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Kernels, inflations, evaluations, and imprimitivity of Mackey functors

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Abstract

Let M be a Mackey functor for a finite group G. By the kernel of M we mean the largest normal subgroup N of G such that M can be inflated from a Mackey functor for G/N. We first study kernels of Mackey functors, and (relative) projectivity of inflated Mackey functors. For a normal subgroup N of G, denoting by $P_{H,V}^{G}$ the projective cover of a simple Mackey functor for G of the form $S_{H,V}^{G}$ we next try to answer the question: how are the Mackey functors $P_{H/N,V}^{G/N}$ and $P_{H,V}^{G}$ related? We then study imprimitive Mackey functors by which we mean Mackey functors for G induced from Mackey functors for proper subgroups of G. We obtain some results about imprimitive Mackey functors of the form $P_{H,V}^{G}$, including a Mackey functor version of Fong's theorem on induced modules of modular group algebras of p-solvable groups. Aiming to characterize subgroups H of G for which the module $P_{H,V}^{G}(H)$ is the projective cover of the simple $\mathbb{K}\overline{N}_G(H)$-module V where the coefficient ring $\mathbb{K}$ is a field, we finally study evaluations of Mackey functors.

Keywords: Mackey functor; Mackey algebra; Inflation; Kernel; Faithful Mackey functor; Projective Mackey functor; Induction; Imprimitive Mackey functor; Fong's theorem; Evaluation

1. Introduction

Let G be a finite group and N be a normal subgroup of G. A basic functor from the category of Mackey functors for G/N to that for G is the inflation functor $\mathrm{Inf}_{G/N}^{G}$. One of the aims of

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this paper is to study Mackey functors for G of the form $M = \mathrm{Inf}_{G/N}^{G} T$ and to seek properties possessed by both of M and T such as relative projectivity. We also try to understand Mackey functors for G that can be induced from Mackey functors for a proper subgroup of G.

Similar topics are well established in finite group representation theory. Here we try to obtain related results for Mackey functors. However, we see that Mackey functor versions of them are completely different.

The concept of Mackey functors was introduced by J.A. Green [4] and A. Dress [2] to study group representation theory in an abstract setting, unifying several notions like representation rings, G-algebras and cohomology. The theory of Mackey functors was developed mainly by J. Thévenaz and P. Webb in [8,9] which are now standard references on the subject. They constructed simple Mackey functors explicitly in [8], and taking representation theory of finite groups as a model they developed a comprehensive theory of representations of Mackey functors in [9]. It is shown in [9] that Mackey functors for G over a field $\mathbb{K}$ can be viewed as modules of a finite dimensional $\mathbb{K}$-algebra $\mu_{\mathbb{K}}(G)$, allowing one to adopt easily many module theoretic constructions.

After recalling some crucial preliminary results about Mackey functors in Section 2, we begin to study inflated Mackey functors in Section 3. Let M be a Mackey functor for G. We observe that the intersection of all minimal subgroups of M is the largest normal subgroup of G such that M can be inflated from a Mackey functor for the quotient group. We refer to this largest normal subgroup as the kernel of M. Our first aim in Section 3 is to describe the kernels of simple and indecomposable Mackey functors. It is easily seen that the kernel of a simple Mackey functor for G of the form $S_{H,V}^{G}$ is equal to the core H_G of H in G. For an indecomposable Mackey functor M for G over a field $\mathbb{K}$ of characteristic $p > 0$, we show by using [9] that the kernel $\mathcal{K}(M)$ of M satisfies:

$$\big(O^{p}(H)\big)_{G} \leqslant \mathcal{K}(M) \leqslant H_G \quad \text{and} \quad O^{p}\big(\mathcal{K}(M)\big) = O^{p}(H_G)$$

where H is a vertex of M.

Some of our main results can be explained as follows. Let N be a normal subgroup of G and T be an indecomposable $\mu_{\mathbb{K}}(G/N)$-module with vertex P/N. We show in Section 3 that P is a vertex of $\mathrm{Inf}_{G/N}^{G} T$ so that $\mathrm{Inf}_{G/N}^{G}$ preserves vertices. However, it may not preserve projectivity. Using some results of [9] we also observe that the functor $\mathrm{Inf}_{G/N}^{G}$ sends projectives to projectives if and only if N is p-perfect where p is the characteristic of the field $\mathbb{K}$.

Denoting by $P_{H,V}^{G}$ the projective cover of the simple $\mu_{\mathbb{K}}(G)$-module of the form $S_{H,V}^{G}$, we also study the relationship between the Mackey functors of the form $P_{H/N,V}^{G/N}$ and $P_{H,V}^{G}$. For example we prove in Section 3 that $\mathrm{Inf}_{G/N}^{G}$ sends $P_{H/N,V}^{G/N}$ to a projective $\mu_{\mathbb{K}}(G)$-module if and only if N is inside the kernel of $P_{H,V}^{G}$, and if this is the case we have

$$P_{H,V}^{G} \cong \mathrm{Inf}_{G/N}^{G} P_{H/N,V}^{G/N}.$$

Moreover, in Section 4 we prove in general that

$$P_{H/N,V}^{G/N} \cong e_N P_{H,V}^{G} / I_N P_{H,V}^{G}$$

as $\mu_{\mathbb{K}}(G/N)$-modules where e_N is a certain idempotent of $\mu_{\mathbb{K}}(G)$ and I_N is a two sided ideal of $e_N \mu_{\mathbb{K}}(G) e_N$.

In Section 5, we deal with inflations of principal indecomposable Mackey functors. For example, we show that $\mathrm{Inf}_{G/N}^{G} P_{H/N,V}^{G/N}$ is isomorphic to the largest quotient of $P_{H,V}^{G}$ that can be inflated from a $\mu_{\mathbb{K}}(G/N)$-module.

Section 6 deals with imprimitive Mackey functors, meaning that Mackey functors induced from Mackey functors for proper subgroups of G. We give a criterion for simple Mackey functors to be primitive. We also obtain a similar result about primitivity of projective Mackey functors for nilpotent groups.

We justify that a version of Fong's theorem on induced modules of modular group algebras of p-solvable groups holds in the context of Mackey functors. Namely, if $\mathbb{K}$ is an algebraically closed field of characteristic $p > 0$ and G is p-solvable then any indecomposable $\mu_{\mathbb{K}}(G)$-module whose vertex is a p'-group (such a $\mu_{\mathbb{K}}(G)$-module is necessarily projective) is induced from a $\mu_{\mathbb{K}}(K)$-module where K is a Hall p'-subgroup of G.

Finally, we study evaluations of Mackey functors in Section 7. We give some results about the structure of $P_{H,V}^{G}(H)$ as $\mathbb{K}\overline{N}_G(H)$-module where $P_{H,V}^{G}$ is a principal indecomposable Mackey functor for G over a field $\mathbb{K}$. For instance, we prove that $P_{H,V}^{G}(H)$ is projective if H is normal in G, and that $P_{H,V}^{G}(H)$ is the projective cover of V if H is a p'-subgroup where p is the characteristic of the field $\mathbb{K}$.

Most of our notations are standard. Let $H \leqslant G \geqslant K$. By the notation $HgK \subseteq G$ we mean that g ranges over a complete set of representatives of double cosets of (H, K) in G. We also write $\overline{N}_G(H)$ for the quotient group $N_G(H)/H$ where $N_G(H)$ is the normalizer of H in G.

Throughout $\mathbb{K}$ is a field and G is a finite group. We consider only finite dimensional Mackey functors.

2. Preliminaries

In this section, we briefly summarize some crucial material on Mackey functors. For the proofs, see Thévenaz and Webb [8,9]. Recall that a Mackey functor for G over a commutative unital ring R is such that, for each subgroup H of G, there is an R-module $M(H)$; for each pair $H, K \leqslant G$ with $H \leqslant K$, there are R-module homomorphisms $r_H^K : M(K) \to M(H)$ called the restriction map and $t_H^K : M(H) \to M(K)$ called the transfer map or the trace map; for each $g \in G$, there is an R-module homomorphism $c_H^g : M(H) \to M({}^gH)$ called the conjugation map. The following axioms must be satisfied for any $g, h \in G$ and $H, K, L \leqslant G$ [1,4,8,9].

(M$_1$) If $H \leqslant K \leqslant L$, $r_H^L = r_H^K r_K^L$ and $t_H^L = t_K^L t_H^K$; moreover $r_H^H = t_H^H = id_{M(H)}$.
(M$_2$) $c_K^{gh} = c_{{}^hK}^{g} c_K^h$.
(M$_3$) If $h \in H$, $c_H^h : M(H) \to M(H)$ is the identity.
(M$_4$) If $H \leqslant K$, $c_H^g r_H^K = r_{{}^gH}^{{}^gK} c_K^g$ and $c_K^g t_H^K = t_{{}^gH}^{{}^gK} c_H^g$.
(M$_5$) (Mackey Axiom) If $H \leqslant L \geqslant K$, $r_H^L t_K^L = \sum_{HgK \subseteq L} t_{H\cap {}^gK}^{H} r_{H\cap {}^gK}^{{}^gK} c_K^g$.

Another possible definition of Mackey functors for G over R uses the Mackey algebra $\mu_R(G)$ [1,9]: $\mu_{\mathbb{Z}}(G)$ is the algebra generated by the elements r_H^K, t_H^K, and c_H^g, where H and K are subgroups of G such that $H \leqslant K$, and $g \in G$, with the relations (M$_1$)–(M$_7$).

(M$_6$) $\sum_{H\leqslant G} t_H^H = \sum_{H \leqslant G} r_H^H = 1_{\mu_{\mathbb{Z}}(G)}$.
(M$_7$) Any other product of r_H^K, t_H^K and c_H^g is zero.

A Mackey functor M for G, defined in the first sense, gives a left module $\widetilde{M}$ of the associative R-algebra $\mu_R(G) = R \otimes_{\mathbb{Z}} \mu_{\mathbb{Z}}(G)$ defined by $\widetilde{M} = \bigoplus_{H \leqslant G} M(H)$. Conversely, if $\widetilde{M}$ is a $\mu_R(G)$-module then $\widetilde{M}$ corresponds to a Mackey functor M in the first sense, defined by $M(H) = t_H^H \widetilde{M}$, the maps t_H^K, r_H^K, and c_H^g being defined as the corresponding elements of the $\mu_R(G)$. Moreover, homomorphisms and subfunctors of Mackey functors for G are $\mu_R(G)$-module homomorphisms and $\mu_R(G)$-submodules, and conversely.

Theorem 2.1. *(See [9].) Letting H and K run over all subgroups of G, letting g run over representatives of the double cosets $HgK \subseteq G$, and letting J runs over representatives of the conjugacy classes of subgroups of $H^g \cap K$, then $t_{^gJ}^H c_J^g r_J^K$ comprise, without repetition, a free R-basis of $\mu_R(G)$.*

Let M be a Mackey functor for G over R. A subgroup H of G is called a minimal subgroup of M if $M(H) \neq 0$ and $M(K) = 0$ for every subgroup K of H with $K \neq H$. Given a simple Mackey functor M for G over R, there is a unique, up to G-conjugacy, minimal subgroup H of M. Moreover, for such an H the $R\overline{N}_G(H)$-module $M(H)$ is simple, where the $R\overline{N}_G(H)$-module structure on $M(H)$ is given by $gH.x = c_H^g(x)$, see [8].

Theorem 2.2. *(See [8].) Given a subgroup $H \leqslant G$ and a simple $R\overline{N}_G(H)$-module V, then there exists a simple Mackey functor $S_{H,V}^G$ for G, unique up to isomorphism, such that H is a minimal subgroup of $S_{H,V}^G$ and $S_{H,V}^G(H) \cong V$. Moreover, up to isomorphism, every simple Mackey functor for G has the form $S_{H,V}^G$ for some $H \leqslant G$ and simple $R\overline{N}_G(H)$-module V. Two simple Mackey functors $S_{H,V}^G$ and $S_{H',V'}^G$ are isomorphic if and only if, for some element $g \in G$, we have $H' = {}^gH$ and $V' \cong c_H^g(V)$.*

We now recall the definitions of restriction, induction and conjugation for Mackey functors [1,7–9]. Let M and T be Mackey functors for G and H, respectively, where $H \leqslant G$, then the restricted Mackey functor $\downarrow_H^G M$ is the $\mu_R(H)$-module $1_{\mu_R(H)} M$ and the induced Mackey functor $\uparrow_H^G T$ is the $\mu_R(G)$-module $\mu_R(G) 1_{\mu_R(H)} \otimes_{\mu_R(H)} T$, where $1_{\mu_R(H)}$ denotes the unity of $\mu_R(H)$. For $g \in G$, the conjugate Mackey functor $|_H^g T = {}^gT$ is the $\mu_R({}^gH)$-module T with the module structure given for any $x \in \mu_R({}^gH)$ and $t \in T$ by $x.t = (\gamma_{g^{-1}} x \gamma_g)t$, where γ_g is the sum of all c_X^g with X ranging over subgroups of H. Obviously, one has $|_L^g S_{H,V}^L \cong S_{{}^gH, c_H^g(V)}^{{}^gL}$. The subgroup $\{g \in N_G(H)\colon\ {}^gT \cong T\}$ of $N_G(H)$ is called the inertia group of T in $N_G(H)$.

Theorem 2.3. *(See [7].) Let H be a subgroup of G. Then $\uparrow_H^G$ is both left and right adjoint of $\downarrow_H^G$.*

Given $H \leqslant G \geqslant K$ and a Mackey functor M for K over R, the following is the Mackey decomposition formula for Mackey algebras, which can be found in [9],

$$\downarrow_H^L \uparrow_K^L M \cong \bigoplus_{HgK \subseteq L} \uparrow_{H \cap {}^gK}^H \downarrow_{H \cap {}^gK}^{{}^gK} |_K^g M.$$

We finally recall some facts from [8] about inflated Mackey functors. Let N be a normal subgroup of G. Given a Mackey functor $\widetilde{M}$ for G/N, we define a Mackey functor $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$ for G, called the inflation of $\widetilde{M}$, as $M(K) = \widetilde{M}(K/N)$ if $K \geqslant N$ and $M(K) = 0$ otherwise.

The maps t_H^K, r_H^K, c_H^g of M are zero unless $N \leqslant H \leqslant K$ in which case they are the maps $\tilde{t}_{H/N}^{K/N}, \tilde{r}_{H/N}^{K/N}, \tilde{c}_{H/N}^{gN}$ of $\widetilde{M}$. Evidently, one has $\mathrm{Inf}_{G/N}^G S_{H/N,V}^{G/N} \cong S_{H,V}^G$.

Given a Mackey functor M for G we define Mackey functors $L_{G/N}^+ M$ and $L_{G/N}^- M$ for G/N as follows:

$$\left(L_{G/N}^+ M\right)(K/N) = M(K) \Big/ \sum_{J \leqslant K:\ J \ngeqslant N} t_J^K\left(M(J)\right),$$

$$\left(L_{G/N}^- M\right)(K/N) = \bigcap_{J \leqslant K:\ J \ngeqslant N} \operatorname{Ker} r_J^K.$$

The maps on these two new functors come from those on M. They are well defined because the maps on M preserve the sum of images of traces and the intersection of kernels of restrictions, see [8].

