A Polyhedral Study of Multiechelon Lot Sizing with Intermediate Demands

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In this paper, we study a multiechelon uncapacitated lot-sizing problem in series (m-ULS), where the output of the intermediate echelons has its own external demand and is also an input to the next echelon. We propose a polynomial-time dynamic programming algorithm, which gives a tight, compact extended formulation for the two-echelon case (2-ULS). Next, we present a family of valid inequalities for m-ULS, show its strength, and give a polynomial-time separation algorithm. We establish a hierarchy between the alternative formulations for 2-ULS. In particular, we show that our valid inequalities can be obtained from the projection of the multicommodity formulation. Our computational results show that this extended formulation is very effective in solving our uncapacitated multi-item two-echelon test problems. In addition, for capacitated multi-item, multiechelon problems, we demonstrate the effectiveness of a branch-and-cut algorithm using the proposed inequalities.

Subject classifications: lot sizing; multiechelon; facets; extended formulation; fixed-charge networks.
Area of review: Optimization.
History: Received May 2011; revisions received August 2011, December 2011, February 2012; accepted February 2012.
Published online in Articles in Advance July 24, 2012.

1. Introduction

Managing inventory can be a challenging task for many enterprises. In particular, this task becomes significantly more complex for firms with multiechelon supply chains, where replenishments of inventory located in multiple tiers must be synchronized. In this paper, we study a multiechelon lot-sizing problem in series and with intermediate demands, which arises frequently for many wholesalers, retail chains, and manufacturers. For example, consider a two-echelon distribution system for a wholesaler that consists of regional and forward distribution centers (DCs). The regional DCs (first echelon) place orders to receive products directly from suppliers and then ship these products to forward DCs (second echelon). The forward DCs fulfill demand for most end-customers. However, the regional DCs may also ship directly to some end-customers in close proximity. Similarly, consider a two-echelon distribution system for a multichannel retailer that consists of DCs and customer-facing stores. The DCs ship to all stores but may also ship directly to end-customers who order online. Finally, consider a two-echelon production system for a vertically integrated manufacturer. The firm produces a part at the first echelon, which is used at the second echelon to assemble the final product. In addition, the same part may also be used to fulfill external demand from the repair or field service business.

In all these examples, demand is dynamic and time-varying, and there are economies of scale in production/shipping of orders. The goal is to determine the production/order plan over a finite horizon to meet the demand at both echelons in each period with the minimum total cost, which includes fixed and variable production/order costs, and variable holding costs at each echelon. This problem can be seen as a fixed-charge network flow problem on a grid (see Figure 1).

In a seminal paper on the single-echelon uncapacitated lot-sizing problem (ULS), Wagner and Whitin (1958) analyze the properties of optimal solutions to ULS, and propose a polynomial-time algorithm. The running time was later improved by Aggarwal and Park (1993), Federgruen and Tzur (1991), Wagelmans et al. (1992). Krarup and Bilde (1977) give an uncapacitated facility location extended formulation for ULS and show that the linear programming (LP) relaxation of this formulation always has an optimal solution with integer setup variables. Barany et al. (1984) give a complete linear description of the ULS polyhedron using the so-called \((\ell, S)\) inequalities. Since then, several extensions of the single-echelon ULS polyhedron have been considered to incorporate backlogging (Pochet and Wolsey 1988, Küçükyavuz and Pochet 2009), uncertainty in demands (Guan et al. 2006a, b), and production or inventory capacities (Pochet and Wolsey 1993, Atamtürk and Muñoz 2004, Atamtürk and Küçükyavuz 2005), among others (see Pochet and Wolsey 2006 for a review). Belvaux and Wolsey (2000,
Figure 1. Two-echelon, four-period uncapacitated lot-sizing network.

In this paper, we are interested in exact methods for \( m \)-ULS based on its polyhedral characterizations. In §2, we give an \( O(n^4) \) dynamic program for 2-ULS. In §3, we propose valid inequalities for \( m \)-ULS and study their strength. We also give a polynomial-time separation algorithm. In §4, we establish a hierarchy of alternative extended formulations for 2-ULS and show that our inequalities can be obtained from the projection of the so-called multicommodity formulation. Our computational results, summarized in §5, illustrate that the multicommodity formulation is very effective in solving a difficult class of uncapacitated multi-item, two-echelon lot-sizing problems. In addition, for capacitated multi-item, multiechelon problems, we demonstrate the effectiveness of a branch-and-cut algorithm using the proposed inequalities.

1.1. Mathematical Model

Let \( d_{ij} \geq 0 \) denote the demand in period \( t \) at the \( i \)th echelon, and \( d_{ij} = \sum_{k=1}^{n} d_{ik} \), with \( d_{ik} = 0 \) if \( i > k \). If we order in period \( t \) at echelon \( i \), we incur a fixed cost \( f_{ij} \) and a variable cost \( c_{ij} \). Let \( h_{ij} \) denote the unit holding cost at echelon \( i \) at the end of period \( t \). Let \( x_{ij} \) be the order quantity at the \( i \)th echelon in period \( t \), \( s_{ij} \) be the inventory at echelon \( i \) at the end of period \( t \), \( y_{ij} \) be the order setup variable at the \( i \)th echelon in period \( t \), where \( y_{ij} = 1 \) if \( x_{ij} > 0 \); \( y_{ij} = 0 \) otherwise. Throughout the paper, we let \( (i, j) \) denote the interval \( [i, j] \) for \( i \leq j \), and \( \emptyset \) for \( i > j \).

Figure 1 depicts a two-echelon four-period uncapacitated lot-sizing network with demand in both echelons, where node \((i, j)\) represents echelon \( j \) and period \( i \). A natural formulation of 2-ULS is

\[
\min \sum_{i=1}^{n} \sum_{t=1}^{n} (f_{ij}y_{ij} + c_{ij}x_{ij} + h_{ij}s_{ij}),
\]

\[
s.t. \quad s_{i+1,j} - s_{ij} = x_{ij} + s_{ij}, \quad t \in [1, n],
\]

\[
s_{i+1,j} - s_{ij} = d_{ij}, \quad t \in [1, n],
\]

\[
s_{ij} = 0, \quad i \in [1, 2],
\]

\[
x_{ij} \leq (d_{in} + d_{in})y_{ij}, \quad t \in [1, n],
\]

\[
x_{ij} \leq d_{in}y_{ij}, \quad t \in [1, n],
\]

\[
y_{ij} \in \{0, 1\}, \quad t \in [1, n], \quad i \in [1, 2],
\]

\[
x_{ij} \geq 0, \quad t \in [1, n], \quad i \in [1, 2],
\]

\[
s_{ij} \geq 0, \quad t \in [1, n], \quad i \in [1, 2].
\]
constraints that force the binary variables \( y_1^t \) and \( y_2^t \) to be 1 if there is a positive order in period \( t \) at the first and second echelon, respectively. Finally, constraints (7)–(9) are variable restrictions. The formulation of \( m \)-ULS for \( m \geq 3 \) follows similarly.

Note that from (2)–(4) the stock variables can be projected out by letting \( s_1^i = \sum_{j=1}^n (x_1^j - x_2^j) - d_{1i}^j \), \( s_2^i = \sum_{j=1}^n x_2^j - d_{2i}^j \) for \( t \in [1, n] \), and we get an alternative formulation:

\[
\begin{align*}
\min & \quad \sum_{i=1}^n \sum_{t=1}^n (f_j y_j^t + c_i x_i^t) - B, \\
\text{s.t.} & \quad (5)–(8), \\
& \quad \sum_{j=1}^n x_1^j = d_{1i}^1 + d_{1i}^2, \quad t \in [1, n], \tag{10} \\
& \quad \sum_{j=1}^n x_2^j = d_{1i}^2, \quad t \in [1, n], \tag{11} \\
& \quad \sum_{j=1}^t x_3^j \geq d_{1i}^1 \quad t \in [1, n], \tag{12} \\
& \quad \sum_{j=1}^t x_3^j \geq t \sum_{j=1}^n x_2^j + d_{1i}^2 \quad t \in [1, n], \tag{13}
\end{align*}
\]

where the unit order costs are updated as \( c_1^i = c_1^i + \sum_{i=1}^n (h_1^i - h_1^j) \), \( c_2^i = c_2^i + \sum_{i=1}^n (h_2^i - h_1^j) \), for \( t \in [1, n] \) and \( B = \sum_{i=1}^n (h_1^i d_{1i}^2 + h_2^i d_{2i}^2) \) is a constant. In the sequel, we drop the constant term \( B \) from the objective function. We also make a realistic assumption that \( c_1^i \) and \( c_2^i \) are nonnegative, and \( h_2^i \geq h_1^i \) for all \( i \in [1, n] \). Thus, \( c_1^i \) and \( c_2^i \) are nonnegative. In addition, we let \( \mathcal{F} \) denote the set of feasible solutions to (5)–(8) and (10)–(13).

### 2. Dynamic Programming Recursion and Reformulation

In this section, we give a dynamic programming (DP) recursion for 2-ULS that generalizes the algorithm of Zangwill (1969) by allowing positive demands at the first echelon. As 2-ULS is a single-source uncapacitated fixed-charge network (SSFCN) flow problem, we can apply the well-known result that the extreme points of SSFCN correspond to a spanning tree (Zangwill 1968, Veinott 1969) to conclude that there exists an optimal basic feasible solution to 2-ULS with \( x_2^i = 0 \) for all \( t \in [1, n] \) and \( i \in [1, 2] \).

For \( 1 \leq i_2 \leq j_2 \leq n \), we define \((i_2, j_2, j_2)\) as a regeneration interval if \( s_2^{i_2} = d_{2i}^{i_2} = 0, x_1^{i_2} = d_{1i}^{i_2} + d_{1i}^{j_2}, \) and \( s_1^{i_2} > 0 \) or \( d_{1i_2}^{i_2} = 0 \) for \( i \in [1, i_2 - 1] \). Similarly, for \( 2 \leq i_2 \leq j_2 \leq n \), we define \((i_2, i_2, j_2)\) as a regeneration interval, if for \( 1 \leq i_2 \leq j_2 \), we have \( s_2^{i_2} = s_2^{i_2} = s_2^{j_2} = 0, x_1^{i_2} = d_{1i_2}^{i_2} + d_{1i_2}^{j_2}, \) and \( s_1^{i_2} > 0 \) or \( d_{1i_2}^{i_2} = 0 \) for \( i \in [i_2, i_2 - 1] \). In addition, we define an interval \((j_2, j_2)\) with \( 1 \leq j_2 \leq n \), \( s_2^{i_2} = s_2^{j_2} = 0, x_1^{i_2} = d_{1i_2}^{j_2}, \) and \( s_1^{i_2} > 0 \) or \( d_{1i_2}^{j_2} = 0 \) for \( j \in [j_2 - 1, j_2) \) as a regeneration subinterval for the second echelon. A regeneration interval can contain several regeneration subintervals or no regeneration subinterval (when \( j_2 = j_2 + 1 \)). In the latter case, the value of \( j_2 \) is equal to that of the preceding regeneration interval. For example, in Figure 2, (1, 3, 1, 5), (4, 4, 6, 5), and (5, 6, 6, 6) are regeneration intervals, (1, 2), (3, 5), and (6, 6) are regeneration subintervals. The regeneration interval (1, 3, 1, 5) contains the regeneration subintervals (1, 2) and (3, 5). However, the regeneration interval (4, 4, 6, 5) contains no regeneration subinterval.

The spanning tree property of SSFCN implies that there exists an optimal basic feasible solution that is a concatenation of regeneration intervals.

