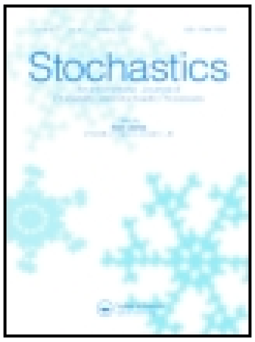

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Hedging portfolio for a market model of degenerate diffusions

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ABSTRACT

We consider a semimartingale market model when the underlying diffusion has a singular volatility matrix and compute the hedging portfolio for a given payoff function. Recently, the representation problem for such degenerate diffusions as a stochastic integral with respect to a martingale has been completely settled. This representation and Malliavin calculus established further for the functionals of a degenerate diffusion process constitute the basis of the present work. Using the Clark–Hausmann–Bismut–Ocone type representation formula derived for these functionals, we prove a version of this formula under an equivalent martingale measure. This allows us to derive the hedging portfolio as a solution of a system of linear equations. The uniqueness of the solution is achieved by a projection idea that lies at the core of the martingale representation at the first place. We demonstrate the hedging strategy as explicitly as possible with some examples of the payoff function such as those used in exotic options, whose value at maturity depends on the prices over the entire time horizon.

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Degenerate diffusion; Malliavin calculus; exotic option; replicating portfolio; Clark–Ocone formula

1. Introduction

An important application of the classical martingale representation theorem is in mathematical finance for calculating the hedging strategy when the risky asset price can be modelled as a diffusion process. For a portfolio of assets that are diffusions in $\mathbb{R}^n$, the volatility is captured by $\sigma \in \mathbb{R}^{n\times d}$, and the hedging strategy is derived under the assumption that the diffusion matrix $\sigma\sigma^*$ is non-singular. This case is studied extensively in prior work for hedging (see, e.g. Refs. [12,22]). On the other hand, the martingale representation of degenerate diffusions recently developed in Ref. [25] makes the calculation of a hedging strategy possible when the volatility matrix $\sigma\sigma^*$ is singular. Such degeneracy can occur for example when the noise dimension is larger than the stock dimension, or when the range of the volatility matrix becomes smaller than the dimension of the stock vector in general. In this paper, we derive the replicating portfolio as a solution of a system of linear equations in the degenerate case based on martingale representation [25] and Malliavin calculus developed in Ref. [26]. Malliavin calculus has already been applied to problems in finance,

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which include minimal variance hedging in incomplete markets, sensitivity analysis and efficient computation of Greeks, optimal portfolio selection with partial information or in an anticipating environment, optimal consumption in a general information setting, and insider trading [6, Preface], [17, Chp. 6].

Let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space, and let $X = \{X_t : 0 \leq t \leq 1\}$ satisfy the stochastic differential equation

$$\mathrm{d}X_t = b\left(X_t\right)\mathrm{d}t + \sigma\left(X_t\right)\mathrm{d}W_t, \tag{1}$$

where $\{W_t : 0 \leq t \leq 1\}$ is an $\mathbb{R}^d$-valued Brownian motion and $\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times d}$ and $b : \mathbb{R}^n \to \mathbb{R}^n$ are measurable maps. We assume that the drift b and the diffusion matrix σ are C^1-Lipschitz and of linear growth to ensure existence of the strong solution of (1). The diffusion X is possibly degenerate in the sense that $\sigma(x)\sigma(x)^*$ can be singular for some $x \in \mathbb{R}^n$. We consider a market model where the risky assets are diffusions described by (1). Using Malliavin calculus and generalizing the Clark representation formula, the hedging portfolio is found in Ref. [18] under the ellipticity condition, which is called non-degeneracy condition, for a similar model. Ellipticity condition, which is given by $\sum_j \sum_k \xi^* \sigma(x)\sigma^*(x)\xi \geq \varepsilon\|\xi\|^2$ for some $\varepsilon > 0$ and for all $\xi \in \mathbb{R}^n$, clearly implies the non-singularity of $\sigma\sigma^*$. Similarly, degenerate diffusions in Ref. [28] refer to those for which ellipticity condition does not hold. A specific stochastic differential equation with a singular diffusion matrix is considered in Ref. [28] for the purpose of establishing an explicit derivative formula for the associated Markov semigroup using Malliavin calculus. The terms degenerate and singular are used interchangeably for diffusion matrices in Ref. [5], arising in several models for option pricing. The value process is characterized as a solution of partial integro-differential equations, for which existence, uniqueness or regularity of solutions may not hold in the case of degeneracy. Therefore, the theory of viscosity solutions is used to deal with a singular diffusion matrix under the assumption that a suitable Lyapunov function exists. The method is demonstrated for Asian option in Heston stochastic volatility model among other examples in Ref. [5].

Malliavin calculus is a probabilistic approach for deriving the hedging strategy in contrast to partial differential equations (PDEs) approach which yields the so-called Δ-hedge formula [6, Sec. 4.4]. The main advantage of Δ-hedge formula is that it does not involve conditional expectations or gradients. As discussed in Ref. [3], in a one-dimensional market where the value process has the form $V^\theta(t) = f(t, X(t))$ for some function $f : \mathbb{R}^2 \to \mathbb{R}$ and payoff function in the form of $G(X_T)$, the two approaches are in some cases more or less equivalent with respect to the differentiability properties of f. The Δ-hedge formula can be derived for more complicated claims such as Asian options when the value process has the form $V^\theta(t) = f(t, X(t), S(t))$, where $S(t)$ is a state variable. However, this method requires that f is a $C^{1,2,2}$ function, although Malliavin calculus yields the same representation under the condition that f is only Lipschitz. The Δ-hedge is based on solving a relevant PDE under the ellipticity condition, which does not hold in the present paper.

Let $\mathcal{F}(X) = \{\mathcal{F}_t(X) : 0 \leq t \leq 1\}$ denote the filtration generated by X. The martingale representation theorem for degenerate diffusions [25, Thm. 2] reveals that an $\mathcal{F}_1$-measurable functional F of X can be represented as

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 P(X_s)\xi_s(X)\cdot \mathrm{d}W_s = \mathbb{E}[F(X)] + \int_0^1 \xi_s(X)\cdot P(X_s)\,\mathrm{d}W_s$$

with an $\mathcal{F}_t(X)$-adapted process ξ taking values in $\mathbb{R}^d$, where dot product is used for simplicity of notation and $P(X_s)$ denotes orthogonal projection to the range space of σ^*, the transpose of σ. In essence, there exists a martingale, given above as $P(X_s)\,\mathrm{d}W_s$ in its infinitesimal Itô form, with respect to which every square integrable $\mathcal{F}_1$-measurable functional can be written as an integral of an $\mathcal{F}(X)$-adapted process. The representation problem for degenerate diffusions has been settled in Ref. [25] as a result. Furthermore, Malliavin calculus for degenerate diffusions is developed in Ref. [26], from which we borrow the results needed for the present paper. Let (W, H, μ) be the classical Wiener space on $\mathbb{R}^d$. For suitable $\mathcal{F}$-measurable functionals F, we first provide the Clark–Hausmann–Bismut–Ocone type formula of Ref. [26, Thm. 8] given by

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 P(X_s)\mathbb{E}[\hat{D}_s F(X) \mid \mathcal{F}_s] \cdot \mathrm{d}W_s, \tag{2}$$

where the operator $\hat{D}$ is defined as the density of $\hat{\nabla}$ with respect to Lebesgue measure and $\hat{\nabla}$ is an operator similar to Gross-Sobolev derivative ∇ for Wiener functionals. Then, we find the hedging strategy given a stock portfolio where the prices are modelled as degenerate diffusions, where the projection $P(X_s)$ plays a crucial role.