Theorem 2.4. *(See [8].) For any normal subgroup N of G, $L_{G/N}^+$ is a left adjoint of $\mathrm{Inf}_{G/N}^G$ and $L_{G/N}^-$ is a right adjoint of $\mathrm{Inf}_{G/N}^G$.*

Theorem 2.5. *(See [8].) For any simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$, we have*

$$S_{H,V}^G \cong \uparrow_{N_G(H)}^G \mathrm{Inf}_{N_G(H)/H}^{N_G(H)} S_{1,V}^{\overline{N}_G(H)}.$$

3. Kernels, inflations, and relative projectivity

In this section, we want to define and study a notion of a kernel of a Mackey functor, and also want to relate this notion to the adjoints of the inflation functor given in 2.4. We also study the relative projectivity of inflated Mackey functors.

Let M be a $\mu_{\mathbb{K}}(G)$-module. We first study the existence of a normal subgroup N of G such that $M \cong \mathrm{Inf}_{G/N}^G T$ for some $\mu_{\mathbb{K}}(G/N)$-module T. There is an obvious such N, namely the trivial subgroup of G. Indeed, we will show that there is a unique largest normal subgroup $\mathcal{K}(M)$ of G such that M is inflated from the quotient $G/\mathcal{K}(M)$.

For any nonzero $\mu_{\mathbb{K}}(G)$-module M we define

$$\mathcal{K}(M) = \bigcap_X X$$

where X ranges over all minimal subgroups of M. Since the set of minimal subgroups of M is closed under taking G-conjugates (as the maps c_H^g are bijective), $\mathcal{K}(M)$ is the unique largest normal subgroup of G satisfying $\mathcal{K}(M) \leqslant H$ for any subgroup H of G with $M(H) \neq 0$.

Remark 3.1. Let N be a normal subgroup of G and $\widetilde{M}$ be a $\mu_{\mathbb{K}}(G/N)$-module. Then, letting $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$ we have $N \subseteq \mathcal{K}(M)$ and $\mathcal{K}(M)/N = \mathcal{K}(\widetilde{M})$.

Proof. This is obvious by the definition of inflated Mackey functors. $\square$

For a $\mu_{\mathbb{K}}(G)$-module M with maps t, r, c we define a $\mu_{\mathbb{K}}(G/\mathcal{K}(M))$-module M^0 (see 3.2) with maps $\tilde{t}, \tilde{r}, \tilde{c}$ as follows:

$$M^0\bigl(H/\mathcal{K}(M)\bigr) = M(H),$$

$$\tilde{t}^{K/\mathcal{K}(M)}_{H/\mathcal{K}(M)} = t^K_H, \quad \tilde{r}^{K/\mathcal{K}(M)}_{H/\mathcal{K}(M)} = r^K_H, \quad \text{and} \quad \tilde{c}^{g\mathcal{K}(M)}_{H/\mathcal{K}(M)} = c^g_H,$$

for any H, K and $g \in G$ with $\mathcal{K}(M) \leqslant H \leqslant K \leqslant G$.

Lemma 3.2. *M^0 is a $\mu_{\mathbb{K}}(G/\mathcal{K}(M))$-module satisfying $M = \operatorname{Inf}^G_{G/\mathcal{K}(M)} M^0$ and $\mathcal{K}(M^0) = 1$.*

Proof. Let H be a subgroup of G. If $M(H) \neq 0$ then $\mathcal{K}(M) \subseteq H$ so that

$$M(H) = \bigl(\operatorname{Inf}^G_{G/\mathcal{K}(M)} M^0\bigr)(H).$$

This shows that $M = \operatorname{Inf}^G_{G/\mathcal{K}(M)} M^0$ as sets. Moreover, it follows by the construction of M^0 that the maps $\tilde{t}, \tilde{r}, \tilde{c}$ of M^0 satisfy the required axioms so that M^0 becomes a Mackey functor because the maps t, r, c satisfy the similar axioms. Therefore M^0 is a well defined $\mu_{\mathbb{K}}(G/\mathcal{K}(M))$-module such that $M = \operatorname{Inf}^G_{G/\mathcal{K}(M)} M^0$. Finally, 3.1 shows that $\mathcal{K}(M^0) = 1$. □

We note that the Mackey functor M^0 constructed above is equal to both of $L^+_{G/\mathcal{K}(M)} M$ and $L^-{}_{G/\mathcal{K}(M)} M$.

Proposition 3.3. *For any $\mu_{\mathbb{K}}(G)$-module M, the set of all normal subgroups N of G such that M is inflated from the quotient G/N has a unique largest element with respect to inclusion. Moreover, this largest element is equal to $\mathcal{K}(M)$.*

Proof. 3.2 implies that M is inflated from the quotient $G/\mathcal{K}(M)$. Suppose that N is a normal subgroup of G such that M is inflated from the quotient G/N. Then N is a subgroup of $\mathcal{K}(M)$ by 3.1. Hence $\mathcal{K}(M)$ is the largest normal subgroup of G such that M is inflated from the quotient $G/\mathcal{K}(M)$. □

It is evident that 3.3 is true for Mackey functors over any commutative ring R, not just over a field $\mathbb{K}$.

It is clear that any $\mu_{\mathbb{K}}(G)$-module M can be inflated from G/N where N is any normal subgroup of G with $N \leqslant \mathcal{K}(M)$.

Let M be a $\mu_{\mathbb{K}}(G)$-module. By the kernel of M we mean the subgroup $\mathcal{K}(M)$. We say that M is faithful if it is not inflated from a proper quotient of G, equivalently $\mathcal{K}(M) = 1$.

For a subgroup H of G, we denote by H_G the core of H in G, that is the largest normal subgroup of G contained in H, equivalently the intersection of all G-conjugates of H.

We now describe the kernels of simple Mackey functors.

Corollary 3.4. *$\mathcal{K}(S^G_{H,V}) = H_G$ for any simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$. In particular, for any normal subgroup N of G contained in H, we have*

$$S^G_{H,V} \cong \operatorname{Inf}^G_{G/N} S^{G/N}_{H/N,V}.$$

Proof. It is clear that $\mathcal{K}(S^G_{H,V}) = H_G$, because the minimal subgroups of $S^G_{H,V}$ are precisely the G-conjugates of H. So 3.3 implies that $S^G_{H,V} \cong \mathrm{Inf}^G_{G/N}\, T$ for some $\mu_{\mathbb{K}}(G/N)$-module T. As Inf is an exact functor, T must be simple which is isomorphic to $S^{G/N}_{H/N,V}$ by the definition of inflated functors. $\square$

As in [9] we denote by $P^G_{H,V}$ the projective cover of the simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$.

Corollary 3.5. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a p-subgroup of G. Then for any simple $\mathbb{K}\overline{N}_G(H)$-module V the $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ is faithful.*

Proof. This follows from [9, (12.2) Corollary] stating that 1 is a minimal subgroup of $P^G_{H,V}$. $\square$

Before going further we need the following.

Lemma 3.6.

(1) *Let M be a $\mu_{\mathbb{K}}(G)$-module and H be a subgroup of G such that $\downarrow^G_H M \neq 0$. Then $\mathcal{K}(M) \leqslant \mathcal{K}(\downarrow^G_H M)$.*
(2) *$\mathcal{K}(M) \leqslant \mathcal{K}(T)$ for any $\mu_{\mathbb{K}}(G)$-module M and any submodule T of M.*
(3) *Let $M \to T$ be an epimorphism of $\mu_{\mathbb{K}}(G)$-modules. Then $\mathcal{K}(M) \leqslant \mathcal{K}(T)$.*
(4) *Let H be a subgroup of G and T be a $\mu_{\mathbb{K}}(H)$-module. Then $\mathcal{K}(\uparrow^G_H T) \leqslant \mathcal{K}(T)$.*
(5) *For any exact sequence*

$$0 \to S \to M \to T \to 0$$

of $\mu_{\mathbb{K}}(G)$-modules, $\mathcal{K}(M) = \mathcal{K}(S) \cap \mathcal{K}(T)$.
(6) *$\mathcal{K}(M_1 \oplus M_2) = \mathcal{K}(M_1) \cap \mathcal{K}(M_2)$ for any $\mu_{\mathbb{K}}(G)$-modules M_1 and M_2.*

Proof. (1) If K is a minimal subgroup of $\downarrow^G_H M$ then $M(K) \neq 0$ so that K contains a minimal subgroup of M. This shows that $\mathcal{K}(M) \leqslant \mathcal{K}(\downarrow^G_H M)$.

(2) Let T be a submodule of M. Then it is clear that any minimal subgroup of T contains a minimal subgroup of M, implying that $\mathcal{K}(M) \leqslant \mathcal{K}(T)$.

(3) Let K be a minimal subgroup of T. As T is an epimorphic image of M, there is a surjective map $M(K) \to T(K)$, implying that $M(K) \neq 0$ because $T(K) \neq 0$. Therefore K contains a minimal subgroup of M. Consequently, $\mathcal{K}(M) \leqslant \mathcal{K}(T)$.

(4) By the Mackey decomposition formula T is a direct summand of $\downarrow^G_H \uparrow^G_H T$. Then parts (1) and (3) imply that

$$\mathcal{K}\big(\uparrow^G_H T\big) \leqslant \mathcal{K}\big(\downarrow^G_H \uparrow^G_H T\big) \leqslant \mathcal{K}(T).$$

(5) Parts (2) and (3) imply that $\mathcal{K}(M) \leqslant \mathcal{K}(S) \cap \mathcal{K}(T)$. For the reverse inclusion, if K is a minimal subgroup of M then it follows from the exactness of the given sequence that $S(K)$ or $T(K)$ is nonzero, implying that $\mathcal{K}(M) \supseteq \mathcal{K}(S) \cap \mathcal{K}(T)$.

(6) Follows by part (5). $\square$

We now note that the inclusions in the previous results may be strict inclusions.

Let $M = S^G_{H,\mathbb{K}}$. Then it is clear that $\downarrow^G_H M = S^H_{H,\mathbb{K}}$. Therefore 3.4 implies that $\mathcal{K}(M) = H_G$ and $\mathcal{K}(\downarrow^G_H M) = H$. So the inclusion in part (1) of 3.6 may be strict.

Let $\mathbb{K}$ be a field of characteristic $p > 0$ and C be a subgroup of G of order p. Then the socle of any principal indecomposable $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{C,V}$ is isomorphic to $S^G_{C,V}$ by [9, (19.1) Lemma]. Therefore if we put $M = P^G_{C,V}$ and $T = S^G_{C,V}$, then T is a subfunctor of M such that $\mathcal{K}(M) = 1$ (by 3.5) and $\mathcal{K}(T) = C_G$. Furthermore, T is an epimorphic image of M. This shows that the inclusions in parts (2) and (3) of 3.6 may be strict.

We next record some commuting relations of induction and restriction with inflation.

Lemma 3.7. *Let N be a normal subgroup of G and H be a subgroup of G.*

(1) *If $N \leqslant H$ then for any $\mu_{\mathbb{K}}(H/N)$-module $\widetilde{T}$,*

$$\mathrm{Inf}^G_{G/N} \uparrow^{G/N}_{H/N} \widetilde{T} \cong \uparrow^G_H \mathrm{Inf}^H_{H/N} \widetilde{T}.$$

(2) *Let $\widetilde{M}$ be a $\mu_{\mathbb{K}}(G/N)$-module. If $\downarrow^G_H \mathrm{Inf}^G_{G/N} \widetilde{M}$ is nonzero then $N \leqslant H$. Moreover, for $N \leqslant H$ we have*

$$\downarrow^G_H \mathrm{Inf}^G_{G/N} \widetilde{M} \cong \mathrm{Inf}^H_{H/N} \downarrow^{G/N}_{H/N} \widetilde{M}.$$

Proof. (1) One may prove the result by using the explicit description of induced Mackey functors given in [7]. Alternatively we prove the result by using the adjointness of functors given in 2.3 and 2.4. From the adjointness of the pairs

$$\left(L^+_{G/N}, \mathrm{Inf}^G_{G/N}\right) \quad \text{and} \quad \left(\downarrow^{G/N}_{H/N}, \uparrow^{G/N}_{H/N}\right)$$

we see that the pair

$$\left(\downarrow^{G/N}_{H/N} L^+_{G/N}, \mathrm{Inf}^G_{G/N} \uparrow^{G/N}_{H/N}\right)$$

is an adjoint pair. Similarly, the adjointness of the pairs

$$\left(\downarrow^G_H, \uparrow^G_H\right) \quad \text{and} \quad \left(L^+_{H/N}, \mathrm{Inf}^H_{H/N}\right)$$

imply that the pair

$$\left(L^+_{H/N} \downarrow^G_H, \uparrow^G_H \mathrm{Inf}^H_{H/N}\right)$$

is an adjoint pair. It is clear by the definition of L^+ (see Section 2) that the functors

$$\downarrow^{G/N}_{H/N} L^+_{G/N} \quad \text{and} \quad L^+_{H/N} \downarrow^G_H$$

are naturally isomorphic. Consequently, the functors

$$\mathrm{Inf}^G_{G/N} \uparrow^{G/N}_{H/N} \quad \text{and} \quad \uparrow^G_H \mathrm{Inf}^H_{H/N},$$

being right adjoints of two isomorphic functors, must be naturally isomorphic.

(2) This is obvious by the definitions of inflated and restricted Mackey functors. □

Part (1) of 3.7 is straightforward, when Mackey functors are viewed as functors on the category of finite G-sets [2]. Induction of Mackey functors corresponds to restriction of G-sets, and inflation of Mackey functors corresponds to fixed points. If X is a G-set, then the G/N-sets $(\mathrm{Res}^G_H X)^N$ and $\mathrm{Res}^{G/N}_{H/N}(X^N)$ are obviously isomorphic. See [1,2].

We also need the following commuting relations between L^+, L^-, Inf and $\uparrow$.

Lemma 3.8. *Let N be a normal subgroup of G and H be a subgroup of G. Given a $\mu_{\mathbb{K}}(G/N)$-module $\widetilde{M}$ and a $\mu_{\mathbb{K}}(H)$-module T we have*

(1) $L^+_{G/N}\operatorname{Inf}^G_{G/N}\widetilde{M}\cong\widetilde{M}$.
(2) $L^-_{G/N}\operatorname{Inf}^G_{G/N}\widetilde{M}\cong\widetilde{M}$.
(3) $L^+_{G/N}\uparrow^G_H T\cong\uparrow^{G/N}_{H/N}L^+_{H/N}T$ *if* $N\leqslant H$.

Proof. (1) We note that $(\operatorname{Inf}^G_{G/N}\widetilde{M})(J)=0$ for any J not containing N. Then the result follows immediately by the definition of L^+.

(2) Follows from part (1), since the functor $L^-_{G/N}\operatorname{Inf}^G_{G/N}$ is right adjoint to the functor $L^+_{G/N}\operatorname{Inf}^G_{G/N}$.