Let \( G(i_2, j_2) \), \( 1 \leq i_2 \leq j_2 \leq n \), denote the minimum cost of satisfying the demand in periods 1 to \( i_2 \) at the first echelon and the demand in periods 1 to \( j_2 \) at the second echelon. In addition, let \( H(j_2, j_2) \), \( 1 \leq j_i \leq n + 1, 0 \leq j_2 \leq n \) be the minimum cost to satisfy the demand in periods \( j_1 \) to \( j_2 \) at the second echelon, where \( H(j_2, j_2) = 0 \) if \( j_2 > j_2 \). For \( 1 \leq i_2 \leq j_2 \leq n \), consider the forward recursions:

\[
\begin{align*}
G(i_2, j_2) &= \min \left\{ \min_{1 \leq i_1 \leq n} \left\{ G(i_1, i_2) + f_i^1 + c_i^1 d_{1i_2}^1 \right\} + c_i^2 d_{2j_2}^2 + H(j_2, j_2) \right\}, \\
&= \min \left\{ \min_{1 \leq i_1 \leq n} \left\{ G(i_1, i_2) + f_i^1 + c_i^1 d_{1i_2}^1 + c_i^2 d_{2j_2}^2 + H(1, j_2) \right\} \right\}, \tag{14}
\end{align*}
\]

**Figure 2.** An optimal solution of a two-echelon, six-period uncapacitated lot-sizing problem.
where for $1 \leq j_1 \leq j_2 \leq n$,
\[
H(j_1, j_2) = \min_{i \in \mathcal{I}} \left\{ H(j_1, j_1 - 1) + f_{j_1}^2 + e_{j_1}^2 d_{j_1}^{l_{j_1}} \right\}.
\] (15)

The minimum total cost over the entire planning horizon for the original problem is given by $G(n, n) - B$.

**Proposition 1.** The dynamic program given by the recursions (14) and (15) solves 2-ULS in $O(n^4)$ time.

**Proof.** Note that the recursion (14) evaluates the minimum cost to satisfy the demand in periods 1 to $i_2$ at the first echelon and the demand in periods 1 to $j_2$ at the second echelon such that the last regeneration interval is $(i_1, i_2, j_1, j_2)$. Similarly, the recursion (15) calculates the minimum cost to satisfy the demand in periods $j_1$ to $j_2$ at the second echelon such that the last regeneration subinterval is $(j_1, j_2)$. As a result, $G(n, n) - B$ gives the optimal objective function value to 2-ULS and is calculated in $O(n^4)$ time. □

In the special case that the intermediate demands at the first echelon are zero, we can drop the index $i_2$ in the recursion (14). Then the resulting recursions for $G(j_2)$ and $H(j_1, j_2)$ are identical to the dynamic programming recursions in Melo and Wolsey (2010).

We note that using the approach proposed by Eppen and Martin (1987) and Martin (1987), we can obtain a tight extended formulation for 2-ULS based on the proposed DP. This formulation has $O(n^5)$ variables and $O(n^5)$ constraints, including nonnegativities.

### 3. Valid Inequalities

#### 3.1. Two-Echelon Inequalities

We define $\beta(T, k)$ as the set of consecutive elements in set $T$ starting from $k$, where if $k \notin T$, $\beta(T, k) = \emptyset$. In other words, if $k \in T$, then $\beta(T, k) = [k, k') \subseteq T$, for some $k'$ such that $k' + 1 \notin T$.

**Theorem 2.** For $0 \leq k \leq l \leq n$, let $T_1 \subseteq [1, k]$, $[k + 1, l] \subseteq T_2 \subseteq [1, l]$ and $T_3 \subseteq T_2$. Then the two-echelon inequality
\[
\sum_{j \in [1, k] \setminus T_1} x_j + \sum_{j \in T_2 \setminus T_3} \sum_{i \in T_3} x_j + \sum_{j \in T_3} \psi_j y_j^2 \geq d_{1k}^l + d_{1l}^0 = \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0
\] (16)

is valid for $\mathcal{P}$, where $\psi_j = \sum_{i \in \beta(T_2, j)} d_i^2$ and $\phi_j = d_{1l}^0 + d_{1j}^0 - \psi_j$.

**Proof.** We prove the validity of inequality (16) considering two cases.

(i) If $y_j = 0$ for all $j \in T_1$, then $x_j = 0$ for all $j \in T_1$. Let $i_1 := \min\{j \in T_2 \setminus T_3; x_j > 0, i \geq k + 1\}$; if $\{j \in T_2 \setminus T_3; x_j > 0, i \geq k + 1\} = \emptyset$, then let $i_1 := l + 1$. Let $i_2 := \min\{j \in T_3; x_j > 0, i \geq k + 1\}$; if $\{j \in T_3; x_j > 0, i \geq k + 1\} = \emptyset$, then let $i_2 := l + 1$. Note that $i_1 \neq i_2$ unless $i_1 = i_2 = l + 1$.

- If $i_1 > i_2$, then $\sum_{j \in [1, k] \setminus T_1} x_j^2 + \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0$ and $\psi_i y_i^2 = \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0$. Summing these two inequalities up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0.
\]

- If $i_1 < i_2$, then $\sum_{j \in [1, k] \setminus T_1} x_j^2 + \sum_{j \in [1, i_2 - 1] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\psi_i y_i^2 = \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0$. Summing these two inequalities up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \sum_{j \in [1, k_2 - 1] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

Note that $(\{i_1, i_2 - 1\} \setminus T_3) \subseteq (T_2 \setminus T_3)$.

- If $i_1 = i_2$, then $\sum_{j \in [1, k] \setminus T_1} x_j^2 + \sum_{j \in [1, i_2 - 1] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\psi_i y_i^2 = \psi_i y_i^2 \geq d_{1l}^0 + d_{1l}^0$. Because all terms on the left-hand side of inequality (16) are nonnegative, inequality (16) is valid if $y_j = 0$ for all $j \in T_1$.

(ii) If there exists $j \in T_2$ such that $y_j^2 = 1$, then let $j_1 := \min\{j \in T_2; y_j^2 = 1\}$.

- If $j_1 \notin T_2$, then $\sum_{j \in 1, k \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\phi_j y_j^2 = \phi_j = d_{1l}^0 + d_{1l}^0$. Summing them up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \phi_j y_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

- If $j_1 \in T_2$, then let $v := \max\{j \in \beta(T_2, j_1)\}$.

- If $x_j^2 = 0$ for all $j \in (\beta(T_2, j_1))$, then $\sum_{j \in [1, k] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\phi_j y_j^2 = \phi_j = d_{1l}^0 + d_{1l}^0$. Summing these two inequalities up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \phi_j y_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

- If $x_j^2 = 0$ for all $j \in \beta(T_2, j_1)$ such that $x_j^2 > 0$, then let $j_2 := \min\{j \in \beta(T_2, j_1); x_j > 0\}$.

- If $j_2 \notin T_3$, then $\sum_{j \in [1, k] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\phi_j y_j^2 = \phi_j = d_{1l}^0 + d_{1l}^0$. Summing them up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \phi_j y_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

- If $j_2 \in T_3 \setminus T_2$, then consider the following two cases:

- If $(\{j \in [j_2 + 1, v] \cap T_3; x_j^2 > 0\} = \emptyset$, then let $j_1 := \min\{j \in [j_2 + 1, v] \cap T_3; x_j^2 > 0\}$.

- $\sum_{j \in [1, k] \setminus T_1} x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\phi_j y_j^2 = \phi_j = d_{1l}^0 + d_{1l}^0$. Summing these two inequalities up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \phi_j y_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

Note that $(\{j_2, j_1 \setminus T_3) \subseteq (T_3 \setminus T_3)$.

- If $(\{j_2, j_1 \setminus T_3) \subseteq (T_3 \setminus T_3)$.

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \sum_{j \in [1, j_1 \setminus T_3] x_j^2 \geq d_{1l}^0 + d_{1l}^0$ and $\phi_j y_j^2 = \phi_j = d_{1l}^0 + d_{1l}^0$. Summing these two inequalities up, we get

\[
\sum_{j \in [1, k] \setminus T_1} x_j^2 + \phi_j y_j^2 \geq d_{1l}^0 + d_{1l}^0.
\]

Note that $(\{j_2, v \setminus T_3) \subseteq (T_3 \setminus T_3)$.
Because all terms on the left-hand side of inequality (16) are nonnegative, inequality (16) is valid if there exists \( j \in T_i \) such that \( y_j^1 > 0 \).

Hence, the inequality (16) is valid. \( \square \)

An alternative proof can be obtained by using the dicut collection inequalities of Rardin and Wolsey (1993). We provide the precise correspondence between the simple dicut collection inequalities and the two-echelon inequalities in Corollary 9.

**Example 1.** To illustrate the two-echelon inequalities, consider a four-period problem as shown in Figure 1 with \( d_i^1 = d_i^2 = 1 \) for \( i \in [1, 4] \). For \( k = 2 \) and \( l = 3 \), we have \( x_1^1 + 3y_1^1 + x_2^1 \geq 5 \) where \( T_i = \{2\} \), \( T_2 = \{3\} \), \( T_3 = \emptyset \). For \( k = l = 3 \), we have \( x_1^1 + 4y_2^1 + y_1^1 + x_3^1 \geq 6 \), where \( T_i = \{2, 3\} \), \( T_2 = \{3\} \), \( T_3 = \emptyset \), and \( x_1^1 + 4y_2^1 + y_1^1 + y_2^1 \geq 6 \), where \( T_i = \{2, 3\} \), \( T_2 = \{3\} \), \( T_3 = \{3\} \). For \( k = 3 \) and \( l = 4 \), we have \( x_1^1 + 4y_2^1 + 3y_1^3 + x_3^2 + x_4^2 \geq 7 \), where \( T_i = \{2, 3\} \), \( T_2 = \{2, 3\} \), \( T_3 = \{2, 4\} \), \( T_4 = \emptyset \), and \( x_1^1 + 4y_2^1 + 3y_1^3 + y_2^1 + x_2^3 \geq 7 \), where \( T_i = \{2, 3\} \), \( T_2 = \{2, 4\} \), \( T_3 = \{2\} \).

Note that for \( k = 0 \), we have \( T_i = \emptyset \), \( T_2 = \{1, l\} \) and \( T_3 \subseteq T_2 \), so inequality (16) is equivalent to the \((\ell, S)\) inequality of Barany et al. (1984) for the second echelon only, where \( \ell = l \) and \( T_2 = S \). For example,

\[
x_1^3 + x_2^3 + y_2^3 \geq 3
\]

is the \((\ell, S)\) inequality for the second echelon only, with \( \ell = 3 \) and \( S = \{3\} \). In addition, for \( l = n \), \( T_i = \{1, n\} \), \( T_3 = \emptyset \), inequality (16) is equivalent to the \((\ell, S)\) inequality of Barany et al. (1984) for the first echelon only, where \( \ell = k \) and \( T_i = S \). For example,

\[
x_1^3 + x_2^3 + y_3^3 \geq 3
\]

is the \((\ell, S)\) inequality for the first echelon only, with \( \ell = 3 \), \( S = \{3\} \). As a result, single echelon \((\ell, S)\) inequalities are valid for 2-ULS, and they are subsumed by the two-echelon inequalities.

Also, for \( k = l \) and \( T_2 = \emptyset \), inequality (16) is equivalent to the \((\ell, S)\) inequality for the aggregation of the two echelons. For example,

\[
x_1^1 + x_2^2 + 2y_1^3 \geq 6
\]

is the \((\ell, S)\) inequality for the aggregation of the two echelons with \( \ell = 3 \), \( S = \{3\} \).

Using a similar argument, we can show that the two-echelon inequalities obtained by aggregating the demands in echelons \([m_1, m_2]\) (echelon 1) and those in \([m_2 + 1, m_3]\) (echelon 2) for \( 1 \leq m_1 < m_2 < m_3 \leq m \), are valid for \(-\)-US for any \( m \geq 2 \). For example, for a four-period five-echelon lot-sizing problem with unit demands in all echelons, letting \( m_1 = 1 \), \( m_2 = 2 \), \( m_3 = 4 \):

\[
x_1^1 + 8y_1^1 + 6y_2^1 + x_3^3 + x_4^3 \geq 14
\]

is a valid two-echelon inequality where \( k = 3 \), \( l = 4 \), \( T_1 = [2, 3] \), \( T_2 = [2, 4] \) and \( T_3 = \emptyset \).

### 3.2. Facet Conditions

Next we give necessary and sufficient conditions for two-echelon inequalities (16) to be facet-defining for \( \text{conv}(\mathcal{F}) \). We assume that \( d_1^1 \) and \( d_1^2 \) are positive for ease of exposition. Note that under this assumption, \( y_1^1 = y_1^2 = 1 \). Denote a feasible point in \( \text{conv}(\mathcal{F}) \) as \((x_1^1, y_1^1, x^2_1, y_1^2)\).

The dimension of \( \text{conv}(\mathcal{F}) \) is \( 4n - 4 \) for \( d_1^1 > 0 \) and \( d_1^2 > 0 \) (see Appendix A).

