn space $\mathcal{K}$ is called *closely embedded* in $\mathcal{H}$ if:

(cek1) There exists a linear manifold $\mathcal{D}$ in $\mathcal{K} \cap \mathcal{H}$ that is dense in $\mathcal{K}$.
(cek2) The canonical embedding $j : \mathcal{D}(\subseteq \mathcal{K}) \to \mathcal{H}$ is closed, as an operator from $\mathcal{K}$ to $\mathcal{H}$.
(cek3) There exists positive/negative uniformly definite linear manifolds $\mathcal{D}_\pm$ in $\mathcal{K}$ such that $\operatorname{Dom}(j) = \mathcal{D}_+ [+] \mathcal{D}_-$.

This definition is a generalization of the concept of closely embedded Hilbert space that allows us to establish the connection with induced Kreĭn spaces. Again, the meaning of the axiom (cek1) is that on $\mathcal{D}$ the algebraic structures of $\mathcal{K}$ and $\mathcal{H}$ coincide.

Proposition 4.1. *If $\mathcal{H}$ is a Hilbert space and $\mathcal{K}$ is a Kreĭn space closely embedded in $\mathcal{H}$, with embedding operator j, then $A = jj^\sharp$ is a selfadjoint operator in $\mathcal{H}$ and $(\mathcal{K}; j^\sharp)$ is a Kreĭn space induced by A.*

Proof. By the R. Phillips Extension Theorem, e.g. see [3], there exists $\mathcal{K} = \mathcal{K}^+ [+] \mathcal{K}^-$ a fundamental decomposition of the Kreĭn space $\mathcal{K}$ such that $\mathcal{D}_\pm \subseteq \mathcal{K}^\pm$, and let J be the associated fundamental decomposition. Then $A = jj^\sharp = jJj^*$ is a selfadjoint operator in the Hilbert space $\mathcal{H}$, where j^* is the adjoint of j with respect to the Hilbert space $\mathcal{H}_+ := (\mathcal{K}; \langle\cdot,\cdot\rangle_J)$. Also, $|A| = jj^*$ and we can apply Proposition 3.1 in [8] in order to conclude that $(\mathcal{H}_+; j^*)$ is a Hilbert space induced by $|A|$. Since $j^\sharp = Jj^*$ this implies that $(\mathcal{K}; j^\sharp)$ is a Kreĭn space induced by A. $\square$

Given $\mathcal{K}$, a Kreĭn space closely embedded in the Hilbert space $\mathcal{H}$, with the closed embedding $j : \operatorname{Dom}(j)(\subseteq \mathcal{K}) \to \mathcal{H}$, we call $A := jj^\sharp$ the *kernel operator* of $\mathcal{K}$. The axiom (cek3) in the definition of a closely embedded Kreĭn space is justified by the anomaly in the indefinite setting that allows closed densely defined operators T between Kreĭn spaces such that $TT^\sharp$ may not be densely defined.

Remark 4.2. From Proposition 3.3 it follows that if T is a closed densely defined operator from a Kreĭn space $\mathcal{G}$ to another Kreĭn space $\mathcal{H}$ such that $\operatorname{Ker}(T)$ is regular, then the Kreĭn space $\mathcal{R}(T)$ is closely embedded in the ambient Kreĭn space $\mathcal{H}$.

We also recall a generalization of the variant of the Lifting Theorem in [18], cf. [9]. This theorem is actually another variant of Theorem 2.3, in view of Proposition 4.1. For the notation of the Kreĭn space $\mathcal{B}_A$ see Remark 3.5.

Theorem 4.3. *Let A and B be two selfadjoint operators in the Hilbert spaces $\mathcal{H}_1$ and respectively $\mathcal{H}_2$. We consider the Kreĭn spaces $\mathcal{B}_A$ and $\mathcal{B}_B$, closely embedded in $\mathcal{H}_1$ and respectively $\mathcal{H}_2$, as well as the closed embeddings $j_A : \mathrm{Dom}(j_A)(\subseteq \mathcal{B}_A) \to \mathcal{H}_1$ and respectively $j_B : \mathrm{Dom}(j_B)(\subseteq \mathcal{B}_B) \to \mathcal{H}_2$. Then, for any operators $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, and $S \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$ such that*

