Choice with Affirmative Action*

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Abstract

A choice rule with affirmative action decides on the recipients of a limited number of identical objects by reconciling two objectives: respecting a priority ordering over the applicants and supporting a minority group. We extend the standard formulation of a choice problem by incorporating a type function and a priority ordering, and introduce *monotonicity* axioms on how a choice rule should respond to variations in these parameters. We show that monotonic and substitutable affirmative action rules are the ones that admit a *bounded reserve representation*. As a prominent class of choice rules that satisfy the *monotonicity* axioms, we characterize *lexicographic affirmative action rules* that are prevalent both in the literature and in practice. Our axiomatic approach provides a novel way to think about reserve systems and uncovers choice rules that go beyond lexicographic affirmative action rules.

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1 Introduction

When allocating scarce resources, it is common that decision makers aim to support a particular minority group. The minority group can be, for example, low socioeconomic status applicants or applicants from the school's neighborhood when allocating school seats, applicants with disabilities when allocating job positions, or highly educated applicants when allocating visas. It is also common that the objective to support the minority group has to be reconciled with an objective to respect a given priority ordering over applicants, such as an exam-score order in school choice, a merit order in job assignment, or time-of-application order in visa assignment. How can these two potentially conflicting objectives be reconciled?

We model this problem as the choice problem of a decision maker who has a limited number of seats (the capacity) and encounters *a set of applicants* together with a *type profile* indicating who is a minority and a *priority ordering* over the applicants. A choice rule, for each choice problem, chooses a subset of the applicants without exceeding the capacity, with the interpretation that each chosen applicant is allocated a seat.

We focus on choice rules that satisfy *substitutability*: a chosen applicant remains chosen in a smaller set of applicants.[1] We introduce axioms that are based on two simple comparative statics: How should a choice rule respond to (1) improving the priority order of a chosen minority applicant or (2) changing the type of a chosen minority applicant? Our central axioms, namely *monotonicity axioms*, require that if the priority order of a chosen minority applicant is improved or the type of a chosen minority applicant is changed (into a majority), then no other minority applicant should be adversely affected,[2] since such changes mean that the set of intended beneficiaries for affirmative action possibly gets smaller.

Our contribution consists of two characterizations. First, in Theorem 1, we show that the class of monotonic and substitutable choice rules for affirmative action is the class of choice rules that admit a *bounded reserve representation*: at each problem, in addition

[1] *Substitutability* of institutions' choice rules is typically indispensable when designing centralized clearinghouses to allocate seats from multiple institutions, since *substitutability* ensures *stability* of the outcome (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008).

[2] That is, all other minority applicants who were chosen before the change should still be chosen after the change.

to all top-tier[3] minority applicants, a certain number of bottom-tier[4] minority applicants, called a *reserve number*, are chosen. It is such that the reserve number of each problem only depends on the type configuration of the top-tier, and the reserve number changes by at most one in response to a priority improvement or a type change. This result provides a fairly general way to think about reserve systems that are common in practice.

Second, in Theorem 2, we characterize a prominent class of monotonic and substitutable choice rules for affirmative action called *lexicographic affirmative action rules* that are prevalent both in the recent resource allocation literature and in practice.[5] These choice rules allocate seats sequentially by following a fixed list of priority orderings that rank applicants either according to a given priority ordering or according to the *affirmative priority ordering* in which the minority applicants are prioritized. The highest priority applicant according to the first priority ordering is chosen. Then, among the remaining applicants, the highest priority applicant according to the second priority ordering is chosen, and so on. The additional axiom in the characterization is *consistency in effective type changes*: if changing the type of a minority applicant is *effective*, i.e., if it results in new chosen minority applicants, then changing the type of any minority applicant with a lower priority must also be *effective* (unless all minority applicants are already chosen).[6]

By providing a novel and flexible way to think about reserve systems, our results also pave the way for discovering new affirmative action rules. To highlight this, we introduce a class of monotonic and substitutable choice rules called *stepwise-adjusted–reserves rules* that are not lexicographic but might be more reasonable to use in applications in which some additional flexibility is desirable. For example, a policy maker can reserve different numbers of seats for affirmative action depending on whether there are *few*, *many*, or *enough* top-tier minority students, while defining these categories in a rather flexible way. This is not possible with a lexicographic affirmative action rule as we discuss in Section 5.1, which indicates that choice with affirmative action goes beyond lexicographic choice.

[3]Applicants who are top-q ranked according to the priority ordering, where q denotes the capacity.

[4]Applicants who are not top-q ranked.

[5]In some applications, a certain group of agents is supported for reasons that do not necessarily include "affirmative action" as it is commonly understood. Nevertheless, our results are applicable as long as respecting a priority ordering is reconciled with supporting a certain group.

[6]We show the independence of the axioms in Appendix B.

1.1 Related Literature

When designing mechanisms to allocate resources from multiple institutions, the design of institutional choice rules may potentially have more importance than designing the rest of the mechanism.[7] However, how to design these institutional choice rules has been relatively under-examined in the literature. A recent exception is Echenique and Yenmez (2015), who provide axiomatic characterizations of different choice rules for schools that face applicants of multiple possible types and aim to achieve *diversity*.[8] Although our basic axioms such as *capacity-filling* and *substitutability* overlap, we differ by incorporating the type profile and the priority ordering as parameters of a choice problem and introduce axioms novel to this context that aim to formulate how to implement *affirmative action* policies. In particular, we introduce the *monotonicity* axioms on how the choice rule should respond to type or priority changes. *Monotonicity* axioms are in the same spirit with the *population monotonicity* axiom, which was introduced by Thomson (1983) in the context of bargaining theory and became one of the standard axioms in the resource allocation literature (Moulin, 1990; Thomson, 1995; Ehlers et al., 2002).

Affirmative action has been an important topic especially in the matching context starting with the seminal paper Abdulkadiroğlu and Sönmez (2003), which includes a section on controlled school choice. Among others, lexicographic choice comes up in school choice rules in several school districts in the US such as in Boston (Dur et al., 2018) and in Chicago (Dur et al., 2020), choice rules for government job positions and seats at publicly funded educational institutions in India (Aygün and Turhan, 2020; Aygün and Turhan, 2017; Sönmez and Yenmez, 2020), H-1B visa allocation for U.S. immigration (Pathak et al., 2020), and choice rules for allocating ventilators during a pandemic (Pathak et al., 2020).[9] A version of lexicographic choice rules has been implemented also in Israeli "Mechinot" gap-year program (Gonczarowski et al., 2019). Although these applications include different institutional constraints and therefore result in different details in how the corresponding choice rules operate, the lexicographic feature remains common.

[7] In a recent study, Leshno and Lo (2021) argue that the choice of priorities can have larger welfare implications than the choice of mechanism.

[8] Imamura (2020) formalizes the trade-off between meritocracy and diversity in the setting of Echenique and Yenmez (2015).

[9] Pathak et al. (2020) use the "sequential reserve matching" terminology.

Decision making based on a lexicographic order has been widely studied in the literature.[10] In the context of allocating multiple identical objects under a capacity constraint, to our knowledge, Kominers and Sönmez (2016) is the first study that includes lexicographic choice rules.[11] Chambers and Yenmez (2017, 2018) are other related recent papers.[12] Chambers and Yenmez (2018) particularly note that lexicographic choice rules satisfy *capacity-filling* and *substitutability*, but do not provide an axiomatic characterization. Doğan et al. (2021) also consider choice rules based on lexicographic choice procedures, but the class of choice rules characterized is different than the class we study here since they include *variable* capacity constraints and do not include a minority group or axioms for affirmative action.

2 The Model and Notation

In what follows, we refer to the decision maker as a *school* that wants to allocate a limited number of *seats*, and refer to the applicants as the *students*.

Let $\mathcal{S}$ be a nonempty finite set of students. A (choice) **problem** is a triple $(S, \tau, \succ)$ such that

i. $S \subseteq \mathcal{S}$ is a set of students,
ii. $\tau : S \to \{0, 1\}$ is a **type function** where 1 denotes the minority type and 0 denotes the majority type (i.e., $s \in S$ is a minority student if $\tau(s) = 1$ and a majority student if $\tau(s) = 0$).
iii. $\succ$ is a (strict) **priority ordering**, which is a complete, transitive, and anti-symmetric binary relation on S.

Let $S^m(\tau)$ denote the set of minority students and $S^M(\tau)$ denote the set of majority students in S with respect to τ. When the type function in question is clear, we simply write S^m and S^M.

Given $S \subseteq \mathcal{S}$, let $\mathcal{T}(S)$ denote the set of all type functions and let $\Pi(S)$ denote the set of

[10]See, among others, Chipman (1960), Fishburn (1974), Blume et al. (1989), Rubinstein (1998), Manzini and Mariotti (2012) and Houy and Tadenuma (2009). Blume et al. (1989) note that the idea of a *lexicographic utility function* was mentioned in Von Neumann and Morgenstern (1947).

[11]Kominers and Sönmez (2016) use the "slot-specific priority" terminology, in the general matching with contracts framework.