In this paper, we not only solve the hedging problem for a semimartingale market model of degenerate diffusions but also find a hedging strategy adapted to $\mathcal{F}(X)$, the filtration of the asset prices themselves, instead of the filtration $\mathcal{F}(W)$ of the driving Wiener process. More explicitly, let the price dynamics of n assets X_t in a market follow (1) and let the equation for the risk-free asset X_t^0 at time t be given by $\mathrm{d}X_t^0 = r_t X_t^0\,\mathrm{d}t$, where r_t is the interest rate at time t, for $t \in [0, 1]$. Let θ_t and θ_t^0 be the number of shares of n risky assets and the risk-free asset, respectively, where θ_t is taken as a row vector. Then, the value process V_t^θ is written as $V_t^\theta = \theta_t X_t + \theta_t^0 X_t^0$. Assuming that the portfolio (θ, θ^0) is self-financing, we find the hedging portfolio that replicates the terminal value function V_1, which is assumed to be specified by an $\mathcal{F}_1$-measurable random variable $G(X)$ called the payoff function.

When the volatility matrix $\sigma\sigma^*$ is singular, the equation

$$\sigma(X_t)u(X_t) = b(X_t) - r_t X_t$$

in u may not have a unique solution. We assume that it has more than one solution so that the market can be used for option pricing. Although the market is incomplete under this assumption, $P(X_t)u(X_t)$ is unique for all solutions u and it can be chosen to construct an equivalent martingale measure $\mathbb{Q}$ uniquely. By denoting the d-dimensional Brownian motion under $\mathbb{Q}$ with $\widetilde{W}$ and assuming that r is deterministic, we show that the hedging strategy is obtained by solving the equation

$$\begin{aligned}
&\sigma^*(X_t)\theta_t^* \\
&\quad = e^{-\int_t^1 r_s\,\mathrm{d}s} P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_t G(X) - G(X)\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right) \cdot \mathrm{d}\widetilde{W}_s \mid \mathcal{F}_t(X)\right]
\end{aligned}$$

for θ, which may not be unique although $\{P(X_t)\theta_t : 0 \leq t \leq 1\}$ is unique for all solutions θ. We prove two fundamental results related to $\hat{\nabla}$ needed in our derivations. Namely, Proposition 2.1 as the chain rule and Lemma 2.1 as the fundamental theorem of calculus are developed as a follow up of Ref. [26], where $\hat{\nabla}$ is shown to satisfy the properties of a derivative operator. Clearly, these properties are adopted from those of ∇, but with care on the projection with $P(X_s)$ and using the cylindrical functions common in the domains of the two operators when necessary. In Theorem 3.1, we derive an equivalent representation to (2) using the equivalent martingale measure $\mathbb{Q}$ and Wiener process $\widetilde{W}$, in view of the properties of the operator $\hat{\nabla}$. Then, we take $F = e^{-\int_0^1 r_s\,\mathrm{d}s}G(X)$ to obtain the equation for the hedging portfolio. As martingale representation and hedging are closely related, a representation result is proved and then applied to incomplete financial markets in general in Ref. [10]. In this paper, we show that some differentiability assumptions on the contingent claim G are sufficient for attainability in an incomplete market that arises from degeneracy due to a singular volatility matrix.

In applications, the sample covariance matrix of stock returns is rarely used to estimate volatility because it may be either singular or ill-conditioned. This can occur for example when the number of stocks n is of the same order of magnitude as the number of historical observations, even if the true covariance matrix is non-singular. An estimate of the covariance matrix of stock returns by an optimally weighted average of the sample covariance matrix and Sharpe's single-index covariance matrix is proposed in Ref. [13]. Then, it is shown to perform well for NYSE and AMEX stock returns from 1972 to 1995 in Markowitz optimal portfolio selection. For the same problem, Moore–Penrose pseudo-inverse is proposed in Ref. [19] and its numerical tractability is demonstrated with financial data from 2008. However, in some models like Asian options with Heston model given in Ref. [5], the diffusion matrix is singular. In these problems, the results of the present paper can be used. Therefore, we demonstrate the hedging strategy as explicitly as possible with some specific examples of the payoff G. Asian options, exotic options, look-back options, and exchange options are considered since their value at maturity depends on the path of the price process over the whole time horizon. It should be pointed out that our expressions involve conditional expectations. The search for more explicit formulas for the hedging strategy might require further specification of the drift and diffusion parameters. In Ref. [3], it is shown that the hedging strategy involves no conditional expectations for the Asian options in the one-dimensional model. Similarly in Ref. [20], explicit expressions are obtained for look-back options under the assumption of constant coefficients. Our formulas can be simplified further with similar assumptions on the coefficients. The results can be useful in several finance and interdisciplinary applications where diffusion processes and hedging are considered (see, e.g. Refs. [4,7–9,23]).

The paper is organized as follows. In Section 2, we review the essential parts of Malliavin calculus for degenerate diffusions and prove the preliminary results useful for the present work. Then, the hedging formula is derived for the degenerate semimartingale market model in Section 3. Special cases of the payoff function are considered in Section 4 to demonstrate hedging and option pricing. Finally, Section 5 concludes the paper.

2. Preliminaries

In this section, we establish the essential properties of the operator $\hat{\nabla}$ for the aim of obtaining the hedging strategy in the sequel. This is closely related to the Gross–Sobolev derivative ∇ defined on the classical Wiener space (W, H, μ) [24]. The random variable

$G : W \to \mathbb{R}$ is called a cylindrical Wiener functional if it is of the form

$$G(\omega) = f\left(\delta h_1(\omega), \ldots, \delta h_m(\omega)\right), \quad h_1, \ldots, h_m \in H, f \in \mathcal{S}\left(\mathbb{R}^m\right)$$

for some $m \in \mathbb{R}^m$, where $\mathcal{S}(\mathbb{R}^m)$ denotes the Schwartz space of rapidly decreasing functions on $\mathbb{R}^m$ and $\delta h = \int_0^1 h'_s \, \mathrm{d}W_s$. The space of cylindrical functions on W will be denoted by $\mathcal{S}(W)$. For $G \in \mathcal{S}(W)$ and $h \in H$, the Gateaux derivative of G in the direction of h is defined as

$$\nabla_h G(\omega) = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon} G(\omega + \epsilon h)\right|_{\epsilon=0}.$$

Then, we define ∇ as

$$\nabla G(\omega) = \sum_{i=1}^{\infty} \nabla_{e_i} G(\omega) e_i \quad \omega \in W,$$

where $\{e_i : i \in \mathbb{N}\}$ is any complete, orthonormal basis in H. The operator ∇ is a closable operator on $L^p(\mu)$ for any $p \geq 1$. Note that ∇G is now well defined for $G \in L^p(\mu)$ and it can be regarded as an H-valued Wiener functional. The density of ∇ with respect to Lebesgue measure is denoted by D, that is, D is defined by

$$\nabla G(t) = \int_0^t D_s G \, \mathrm{d}s$$

for each $t \in [0, 1]$.

Now, consider $\mathcal{F}_1$-measurable functionals of the diffusion process X. By Ref. [17, Thm. 2.2.1], ∇X is well defined in view of Lipschitz and linear growth assumptions on the coefficients of (1). In particular for X_t, we define

$$\hat{\nabla}_h X_t := \mathbb{E}[\nabla_h X_t \mid \mathcal{F}_1(X)], \tag{3}$$

where $\nabla_h X_t = (\nabla_h X_t^1, \ldots, \nabla_h X_t^n)$. Let $\mathcal{S}(X)$ denote the set of cylindrical functions given by

$$\begin{aligned}\mathcal{S}(X) = \Big\{ & f\left(X_{t_1}^1, \ldots, X_{t_1}^n, \ldots, X_{t_m}^1, \ldots, X_{t_m}^n\right) : \\ & 0 \leq t_1 < \cdots < t_m \leq 1, f \in \mathcal{S}\left(\mathbb{R}^{nm}\right), m \geq 1 \Big\},\end{aligned}$$

where $\mathcal{S}(\mathbb{R}^{nm})$ denotes the space of rapidly decreasing smooth functions of Laurent Schwartz. As in Ref. [26], for $h \in H$ and $F(X) \in \mathcal{S}(X)$, the operator $\hat{\nabla}_h$ is defined as

$$\hat{\nabla}_h F(X) = \sum_{j=1}^{m} \sum_{i=1}^{n} \partial_{(j-1)n+i} f(X_{t_1}^1, \ldots, X_{t_1}^n, \ldots, X_{t_m}^1, \ldots, X_{t_m}^n) \hat{\nabla}_h X_{t_j}^i$$

in view of (3). The following theorem gathers the basic properties of $\hat{\nabla}$ from Ref. [26, Thm. 5,Cor. 2], including the result that it is a closable operator on $L^2(\nu)$, where ν denotes the image of μ under X, that is, the probability law of X.