$$\langle Bx, Ty\rangle_{\mathcal{H}_2} = \langle Sx, Ay\rangle_{\mathcal{H}_1}, \quad x \in \mathrm{Dom}(B),\ y \in \mathrm{Dom}(A),$$

there exist uniquely determined operators $\widetilde{T} \in \mathcal{L}(\mathcal{B}_A, \mathcal{B}_B)$ and $\widetilde{S} \in \mathcal{L}(\mathcal{B}_B, \mathcal{B}_A)$ such that $\widetilde{T} j_A^\sharp x = j_B^\sharp Tx$ for all $x \in \mathrm{Dom}(A)$, $\widetilde{S} j_B^\sharp y = j_A^\sharp Sy$, for all $y \in \mathrm{Dom}(B)$, and

$$\langle \widetilde{S}h, k\rangle_{\mathcal{B}_A} = \langle h, \widetilde{T}k\rangle_{\mathcal{B}_B}, \quad h \in \mathcal{B}_B,\ k \in \mathcal{B}_A.$$

4.2. Uniqueness of closely embedded Kreĭn spaces

We can now approach the question on uniqueness of a closely embedded Kreĭn space with respect to the kernel operator. We first write down the interplay between closely embedded Kreĭn spaces and kernel operators.

Proposition 4.4. *Let $\mathcal{K}$ be a Kreĭn space closely embedded in the Hilbert space $\mathcal{H}$, with the corresponding closed embedding j and kernel operator $A = jj^\sharp$. Then*:

(i) $\langle jx, y\rangle_{\mathcal{H}} = [x, Ay]_{\mathcal{K}}$ *for all* $x \in \mathrm{Dom}(j)$ *and all* $y \in \mathrm{Dom}(A)$.
(ii) $\mathrm{Ran}(A) \subseteq \mathcal{K}$ *is dense in* $\mathcal{K}$.

Proof. We first observe that $\mathrm{Ran}(A) = \mathrm{Ran}(jj^\sharp) \subseteq \mathrm{Ran}(j) = \mathrm{Dom}(j) \subseteq \mathcal{K}$, hence $\mathrm{Ran}(A) \subseteq \mathcal{K}$.

Let $x \in \mathrm{Dom}(j)$ and $y \in \mathrm{Dom}(A)$. Then, since $\mathrm{Dom}(A) \subseteq \mathrm{Dom}(j^\sharp)$ we have $\langle jx, y\rangle_{\mathcal{H}} = [x, j^\sharp y]_{\mathcal{H}}$. Taking into account that $j^\sharp y \in \mathrm{Dom}(j)$ and that j is simply the identity operator, it follows that

$$\langle jx, y\rangle_{\mathcal{H}} = \big[x, j^\sharp y\big]_{\mathcal{K}} = \big[x, jj^\sharp y\big]_{\mathcal{H}} = [x, Ay]_{\mathcal{K}}.$$

To see that $\mathrm{Ran}(A)$ is dense in $\mathcal{K}$, it is sufficient to prove that it is dense in $\mathrm{Dom}(j)$. To this end, let $x \in \mathrm{Dom}(j)$ that is $[\cdot,\cdot]_{\mathcal{K}}$-orthogonal on $\mathrm{Ran}(A)$. Then, for all $y \in \mathrm{Dom}(A)$ we have $0 = [x, Ay]_{\mathcal{K}} = \langle jx, y\rangle_{\mathcal{H}}$, hence, since $\mathrm{Dom}(A)$ is dense in $\mathcal{H}$ it follows that $0 = jx = x$. □

Proposition 4.4 shows that for any selfadjoint operator A in Hilbert space $\mathcal{H}$, the linear manifold $\mathrm{Ran}(A)$ lies densely in all Kreĭn spaces closely embedded in $\mathcal{H}$ and with kernel operator A. We say that A has the *uniqueness closely embedded Kreĭn space property* if, for any two Kreĭn spaces $\mathcal{K}_i$, closely embedded in $\mathcal{H}$, and with closed embedding j_i, $i = 1, 2$, the identity operator on $\mathrm{Ran}(A)$ extends (uniquely) to a unitary operator from $\mathcal{K}_1$ to $\mathcal{K}_2$.