**Proposition 3.** For \( d_1^1 > 0 \) and \( d_1^2 > 0 \), inequality (16) is facet-defining for \( \text{conv}(\mathcal{F}) \) if and only if either

1. \( 1 \not\in T_i \);
2. \( 1 \not\in T_j \) if \( k \neq 0 \);
3. \( 1 \not\in T_i \) if \( k = 0 \);
4. \( k \neq 1 \);
5. if \( k = 0 \), \( l = n \), then \( |T_i| = 1 \);
6. for every \( j \in T_i \cap [2, k] \), there exists \( i \in T_i \) such that \( j \in T_i \cap \beta(T_i, i) \);
7. if \( 2 \leq k \leq l = n \) with \( T_i \neq \emptyset \), then \( T_i \cap [k + 1, n] = \emptyset \) and for each \( j \in T_i \cap [2, k] \), there exists \( j^* \in [j + 1, k] \) such that \( j^* \not\in T_j \);
8. if \( 2 \leq k \leq l < n \), then there exists \( j \in [p^1, k] \) such that \( j \not\in T_j \);
9. if \( k = l = n \), then either \( T_2 = \emptyset \) with \( |T_i| = 1 \), or \( T_2 \neq \emptyset \) is a consecutive set with \( p^2 = p^1 \) and \( [p^1, w^1] \subset T_2 \cap [p^1, w^2] \subset [p^1, n] \);
10. if \( k = 0 \), then \( T_i \neq \emptyset \); if \( k = 0 \), then \( T_i \neq \emptyset \); where

\[
p^1 := \min \{j \in T_i\}, \quad w^1 := \max \{j \in T_i\},
\]

\[
p^2 := \min \{j \in T_2\}, \quad w^2 := \max \{j \in T_2\}.
\]

**Proof.** See Appendix B. \( \square \)

Using the facet conditions, we see that \((\ell, S)\) inequalities for the second echelon only and for the aggregation of two echelons are facet-defining for 2-ULS problem, such as inequalities (17) and (19). But \((\ell, S)\) inequality for the first echelon only, such as inequality (18), is not facet-defining because it violates facet condition (2).

Based on our experiments with PORTA (Christof and Löbel 2008), in a three-period two-echelon lot-sizing problem with unit demands in both echelons, all facets of the convex hull of 2-ULS solutions are defined by the two-echelon inequalities. However, in a four-period problem with unit demands in both echelons, 65 out of the 81 facets are defined by the two-echelon inequalities. Four out of these 65 facets are \((\ell, S)\) inequalities for the aggregation of the first and second echelons, and 4 out of these 65 facets are \((\ell, S)\) inequalities for the second echelon only.

### 3.3. Separation

**Proposition 4.** Given a fractional point \((x_1^1, y_1^1, x_2^2, y_1^2) \in \mathbb{R}^{8n}\), there is an \( O(n^3) \) algorithm to find the most violated inequality (16), if any.

**Proof.** As stated earlier, when \( k = 0 \), two-echelon inequalities are \((\ell, S)\) inequalities of Barany et al. (1984) for
the second echelon, which have an $O(n \log n)$ separation algorithm (c.f., Pochet and Wolsey 2006). When $k = 1$, the two-echelon inequalities are not facet-defining due to facet condition (4). Next, for given $k$ and $l$ such that $2 \leq k \leq l \leq n$, we give an $O(n^2)$ algorithm that minimizes the left-hand side of inequality (16). Note that for a given $k$ and $l$, the right-hand side of inequality (16) is fixed, so this algorithm maximizes the violation, if any.

Note that by definition, $[k, 1, l] \subseteq T_j$. To minimize $\sum_{i \in T_i \cap [k, 1, l]} \sum_{i = 1}^{n} x_i^l + \sum_{i \in T_i \cap [k, 1, l]} \sum_{j = 1}^{n} y_{ij}^l$, let $T_i \cap [k, 1, l] := \{ j \in [k + 1, l] : x_i^j \geq d_i^j y_{ij}^l \}$. This takes $O(n)$ time. Now we need to determine the sets $T_1, T_2 \cap [k, 1, l]$ and $T_3 \cap [k, 1, l]$. Note that the coefficients of the variables in $T_1$ depend on the choice of $T_2$, because they contain the term $\psi_j = \sum_{i \in T_2 \cap T_j} d_i^j$.

Consider a shortest-path network $G = (V, A)$. For example, Figure 3 is the shortest path network for separating a two-echelon inequality (16) with $k = 4$. The node set is $V = \{ i \} \cup \{ i : i \in [2, k + 1] \} \cup \{ i' : i \in [2, k] \}$, where $(k + 1)$ is the sink node. Node $i'$ represents $i \not\in T$, and node $i$ represents $i \in T$. By definition, we know that if $k \not\in T$, then $(k + 1) \in T$. From the facet conditions, we know that $i \not\in T$. The arc set is $A = \{(i', i + 1) : i \in [1, k] \} \cup \{(i', i + 1) : i \in [1, k - 1] \} \cup \{(i, (l + 1)) : i \in [2, k] \}$.

(1) A shortest path visiting the arc $(i', i + 1)$ for $i \in [1, k]$ implies that to minimize the left-hand side of inequality (16), we let $i \not\in T$, and $(i + 1) \in T$. The cost on this arc is $c_{i', i + 1} = \min \{ x_i^l, \sum_{i = 1}^{n} d_i^k + d_i^j y_{ij}^l \}$. Note that when $i \not\in T$, $\phi_i = d_i^k + d_i^l$. Therefore, if $x_i^l \leq (d_i^k + d_i^l) y_{ij}^l$, then we let $i \not\in T$, else we let $i \in T$.

(2) A shortest path visiting the arc $(i', i + 1)$ for $i \in [1, k] - 1$ implies that to minimize the left-hand side of inequality (16), we let $i \not\in T$, and $(i + 1) \not\in T$. The cost on this arc is $c_{i', i + 1} = \min \{ x_i^l, (d_i^k + d_i^j) y_{ij}^l \}$. If $x_i^l \leq (d_i^k + d_i^j) y_{ij}^l$, then we let $i \not\in T$, else we let $i \in T$.

(3) A shortest path visiting the arc $(i, (v + 1))$ for $i \in [2, k] - 1$ and $v \in [i, (i + 1)]$ implies that to minimize the left-hand side of inequality (16), we let $i \not\in T$ and $(i + 1) \not\in T$ and $(v + 1) \not\in T$. As a result, $\beta(T_2, j) = [j, v]$ for all $j \in [i, v]$, and the decision on which elements to include in $T_1 \cap [i, v]$ can be made easily as the coefficients $\phi_i$ depend on $\beta(T_2, j)$. The cost on this arc is $c_{i, (v + 1)} = \sum_{i \in T_i} \min \{ x_i^l, (d_i^k + d_i^j) y_{ij}^l \} + \sum_{i \in T_i} \min \{ x_i^l, (d_i^k + d_i^j) y_{ij}^l \}$. As before, if $x_i^l \leq (d_i^k + d_i^j) y_{ij}^l$, then we let $i \not\in T$; else, we let $i \in T$. Similarly, if $x_i^l \leq (d_i^k + d_i^j) y_{ij}^l$, then we let $i \in T \setminus T_j$; else, we let $i \in T_j$.

(4) A shortest path visiting the arc $(i, (k + 1))$ for $i \in [2, k]$ represents $[i, l] \subseteq T_2, (i - 1) \not\in T_2$, and $(k + 1) \not\in T_2$ if $k < l$. As a result, $\beta(T_2, j) = [j, l]$ for all $j \in [i, k]$. Hence, the cost on this arc is $c_{i, (k + 1)} = \sum_{i \in T_i} \min \{ x_i^l, (d_i^k + d_i^j) y_{ij}^l \} + \sum_{i \in T_i} \min \{ x_i^l, (d_i^k + d_i^j) y_{ij}^l \}$. As before, if $x_i^l \leq (d_i^k + d_i^j) y_{ij}^l$, then we let $i \not\in T$; else, we let $i \in T$. Similarly, if $x_i^l \leq (d_i^k + d_i^j) y_{ij}^l$, then we let $i \in T \setminus T_j$; else, we let $i \in T_j$.

Note that there are $O(n)$ nodes and $O(n^2)$ arcs in this network. In addition, $G$ is directed acyclic. Hence, the shortest-path problem for a given $k$ and $l$ can be solved in $O(n^2)$ time. Overall, this separation algorithm takes $O(n^2)$ time considering all $k, l$ such that $0 \leq k < l \leq n$. □

4. Alternative Extended Formulations for 2-ULS

A tight and compact extended formulation for 2-ULS can be obtained from the dynamic program given in §2. However, the size of this formulation is large, and its projection is nontrivial. In this section, we consider alternative extended formulations obtained by adapting those for $m$-ULS-F from the literature, such as the multicommodity formulation (Kraup and Bilde 1977, Rardin and Wolsey 1993) and the echelon stock formulation (Wolsey 2002, Belváx and Wolsey 2001) (see also Pochet and Wolsey 2006). We establish a hierarchy of formulations by studying their relative strength.

4.1. Multicommodity Formulation

In this section, we propose a multicommodity extended formulation similar to that of Pochet and Wolsey (2006) for $m$-ULS-F. Let $z_{11}^u$ be the order quantity in period $u$ at the first echelon to satisfy the intermediate demand in period $t$, $z_{12}^u$ be the order quantity in period $u$ at the first echelon to satisfy the demand at the second echelon in period $t$, and $z_{22}^u$ be the order quantity in period $u$ at the second echelon to satisfy the demand at the second echelon in period $t$ for $1 \leq u \leq t \leq n$. Using these additional variables, we can model 2-ULS as follows:

$$\min \sum_{i = 1}^{n} \sum_{j = 1}^{m} (f_{ij}^u + c_{ij}^u) x_{ij}^u,$$

s.t. $\sum_{u = 1}^{n} z_{11}^u = d_t^1 \quad t \in [1, n], \quad (21)$

$\sum_{u = 1}^{n} z_{12}^u = d_t^2 \quad t \in [1, n], \quad (22)$

$\sum_{u = 1}^{n} z_{22}^u = d_t^3 \quad t \in [1, n], \quad (23)$

$\sum_{u = 1}^{n} z_{22}^u \geq \sum_{i = 1}^{n} z_{12}^u \quad t \in [1, n], \quad (24)$

$z_{12}^u \geq z_{11}^u \quad t \in [1, n], \quad u \in [1, t], \quad (25)$

$z_{12}^u \geq z_{22}^u \quad t \in [1, n], \quad u \in [1, t], \quad (26)$

Note that $z_{11}^u = \sum_{i = 1}^{n} x_{ij}^u$.
\[ d_i^2 y_i^2 \geq z_{ii}^{22} \quad t \in [1, n], \quad u \in [1, t], \tag{27} \]
\[ x_i^1 = \sum_{t=1}^{n} (z_{i1}^{11} + z_{i1}^{12}) \quad t \in [1, n], \tag{28} \]
\[ x_i^2 = \sum_{t=1}^{n} z_{i2}^{22} \quad t \in [1, n], \tag{29} \]
\[ z_{ii}^{11} + z_{ii}^{12} + z_{ii}^{22} \geq 0 \quad t \in [1, n], \quad u \in [1, t], \tag{30} \]
\[ y_i^j \in [0, 1] \quad t \in [1, n], \quad i \in [1, 2]. \tag{31} \]

Here constraints (21)–(24) ensure that the demand is satisfied on time. In particular, constraints (24) enforce that the order quantity at the second echelon until period j to satisfy the second echelon demand in period t cannot be larger than the order quantity at the first echelon until period j to satisfy the second echelon demand in period t. Constraints (25)–(27) ensure that there are no orders in periods with no order setup. Constraints (28) and (29) relate the values of the order variables in the natural formulation with the additional variables in the extended formulation. We refer to the formulation (21)–(31) as the multicommodity (MC) formulation.

### 4.1.1. Comparison of MC Formulation with the Natural Formulation Strengthened with Two-Echelon Inequalities

Here we prove that the LP relaxation of MC formulation is at least as strong as the natural formulation strengthened with two-echelon inequalities. It is easy to see that the constraints of the natural formulation (5)–(8), (10)–(13) are implied by MC formulation. Next, we show that the two-echelon inequalities are implied by MC formulation. To do this, we study the projection of the feasible set of MC formulation onto the space of order and setup variables.