[12]To our knowledge, an earlier working paper version of Chambers and Yenmez (2017) is the first paper to use the lexicographic choice terminology in this context.

all possible priority orderings over S. Note that $(S, \tau, \succ)$ is a problem if and only if $S \subseteq \mathcal{S}$, $\tau \in \mathcal{T}(S)$, and $\succ \in \Pi(S)$.

Given $S \subseteq \mathcal{S}$ and a type function $\tau \in \mathcal{T}(S)$, for each $S' \subset S$, let $\tau|_{S'}$ be the **restriction** of τ to S', i.e., $\tau|_{S'} \in \mathcal{T}(S')$ and for each $s \in S'$, $\tau|_{S'}(s) = \tau(s)$. Similarly, given $S \subseteq \mathcal{S}$ and a priority ordering $\succ \in \Pi(S)$, let $\succ|_{S'}$ be the **restriction** of $\succ$ to S'.

Given $S \subseteq \mathcal{S}$, $\succ \in \Pi(S)$, and $k \in \{1, \ldots, |S|\}$, let $r_k(\succ)$ denote the k-th ranked student in $\succ$, i.e., $s = r_k(\succ)$ if and only if $|\{s' \in S \setminus \{s\} : s' \succ s\}| = k - 1$. Also, let $rank(s, \succ)$ denote the rank of s at $\succ$, i.e., $rank(s, \succ) = k$ if and only if $s = r_k(\succ)$.

A **choice rule** C for the school maps each problem $(S, \tau, \succ)$ to a nonempty subset $C(S, \tau, \succ) \subseteq S$. Let $q \geq 3$ denote the **capacity** of the school. It is assumed that for each problem $(S, \tau, \succ)$, $|C(S, \tau, \succ)| \leq q$. We use $C^m(S, \tau, \succ)$ and $C^M(S, \tau, \succ)$ to denote the chosen minority and majority students, respectively.

Given $S \in \mathcal{S}$, a pair of type functions $\tau, \tau' \in \mathcal{T}(S)$, and a student $s \in S$, we say that τ' is **obtained from τ by changing the type** of s if the type of s is different while the types of the other students stay the same. That is, $\tau(s) \neq \tau'(s)$ and for each $s' \in S \setminus \{s\}$, $\tau(s) = \tau'(s)$.

Given $S \in \mathcal{S}$, a pair of priority orderings $\succ, \succ' \in \Pi(S)$, and a student $s \in S$, we say that $\succ'$ is an **improvement over** $\succ$ for s if, when we move from $\succ$ to $\succ'$, the priority order of s weakly improves relative to each other student and strictly improves relative to at least one student, while the priority relation within other students stays the same. That is, for each $s' \in S$, if $s \succ s'$ then $s \succ' s'$; there exists $s' \in S \setminus \{s\}$ such that $s' \succ s$ and $s \succ' s'$; and for each $s', s'' \in S \setminus \{s\}$, $s' \succ s''$ if and only if $s' \succ' s''$.

3 Axioms

3.1 Axioms for substitutable affirmative action

Capacity-filling requires that a student is rejected only if the capacity is full.[13] That is, *capacity-filling* ensures that no resource is wasted, even if there are not enough minority

[13]In the literature, *capacity-filling* is also called *acceptance*. Alkan (2001) is the first study which uses the *filling* terminology where he uses the term *quota filling*.

students with relatively low priorities who are in need of affirmative action.[14]

Capacity-filling: For each problem $(S, \tau, \succ)$, $|C(S, \tau, \succ)| = \min\{|S|, q\}$.

Neutrality requires that the choice only depends on the types of the students and the relative priority ordering of the students in the choice set, and not on other characteristics such as their names.

Neutrality: Let $(S, \tau, \succ)$ and $(S', \tau', \succ')$ be a pair of problems such that $|S| = |S'|$ and for each $k \in \{1, \ldots, |S|\}$, the types of the k-th ranked students are the same at both problems, i.e., $\tau(r_k(\succ)) = \tau'(r_k(\succ'))$. Then, for each $k \in \{1, \ldots, |S|\}$, $r_k(\succ) \in C(S, \tau, \succ)$ if and only if $r_k(\succ') \in C(S', \tau', \succ')$.

Our next axiom, *(affirmative) priority-compatibility*, requires that a student is chosen over a higher priority student only if the former student is a minority student and the latter is a majority student.

Priority-compatibility: For each problem $(S, \tau, \succ)$, if $s \in C(S, \tau, \succ)$, $s' \notin C(S, \tau, \succ)$, and $s' \succ s$, then $s \in S^m$ and $s' \in S^M$.

Since the above axioms constitute a minimal set of requirements for a choice rule with affirmative action, in what follows we call a choice rule an **affirmative action rule** if it satisfies *capacity-filling*, *neutrality*, and *priority-compatibility*.

Substitutability requires that a chosen student remains chosen when the set of students shrinks, everything else the same.[15]

Substitutability: For each problem $(S, \tau, \succ)$ and a chosen student $s \in C(S, \tau, \succ)$, we have $s \in C(S', \tau|_{S'}, \succ|_{S'})$ for any $S' \subseteq S$ such that $s \in S'$.

Substitutability of institutions' choice rules is crucial when designing centralized clearing-houses to allocate seats from multiple institutions, since *substitutability* ensures *stability* of the outcome (Hatfield and Milgrom, 2005; Hatfield and Kojima, 2008). In the remainder of this paper, we study *substitutable affirmative action rules*.

[14] There are applications where some seats are exclusively reserved for minority applicants and cannot be used for majority applicants even if the number of minority applicants is less than the number of reserved seats. Choice rules in such applications might fail capacity-filling.

[15] *Substitutability* was first introduced in the choice literature by Chernoff (1954). It has been studied in the literature under different names such as *Chernoff's axiom*, *Sen's* α, *contraction consistency*, or *gross substitutes* (Kelso and Crawford, 1982).

3.2 Monotonicity axioms

We introduce axioms on how a choice rule with affirmative action should respond to changes in the types of students or in the priority orderings of the students. The general principle underlying our axioms is that affirmative action is a limited resource where the minority students are the potential beneficiaries and the *intended beneficiaries* are those minority students with relatively low priorities who are in need of affirmative action.

Our first monotonicity axiom, *monotonicity in priority improvements*, is concerned with an improvement in the priority rank of a minority student, other things equal. Since intended beneficiaries are those minority students who are relatively low ranked, such a change means a potential decrease in the number of intended beneficiaries, and therefore no minority student should be adversely affected: all minority students who were chosen before the change should still be chosen after the change.

Monotonicity in priority improvements (MPI): For each problem $(S, \tau, \succ)$, each $s \in C^m(S, \tau, \succ)$, and each priority ordering $\succ'$ that is an improvement over $\succ$ for s, we have $C^m(S, \tau, \succ) \subseteq C^m(S, \tau, \succ')$.

The second monotonicity axiom, *monotonicity in type changes*, is concerned with a change in the type of a chosen minority student (into a majority student), other things equal. Since such a change again means a potential decrease in the number of intended beneficiaries, no other minority student should be adversely affected: all other minority students who were chosen before the change should still be chosen after the change.

Monotonicity in type changes (MTC): For each problem $(S, \tau, \succ)$, each $s \in C^m(S, \tau, \succ)$, and each type function τ' that is obtained from τ by changing the type of s, we have $C^m(S, \tau, \succ) \setminus \{s\} \subseteq C^m(S, \tau', \succ)$.

4 Monotone and Substitutable Affirmative Action

In this section, we provide a representation for the substitutable affirmative action rules that satisfy the monotonicity axioms. As a stepping stone, first, we introduce a general and intuitive representation for substitutable affirmative action rules, called *reserve representation*.

Given a problem, we refer to the top-q ranked students (recall that q is the capacity) as the **top-tier** students and the remaining students as the **bottom-tier** students. Note that the top-tier students are the ones who would be chosen if there was no affirmative action. Types and rankings of the top-tier students are critical in a *reserve representation*, in that, at each problem, in addition to all the top-tier minority students, a certain number of bottom-tier minority students, called a *reserve number*, are chosen instead of an equal number of top-tier majority students.

Formally, a **reserve function** R associates with each problem $(S, \tau, \succ)$, a non-negative integer $R(S, \tau, \succ)$ that is less than or equal to the number of top-tier majority students. We require that $R(S, \tau, \succ)$ depend only on the types and the rankings of the top-tier students, that is, for each $(S, \tau, \succ)$ and $(S', \tau', \succ')$, if the k-th ranked students in $(S, \tau, \succ)$ and $(S', \tau', \succ')$ are of the same type for each $k \in \{1, \ldots, q\}$, then $R(S, \tau, \succ) = R(S', \tau', \succ')$.

A choice rule C admits a **reserve representation** via a reserve function R if for each problem $(S, \tau, \succ)$, $C(S, \tau, \succ)$ is obtainable as follows:

- choose all the top-tier minority students,
- choose the highest priority bottom-tier minority students until $R(S, \tau, \succ)$ of them are chosen or none of them is left,
- and then choose the highest priority majority students until all seats are filled or no student is left.