Another natural question in connection with the uniqueness matter is whether it is always possible to produce closely embedded Kreĭn spaces from induced Kreĭn spaces. In this respect, Proposition 3.3 describes a canonical construction on how to do this, in case the kernel operator A is selfadjoint.

Lemma 4.5. *Let $(\mathcal{K}; \Pi)$ be a Kreĭn space induced by a selfadjoint operator A in the Hilbert space $\mathcal{H}$. Then $\mathcal{R}(\Pi^\sharp)$ is a closely embedded Kreĭn space in $\mathcal{H}$.*

Proof. Observe that $\Pi^\sharp$ is injective, hence the construction of the Kreĭn space as in (3.1) and (3.2), as well as of the embedding operator j_T, defined as in (3.3), make sense. Then we can apply Proposition 3.3. □

Our aim is to reduce the uniqueness of closely embedded Kreĭn spaces to the uniqueness of induced Kreĭn space, as stated in Theorem 2.4. In this respect, Lemma 4.5 is the first step. The second step is to perform this reduction and, in the same time, to clarify the meaning of "uniqueness" for closely embedded Kreĭn spaces.

Lemma 4.6. *Let A be a selfadjoint operator in the Hilbert space $\mathcal{H}$ and two Kreĭn spaces $(\mathcal{K}_1; \Pi_1)$ and $(\mathcal{K}_2; \Pi_2)$ induced by the same operator A. The following assertions are equivalent*:

(i) *The induced Kreĭn spaces $(\mathcal{K}_1; \Pi_1)$ and $(\mathcal{K}_2; \Pi_2)$ are unitarily equivalent.*
(ii) *The identity operator on* $\mathrm{Ran}(A)$ *extends* (*uniquely*) *to a unitary operator* $U \in \mathcal{L}(\mathcal{R}(\Pi_1^\sharp), \mathcal{R}(\Pi_2^\sharp))$.

Proof. Let $V \in \mathcal{L}(\mathcal{K}_2, \mathcal{K}_1)$ be a unitary operator (of Kreĭn spaces) such that $V\Pi_2 = \Pi_1$. Then $\Pi_1^\sharp = \Pi_2^\sharp V^\sharp$ and hence $\mathrm{Ran}(\Pi_1^\sharp) = \mathrm{Ran}(\Pi_2^\sharp) =: \mathcal{D}$. Let G_i be a fundamental symmetry in $\mathcal{K}_i$, $i = 1, 2$, be such that $VG_2 = G_1$. Then, the norms $\|\cdot\|_{T_1}$ and $\|\cdot\|_{T_2}$ defined by (3.2), and using norms $\|\cdot\|_{G_1}$ and $\|\cdot\|_{G_2}$, respectively, coincide on $\mathcal{D} \supseteq \mathrm{Ran}(A)$. Also, the inner products $[\cdot,\cdot]_{T_1}$ and $[\cdot,\cdot]_{T_2}$, as defined by (3.1), coincide. Therefore, the identity operator on $\mathcal{D}$, and hence, the identity operator on $\operatorname{Ran}(A)$, extends (uniquely) to a unitary operator $U \in \mathcal{L}(\mathcal{R}(\Pi_1^\sharp), \mathcal{R}(\Pi_2^\sharp))$.

Conversely, let $U \in \mathcal{L}(\mathcal{R}(\Pi_1^\sharp), \mathcal{R}(\Pi_2^\sharp))$ be the unitary operator of Kreĭn spaces that is the identity operator when restricted to $\operatorname{Ran}(A)$. Let $U_{\Pi_i^\sharp} \in \mathcal{L}(\mathcal{K}_i, \mathcal{R}(\Pi^\sharp))$ be the Kreĭn space unitary operator, as in Proposition 3.1, such that $\Pi_i^\sharp = j_{\Pi_i^\sharp} U_{\Pi_i^\sharp}$. Then $V := U^\sharp_{\Pi_2^\sharp} U U_{\Pi_1^\sharp}$ is a unitary operator between the Kreĭn spaces $\mathcal{K}_1$ and $\mathcal{K}_2$ such that $\Pi_1 = \Pi_2 V$, that is, the Kreĭn spaces $\mathcal{K}_1$ and $\mathcal{K}_2$, induced by the same selfadjoint operator A, are unitarily equivalent. □

From Lemmas 4.5, 4.6, and Theorem 2.4, we get the characterization of those selfadjoint operators in a Hilbert space A that have the unique closely embedded Kreĭn space property, which is a generalization of results in [22,11,17,14].