(3) Firstly it is easy to see from the definitions of $\downarrow$ and Inf that the functors

$$\downarrow^G_H\operatorname{Inf}^G_{G/N}\quad\text{and}\quad\operatorname{Inf}^H_{H/N}\downarrow^{G/N}_{H/N}$$

are naturally isomorphic. Therefore their left adjoints must be naturally isomorphic. As in the proof of the previous result we see using the adjoint functors given in 2.3 and 2.4 that the respective left adjoints of the functors

$$\downarrow^G_H\operatorname{Inf}^G_{G/N}\quad\text{and}\quad\operatorname{Inf}^H_{H/N}\downarrow^{G/N}_{H/N}$$

are

$$L^+_{G/N}\uparrow^G_H\quad\text{and}\quad\uparrow^{G/N}_{H/N}L^+_{H/N}.\qquad\square$$

Now we can study the relative projectivity of inflated Mackey functors. An indecomposable Mackey functor M for G over $\mathbb{K}$ is said to be H-projective for some subgroup H of G if M is a direct summand of $\uparrow^G_H\downarrow^G_H M$, equivalently M is a direct summand of $\uparrow^G_H T$ for some Mackey functor T for H. For an indecomposable Mackey functor M, up to conjugacy there is a unique minimal subgroup H of G, called the vertex of M, so that M is H-projective, see [7].

Although the definition of relative projectivity of Mackey functors is similar to the that of modules of group algebras, there are some differences. Any principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ has vertex H. If M is an indecomposable $\mu_{\mathbb{K}}(G)$-module and $\mathbb{K}$ is of characteristic $p>0$, then vertices of M are not necessarily p-subgroups of G in which case we have $\downarrow^G_P M=0$ where P is a Sylow p-subgroup of G. For more details see [9].

Remark 3.9. Let H be a subgroup of G and M be an indecomposable H-projective $\mu_{\mathbb{K}}(G)$-module. Then $\mathcal{K}(M)\leqslant H_G$.

Proof. M is a direct summand of $\uparrow_H^G \downarrow_H^G M$. Thus $\downarrow_H^G M \neq 0$. So we may find a minimal subgroup of M contained in H. This shows that $\mathcal{K}(M) \leqslant H$. The result follows by the normality of $\mathcal{K}(M)$ in G. $\square$

Note that by their definitions all of the functors Inf, L^+, and L^- commute with finite direct sums. Indeed, by 2.4 we see that L^+ and Inf commute with arbitrary direct sums, while L^- commutes with arbitrary direct products.

Lemma 3.10. *Let N be a normal subgroup of G and $\widetilde{M}$ be a $\mu_{\mathbb{K}}(G/N)$-module. Then*

(1) $\mathrm{Inf}_{G/N}^G \widetilde{M}$ *is indecomposable if and only if $\widetilde{M}$ is indecomposable.*
(2) *If* $\mathrm{Inf}_{G/N}^G \widetilde{M}$ *is projective then $\widetilde{M}$ is projective.*

Proof. We let $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$.

(1) It is clear by the definition of the functor Inf that $\mathrm{End}_{\mu_{\mathbb{K}}(G)}(M) \cong \mathrm{End}_{\mu_{\mathbb{K}}(G/N)}(\widetilde{M})$ as $\mathbb{K}$-algebras. Then, the result follows, because a module is indecomposable if and only if the identity is a primitive idempotent of its endomorphism algebra.

(2) By the functorial properties of the functors $L_{G/N}^+$ and $\mathrm{Inf}_{G/N}^G$ given in 2.4, we see that $L_{G/N}^+$ sends projectives to projectives. Hence, if M is projective then $L_{G/N}^+ M$, which is isomorphic to $\widetilde{M}$ by 3.8, is projective. $\square$

In the next result we show that inflation preserves the vertices of Mackey functors, which is not the case for modules of group algebras.

Theorem 3.11. *Let N be a normal subgroup of G, let $\widetilde{M}$ be an indecomposable $\mu_{\mathbb{K}}(G/N)$-module, and let $M = \mathrm{Inf}_{G/N}^G \widetilde{M}$. If Q is a vertex of M and P/N is a vertex of $\widetilde{M}$ then $Q =_G P$.*

Proof. As P/N is a vertex of $\widetilde{M}$, there is a $\mu_{\mathbb{K}}(P/N)$-module $\widetilde{T}$ such that $\widetilde{M}$ is a direct summand of $\uparrow_{P/N}^{G/N} \widetilde{T}$. Since Inf commutes with direct sums, M is a direct summand of

$$\mathrm{Inf}_{G/N}^G \uparrow_{P/N}^{G/N} \widetilde{T}$$

which is by 3.7 isomorphic to

$$\uparrow_P^G \mathrm{Inf}_{P/N}^P \widetilde{T}.$$

So M is P-projective. This implies that $Q \leqslant_G P$ because M is indecomposable.

Moreover, having Q as a vertex, M is a direct summand of $\uparrow_Q^G T$ for some $\mu_{\mathbb{K}}(Q)$-module T. Then, for $L_{G/N}^+$ commutes with finite direct sums, we see that $L_{G/N}^+ M$ is a direct summand of

$$L_{G/N}^+ \uparrow_Q^G T,$$

isomorphic to

$$\uparrow_{Q/N}^{G/N} L_{G/N}^+ T$$

by 3.8, where we also use 3.9 to see that $N \leqslant Q$. Hence $L^+_{G/N}M$ is Q/N-projective. It follows by 3.8 that

$$L^+_{G/N}M = L^+_{G/N}\operatorname{Inf}^G_{G/N}\widetilde{M} \cong \widetilde{M}.$$

Consequently $P/N \leqslant_{G/N} Q/N$, or $P \leqslant_G Q$. $\square$

Almost the whole proof of 3.11 holds for modules over group algebras, the only difference is the point where we use 3.9 to see that $N \leqslant Q$.

We next give a result about inflations of principal indecomposable Mackey functors.

Corollary 3.12. *Let $P^G_{H,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G)$-module. If N is a normal subgroup of G in the kernel of $P^G_{H,V}$ then*

$$P^G_{H,V} \cong \operatorname{Inf}^G_{G/N} P^{G/N}_{H/N,V}.$$

Proof. We may write

$$P^G_{H,V} \cong \operatorname{Inf}^G_{G/N}\widetilde{M}$$

for some $\mu_{\mathbb{K}}(G/N)$-module $\widetilde{M}$. Then 3.10 implies that $\widetilde{M}$ is isomorphic to a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module, say $\widetilde{M} \cong P^{G/N}_{K/N,W}$. We may assume that $H = K$ because $H =_G K$ by 3.11. As $\operatorname{Inf}^G_{G/N}$ is an exact functor and $P^{G/N}_{K/N,W}$ is the projective cover of $S^{G/N}_{K/N,W}$, there is a $\mu_{\mathbb{K}}(G)$-module epimorphism

$$P^G_{H,V} \to \operatorname{Inf}^G_{G/N} S^{G/N}_{H/N,W} \cong S^G_{H,W}.$$

This shows that $S^G_{H,V} \cong S^G_{H,W}$, and hence $V \cong W$. $\square$

The previous result shows that inflation of some projective Mackey functors are still projective, which is not true for some other projective Mackey functors. Therefore, given a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module $P^{G/N}_{H/N,V}$ it is not true in general that

$$P^G_{H,V} \cong \operatorname{Inf}^G_{G/N} P^{G/N}_{H/N,V}.$$

For example, let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a p-group. If the above isomorphisms holds then considering kernels of both sides we get $1 = N$ (see 3.5 and 3.1).

Lemma 3.13. *Let N be a normal subgroup of G. If $P^{G/N}_{H/N,V}$ is a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module such that $M = \operatorname{Inf}^G_{G/N} P^{G/N}_{H/N,V}$ is a projective $\mu_{\mathbb{K}}(G)$-module, then $M \cong P^G_{H,V}$.*

Proof. Being an exact functor, $\operatorname{Inf}^G_{G/N}$ induces a $\mu_{\mathbb{K}}(G)$-module epimorphism

$$M \to \operatorname{Inf}^G_{G/N} S^{G/N}_{H/N,V} \cong S^G_{H,V}.$$

Then by 3.10 M is indecomposable. Since it is also projective, M is isomorphic to the projective cover $P^G_{H,V}$ of $S^G_{H,V}$. $\square$

For any group X, we denote by $P_X(\,)$ the projective cover of its argument which is a $\mu_{\mathbb{K}}(X)$-module. We also denote by $J(\,)$ the radical of its argument.

By the following we can easily describe the image of a principal indecomposable $\mu_{\mathbb{K}}(G)$-module under the functor L^+.

Theorem 3.14. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then*

(1) $L^+_{G/N} P_G(M) \cong P_{G/N}(L^+_{G/N} M)$.
(2) $L^+_{G/N} P_G(M)$ *is nonzero if and only if $M/J(M)$ has a simple summand with kernel containing N.*

Proof. It follows by 2.4 that L^+ sends projectives to projectives. Letting $M_1 = L^+_{G/N} P_G(M)$ and $M_2 = P_{G/N}(L^+_{G/N} M)$, we will show that $M_1/J(M_1) \cong M_2/J(M_2)$. This clearly shows that $M_1 \cong M_2$ because both are projective.

For any simple $\mu_{\mathbb{K}}(G/N)$-module $T = S^{G/N}_{H/N,V}$, by the adjointness of the pair (L^+, Inf) given in 2.4, we have the following $\mathbb{K}$-space isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M_1/J(M_1), T\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}(M_1, T) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(P_G(M), S^G_{H,V}\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(P_G(M)/J\big(P_G(M)\big), S^G_{H,V}\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(M/J(M), S^G_{H,V}\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(M, S^G_{H,V}\big).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
\mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M_2/J(M_2), T\big) &\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(L^+_{G/N} M/J\big(L^+_{G/N} M\big), T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(L^+_{G/N} M, T\big) \\
&\cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(M, S^G_{H,V}\big).
\end{aligned}$$

Consequently,

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M_1/J(M_1), S\big) \cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M_1/J(M_1), S\big)$$

for any simple $\mu_{\mathbb{K}}(G/N)$-module S. This proves that $M_1/J(M_1) \cong M_2/J(M_2)$.

Finally, from

$$\mathrm{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M_1/J(M_1), S^{G/N}_{H/N,V}\big) \cong \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(M, S^G_{H,V}\big),$$

it follows that $M_1 \neq 0$ if and only if $\mathrm{Hom}_{\mu_{\mathbb{K}}(G)}(M, S^G_{H,V}) \neq 0$, equivalently $M/J(M)$ has a simple summand of the form $S^G_{H,V}$ with $N \leqslant H$. Then part (2) follows by 3.4. $\square$

Corollary 3.15. *Let N be a normal subgroup of G and $P^G_{H,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G)$-module. Then $L^+_{G/N}P^G_{H,V}$ is nonzero if and only if $N \leqslant H$. Moreover, if $N \leqslant H$ then*

$$L^+_{G/N}P^G_{H,V} \cong P^{G/N}_{H/N,V}.$$

Proof. Letting $M = S^G_{H,V}$, it follows by 3.14 that

$$L^+_{G/N}P^G_{H,V} \cong P_{G/N}\big(L^+_{G/N}S^G_{H,V}\big),$$

and also that it is nonzero if and only if $N \leqslant K(M) \leqslant H$. Suppose now that $N \leqslant H$. Then 3.4 implies

$$S^G_{H,V} \cong \operatorname{Inf}^G_{G/N} S^{G/N}_{H/N,V}.$$

Finally, applying the functor $L^+_{G/N}$ to the both sides of the latest isomorphism, by 3.8 we obtain

$$L^+_{G/N}S^G_{H,V} \cong L^+_{G/N}\operatorname{Inf}^G_{G/N} S^{G/N}_{H/N,V} \cong S^{G/N}_{H/N,V}.$$

This finishes the proof. $\square$

We also have the following obvious consequence of 3.14.

Corollary 3.16. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then, $L^+_{G/N}M$ is nonzero if and only if $M/J(M)$ has a simple summand with kernel containing N.*

Although it is clear by the definition of L^+, the proof of 3.15 shows that

$$L^+_{G/N}S^G_{H,V} \cong S^{G/N}_{H/N,V}$$

if $N \leqslant H$ (and 0 otherwise).

Given a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module $P^{G/N}_{H/N,V}$, it follows by 3.12 and 3.13 that $\operatorname{Inf}^G_{G/N} P^{G/N}_{H/N,V}$ is projective if and only if $N \leqslant \mathcal{K}(P^G_{H,V})$. However, for the projective cover of an inflated Mackey functor we have the following.

Proposition 3.17. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G/N)$-module. Then*

$$P_G\big(\operatorname{Inf}^G_{G/N} M\big) \cong P_G\big(\operatorname{Inf}^G_{G/N} P_{G/N}(M)\big).$$

Proof. Letting $M_1 = P_G(\operatorname{Inf}^G_{G/N} M)$ and $M_2 = P_G(\operatorname{Inf}^G_{G/N} P_{G/N}(M))$, it suffices to show that

$$\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_1, S) \cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_2, S)$$

for any simple $\mu_{\mathbb{K}}(G)$-module S because M_1 and M_2 are projective.

Take any simple $\mu_{\mathbb{K}}(G)$-module S. If $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_i, S) \neq 0$ for $i = 1$ or $i = 2$, then we first observe that S can be inflated from the quotient G/N. Indeed, if

$$\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_i, S) \cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big(M_i/J(M_i), S\big) \neq 0$$

then it follows by part (3) of 3.6 that $\mathcal{K}(M_i/J(M_i)) \leqslant \mathcal{K}(S)$. As

$$M_1/J(M_1) \cong \operatorname{Inf}_{G/N}^{G} M/J\big(\operatorname{Inf}_{G/N}^{G} M\big),$$

part (3) of 3.6 implies that

$$N \leqslant \mathcal{K}\big(\operatorname{Inf}_{G/N}^{G} M\big) \leqslant \mathcal{K}\big(\operatorname{Inf}_{G/N}^{G} M/J\big(\operatorname{Inf}_{G/N}^{G} M\big)\big) = \mathcal{K}\big(M_1/J(M_1)\big).$$

Similarly, we can deduce that $N \leqslant \mathcal{K}(M_2/J(M_2))$. Thus we may assume that $N \leqslant \mathcal{K}(S)$.

As $N \leqslant \mathcal{K}(S)$, by the proof of 3.15 the $\mu_{\mathbb{K}}(G/N)$-module $L_{G/N}^{+} S$ is simple and

$$S \cong \operatorname{Inf}_{G/N}^{G} L_{G/N}^{+} S.$$

Now by using the adjointness of the pair $(L^{+}, \operatorname{Inf})$ and part (1) of 3.8 we obtain

$$\begin{aligned}
\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_1, S) &\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big(M_1/J(M_1), S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big(\operatorname{Inf}_{G/N}^{G} M/J\big(\operatorname{Inf}_{G/N}^{G} M\big), S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big(\operatorname{Inf}_{G/N}^{G} M, S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big(\operatorname{Inf}_{G/N}^{G} M, \operatorname{Inf}_{G/N}^{G} L_{G/N}^{+} S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(L_{G/N}^{+} \operatorname{Inf}_{G/N}^{G} M, L_{G/N}^{+} S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M, L_{G/N}^{+} S\big).
\end{aligned}$$

In a similar way we obtain also that

$$\begin{aligned}
\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M_2, S) &\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(P_{G/N}(M), L_{G/N}^{+} S\big)\\
&\cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G/N)}\big(M, L_{G/N}^{+} S\big)
\end{aligned}$$

where the last isomorphism follows from the simplicity of $L_{G/N}^{+} S$. $\square$

The argument of the proof of 3.17 uses 3.6 which implies that if $\operatorname{Hom}_{\mu_{\mathbb{K}}(G)}(M, S) \neq 0$ for a simple $\mu_{\mathbb{K}}(G)$-module S and a $\mu_{\mathbb{K}}(G)$-module M with $N \leqslant \mathcal{K}(M)$ then $N \leqslant \mathcal{K}(S)$ so that $L_{G/N}^{+} S$ is simple and $S \cong L_{G/N}^{+} \operatorname{Inf}_{G/N}^{G} S$. As in the proof of 3.17 we can conclude by using the adjointness of the pair $(L^{+}, \operatorname{Inf})$ that

$$\operatorname{Inf}_{G/N}^{G} T/J\big(\operatorname{Inf}_{G/N}^{G} T\big) \cong \operatorname{Inf}_{G/N}^{G}\big(T/J(T)\big) \cong \operatorname{Inf}_{G/N}^{G} T/\operatorname{Inf}_{G/N}^{G} J(T)$$

for any $\mu_{\mathbb{K}}(G/N)$-module T. In particular, $\operatorname{Inf}_{G/N}^{G} T$ is semisimple if and only if T is semisimple.