Given a choice rule and a problem, consider the set of bottom-tier minority students who are chosen. We say that these minority students are **chosen via affirmative action**, since they would not be chosen if the choice was solely based on the priority. Thus, in a reserve representation, the reserve number of each problem determines the number of minority students chosen via affirmative action in that problem.

Next, we introduce the *bounded reserve representation*, which additionally requires that the reserve number changes by at most one in response to priority or type changes. Formally, a choice rule C has a **bounded reserve representation** if C admits a reserve representation via a reserve function R that satisfies the following condition, which provides the bounds for the representation by requiring that the reserve number either stays the same or increases by one in response to changing the type of a minority student, improving the priority rank of a minority student, or worsening the priority rank of a majority student.

Condition B: For each problem$(S, \tau, \succ)$,

B1. if τ' is obtained from τ by changing the type of a top-tier minority student, then either $R(S, \tau', \succ) = R(S, \tau, \succ)$ or $R(S, \tau', \succ) = R(S, \tau, \succ) + 1$.

B2. if $\succ'$ is obtained from $\succ$ by improving the priority rank of a top-tier minority student, then $R(S, \tau, \succ') = R(S, \tau, \succ)$ or $R(S, \tau, \succ') = R(S, \tau, \succ) + 1$.

B3. if $\succ$ is obtained from $\succ'$ by improving the priority rank of a top-tier majority student,
(i) $R(S, \tau, \succ') = R(S, \tau, \succ)$ or $R(S, \tau, \succ') = R(S, \tau, \succ) + 1$,
(ii) $R(S, \tau, \succ') = R(S, \tau, \succ)$ if the student is one of the $R(S, \tau, \succ)$ lowest-ranked top-tier majority students.[16]

Note that B2 and B3 require that the reserve number changes at most by one, independent of the particular improvement level in the priority rank of a top-tier student. For example, suppose that $\succ'$ is obtained from $\succ$ by improving the priority rank of a top-tier minority student just by one and as a result the reserve number increases by one. Then, suppose that $\succ''$ is obtained from $\succ'$ by improving the priority rank of the same top-tier minority student by one again. Now, it follows from B2 that the reserve number cannot increase further, since $\succ''$ can be obtained from $\succ$ by improving the priority rank of a top-tier minority student by two.

Theorem 1. *A choice rule is a substitutable affirmative action rule that satisfies monotonicity in priority improvements and monotonicity in type changes if and only if it admits a bounded reserve representation.*

The proof is in Appendix A.1. It is not difficult to see that each substitutable affirmative action rule admits a reserve representation where the reserve is independent from the configuration of the bottom-tier. The crucial insight here is that the reserve can increase by *at most one* in response of a type or a priority change.

The proof is build on the observation that the effects of $B1 - B3$ can be imitated by a removal of a student. For some intuition, consider a choice rule C and a problem $(S, \tau, \succ)$ such that there is only one top-tier minority student, say s. Suppose that the type of s is changed into a majority. Now, if the $(q+1)$-th ranked student is a majority student, then this type change can be imitated by removing s from the set of applicants S. To see this,

[16] That is, he is a top-tier majority student and the number of top-tier majority students who have lower priority than him is less than $R(S, \tau, \succ)$.

note that after both changes all top-tier students are majority students in the resulting problems, and therefore the types and rankings of the top-tiers are the same. Now, to see that the change in reserves must be limited by one in case of a type change, note that, by *substitutability*, all the chosen students will remain chosen after the removal of s from S. Then, *capacity-filling* limits the change in reserves by one.

5 Lexicographic Affirmative Action Rules

A prominent class of substitutable affirmative choice rules is the class of rules that admit a *lexicographic representation*. We call these choice rules **lexicographic affirmative action rules**. A choice rule admits a **lexicographic representation** if there exists a function $l : \{1, \ldots, q\} \to \{0, 1\}$, called a **lexicographic order**, such that for each choice problem $(S, \tau, \succ)$, $C(S, \tau, \succ)$ is obtainable by the following procedure. First, let $\succ^a$ denote the **affirmative priority ordering** obtained from $\succ$ by moving the minority students to the top of $\succ$, while keeping the relative orderings among the minority students and among the majority students the same.

Lexicographic Procedure:

Step 1*:* If $l(1) = 0$, then choose the highest priority student. If $l(1) = 1$, then choose the highest $\succ^a$-priority student.

Steps $k \geq 2$*:* If $l(k) = 0$, then choose the highest priority student among the remaining students. If $l(k) = 1$, then choose the highest $\succ^a$-priority student among the remaining students.

An interpretation of the lexicographic order is that it fixes an ordering of the available seats, from Seat 1 to Seat q, and labels each seat k either as an **open seat** ($l(k) = 0$) which will be allocated based on the given priority ordering $\succ$ or as a **reserve seat** ($l(k) = 1$) which will be allocated based on the *affirmative priority ordering* $\succ^a$. In turn, the lexicographic procedure allocates seats sequentially according to the lexicographic order.

5.1 An axiomatization of lexicographic affirmative action

We introduce a new axiom and characterize the whole class of lexicographic affirmative action rules. First, remember that the monotonicity axioms require the following. If the type of a chosen minority student is changed or his priority rank is improved, no minority student is adversely affected, i.e., if such a change affects the choice, it should only be by choosing new minority students. Our next axiom requires consistency in the way such changes affect the choice.

Given a choice problem, changing the type of a chosen minority student is called *effective* if it results in new chosen minority students. That is, given a problem $(S, \tau, \succ)$ and $s \in C^m(S, \tau, \succ)$, changing the type of s is **effective** if $C^m(S, \tau', \succ) \setminus C^m(S, \tau, \succ) \neq \emptyset$ where τ' that is obtained from τ by changing the type of s.

The next axiom, *consistency in effective type changes*, requires that if changing the type of a chosen minority student is *effective*, then a following change in the type of a lower ranked chosen minority student is also *effective* (unless all minority students are already chosen).

Consistency in effective type changes (CTC): Let $(S, \tau, \succ)$ be a given problem and $s, s' \in C^m(S, \tau, \succ)$ be such that $s \succ s'$. Let τ' be obtained from τ by changing the type of s and τ'' be obtained from τ by changing the types of both s and s'. If $C^m(S, \tau', \succ) \setminus C^m(S, \tau, \succ) \neq \emptyset$, then we have $C^m(S, \tau'', \succ) \setminus C^m(S, \tau', \succ) \neq \emptyset$ unless $C^m(S, \tau', \succ) = S^m(\tau')$.

The motivation behind *CTC* is the following. The fact that changing the type of a chosen minority student is *effective* reveals that every chosen minority student with a lower priority is chosen via affirmative action both before and after the type change. Hence, a further change in the type of a lower priority minority student should be *effective* as well.

Theorem 2. *A choice rule is a substitutable affirmative action rule that satisfies monotonicity in priority improvements, monotonicity in type changes, and consistency in effective type changes if and only if it admits a lexicographic representation. The lexicographic representation is unique.*

To prove Theorem 2, we first construct a bounded reserve representation for a given lexicographic affirmative action rule (see the *lexicographic-to-reserve algorithm* in Appendix

A.2). Then, we use this construction to prove that a choice rule that satisfies the *monotonicity* axioms and *CTC* must coincide with a lexicographic affirmative action rule. The proof of Theorem 2 is in Appendix A.2. Here, we provide an example of a choice rule that admits a bounded reserve representation, but violates the CTC axiom and therefore does not admit a lexicographic representation. In our discussion of the example, we provide an intuition for the role that CTC plays in the proof of Theorem 2. Then, building on this example, we formulate a new class of choice rules demonstrating that substitutable affirmative action rules go beyond lexicographic choice rules in a way that can be relevant for applications.

Example 1. Let $q = 100$. Let C be the choice rule with a reserve representation via the following reserve function R. For each problem $(S, \tau, \succ)$, if the number of top-tier minority students is

- at most 10, then $R(S, \tau, \succ) = 2$;
- more than 10 but at most 20, then $R(S, \tau, \succ) = 1$;
- more than 20, then $R(S, \tau, \succ) = 0$.

It is easy to check that this is a bounded reserve representation. The choice rule C has a simple structure and a clear interpretation: two, one, or zero minority students are chosen via affirmative action if there are *few*, *many*, or *enough* top-tier minority students, respectively, where the few, many, and enough categories are defined by the thresholds 10 and 20.

To see that C violates *CTC*, consider a problem $(S, \tau, \succ)$ such that there are exactly 11 top-tier minority students (their exact ranks are not important) and there are at least two bottom-tier minority students. Let τ' be obtained from τ by changing the type of the minority student with the highest priority. Now, there are 10 top-tier minority students and the reserve number increases from 1 to 2. Therefore, this type change is *effective*, since one of the bottom-tier minority students becomes chosen via affirmative action. Next, consider τ'' that is obtained from τ' by a following change in the type of another top-tier minority student. Now, there are 9 top-tier minority students and the reserve number remains as 2. Therefore, the following change is not *effective*, since no additional bottom-tier minority student is chosen. Hence, C violates *CTC* and does not admit a lexicographic representation.