Theorem 2.1: *The process* $\{\hat{\nabla}_h X_t : 0 \le t \le 1\}$ *satisfies the equation*

$$\hat{\nabla}_h X_t = \int_0^t \partial\sigma(X_s)\hat{\nabla}_h X_s P(X_s)\,\mathrm{d}W_s + \int_0^t \partial b(X_s)\hat{\nabla}_h X_s\,\mathrm{d}s + \int_0^t \sigma(X_s)h'_s\,\mathrm{d}s$$

μ*-almost surely. Moreover, if* $(F_k(X), k \ge 1) \subset \mathcal{S}(X)$ *converges to zero in* $L^2(\mathcal{F}(X))$ *and* $(\hat{\nabla}_h F_k(X), k \ge 1)$ *is Cauchy in* $L^2(\mathcal{F}(X))$*, then*

$$\lim_{k\to\infty} \hat{\nabla}_h F_k(X) = 0$$

μ*-a.s.*

Proof: For $i \in \{1, \dots, n\}$, we have

$$\begin{aligned}\nabla_h X_t^i &= \sum_{j=1}^d \sum_{k=1}^n \int_0^t \partial_k \sigma_{ij}(X_s)\nabla_h X_t^k\,\mathrm{d}W_s^j + \sum_{j=1}^d \int_0^t \sigma_{ij}(X_s)(h_s^j)'\,\mathrm{d}s \\ &\quad + \sum_{k=1}^n \int_0^t \partial_k b_i(X_s)\nabla_h X_t^k\,\mathrm{d}s \\ &= \int_0^t J_{\sigma_i}(X_s)\nabla_h X_s \cdot \mathrm{d}W_s + \int_0^t \partial b_i(X_s)\cdot \nabla_h X_s\,\mathrm{d}s + \int_0^t \sigma_i(X_s)h'_s\,\mathrm{d}s, \end{aligned} \tag{4}$$

where σ_i is the i^{th} row of matrix σ, J_{σ_i} denotes the Jacobian matrix of σ_i, and ∂b_i denotes the gradient of b_i. We use the compact notations $\partial\sigma$ and ∂b for the respective tensor and matrix identified in (4). Taking the conditional expectation of both sides of this equation with respect to $\mathcal{F}_1(X)$, the term with stochastic integral follows from Ref. [25, Thm. 3] and the Lebesgue integral follows from

$$\mathbb{E}[\nabla_h X_t \mid \mathcal{F}_1(X)] = \mathbb{E}[\nabla_h X_t \mid \mathcal{F}_t(X)]$$

due to the fact that $\mathcal{F}_t(W) := \sigma(W_s, s \le t)$ is independent of the future increments of the Brownian motion after t.

Let $\eta = \eta(X)$ be the limit of $(\hat{\nabla}_h F_k(X), k \ge 1)$. Then, using Ref. [25, Th. 3], we have, for any cylindrical $G(X) \in \mathcal{S}(X)$,

$$\begin{aligned}\mathbb{E}[\eta(X)\,G(X)] &= \lim_k \mathbb{E}[\hat{\nabla}_h F_k(X) G(X)] = \lim_k \mathbb{E}[\nabla_h F_k(X) G(X)] \\ &= \lim_k \mathbb{E}[F_k(X)(-\nabla_h G(X) + G(X)\delta h)] \\ &= \lim_k \mathbb{E}[F_k(X)(-\hat{\nabla}_h G(X) + G(X)\delta(\mathbf{P}(X)h))] = 0,\end{aligned}$$

where $\delta(\mathbf{P}(X)h) = \int_0^1 P(X_s)h'_s \cdot \mathrm{d}W_s$. ■

Similar to ∇, we can define the operator $\hat{\nabla}$ by

$$\hat{\nabla} F(X) = \sum_{i=1}^{\infty} \hat{\nabla}_{e_i} F(X) e_i$$

for $F(X) \in \mathcal{S}(X)$, where $\{e_i, i \geq 1\}$ is a complete, orthonormal basis in the Cameron–Martin space H. It follows that $\hat{\nabla}$ is a closable operator from $L^p(\nu)$ to $L^p(\nu; H)$, where H is indicated to specify the range of $\hat{\nabla}$.

The norm defined by

$$\|F(X)\|_{p,1} := \|F(X)\|_{L^p(\mu)} + \|\hat{\nabla} F(X)\|_{L^p(\mu;H)}$$

is used for the completion of $S(X)$, which will be denoted by $\mathbb{M}_{p,1}$. Note that we use $|\cdot|$ for Euclidean norm, $\|\cdot\|$ for $L^2([0,1])$-norm, and for all others we specify the space in the notation. For $F(X) \in \mathbb{M}_{2,1}$, define $\hat{D}$ similar to D by $\hat{\nabla} F(X)(t) = \int_0^t \hat{D}_s F(X)\, \mathrm{d}s, \forall\, t \in [0,1]$. Note that $\hat{D}_s F(X)$ is $\mathrm{d}s \times \mathrm{d}\mu$-almost everywhere well defined. Then, we have the following relation:

$$\hat{\nabla}_h F(X) = \langle \hat{\nabla} F(X), h \rangle_H = \int_0^1 \hat{D}_s F(X) \cdot h'_s\, \mathrm{d}s = \langle \hat{D} F(X), h' \rangle_{L^2([0,1])}.$$

The following theorem is borrowed from Ref. [26, Thm. 8], as a Clark–Hausmann–Bismut–Ocone formula for degenerate diffusions.

Theorem 2.2: *Assume that $F(X) \in \mathbb{M}_{2,1}$, then it can be represented as*

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 P(X_s)\, \mathbb{E}\left[\hat{D}_s F(X) \mid \mathcal{F}_s(X)\right] \cdot \mathrm{d}W_s,$$

where $\hat{D}_s F(X)$ is defined as $\hat{\nabla} F(X)(\cdot) = \int_0^{\cdot} \hat{D}_s F(X)\, \mathrm{d}s$.

Proof: We know from Ref. [25, Thm. 2] that $F(X)$ can be represented as

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 P(X_s)\, \alpha_s(X) \cdot \mathrm{d}W_s$$

for some $\alpha(X) \in L_a^2(\mathrm{d}t \times \mathrm{d}\mu, \mathbb{R}^d)$, adapted to the filtration $(\mathcal{F}_t(X), t \in [0,1])$. Moreover, if $F(X) \in \mathcal{S}(X)$, then from the Gaussian case, we have

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 \mathbb{E}\left[D_s F(X) \mid \mathcal{F}_s(W)\right] \cdot \mathrm{d}W_s, \tag{5}$$

where $\{\mathcal{F}_s(W) : 0 \leq s \leq 1\}$ is the filtration of the Brownian motion. It follows from Ref. [25, Thm. 3], taking the conditional expectations of both sides of (5), that

$$F(X) = \mathbb{E}[F(X)] + \int_0^1 P(X_s)\, \mathbb{E}\left[D_s F(X) \mid \mathcal{F}_s(X)\right] \cdot \mathrm{d}W_s.$$

Hence in this case $P(X_s)\alpha_s(X) = P(X_s)\mathbb{E}[D_s F(X) \mid \mathcal{F}_s(X)] = P(X_s)\mathbb{E}[\mathbb{E}[D_s F(X) | \mathcal{F}_1(X) | \mathcal{F}_s(X)]]$, which is equal to $P(X_s)\mathbb{E}[\hat{D}_s F(X) \mid \mathcal{F}_s(X)]$, $\mathrm{d}s \times \mathrm{d}\mu$ -a.s. Choose now a sequence

$(F_n(X), n \geq 1) \subset \mathcal{S}(X)$ which approximates $F(X)$ in $\mathbb{M}_{2,1}$. Then $(\hat{\nabla} F_n(X), n \geq 1)$ converges to $\hat{\nabla} F(X)$ in $L^2(\mu, H)$ (in other words $(\hat{\nabla} F_n)$ converges to $\hat{\nabla} F$ in $L^2(\nu, H)$). Consequently,

$$\lim_n \mathbb{E} \int_0^1 \left| P(X_s)\, \mathbb{E}\left[\hat{D}_s F_n(X) - \hat{D}_s F(X) \mid \mathcal{F}_s(X)\right] \right|^2 \mathrm{d}s = 0.$$

Hence, the result follows for any $F(X) \in \mathbb{M}_{2,1}$. ■

In the next section, we will need the representation of a payoff function of the stock price X with respect to a Wiener process under the equivalent martingale measure. We will derive such a representation on the basis of Theorem 2.2. In view of this, we prove further properties of $\hat{\nabla}$ in the following proposition and lemma.