The following is immediate from 3.17.

Corollary 3.18. *Let N be a normal subgroup of G and $P^{G/N}_{H/N,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module. Then*

$$P_G\big(\mathrm{Inf}^G_{G/N} P^{G/N}_{H/N,V}\big) \cong P^G_{H,V}.$$

We are aiming to characterize the normal subgroups N of G such that the functor $\mathrm{Inf}^G_{G/N}$ sends projectives to projectives. The example given before 3.13 shows that this problem is related to the problem of finding kernels of principal indecomposable $\mu_{\mathbb{K}}(G)$-modules.

For any prime p and group H, we denote by $O^p(H)$ the minimal normal subgroup of H such that the quotient $H/O^p(H)$ is a p-group. If $H = O^p(H)$ then H is said to be p-perfect.

The following is an immediate consequences of some results proved in Section 9 of [9], by analyzing the action of the Burnside ring on a Mackey functor.

Lemma 3.19. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a subgroup of G. Then, for any indecomposable $\mu_{\mathbb{K}}(G)$-module M with vertex H,*

$$\big(O^p(H)\big)_G \leqslant \mathcal{K}(M) \leqslant H_G.$$

Proof. The inclusion $\mathcal{K}(M) \leqslant H_G$ follows by 3.9. According to the results of [9] mentioned above, if $M(X)$ is nonzero then $O^p(H) \leqslant_G X$. Therefore $(O^p(H))_G \leqslant \mathcal{K}(M)$. □

Since any principal indecomposable $\mu_{\mathbb{K}}(G)$-module of the form $P^G_{H,V}$ has vertex H, the previous result applies to $P^G_{H,V}$.

Lemma 3.20. *Let N be a normal subgroup of G. If the functor $\mathrm{Inf}^G_{G/N}$ sends projectives to projectives then the same is true for the functor $\mathrm{Inf}^H_{H/N}$ where H is any subgroup of G containing N.*

Proof. Let M be a projective $\mu_{\mathbb{K}}(H/N)$-module. By 2.3 both of the functors $\downarrow$ and $\uparrow$ send projectives to projectives. Therefore the $\mu_{\mathbb{K}}(G)$-module

$$\mathrm{Inf}^G_{G/N} \uparrow^{G/N}_{H/N} M \cong \uparrow^G_H \mathrm{Inf}^H_{H/N} M$$

is projective, where we use 3.7 for the isomorphism. It follows by the Mackey decomposition formula that $\mathrm{Inf}^H_{H/N} M$ is a direct summand of the projective $\mu_{\mathbb{K}}(H)$-module

$$\downarrow^G_H \uparrow^G_H \mathrm{Inf}^H_{H/N} M.$$

Therefore $\mathrm{Inf}^H_{H/N} M$ is projective. □

We now characterize the normal subgroups N of G for which the right adjoint $L^-_{G/N}$ of the functor $\mathrm{Inf}^G_{G/N}$ is exact.

Theorem 3.21. *Let $\mathbb{K}$ be a field of characteristic $p > 0$, and N be a normal subgroup of G. Then, the functor $\mathrm{Inf}^G_{G/N}$ sends projectives to projectives if and only if N is p-perfect.*

Proof. Suppose that the functor $\mathrm{Inf}_{G/N}^{G}$ sends projectives to projectives. Then the same is true for the functor $\mathrm{Inf}_{N/N}^{N}$ by the virtue of 3.20. Thus, inflating the following isomorphic projective $\mu_{\mathbb{K}}(N/N)$-modules

$$P_{N/N,\mathbb{K}}^{N/N} \cong S_{N/N,\mathbb{K}}^{N/N},$$

we get the following isomorphic projective $\mu_{\mathbb{K}}(N)$-modules

$$\mathrm{Inf}_{N/N}^{N} P_{N/N,\mathbb{K}}^{N/N} \cong \mathrm{Inf}_{N/N}^{N} S_{N/N,\mathbb{K}}^{N/N} \cong S_{N,\mathbb{K}}^{N}.$$

Then 3.13 implies that

$$P_{N,\mathbb{K}}^{N} \cong S_{N,\mathbb{K}}^{N}.$$

Therefore $S_{N,\mathbb{K}}^{N}$ is a projective simple $\mu_{\mathbb{K}}(N)$-module. [9, (13.2) Corollary] states that for a simple Mackey functor $S_{H,V}^{G}$ to be projective it is necessary that H is p-perfect. This result allow us to deduce that N is p-perfect.

Conversely, we assume that N is p-perfect. We take any principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module $P_{H/N,V}^{G/N}$. We want to show that $\mathrm{Inf}_{G/N}^{G} P_{H/N,V}^{G/N}$ is a projective $\mu_{\mathbb{K}}(G)$-module. As $O^{p}(H)$ is a normal subgroup of H and H contains N, we see that $N \cap O^{p}(H)$ is a normal subgroup of N. Then, from

$$N/N \cap O^{p}(H) \cong NO^{p}(H)/O^{p}(H) \leqslant H/O^{p}(H),$$

we obtain that $N = N \cap O^{p}(H)$ because N is p-perfect. Hence

$$N \leqslant O^{p}(H) \leqslant H$$

implying by the normality of N in G that

$$N \leqslant \bigl(O^{p}(H)\bigr)_{G} \leqslant H.$$

Now, 3.19 yields $N \leqslant \mathcal{K}(P_{H,V}^{G})$. Finally, from 3.12 we get

$$P_{H,V}^{G} \cong \mathrm{Inf}_{G/N}^{G} P_{H/N,V}^{G/N}.$$

This proves that $\mathrm{Inf}_{G/N}^{G} P_{H/N,V}^{G/N}$ is projective. $\square$

The proof of 3.21 suggests the following result connected to 3.19.

Proposition 3.22. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a subgroup of G. Then, any indecomposable $\mu_{\mathbb{K}}(G)$-module M with vertex H satisfies*

$$O^{p}\bigl(\mathcal{K}(M)\bigr) = O^{p}(H_{G}).$$

Proof. Let N be any normal subgroup of G with $N \leqslant H$. Then we see that $N \cap O^p(H)$ is a normal subgroup of N, and the corresponding quotient is isomorphic to a subgroup of $H/O^p(H)$, and so

$$O^p(N) \leqslant N \cap O^p(H) \leqslant O^p(H).$$

Since $O^p(N)$ is a normal subgroup of G contained in $O^p(H)$, it follows by 3.19 that

$$O^p(N) \leqslant \big(O^p(H)\big)_G \leqslant \mathcal{K}(M) \leqslant H_G.$$

Letting $N = H_G$ we obtain

$$O^p\big(\mathcal{K}(M)\big) = O^p(H_G). \qquad \square$$

Let M be an indecomposable $\mu_{\mathbb{K}}(G)$-module with vertex H and $\mathbb{K}$ be field of characteristic $p > 0$. We know that H is a p-group if and only if $\downarrow^G_S M \neq 0$ where S is a Sylow p-subgroup of G, see [9]. A slight stronger form of this is the following.

Remark 3.23. Let $\mathbb{K}$ be a field of characteristic $p > 0$ and M be an indecomposable $\mu_{\mathbb{K}}(G)$-module with vertex H. Then, H/H_G is a p-group (equivalently, $O^p(H) \trianglelefteq G$) if and only if

$$\downarrow^G_{\mathcal{K}(M)S} M \neq 0$$

where S is a Sylow p-subgroup of G.

Proof. There is a $\mu_{\mathbb{K}}(G/N)$-module $\widetilde{M}$ such that

$$M = \mathrm{Inf}^G_{G/N} \widetilde{M},$$

where $N = \mathcal{K}(M)$. By 3.10 and 3.11, $\widetilde{M}$ is indecomposable and has vertex H/N. Then using 3.7 we get

$$\downarrow^G_{NS} M = \mathrm{Inf}^{NS}_{NS/N} \downarrow^{G/N}_{NS/N} \widetilde{M}.$$

Since NS/N is a Sylow p-subgroup of G/N, it follows by the explanation above that H/N is a p-group if and only if

$$\downarrow^{G/N}_{NS/N} \widetilde{M} \neq 0.$$

Finally, from the proof of 3.22 we see that

$$O^p(H_G) \leqslant \mathcal{K}(M) = N \leqslant H_G \leqslant H,$$

and so $H/\mathcal{K}(M)$ is a p-group if and only if H/H_G is a p-group. $\square$

In the situation of 3.23 we can find the kernels of some principal indecomposable $\mu_{\mathbb{K}}(G)$-modules.

Remark 3.24. Let $\mathbb{K}$ be a field of characteristic $p>0$. If H/H_G is a p-group then $\mathcal{K}(P^G_{H,V})=O^p(H)$.

Proof. Let $M=P^G_{H,V}$ and $N=O^p(\mathcal{K}(M))$. Then N is a p-perfect normal subgroup of G, and so from 3.21 and 3.13 we get $M=\mathrm{Inf}^G_{G/N}\widetilde{M}$ where $\widetilde{M}=P^{G/N}_{H/N,V}$. By the proof of 3.22,

$$O^p(H_G)=O^p\big(\mathcal{K}(M)\big)=N\leqslant\big(O^p(H)\big)_G\leqslant\mathcal{K}(M)\leqslant H_G\leqslant H.$$

Thus, if H/H_G is a p-group then H/N is a p-group and 3.5 implies that $\mathcal{K}(\widetilde{M})=N/N$. Consequently, $\mathcal{K}(M)=N$ from 3.1 and it is then easy to see that $N=O^p(H)$. $\square$

Another case for which we can find the kernel of $P^G_{H,V}$ is explained in the next result.

Proposition 3.25. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p>0$. If G is nilpotent, then for any principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ we have $\mathcal{K}(P^G_{H,V})=(O^p(H))_G$.*

Proof. Since G is nilpotent, $O^p(X)=X_{p'}$ for any subgroup X of G where $X_{p'}$ is the unique Hall p'-subgroup of X. If G is nilpotent then by Section 7 of [12] the Mackey algebra $\mu_{\mathbb{K}}(G)$ admits a tensor product decomposition $\mu_{\mathbb{K}}(G_{p'})\otimes_{\mathbb{K}}\mu_{\mathbb{K}}(G_p)$. It is easy to see that under this identification of $\mu_{\mathbb{K}}(G)$, the module $P^G_{H,V}$ corresponds to the module

$$S^{G_{p'}}_{H_{p'},V}\otimes_{\mathbb{K}}P^{G_p}_{H_p,\mathbb{K}},$$

see [12] and the proof of 6.11. Then the result follows because the functor $\mathrm{Inf}^G_{G/N}$ corresponds to

$$\mathrm{Inf}^{G_{p'}}_{G_{p'}/N_{p'}}\otimes_{\mathbb{K}}\mathrm{Inf}^{G_p}_{G_p/N_p}. \qquad\square$$

4. Projective covers of Mackey functors for quotient groups

We devote this section to obtaining a relationship between principal indecomposable Mackey functors of the form $P^{G/N}_{H/N,V}$ and $P^G_{H,V}$. Let V be a simple $\mathbb{K}G$-module and N be a normal subgroup of G acting on V trivially. Then it is well known that the $\mathbb{K}G$ and $\mathbb{K}(G/N)$-module projective covers $P_G(V)$ and $P_{G/N}(V)$ of V satisfy:

$$P_{G/N}(V)\cong P_G(V)/J(\mathbb{K}N)P_G(V).$$

We mainly want to obtain a similar result for Mackey functors, see 4.9.

For any normal subgroup N of G we put

$$e_N=\sum_{X\leqslant G:X\geqslant N}t^X_X.$$

It is clear that e_N is an idempotent of $\mu_{\mathbb{K}}(G)$ with the property that, for a $\mu_{\mathbb{K}}(G)$-module M, $e_NM=M$ if and only if $N\leqslant\mathcal{K}(M)$.

We now record some well-known (and widely used in representation theory of symmetric groups) basic facts about the modules of an algebra A and its corner subalgebra eAe where e is an idempotent of A. We have the following functors some of whose properties are recalled in the next result:

$$R_e : \mathrm{Mod}(A) \to \mathrm{Mod}(eAe) \quad \text{and} \quad C_e, I_e : \mathrm{Mod}(eAe) \to \mathrm{Mod}(A)$$

given on the objects by

$$R_e(V) = eV, \qquad C_e(W) = \mathrm{Hom}_{eAe}(eA, W) \quad \text{and} \quad I_e(W) = Ae \otimes_{eAe} W.$$

The definitions on morphisms of these functors are obvious (and well known).

Remark 4.1. Let A be a finite dimensional algebra over a field and e be an idempotent of A. Then:

(1) I_e and C_e are full and faithful linear functors such that both of the functors R_eI_e and R_eC_e are naturally isomorphic to the identity functor.
(2) (I_e, R_e) and (R_e, C_e) are adjoint pairs.
(3) Both of I_e and C_e send indecomposable modules to indecomposable modules.
(4) Any simple eAe-module is of the form eS for some simple A-module S, and conversely for any simple A-module S the eAe-module eS is either zero or simple.
(5) Given simple A-modules S and S' that are not annihilated by e, one has $S \cong S'$ as A-modules if and only if $eS \cong eS'$ as eAe-modules.
(6) Given a simple eAe-module T, the A-module $I_e(T)$ has a unique maximal A-submodule J_T and one has $R_e(I_e(T)/J_T) \cong T$ and J_T is the sum of all A-submodules of $I_e(T)$ annihilated by e.

The above fact is well known, and can be found in [5, pp. 83–87].

Let $P^G_{H,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G)$-module. If N is a normal subgroup of G such that $e_N S^G_{H,V} \neq 0$, then by an application of the following result $e_N P^G_{H,V}$ is the projective cover of the simple $e_N\mu_{\mathbb{K}}(G)e_N$-module $e_N S^G_{H,V}$, and $I_{e_N} R_{e_N}(P^G_{H,V}) \cong P^G_{H,V}$.