Note that because \( c^1 \) and \( c^2 \) are nonnegative, equality (22) for a given \( k \) can be relaxed as \( \sum_{i=1}^{n} z_{ii}^{22} \geq d_i^2 \), which is implied by inequality (23) for that \( t \) and inequality (24) for \( j = t \). We associate dual variables \( \alpha_i^1, \alpha_i^2, \rho_{ij}, \gamma_{ii}^{11}, \gamma_{ii}^{12}, \gamma_{ii}^{22}, \sigma_i^1, \) and \( \sigma_i^2 \) to constraints (21) and (23)–(29), respectively. From Farkas’ lemma, for a given \((x^1, y^1, x^2, y^2)\) satisfying these constraints, the LP relaxation of MC formulation has a solution if and only if

\[
\sum_{i=1}^{n} \alpha_i^1 x_i^1 + \sum_{i=1}^{n} \sigma_i^2 y_i^2 + \sum_{i=1}^{n} (\gamma_{ii}^{11} d_i^1 + \gamma_{ii}^{12} d_i^2) y_i^1
\]
\[ + \sum_{i=1}^{n} \gamma_{ii}^{22} d_i^2 y_i^2 \geq \sum_{t=1}^{n} (d_i^1 \alpha_i^1 + d_i^2 \alpha_i^2) \tag{32} \]

for all \((\alpha_i^1, \sigma_i^2, \gamma_{ii}^{11}, \gamma_{ii}^{12}, \gamma_{ii}^{22}, \alpha_i^1, \alpha_i^2, \rho_{ij})\) satisfying

\[ \gamma_{ii}^{11} + \sigma_i^1 \geq \alpha_i^1 \quad 1 \leq u \leq t \leq n, \tag{33} \]
\[ \gamma_{ii}^{12} + \sigma_i^1 \geq \sum_{j=1}^{t} \rho_{ij} \quad 1 \leq u \leq t \leq n, \tag{34} \]
\[ \gamma_{ii}^{22} + \sigma_i^2 \geq \alpha_i^2 - \sum_{j=1}^{t} \rho_{ij} \quad 1 \leq u \leq t \leq n. \tag{35} \]

### Proposition 5

If a projection inequality (32) defined by a nonnegative extreme ray \((\alpha^1, \sigma^2, \gamma^{11}, \gamma^{12}, \gamma^{22}, \alpha^1, \alpha^2, \rho)\) of the projection cone with equal positive entries is not dominated, then it has the following form:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1 \setminus S_1} \hat{\psi}_u y_u^1 + \sum_{u \in A_2 \setminus S_2} \hat{\psi}_u y_u^2 \geq d_{i1}^1 + d_{i2}^2, \tag{36} \]

where \( 0 \leq i_1 \leq i_2 \leq n, \ A_1 = [1, i_1], \ A_2 = [1, i_2], \ S_1 \subseteq A_1, \ S_1 \subseteq A_2, \ j(1) \in [0, 1], \ j(t+1) \in \{j(t), t+1\} \) for all \( t \in A_2, \ t \leq n-1, \ j(t) \leq t \) for all \( t \in A_2, \ A_2 \subseteq A_1, \ j(1) \in [0, 1], \ j(t+1) \in \{j(t), t+1\} \) for all \( t \in A_2, \ t \leq n, \ (j(t), t+1) \) for all \( t \in A_2, \ t \leq n, \end{itemize}

**Proof.** See Appendix C.

### Proposition 6

If a projection inequality (32) defined by a nonnegative extreme ray of the projection cone with equal positive entries is not dominated, then it is a two-echelon inequality (16).

**Proof.** Let \( 0 \leq i_1 \leq i_2 \leq n, \ A_1 = [1, i_1], \ A_2 = [1, i_2], \ S_1 \subseteq A_1, \ S_1 \subseteq A_2, \ j(1) \in [0, 1], \ j(t+1) \in \{j(t), t+1\} \) for all \( t \in A_2, \ t \leq n-1, \ j(t) \leq t \) for all \( t \in A_2, \ A_2 \subseteq A_1, \ j(1) \in [0, 1], \ j(t+1) \in \{j(t), t+1\} \) for all \( t \in A_2, \ t \leq n, \ (j(t), t+1) \) for all \( t \in A_2, \ t \leq n, \)

Define \( k = t_1, \ l = t_2, \) and \( C = \{t \in [1, k] : j(t) \neq l\}. \) Let \( T_2 = C \cup \{k+1, l\}. \) As \( j(t) \leq k \) for \( t \in A_2, \ T_2 = \{t \in A_2 : j(t) \neq l\} \). Let \( T_1 = A_1 \setminus T_2 \) and \( T_1 \subseteq A_1 \setminus S_2.

Let \( u \in A_1 \setminus S_1. \) If \( u \notin T_2, \) then \( \psi_u = 0 = \hat{\psi}_u \) and we let \( u \in T_3. \) If \( u \notin T_3, \) then \( \psi_u = \hat{\psi}_u, \) and we let \( u \in T_3.

Let \( u \in T_3 = A_1 \setminus S_1. \) Then \( \psi_u = d_u^1 + d_u^2 - \sum_{t \in j(T_2, u)} \hat{\psi}_u d_t^1 \). If \( u \notin T_2, \) then \( j(u) = u, \) and for all \( t \in A_2, \ t \geq u, \) we have \( j(t) \geq j(u). \) Hence \( \sum_{t \in j(T_2, u)} d_t^1 = d_u^1 \) and \( \psi_u = d_u^1 + d_u^2 - \hat{\psi}_u. \) If \( u \in T_2, \) then \( j(u) \neq u. \) Let \( u \) be the smallest index greater than \( u \) with \( j(u') = u'. \) We have \( \sum_{t \in j(T_2, u')} d_t^2 = d_u^2, \) and as \( d_u^2 = d_u^2 - d_{u-1}^2, \) \( \psi_u = \hat{\psi}_u. \)

The resulting two-echelon inequality is

\[
\sum_{u \in S_2} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in T_1} \phi_u^1 y_u^1 + \sum_{u \in T_2} \phi_u^2 y_u^2 \geq d_{i1}^1 + d_{i2}^2, \tag{37} \]

and dominates the projection inequality if there exists \( u \in S_2 \) with \( j(u) = u. \) □

### Proposition 7

Inequalities (16) can be obtained by projecting the MC formulation onto the \((x^1, y^1, x^2, y^2)\) space.

**Proof.** Consider the two-echelon inequality (16) defined by \( 0 \leq k \leq l \leq n, \ T_1 \subseteq [1, k], \ [k+1, l] \subseteq T_2 \subseteq [1, l], \ C = T_2 \cap [1, k], \) and \( T_3 \subseteq T_2. \) Let \( T_2 = \bigcup_{t=1}^{n} T_2^t \) where \( T_2^t \) is a maximal consecutive component, i.e., \( T_2^t = \{a(s), b(s)\} \subseteq T_2 \) with \( a(s) - 1 \notin T_2 \) and \( b(s) + 1 \notin T_2 \) for each
$s = 1, \ldots, r$ and $r$ is the number of maximal consecutive components comprising $T_2$.

Now define $t^k = k, t^i = i, A_1 = [1, k], A_2 = [1, l]$, $S_1 = [1, k]\setminus T_1$, $S_2 = T_2\setminus T_1$ and $j(t) = t$ for $t \in [1, k]\setminus C$ and $j(t) = a(s) - 1$ if $t \in T_2'$ for $s = 1, \ldots, r$.

For $u \in A_1 \setminus S_1$, $\hat{\psi}_u = d^2_{u^1} + \sum_{t \in T_2'} d^2_{u^1, j(t) < u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2} + \sum_{t \in T_2'} d^2_{a(u^1), j(t) = a(u^1) - 1}$. If $u \notin T_2'$, then $\sum_{t \in T_2'} d^2_{u^1, j(t) < u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2} + \sum_{t \in T_2'} d^2_{a(u^1), j(t) = a(u^1) - 1} - d^2_{u^2}$. If $u \in T_2'$, let $\bar{s}$ be the interval that $u$ falls into, i.e., $u \in T_2'$. Then $\sum_{t \in [1, k]\setminus C^r \setminus u^1} d^2_{u^1} = d^2_{u^1, j(t) = u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2} + d^2_{a(u^1), j(t) = a(u^1) - 1} - d^2_{u^2}$ for all cases.

Let $u \in A_2 \setminus S_2$. Then

\[
\hat{\psi}_u = \sum_{t \in A_2 \setminus S_2} d^2_{u^1} = \sum_{t \in A_2 \setminus S_2} d^2_{t^1} + \sum_{t \in A_2 \setminus S_2} d^2_{t^2} + \sum_{t \in T_2'} d^2_{u^1, j(t) < u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2} + \sum_{t \in T_2'} d^2_{u^1, a(u^1) - 1 < u^1} + \sum_{t \in T_2'} d^2_{u^2, a(u^1) - 1 < u^1}.
\]

Observe that $\sum_{t \in [1, k]\setminus C^r \setminus u^1} d^2_{u^1} = 0$. If $u \notin T_2'$, then

\[
\sum_{t \in T_2'} d^2_{u^1, j(t) < u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2} + \sum_{t \in T_2'} d^2_{u^1, a(u^1) - 1 < u^1} = 0.
\]

If $u \in T_2'$, then $\sum_{t \in T_2'} d^2_{u^1, a(u^1) - 1 < u^1} = d^2_{u^1, j(t) = u^2} + \sum_{t \in T_2'} d^2_{u^2, j(t) = u^2}$. If $u \notin T_2'$, then $\hat{\psi}_u = \psi_u = 0$ if $u \notin T_2$ and $\hat{\psi}_u = \psi_u$ if $u \in T_2$.

As a result, the projection inequality for these choices is the same as the two-echelon inequality (16). □

Using the Propositions (6) and (7), we have the following theorem.

**Theorem 8.** The formulation obtained by adding the projection inequalities (32) to the nonnegative extreme rays with equal positive entries has the same strength as the formulation obtained by adding all two-echelon inequalities (16).

Rardin and Wolsey (1993) give a class of dicut collection inequalities for single-source uncapacitated fixed-charge networks, which are obtained by projecting the multi-commodity extended formulation to the original space. Dicut collection inequalities are written implicitly as a function of a collection of dicuts in a graph. Therefore, there are no known explicit conditions for dicut collection inequalities to be facet-defining, and as a result, many of these inequalities are dominated. In addition, there are no known combinatorial separation algorithms for them.

**Corollary 9.** Two-echelon inequalities are special cases of dicut collection inequalities.

**Proof.** This follows from Theorem 8. Here we give the dicut collection that corresponds to the two-echelon inequalities. For $t \in [1, n]$ and $i \in [1, 2]$, $\Gamma_i$ is a collection of variables such that removing the arcs corresponding to these variables will disconnect the flows from source node to nodes $(t, i)$ in the single-source network depicted in Figure 1. To yield the two-echelon inequality $(T_1, T_2, T_1, k, l)$, the required dicut collection $\Gamma = \{\Gamma_i\}_{i=1}^n \cup \{\Gamma_{ij}\}_{i=1}^n$ has each $\Gamma_i$ as a singleton $\{Q_i\}$ for $t \in [1, n]$ and $j \in [1, 2]$. We define $\beta^{-1}(T, \cdot)$ as the inverse function of $\beta(T, \cdot)$, i.e., $\beta^{-1}(T, t)$ if and only if $t \in \beta^{-1}(T, \cdot)$. Then the dicut collection that gives the two-echelon inequality is

1. For $t \in [1, k]$, $\Gamma_i = \{Q_i\} = \{x_i^1: i \in [1, t]\setminus T_1\} \cup \{y_i^1: i \in [1, l]\setminus T_1\}$.
2. For $t \in [1, l]$, $\Gamma_i = \{Q_i\} = \{x_i^2: i \in [1, t]\setminus T_1\} \cup \{x_i^2: i \in [1, l]\setminus T_1\} \cup \{y_i^2: i \notin \beta^{-1}(T_2, l) \cap \{T_2\} \cap \{T_2\}\}$.
3. For $t \in [k + 1, n]$, $\Gamma_i = \emptyset$.
4. For $t \in [l + 1, n]$, $\Gamma_i = \emptyset$.

We refer the reader to Rardin and Wolsey (1993) for further details on the dicut collection inequalities.

Nevertheless, as two-echelon inequalities are in closed form, we are able to show that they are facet-defining under certain conditions (Proposition 3) and give a combinatorial separation algorithm for them (Proposition 4).