Proposition 2.1: *Assume $F \in \mathbb{M}_{p,1}(\mathbb{R}^d)$, $g : \mathbb{R}^d \to \mathbb{R}$ is a continuous function.*

(i) *If g is Lipschitz continuous, then $g \circ F \in \mathbb{M}_{p,1}$.*
(ii) *If g is C^1-function such that*

$$\mathbb{E}\left[|g \circ F|^q + \sum_i \left| \partial_i g \circ F\, \hat{\nabla} F_i \right|^p \right] < \infty,$$

then $g \circ F \in \mathbb{M}_{r,1}$ for any $r < p \wedge q$, where p and q are conjugates.

Proof: (i) is evident from Mazur Lemma which says that closure of a convex set is the same under any topology of the dual pair and from the fact that the graph of $\hat{\nabla}$ is convex in any $L^n(\nu)$, for any $n \geq 1$.

(ii) Let θ be a smooth function of compact support on $\mathbb{R}^d$, $\theta(0) = 1$. Let $\theta_n(x) = \theta(x/n)$. Then

$$\mathbb{E}\left[|\hat{\nabla}(\theta_n g) \circ F|^r \right] \leq 2^{r-1} \sum_i \mathbb{E}\left[|g \circ F|^r |\hat{\nabla} F_i|^r + K |\partial_i g \circ F|^r |\hat{\nabla} F_i|^r \right]$$

and we have

$$\mathbb{E}\left[|g \circ F|^r |\hat{\nabla} F_i|^r \right] \leq \mathbb{E}\left[|g \circ F|^q \right]^{r/q} \mathbb{E}\left[|\hat{\nabla} F_i|^p \right]^{r/p},$$

where K is an upper bound for θ and the term with θ_n' does not contribute. So, $(\theta_n g \circ F, n \geq 1)$ is bounded in $\mathbb{M}_{r,1}$; hence it has a subsequence which converges weakly and this implies that $\lim_n \theta_n g \circ F = g \circ F$ belongs to $\mathbb{M}_{r,1}$. ■

Lemma 2.1: *Let $u(X_s) \in \mathbb{M}_{2,1}(L^2([0,1]))$ be adapted to $\mathcal{F}(X)$. Then, we have*

$$\begin{aligned} &\hat{\nabla}_h \int_0^1 P(X_s) u_s(X) \cdot \mathrm{d}W_s \\ &\quad = \int_0^1 P(X_s) \hat{\nabla}_h u_s(X) \cdot \mathrm{d}W_s + \int_0^1 P(X_s) \partial P(X_s) \hat{\nabla}_h X_s u_s(X) \cdot \mathrm{d}W_s \\ &\qquad + \int_0^1 P(X_s) u_s(X) \cdot h_s' \, \mathrm{d}s. \end{aligned}$$

Proof: Assume that (u_s) is a step process, then

$$\begin{aligned}\int_0^1 P(X_s)u_s(X)\cdot \mathrm{d}W_s &= \int_0^1 u_s(X)\cdot P(X_s)\,\mathrm{d}W_s\\ &= \sum_i u_{s_i}(X)\cdot (M_{s_{i+1}} - M_{s_i}),\end{aligned}$$

where $M_t = \int_0^t P(X_s)\,\mathrm{d}W_s$ by the martingale representation theorem [25, Thm. 2]. Therefore, we have

$$\begin{aligned}\hat{\nabla}_h \int_0^1 P(X_s)u_s(X)\cdot \mathrm{d}W_s &= \sum_i \hat{\nabla}_h u_{s_i}(X)\cdot (M_{s_{i+1}} - M_{s_i})\\ &\quad + \sum_i u_{s_i}(X)\cdot (\mathbf{P}(X_{s_{i+1}})h_{s_{i+1}} - \mathbf{P}(X_{s_i})h_{s_i})\\ &\quad + \mathbb{E}\left[\sum_i u_{s_i}(X)\cdot \left(\int_{s_i}^{s_{i+1}} \partial P(X_s)\nabla_h X_s\cdot \mathrm{d}W_s\right) \mid \mathcal{F}_1(X)\right],\end{aligned}$$

where we define the action of $\mathbf{P}(X)$ on the Cameron–Martin space H as $\mathbf{P}(X_t)h_t = \int_0^t P(X_s)h'_s\,\mathrm{d}s$ and use Ref. [24, Prop. II.2] for ∇M_t. It follows from Ref. [25, Thm. 3] that the last term is equal to

$$\sum_i u_{s_i}(X)\cdot \left(\int_{s_i}^{s_{i+1}} P(X_s)\partial P(X_s)\hat{\nabla}_h X_s\cdot \mathrm{d}W_s\right).$$

Then, we pass to the limit in L^2 as the mesh of partition goes to zero. For the other terms, the result is straightforward. ■

Remark 2.1: Suppose $u(X_s)$ satisfies the hypothesis of Lemma 2.1. Then, we have

$$\hat{D}_t \int_0^1 P(X_s)u(X_s)\cdot \mathrm{d}W_s = \int_0^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}W(s) + P(X_t)u(X_t)$$

$\mathrm{d}t\times\mu$-almost everywhere, as $\hat{\nabla}_h F = \int_0^1 \hat{D}_s F h'_s\,\mathrm{d}s$.

Lemma 2.2: *Let $u(X_s)\in\mathbb{M}_{2,1}(L^2([0,1]))$ be adapted to $\mathcal{F}(X)$. Then, $\|u(X_s)\|^2\in\mathbb{M}_{2,1}$ and*

$$\hat{\nabla}_h\|u(X_s)\|^2 = \hat{\nabla}_h\|U(X)\|_H^2 = 2\langle\hat{\nabla}_h u, U(X)\rangle_H,$$

where $U_t(X) = \int_0^t u(X_s)\,\mathrm{d}s$.

Proof: The proof is similar to that of Lemma 2.1. ■

Remark 2.2: Suppose $u(X_s)$ satisfies the assumption of Lemma 2.2. Then, $\int_0^1 |P(X_s)u(X_s)|^2\,\mathrm{d}s\in\mathbb{M}_{2,1}$ and

$$\hat{D}_t\int_0^1 |u(X_s)|^2\,\mathrm{d}s = 2\int_0^1 \hat{D}_t u(X_s)\cdot u(X_s)\,\mathrm{d}s.$$

3. Hedging a stock portfolio

We consider a semimartingale market model with n risky assets with price $X_t = (X_t^1, \ldots, X_t^n)^*$ and a risk-free asset X_t^0 at time t. The asset prices X_t and X_t^0 are determined by the system of stochastic differential equations

$$\begin{aligned} \mathrm{d}X_t &= b(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \\ \mathrm{d}X_t^0 &= r_t X_t^0\,\mathrm{d}t, \end{aligned} \tag{6}$$

where $W_t = (W_t^1, \ldots, W_t^d)^*$ and r_t is the deterministic interest rate, $t \in [0, 1]$. We assume that the drift b and the diffusion matrix σ satisfy the linear growth and Lipschitz conditions for the existence and uniqueness of a strong solution [29, Thm. 3.1] and they are assumed to be C^1-functions for applicability of the results of Section 2. In (6), the arguments of b and *sigma* can include time t separately and the analysis of this section will be still valid as this is allowed in our basic reference [25], but omitted for the sake of brevity. Examples where the coefficients are functions of only time t are included in the next section among others. In this section, we will derive the hedging strategy for a given payoff.