Lemma 4.2. *Let A be a finite dimensional algebra over a field and e be an idempotent of A. Suppose that S is a simple A-module such that $eS \neq 0$. If P is the projective cover of the A-module S, then eP is the projective cover of the eAe-module eS.*

Proof. Let P' be the projective cover of the simple eAe-module eS. By 4.1 the functor I_e, which is right exact, preserves indecomposability and projectivity. Therefore, by parts (5) and (6) of 4.1, we have an A-module epimorphism $I_e(P') \to I_e(eS) \to I_e(eS)/J_{eS} \cong S$. Consequently, $I_e(P') \cong P$ proving that $P' \cong eP$. $\square$

To make use of the existing results about functors between module categories it may be useful to identify the inflation functor $\mathrm{Inf}^G_{G/N}$ given in Section 2 with the functor

$${}_{\mu_{\mathbb{K}}(G)}\mu_{\mathbb{K}}(G/N) \otimes_{\mu_{\mathbb{K}}(G/N)} -.$$

Let N be a normal subgroup of G. Then the Mackey algebra $\mu_{\mathbb{K}}(G/N)$ can be regarded as a left $\mu_{\mathbb{K}}(G)$-module if we identify $\mu_{\mathbb{K}}(G/N)$ with the inflated $\mu_{\mathbb{K}}(G)$-module $\operatorname{Inf}_{G/N}^{G}\mu_{\mathbb{K}}(G/N)$. Therefore, the left $\mu_{\mathbb{K}}(G)$-module action on $\mu_{\mathbb{K}}(G/N)$ is given by

$$t_{g_J}^{H} c_J^g r_J^K .x = t_{gN_{J/N}}^{H/N} c_{J/N}^{gN} r_{J/N}^{K/N} x$$

if $N \leqslant J$ and 0 otherwise. In a similar way $\mu_{\mathbb{K}}(G/N)$ has a right $\mu_{\mathbb{K}}(G)$-module structure.

Lemma 4.3. *Let N be a normal subgroup of G. Then the functors*

$$\operatorname{Inf}_{G/N}^{G} \quad \textit{and} \quad {}_{\mu_{\mathbb{K}}(G)}\mu_{\mathbb{K}}(G/N) \otimes_{\mu_{\mathbb{K}}(G/N)} -$$

are naturally isomorphic.

Proof. Although this is evident from the definition of inflated Mackey functors given in Section 2, one may also deduce the result from the characterization of additive right exact covariant functors commuting with direct sums between module categories given in [10, Theorem 1]. This results says that if F is such a functor from modules of a ring A to modules of a ring B then F is naturally isomorphic to the functor ${}_BF(A) \otimes_A -$. □

It is now clear that the inflation functor $\operatorname{Inf}_{G/N}^{G}$ can be identified with the restriction functor along the unital morphism of algebras $\mu_{\mathbb{K}}(G) \to \mu_{\mathbb{K}}(G/N)$ given by

$$t_{g_J}^{H} c_J^g r_J^K \mapsto t_{gN_{J/N}}^{H/N} c_{J/N}^{gN} r_{J/N}^{K/N}$$

if $N \leqslant J$ and 0 otherwise. The left and right $\mu_{\mathbb{K}}(G)$-module structures on $\mu_{\mathbb{K}}(G/N)$ come from this algebra homomorphism. See also the algebra homomorphism ψ_N introduced after 4.5.

We can also make similar identifications for the functors L^- and L^+. Let A and B be algebras, and let ${}_AU_B$ be an (A, B)-bimodule. It is well known that the pair

$$\big({}_AU \otimes_B -, \operatorname{Hom}_A({}_AU_B, -)\big)$$

is an adjoint pair, and in the case U_B is finitely generated and projective the pair

$$\big({}_B\operatorname{Hom}_B({}_AU_B, {}_BB_B) \otimes_A -, {}_AU \otimes_B -\big)$$

is an adjoint pair.

Remark 4.4. Let N be a normal subgroup of G. We have the following natural isomorphisms of the functors:

$$L_{G/N}^{+} \cong {}_{\mu_{\mathbb{K}}(G/N)}\mu_{\mathbb{K}}(G/N) \otimes_{\mu_{\mathbb{K}}(G)} - \quad \text{and} \quad L_{G/N}^{-} \cong \operatorname{Hom}_{\mu_{\mathbb{K}}(G)}\big({}_{\mu_{\mathbb{K}}(G)}\mu_{\mathbb{K}}(G/N), -\big).$$

Proof. Letting $A = \mu_{\mathbb{K}}(G)$ and $U = B = \mu_{\mathbb{K}}(G/N)$ and noting that $\operatorname{Hom}_B({}_AB_B, {}_BB_B)$ and ${}_BB_A$ are isomorphic as (B, A)-bimodules, the result follows by the explanation given above. □

We want to relate the functor $\operatorname{Inf}_{G/N}^{G}$ with the functors R_{e_N}, C_{e_N}, and I_{e_N} where A is the algebra $\mu_{\mathbb{K}}(G)$.

Given a normal subgroup N of G we define the following function

$$\varphi_N : \mu_{\mathbb{K}}(G/N) \to e_N \mu_{\mathbb{K}}(G) e_N$$

whose image at a nonzero element x of $\mu_{\mathbb{K}}(G/N)$ of the form

$$x = t_{gN\,J/N}^{H/N} c_{J/N}^{gN} r_{J/N}^{K/N}$$

is given by

$$\varphi_N(x) = t_{g\,J}^{H} c_J^{g} r_J^{K}.$$

Proposition 4.5. *The $\mathbb{K}$-linear extension of the map φ_N is a unital $\mathbb{K}$-algebra monomorphism, and we have the direct sum decomposition*

$$e_N \mu_{\mathbb{K}}(G) e_N = \operatorname{Im} \varphi_N \oplus I_N$$

where I_N is a two sided ideal of $e_N \mu_{\mathbb{K}}(G) e_N$ having the elements of the form $t_{g\,J}^{H} c_J^{g} r_J^{K}$ with $H \geqslant N \leqslant K$ and J not contain N as $\mathbb{K}$-basis.

Proof. This follows easily by the basis Theorem 2.1 and by the axioms in the definition of Mackey algebras. $\square$

By the previous result, we have a $\mathbb{K}$-algebra epimorphism

$$\psi_N : e_N \mu_{\mathbb{K}}(G) e_N \to \mu_{\mathbb{K}}(G/N)$$

whose kernel is equal to the ideal I_N. Thus its image at a nonzero element of $e_N \mu_{\mathbb{K}}(G) e_N$ of the form

$$y = e_N t_{g\,J}^{H} c_J^{g} r_J^{K} e_N$$

is given by

$$\psi_N(y) = t_{gN\,J/N}^{H/N} c_{J/N}^{gN} r_{J/N}^{K/N}$$

if $N \leqslant J$ and 0 otherwise. We note that y is nonzero if and only if $N \leqslant H \cap K$, and in this case $y = t_{g\,J}^{H} c_J^{g} r_J^{K}$. Furthermore, the left $\mu_{\mathbb{K}}(G)$-module action on $\mu_{\mathbb{K}}(G/N)$ described before 4.3 satisfies $a.x = \psi_N(e_N a e_N)x$.

The $\mathbb{K}$-algebra epimorphism ψ_N induces the following well-known functors, some of whose properties are recalled in the next result, between the module categories of the algebras $e_N \mu_{\mathbb{K}}(G) e_N$ and $\mu_{\mathbb{K}}(G/N)$:

$$\begin{aligned} &\operatorname{Res}_{e_N} = {}_{e_N A e_N} B \otimes_B -, \qquad \operatorname{Ind}_{e_N} = {}_B B \otimes_{e_N A e_N} -, \quad \text{and} \\ &\operatorname{Coind}_{e_N} = \operatorname{Hom}_{e_N A e_N}({}_{e_N A e_N} B, -), \end{aligned}$$

where $A = \mu_{\mathbb{K}}(G)$ and $B = \mu_{\mathbb{K}}(G/N)$.

Remark 4.6.

(1) $(\mathrm{Ind}_{e_N}, \mathrm{Res}_{e_N})$ and $(\mathrm{Res}_{e_N}, \mathrm{Coind}_{e_N})$ are adjoint pairs.
(2) Both of the functors $\mathrm{Ind}_{e_N} \mathrm{Res}_{e_N}$ and $\mathrm{Coind}_{e_N} \mathrm{Res}_{e_N}$ are naturally isomorphic to the identity functor.

The above result is well known, and its second part follows from the surjectivity of ψ_N.

Lemma 4.7. *The functor* Ind_{e_N} *is naturally isomorphic to the functor*

$$\mathrm{Mod}(e_N A e_N) \to \mathrm{Mod}(B), \quad M \to M/I_N M$$

where N is a normal subgroup of G, $A = \mu_{\mathbb{K}}(G)$, and $B = \mu_{\mathbb{K}}(G/N)$.

Proof. As in the proof of 4.3 the latter functor is naturally isomorphic to the functor

$${}_B(e_N A e_N / I_N e_N A e_N) \otimes_{e_N A e_N} -.$$

Then from 4.5 we see that

$$e_N A e_N / I_N e_N A e_N \cong \mathrm{Im}\,\varphi_N \cong B$$

as $(B, e_N A e_N)$-bimodules. □

Proposition 4.8. *Let N be a normal subgroup of G. Then we have the following natural isomorphisms of functors*:

$$\mathrm{Res}_{e_N} \cong R_{e_N} \mathrm{Inf}^G_{G/N}, \qquad \mathrm{Ind}_{e_N} \cong L^+_{G/N} I_{e_N}, \quad \text{and} \quad \mathrm{Coind}_{e_N} \cong L^-_{G/N} C_{e_N}.$$

Proof. $\mathrm{Inf}^G_{G/N}$ and ${}_A B \otimes_B -$ are naturally isomorphic by 4.3, where $A = \mu_{\mathbb{K}}(G)$ and $B = \mu_{\mathbb{K}}(G/N)$. Since A-module action on B is given by $a.x = \psi_N(e_N a e_N)x$, it follows that Res_{e_N} and $R_{e_N} \mathrm{Inf}^G_{G/N}$ are naturally isomorphic. By the uniqueness of adjoints, the other isomorphisms of functors follow from 4.6, 4.1 and 2.4. □

Theorem 4.9. *Let N be a normal subgroup of G. For any principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module $P^{G/N}_{H/N,V}$, one has*

$$P^{G/N}_{H/N,V} \cong e_N P^G_{H,V} / I_N P^G_{H,V}.$$

Proof. Noting that $e_N S^G_{H,V} \neq 0$, we have by 4.2

$$I_{e_N} R_{e_N}\left(P^G_{H,V}\right) \cong P^G_{H,V}.$$

Then using 4.7 and 4.8 we obtain

$$\begin{aligned} e_N P^G_{H,V} / I_N P^G_{H,V} &\cong \operatorname{Ind}_{e_N} R_{e_N}\big(P^G_{H,V}\big) \\ &\cong L^+_{G/N} I_{e_N} R_{e_N}\big(P^G_{H,V}\big) \\ &\cong L^+_{G/N} P^G_{H,V} \\ &\cong P^{G/N}_{H/N,V}, \end{aligned}$$

where we use 3.15 for the latest isomorphism. □

5. Inflations of projective covers

This section concerns inflations of principal indecomposable Mackey functors. Given a $\mu_{\mathbb{K}}(G)$-module M and a normal subgroup N of G we first want to study the relationship between the $\mu_{\mathbb{K}}(G)$-modules

$$\operatorname{Inf}^G_{G/N} L^+_{G/N} M, \quad \operatorname{Inf}^G_{G/N} L^-_{G/N} M, \quad \text{and} \quad M.$$

In fact, we will observe that the first two are isomorphic to a quotient module and a submodule of M, respectively. These results will allow us to relate the $\mu_{\mathbb{K}}(G)$-modules of the form $\operatorname{Inf}^G_{G/N} P^{G/N}_{H/N,V}$ and $P^G_{H,V}$.

We begin with recalling from [8] that if χ is a family of subgroups of G closed under taking subgroups and taking G-conjugates then any Mackey functor M for G has the following two subfunctors defined by:

$$\big(\operatorname{Im} t^M_\chi\big)(K) = \sum_{X \in \chi : X \leqslant K} t^K_X\big(M(X)\big),$$

$$\big(\operatorname{Ker} r^M_\chi\big)(K) = \bigcap_{X \in \chi : X \leqslant K} \operatorname{Ker}\big(r^K_X : M(K) \to M(X)\big).$$

For any normal subgroup N of G we denote by $\mathcal{Y}_N$ the set of all subgroups of G not containing N. That is,

$$\mathcal{Y}_N = \{J \leqslant G:\ N \text{ is not in } J\}.$$

It is obvious that $\mathcal{Y}_N$ is closed under taking subgroups and taking G-conjugates.

Lemma 5.1. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then we have the following $\mu_{\mathbb{K}}(G)$-module isomorphisms*:

$$\operatorname{Inf}^G_{G/N} L^+_{G/N} M \cong M / \operatorname{Im} t^M_{\mathcal{Y}_N} \quad \textit{and} \quad \operatorname{Inf}^G_{G/N} L^-_{G/N} M \cong \operatorname{Ker} r^M_{\mathcal{Y}_N}.$$

Proof. For a subgroup X of G, if X does not contain N then $X \in \mathcal{Y}_N$ so that by their definitions $\operatorname{Im} t^M_{\mathcal{Y}_N}(X) = M(X)$ and $\operatorname{Ker} r^M_{\mathcal{Y}_N}(X) = 0$. Suppose now that X is a subgroup of G containing N. Then the set $\{J \in \mathcal{Y}_N:\ J \leqslant X\}$ consists of all subgroups J of X not containing N. Therefore, the result follows by the definitions of L^+, L^-, and Inf given in Section 2. □

Lemma 5.2. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. For a $\mu_{\mathbb{K}}(G)$-submodule T of M we have*:

(1) *If the kernel of M/T contains N, then* $\operatorname{Im} t^M_{\mathcal{Y}_N} \leqslant T$.
(2) *If the kernel of T contains N, then* $T \leqslant \operatorname{Ker} r^M_{\mathcal{Y}_N}$.

Proof. (1) By its definition it is clear that $\operatorname{Im}^M_{\mathcal{Y}_N}$ is the minimal subfunctor of M such that $\operatorname{Im}^M_{\mathcal{Y}_N}(X) = M(X)$ for all $X \in \mathcal{Y}_N$. Therefore, it is enough to show that $T(X) = M(X)$ for any $X \in \mathcal{Y}_N$. Suppose that $T(X) \neq M(X)$ for some subgroup X of G. Then $(M/T)(X) \neq 0$ and so $X \geqslant \mathcal{K}(M/T) \geqslant N$, implying that $X \notin \mathcal{Y}_N$.

(2) By the definition of $\operatorname{Ker} r^M$ subfunctor, $T \leqslant \operatorname{Ker}^M_{\mathcal{Y}_N}$ if and only if $r^K_X(T(K)) = 0$ for all $K \leqslant G$ and $X \in \mathcal{Y}_N$ with $X \leqslant K$. Indeed, if $T(X) \neq 0$ for some subgroup X of G then $X \geqslant \mathcal{K}(T) \geqslant N$ implying that $X \notin \mathcal{Y}_N$. Consequently, for any subgroup K of G and an element X of $\mathcal{Y}_N$ with $X \leqslant K$, we have $r^K_X(T(K)) \subseteq T(X) = 0$. □

The previous two results suggest the following.

Proposition 5.3. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then*

(1) *M has a unique smallest $\mu_{\mathbb{K}}(G)$-submodule $J_N(M)$ such that $M/J_N(M)$ has kernel containing N. Moreover, $J_N(M)$ is equal to* $\operatorname{Im} t^M_{\mathcal{Y}_N}$.
(2) *M has a unique largest $\mu_{\mathbb{K}}(G)$-submodule $S_N(M)$ such that $S_N(M)$ has kernel containing N. Moreover, $S_N(M)$ is equal to* $\operatorname{Ker} r^M_{\mathcal{Y}_N}$.