**Example 1 (Continued).** Based on our experiments with PORTA (Christof and Löbel 2008), the LP relaxation of MC formulation is not tight for 2-ULS with more than three periods. Consider the four-period 2-ULS problem with $d_1 = d_2 = (1, 1, 1, 1)$. As stated before, 65 out of 81 facets are defined by two-echelon inequalities. Besides these 65 facets, 3 out of the 16 remaining facets are defined by the projection of MC formulation. For example, $x_i^1 + x_i^2 + 2y_i^1 - x_i^1 - 2y_i^2 \geq 6$ is a projection inequality, but it is clearly not a two-echelon inequality because of the negative coefficients of $x_i^2$ and $y_i^2$. Thus, the MC formulation is strictly contained in the natural formulation with two-echelon inequalities.

Let $h^1 = h^2 = (0, 0, 0, 0), f^1 = (0, 2, 2, 2), f^2 = (0, 0, 2, 0), c^1 = (8, 7, 6, 5), c^2 = (0, 0, 2, 2)$. The solution to the linear relaxation of the MC formulation is $x_i^1 = (3, 2.5, 1.5, 1), x_i^2 = (1.5, 1.5, 0.5, 0.5), y_i^1 = (1, 0.5, 0.5, 0.5), y_i^2 = (1, 0.5, 1, 1)$. Because binary variables $y_i^1$ and $y_i^2$ are fractional at the optimal solution, the MC formulation is not tight in this example. So we conclude that the exact DP-based formulation is stronger than the MC formulation.

**4.2. Echelon Stock Reformulation**

Pochet and Wolsey (2006) derive an alternative formulation for $m$-ULS-F using the so-called “echelon stock variables.” Here we adapt this formulation to our problem. The first echelon stock variable $e_i^1 = s_i^1 + x_i^1$ is the total inventory at the first echelon at the end of period $t$, and the second echelon stock variable $e_i^1 = s_i^1$ is the total inventory at the second echelon at the end of period $t$. Using these variables, we obtain the following model:

\[
\min \sum_{i=1}^n \sum_{t=1}^n (f_i^1 y_i^1 + c_i^1 x_i^1),
\]
inequalities for each echelon. Let structure as that of ULS. Now, we can generate programming relaxation bound as the natural formulation. However, if we consider the variables and the constraints which is the same as the echelon stock inequality (37). We refer to inequalities (37) and (38) as echelon stock inequalities.

4.3. Hierarchy of Formulations

A formulation of a mixed-integer program is formally defined as the polyhedron given by the linear programming relaxation of its constraints (Definition 1.2 of Wolsey 1998). From §§2, 3, 4.1, and 4.2, we establish a hierarchy of formulations for 2-ULS, in its natural space, from stronger to weaker as: projection of the DP-based exact extended formulation; projection of the MC formulation; natural formulation with two-echelon inequalities (16); echelon stock formulation with echelon stock inequalities; natural formulation. Also, the inclusion in each case is strict. For example, we know that not all projection inequalities of MC formulation are two-echelon inequalities (16).

5. Computations

In this section, we report our computational experiments with a class of multi-item, multiechelon lot-sizing problems with mode constraints. In these problems, we have $n$ time periods, $m$ echelons, and $r$ items. The mode constraints allow at most $\kappa$ orders to be placed in each period and each echelon. Let $M_{ai}$ be the order capacity of item $a$ at echelon $i$ in period $t$, $1 \leq i \leq m$, $1 \leq a \leq r$, and $1 \leq t \leq n$. Let $d_{ai}$ be the demand of item $a$ in period $t$ at echelon $i$, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$. Define $d_{ai}^{\hat{a}} := \sum_{i=1}^{t} d_{ai}$.

Let $x_{ai}^{\hat{a}}$ denote the total order quantity of item $a$ in period $t$ at echelon $i$, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$. The mixed-integer programming formulation of capacitated multi-item lot-sizing problem with mode constraint is as follows:

$$\min \sum_{a=1}^{r} \sum_{i=1}^{m} \sum_{j=1}^{n} (f_{ai}^{j} x_{ai}^{j} + e_{ai}^{j} x_{ai}^{j}).$$

subject to $\sum_{j=1}^{n} x_{ai}^{j} = \sum_{j=1}^{n} d_{ai}^{j}$, $1 \leq i \leq m$, $1 \leq a \leq r$, $\sum_{j=1}^{t} x_{ai}^{j} \geq \sum_{j=1}^{t} x_{ai}^{j-1} + d_{ai}^{t}$, $1 \leq i \leq m - 1$, $1 \leq a \leq r$, $1 \leq t \leq n$, $\sum_{j=1}^{t} x_{ai}^{j} \geq \sum_{i=1}^{t} d_{ai}^{i}$, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$, $\sum_{a=1}^{r} y_{ai}^{j} \leq M_{ai} y_{ai}^{j}$, $1 \leq i \leq m$, $1 \leq a \leq r$, $1 \leq t \leq n$, $\sum_{a=1}^{r} y_{ai}^{j} \leq \kappa$, $1 \leq t \leq n$, $1 \leq i \leq m$, $1 \leq a \leq r$, $y_{ai}^{j} \in \{0, 1\}$, $1 \leq i \leq m$, $1 \leq t \leq n$, $1 \leq a \leq r$.

Let $z_{ai}^{ij}$ denote the order quantity of item $a$ in period $u$ at echelon $i$ to satisfy the demand in period $t$ at echelon $j$, $1 \leq i \leq j \leq m$, $1 \leq u \leq t \leq n$, $1 \leq a \leq r$. The multi-
commodity formulation of capacitated multi-item lot-sizing problem with mode constraint is as follows:

\[ \min \sum_{a=1}^{r} \sum_{i=1}^{m} \sum_{t=1}^{n} \left( f_{at} x_{at} + c_{at} y_{at} \right), \]

s.t. \( \sum_{i=1}^{m} \sum_{j=1}^{m} z_{au} = d_{at} \quad 1 \leq i \leq j \leq m, 1 \leq a \leq r, 1 \leq t \leq n, \)

\[ \sum_{i=1}^{k} z_{au} \geq \sum_{i=1}^{k} z_{ai}^{(i+1)} \quad 1 \leq i < j \leq m, 1 \leq a \leq r, 1 \leq k \leq t \leq n, \]

\[ x_{at} = \sum_{i=1}^{m} \sum_{j=1}^{m} x_{ij}^{au} \quad 1 \leq i \leq m, 1 \leq a \leq r, 1 \leq u \leq n, \]

\[ z_{ij}^{au} \leq d_{at} x_{at} \quad 1 \leq i \leq j \leq m, 1 \leq a \leq r, 1 \leq u \leq t \leq n, \]

\[ x_{at} \leq M_{at}^{i} y_{at} \quad 1 \leq i \leq m, 1 \leq a \leq r, 1 \leq t \leq n, \]

\[ \sum_{a=1}^{r} y_{at} \leq \kappa \quad 1 \leq i \leq m, 1 \leq t \leq n, \]

\[ z_{au} \geq 0 \quad 1 \leq i \leq m, 1 \leq a \leq r, 1 \leq u \leq t \leq n, \]

\[ y_{at} \geq 0 \quad 1 \leq i \leq m, 1 \leq a \leq r, 1 \leq t \leq n, \]

\[ y_{at} \in \{0,1\} \quad 1 \leq i \leq m, 1 \leq a \leq r, 1 \leq t \leq n. \]

We conduct all the experiments on a 1-GHz dual-core AMD Opteron(tm) processor 1218 with 2 GB RAM. We use IBM ILOG CPLEX 12.0 as the MIP solver.

### 5.1. Strength of Alternative Formulations for Uncapacitated Multi-Item Two-Echelon Instances

In this subsection, we investigate the strength of alternative formulations and cuts. We limit ourselves to uncapacitated instances with 30 periods and 2 echelons, where \( M_{at}^{i} = \sum_{j=1}^{m} d_{at}^{j} \) for \( 1 \leq i \leq m, 1 \leq t \leq n, 1 \leq a \leq r \). The variable costs of the first and second echelons are generated using a discrete uniform distribution in the interval \([0, 50]\) and \([0, 100]\), respectively. Unit inventory costs of both echelons are generated using a discrete uniform distribution in the interval \([0, 6]\). Let \( \delta \) be the ratio of fixed and unit order costs. For various values of \( r, \kappa, \) and \( \delta \), we generate five instances and report the averages in Table 1.

For each formulation, we report the average percentage duality gap (rounded to two significant digits) and the average number of cuts added (if applicable). First, we solve the LP relaxations of the natural and multicommodity formulations, which we refer to as NF and MCF, respectively. The gap reported for NF and MCF is calculated as \( 100 \times (z_{ub} - z_{lb}) / z_{ub} \), where \( z_{ub} \) is objective function value of the optimal solution and \( z_{lb} \) is the optimal value of the initial LP relaxation. The MCF is very strong and has zero gap for all the instances considered, whereas the initial gap of NF can be as high as 25%. Next, we solve NF by letting CPLEX generate its cuts and report the root gap and the average number of cuts generated before branching. The root gap is calculated similarly by letting \( z_{lb} \) be the optimal value of the LP relaxation strengthened by cutting planes. We refer to the natural formulation with CPLEX cuts as CPX. We observe that CPLEX can close a big portion of the gap. Finally, using cutting plane algorithms, we solve the LP relaxations of the natural formulation strengthened with the two-echelon inequalities (referred to as 2ULS) and the echelon stock formulation with echelon stock inequalities (referred to as ES). We can see that the echelon stock inequalities reduce the duality gap significantly but the remaining gaps are slightly higher than those with CPLEX cuts. The two-echelon inequalities, however, close almost all the gap, with the average gap being below 0.5%. This comparison shows that using two-echelon inequalities, we obtain a formulation that is almost as strong as the multicommodity formulation and significantly stronger than the formulation obtained by adding only the echelon stock inequalities. Because our goal in this experiment is to test the strength of 2ULS empirically, we do not report the solution times. The exact separation

<table>
<thead>
<tr>
<th>n.m.r.δ</th>
<th>NF Gap (%)</th>
<th>CPX Gap (%)</th>
<th>Cuts</th>
<th>2ULS Gap (%)</th>
<th>Cuts</th>
<th>ES Gap (%)</th>
<th>Cuts</th>
<th>MCF Gap (%)</th>
</tr>
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<tbody>
<tr>
<td>30.2.5.2.500</td>
<td>25.40</td>
<td>3.66</td>
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<td>5,990.2</td>
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<td>1,637.0</td>
<td>0</td>
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<td>208.8</td>
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<td>4.96</td>
<td>4,188.2</td>
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<td>11,563.4</td>
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<td>2,894.6</td>
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<tr>
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<td>4.71</td>
<td>62.6</td>
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<td>3,608.4</td>
<td>5.30</td>
<td>1,279.2</td>
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</tr>
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<td>3,791.6</td>
<td>0.02</td>
<td>1,328.4</td>
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</table>
of the two-echelon inequalities can be quite time consuming in practice due to its $O(n^3)$ time complexity. In the next subsection, we employ a heuristic separation to make 2ULS practicable.

In our computational experience, MCF is also highly effective in solving uncapacitated multi-item lot-sizing instances for more echelons with $2 \leq m \leq 5$. However, in the next subsection, we show that for capacitated instances a branch-and-cut algorithm using our proposed inequalities is more effective than the MCF formulation.

### 5.2. Effectiveness of Two-Echelon Inequalities for Capacitated Multi-Item Multiechelon Instances

In this subsection, we test the multicommodity formulation and three alternative branch-and-cut methods on capacitated multi-item, multiechelon lot-sizing problem with mode constraints:

Algorithm 1. Multicommodity formulation with all CPLEX cuts (denoted by MCF).

Algorithm 2. Echelon stock formulation with echelon stock inequalities (37)–(38) and all CPLEX cuts (denoted by ES).

Algorithm 3. Natural formulation with a subset of two-echelon inequalities and all CPLEX cuts (denoted by 2ULS).

Algorithm 4. Natural formulation with all CPLEX cuts (denoted by CPX).

Note that echelon stock inequalities are special cases of two-echelon inequalities. We impose an hour time limit for all algorithms.