Theorem 4.7. *Let A be a selfadjoint operator in a Hilbert space $\mathcal{H}$. The following assertions are equivalent*:

(i) *There exists $\epsilon > 0$ such that either $(-\epsilon, 0)$ or $(0, \epsilon)$ are in the resolvent set $\rho(A)$.*
(ii) *A has the uniqueness closely embedded Kreĭn space property.*

Proof. (i) ⇒ (ii). Assume that there exists $\epsilon > 0$ such that either $(-\epsilon, 0)$ or $(0, \epsilon)$ are in $\rho(A)$ and let $\mathcal{K}$ be a Kreĭn space closely embedded in $\mathcal{H}$, with embedding operator j and kernel operator A. By Proposition 4.1 we have $A = jj^\sharp$ and $(\mathcal{K}; j^\sharp)$ is a Kreĭn space induced by A. By Theorem 2.4 and Lemma 4.6 we get that the identity operator on $\operatorname{Ran}(A)$ extends uniquely to a unitary operator $U : \mathcal{R}(\Pi^\sharp_{\mathcal{B}_A}) \to \mathcal{R}(j^\sharp) = \mathcal{K}$, and hence A has the uniqueness closely embedded Kreĭn space property.

(ii) ⇒ (i). Conversely, assume that A has the uniqueness closely embedded Kreĭn space property but A does not have the lateral spectral gap property, that is, 0 is an accumulation point in the spectrum of A from both sides. By Theorem 2.4, there exist two Kreĭn spaces $(\mathcal{K}_1; \Pi_1)$ and $(\mathcal{K}_2; \Pi_2)$ induced by A that are not unitarily equivalent. By Lemma 4.5, $\mathcal{R}(\Pi_i^\sharp)$, $i = 1, 2$ are closely embedded in $\mathcal{H}$ hence, the identity operator on $\operatorname{Ran}(A)$ extends uniquely to a unitary operator $U : \mathcal{R}(\Pi_1^\sharp) \to \mathcal{R}(\Pi_2^\sharp)$ of Kreĭn spaces, and this contradicts the fact that the two Kreĭn spaces $(\mathcal{K}_1; \Pi_1)$ and $(\mathcal{K}_2; \Pi_2)$ induced by A are not unitarily equivalent. □

5. Closely embedded Kreĭn spaces associated to Dirac operators

One of the motivations for introducing the concept of closely embedded Kreĭn space comes from a convenient energy space representation, in the sense of K. Friedrichs [21], for Dirac operators. The case of the free Dirac operator corresponding to a particle with nontrivial mass have been obtained in [9] but here we can add the uniqueness of the energy space and consider the case of the massless particle as well. For this reason and for a better comparison between the mass and the massless particle cases we here explicitly consider both cases.

In this section we will use the definitions and basic properties of Sobolev spaces, as in R.A. Adams [1] and V.G. Maz'ja [26]. In addition, some basic facts on Dirac operators and their spectral theory that will be used can be found in B. Thaller [31].

Below the following notations are systematically used. We let $L_2(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes L_2(\mathbb{R}^n)$ the space of all square summable $\mathbb{C}^m$-valued functions on $\mathbb{R}^n$. By $\hat{u}(\xi)$ we denote the Fourier transform of $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$:

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{n/2}} \int u(x) e^{\mathrm{i}\langle x,\xi\rangle}\, \mathrm{d}x,$$

in which $\langle x, \xi\rangle$ designates the scalar product of all elements $x, \xi \in \mathbb{R}^n$. Here and in what follows $\int := \int_{\mathbb{R}^n}$. The norm in $\mathbb{R}^n$ (or $\mathbb{C}^m$) will be denoted simply by $|\cdot|$. The operator norm of $m \times m$ matrices corresponding to the norm $|\cdot|$ in $\mathbb{C}^m$ will be denoted by $|\cdot|$, as well. We will also need two more Hilbert spaces. $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes W_2^{-1/2}(\mathbb{R}^n)$ is defined as the completion of $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with respect to the norm

$$\|u\|^2_{W_2^{-1/2}} := \int \left(1 + |\xi|^2\right)^{-1/2} \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi. \tag{5.1}$$