The critical feature of the above example is that, when we start from a problem with no top-tier minority student and change the types of the majority students to minority *one by one*, the reserve number stays constant until a threshold, and decreases after the threshold. The monotonicity axioms, in particular MTC, requires the reserve number to change by at most one whenever it changes, but does not preclude it from staying constant in some intervals. On the other hand, CTC rules out the possibility that the reserve number increases in response to the type change of a chosen minority, yet remains constant in response to a following change in the type of a lower ranked chosen minority student.

Building on Example 1, we obtain a general class of choice rules that are not lexicographic but admit a bounded reserve representation. The common feature is that reserves are stepwise adjusted by increments of one.

Stepwise-adjusted–reserves rule: A choice rule C admits a stepwise-adjusted–reserves representation if it admits a reserve representation via a reserve function R such that there exist thresholds $0 \leq t_1 \leq \ldots \leq t_r \leq q$, and for each problem $(S, \tau, \succ)$,

- if the number of top-tier minority students is in $[0, t_1]$, then $R(S, \tau, \succ) = r$,
- for each $i \in \{1, \ldots, r-1\}$, if the number of top-tier minority students is in $(t_i, t_{i+1}]$, then $R(S, \tau, \succ) = r - i$,
- if the number of top-tier minority students is in $(t_r, q]$, then $R(S, \tau, \succ) = 0$.[17]

The class of stepwise-adjusted–reserves rules include some lexicographic choice rules. For example, let $q = 100$ and consider C that admits a stepwise-adjusted–reserve representation with thresholds $0, 1, 2, 3, 4$. Observe that this choice rule is equivalent to the lexicographic affirmative action rule where the reserve seats are the first 5 seats. In fact, the stepwise-adjusted–reserves rules are generalizations of lexicographic affirmative action rules where reserve seats precede open seats: they share the common feature that the number of top-tier minority students, irrespective of their exact ranks, determines the number of minority students to be chosen via affirmative action. The generality comes from the fact that, if a lexicographic affirmative action rule admits a stepwise-adjusted–reserves representation, then the thresholds in the representation must be *consecutive*. As we have illustrated above, this is not a requirement for substitutable affirmative action. In fact, in some applications

[17] Note that, by definition of a reserve representation, the reserve number can not exceed the number of top-tier majority students. Therefore, the thresholds must be such that the number of chosen students does not exceed the capacity. Formally, for each $i \in \{0, \ldots, r-1\}$, we must have $r - i \leq q - t_{i+1}$.

where there are many available seats, it might be more natural, and possibly simpler, to have non-consecutive thresholds as in Example 1.

6 Conclusion

We have analyzed choice rules with affirmative action to allocate scarce resources. By their very nature, an important step in designing an institutional choice rule and communicating it with public is to understand how these choice rules respond to changes in the priorities of the applicants or in their types. We have extended the standard formulation of a choice problem by incorporating a type function and a priority ordering, and introduced axioms on how a choice rule should respond to variations in these parameters. Our axiomatic approach uncovers the two intentions behind substitutable affirmative action rules: (1) to reserve some seats for affirmative action and (2) to use them only for the intended, relatively low-priority, beneficiaries. As a consequence of our axiomatic approach, we provide a foundation for lexicographic affirmative action rules. Moreover, we introduce a new class of choice rules that goes beyond lexicographic affirmative action. We believe that our new formulation of a choice problem and the general principles behind our new axioms may facilitate analysis of choice rules with affirmative action in different applications with specific institutional constraints.

A natural extension of our model is to consider multiple minority types. However, it is not trivial how our axioms will extend to this setting and there are multiple different possible extensions depending on the nature of the application. For example, what *priority-compatibility* should require when there are multiple minority types is not straightforward and in particular depends on whether there is an additional priority order over different minority types. Our *monotonicity* axioms naturally extend by considering all minority types as a single minority class, however there are multiple plausible directions to pursue in terms of how a choice rule should respond to changing the type of a minority applicant to another minority type. We leave the analysis of these extensions for future research.

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Appendix A Proofs

We start with introducing some notation. Let $T(S, \tau, \succ)$ denote the top-tier students in S at $(S, \tau, \succ)$. Let $T^m(S, \tau, \succ) = S^m \cap T(S, \tau, \succ)$ and $T^M(S, \tau, \succ) = S^M \cap T(S, \tau, \succ)$ denote the top-tier minority students and majority students, respectively. Let $B(S, \tau, \succ) = S \backslash T(S, \tau, \succ)$ denote the bottom-tier students in S at $(S, \tau, \succ)$. Let $B^m(S, \tau, \succ) = S^m \cap B(S, \tau, \succ)$ and $B^M(S, \tau, \succ) = S^M \cap B(S, \tau, \succ)$ denote the bottom-tier minority students and majority students, respectively.

Given a pair of choice problems $(S, \tau, \succ)$ and $(S', \tau', \succ')$, we say that **top-tiers coincide** if $T(S, \tau, \succ) = T(S', \tau', \succ') = S^*$ and the relative ordering of the students in S^* with respect to $\succ$ and $\succ'$ are the same (when **bottom-tiers coincide** is defined similarly).

Given a choice rule C and a problem $(S, \tau, \succ)$, let $C^A(S, \tau, \succ) = C^m(S, \tau, \succ) \cap B(S, \tau, \succ)$ denote the set of minority students who are chosen via affirmative action.

A q**-ordering** is a priority ordering over a set of q students of two possible types: minority or majority. More precisely, a q-ordering is a problem $\rho = (S^\rho, \tau^\rho, \succ^\rho)$ such that $|S^\rho| = q$. Let Q denote the set of all q-orderings. Given a q-ordering $\rho = (S^\rho, \tau^\rho, \succ^\rho) \in Q$, we will simply write $s \in \rho$ instead of $s \in S$, $s \ \rho \ s'$ instead of $s \succ^\rho s'$, and denote the set of minority and majority students in ρ by ρ_m and ρ_M, respectively. Also, for each $k \in \{1, \ldots, q\}$, let $r_k(\rho)$ denote the k-th ranked student in ρ, i.e., $s = r_k(\rho)$ if and only if $|\{s' \in \rho \setminus \{s\} : s' \ \rho \ s\}| = k - 1$, and $rank(s, \rho)$ denote the rank of student s in ρ.

Given a problem $(S, \tau, \succ)$, we say that the top-tier of S at $(S, \tau, \succ)$ **coincides** with the q-ordering ρ if the sets of students in $T(S, \tau, \succ)$ and ρ, and their relative orderings in $\succ$ and ρ, are the same. Recall that we require $R(S, \tau, \succ)$ depend only on the types and the rankings of the top-tier students. Therefore, we can equivalently define a reserve function as a mapping $R : Q \to \{0, \ldots, q\}$ that associates with each q-ordering a number that is less than or equal to the number of majority students in the q-ordering such that for each $\rho, \rho' \in Q$, if for each $k \in \{1, \ldots, q\}$, the k-th ranked students in ρ and ρ' are of the same type (minority or majority), then $R(\rho) = R(\rho')$.

A.1 Proof of Theorem 1

Proof. ($\Longrightarrow$) Suppose that C is a substitutable affirmative action rule that satisfies *the monotonicity* axioms. We first show, in Lemma 1, that the priority ordering at the bottom-tier is irrelevant for the number of minority students to be chosen.

Lemma 1. *Suppose that C is a substitutable affirmative action rule. Let $(S, \tau, \succ)$ and $(S, \tau, \succ')$ be a pair of choice problems such that $T(S, \tau, \succ)$ coincides with $T(S, \tau, \succ')$. Then, the same number of minority students are chosen at both problems, i.e., $|C^m(S, \tau, \succ)| = |C^m(S, \tau, \succ')|$.*

Proof. By *priority-compatibility*, any majority student who is ranked below rank q at $(S, \tau, \succ)$ or $(S, \tau, \succ')$ is not chosen. Let S' be obtained from S by removing all the majority students who are ranked below rank q at $(S, \tau, \succ)$, i.e., $S' = S^m \cup T(S, \tau, \succ)$. By *capacity-filling* and *substitutability*, $C(S, \tau, \succ) = C(S', \tau|_{S'}, \succ|_{S'})$ and $C(S, \tau, \succ') = C(S', \tau|_{S'}, \succ'|_{S'})$. Now, note that for each $k \in \{1, \ldots, |S'|\}$, the types of the k-th ranked students are the same at problems $(S', \tau|_{S'}, \succ|_{S'})$ and $(S', \tau|_{S'}, \succ'|_{S'})$. Then, it follows from *neutrality* that $|C^m(S', \tau|_{S'}, \succ|_{S'})| = |C^m(S', \tau|_{S'}, \succ'|_{S'})|$. Hence, $|C^m(S, \tau, \succ)| = |C^m(S, \tau, \succ')|$. $\square$

Consider the following reserve function R. For each $\rho \in Q$, consider a problem $(S, \tau, \succ)$ such that $|S^m| = q$ and ρ coincides with the top-tier of S. Let $R(\rho) = |C^A(S, \tau, \succ)|$, the number of minority students who are chosen via affirmative action. Note that, by *neutrality* of C, the reserve function R is well-defined. By Lemma 1, C admits a reserve representation via R. We will show that R satisfies condition A.