In 2ULS, we generate a subset of the violated two-echelon inequalities at the root node only. We add all violated echelon stock inequalities for a single echelon obtained by aggregating the echelons $[m_1, m]$ for $m_1 \in [1, m]$. To apply the two-echelon inequalities in the multiechelon setting, we aggregate echelons $[m_1, m_2]$ and treat as echelon 1, and we aggregate echelons $[m_2 + 1, m_3]$ and treat as echelon 2, for certain choices of $m_1$, $m_2$, $m_3$, where $1 \leq m_1 \leq m_2 < m_3 \leq m$. In particular, we consider only the facet-defining two-echelon inequalities for the following cases:

(a) echelons $[m_1, m-1]$ aggregated as echelon 1 and $[m, m]$ aggregated as echelon 2 (i.e., $m_2 = m - 1$, $m_3 = m$) for all $k, l$ with $2 \leq k < l = n$.

(b) echelon $m_1$ used as echelon 1 and $[m_1 + 1, m]$ aggregated as echelon 2 (i.e., $m_2 = m_1$, $m_3 = m$) for all $k, l$ with $k = l = n$.

We add all the cuts aggressively, and we force CPLEX to start branching if the improvement of lower bound at the root node is less than 0.01% after adding all cuts generated in one iteration.

In our experimental setup, the demands, fixed costs, variable costs, and holding cost of each item in each echelon and each period are generated using a discrete uniform distribution in the intervals $[0, 50]$, $[1, 100]$, $[20], [0, 20]$, and $[0, 6]$, respectively. The capacity $M_{ij}$ is set to be $3[d_{ij}/n]$ for $i \in [1, m]$, $a \in [1, r]$, and $t \in [1, n]$.

We report our results in Table 2 for various settings $n, m, r, k$. For each setting, we generate five instances and report the averages. In column $\text{RGap}(\text{noint})$, we report the average percentage integrality gap at the root node just before branching, which is $100 \times (z_{\text{ub}} - z_{\text{best}})/z_{\text{ub}}$, where $z_{\text{ub}}$ is objective function value of the best integer solution obtained within time limit and $z_{\text{best}}$ is the best lower bound obtained at the root node. The number of instances without integer solutions obtained within time limit is given in parentheses in cases where not all five instances are solved with integer solutions. In column $\text{GClos}(\text{noint})$, we report the average percentage closure of the integrality gap at the root node before branching, which is $100 \times (z_{\text{ub}} - z_{\text{best}})/z_{\text{ub}}$, and in parentheses, we give the number of instances with no feasible integer solutions obtained within time limit. In columns $\text{EGap}(\text{noint})$, we report the average percentage end gap at termination output by CPLEX, which is $100 \times (z_{\text{ub}} - z_{\text{best}})/z_{\text{ub}}$, and in parentheses, we give the number of instances without integer solutions obtained within the time limit in parentheses. Columns $\text{Time(unsolv)}$ report the average solution time in seconds and the number of unsolved instances in parentheses in cases where not all five instances are solved to optimality within time limit. Columns $\text{Nodes(nobr)}$ report the average number of branch-and-cut tree nodes explored and the number of instances without branching in parentheses in cases where not all five instances start branching. In columns $\text{Cuts}$, we report the average number of CPLEX cuts and user inequalities (echelon stock inequalities for ES and two-echelon inequalities for 2-ULS) added separately.

The branch-and-cut method with the MC formulation was not able to obtain any integer feasible solutions for any of the five instances from 30.5.5.3 setting within an hour. Therefore, the gap closure and the end gap for the MC formulation is not calculated. Also, for all five instances from 20.5.5.3 and 30.5.3.2 settings, the MC formulation was not able to start branching, although it was able to solve the initial LP relaxation, add CPLEX default cuts at the root node and even obtained integer feasible solutions in all but one instance of the 30.5.3.2 setting. These experiments demonstrate that the MC formulation might not scale up for capacitated problems as the number of echelons, items or periods increase. Overall, two-echelon inequalities are the most effective method in obtaining optimal solutions in shortest time, or solutions with the smallest end gaps within an hour.

### 6. Conclusions

In this paper, we studied an $m$-echelon lot-sizing problem with intermediate demands ($m$-ULS). We gave a
Table 2. Comparison of MCF and alternative branch-and-cut methods for capacitated multi-item, multiechelon lot-sizing problems.

<table>
<thead>
<tr>
<th>n.m.r.k</th>
<th>Alg.</th>
<th>RGap</th>
<th>GClos (noint)</th>
<th>Time (unsolv)</th>
<th>Nodes (nobl)</th>
<th>Cuts</th>
<th>EGap (noint)</th>
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<td>20.2.5.3</td>
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<td>1.19%</td>
<td>47.11%</td>
<td>$\geq 3,600$</td>
<td>36,344.4</td>
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<td>ES</td>
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<td>898.0</td>
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<td>45.80%</td>
<td>530.52 (4)</td>
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<td>156.4</td>
<td>85.2</td>
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<td>10.78%</td>
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<td>(5)</td>
<td>68.4</td>
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Operations Research 60(4), pp. 918–935, © 2012 INFORMS

polynomial-time dynamic program, which implies a tight and compact extended formulation to solve 2-ULS. In addition, we presented a class of valid inequalities for m-ULS, which are separable in polynomial time. Our computational experience with these inequalities demonstrates the effectiveness of these inequalities for multi-item, multiechelon instances. We conjecture that these inequalities are enough to give the convex hull of solutions to 2-ULS for $n = 3$. However, they are not enough to give the convex hull for $n > 3$. In addition, we compared the theoretical strength of alternative formulations such as the multicommodity and echelon stock reformulations, and established a hierarchy between them. Finally, we presented our computational experiments with the multicommodity formulation and our valid inequalities. The multicommodity formulation performs extremely well for capacitated problems and the branch and cut algorithm outperforms the multicommodity formulation when capacity constraints are introduced.

Appendix A. Dimension of $\text{conv} (\mathcal{F})$

Let $\eta_j \in \mathbb{R}^n$ and $\epsilon_j \in \mathbb{R}^n$, $j \in \{1, \ldots, n\}$, $i \in \{1, 2\}$, be the unit vectors corresponding to the variables $x_i^j$ and $y_i^j$. The component of $\eta_j$, which has the same position with $x_i^j$ in the feasible solution, is 1; all other components of $\eta_j$ are 0. The component of $\epsilon_j$, which has the same position with $y_i^j$ in the feasible solution, is 1; all other components of $\epsilon_j$ are 0.
Appendix B. Proof of Proposition 3

Proposition 3. For $d^1 > 0$ and $d^2 > 0$, inequality (16) is facet-defining for $\text{conv}(\mathcal{F})$ if and only if

1. $l \notin T_1$;
2. $1 \notin T_2$, if $k \neq 0$;
3. $1 \notin T_2$, if $k = 0$;
4. $(k) \neq 1$;
5. if $k = 0$, $l = n$, then $|T_1| = 1$;
6. for every $j \in T_2 \cap [2,k]$, there exists $i \in T_1$ such that $j \in T_j(j)$;
7. if $2 \leq k \leq l = n$ with $T_3 \neq \emptyset$, then $T_1 \cap [k+1,n] = \emptyset$ and for each $j \in T_2 \cap [2,k]$, there exists $j^* \in [j+1,k]$ such that $j^* \notin T_j(j^*)$;
8. if $2 \leq k \leq l < n$, then there exists $j \in [p^1,k]$ such that $j \notin T_j$;
9. if $k = l = n$, then $T_1 \neq \emptyset$ and either $T_2 = \emptyset$ with $|T_1| = 1$, or $T_2 \neq \emptyset$ is a consecutive set with $p^2 = p^1$ and $[p^1,w^1] \subseteq T_2 = [p^1,w^1] \subseteq [p^1,w^2]$;
10. if $k \neq 0$, then $T_1 \neq \emptyset$; if $k = 0$, then $T_1 \neq \emptyset$.

Proof.

Necessity. For simplicity, we denote the two-echelon inequality (16) with the particular choice of $T_1, T_2, T_3, k, l$, by $(T_1, T_2, T_3, k, l)$. Note that $(x^1, y^1, x^2, y^2, T) \geq 0$.

1. Suppose that $l \in T_1$. Because $\gamma_i^1 = 1$, $x_j^1 \geq 0$ and $y_j^1 \geq 0$ for $j \in [1,n]$, $i \in [1,2]$, then the two-echelon inequality $(T_1, T_2, T_3, k, l)$ is dominated by the inequality $\gamma_i^1 \geq 1$ and two-echelon inequality $(\emptyset, T_2(T_1), 1)$.
2. Suppose that $1 \notin T_1$ and $1 \notin T_2$ with $k \neq 0$. Because $x_i^2 > 0$ and $y_i^2 = 1$, the two-echelon inequality $(T_2, T_3, k, l)$ is dominated by the two-echelon inequality $(T_2, T_3 \cup \{1\}, k, l)$.
3. Note that if $k = 0$, then $T_1 \neq \emptyset$ and $T_2 = [1,l]$. Suppose $1 \in T_1$. Then the two-echelon inequality $(\emptyset, T_2, T_3, 0, l)$ is dominated by the inequality $\gamma_i^2 \geq 1$.
4. By facet conditions (1)–(2) and the fact that $x_j^1 \geq d_1^1$, if $k = 1$, then the two-echelon inequality $(\emptyset, T_2, T_3, 1, l)$ is dominated by the two-echelon inequality $(\emptyset, T_2, T_3, 0, l)$.
5. Suppose that $k = 0$, $l = n$. In this case, $T_3 = [1,n]$. If $T_1 = \emptyset$, then the face defined by two-echelon inequality $(\emptyset, T_2, T_3, 0, l)$ is equivalent to the flow balance equation (11), so it is not proper. If $|T_1| > 1$, then the two-echelon inequality $(\emptyset, T_2, T_3, 0, n)$ is dominated by the two-echelon inequalities $(\emptyset, T_2, \{j\}, 0, n)$, $j \in T_1$. Note that when $T_1 = \{j\}$ for some $j \in [1,n]$, the two-echelon inequality $(\emptyset, T_2, T_3, 0, n)$ is equivalent to the variable upper-bound constraint $\gamma_i^2 \leq d_2^0 + d_2^1 \gamma_i^1$ given by (6).

6. Suppose that there exists $j \in T_1$ such that $j \notin \beta(T_1, i)$ for all $i \in T_2$, then the two-echelon inequality $(T_1, T_2, T_3, k, l)$ is dominated by the two-echelon inequality $(T_1, T_2 \setminus \{j\}, T_3 \setminus \{j\}, k, l)$.

(7) Suppose that $2 \leq k \leq l = n$ and $T_3 \neq \emptyset$. If there exists $j \in T_1 \cap [2,k]$ such that $j \notin \beta(T_1, i)$ for all $i \in T_2$, then the two-echelon inequality $(T_1, T_2, T_3, k, n)$ is dominated by the two-echelon inequality $(T_1, T_2, T_3 \setminus \{j\}, k, n)$ and inequality $\gamma_i^2 \leq d_2^0 + d_2^1 \gamma_i^1$.

(8) Suppose that $k \leq l < n$ and $[p^1,k] \subseteq T_2$. Note that in this case, the coefficients $\phi_j, j \in T_1$ of the two-echelon inequality $(T_1, T_2, T_3, k, l)$ are the same with the coefficients $\phi_j, j \in T_1$ of the two-echelon inequality $(T_1, T_2 \cup \{j\}, n, \emptyset, k, l)$. Then the two-echelon inequality $(T_1, T_2, T_3, k, l)$ is dominated by the two-echelon inequalities $(T_1, T_2 \cup \{j\}, n, \emptyset, k, l)$ and $(\emptyset, [1,l], T_3, 0, l)$, because the sum of inequalities $(T_1, T_2 \cup \{j\}, \emptyset, k, n)$, $(\emptyset, [1,l], T_3, 0, l)$ is equal to the sum of two-echelon inequality $(T_1, T_2, T_3, k, l)$ and flow balance equation (11).

(9) It is easy to see that for $k = l = n$, we cannot have $T_1 = \emptyset$ in a facet-defining inequality. Suppose that $k = l = n$ and $T_2 = \emptyset$. If $|T_1| > 1$, then the two-echelon inequality $(T_1, \emptyset, \emptyset, n, n)$ is dominated by the two-echelon inequalities $(\{j\}, \emptyset, \emptyset, n, n)$, $j \in T_1$. Next, suppose that $k = l = n$, $T_2 \neq \emptyset$, $w^1 \leq w^2$ and there exists $j \in [p^1,w^2]$ such that $j \notin T_1$. Let $j^* := \min(j \in [p^1,w^2], j \notin T_1)$.