In addition, $W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes W_2^{1/2}(\mathbb{R}^n)$ is defined to be the Sobolev space of all $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ with norm

$$\|u\|^2_{W_2^{1/2}} := \int \left(1 + |\xi|^2\right)^{1/2} \left|\hat{u}(\xi)\right|^2 \mathrm{d}\xi < \infty. \tag{5.2}$$

5.1. A general Dirac operator for a free particle with nontrivial mass

Let H denote the free Dirac operator defined in the space $L_2(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes L_2(\mathbb{R}^n)$ by

$$H = \sum_{k=1}^{n} \alpha_k \otimes D_k + \alpha_0 \otimes I, \tag{5.3}$$

where $D_k = \mathrm{i}\partial/\partial x_k$ for $(k = 1, \dots, n)$, α_k for $(k = 0, 1, \dots, n)$ are $m \times m$ Hermitian matrices which satisfy the Clifford's anticommutation relations

$$\alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} I_m \quad (j, k = 0, 1, \dots, n), \tag{5.4}$$

$m = 2^{n/2}$ for n even and $m = 2^{(n+1)/2}$ for n odd, δ_{jk} denotes the Kronecker symbol, I_m is the $m \times m$ unit matrix, and I is the identity operator on $L_2(\mathbb{R}^n)$.

We consider the operator H defined on its maximal domain, the Sobolev space $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$, and viewed in this way it is a selfadjoint operator. Note that

$$\begin{aligned} H^2 &= \sum_{k=1}^{n} \alpha_k^2 \otimes D_k^2 + \sum_{j \neq k} (\alpha_j \alpha_k + \alpha_k \alpha_j) \otimes D_j D_k + \sum_{k=1}^{n} (\alpha_0 \alpha_k + \alpha_k \alpha_0) \otimes D_k + \alpha_0^2 \otimes I \\ &= \sum_{k=1}^{n} I_m \otimes D_k^2 + I_m \otimes I = I_m \otimes (-\Delta + I), \end{aligned}$$

that is,

$$H^2 = I_m \otimes (-\Delta + I), \tag{5.5}$$

where Δ denotes the Laplace operator on $\mathbb{R}^n$.

In the following we want to construct the space $\mathcal{R}(T)$ as in Section 3. One of the difficulties encountered in pursuing this way is related to making explicit and computable the operator $|H|^{1/2}$. Thus, we consider the polar decomposition of the Dirac operator H writing $H = S|H|$ with the selfadjoint and positive operator $|H|$ (the modulus of H) defined on $\mathrm{Dom}(|H|) = \mathrm{Dom}(H)$ and $S = \mathrm{sgn}(H)$. By (5.5) we have

$$|H| = I_m \otimes (-\Delta + I)^{1/2} \quad \text{and} \quad S = H\big(I_m \otimes (-\Delta + I)^{-1/2}\big).$$

Further on, we let

$$T = |H|^{1/2} = I_m \otimes (-\Delta + I)^{1/4} \tag{5.6}$$

by considering T defined in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with domain $\mathrm{Dom}(T) := W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$. The operator T represents on this domain a positive definite selfadjoint operator. In particular, T is a boundedly invertible operator, and its inverse T^{-1} is the (vector-valued) Bessel potential $I_m \otimes (I - \Delta)^{-1/4}$ of order $l = 1/2$ (cf. E.M. Stein [30]).