Satisfies B1: Let $\rho, \rho' \in Q$ be such that ρ' is obtained from ρ by changing the type of a minority $m \in \rho$. By *MTC*, $R(\rho') \geq R(\rho)$. We will show that $R(\rho') \leq R(\rho) + 1$. Let $(S, \tau, \succ)$ be a problem such that $T(S, \tau, \succ)$ coincides with ρ, the $(q+1)$-th ranked student is a majority student M, and $|S^m| = q$.

Let $S' = S \setminus \{m\}$ and consider the problem $(S', \tau|_{S'}, \succ|_{S'})$. By *substitutability*, $C(S, \tau, \succ) \setminus \{m\} \subseteq C(S', \tau|_{S'}, \succ|_{S'})$, and by *capacity-filling*, at most one new minority student can be chosen via affirmative action at $(S', \tau|_{S'}, \succ|_{S'})$, i.e., $|C^A(S', \tau|_{S'}, \succ|_{S'})| \leq |C^A(S, \tau, \succ)| + 1$.

Now note that M has rank q at $(S', \tau|_{S'}, \succ|_{S'})$. Consider the priority ordering $\succ' \in \Pi(S')$ obtained from $\succ|_{S'}$ by improving the rank of M to the rank of m at $\succ$, while keeping the relative rankings of the other students the same. By *neutrality* and *MPI*,[18] $|C^A(S',\tau|_{S'},\succ')| \leq |C^A(S,\tau|_{S'},\succ|_{S'})|$. Therefore, $|C^A(S',\tau|_{S'},\succ')| \leq |C^A(S,\tau,\succ)|+1$. Now, since $T(S',\tau|_{S'},\succ')$ coincides with ρ' and $T(S,\tau,\succ)$ coincides with ρ, we get $R(\rho') \leq R(\rho)+1$.

Satisfies B2: Let $\rho,\rho' \in Q$ be such that ρ' is an improvement over ρ for a minority student $m \in \rho$. By *MPI*, $R(\rho') \geq R(\rho)$. We will show that $R(\rho') \leq R(\rho)+1$. Let $(S,\tau,\succ)$ be a problem such that $T(S,\tau,\succ)$ coincides with ρ, the $(q+1)$-th ranked student is a majority student M, and $|S^m| = q$. Let $\succ' \in \Pi(S)$ be obtained from $\succ$ by improving the rank of m to its rank at ρ', while keeping the relative rankings of the other students the same. Note that $|C^A(S,\tau,\succ)| = R(\rho)$ and $|C^A(S,\tau,\succ')| = R(\rho')$.

Let $S' = S \setminus \{m\}$, and consider the problems $(S',\tau|_{S'},\succ|_{S'})$ and $(S',\tau|_{S'},\succ'|_{S'})$. By *substitutability*, $C(S,\tau,\succ) \setminus \{m\} \subseteq C(S',\tau|_{S'},\succ|_{S'})$, and by *capacity-filling*, at most one new minority student can be chosen via affirmative action when we move from $(S,\tau,\succ)$ to $(S',\tau|_{S'},\succ|_{S'})$, i.e., $|C^A(S',\tau|_{S'},\succ|_{S'})| \in \{R(\rho),R(\rho)+1\}$. By similar arguments, we have $|C^A(S',\tau|_{S'},\succ'|_{S'})| \in \{R(\rho'),R(\rho')+1\}$. Moreover, $|C^A(S',\tau|_{S'},\succ|_{S'})| = |C^A(S',\tau|_{S'},\succ'|_{S'})|$ since $T(S',\tau|_{S'},\succ|_{S'})$ and $T(S',\tau|_{S'},\succ'|_{S'})$ coincide. Hence, $R(\rho') \leq R(\rho)+1$.

Satisfies B3-i: Let $\rho,\rho' \in Q$ be such that ρ is an improvement over ρ' for a majority student $M \in \rho$. By *MPI*, $R(\rho) \leq R(\rho')$.[19] Next, we show that $R(\rho') \leq R(\rho)+1$. Let $(S,\tau,\succ)$ be a problem such that $T(S,\tau,\succ)$ coincides with ρ, the $(q+1)$-th ranked student is a majority student M', and $|S^m| = q$. Let $\succ' \in \Pi(S)$ be obtained from $\succ$ by moving M down to its rank at ρ', while keeping the relative rankings of the other students the same. Note that $|C^A(S,\tau,\succ')| = R(\rho')$ and $|C^A(S,\tau,\succ)| = R(\rho)$.

Let $S' = S \setminus \{M\}$ and consider the problems $(S',\tau|_{S'},\succ|_{S'})$ and $(S',\tau|_{S'},\succ'|_{S'})$. By *substitutability*, $C(S,\tau,\succ) \setminus \{M\} \subseteq C(S',\tau|_{S'},\succ|_{S'})$, and by *capacity-filling*, at most one new minority student can be chosen via affirmative action when we move from $(S,\tau,\succ)$ to $(S',\tau|_{S'},\succ|_{S'})$, i.e., $|C^A(S',\tau|_{S'},\succ|_{S'})| \in \{R(\rho),R(\rho)+1\}$. By similar arguments, we have $|C^A(S',\tau|_{S'},\succ'|_{S'})| \in \{R(\rho'),R(\rho')+1\}$. Moreover, $|C^A(S',\tau|_{S'},\succ|_{S'})| = |C^A(S',\tau|_{S'},\succ'|_{S'})|$, since $T(S',\tau|_{S'},\succ|_{S'})$ and $T(S',\tau|_{S'},\succ'|_{S'})$ coincide. Hence, $R(\rho') \leq R(\rho)+1$.

Satisfies B3-ii: Suppose that ρ' is obtained from ρ by worsening the priority rank of

[18] Note that, under *neutrality*, improvement of the priority order of M can be expressed as a consecutive worsening of the priority orderings of several chosen minority students.

[19] Like before, under *neutrality*, improvement of the priority order of M can be expressed as a consecutive worsening of the priority orderings of several chosen minority students.

a majority student $M \notin C(S,\tau,\succ)$, where $(S,\tau,\succ)$ is a problem such that $T(S,\tau,\succ)$ coincides with ρ. Let $(S,\tau,\succ')$ be a problem such that $T(S,\tau,\succ')$ coincides with ρ'. By *MPI*, $R(\rho) \leq R(\rho')$. Then, since ρ is an improvement over ρ' for M and $M \notin C(S,\tau,\succ)$, we have $M \notin C(S,\tau,\succ')$.

Let $S' = S \setminus \{M\}$ and consider the problems $(S',\tau|_{S'},\succ|_{S'})$ and $(S',\tau|_{S'},\succ'|_{S'})$. Then, by *substitutability* and *capacity-filling*, $C(S,\tau,\succ) = C(S',\tau|_{S'},\succ|_{S'})$ and $|C^A(S',\tau|_{S'},\succ|_{S'})| = R(\rho)$. By similar arguments, we have $|C^A(S',\tau|_{S'},\succ'|_{S'})| = R(\rho')$. Since we showed, in proving that R satisfies B3-i, that $|C^A(S',\tau|_{S'},\succ|_{S'})| = |C^A(S',\tau|_{S'},\succ'|_{S'})|$, we get $R(\rho') = R(\rho)$.

$(\Longleftarrow)$ Suppose that C admits a reserve representation via a reserve function R satisfying conditions $B1 - B3$. It is straightforward to see that C satisfies all the axioms except for *substitutability*. To see that C satisfies *substitutability*, let $(S,\tau,\succ)$ be a problem and let $s \in C(S,\tau,\succ)$, $s' \in S \setminus \{s\}$. Let $S' = S \setminus \{s'\}$ and consider the problem $(S',\tau|_{S'},\succ|_{S'})$. We want to show that $s \in C(S',\tau|_{S'},\succ|_{S'})$. Let $\rho \in Q$ be the q-ordering that coincides with $T(S,\tau,\succ)$ and $\rho' \in Q$ be the q-ordering that coincides with $T(S',\tau|_{S'},\succ|_{S'})$.

If $s' \in B(S,\tau,\succ)$, then $\rho = \rho'$ and $R(\rho) = R(\rho')$, and clearly $s \in C(S',\tau|_{S'},\succ|_{S'})$. So suppose that $s' \in T(S,\tau,\succ)$, equivalently $s' \in \rho$. Then, there exists $s^* \in B(S,\tau,\succ)$ that moves to the top when s' is removed from S, i.e., $\{s^*\} = T(S',\tau|_{S'},\succ|_{S'}) \setminus T(S,\tau,\succ)$.