Recall that the value process V_t^θ is given by

$$V_t^\theta = \theta_t^0 X_t^0 + \theta_t X_t, \tag{7}$$

where by θ_t, θ_t^0 denote the number of shares of n assets and risk-free asset, respectively, and θ_t is taken as a row vector for simplicity of notation. The portfolio (θ, θ^0) is considered to be self-financing, that is, V_t^θ satisfies

$$\mathrm{d}V_t^\theta = \theta_t^0\,\mathrm{d}X_t^0 + \theta_t\,\mathrm{d}X_t. \tag{8}$$

Since from (7), we have $\theta_t^0 = (V_t^\theta - \theta_t X_t)/X_t^0$, we rewrite (8) as

$$\begin{aligned} \mathrm{d}V_t^\theta &= r_t\left(V_t^\theta - \theta_t X_t\right)\mathrm{d}t + \theta_t\,\mathrm{d}X_t \\ &= \left[r_t V_t^\theta + \theta_t b(X_t) - r_t\theta_t X_t\right]\mathrm{d}t + \theta_t\sigma(X_t)\,\mathrm{d}W_t. \end{aligned}$$

Assume that the equation below has a solution

$$\sigma(X_t)u(X_t) = b(X_t) - r_t X_t. \tag{9}$$

Although this equation may have several solutions u, the orthogonal projection by $P(X_t)$ of these solutions onto the range space of $\sigma^*(X_t)$ is unique, which can be verified by simple algebra. Then, the solution $P(X_t)u(X_t)$, called *market price of risk process*, satisfies

$$\sigma(X_t)P(X_t)u(X_t) = b(X_t) - r_t X_t. \tag{10}$$

Note that when (9) does not have a solution, then the market is not arbitrage-free and it cannot be used for pricing [22, p. 228]. Since we have assumed (9) has a solution, we can proceed with (10) for u. By definition of the projection operator $P(X_t)$, there exists an $\mathbb{R}^n$-valued random variable ξ such that $\sigma^*(X_t)\xi = P(X_t)u(X_t)$ and (10) can be rewritten as

$$\sigma(X_t)\sigma^*(X_t)\,\xi = b(X_t) - r_t X_t.$$

Now, it is clear that (9) would have a unique solution if the diffusion matrix $\sigma\sigma^*$ was invertible.

Assume also that the d-dimensional market price of risk process $u(X_t)$ satisfies

$$\int_0^1 |P(X_t)u(X_t)|^2\, \mathrm{d}t < \infty$$

almost surely and the positive local martingale

$$Z_t \triangleq \exp\left\{-\int_0^t P(X_s)u(X_s)\cdot \mathrm{d}W_s - \frac{1}{2}\int_0^t |P(X_s)u(X_s)|^2\, \mathrm{d}s\right\} \tag{11}$$

satisfies $\mathbb{E}Z_1 = 1$. Then, Z is a martingale with respect to the filtration generated by X, $\mathcal{F}(X)$, in view of the converse statement in the martingale representation theorem [25, Thm. 2]. Now, define $\widetilde{W}_t$ by

$$\widetilde{W}_t = W_t + \int_0^t P(X_s)u(X_s)\, \mathrm{d}s. \tag{12}$$

Then, $\{\widetilde{W}_t,\ 0 \le t \le 1\}$ is a Brownian motion under the probability measure $\mathbb{Q}$ on $F_1(W)$ given by

$$\mathbb{Q}(A) = \mathbb{E}[Z_1 1_A] \quad \forall\, A \in \mathcal{F}_1(W).$$

Using (10) and(12), we can write the price dynamics (6) using $\widetilde{W}$ as

$$\mathrm{d}X_t = r_t X_t\, \mathrm{d}t + \sigma(X_t)\, \mathrm{d}\widetilde{W}_t.$$

Similarly, the discounted price $S_t := \exp(-\int_0^t r_s\, \mathrm{d}s)X_t$ satisfies

$$\mathrm{d}S_t = e^{-\int_0^t r_s\, \mathrm{d}s}\sigma(X_t)\, \mathrm{d}\widetilde{W}_t.$$

Note that $\widetilde{W}$ cannot be written in terms of the price process S since the diffusion matrix is not invertible. Moreover, in view of (7) and (12), we can rewrite the value process as

$$\mathrm{d}V_t^\theta = r_t V_t^\theta\, \mathrm{d}t + \theta_t\sigma(X_t)\, \mathrm{d}\widetilde{W}_t. \tag{13}$$

Define the discounted value process $U_t^\theta := e^{-\int_0^t r_s\, \mathrm{d}s}V_t^\theta$. Let $G(X)$ be an $\mathcal{F}_1(X)$-measurable and integrable payoff function. After setting $V_1^\theta = G(X)$ to find the hedging strategy, the equation

$$\mathrm{d}U_t^\theta = e^{-\int_0^t r_s\, \mathrm{d}s}\theta_t\sigma(X_t)\, \mathrm{d}\widetilde{W}_t \tag{14}$$

can be considered as a backward stochastic differential equation with final condition

$$U_1^\theta = e^{-\int_0^1 r_s\, \mathrm{d}s}G(X). \tag{15}$$

Clearly, both the discounted price process and the value process are martingales under $\mathbb{Q}$ when we assume $\int_0^1 \sigma(X_t)\sigma^*(X_t)\, \mathrm{d}t$ is finite a.s. We further assume that

$$\int \theta_t\sigma(X_t)\sigma^*(X_t)\theta_t^*\, \mathrm{d}t < \infty \quad \text{a.s.}$$

to have an admissible strategy θ [12, p. 302]. In the following theorem, we will derive the representation of an $\mathcal{F}_1(X)$-measurable function F with respect to $\widetilde{W}$.

Theorem 3.1: *Suppose $F \in \mathbb{M}_{2,1}$ is $\mathcal{F}_1(X)$-measurable and the conditions*

$$\mathbb{E}[Z_1^2 F^2] < \infty, \quad \mathbb{E}\left[Z_1^2 \|\hat{D}F\|^2\right] < \infty,$$

$$\mathbb{E}\left[Z_1^2 F^2 \left\| \int_t^1 P(X_s)\hat{D}\left(P(X_s)u(X_s)\right) \cdot \mathrm{d}W(s) + P(X_t)u(X_t) \right.\right.$$
$$\left.\left. + \int_t^1 \hat{D}\left(P(X_s)u(X_s)\right) \cdot P(X_s)u(X_s)\,\mathrm{d}s \right\|^2\right] < \infty.$$

Then, we have $Z_1 F \in \mathbb{M}_{2,1}$ and

$$F = \mathbb{E}_{\mathbb{Q}}[F] + \int_0^1 P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_t F - F\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s \mid \mathcal{F}_t(X)\right]\mathrm{d}\widetilde{W}_t.$$

Proof: We will show that $Z_1 F \in \mathbb{M}_{2,1}$ first. Remember that $Z_1 = e^{-K}$, where

$$K = \int_0^1 P(X_s)u(X_s)\cdot \mathrm{d}W_s + \frac{1}{2}\int_0^1 |P(X_s)u(s,X)|^2\,\mathrm{d}s.$$

Lemma 2.1 implies that $\int_0^1 P(X_s)u(X_s)\cdot \mathrm{d}W_s \in \mathbb{M}_{2,1}$ and Lemma 2.2 implies that $\int_0^1 |P(X_s)u(X_s)|^2\,\mathrm{d}s \in \mathbb{M}_{2,1}$. Hence, $K \in \mathbb{M}_{2,1}$. Since $\mathbb{E}[F^2 e^{2K}]$, $\mathbb{E}[e^{2K}\|\hat{D}F\|^2]$ and $\mathbb{E}[F^2\, e^{2K}\, \|\hat{D}K\|^2]$ are finite by the given assumptions, Proposition 2.1 implies that $Z_1 F \in \mathbb{M}_{2,1}$ satisfying

$$\hat{D}_t Z_1 F = Z_1\left(\hat{D}_t F - F(\hat{D}_t K)\right)$$

and

$$\hat{D}_t K = -\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}W_s - P(X_t)u(X_t)$$
$$- \int_t^1 \hat{D}_t(P(X_s)u(X_s))\cdot P(X_s)u_s\,\mathrm{d}s.$$