Proof. Let M_1 and M_2 be $\mu_{\mathbb{K}}(G)$-submodules of M.

(1) Since $M/M_1 \cap M_2$ is isomorphic to a submodule of $M/M_1 \oplus M/M_2$, it follows by 3.6 that if M/M_i has kernel containing N for $i = 1, 2$ then the kernel of $M/M_1 \cap M_2$ contains N. Therefore, $J_N(M)$ is the intersection of all $\mu_{\mathbb{K}}(G)$-submodules M' of M such that M/M' has kernel containing N. Finally it follows by 5.1 that $J_N(M) \leqslant \operatorname{Im}^M_{\mathcal{Y}_N}$. The reverse inclusion follows from 5.2.

(2) We have an exact sequence

$$0 \to M_1 \to M_1 + M_2 \to M_2/M_1 \cap M_2 \to 0$$

of $\mu_{\mathbb{K}}(G)$-modules, from which we can conclude by 3.6 that $\mathcal{K}(M_1) \cap \mathcal{K}(M_2) \leqslant \mathcal{K}(M_1 + M_2)$. Therefore, $S_N(M)$ is the sum of all $\mu_{\mathbb{K}}(G)$-submodules of M with kernel containing N. Finally, 5.1 and 5.2 imply that $S_N(M) = \operatorname{Ker} r^M_{\mathcal{Y}_N}$. □

Theorem 5.4. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then*

(1) $\operatorname{Inf}^G_{G/N} L^+_{G/N} M$ *is isomorphic to the largest quotient of M with kernel containing N.*
(2) $\operatorname{Inf}^G_{G/N} L^-_{G/N} M$ *is isomorphic to the largest submodule of M with kernel containing N.*

Proof. Immediate from 5.1 and 5.3. □

We can also give the following identifications of the subfunctors $\operatorname{Im} t^M_{\mathcal{Y}_N}$ and $\operatorname{Ker}^M_{\mathcal{Y}_N}$ whose proof is easy and hence omitted.

Remark 5.5. Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then

$$\operatorname{Im}^{M}_{\mathcal{Y}_N} = J_N(M) = \mu_{\mathbb{K}}(G)(1-e_N)M,$$

$$\operatorname{Ker} r^{M}_{\mathcal{Y}_N} = S_N(M) = \{m \in M\colon\ e_N a m = a m\ \forall a \in \mu_{\mathbb{K}}(G)\}.$$

Theorem 5.6. *Let N be a normal subgroup of G and M be a $\mu_{\mathbb{K}}(G)$-module. Then, the largest quotient of $P_G(M)$ with kernel containing N is isomorphic to*

$$\operatorname{Inf}^{G}_{G/N} P_{G/N}\big(L^{+}_{G/N} M\big).$$

Proof. By 5.4 the largest quotient of $P_G(M)$ with kernel containing N is isomorphic to

$$\operatorname{Inf}^{G}_{G/N} L^{+}_{G/N} P_G(M) \cong \operatorname{Inf}^{G}_{G/N} P_{G/N}\big(L^{+}_{G/N} M\big)$$

where we use 3.14 for the isomorphism. □

Corollary 5.7. *Let N be a normal subgroup of G and $P^{G/N}_{H/N,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G/N)$-module. Then, $\operatorname{Inf}^{G}_{G/N} P^{G/N}_{H/N,V}$ is isomorphic to the largest quotient of $P^{G}_{H,V}$ with kernel containing N.*

Proof. Letting $M = P^{G}_{H,V}$, it follows by 3.15 that $L^{+}_{G/N} M \cong P^{G/N}_{H/N,V}$. Then the result follows from 5.4. □

6. Imprimitive Mackey functors

A $\mu_{\mathbb{K}}(G)$-module M is called imprimitive if there is a subgroup H of G with $H \neq G$ and a $\mu_{\mathbb{K}}(H)$-module T such that $M \cong \uparrow^{G}_{H} T$. If M is not imprimitive then it is called primitive. Our aim in this section is to study imprimitive Mackey functors.

Lemma 6.1. *Let K be a subgroup of G and T be a $\mu_{\mathbb{K}}(K)$-module. Then*

(1) *If $\uparrow^{G}_{K} T$ is simple then T is simple.*
(2) *If $\uparrow^{G}_{K} T$ is indecomposable then T is indecomposable.*
(3) *If $\uparrow^{G}_{K} T$ is projective then T is projective.*
(4) *If $\uparrow^{G}_{K} T$ is simple (respectively, indecomposable) then $\uparrow^{L}_{K} T$ is simple (respectively, indecomposable) for any L with $K \leqslant L \leqslant G$.*
(5) *If $M = \uparrow^{G}_{K} T$ is indecomposable then M and T have a vertex in common.*
(6) *The minimal subgroups of $\uparrow^{G}_{K} T$ are precisely the G-conjugates of the minimal subgroups of T.*

Proof. We first note that if $\uparrow^{G}_{K} T = 0$ then $T = 0$, because by the Mackey decomposition formula T is a direct summand of $\downarrow^{G}_{K}\uparrow^{G}_{K} T$.

(1) Let T' be a $\mu_{\mathbb{K}}(K)$-submodule of T. By the exactness of the functor $\uparrow^{G}_{K}$ (see 2.3), we get an exact sequence

$$0 \to \uparrow^{G}_{K} T' \to \uparrow^{G}_{K} T \to \uparrow^{G}_{K} T/T' \to 0$$

of $\mu_{\mathbb{K}}(G)$-modules. Since $\uparrow_K^G T$ is simple, it follows that either $\uparrow_K^G T'$ or $\uparrow_K^G T/T'$ is zero, implying that $T'=0$ or $T=T'$. Hence T is simple.

(2) For the functor $\uparrow_K^G$ commutes with direct sums.

(3) As the functor $\downarrow_K^G$ sends projectives to projectives by 2.3, the result is clear from the Mackey decomposition formula implying that T is a direct summand of the projective $\mu_{\mathbb{K}}(K)$-module $\downarrow_K^G\uparrow_K^G T$.

(4) This is obvious because we may write $\uparrow_K^G T \cong \uparrow_L^G\uparrow_K^L T$ and use parts (2) and (1).

(5) Let P and Q be vertices of M and T, respectively. Then there are $\mu_{\mathbb{K}}(P)$ and $\mu_{\mathbb{K}}(Q)$-modules M' and T' such that M and T are respective direct summands of $\uparrow_P^G M'$ and $\uparrow_Q^K T'$. From $M=\uparrow_K^G T$ we see that M is a direct summand of $\uparrow_Q^G T'$. This shows that $P \leqslant_G Q$. On the other hand, from the Mackey decomposition formula T is a direct summand of $\downarrow_K^G M$ which is a direct summand of

$$\downarrow_K^G\uparrow_P^G M' \cong \bigoplus_{KgP\subseteq G} \uparrow_{K\cap {}^gP}^K \downarrow_{K\cap {}^gP}^{{}^gP} |_P^g M'.$$

This shows that $Q \leqslant_K K\cap {}^gP$ for some $g\in G$. Consequently $Q =_G P$.

(6) We use the following explicit formula for the induced Mackey functors from [7], see also [8],

$$\big(\uparrow_K^G T\big)(H) = \bigoplus_{HgK\subseteq G} T\big(K\cap H^g\big).$$

If Y is a minimal subgroup of $\uparrow_K^G T$, then $T(K\cap Y^g)\neq 0$ for some $g\in G$ implying the existence of a minimal subgroup X of T such that $X \leqslant K\cap Y^g \leqslant_G Y$. Moreover, for any minimal subgroup X' of T, from

$$0\neq T(X') \subseteq \bigoplus_{X'gK\subseteq G} T\big(K\cap X'^g\big) = \big(\uparrow_K^G T\big)(X'),$$

we see that there is a minimal subgroup Y' of $\uparrow_K^G T$ such that $Y' \leqslant X'$. Evidently, these imply the result. □

The last part of the previous result implies

Remark 6.2. Let K be a subgroup of G. Then $\mathcal{K}(\uparrow_K^G T) = (\mathcal{K}(T))_G$ for any $\mu_{\mathbb{K}}(K)$-module T.

We now study primitive simple Mackey functors. The next result is an immediate consequence of explicit construction of simple Mackey functors given in [8].

Remark 6.3. Let $S_{H,V}^G$ be a simple $\mu_{\mathbb{K}}(G)$-module. If $S_{H,V}^G$ is primitive then H is a normal subgroup of G.

Proof. It is clear from 2.5. □

Lemma 6.4. *If $S^G_{H,V}$ is a simple $\mu_{\mathbb{K}}(G)$-module satisfying $S^G_{H,V} \cong \uparrow^G_K S$ for a subgroup K of G and a $\mu_{\mathbb{K}}(K)$-module S, then $S \cong S^K_{{}^gH,W}$ for some $g \in G$ with ${}^gH \leqslant K$ and for some simple $\overline{N}_K({}^gH)$-module W.*

Proof. Follows from parts (1) and (6) of 6.1. $\square$

For future use we record the following from [11, Lemma 3.4 and Proposition 3.5].

Lemma 6.5. *Let $H \leqslant K \leqslant G$ be such that ${}^gH \leqslant K$ for every $g \in G$. Given a simple $\mu_{\mathbb{K}}(K)$-module $S^K_{H,W}$ we put*

$$S = \uparrow^G_K S^K_{H,W} \quad \textit{and} \quad V = \uparrow^{\overline{N}_G(H)}_{\overline{N}_K(H)} W.$$

Then $S(H) \cong V$, and S is a simple $\mu_{\mathbb{K}}(G)$-module if and only if V is a simple $\mathbb{K}\overline{N}_G(H)$-module. Moreover $S \cong S^G_{H,V}$ if V is simple.

Theorem 6.6. *Let $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module. Then, $S^G_{H,V}$ is imprimitive if and only if either H is a nonnormal subgroup of G, or H is a normal subgroup of G different from G and V is an imprimitive $\mathbb{K}G/H$-module.*

Proof. Suppose that $S^G_{H,V}$ is imprimitive. There is a $K \leqslant G$ with $K \neq G$ and a $\mu_{\mathbb{K}}(K)$-module T such that $S^G_{H,V} \cong \uparrow^G_K T$. Assume that H is a normal subgroup of G. From 6.1 we get $H \leqslant K$, and so H is different from G. Now 6.4 implies that $T \cong S^K_{H,W}$ for some simple $\mathbb{K}K/H$-module W. Since H is normal, we may apply 6.5 to deduce that $V \cong \uparrow^{G/H}_{K/H} W$. Therefore V is an imprimitive $\mathbb{K}G/H$-module.

Conversely, suppose that a simple $\mu_{\mathbb{K}}(G)$-module $S^G_{H,V}$ is given. If H is a nonnormal subgroup of G then 6.3 implies that $S^G_{H,V}$ is imprimitive. Thus we assume that H is a normal subgroup of G different from G and V is an imprimitive $\mathbb{K}G/H$-module. Then there is a subgroup K with $H \leqslant K \leqslant G$ and $K \neq G$ such that $V \cong \uparrow^{G/H}_{K/H} W$ for some necessarily simple $\mathbb{K}K/H$-module W. Now it follows by 6.5 that $S^G_{H,V} \cong \uparrow^G_K S^K_{H,W}$. Consequently, $S^G_{H,V}$ is imprimitive. $\square$

Corollary 6.7. *Let $\mathbb{K}$ be algebraically closed, G be a nilpotent group, and $S^G_{H,V}$ be a simple $\mu_{\mathbb{K}}(G)$-module. Then, $S^G_{H,V}$ is primitive if and only if H is a normal subgroup of G and V is one dimensional.*

Proof. Suppose that $S^G_{H,V}$ is primitive. Then H is a normal subgroup of G by 6.3. Since G/H is nilpotent and $\mathbb{K}$ is algebraically closed, the simple $\mathbb{K}G/H$-module V must be monomial, see [6, Theorem 3.7, p. 205]. Hence $V \cong \uparrow^{G/H}_{K/H} W$ for some subgroup K with $H \leqslant K \leqslant G$ and some one dimensional $\mathbb{K}K/H$-module W. Now 6.5 implies that $S^G_{H,V} \cong \uparrow^G_K S^K_{H,W}$. Hence, if V is not one dimensional then $K \neq G$, implying that $S^G_{H,V}$ is imprimitive.

The converse statement follows from 6.6 because H is a normal subgroup of G and, being one dimensional, V is primitive. $\square$

The above result follows also from [11, Corollary 3.9].

Corollary 6.8. *Any faithful simple $\mu_{\mathbb{K}}(G)$-module M whose minimal subgroup is different from* 1 *is imprimitive.*

Proof. Let $M = S^G_{H,V}$ with $H \neq 1$. By the condition $\mathcal{K}(M) = H_G = 1$, the subgroup H is not normal in G, and so the result follows by 6.6. □

We next investigate primitive principal indecomposable $\mu_{\mathbb{K}}(G)$-modules. However, except for nilpotent groups we have no criteria for a $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ to be primitive. Obviously, imprimitivity of an indecomposable $\mu_{\mathbb{K}}(G)$-module is related to indecomposability of an induced $\mu_{\mathbb{K}}(G)$-module from a proper subgroup of G. A classical result about indecomposability of an induced module in the context of group algebras is Green's theorem. It is shown in [11, Theorem 6.3] that an analogue of Green's theorem works in the context of Mackey functors. Namely, letting $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and N be a normal subgroup of G, for an indecomposable $\mu_{\mathbb{K}}(N)$-module T the induced module $\uparrow^G_N T$ is indecomposable if and only if L/N is a p-group where L is the inertia group of T in G.

Let M be a simple $\mu_{\mathbb{K}}(G)$-module and $P_G(M)$ be its projective cover. One may wonder about the connections between primitivity of $\mu_{\mathbb{K}}(G)$-modules M and $P_G(M)$. For example, suppose that G is a p-group and $\mathbb{K}$ is an algebraically closed field of characteristic $p > 0$. Then, $S^G_{H,\mathbb{K}}$ is primitive if and only if H is a normal subgroup of G (by 6.7), while $P^G_{H,\mathbb{K}}$ is primitive if and only if $H = G$ (by Green's theorem). Therefore, in general primitivity of one of M or $P_G(M)$ does not imply primitivity of the other. However we have the following trivial result.

Remark 6.9. Let K be a subgroup of G. For a simple $\mu_{\mathbb{K}}(K)$-module T and a simple $\mu_{\mathbb{K}}(G)$-module M, if $P_G(M) \cong \uparrow^G_K P_K(T)$ then $\uparrow^G_K T/J(\uparrow^G_K T) \cong M$.

Proof. Since the functor $\uparrow^G_K$ is exact and sends projectives to projectives (see 2.3), $P_G(\uparrow^G_K T)$ is a direct summand of $\uparrow^G_K P_K(T)$ which is indecomposable. Therefore $P_G(\uparrow^G_K T) \cong P_G(M)$, proving the result. □

We next provide a result about principal indecomposable $\mu_{\mathbb{K}}(G)$-modules induced from p-subgroups of G where p is the characteristic of $\mathbb{K}$.