We consider on $\mathrm{Ran}(T) = L_2(\mathbb{R}^n; \mathbb{C}^m)$ an inner product by setting

$$\langle Tf, Tg \rangle := \langle f, g \rangle_{L_2}, \quad f, g \in W_2^{1/2}\big(\mathbb{R}^n; \mathbb{C}^m\big).$$

We can choose for the completion of $L_2(\mathbb{R}^n; \mathbb{C}^m)$ with respect to the corresponding norm $\|\cdot\|_T$ the space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ that is not entirely made up of functions, but at least of distributions $\mathbb{C}^m$-valued distributions. Keeping the notations made in Section 3 we have $\mathcal{R}(T) = W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$, for T defined as in (5.6). Since S commutes with H, it follows from Theorem 4.3 that the operator S extends uniquely to a symmetry J_T in the space $\mathcal{R}(T)$, and hence $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be regarded as a Kreĭn space with respect to the fundamental symmetry J_T. The corresponding indefinite inner product is defined by

$$[u, v]_T = \langle J_T u, v \rangle_{W_2^{-1/2}}, \quad u, v \in W_2^{-1/2}\big(\mathbb{R}^n; \mathbb{C}^m\big). \tag{5.7}$$

According to the results discussed in Section 4, $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)(= \mathcal{R}(T))$ is closely (but not continuously) embedded in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$. The canonical embedding operator j_T of $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is defined on the domain $\mathrm{Dom}(j_T) = L_2(\mathbb{R}^n; \mathbb{C}^m)$, and since the kernel operator of this closed embedding is H (cf. Proposition 4.1), we get the following factorization

$$H = j_T j_T^\sharp = j_T J_T j_T^*. \tag{5.8}$$

Concerning the symmetry S, the space $\mathcal{H} := L_2(\mathbb{R}^n; \mathbb{C}^m)$ can be decomposed into an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-,$$

where $\mathcal{H}_\pm = S_\pm \mathcal{H}$ and $S_\pm = \frac{1}{2}(I \pm S)$, that is, $S = S_+ - S_-$ is the Jordan decomposition of S. This provides the Jordan decomposition of $H = H_+ - H_-$, where

$$H_+ := S_+ H S_+ = S_+ S|H| S_+ = S_+ |H| S_+ \geqslant 0,$$

and

$$H_- := S_- H S_- = S_- S|H| S_- = -S_- |H| S_- \leqslant 0$$

on $\operatorname{Dom}(H)$. In this respect, we note that both operators H_+ and H_- are positive definite selfadjoint in $\mathcal{H}$, and that $\sigma(H_-) = (-\infty, -1]$ and $\sigma(H_+) = [1, +\infty)$ (cf. 5.5) and, of course, $\sigma(H) = \sigma(H_-) \cup \sigma(H_+) = (-\infty, -1] \cup [1, +\infty)$. Since H has a spectral gap about 0, we can apply the uniqueness Theorem 4.7.

Summing up we can formulate the following.

Theorem 5.1.

(i) *The space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ defined by* (5.1) *can be organized as a Kreĭn space by extending uniquely the symmetry $S = \operatorname{sgn}(H)$ to a fundamental symmetry J_T on the space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$.*

(ii) *The space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ endowed with the indefinite inner product* (5.7) *is a Kreĭn space closely, but not continuously, embedded in $L_2(\mathbb{R}^n; \mathbb{C}^m)$, with canonical embedding operator j_T having the domain $L_2(\mathbb{R}^n; \mathbb{C}^m)$, and the kernel operator of this canonical embedding j_T is the Dirac operator H.*

(iii) *The closely embedded Kreĭn space $W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$, organized as before, is uniquely determined by its kernel operator H.*

(iv) *The Dirac operator H admits the factorization* (5.8).

According to the K. Friedrichs interpretation of the energy space associated to a Hamiltonian, the Kreĭn space $\mathcal{K} = W_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be regarded as the energy space associated to the Dirac operator H. This space consists of distributions in which the function space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is dense. The Kreĭn space structure of $\mathcal{K}$ shows that there exist some vectors u of positive energy $[u, u]_{\mathcal{K}} > 0$, some vectors v of negative energy $[v, v]_{\mathcal{K}} < 0$, as well as nontrivial vectors w of null energy $[w, w]_{\mathcal{K}} = 0$. The fundamental symmetry J_T defined as the lifting of the symmetry S from $\mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^m)$ to $\mathcal{K}$ through the lifting Theorem 4.3, has a special role, because the associated fundamental symmetry $\mathcal{K} = \mathcal{K}_-[+]\mathcal{K}_+$ has the remarkable property that $\mathcal{H}_\pm$ are, respectively, dense in $\mathcal{K}_\pm$. Thus, even though some of the elements in $\mathcal{K}_\pm$ are distributions, they can be approximated by functions in $\mathcal{H} = L_2(\mathbb{R}^n; \mathbb{C}^m)$ with respect to the norm (5.1), of the same type (that is, positive or, respectively, negative).