Let $Y_t = \mathbb{E}_{\mathbb{Q}}[F \mid \mathcal{F}_t(X)]$ and note that

$$\begin{aligned} Z_t^{-1} &= \exp\left\{\int_0^t P(X_s)u(X_s)\cdot \mathrm{d}W_s + \frac{1}{2}\int_0^t |P(X_s)u(s,X)|^2\,\mathrm{d}s\right\} \\ &= \exp\left\{\int_0^t P(X_s)u(X_s)\cdot \mathrm{d}\widetilde{W}_s - \frac{1}{2}\int_0^t |P(X_s)u(s,X)|^2\,\mathrm{d}s\right\}. \end{aligned} \tag{16}$$

Then, we get

$$\begin{aligned} Y_t &= Z_t^{-1}\mathbb{E}\left[Z_1 F \mid \mathcal{F}_t(X)\right] \\ &= Z_t^{-1}\left\{\mathbb{E}[Z_1 F] + \int_0^1 P(X_s)\mathbb{E}\left[\hat{D}_s \mathbb{E}\left[Z_1 F \mid \mathcal{F}_t(X)\right] \mid \mathcal{F}_s(X)\right]\cdot \mathrm{d}W_s\right\} \\ &= Z_t^{-1}\left\{\mathbb{E}[Z_1 F] + \int_0^t P(X_s)\mathbb{E}\left[\hat{D}_s(Z_1 F) \mid \mathcal{F}_s(X)\right]\cdot \mathrm{d}W_s\right\} =: Z_t^{-1}A_t, \end{aligned} \tag{17}$$

where we have applied the formula of Ref. [26, Thm. 8] to $\mathbb{E}[Z_1 F \mid \mathcal{F}_t(X)]$ and used the fact $\hat{D}_s \mathbb{E}[Z_1 F \mid \mathcal{F}_t(X)]$ is $\mathcal{F}_t(X)$-measurable for $t > s$ and equal to 0 otherwise. From (16) and (17), we get

$$\begin{aligned}
\mathrm{d}Z_t^{-1} &= Z_t^{-1}P(X_t)u(X_t)\cdot \mathrm{d}\widetilde{W}_t,\\
\mathrm{d}A_t &= P(X_t)\mathbb{E}\left[\hat{D}_t(Z_1F)\mid \mathcal{F}_t\right]\cdot \mathrm{d}W_t,\\
\mathrm{d}A_t\,\mathrm{d}Z_t^{-1} &= Z_t^{-1}P(X_t)u(X_t)\cdot P(X_t)\mathbb{E}\left[\hat{D}_t(Z_1F)\mid \mathcal{F}_t\right]\mathrm{d}t.
\end{aligned}$$

Since $dY_t = A_t\,\mathrm{d}Z_t^{-1} + Z_t^{-1}\,\mathrm{d}A_t + \mathrm{d}A_t\,\mathrm{d}Z_t^{-1}$, it follows that

$$\begin{aligned}
\mathrm{d}Y_t &= \left\{\mathbb{E}[Z_1F] + \int_0^t P(X_s)\mathbb{E}\left[\hat{D}_s(Z_1F)\mid \mathcal{F}_s(X)\right]\cdot \mathrm{d}W_s\right\}Z_t^{-1}P(X_t)u(X_t)\cdot \mathrm{d}\widetilde{W}_t\\
&\quad + Z_t^{-1}P(X_t)\mathbb{E}\left[\hat{D}_t(Z_1F)\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}W_t\\
&\quad + Z_t^{-1}P(X_t)u_t\cdot P(X_t)\mathbb{E}\left[\hat{D}_t(Z_1F)\mid \mathcal{F}_t(X)\right]\mathrm{d}t\\
&= Y_tP(X_t)u_t\cdot \mathrm{d}\widetilde{W}_t + Z_t^{-1}P(X_t)\mathbb{E}\left[\hat{D}_t(Z_1F)\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t\\
&= P(X_t)u_t\mathbb{E}_{\mathbb{Q}}\left[F\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t\\
&\quad + P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_tF\mid \mathcal{F}_t(X)\right]\cdot \widetilde{W}_t\\
&\quad + P(X_t)\mathbb{E}_{\mathbb{Q}}\left[F\left\{-\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u_s\right)\cdot \mathrm{d}W(s) - P(X_t)u_t\right\}\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t\\
&\quad + P(X_t)\mathbb{E}_{\mathbb{Q}}\left[F\left\{-\int_t^1 \hat{D}_t(P(X_s)u(X_s))\cdot P(X_s)u(X_s)\,\mathrm{d}s\right\}\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t\\
&= P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_tF - F\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t.
\end{aligned}$$

In view of $Y_1 = \mathbb{E}_{\mathbb{Q}}[F\mid \mathcal{F}_1] = F$ and $Y_0 = \mathbb{E}_{\mathbb{Q}}[F\mid \mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}}[F]$, we get

$$F = \mathbb{E}_{\mathbb{Q}}\left[F\right] + \int_0^1 P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_tF - F\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s\mid \mathcal{F}_t(X)\right]\cdot \mathrm{d}\widetilde{W}_t.$$

■

Now, we are ready to find the hedging strategy for our market model, when a payoff function G is given. Letting $F := U_1^\theta = e^{-\int_0^1 r_s\,\mathrm{d}s}G(X)$ in (15), which needs to hold for the aim of finding a replicating portfolio, and substituting F in the result of Theorem 3.1 with the assumption that its conditions are satisfied, we get

$$\begin{aligned}
U_1^\theta = \mathbb{E}_{\mathbb{Q}}[e^{-\int_0^1 r_s\,\mathrm{d}s}G] + \int_0^1 P(X_t)\mathbb{E}_{\mathbb{Q}}\Big[&\hat{D}_t(e^{-\int_0^1 r_s\,\mathrm{d}s}G)\\
&-e^{-\int_0^1 r_s\,\mathrm{d}s}G\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s\mid \mathcal{F}_t(X)\Big]\mathrm{d}\widetilde{W}_t. \qquad (18)
\end{aligned}$$

On the other hand, in view of (14) and as $U_0^\theta = V_0^\theta$ by definition, we have

$$U_1^\theta = V_0^\theta + \int_0^1 e^{-\int_0^t r_s\,\mathrm{d}s}\theta_t \sigma(X_t)\,\mathrm{d}\widetilde{W}_t.$$

In comparison with (18), we conclude that

$$V_0^\theta = \mathbb{E}_{\mathbb{Q}}[e^{-\int_0^1 r_s\,\mathrm{d}s} G(X)] \tag{19}$$

and the hedging strategy θ_t solves

$$\begin{aligned}\sigma^*(X_t)\theta_t^* = e^{\int_0^t r_s\,\mathrm{d}s} P(X_t)\mathbb{E}_{\mathbb{Q}}\Big[\hat{D}_t(e^{-\int_0^1 r_s\,\mathrm{d}s} G(X)) \\ - e^{-\int_0^1 r_s\,\mathrm{d}s} G(X) \int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s \mid \mathcal{F}_t(X)\Big]\end{aligned} \tag{20}$$

at each time $t \in [0,1]$. Note that u appears in $\mathbb{Q}$ and $\widetilde{W}$ in the above equation, and we can obtain a unique and adapted strategy if we replace θ_t by $P(X_t)\theta_t$. Since the interest rate is deterministic, Equation (20) reduces to

$$\begin{aligned}\sigma^*(X_t)\theta_t^* &= e^{-\int_t^1 r_s\,\mathrm{d}s} P(X_t) \\ &\times \mathbb{E}_{\mathbb{Q}}\Big[\hat{D}_t G(X) - G(X)\int_t^1 P(X_s)\hat{D}_t\left(P(X_s)u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s \mid \mathcal{F}_t(X)\Big].\end{aligned} \tag{21}$$

Remark 3.1: The interest rate r_t can be taken as random only as a function of the asset prices, denoted by $r(X_t)$. The solution of (9) needs to be in the form of $u(X_t)$ so that u is adapted to $\mathcal{F}(X)$ and Z_t in (11) can be a martingale.