Case 1: $s' \in S^m$ and $s^* \in S^m$. By Condition $B2$, $R(\rho') \in \{R(\rho)-1, R(\rho)\}$. Now, if $s \in S^M$, then $s \in C(S',\tau|_{S'},\succ|_{S'})$, since $R(\rho') \leq R(\rho)$ implies no more minority student is chosen via affirmative action. If $s \in S^m$, then $s \in C(S',\tau|_{S'},\succ|_{S'})$. To see this, note that $R(\rho') \geq R(\rho) - 1$ implies there is at most one minority student who is chosen via affirmative action at $(S,\tau,\succ)$ but is not chosen via affirmative action at $(S',\tau|_{S'},\succ|_{S'})$. Since, we already have s^*, who is now chosen from the top-tier, as such a student, there can not be a second one.

Case 2: $s' \in S^m$ and $s^* \in S^M$. By Conditions $B1$ and $B2$, $R(\rho') \in \{R(\rho)-1, R(\rho), R(\rho)+1\}$.[20] Also, by Conditions $B1$ and $B3-i$, $R(\rho') \in \{R(\rho), R(\rho)+1, R(\rho)+2\}$.[21] Hence, $R(\rho') \in \{R(\rho), R(\rho)+1\}$.

[20] By Condition B2, moving s' down to rank q decreases R by at most one, and then, by Condition B1, changing the type of s' to majority increases R by at most one.

[21] By Condition B1, changing the type of s' to majority increases R by at most one, and then, by Condition $B3-i$, moving s' down to rank q increases R by at most one.

Now, if $s \in S^m$, then clearly $s \in C(S', \tau|_{S'}, \succ|_{S'})$ since $R(\rho') \geq R(\rho)$. If $s \in S^M$, then $s \in C(S', \tau|_{S'}, \succ|_{S'})$. To see this, first note that $R(\rho') \leq R(\rho) + 1$ implies at most one more minority student can be chosen via affirmative action at $(S', \tau|_{S'}, \succ|_{S'})$. Then, since a minority student, s', who was previously chosen from $T(S, \tau, \succ)$ is now removed, all previously chosen majority students are still chosen.

Case 3: $s' \in S^M$ and $s^* \in S^m$. By Conditions $B1$ and $B3-i$, $R(\rho') \in \{R(\rho) - 1, R(\rho), R(\rho) + 1\}$. Also, by Conditions $B1$ and $B2$, $R(\rho') \in \{R(\rho) - 2, R(\rho) - 1, R(\rho)\}$. Hence, $R(\rho') \in \{R(\rho) - 1, R(\rho)\}$.

Now, if $s \in S^M$, then $s \in C(S', \tau|_{S'}, \succ|_{S'})$, since $R(\rho') \leq R(\rho)$ implies no more minority student is chosen via affirmative action. If $s \in S^m$, then $s \in C(S', \tau|_{S'}, \succ|_{S'})$. To see this, first note that $R(\rho') \geq R(\rho) - 1$ implies there is at most one minority student who is chosen via affirmative action at $(S, \tau, \succ)$ but is not chosen via affirmative action at $(S', \tau|_{S'}, \succ|_{S'})$. Since, we already have s^*, who is now chosen from the top-tier, as such a student, there can not be a second one.

Case 4: $s' \in S^M$ and $s^* \in S^M$. Suppose that $s' \notin C(S, \tau, \succ)$. Since R represents C, by Condition B, $R(\rho') = R(\rho)$. It follows that $C(S, \tau, \succ) = C(S', \tau|_{S'}, \succ|_{S'})$, which implies $s \in C(S', \tau|_{S'}, \succ|_{S'})$.

Suppose that $s' \in C(S, \tau, \succ)$. Then, by Condition $B3-i$, $R(\rho') \in \{R(\rho), R(\rho) + 1\}$. Now, if $s \in S^m$, then $s \in C(S', \tau|_{S'}, \succ|_{S'})$, since $R(\rho') \geq R(\rho)$. If $s \in S^M$, then $s \in C(S', \tau|_{S'}, \succ|_{S'})$. To see this, first note that $R(\rho') \leq R(\rho) + 1$ implies at most one less majority student is chosen at $(S', \tau|_{S'}, \succ|_{S'})$. Since a majority student, s', who was chosen at $(S, \tau, \succ)$ is now removed, all other previously chosen majority students are still chosen □

A.2 Proof of Theorem 2

Proof. We first show that a choice rule that admits a lexicographic representation also admits a reserve representation.

Lemma 2. *If a choice rule C admits a lexicographic representation via a lexicographic order l, then C admits a reserve representation via a reserve function R defined through the following algorithm.*

Lexicographic-to-reserve algorithm: *For each $\rho \in Q$,* *Step 1: Let t_1 be the first rank such that l and ρ differ in types; if there is no such rank, then stop. Let $m_1 = r_{t_1}(\rho)$, and proceed as follows:*

(i) If $l(t_1) = 0$, then change the type of m_1 to obtain ρ^1.

(ii) If $l(t_1) = 1$, then improve the priority rank of m_1 to rank t_1 to obtain ρ^1.

Steps $k \geq 2$: Let t_k be the first rank such that l and ρ^{k-1} differ in types; if there is no such rank, then stop (note that $t_k > t_{k-1}$). Let $m_k = r_{t_k}(\rho^{k-1})$ and proceed as follows:

(i) If $l(t_k) = 0$, then change the type of m_k to obtain ρ^k.

(ii) If $l(t_k) = 1$, then improve the priority rank of m_k to rank t_k to obtain ρ^k.

Note that the algorithm stops at most in q steps. Let ρ^* be the final q-ordering. Set $R(\rho) = \sum_{k\in\{1,\dots,q\}} l(k) - |\rho^*_m|$.

Proof. Let $(S, \tau, \succ)$ be an arbitrary problem. Let $\rho \in Q$ coincide with $T(S, \tau, \succ)$. Note that in the *lexicographic-to-reserve algorithm* at ρ, if the type of a minority student is changed at any step, then that minority student must be allocated an open seat at the lexicographic procedure at $(S, \tau, \succ)$; if the priority rank of a minority student is improved at any step, then that minority student must be allocated a reserve seat at the Lexicographic procedure at $(S, \tau, \succ)$. Therefore, ρ^*_m is the set of minority students in the top-tier who are allocated reserve seats. Now, since there are $\sum_{k\in\{1,\dots,q\}} l(k)$ reserve seats in the lexicographic order, $\sum_{k\in\{1,\dots,q\}} l(k) - |\rho^*_m|$ minority students from the bottom-tier will be chosen (if there are fewer minority students at the bottom-tier, all will be chosen). Hence, C admits a reserve representation via R. $\square$

We are now ready to prove Theorem 2.

($\Longrightarrow$) Suppose that C substitutable affirmative action rule that satisfies the *monotonicity* axioms and *CTC*. By Proposition 1, C admits a reserve representation via a reserve function R satisfying Condition B. In the following lemma, by using the language of reserve representation, we show that if improving the priority of a chosen minority student is *effective*, then a following change in the type of a lower ranked chosen minority student is also *effective* (unless all minority students are already chosen). In the vein of *CTC*, this requirement can be thought of as consistency in effective priority improvements.

Lemma 3. *Let $\rho, \rho', \rho'' \in Q$ such that ρ' is an improvement over ρ for a minority student s and ρ'' is obtained from ρ' by changing the type of another minority student s' such that $s \ \rho \ s'$. If $R(\rho') = R(\rho) + 1$ then $R(\rho'') = R(\rho') + 1$.*

Proof. Since $R(\rho') = R(\rho) + 1$, by *neutrality*, there exists a majority student M such that the unit increase in the reserve number occurs, when s moves right on top of M. Since, for each problem we require the reserve number depend only on the types and the rankings of the top-tier students, we can assume without loss of generality that s and M are ranked consecutively at ρ and ρ' (in reverse orders).

Next, let ρ^* be the q-ordering obtained from ρ by changing the type of M into minority. Then, since R satisfies $B1$, we have $R(\rho) \geq R(\rho^*)$. Therefore, $R(\rho') > R(\rho^*)$. Now, note that ρ' is obtained from ρ^* by changing the type of s. Since $R(\rho') > R(\rho^*)$, this type change is *effective*. Then, since ρ'' is obtained from ρ' by changing the type of s', by *CTC*, this type must also be *effective*. Thus, we conclude that $R(\rho'') = R(\rho') + 1$. □

Lemma 4 (Type-supermodularity)**.** *Let $\rho, \rho', \rho'' \in Q$. If ρ' is obtained from ρ by changing the type of a minority student s', ρ'' is obtained from ρ by changing the type of another minority student s'' with $s' \succ s''$, and ρ''' is obtained from ρ by changing the types of both s' and s'', then*

$$R(\rho''') - R(\rho) \geq [R(\rho') - R(\rho)] + [R(\rho'') - R(\rho)]. \tag{TS}$$

Proof. Since R satisfies condition $B1$, we have $R(\rho'), R(\rho'') \in \{R(\rho), R(\rho) + 1\}$. Also by $B1$, we have $R(\rho''') \in \{R(\rho'), R(\rho') + 1\}$ and $R(\rho''') \in \{R(\rho''), R(\rho'') + 1\}$. Note that these already imply that if $[R(\rho') - R(\rho)] + [R(\rho'') - R(\rho)] \in \{0, 1\}$, then TS holds. Suppose that $R(\rho') - R(\rho) = 1$ and $R(\rho'') - R(\rho) = 1$, but $R(\rho''') - R(\rho) < 2$.