Proposition 6.10. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. If a principal indecomposable $\mu_{\mathbb{K}}(G)$-module M is induced from a p-subgroup of G then $N_G(H)$ is a p-group where H is the vertex of M.*

Proof. Let $M = P^G_{H,V}$ be induced from a p-subgroup K of G, and write $M = \uparrow^G_K T$ for some $\mu_{\mathbb{K}}(K)$-module T. Using 6.1 we see that T is isomorphic to a $\mu_{\mathbb{K}}(K)$-module of the form $P^K_{{}^gH,\mathbb{K}}$ for some $g \in G$ with ${}^gH \leqslant K$. We first show that $f = t^{{}^gH}_{{}^gH}$ is a primitive idempotent of $\mu_{\mathbb{K}}(G)$.

Now, it follows by [12, Proposition 3.1] that f is a primitive idempotent of $\mu_{\mathbb{K}}(K)$ and $T \cong \mu_{\mathbb{K}}(K)f$. Then

$$M = \uparrow^G_K T \cong \mu_{\mathbb{K}}(G) \otimes_{\mu_{\mathbb{K}}(K)} \mu_{\mathbb{K}}(K)f \cong \mu_{\mathbb{K}}(G)f,$$

implying from the indecomposability of M that f is a primitive idempotent of $\mu_{\mathbb{K}}(G)$.

For any subgroup L of G, it is shown in [11, Proposition 5.17] that if t_L^L is a primitive idempotent of $\mu_{\mathbb{K}}(G)$ then $N_G(L)$ is a p-group. This implies that $N_G(H)$ is a p-group. $\square$

For any group X and prime p, we denote by X_p and $X_{p'}$ the respective largest normal p and normal p'-subgroups of G.

Let G be nilpotent. Since the subgroups X of G satisfy $X \cong X_p \times X_{p'}$, it follows by the basis Theorem 2.1 that the Mackey algebra $\mu_{\mathbb{K}}(G)$ admits a tensor product decomposition

$$\mu_{\mathbb{K}}(G) \cong \mu_{\mathbb{K}}(G_p) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_{p'}), \qquad t_{{}^gJ}^H c_J^g r_J^K \leftrightarrow t_{{}^{g_p}J_p}^{H_p} c_{J_p}^{g_p} r_{J_p}^{K_p} \otimes t_{{}^{g_{p'}}J_{p'}}^{H_{p'}} c_{J_{p'}}^{g_{p'}} r_{J_{p'}}^{K_{p'}},$$

see Section 7 of [12]. Therefore, if $\mathbb{K}$ is algebraically closed, then any $\mu_{\mathbb{K}}(G)$-module M can be identified with a $\mu_{\mathbb{K}}(G_p) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_{p'})$-module of the form $M' \otimes_{\mathbb{K}} M''$ for some $\mu_{\mathbb{K}}(G_p)$ and $\mu_{\mathbb{K}}(G_{p'})$-modules M' and M''. Moreover, if X is a subgroup of G, if M' is a $\mu_{\mathbb{K}}(X_p)$-module, and M'' is a $\mu_{\mathbb{K}}(X_{p'})$-module, then considering $M' \otimes_{\mathbb{K}} M''$ as a $\mu_{\mathbb{K}}(X)$-module we have a $\mu_{\mathbb{K}}(G)$-module isomorphism

$$\uparrow_{X_p}^{G_p} M' \otimes_{\mathbb{K}} \uparrow_{X_{p'}}^{G_{p'}} M'' \cong \uparrow_X^G (M' \otimes_{\mathbb{K}} M'') \quad \text{given by}$$

$$\left(t_{{}^{g_p}J_p}^{H_p} c_{J_p}^{g_p} r_{J_p}^{K_p} \otimes m'\right) \otimes \left(t_{{}^{g_{p'}}J_{p'}}^{H_{p'}} c_{J_{p'}}^{g_{p'}} r_{J_{p'}}^{K_{p'}} \otimes m''\right) \leftrightarrow t_{{}^gJ}^H c_J^g r_J^K \otimes (m' \otimes m'').$$

Theorem 6.11. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$. Suppose that G is nilpotent. Then, a principal indecomposable $\mu_{\mathbb{K}}(G)$-module $P_{H,V}^G$ is primitive if and only if H is a normal subgroup of G such that G/H is a p'-group and* $\dim_{\mathbb{K}} V = 1$.

Proof. Under the tensor product decomposition of $\mu_{\mathbb{K}}(G)$, the simple $\mu_{\mathbb{K}}(G)$-module $S_{H,V}^G$ corresponds to the simple $\mu_{\mathbb{K}}(G_p) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_{p'})$-module

$$S_{H_p,\mathbb{K}}^{G_p} \otimes_{\mathbb{K}} S_{H_{p'},V}^{G_{p'}},$$

see the proof of [12, Corollary 7.4]. Therefore, the projective cover $P_{H,V}^G$ of $S_{H,V}^G$ corresponds to the principal indecomposable $\mu_{\mathbb{K}}(G_p) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_{p'})$-module

$$P_{H_p,\mathbb{K}}^{G_p} \otimes_{\mathbb{K}} S_{H_{p'},V}^{G_{p'}},$$

where we also use the semisimplicity of the algebra $\mu_{\mathbb{K}}(G_{p'})$ from [8]. Therefore, it follows by the explanation given above that $P_{H,V}^G$ is primitive if and only if both of $P_{H_p,\mathbb{K}}^{G_p}$ and $S_{H_{p'},V}^{G_{p'}}$ are primitive. Green's indecomposability theorem for Mackey algebras implies that $P_{H_p,\mathbb{K}}^{G_p}$ is primitive if and only if $H_p = G_p$. Thus, the result follows from 6.7. $\square$

When $\mathbb{K}$ is an algebraically closed field of characteristic $p > 0$ and G is p-solvable, by a theorem of Fong [3, Theorem (2D)], any principal indecomposable $\mathbb{K}G$-module V is isomorphic to an induced module $\uparrow_K^G W$ of a $\mathbb{K}K$-module W where K is a Hall p'-subgroup of G. We next try to obtain a similar result for Mackey functors.

Let K be a subgroup of G and T be a simple $\mu_{\mathbb{K}}(K)$-module with minimal subgroup H where $\mathbb{K}$ is any field. In the next result we relate the $\mathbb{K}\overline{N}_G(H)$-modules $(\uparrow_K^G T)(H)$ and $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} T(H)$. A similar result appears also in part (iv) of [11, Lemma 3.4] where it is assumed also that ${}^gH \leqslant K$ for every $g \in G$ which is necessary only for the other parts of that result (see also 6.5). For convenience, we give its similar justification.

Lemma 6.12. *Let $H \leqslant K \leqslant G$. For any simple $\mu_{\mathbb{K}}(K)$-module $T = S_{H,W}^K$ we have the following $\mathbb{K}\overline{N}_G(H)$-module isomorphism*:

$$\big(\uparrow_K^G T\big)(H) \cong \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Proof. Because of $T = S_{H,W}^K$, for a $g \in G$ we see that $T(K \cap H^g) \neq 0$ if and only if $K \cap H^g$ is equal to H^g and H^g is a K-conjugate of H, which is equivalent to $g \in N_G(H)K$. Moreover, $T(K \cap H^g) = c_H^{g^{-1}}(W)$ if $g \in N_G(H)K$ where c is the conjugation map for T. Then using the explicit formula for the induced Mackey functors given in [7,8] we obtain

$$\big(\uparrow_K^G T\big)(H) = \bigoplus_{gK \subseteq N_G(H)K} c_H^{g^{-1}}(W).$$

If $\tilde{c}$ denotes the conjugation map for $\uparrow_K^G T$ then $k \in N_G(H)$ acts on an element

$$x = \bigoplus_{gK \subseteq N_G(H)K} x_g \in \big(\uparrow_K^G T\big)(H) \quad \text{as}$$

$$k.x = \tilde{c}_H^k(x) = \bigoplus_{gK \subseteq N_G(H)K} \big(\tilde{c}_H^k(x)\big)_g \quad \text{where } \big(\tilde{c}_H^k(x)\big)_g = x_{k^{-1}g},$$

see [7,8]. Therefore $\overline{N}_G(H)$ permutes the summands $c_H^{g^{-1}}(W)$ of $(\uparrow_K^G T)(H)$ transitively, and the stabilizer of the summand $c_H^1(W) = W$ is $N_G(H) \cap K = N_K(H)$. This proves the result. □

For simple Mackey functors, part (6) of 6.1 has the following stronger form.

Lemma 6.13. *Let K be a subgroup of G and $T = S_{H,W}^K$ be a simple $\mu_{\mathbb{K}}(K)$-module. Then for any $\mu_{\mathbb{K}}(G)$-submodule M of $\uparrow_K^G T$ the minimal subgroups of M are precisely the G-conjugates of H.*

Proof. Let X be a minimal subgroup of M. Then $M'(X) \neq 0$ where $M' = \uparrow_K^G T$. Therefore there is a minimal subgroup of M' contained in X, implying from part (6) of 6.1 that $H \leqslant_G X$. Using the adjointness of the pair $(\downarrow_K^G, \uparrow_K^G)$ we get

$$0 \neq \mathrm{Hom}_{\mu_{\mathbb{K}}(G)}\big(M, \uparrow_K^G T\big) \cong \mathrm{Hom}_{\mu_{\mathbb{K}}(K)}\big(\downarrow_K^G M, T\big).$$

As T is simple, the above isomorphism of $\mathbb{K}$-spaces implies the existence of a $\mu_{\mathbb{K}}(K)$-module epimorphism $\downarrow_K^G M \to T$. Therefore it follows from $T(H) \neq 0$ that $M(H) \neq 0$. This shows that $X =_G H$. □

Over characteristic 0 fields, in the next result we observe that dropping some conditions from the hypothesis of 6.5 does not alter the conclusion.

Proposition 6.14. *Let $\mathbb{K}$ be a field of characteristic* 0 *and K be a subgroup of G. For any simple $\mu_{\mathbb{K}}(K)$-module $S^K_{H,W}$, the induced module $\uparrow^G_K S^K_{H,W}$ is simple if and only if the $\mathbb{K}\overline{N}_G(H)$-module*

$$V = \uparrow^{\overline{N}_G(H)}_{\overline{N}_K(H)} W$$

is simple. And if this is the case then

$$\uparrow^G_K S^K_{H,W} \cong S^G_{H,V}.$$

Proof. Since the Mackey algebra $\mu_{\mathbb{K}}(G)$ is semisimple in this case [8], the $\mu_{\mathbb{K}}(G)$-module $M = \uparrow^G_K S^K_{H,W}$ is semisimple. By 6.13 the minimal subgroups of the simple summands of M are all (G-conjugate to) H, and so the result follows from 6.12. □

We next obtain a result about indecomposability of induced Mackey functors from p'-subgroups where p is the characteristic of the coefficient field.

Theorem 6.15. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and K be a p'-subgroup of G. For any simple $\mu_{\mathbb{K}}(K)$-module $S^K_{H,W}$, the induced module $\uparrow^G_K S^K_{H,W}$ is indecomposable if and only if the $\mathbb{K}\overline{N}_G(H)$-module*

$$U = \uparrow^{\overline{N}_G(H)}_{\overline{N}_K(H)} W$$

is indecomposable. Moreover, if this is the case then U is a principal indecomposable $\mathbb{K}\overline{N}_G(H)$-module and

$$\uparrow^G_K S^K_{H,W} \cong P^G_{H,V}$$

where $V = U/J(U)$.

Proof. $T = S^K_{H,W}$ is a projective $\mu_{\mathbb{K}}(K)$-module because $\mu_{\mathbb{K}}(K)$ is a semisimple algebra by [8]. Since the functor $\uparrow^G_K$ sends projectives to projectives, $\uparrow^G_K T$ is a projective $\mu_{\mathbb{K}}(G)$-module. Moreover, as H is a vertex of T, the indecomposable summands of $\uparrow^G_K T$ are all projective $\mu_{\mathbb{K}}(G)$-modules which are H-projective. Therefore if M is an indecomposable summand of $\uparrow^G_K T$ then M is of the form $P^G_{X,W'}$ for some subgroup X with $X \leqslant H$ and some simple $\mathbb{K}\overline{N}_G(X)$-module W'. On the other hand, it follows from 6.13 that H is a minimal subgroup of $P^G_{X,W'}$. This gives $X = H$ because $P^G_{X,W'}(X) \neq 0$. Consequently we may write

$$\uparrow^G_K T \cong \bigoplus_{V'} n_{V'} P^G_{H,V'}$$

where $n_{V'}$ are nonnegative integers and V' ranges over a complete set of isomorphism classes of simple $\mathbb{K}\overline{N}_G(H)$-modules. Then by 6.12

$$U \cong \left(\uparrow_K^G T\right)(H) \cong \bigoplus_{V'} n_{V'} P_{H,V'}^G(H).$$

Noting that each $P_{H,V'}^G(H)$ is nonzero, we see that if U is indecomposable then $\uparrow_K^G T$ is indecomposable.

Conversely, suppose that $\uparrow_K^G T$ is indecomposable. Then $\uparrow_K^G T = P_{H,V'}^G$ for some simple $\mathbb{K}\overline{N}_G(H)$-module V' so that $U \cong P_{H,V'}^G(H)$. Using the fact that H is a p'-group, one can prove that U is indecomposable. (Indeed, in 7.5 we will prove that it is a projective indecomposable $\mathbb{K}\overline{N}_G(H)$-module.)

Now suppose that U is indecomposable. Then we have

$$\uparrow_K^G T \cong P_{H,V'}^G$$

for some simple $\mathbb{K}\overline{N}_G(H)$-module V'. This shows

$$\uparrow_K^G T / J\left(\uparrow_K^G T\right) \cong S_{H,V'}^G$$

from which we may obtain by evaluation at the group H that

$$U/J \cong V'$$

where $J = (J(\uparrow_K^G T))(H)$. We next see that U is a principal indecomposable $\mathbb{K}\overline{N}_G(H)$-module because $U = \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W$, because $\mathbb{K}\overline{N}_K(H)$ is a semisimple algebra and the functor $\uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)}$ sends projectives to projectives. Hence, being a principal indecomposable module, U has a unique maximal submodule $J(U)$, implying from the isomorphism $U/J \cong V'$ and from the simplicity of V' that $J = J(U)$. Hence $V' \cong U/J(U)$. $\square$

We now provide a Mackey algebra version of Fong's theorem on induced modules.

Theorem 6.16. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$, let G be a p-solvable group, and let K be a Hall p'-subgroup of G. Then, for any principal indecomposable K-projective $\mu_{\mathbb{K}}(G)$-module M, there is a $\mu_{\mathbb{K}}(K)$-module T such that $M \cong \uparrow_K^G T$.*

Proof. Let $M = P_{H,V}^G$ with $H \leqslant K$ be given. Let U be the projective cover of the simple $\mathbb{K}\overline{N}_G(H)$-module V. Evidently, $\overline{N}_G(H)$ is p-solvable and $\overline{N}_K(H)$ is a Hall p'-subgroup of $\overline{N}_G(H)$. Then by Fong's theorem for group algebras there is a simple $\mathbb{K}\overline{N}_K(H)$-module W such that

$$U \cong \uparrow_{\overline{N}_K(H)}^{\overline{N}_G(H)} W.$$

Now, letting $T = S_{H,W}^K$ the result follows from 6.15. $\square$

Before stating a consequence of 6.16, we observe the projectivity of any indecomposable $\mu_{\mathbb{K}}(G)$-module having a p'-group as vertex where p is the characteristic of the field $\mathbb{K}$ (this follows easily from the semisimplicity of the Mackey algebras of p'-groups over $\mathbb{K}$, see also [12, Remark 4.4]).