Remark 3.2: If σ is non-degenerate, then the projection map $P(X_s)$ is just the identity map, $\mathcal{F}(X) = \mathcal{F}(W)$ and $\hat{\nabla} = \nabla$. Assuming that σ is a square matrix for simplicity, we can rewrite (21) as

$$\begin{aligned}\theta_t^* = \sigma^*(X_t)^{-1}\, e^{\int_0^t r_s\,\mathrm{d}s}\mathbb{E}_{\mathbb{Q}}\Big[D_t(e^{-\int_0^1 r_s\,\mathrm{d}s} G(X)) \\ - e^{-\int_0^1 r_s\,\mathrm{d}s} G(X)\int_t^1 D_t\left(u(X_s)\right)\cdot \mathrm{d}\widetilde{W}_s \mid \mathcal{F}_t(W)\Big],\end{aligned}$$

which is the same as Ref. [18, Eq. (3.10)].

4. Examples for payoff function

In this section, the hedging strategy is worked out for some examples of the payoff function to demonstrate the formulas. Here, we indicate the terminal time by T. From the point of view of option pricing with a claim $G(X)$, the analysis of a hedging strategy θ can readily be used. The claim $G(X)$ is attainable if $\mathbb{E}[G(X)] < \infty$ and there exists an admissable strategy θ_t, $0 \leq t \leq 1$ [12, p. 303]. Then, the price of the claim at time t is given by

$$e^{\int_t^1 r_s \, \mathrm{d}s} \mathbb{E}_{\mathbb{Q}}[G \mid \mathcal{F}_t(X)]$$

with the assumption that the interest rate r is deterministic, and in particular at time 0, the price is equal to (19). We consider various claims below as suitable for demonstration of our results.

4.1. Vanilla options

Consider the one-dimensional Black–Scholes model

$$\begin{aligned} \mathrm{d}X_t^0 &= \rho X_t^0 \, \mathrm{d}t, \quad X_0^0 = 1, \\ \mathrm{d}X_t^1 &= \mu X_t^1 \, \mathrm{d}t + \sigma X_t^1 \, \mathrm{d}W_t^1, \quad X_0^1 > 0, \end{aligned}$$

where $\rho, \mu, \sigma > 0$. The equivalent martingale measure for this one-dimensional model is $\mathbb{R}(A) = \mathbb{E}[Y_T 1_A]$, where

$$Y_t = \exp\left\{ -\int_0^t u \, \mathrm{d}W_s^1 - \frac{1}{2}\int_0^t u^2 \, \mathrm{d}s \right\}, \quad 0 \leq t \leq T$$

and $u = (\mu - \rho)/\sigma$. For this model, the hedging strategy is given by

$$\theta_t = e^{\rho t} \sigma^{-1} (X_t^1)^{-1} \mathbb{E}_{\mathbb{R}} \left[D_t G \mid \mathcal{F}_t(W^1) \right], \tag{22}$$

where $D_t G = (\mathrm{d}/\mathrm{d}t)\nabla G$ and ∇ is Gross–Sobolev derivative defined for the functionals of W^1 [6, Ex. 4.11]. Clearly, this model is not an example of the degenerate case. However, we can rewrite it as a degenerate model by artificially taking $X^2 := X^0$ as a toy example to demonstrate our formulas. Degeneracy of the diffusion parameter appears intrinsically in this example of a single risky asset. In (6), take $r_t = \rho$

$$b(X_t) = (\mu X_t^1, \rho X_t^2), \quad \sigma(X_t) = \begin{bmatrix} \sigma X_t^1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Observe that

$$P(X_t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P(X_t) u(X_t) = \begin{bmatrix} \dfrac{\mu - \rho}{\sigma} \\ 0 \end{bmatrix}, \quad \hat{D}_t P(X_t) u(X_t) = (0, 0).$$

If we substitute these in (21), we get

$$\begin{bmatrix} \sigma X_t^1 & 0 \\ 0 & 0 \end{bmatrix} \theta_t^* = e^{\rho t} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbb{E}_{\mathbb{Q}} \left[\hat{D}_t G \mid \mathcal{F}_t(X) \right].$$

Moreover, it is easy to see that $\mathcal{F}_t(X) = \mathcal{F}_t(W^1)$, $\mathbb{Q}(A) = \mathbb{R}(A)$, for each $A \in \mathcal{F}_t(X)$, and $\hat{D}_t G = (D_t G, 0)$, where derivative $D_t G$ is taken in the sense of Malliavin calculus for Brownian motion. Hence, the hedging strategy is $\theta_t^* = [\theta_t, 0]$ with θ_t of (22). When G is taken to be the European option $G = (X_T^1 - K)^+$, we have $G \in \mathbb{M}_{2,1}$ by Proposition 2.1. The hedging portfolio for G is given by

$$\theta_t^* = \begin{bmatrix} e^{\rho t}\sigma^{-1}(X_t^1)^{-1}\mathbb{E}_{\mathbb{R}}[D_t G \mid \mathcal{F}_t(X)] \\ 0 \end{bmatrix}$$

in this case, equivalent to the result in Ref. [6, Ex. 4.11].

An exchange option gives the right to put a predefined risky asset and call the other risky asset, as introduced in Ref. [15]. The price and hedging strategy have been calculated in Ref. [16] via Malliavin calculus in the non-degenerate case. To find the hedging strategy, we need to compute $\hat{D}_t G(X)$. For the payoff function $G(X) = (X_T^1 - X_T^2)^+$, it is given by $\hat{D}_t G(X) = 1_A(\hat{D}_t X_T^1 - \hat{D}_t X_T^2)$ for $\mathrm{d}t \times \mathrm{d}\mu$-a.e., where $A = \{X_T^1 - X_T^2 > 0\}$.

A power option depends on the underlying asset price raised to some power [31]. The payoff of call power option is given by $G(X) = \sum_{i=1}^n ((X_T^i)^k - K_i^k)^+$. Then, we have $\hat{D}_t G(X) = \sum_{i=1}^n k\, 1_{A_i}(X_T^i)^{k-1}\hat{D}_t X_T^i$ for $\mathrm{d}t \times \mathrm{d}\mu$-a.e., where $A_i = \{(X_T^i)^k - K_i^k > 0\}$.

4.2. *Exotic options*

Exotic options are a class of options contracts in that the value of an option and the expiration date depend on the prices of the assets on the whole period [12]. Since exotic options can be customized to the needs of the investor, it provides various investment alternatives. We will examine exotic options in a two-dimensional market model with terminal time T. Without loss of generality, we assume that σ_{11} in (6) is away from zero. Let $\Delta = \sigma_{11}(X_s)\sigma_{22}(X_s) - \sigma_{12}(X_s)\sigma_{21}(X_s)$. When $\Delta = 0$, the projection map can be written as

$$P(X_s) = \frac{1}{\sigma_{11}^2(X_s) + \sigma_{12}^2(X_s)} \begin{bmatrix} \sigma_{11}^2(X_s) & \sigma_{11}(X_s)\sigma_{12}(X_s) \\ \sigma_{11}(X_s)\sigma_{12}(X_s) & \sigma_{12}^2(X_s) \end{bmatrix}$$

and the projected market price of risk process is given by

$$P(X_s)\,u(X_s) = \frac{b_1(X_s) - r_s X_s^1}{\sigma_{11}^2(X_s) + \sigma_{12}^2(X_s)} \begin{bmatrix} \sigma_{11}(X_s) \\ \sigma_{12}(X_s) \end{bmatrix}$$

in view of (9). Suppose b and σ have bounded partial derivatives, then $P(X_s)u(X_s)$ has the derivative $\hat{D}_t P(X_s)u(X_s) = J_f(X_s)(\hat{D}_t X_s^1, \hat{D}_t X_s^2)$, where J_f is the Jacobian of

$$f(x,y) = \frac{b_1(x,y) - r_s x}{\sigma_{11}^2(x,y) + \sigma_{12}^2(x,y)} \begin{bmatrix} \sigma_{11}(x,y) \\ \sigma_{12}(x,y) \end{bmatrix}$$

and $\hat{D}_t X_s$ solves

$$\hat{D}_t X_s = \int_t^s J_b(X_r)\hat{D}_t(X_r)\,\mathrm{d}r + \int_t^s J_{\sigma_i}(X_r)\hat{D}_t(X_r)\cdot P(X_r)\,\mathrm{d}W_r + \sigma(t, X_t)$$

$\mathrm{d}t \times \mathrm{d}\mu$-a.e. [26, Thm. 5], where σ_i is ith row of matrix $\sigma(X_s)$ and J_b, J_{σ_i} denote the Jacobian matrices of b and σ_i, respectively.