Let $(S, \tau, \succ)$ be a problem such that $T(S, \tau, \succ)$ coincides with ρ and $B(S, \tau, \succ)$ consists of $R(\rho) + 2$ minority students, where the lowest ranked two minority students are m_1 and m_2, with $m_1 \succ m_2$. Note that $C^m(S, \tau, \succ) = S^m \setminus \{m_1, m_2\}$. Now, let τ' be obtained from τ by changing the type of s'. Note that $T(S, \tau', \succ)$ coincides with ρ' and $m_1 \in C^m(S, \tau', \succ)$, i.e., changing the type of s' is *effective* at $(S, \tau', \succ)$. Let τ'' be obtained from τ' be changing the type of s''. By *CTC*, this change should still be effective, and therefore $m_2 \in C^m(S, \tau'', \succ)$. Since $T(S, \tau'', \succ)$ coincides with ρ''', this contradicts that $R(\rho''') - R(\rho) < 2$. Hence, TS holds. □

Lemma 5 (Priority-supermodularity)**.** *Let $\rho, \rho', \rho'' \in Q$. If ρ' is an improvement over ρ for a minority student s and ρ'' is obtained from ρ by changing the type of another minority student s' such that $s \succ s'$, and ρ''' is obtained from ρ' by changing the type of s', then*

$$R(\rho''') - R(\rho) \geq [R(\rho') - R(\rho)] + [R(\rho'') - R(\rho)]. \tag{PS}$$

Proof. Follows by similar arguments as in the proof of Lemma 4, with the modifications that Condition $B2$ is invoked instead of $B1$ and Lemma 3 is invoked instead of CTC. $\square$

Next, we introduce a definition that is crucial for the rest of the proof. A q-ordering $\rho \in Q$ is **critical** if changing the type or improving the priority order of any minority student increases the reserve.[22] That is, for each $\rho' \in Q$ that is obtained from ρ by changing the type of a minority student or is an improvement over ρ for a minority student, we have $R(\rho') = R(\rho) + 1$. Let $\mathcal{C}$ denote the set of all critical q-orderings.

Lemma 6. *Let $\rho \in \mathcal{C}$ and $k \in \{1, \ldots, q\}$. If $\rho' \in Q$ is obtained from ρ by changing the type of the lowest ranked minority student, then $\rho' \in \mathcal{C}$ as well.*

Proof. Note that at ρ', changing the type or improving the priority order of any minority student still increases the reserve by *type-supermodularity* and *priority-supermodularity*, respectively. Therefore, $\rho' \in \mathcal{C}$. $\square$

Lemma 7. *For each $\rho, \rho' \in \mathcal{C}$ and $t \in \{1, \ldots, q\}$, if t is the first rank at which ρ has a minority student while ρ' has a majority student, then ρ' does not have any minority student after rank t, i.e., for each rank $k > t$, $r_k(\rho') \in \mathcal{S}^M$.*

Proof. Suppose otherwise. Let $k^* > t$ be the lowest rank such that $r_{k^*}(\rho') \in \mathcal{S}^m$. Let ρ^* be obtained from ρ by changing the type of each minority student m with rank $k > t$ in ρ. Let ρ^{**} be obtained from ρ' by changing the type of each minority student m with rank $k > k^*$ in ρ'. By successive application of Lemma 6, such that the type of the lowest ranked minority student is changed at each step, we get $\rho^*, \rho^{**} \in \mathcal{C}$.

Next, we show that $R(\rho^*) = R(\rho^{**})$. To see this, let ρ'' be obtained from ρ^* by changing the type the minority student with rank t in ρ^*. Since $\rho^* \in \mathcal{C}$, $R(\rho'') = R(\rho^*) + 1$. Next

[22]Note that if ρ does not have any minority student, then it is critical.

note that ρ'' can be also obtained from ρ^{**} by changing the type of the minority student with rank k^* in ρ^*. Since $\rho^{**} \in \mathcal{C}$, $R(\rho'') = R(\rho^{**}) + 1$. Therefore, $R(\rho^*) = R(\rho^{**})$.

Now, let ρ^{***} be obtained from ρ^{**} by improving the rank of $r_{k^*}(\rho')$ to t. Since $\rho^{**} \in \mathcal{C}$, $R(\rho^{***}) = R(\rho^{**}) + 1$. Since for each $k \in \{1, \ldots, q\}$, the k-th ranked students in ρ^* and ρ^{***} are of the same type, it follows from *neutrality* that $R(\rho^*) = R(\rho^{***})$, which contradicts to $R(\rho^*) = R(\rho^{**})$. □

Lemma 8. *For each pair $\rho, \rho' \in \mathcal{C}$, if ρ and ρ' have the same number of minority students, i.e., $|\rho_m| = |\rho'_m|$, then ρ and ρ' have the same type configuration, i.e., for each $k \in \{1, \ldots, q\}$, the k-th ranked students in ρ and ρ' are of the same type.*

Proof. Follows from Lemma 7. □

Lemma 9. *The sum of the number of minority students in a critical q-ordering and the reserve of the critical q-ordering is constant among all critical q-orderings. That is, for each $\rho, \rho' \in \mathcal{C}$, $|\rho_m| + R(\rho) = |\rho'_m| + R(\rho')$.*

Proof. Let $\rho, \rho' \in \mathcal{C}$. If $|\rho_m| = |\rho'_m|$, then it follows from Lemma 8 that $R(\rho) = R(\rho')$. So suppose, without loss of generality, that $|\rho_m| > |\rho'_m|$. Let $\rho^* \in Q$ be obtained from ρ by changing the types of all minority students, and $\rho^{**} \in Q$ be obtained from ρ' by changing the types of all minority students. Note that, by Lemma 6, if we start with a critical q-ordering ρ^0, then by changing the type of the lowest ranked minority student, we obtain another critical q-ordering ρ^1 with $R(\rho^1) = R(\rho^0) + 1$. Therefore, $\rho^*, \rho^{**} \in \mathcal{C}$ and $R(\rho^*) = R(\rho) + |\rho_m|$ and $R(\rho^{**}) = R(\rho') + |\rho'_m|$. Now, since ρ^* and ρ^{**} have the same type configuration, we get $R(\rho^*) = R(\rho^{***})$. It follows that $R(\rho) + |\rho_m| = R(\rho') + |\rho'_m|$. □

Constructing the lexicographic order: Let κ denote the constant such that for each $\rho \in \mathcal{C}$, $|\rho_m| + R(\rho) = \kappa$. Let $l : \{1, \ldots, q\} \to \{0, 1\}$ be the function defined as follows: for each $k \in \{1, \ldots, q\}$, $l(k) = 1$ if and only if there exists a $\rho \in \mathcal{C}$ such that $\tau(r_k(\rho)) = 1$. Note that for each $\rho \in \mathcal{C}$ that has exactly κ minority students, which is the maximum number of minority students a critical q-ordering can have, by Lemma 9, for each $k \in \{1, \ldots, q\}$, $l(k) = \tau(r_k(\rho))$. Therefore, $\sum_{k \in \{1, \ldots q\}} l(k) = \kappa$.

Lemma 10. *Let $\rho \in Q$. If there exists $t \in \{1, \ldots, q\}$ such that for each $k < t$, $\tau(r_k(\rho)) = l(k)$ and for each $k \geq t$, $\tau(r_k(\rho)) = 0$, then $\rho \in \mathcal{C}$.*

Proof. Let $\rho' \in Q$ be obtained from ρ such that for each $k \geq t$, the type of the majority student $r_k(\rho)$ is converted to minority if and only if $l(k) = 1$. It follows from the construction of l that $\rho' \in \mathcal{C}$. Then, by successive application of Lemma 6, such that the type of the lowest ranked minority student in ρ' is changed at each step until the minority student with rank k is reached, we get $\rho \in \mathcal{C}$. $\square$

Lemma 11. *Let $\rho \in Q$ and $m \in \rho_m$ with $m = r_t(\rho)$ for some $t \in \{1, \dots, q\}$ be such that*

- *$l(t) = 0$, and*
- *the type configuration of ρ coincides with l up to rank t, i.e., for each $k \in \{1, \dots, t-1\}$, $\tau(r_k(\rho)) = l(k)$.*

If $\rho' \in Q$ is obtained from ρ by changing the type of m, then $R(\rho') = R(\rho)$.