• If $j^* \notin T_1$, then the two-echelon inequality $(T_1, T_2, T_3, n, n)$ is dominated by the two-echelon inequalities $(T_1 \cup \{j^*\}, n, \emptyset, n, n)$, $(\emptyset, \emptyset, n, n)$.

• If $j^* \in T_1$, then the two-echelon inequality $(T_1, T_2, T_3, n, n)$ is dominated by the two-echelon inequalities $(T_1 \cup \{j^*\}, n, \emptyset, n, n)$, $(\emptyset, \emptyset, n, n)$.

Lastly, suppose that $k = l = n$, $T_2 \neq \emptyset$ and $w^2 > w^1$. Let $j^* := \min(j \in T_1; j > w^2)$. Then the two-echelon inequality $(T_1, T_2, T_3, n, n)$ is dominated by the two-echelon inequality $(T_1 \cup \{j^*\}, n, \emptyset, n, n)$.

(10) Suppose that $k \neq 0$ and $T_1 = \emptyset$. It is easy to see that if $k = l = n$, then we cannot have $T_1 = \emptyset$ in a facet-defining inequality. Therefore, we assume that $k < n$. Then the two-echelon inequality $(\emptyset, \emptyset, T_3, k, l)$ is dominated by two-echelon inequality $(k \in T_1, T_2, T_3, k, l)$ and inequality $\gamma_i^2 \leq 1$. Suppose that $k = 0$ and $T_1 = \emptyset$. From facet condition (5), we must have $l < n$ in this case. Note that for $k = 0$, $T_2 = \emptyset$ is a consecutive set $[1,l]$ by its definition in Theorem 2. Then the two-echelon inequality $(\emptyset, \emptyset, 0, l)$ is dominated by two-echelon inequality $(\emptyset, [1,n], \emptyset, 0, n)$ and inequalities $\gamma_i^2 \leq 1$ for $j \in [1,n] \setminus T_2$.
It is easy to see that for $j \in [2, n]$, the points $\{\mathbf{a}_i\}_{i=1}^{n-k}$ are in $\text{conv}(\mathcal{F})$ and affinely independent. Thus, the inequalities (16) are facet-defining for 2-ULS when $k = 0$.

From facet condition (4), we have $k \neq 1$ for the two-eelon inequality to be facet-defining. So we assume $k \geq 2$ in the rest of the proof. Note, from facet condition (10), that $T_i \neq \emptyset$ in this case. By facet condition (6), we define $g(j) := \max\{i \in T_i: j \in \beta(T_i, i)\}$ for $j \in T_i \cap [2, k]$. In addition, let $r(j) = \max\{i \in \beta(T_j, i) \}$ if $\beta(T_j, j) \neq \emptyset$, and $r(j) = j - 1$, otherwise.

Consider the point
\[
u_0 = (d_{1t} + d_{1s} + d_{1j})\eta_1 + e_1 + (d_{1j+1} + d_{1j+2})\eta_{j+1} + e_{j+1},
\]
on the face defined by the two-eelon condition (16). Based on $\nu_0$, we can generate $4n - 4$ points as follows.

For $j \in [k + 2, n]$, consider the points
\[
u_j^1 = \begin{cases} 
\nu_0 + (d_{jt} + d_{jt+1} + d_{jt+2})\eta_j + e_j, & \text{if } j \in [k + 2, \min(1, n)], \\
\nu_0 + d_{jt}\eta_j - d_{jt+1}\eta_{j+1} + e_j, & \text{if } j \in [l + 2, n],
\end{cases}
\]
and $\tilde{\nu}_j^1 = \nu_0 + e_j$.

For $j \in [l + 2, n]$, consider the points $\nu_j^1 = \nu_0 + d_{jt}\eta_j - d_{jt+1}\eta_{j+1} + e_j$ and $\tilde{\nu}_j^1 = \nu_0 + e_j$.

For $j \in [k, n]$, consider the points $u_j = \nu_0 + d_{jt}\eta_j - d_{jt+1}\eta_{j+1} + e_j$ and $\bar{u}_j = \nu_0 + e_j$.

For $j \in T_i$, note that either $r(j) < k$ or $r(j) = l$. Also note that $j \neq 1$ from facet condition (1). Consider the points
\[
u_j^1 = \begin{cases} 
\nu_0 + \phi_1 \eta_1 - \phi_1 \eta_2 - d_{1j+1}\eta_{j+1} + d_{1j+1}\eta_{j+1} + d_{1j+1}\eta_{j+1} + e_{j+1}, & \text{if } j \leq \min(1, n), \\
\nu_0 + \phi_1 \eta_1 - \phi_1 \eta_2 - d_{1j+1}\eta_{j+1} + d_{1j+1}\eta_{j+1} + d_{1j+1}\eta_{j+1} + e_{j+1}, & \text{if } j \leq \min(l + 2, n),
\end{cases}
\]
and $\bar{u}_j^1 = \nu_0 + e_j$.

For $j \in T_i$, consider the points $u_j^1 = \nu_0 + d_{jt}\eta_j - d_{jt+1}\eta_{j+1} + e_j$ and $\tilde{u}_j^1 = \nu_0 + e_j$.

For $j \in T_i \cap [2, k]$, either $r(j) < k$ or $r(j) = l$. Consider the points:
\[
u_j^2 = \begin{cases} 
\nu_0 + (\phi_2 + d_{j+1}\eta_{j+1} - (\phi_2 + d_{j+1}\eta_{j+1} - d_{j+1}\eta_{j+1} + d_{j+1}\eta_{j+1} + d_{j+1}\eta_{j+1} + e_{j+1}, & \text{if } j \in T_i \cap [2, k], r(j) < k, \\
\nu_0 + (\phi_2 + d_{j+1}\eta_{j+1} - (\phi_2 + d_{j+1}\eta_{j+1} - d_{j+1}\eta_{j+1} + d_{j+1}\eta_{j+1} + d_{j+1}\eta_{j+1} + e_{j+1}, & \text{if } j \in T_i \cap [2, k], r(j) = l, \\
\end{cases}
\]
and $\bar{u}_j^2 = \nu_0 + e_j$. Note that $j \neq 1$ from facet condition (2) and if $j \in T_i \cap \{k + 1, l\}$, then $r(j) = l$.

(1) If $l \neq n$, three more points, $u_{k+1}^1$, $u_{k+1}^2$, and $\bar{u}_{k+1}^2$, are to be considered. Let $q := \max\{j \in [1, k]: j \notin T\}$ and $\bar{q} := \max\{j \in T; j \leq \bar{q}\}$. By facet condition (8), $\bar{q}$ exists.

(a) If $k = l < n$.

(2) If $k < l < n$.

(3) If $k < l < n$, one more point $u_{k+1}^1$ is to be considered.
Next, for the case of $k = l = n$ with $|T_l| = 1$ ($T_l = \{p^l\}$), we show that the $4n - 4$ points $\{u_0, [u^1, \hat{u}^1], [u_0, \hat{u}^1], [u_0, \hat{u}^1]_c \}_{c \in \mathbb{N}} \setminus \{[u^1]_c\}$ are affinely independent. For all other cases, we show that the $4n - 4$ points $\{u_0, \hat{u}^1, [u^1, \hat{u}^1], [u_0, \hat{u}^1]_c \}_{c \in \mathbb{N}} \setminus \{[u^1]_c\}$ are affinely independent.

We assume that the $4n - 4$ points associated with a particular choice of $(T_1, T_2, T_k, l)$ lie on the hyperplane $\sum_{j=1}^{n} (\lambda_j x_j + \lambda_{j}^1 \bar{x}_j^1 + \theta_1 y_j + \theta_2^1 y_j^2) = \eta_0$.

(1) For the case that $k = l = n$ with $T_2 \neq \emptyset$, by facet condition (9), we have $|U_l| = 1$. Comparing $u_0$ with $\hat{u}^1$ for $j \in [2, n] \setminus T_1$ and $\hat{u}^2_i$ for $i \in [2, n]$, we get $\theta_1^2 = \theta_2^2 = 0$ for $j \in [2, n] \setminus T_1$ and $i \in [2, n]$. Comparing $u_0^1$ and $\hat{u}^1$ for $j \in [2, n] \setminus T_1$, we get $\lambda_j^1 = \lambda_j$ for $j \in [2, n] \setminus T_1$. Therefore, $\lambda_j^1 = \lambda_j$ for $j \in [2, n] \setminus T_1$. So, $\lambda_j^1 = \lambda_j$ for $j \in [2, n] \setminus T_1$. Thus, the hyperplane is of the form $\sum_{j=1}^{n} (\lambda_j x_j + \lambda_{j}^1 \bar{x}_j^1 + \theta_1 y_j + \theta_2^1 y_j^2) = \eta_0$.

(2) Now consider the cases $k = l = n$ with $T_2 \neq \emptyset$, or $k = l = n$, or $k < l < n$. Comparing $u_0^1$ with $\hat{u}^1_j$ for $j \in \{[2, k] \setminus T_1\} \cup \{k + 2, n\}$ and $\hat{u}^2_i$ for $i \in \{[2, l] \setminus T_1\} \cup \{l + 2, n\}$, we get $\theta_1^{2} = 0$ for $j \in \{[2, k] \setminus T_1\} \cup \{k + 2, n\}$ and $i \in \{[2, l] \setminus T_1\} \cup \{l + 2, n\}$. Comparing $u_0^1$ with $\hat{u}^1_j$ for $j \in \{[2, k] \setminus T_1\}$, we get $\lambda_j^1 = \lambda_j$ for $j \in \{[2, k] \setminus T_1\}$. Similarly, $\lambda_j^1 = \lambda_j$ for $j \in [2, n] \setminus T_1$. Comparing $u_0$ and $u^1_j$ for $j \in T_1$, we get $\theta_1^j = \phi_1(\lambda_j^1 - \lambda_j)$. Therefore, the hyperplane is of the form

$$\sum_{j \in [1, k], T_1} \lambda_j x_j + \sum_{j \in [k+1, n] \setminus T_1} \lambda_j x_j + \lambda_j^1 \bar{x}_j^1 + \sum_{j = 1}^{n} \lambda_j \bar{x}_j = \sum_{j = 1}^{n} \lambda_j \bar{x}_j + \lambda_j^1 \bar{x}_j^1 + \sum_{j \in [1, k], T_1} \phi_1(\lambda_j^1 - \lambda_j) x_j + \sum_{j \in [k+1, n] \setminus T_1} \phi_1(\lambda_j^1 - \lambda_j) x_j.$$

Therefore, the hyperplane is of the form

$$\lambda_j^1 \sum_{j \in [1, k], T_1} x_j + \sum_{j \in [k+1, n] \setminus T_1} x_j + \lambda_j^2 \sum_{j = 1}^{n} \phi_j(\lambda_j^1 - \lambda_j) x_j + \lambda_j \bar{x}_j + \sum_{j \in [1, k], T_1} \phi_1(\lambda_j^1 - \lambda_j) x_j + \sum_{j \in [k+1, n] \setminus T_1} \phi_1(\lambda_j^1 - \lambda_j) x_j.$$

Hence these points define the two-echelon inequality (16) up to a multiple $\theta_1$ of $y_j^1 = 1$; a multiple $\theta_1$ of $y_j^2 = 1$; a multiple $\lambda_j^1$ of $\sum_{j = 1}^{n} x_j = d_{1,n}x_{1,n}$; and a multiple $\lambda_j^2$ of $\sum_{j = 1}^{n} x_j = d_{1,n}x_{1,n}$ in addition, if $k = 0$. Then, by facet condition (10), $T_2 \neq \emptyset$, and by facet condition (3), $1 \not\in T_2$, thus the point $(d_{1,n}x_{1,n} + d_0^1 \bar{x}_1^1 + e_1 + e_1^2 + \sum_{j \in T_1} e_j)$ is not the face defined by the two-echelon inequality. If $k = l = n$, by facet conditions (1) and (10), $1 \not\in T_2 \neq \emptyset$, then the point $(d_{1,n}x_{1,n} + d_0^1 \bar{x}_1^1 + e_1 + e_1^2 + \sum_{j \in T_1} e_j)$ is not the face defined by the two-echelon inequality. For other cases, we have $1 \leq k < n$ or $1 \leq l < n$, and the point $(d_{1,n}x_{1,n} + d_0^1 \bar{x}_1^1 + e_1 + e_1^2 + \sum_{j \in T_1} e_j)$ is not the face defined by the two-echelon inequality. Hence, the face is proper.
Lemma 14. If $A_1 \neq \emptyset$ and there exists $i < t'$ with $i \notin A_1$, then inequality (C1) is dominated by other inequalities (C1).