Corollary 6.17. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and G be a p-solvable group whose order is divisible by p. Then any indecomposable $\mu_{\mathbb{K}}(G)$-module whose vertex is a p'-group is imprimitive.*

Proof. Follows from 6.16 because any p'-subgroup of G is contained in a Hall p'-subgroup K of G (so that an indecomposable $\mu_{\mathbb{K}}(G)$-module whose vertex is a p'-group is K-projective) and $K \neq G$. □

7. Evaluations

Let M be a Mackey functor for G over $\mathbb{K}$. For any subgroup H of G, the coordinate module $M(H)$ becomes a $\mathbb{K}\overline{N}_G(H)$-module via the action $gH.x = c_H^g(x)$. The aim of this section is to give some results about properties of the coordinate module $P_{H,V}^G(H)$ considered as a $\mathbb{K}\overline{N}_G(H)$-module where $P_{H,V}^G$ is a principal indecomposable Mackey functor for G. We mainly investigate subgroups H of G such that $P_{H,V}^G(H)$ is the projective cover of the $\mathbb{K}\overline{N}_G(H)$-module V. By the methods of [9, Section 12] one may deduce some of the results presented here, except possibly for 7.5, 7.7, and 7.10.

Lemma 7.1. *Let $P_{H,V}^G$ be a principal indecomposable $\mu_{\mathbb{K}}(G)$-module. For any subgroup K of G with $S_{H,V}^G(K) \neq 0$, the module $P_{H,V}^G(K)$ is the projective cover of $S_{H,V}^G(K)$ as $t_K^K \mu_{\mathbb{K}}(G) t_K^K$-modules.*

Proof. This is immediate from 4.2 by taking $A = \mu_{\mathbb{K}}(G)$, $S = S_{H,V}^G$, and $e = t_K^K$. □

As we want to study coordinate modules like $P_{H,V}^G(K)$ as $\mathbb{K}\overline{N}_G(K)$-modules, we next give a result about the algebras like $t_K^K \mu_{\mathbb{K}}(G) t_K^K$ and $\mathbb{K}\overline{N}_G(K)$.

Lemma 7.2. *For any subgroup K of G we have the direct sum decomposition*

$$t_K^K \mu_{\mathbb{K}}(G) t_K^K = A_K \oplus J_K$$

where A_K is a unital subalgebra of $t_K^K \mu_{\mathbb{K}}(G) t_K^K$ isomorphic to the group algebra $\mathbb{K}\overline{N}_G(K)$ and J_K is a two sided ideal of $t_K^K \mu_{\mathbb{K}}(G) t_K^K$. Moreover, the elements c_K^g with $gK \subseteq N_G(K)$ form a $\mathbb{K}$-basis of A_K, and the set of elements of the form $t_{{}^gJ}^K c_J^g r_J^K$ with $KgK \subseteq G$, $J \leqslant K^g \cap K$, $J \neq K$ contains a $\mathbb{K}$-basis of J_K.

Proof. The basis Theorem 2.1 implies that

$$t_K^K \mu_{\mathbb{K}}(G) t_K^K = \left(\bigoplus_{gK \subseteq N_G(K)} \mathbb{K} c_K^g \right) \oplus J_K$$

as $\mathbb{K}$-spaces, where J_K is the $\mathbb{K}$-subspace with basis elements of the desired form. We see easily that

$$\bigoplus_{gK \subseteq N_G(K)} \mathbb{K}c_K^g \quad \text{and} \quad \mathbb{K}\overline{N}_G(K)$$

are isomorphic algebras with isomorphism given by $c_K^g \leftrightarrow gK$. Finally, using the axioms in the definition of Mackey algebras we observe that J_K is a two sided ideal of $t_K^K \mu_{\mathbb{K}}(G) t_K^K$. □

7.1 and 7.2 imply

Theorem 7.3. *Let $P_{H,V}^G$ be any principal indecomposable $\mu_{\mathbb{K}}(G)$-module. For any subgroup K of G with $S_{H,V}^G(K) \neq 0$, if $J_K S_{H,V}^G(K) = 0$ then the module*

$$P_{H,V}^G(K)/J_K P_{H,V}^G(K)$$

is the projective cover of the $\mathbb{K}\overline{N}_G(K)$-module $S_{H,V}^G(K)$.

We note that the condition $J_K S_{H,V}^G(K) = 0$ in the statement of 7.3 is equivalent to the condition $J_K P_{H,V}^G(K) \neq P_{H,V}^G(K)$.

The following is obtained at once from 7.3 by putting $K = H$.

Corollary 7.4. *For any subgroup H of G and for any simple $\mathbb{K}\overline{N}_G(H)$-module V, the module*

$$P_{H,V}^G(H)/J_H P_{H,V}^G(H)$$

is the projective cover of the $\mathbb{K}\overline{N}_G(H)$-module V. In particular, $P_{H,V}^G(H)$ is the projective cover of the $\mathbb{K}\overline{N}_G(H)$-module V if and only if $J_H P_{H,V}^G(H) = 0$.

We next single out a case in which the ideal J_H annihilates the module $P_{H,V}^G(H)$.

Proposition 7.5. *Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a p'-subgroup of G. Then for any simple $\mathbb{K}\overline{N}_G(H)$-module V, the $\mathbb{K}\overline{N}_G(H)$-module $P_{H,V}^G(H)$ is the projective cover of V.*

Proof. There is an indecomposable $\mu_{\mathbb{K}}(H)$-module T such that $P_{H,V}^G$ is a direct summand of $\uparrow_H^G T$. By the transitivity of induction it follows that H is a vertex of T. Using the semisimplicity of the algebra $\mu_{\mathbb{K}}(H)$ we conclude that $T \cong S_{H,\mathbb{K}}^H$. Then $P_{H,V}^G$ is a direct summand of $\uparrow_H^G S_{H,\mathbb{K}}^H$. By part (6) of 6.1 the minimal subgroups of $\uparrow_H^G S_{H,\mathbb{K}}^H$ are precisely the G-conjugates of H. Consequently, any $\mathbb{K}$-basis element $t_{gJ}^H c_J^g r_J^H$ of the ideal J_H annihilates the module $\uparrow_H^G S_{H,\mathbb{K}}^H$ because $J \neq H$. So J_H annihilates the direct summand $P_{H,V}^G$ of $\uparrow_H^G S_{H,\mathbb{K}}^H$. Finally, as $J_H P_{H,V}^G = J_H P_{H,V}^G(H) = 0$, the result follows by 7.4. □

Needless to say, the conclusion of 7.5 is not true in general. It may happen that $P_{H,V}^G(H)$ is not indecomposable, and not projective as $\mathbb{K}\overline{N}_G(H)$-module. See 7.8, 7.9.

Lemma 7.6. *Let $P^G_{H,V}$ be a principal indecomposable $\mu_{\mathbb{K}}(G)$-module, and let K and X be subgroups of G such that $S^G_{H,V}(X)$ is nonzero. Suppose that $P^G_{H,V}(K)$ is nonzero. Then*

(1) *$P^G_{H,V}(K)$ is a direct summand of $t^K_K\mu_{\mathbb{K}}(G)t^X_X$ as $t^K_K\mu_{\mathbb{K}}(G)t^K_K$-modules.*
(2) *If K is a normal subgroup of G, then $P^G_{H,V}(K)$ is a direct summand of*

$$n\left(\uparrow^{G/K}_{KX/K}\mathbb{K}_{KX/K}\right)$$

as $\mathbb{K}(G/K)$-modules where $\mathbb{K}_{KX/K}$ is the trivial $\mathbb{K}(KX/K)$-module and n is the number of conjugacy classes of subgroups of $K\cap X$.

Proof. Let A be a finite dimensional algebra over a field and e be an idempotent of A. If $eS\neq 0$ for a simple A-module S, then it is obvious and well known that there is a primitive idempotent ϵ of A such that the projective cover P of S is isomorphic to $A\epsilon$ and that $e\epsilon=\epsilon=\epsilon e$. Thus P is a direct summand of Ae. Consequently for any idempotent f of A, the fAf-module fP (if nonzero) is a direct summand of fAe.

(1) This follows from the above explanation by putting $A=\mu_{\mathbb{K}}(G)$, $S=S^G_{H,V}$, $e=t^X_X$ and $f=t^K_K$.

(2) By the first part $P^G_{H,V}(K)$ is a direct summand of $t^K_K\mu_{\mathbb{K}}(G)t^X_X$. Using the normality of K, the basis Theorem 2.1 implies

$$t^K_K\mu_{\mathbb{K}}(G)t^X_X=\bigoplus_{J\leqslant_{K\cap X}K\cap X}\ \bigoplus_{gKX\subseteq G}\mathbb{K}c^g_Kt^K_Jr^X_J.$$

Now it is easy to see that for any $J\leqslant K\cap X$, the $\mathbb{K}$-space

$$\bigoplus_{gKX\subseteq G}\mathbb{K}c^g_Kt^K_Jr^X_J$$

is isomorphic to $\uparrow^{G/K}_{KX/K}\mathbb{K}_{KX/K}$ as $\mathbb{K}(G/K)$-module. $\square$

Proposition 7.7. *Let H be a normal subgroup of G. Then, $P^G_{H,V}(H)$ is a projective $\mathbb{K}(G/H)$-module for any simple $\mathbb{K}(G/H)$-module V.*

Proof. Letting $K=X=H$ in part (2) of 7.6, the result follows. $\square$

Remark 7.8. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p>0$ and G be a p-group. For a subgroup H of G, the module $P^G_{H,\mathbb{K}}(H)$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module if and only if $H=1$.

Proof. By [12, Proposition 3.1] $P^G_{H,\mathbb{K}}\cong\mu_{\mathbb{K}}(G)t^H_H$ so that $P^G_{H,\mathbb{K}}(H)$ is isomorphic to $t^H_H\mu_{\mathbb{K}}(G)t^H_H$ as $t^H_H\mu_{\mathbb{K}}(G)t^H_H$-modules. Therefore, it follows by 7.2 that

$$P^G_{H,\mathbb{K}}(H)\cong\mathbb{K}\overline{N}_G(H)\oplus J_H$$

as $\mathbb{K}\overline{N}_G(H)$-modules. Consequently, $P^G_{H,\mathbb{K}}(H)$ is indecomposable if and only if $J_H = 0$, which is equivalent to $H = 1$. $\square$

Example 7.9. Let $\mathbb{K}$ be an algebraically closed field of characteristic 2, and G be the dihedral group of order 8. Take a nonnormal subgroup H of G of order 2. Then $P^G_{H,\mathbb{K}}(H)$ is a nonprojective $\mathbb{K}\overline{N}_G(H)$-module.

Proof. Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and G be a p-group. For any subgroup H of G, if $P^G_{H,\mathbb{K}}(H)$ is a projective $\mathbb{K}\overline{N}_G(H)$-module, then (since $\mathbb{K}\overline{N}_G(H)$ is a local algebra) it follows that the order of $\overline{N}_G(H)$ divides the dimension of $P^G_{H,\mathbb{K}}(H) \cong t^H_H \mu_{\mathbb{K}}(G) t^H_H$.

In this special case, using the basis Theorem 2.1 we easily calculate the dimension of $t^H_H \mu_{\mathbb{K}}(G) t^H_H$ as 5. Since 5 is not divisible by 2 which is the order of $\overline{N}_G(H)$, $P^G_{H,\mathbb{K}}(H)$ cannot be projective. $\square$

For nilpotent groups, using the tensor product decomposition of Mackey algebras we may give the following general form of 7.8.

Proposition 7.10. *Let $\mathbb{K}$ be an algebraically closed field of characteristic $p > 0$ and G be a nilpotent group. For a subgroup H of G, the module $P^G_{H,V}(H)$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module if and only if H is a p'-group.*

Proof. As in the proof of 6.11, under the tensor product decomposition of $\mu_{\mathbb{K}}(G)$ the $\mu_{\mathbb{K}}(G)$-module $P^G_{H,V}$ corresponds to the

$$\mu_{\mathbb{K}}(G_p) \otimes_{\mathbb{K}} \mu_{\mathbb{K}}(G_{p'})\text{-module } P^{G_p}_{H_p,\mathbb{K}} \otimes_{\mathbb{K}} S^{G_{p'}}_{H_{p'},V}.$$

Therefore, the $\mathbb{K}\overline{N}_G(H)$-module $P^G_{H,V}(H)$ corresponds to the

$$\mathbb{K}\overline{N}_{G_p}(H_p) \otimes_{\mathbb{K}} \mathbb{K}\overline{N}_{G_{p'}}(H_{p'})\text{-module } P^{G_p}_{H_p,\mathbb{K}}(H_p) \otimes_{\mathbb{K}} V.$$

Consequently, $P^G_{H,V}(H)$ is an indecomposable $\mathbb{K}\overline{N}_G(H)$-module if and only if $P^{G_p}_{H_p,\mathbb{K}}(H_p)$ is an indecomposable $\mathbb{K}\overline{N}_{G_p}(H_p)$-module. The result now follows from 7.8. $\square$

Using the results obtained about the kernels of $P^G_{H,V}$ we have

Remark 7.11. Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a p-perfect normal subgroup of G. Then, $P^G_{H,V}(H)$ is the projective cover of the $\mathbb{K}(G/H)$-module V for any simple $\mathbb{K}(G/H)$-module V.

Proof. From 3.19 the kernel of $P^G_{H,V}$ is H so that we may write

$$P^G_{H,V} \cong \operatorname{Inf}^G_{G/H} P^{G/H}_{1,V}.$$

Evaluating at H we get a $\mathbb{K}(G/H)$-module isomorphism $P^G_{H,V}(H) \cong P^{G/H}_{1,V}(1)$. Finally, it follows by 7.5 that $P^{G/H}_{1,V}(1)$ is the projective cover of the $\mathbb{K}(G/H)$-module V. $\square$

If N is a p-perfect normal subgroup of G where p is the characteristic of the field $\mathbb{K}$, then $\mathrm{Inf}^G_{G/N}$ induces an isomorphism between some full subcategories of Mackey functor categories, see [9, Section 10]. 7.11 may also be deduced easily by using this category isomorphism.

There are some similar results obtained in [9] about evaluations of Mackey functors. In the next result we summarize these related results from [9, Sections 12 and 13].

Remark 7.12. (See [9].) Let $\mathbb{K}$ be a field of characteristic $p > 0$ and H be a subgroup of G.

(1) $P^G_{H,V}(1) \neq 0$ if and only if H is a p-group.
(2) If H is a p-group then $P^G_{H,V}(1)$ is an indecomposable direct summand of $\uparrow^G_H \mathbb{K}_H$.
(3) If H is a normal p-subgroup of G then the $\mathbb{K}(G/H)$-module $P^G_{H,V}(1)$ is the projective cover of the $\mathbb{K}(G/H)$-module V.
(4) $P^G_{1,V}$ is isomorphic to the fixed point functor $FP^G_{P_V}$ where P_V is the projective cover of the $\mathbb{K}G$-module V.

For example we note the similarity of parts (2) of 7.12 and 7.6.

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