Consider the linear case, that is, $b(X_s) = (b_1(s)X_s^1, b_2(s)X_s^2)$ and

$$\sigma = \begin{bmatrix} \sigma_{11}(s)X_s^1 & \sigma_{12}(s)X_s^1 \\ \sigma_{21}(s)X_s^2 & \sigma_{22}(s)X_s^2 \end{bmatrix}.$$

Then, we have

$$P\,(X_s)\,u\,(X_s) = \frac{b_1\,(s) - r_s}{\sigma_{11}^2\,(s) + \sigma_{12}^2\,(s)} \begin{bmatrix} \sigma_{11}\,(s) \\ \sigma_{12}\,(s) \end{bmatrix}.$$

Clearly, $P(X_s)u(X_s)$ is deterministic, which implies $\hat{D}_t P(X_s)u(X_s) = 0$ and the hedging strategy θ_t at time t solves

$$\sigma^*(X_t)\theta_t^* = e^{-\int_t^T r_s\,\mathrm{d}s} P(X_t)\mathbb{E}_{\mathbb{Q}}\left[\hat{D}_t G \mid \mathcal{F}_t(X)\right]$$

for given payoff function G by (21).

Asian options are options where the payoff depends on the average of the underlying assets. Pricing of Asian options has been performed in Refs. [21,27] by the use of PDEs, but hedging has not been studied. In Ref. [11], Edgeworth expansion is applied to approximate the hedging strategy for Asian options, and in Ref. [1], it is approximated with the strategy for European options. In Ref. [30], Malliavin calculus is used to derive the hedging strategy and the price of Asian option. All these studies have considered one-dimensional market. We will consider Asian call option with floating strike, which pays at time T, the payoff $G(X) = ((1/T)\int_0^T X_s^1\,\mathrm{d}s - KX_T^2)^+$. Proposition 2.1 implies $G \in \mathbb{M}_{2,1}$ and $\hat{D}_t G(X) = 1_A((1/T)\int_t^T \hat{D}_t X_s^1\,\mathrm{d}s - K\hat{D}_t X_T^2)$ for $\mathrm{d}t \times \mathrm{d}\mu$-a.e., where $A = \{(1/T)\int_0^T X_s^1\,\mathrm{d}s - KX_T^2 > 0\}$ and

$$\hat{D}_t X_s^i = \begin{bmatrix} \sigma_{i1}\,(s) \\ \sigma_{i2}\,(s) \end{bmatrix} \exp\left(\int_t^s P(X_r)\sigma_i(r)\cdot \mathrm{d}W_r + \int_t^s b_i(r)\,\mathrm{d}r - \frac{1}{2}\int_t^s |P(X_r)\sigma_i(r)|^2\,\mathrm{d}r\right).$$

Look-back options are a particular type of path-dependent options so that the value of the payoff function depends on the minimum or maximum of the underlying asset price. The price of option and the hedging strategy have been derived in Refs. [2,20] with Malliavin calculus. Defining $M_{0,T}^{X^1} = \sup_{0\le t\le T} X_t^1$, we consider the payoff $G(X) = (M_{0,T}^{X^1} - KX_T^2)^+$, which has the derivative

$$\hat{D}_t G(X) = 1_A\left(\hat{D}_t M_{0,T}^{X^1} - K\hat{D}_t X_T^2\right)$$

with

$$\hat{D}_t M_{0,T}^{X^1} = \begin{bmatrix} \sigma_{i1}(\tau) \\ \sigma_{i2}(\tau) \end{bmatrix} \exp\left(\int_t^\tau P(X_s)\sigma_i(s)\cdot \mathrm{d}W_s + \int_t^\tau b_i(s)\,\mathrm{d}s - \frac{1}{2}\int_t^\tau |P(X_s)\sigma_i(s)|^2\,\mathrm{d}s\right)$$

for $\mathrm{d}t \times \mathrm{d}\mu$-a.e., where $\tau = \inf\{t:\ X_t^1 = M_{0,T}^{X^1}\}$ and $A = \{M_{0,T}^{X^1} - KX_T^2 > 0\}$. Here, we have used the approach in Ref. [14, p. 55] to calculate the derivative of $M_{0,T}^{X^1}$ as follows. The model considered in Ref. [14] is one-dimensional, has constant volatility and Malliavin calculus for Brownian motion is used to calculate the hedging portfolio. However, the idea can be applied easily to our case. For each $m \in \mathbb{N}$, choose a partition $\pi_m =$

$\{0 = s_1 < \cdots < s_m = T\}$ so that $\pi_m \subseteq \pi_{m+1}$ and $\cup_m \pi_m$ is dense in $[0, T]$. Define φ_m by $\varphi_m(\underline{x}) = \max_{1 \le i \le m} x_i$. Then, $\varphi_m(X^1_{\pi_m}) \to M^{X^1}_{0,T}$ in L^2. Since the function φ_m is Lipschitz, $\varphi_m(X^1_{\pi_m}) \in \mathbb{M}_{2,1}$ by Remark 2.2. Moreover, $\sup_m \|\hat{D}_t \varphi_m(X^1_{\pi_m})\|^2 \le \sup_t \sigma_t T \|M^{X^1}_{0,T}\|^2_{L^2(\mu)}$ and this implies $M^{X^1}_{0,T} \in \mathbb{M}_{p,1}$. Note that

$$\hat{D}_t \varphi_m(X^1_{\pi_m}) = \begin{bmatrix} \sigma_{i1}(\tau_m) \\ \sigma_{i2}(\tau_m) \end{bmatrix} \exp\left(\int_t^{\tau_m} P(X_s)\sigma_i(s) \cdot \mathrm{d}W_s \right.$$
$$\left. + \int_t^{\tau_m} b_i(s)\,\mathrm{d}s - \frac{1}{2}\int_t^{\tau_m} |P(X_s)\sigma_i(s)|^2\,\mathrm{d}s \right),$$

where $\tau_m = \min\{t_j : X^1_{t_j} = \varphi_m(X^1_{\pi_m})\}$. For each m, τ_m is a measurable function and $\tau_m \to \tau$ a.s. When σ_{i1} is assumed to be continuous, the result follows as $\hat{D}$ is a closed operator.

5. Conclusion

We have used Malliavin calculus for degenerate diffusions to derive the hedging portfolio for a given payoff function in a semimartingale market model. The prices are assumed to follow a multidimensional diffusion process with a singular volatility matrix $\sigma\sigma^*$, where σ is taken to be a function of the prices with no extra randomness. In applications, the estimation of the volatility $\sigma\sigma^*$ is crucial from financial data, which may be accomplished for example through estimation of the covariance of the price time series, yielding an estimate for $\sigma\sigma^*$. In the case of degeneracy of the estimate, this can now be taken care of by the results of the present paper.

From a theoretical point of view, the projection operator P to the range space of σ^* plays a crucial role in our results. We have shown that the hedging portfolio can be obtained uniquely as a solution to a system of linear equations by projecting any solution of the system with P. For this purpose, a version of the Clark–Ocone type formula for functionals of degenerate diffusions is proved under an equivalent change of measure. As demonstration of our results, intermediate calculations for the Gross–Sobolev type derivative of the payoff function of the prices are given in the case of exotic options in a two-dimensional linear model.

As future work, the stochastic volatility models can be considered on the basis of the results established in the present paper. Moreover, the hedging strategy can be made more explicit under simplifying assumptions on the coefficients as in previous studies.

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