Proof. Suppose that $A_1 \neq \emptyset$ and there exists $i < t'$ with $i \notin A_1$. Then we would like to show that the projection inequality defined by sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ is dominated. Consider the projection inequalities (C1) for sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ and $(A_1, A_2, S_1, S_2, R, G_{11}^*, G_{12}^*, G_{22}^*)$ where $A_1^* = A_1 \cup \{i\}$, $A_2^* = A_2 \cup \{i\}$, $G_{11}^* = G_{11} \cup \{(u, t): u \in S_1, t \leq i\}$, and $G_{12}^* = G_{12} \cup \{(u, t): u \in S_1, t \leq i\}$. The first inequality is

$$\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1} \left( \sum_{(r, u) \in G_{11}} d_r^i + \sum_{(r, u) \in G_{12}} d_r^i \right) y_u^i$$

and the second inequality is

$$\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1} \left( \sum_{(r, u) \in G_{11}} d_r^i + \sum_{(r, u) \in G_{12}} d_r^i \right) y_u^i$$

Multiplying the first inequality with $d_r^i$, the second with $d_r^i$, and dividing the sum by $d_r^i + d_r^i$, we obtain

$$\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1} \left( \sum_{(r, u) \in G_{11}} d_r^i + \sum_{(r, u) \in G_{12}} d_r^i \right) y_u^i$$

As $t' > i$, we have $\sum_{u \notin S_1, u \notin S_2} y_u^i \geq \sum_{u \notin S_1, u \notin S_2} y_u^i$. As a result, the above inequality dominates the projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$.

Lemma 15. If $A_1 \neq \emptyset$ and there exists $i < t'$ with $i \notin A_2$, then inequality (C1) is dominated by other inequalities (C1).

Proof. Consider the projection inequality defined by sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ and suppose that $A_2 \neq \emptyset$ and there exists $i < t'$ with $i \notin A_2$. Let $A_1 = A_2 \cap \{i\}$, $A_2 = A_2 \setminus \{i\}$, $R = R \cup \{(i, t')\}$, $G_{11} = G_{11} \cup \{(u, i): u \leq i, (u, t') \in G_{12}\}$, $G_{12} = G_{12} \cup \{(u, t'): u \leq i, (u, t') \in G_{12}\}$, $G_{22} = G_{22} \setminus \{(u, t'): u \leq i\}$. First observe that sets $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12})$ and $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12})$ give valid projection inequalities. The projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ is

$$(\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1, u \notin S_2} \left( \sum_{(r, u) \in G_{11}^i} d_r^i + \sum_{(r, u) \in G_{12}^i} d_r^i \right) y_u^i$$

and the projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$ is

$$(\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1, u \notin S_2} \left( \sum_{(r, u) \in G_{11}^i} d_r^i + \sum_{(r, u) \in G_{12}^i} d_r^i \right) y_u^i$$

Again, multiplying the first inequality with $d_r^i$, the second with $d_r^i$, and dividing the sum by $d_r^i + d_r^i$, we obtain

$$(\sum_{u \in S_1} x_u^i + \sum_{u \in S_2} x_u^i + \sum_{u \notin S_1, u \notin S_2} \left( \sum_{(r, u) \in G_{11}^i} d_r^i + \sum_{(r, u) \in G_{12}^i} d_r^i \right) y_u^i$$

As $t' > i$, we have $\sum_{u \notin S_1, u \notin S_2} y_u^i \geq \sum_{u \notin S_1, u \notin S_2} y_u^i$. As a result, the above inequality dominates the projection inequality for $(A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})$.

Lemma 16. If $t' > t^*$, then inequality (C1) is dominated by other inequalities (C1).
Proof. Consider the projection inequality defined by sets \((A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22})\) with \(t^* > t^1\).

The projection inequality for \(\{A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22}\}\), where \(A_1 = A_1 \setminus \{t^1\}\), \(G_{11} = G_{11} \setminus \{(u, t^1)\}\) is:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1 \setminus S_1} \left( \sum_{r \in (u, t^1)G_{11}} d_r^1 + \sum_{r \in (u, t^1)G_{12}} d_r^2 \right) y_u^1
+ \sum_{u \notin S_1} \sum_{r \in (u, t^1)G_{22}} d_r^2 y_u^2 \geq \sum_{u \in A_1} d_u^1 + \sum_{d_r^2 \neq 0} \left( 1 - \sum_{u \notin S_1} y_u^1 \right). 
\]

The projection inequality for \(\{A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22}\}\), where \(A_1 = A_1 \cup \{t^2 + 1\}\), \(R = R \cup \{(t^2 + 1, t^1)\}\) and \(G_{12} = G_{12} \cup \{(u, t^2 + 1): u \notin S_1, u \in t^2 + 1\}\) is:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1 \setminus S_1} \left( \sum_{r \in (u, t^1)G_{11}} d_r^1 + \sum_{r \in (u, t^1)G_{12}} d_r^2 \right) y_u^1
+ \sum_{u \notin S_1} \sum_{r \in (u, t^1)G_{22}} d_r^2 y_u^2 \geq \sum_{u \in A_1} d_u^1 + \sum_{d_r^2 \neq 0} \left( 1 - \sum_{u \notin S_1} y_u^1 \right).
\]

Now we multiply the first inequality with \(d_{t^2 + 1}^2\), the second inequality with \(d_{t^1}^1\), add them up, and divide by \(d_{t^2 + 1}^2 + d_{t^1}^1\) to obtain:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1 \setminus S_1} \left( \sum_{r \in (u, t^1)G_{11}} d_r^1 + \sum_{r \in (u, t^1)G_{12}} d_r^2 \right) y_u^1
+ \sum_{u \notin S_1} \sum_{r \in (u, t^1)G_{22}} d_r^2 y_u^2 \geq \sum_{u \in A_1} d_u^1 + \sum_{d_r^2 \neq 0} \left( \frac{\sum_{u \notin S_1} y_u^1}{d_{t^2 + 1}^2} - \frac{\sum_{u \notin S_1} y_u^2}{d_{t^1}^1} \right).
\]

This inequality dominates the projection inequality for \(\{A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22}\}\) since \(\sum_{u \notin S_1} \sum_{r \in (u, t^1)G_{22}} y_u^1 \geq \sum_{u \notin S_1} \sum_{r \in (u, t^2 + 1)G_{22}} y_u^2 \geq 0\). □

Now we limit our investigation to the projection inequalities defined by sets \(A_1\) and \(A_2\) of the form \(A_1 = [1, t^1]\) and \(A_2 = [1, t^2]\) with \(t^*> t^1 > t^2 > 0\). Note that if \(S_1\) or \(S_2\) has an element larger than \(t^2\), then the resulting inequality is dominated. Hence, \(S_1 \subseteq A_2\) and \(S_2 \subseteq A_1\). The projection inequalities under consideration have the form:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1 \setminus S_1} \hat{d}_u x_u^1 + \sum_{u \in A_1 \setminus S_2} \hat{d}_u x_u^2 \geq d_{t^1}^1 + d_{t^2}^2, \tag{C2}
\]

where \(\hat{d}_u = d_{t^1}^1 + \sum_{r \in (u, t^1)G_{11}} d_r^1 + \sum_{r \in A_2 \setminus S_1} d_r^2\) for \(u \in A_2 \setminus S_1\) and \(\hat{d}_u = \sum_{r \in (u, t^1)G_{12}} d_r^1 + \sum_{r \in A_1 \setminus S_2} d_r^2\) for \(u \in A_1 \setminus S_2\).

**Lemma 17.** If there exists \(\hat{t} \in A_2\) with \(j(\hat{t}) > t^1\), then inequality (C2) is dominated by other inequalities (C2).

**Proof.** If there exists \(\hat{t} \in A_2\) with \(j(\hat{t}) > t^1\), then consider the projection inequalities defined by \(\{A_1, A_2, S_1, S_2, R, G_{11}', G_{12}', G_{22}\}\), where \(A_1' = A_1 \cup \{j(\hat{t})\}\) and \(G_{11}' = G_{11} \cup \{(u, j(\hat{t}) + 1)\}: u \notin S_1, u \leq j(\hat{t})\}\) and \(A_1 = A_1 \setminus \{\hat{t}\}\), \(R = R \cup \{(j(\hat{t}), \hat{t})\}\), \(G_{12}' = G_{12} \cup \{(u, \hat{t})\}: u \notin S_1, u \leq \hat{t}\}\) and \(G_{22}' = G_{22} \cup \{(u, \hat{t})\}: u \notin S_2, u \leq \hat{t}\). These inequalities are:

\[
\sum_{u \in S_1} x_u^1 + \sum_{u \in S_2} x_u^2 + \sum_{u \in A_1} \left( \sum_{r \in (u, t^1)G_{11}} d_r^1 + \sum_{r \in (u, t^1)G_{12}} d_r^2 \right) y_u^1
+ \sum_{u \notin S_1} \sum_{r \in (u, t^1)G_{22}} d_r^2 y_u^2 \geq \sum_{u \in A_1} d_u^1 + \sum_{d_r^2 \neq 0} \left( 1 - \sum_{u \notin S_1} y_u^1 \right).
\]

Hence, this inequality dominates the projection inequality (C2) for \(\{A_1, A_2, S_1, S_2, R, G_{11}, G_{12}, G_{22}\}\). □

If \(j(t) \leq t^1\) for all \(t \in A_2\), then for \(u \in A_1 \setminus A_1\), \(\hat{d}_u = 0\). Hence the projection inequality (C2) simplifies to inequality (36) with \(\hat{d}_u = d_{t^1}^1 + \sum_{u \in S_1} d_{t^1}^2\) for \(u \in A_1 \setminus S_1\) and \(\hat{d}_u = \sum_{u \in A_1 \setminus S_1} d_{t^1}^2\) for \(u \in A_2 \setminus S_2\). Finally, observe that if there exists \(u \in S_1\) with \(u > t^1\), as \(j(t) \leq t^1\) for all \(t \in A_2\), removing \(u\) from \(S_1\) yields a stronger inequality. As a result, the interesting projection inequalities are defined by \(0 \leq t^1 \leq t^2 \leq n\), \(A_1 = [1, t^1]\), \(A_2 = [1, t^2]\), \(S_1 \subseteq A_2\), \(S_2 \subseteq A_1\) and \(j(t) \in [0, \min\{t, t^1\}]\) for \(t \in A_2\).

**Lemma 18.** In a nondominated projection inequality (36), \(j(1) \in [0, 1]\) and \(j(t + 1) \in \{j(t), t + 1\}\) for all \(t \in A_2\) with \(t \leq n - 1\).
Proof. Suppose that \(0 \leq t^1 \leq t^2 \leq n\), \(A_1 = [1, t^1]\), \(A_2 = [1, t^2]\), \(S_t \subseteq A_1\) and \(S_t \subseteq A_2\) are given. Define

\[
\Gamma_j(j) = \sum_{u \in A_1 \setminus S_t} y_u^j + \sum_{u \notin A_1 \setminus S_t, j \leq t} y_u^j.
\]

Then the left-hand side of inequality (C2) is equal to \(\sum_{u \in S_t} y_u^1 + \sum_{u \notin A_1 \setminus S_t} d_{u,0} y_u^0 + \sum_{u \in A_2 \setminus S_t} y_u^2\). So for a given vector \((x_1, y_1, x_2, y_2)\) and fixed \(A_1\), \(A_2\), \(S_t\), and \(S_2\), the best \(j(t)\) choices are those with minimum \(\Gamma_j(j)\) values for each \(t \in A_2\). Now let \(t \in A_2\) with \(t \leq n - 1\) and observe that for a given \(j \in [0, t]\), \(\Gamma(t + 1, j) = \Gamma(t, j) + \sum_{u \in A_1 \setminus S_t} y_u^2\). This implies that

\[
\arg \min \{\Gamma(j, t)\} = \arg \min \{\Gamma(t, j)\} = \arg \min \{\Gamma(j, t + 1)\}.
\]

\(\Box\)

Acknowledgments

The authors thank the two referees for their constructive comments that improved this paper. Minjiao Zhang and Simge Küçükaydın are supported, in part, by the National Science Foundation [NSF-CMMI Grant 0917952].

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