5.2. The massless free particle Dirac operator

The particle with negligible mass is described by the Dirac operator obtained from the operator (5.3) without the last term $\alpha_0 \otimes I$. We denote this operator by H_0. More precisely, we consider the Dirac operator H_0 defined in $L_2(\mathbb{R}^n; \mathbb{C}^m)$, with domain $W_2^1(\mathbb{R}^n; \mathbb{C}^m)$, by

$$H_0 = \sum_{k=1}^{n} \alpha_k \otimes D_k. \tag{5.9}$$

The operator H_0 is selfadjoint in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ and its spectrum covers the whole real line, $\sigma(H_0) = \mathbb{R}$. Similarly as for (5.5), it can be checked that

$$H_0^2 = I_m \otimes (-\Delta). \tag{5.10}$$

Analogously, as in the previous subsection, we proceed from the polar decomposition of the operator $H_0 = S_0|H_0|$, where $|H_0| = (H_0^2)^{1/2}$ and $S_0 = \operatorname{sgn}(H_0)$. It is easily seen from (5.10) that

$$|H_0| = I_m \otimes (-\Delta)^{1/2}.$$

Note that S_0 can be regarded as a pseudodifferential operator in L_2 having the symbol $S(\xi) = |\xi|^{-1} h_0(\xi)$, $\xi \in \mathbb{R}^n$, where

$$h_0(\xi) := \sum_{k=1}^{n} \xi_k \alpha_k,$$

that is

$$(S_0 u)(x) = \frac{1}{(2\pi)^{n/2}} \int |\xi|^{-1} h_0(\xi) \hat{u}(\xi) e^{-\mathrm{i}\langle x, \xi \rangle}\, \mathrm{d}\xi, \quad x \in \mathbb{R}^n,$$

for each $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$.

Further on, let $T_0 = |H_0|^{1/2}$, that is

$$T_0 = I_m \otimes (-\Delta)^{1/4} \tag{5.11}$$

defined on its maximal domain $\mathrm{Dom}(T_0) := W_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$. Clearly, under the Fourier transform (in the momentum space), T_0 turns into the multiplication operator by $|\xi|^{1/2}$. It follows that T_0 is a selfadjoint and injective operator. The inverse operator T_0^{-1} is the (vector-valued) M. Riesz potential $I_m \otimes (-\Delta)^{-1/4}$ of order $l = 1/2$ (cf. E.M. Stein [30]), and it is well known that it represents a unbounded operator in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$.

By definition, the space $\mathcal{R}(T_0)$ is obtained as the completion of $\mathrm{Ran}(T_0)$ with respect to the norm

$$\|u\|_{T_0} = \big\|(-\Delta)^{-1/4} u\big\|_{L_2}, \quad u \in \mathrm{Ran}(T_0), \tag{5.12}$$

hence, $\mathcal{R}(T_0)$ is naturally identified with the dual space $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes \mathcal{H}_2^{-1/2}(\mathbb{R}^n)$ of the homogeneous Sobolev space $\mathcal{H}_2^{1/2}(\mathbb{R}^n; \mathbb{C}^m) = \mathbb{C}^m \otimes \mathcal{H}_2^{1/2}(\mathbb{R}^n)$ consisting of all functions $u \in W_{2,\mathrm{loc}}^{1/2}(\mathbb{R}^n; \mathbb{C}^m)$ for which

$$\|u\|^2_{\mathcal{H}_2^{1/2}} := \int \left(\big|(\nabla_l u)(x)\big|^2 + |x|^{-1} \big|u(x)\big|^2 \right) \mathrm{d}x < \infty, \quad l = \frac{1}{2},$$

where, by defintion,

$$\int \big|\nabla_l u(x)\big|^2 \,\mathrm{d}x = \int |\xi|^{2l} \big|\hat{u}(\xi)\big|^2 \,\mathrm{d}\xi$$

($\hat{u}$ the Fourier transformation of u). Again, $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is not entirely made up of functions but, it is easily seen that it is made up of $\mathbb{C}^m$-valued distributions on $\mathbb{R}^n$. The operator S_0 determines uniquely a symmetry J_{T_0} on $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$. The Kreĭn space so obtained is closely (but not continuously) embedded in the space $L_2(\mathbb{R}^n; \mathbb{C}^m)$. The canonical embedding operator j_{T_0} of $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ has domain the set $\mathrm{Ran}(T_0)$ consisting of all vector-valued functions $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ which admit representations $u = I_m \otimes (-\Delta)^{1/2} f$ for some $f \in L_2(\mathbb{R}^n; \mathbb{C}^m)$, and its kernel operator is the Dirac operator H_0. Therefore, on the domain $\mathrm{Dom}(H_0)(= W_2^1(\mathbb{R}^n; \mathbb{C}^m))$ there holds the factorization

$$H_0 = j_{T_0} j_{T_0}^{\sharp} = j_{T_0} J_{T_0} j_{T_0}^{*}. \tag{5.13}$$

Note that the space $\mathcal{H} := L_2(\mathbb{R}^n; \mathbb{C}^m)$ can be also decomposed with respect to S_0 into an orthogonal sum $\mathcal{H} = \mathcal{H}_+^0 \oplus \mathcal{H}_-^0$, where $\mathcal{H}_\pm^0 = P_\pm^0 \mathcal{H}$ and $P_\pm^0 := \frac{1}{2}(I \pm S_0)$, that is, $S_0 = P_+^0 - P_-$ is the Jordan decomposition of S_0. In turns out that the Dirac operator H_0 has the Jordan decomposition $H_0 = H_+^0 \oplus H_-^0$, where $H_+^0(H_-^0)$ is a nonnegative (nonpositive) selfadjoint operator in $\mathcal{H}$. Moreover, $\sigma(H_-^0) = (-\infty, 0]$, $\sigma(H_+^0) = [0, \infty)$ and $\sigma(H_0) = \sigma(H_-^0) \cup \sigma(H_+^0) = \mathbb{R}$. This shows that we can apply Theorem 4.7 in order to see that there is no uniqueness of the closely embedded Kreĭn space associated to H_0, in this case.

Summing up we have the following

Theorem 5.2.

(i) *The space $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be organized as a Kreĭn space by considering the operator $S_0 = \mathrm{sgn}(H_0)$ on the space consisting of all vector-valued functions $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ which admit representations $u = I_m \otimes (-\Delta)^{1/2} f$ for some $f \in L_2(\mathbb{R}^n; \mathbb{C}^m)$, and extending S_0 to the fundamental symmetry J_{T_0} on $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$.*

(ii) *The Kreĭn space $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is closely, but not continuously, embedded in $L_2(\mathbb{R}^n; \mathbb{C}^m)$, with canonical embedding operator j_{T_0} having domain the space of all vector-valued functions $u \in L_2(\mathbb{R}^n; \mathbb{C}^m)$ which admit representations $u = I_m \otimes (-\Delta)^{1/2} f$ for some $f \in L_2(\mathbb{R}^n; \mathbb{C}^m)$.*

(iii) *The kernel operator of the closed embedding of $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ in $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is the Dirac operator H_0, in particular, the factorization* (5.13) *holds.*

(iv) *The closely embedded Kreĭn space $\mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ is not uniquely determined by its kernel operator H_0.*

The analog of the energy space interpretation, in the sense of K. Friedrichs, that was obtained for the Dirac operator H in the previous subsection, holds for the Dirac operator H_0 as well. More precisely, the Kreĭn space $\mathcal{K}_0 = \mathcal{H}_2^{-1/2}(\mathbb{R}^n; \mathbb{C}^m)$ can be regarded as the energy space associated to the Dirac operator H_0. However, there are two major differences: first is that in the massless particle case there is no uniqueness, and second is that the function space $L_2(\mathbb{R}^n; \mathbb{C}^m)$ is not entirely included in $\mathcal{K}_0$, it only has a linear submanifold that is dense in both $L_2(\mathbb{R}^n; \mathbb{C}^m)$ and $\mathcal{K}_0$.

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