Proof. By contradiction suppose that $R(\rho') = R(\rho) + 1$. Let ρ^* be obtained from ρ by changing the types of all minority students whose rank are greater than t, i.e., for each $k > t$, $\tau(\rho^*_k) = 0$. Let ρ^{**} be obtained from ρ^* by changing the type of m. Since $R(\rho') = R(\rho) + 1$, by Lemma 4, successive application of TS implies $R(\rho^{**}) \geq R(\rho^*) + 1$. Since R satisfies $B1$, by Proposition 1, we also have $R(\rho^{**}) \leq R(\rho^*) + 1$. Therefore, $R(\rho^{**}) = R(\rho^*) + 1$.

Now, note that for each $k \in \{1, \dots, q\}$, $k < t$ implies $\tau(r_k(\rho^{**})) = l(k)$ and $k \geq t$ implies $\tau(r_k(\rho^{**})) = 0$. Then, it follows from Lemma 10 that $\rho^{**} \in \mathcal{C}$. Next, we investigate two cases separately.

Case 1: For each $k > t$, $l(k) = 0$. Since $\sum_{k \in \{1,\dots q\}} l(k) = \kappa$, this implies $\sum_{k \in \{1,\dots t\}} l(k) = \kappa$. Therefore, $|\rho^*_m| = \kappa + 1$. Consider ρ^{***} which is obtained from ρ^* by successively changing the types of all minority students such that the type of the lowest ranked minority student is changed at each step.

Note that at the first step, the type of m is changed and $\rho^{**} \in \mathcal{C}$ is obtained. As we showed before, $R(\rho^{**}) = R(\rho^*) + 1$. By proceeding similarly, at each following step, since we change the type of a minority student at a critical ordering, the reserve increases by one, and by Lemma 6 we obtain another critical ordering. Hence, we eventually get $\rho^{***} \in \mathcal{C}$ with $R(\rho^{***}) = R(\rho^*) + \kappa + 1$. But since $\rho^{***} \in \mathcal{C}$, by Lemma 9, $R(\rho^{***}) \leq \kappa$. Thus, we get a contradiction.

Case 2: There exists $k^* > t$ such that $l(k^*) = 1$. Suppose, without loss of generality, that k^* is the smallest such rank, i.e, for each $k \in \{t, \dots, k^* - 1\}$, $l(k) = 0$.

Consider $\rho^{*\downarrow}$ which is obtained from ρ^* by moving m down to rank k^*. First, note that for each $k \in \{1, \dots, q\}$, $k < k^* + 1$ implies $\tau(r_k(\rho^{*\downarrow})) = l(k)$ and $k \geq k^* + 1$ implies $\tau(r_k(\rho^{*\downarrow})) = 0$. Then, it follows from Lemma 10 that $\rho^{*\downarrow} \in \mathcal{C}$. Since $\rho^{**}, \rho^{*\downarrow} \in \mathcal{C}$ and $|\rho^{**}_m| = |\rho^{*\downarrow}_m| - 1$, it follows from Lemma 9 that $R(\rho^{**}) = R(\rho^{*\downarrow}) + 1$.

Since $\rho^{*\downarrow} \in \mathcal{C}$ and ρ^* is obtained from $\rho^{*\downarrow}$ by improving the priority rank of a minority student, we have $R(\rho^*) = R(\rho^{*\downarrow}) + 1$. But, since we also have $R(\rho^{**}) = R(\rho^{*\downarrow}) + 1$, this implies $R(\rho^{**}) = R(\rho^*)$, a contradiction. $\square$

Lemma 12. *Let $\rho \in Q$ and $m \in \rho_m$ with $m = r_t(\rho)$ for some $t \in \{1, \dots, q\}$ be such that*

- *there exists $k^* \in \{1, \dots, t\}$ such that the type configuration of ρ coincides with l up to rank k^*, i.e., for each $k \in \{1, \dots, k^* - 1\}$, $\tau(r_k(\rho)) = l(k)$,*
- *$\tau(r_{k^*}(\rho)) = 0$ and $l(k^*) = 1$, and*
- *for each rank $k \in \{k^* + 1, \dots, t - 1\}$, $\tau(r_k(\rho)) = 0$.*

If ρ' is obtained from ρ by improving the priority rank of m to rank k^*, then $R(\rho') = R(\rho)$.

Proof. By contradiction, suppose that $R(\rho') = R(\rho) + 1$. Let ρ^* be obtained from ρ by changing the types of all minority students whose rank are greater than t, i.e., for each $k > t$, $\tau(\rho^*_k) = 0$. Let $\rho^{*\uparrow}$ be obtained from ρ^* by improving the priority rank of m to rank k^*.

First, note that since $R(\rho') = R(\rho) + 1$, by Lemma 5, successive application of PS implies $R(\rho^{*\uparrow}) \geq R(\rho^*) + 1$. Since R satisfies $B2$, by Proposition 1, we also have $R(\rho^{*\uparrow}) \leq R(\rho^*) + 1$. Therefore, $R(\rho^{*\uparrow}) = R(\rho^*) + 1$. Moreover, note that for each $k \in \{1, \dots, q\}$, $k < k^* + 1$ implies $\tau(r_k(\rho^{*\uparrow})) = l(k)$ and $k \geq k^* + 1$ implies $\tau(r_k(\rho^{*\uparrow})) = 0$. Then, it follows from Lemma 10 that $\rho^{*\uparrow} \in \mathcal{C}$.

Finally, let ρ^{**} be obtained from ρ^* by changing the type of m. Note that ρ^{**} can also be obtained from $\rho^{*\uparrow}$ by changing the type of m. Since $\rho^{*\uparrow} \in \mathcal{C}$, it follows that $R(\rho^{**}) = R(\rho^{*\uparrow}) + 1$. Since we have $R(\rho^{*\uparrow}) = R(\rho^*) + 1$, this implies $R(\rho^{**}) = R(\rho^*) + 2$. But by Proposition 1, this contradicts that R satisfies $B1$. $\square$

Now, consider the choice rule C' that admits a lexicographic representation via l. We conclude by showing that C coincides with C'. By Proposition 2, C' admits a reserve representation via R' which is defined through the lexicographic-to-reserve algorithm.

Let $(S, \tau, \succ)$ be an arbitrary problem. Let $\rho \in Q$ coincide with $T(S, \tau, \succ)$. Consider the steps of the lexicographic-to-reserve algorithm at ρ.

Step 1: Let t_1 be the first rank such that l and ρ differ in types, i.e., $l(t_1) \neq \tau(r_{t_1}(\rho))$. Let $m_1 = r_{t_1}(\rho)$, and then proceed as follows:

(i) If $l(t_1) = 0$, then change the type of m_1 to obtain ρ^1. Note that, by Lemma 11, $R(\rho^1) = R(\rho)$.

(ii) If $l(t_1) = 1$, then improve the priority rank of m_1 to rank t_1 to obtain ρ^1. Note that, by Lemma 12, $R(\rho^1) = R(\rho)$.

Steps $k \geq 2$: Let t_k be the first rank such that l and ρ^{k-1} differ in types (note that $t_k > t_{k-1}$). Let $m_k = r_{t_k}(\rho^{k-1})$, and then proceed as follows:

(i) If $l(t_k) = 0$, then change the type of m_k to obtain ρ^k. Note that, by Lemma 11, $R(\rho^k) = R(\rho^{k-1})$.

(ii) If $l(t_k) = 1$, then improve the priority rank of m_k to rank t_k to obtain ρ^k. Note that, by Lemma 12, $R(\rho^k) = R(\rho^{k-1})$.

Let ρ^* be the final value. By the construction of the algorithm, we have $R(\rho) = R(\rho^*)$. Moreover, observe that $\rho^* \in \mathcal{C}$ and therefore $R(\rho) = R(\rho^*) = \kappa - |\rho^*_m|$. By Lemma 2, $R'(\rho) = \kappa - |\rho^*_m|$. Hence, $R(\rho) = R'(\rho)$, and since ρ was arbitrary, C coincides with C'.

Uniqueness: To see that the lexicographic representation is unique, by contradiction suppose C has two lexicographic representations via distinct l and l'. First, note that $\sum_{k \in \{1,\ldots,q\}} l(k) = \sum_{k \in \{1,\ldots,q\}} l'(k)$ since otherwise, it is easy to construct a problem where there is no minority student at the top-tier and different numbers of minority students are chosen by following the lexicographic procedures with respect to l and l'. Let ρ and ρ' be two q-orderings that coincide with l and l', respectively, that is, for each $k \in \{1, \ldots, q\}$, $\tau(r_k(\rho)) = 1$ if and only if $l(k) = 1$ and $\tau(r_k(\rho')) = 1$ if and only if $l'(k) = 1$. Note that $\rho, \rho' \in \mathcal{C}$ and $|\rho_m| = |\rho'_m|$. But then it follows from Lemma 8 that ρ and ρ' have the same type configuration, which implies that l and l' are the same, a contradiction.

$(\Longleftarrow)$ Suppose that C admits a lexicographic representation via a lexicographic order l. It is easy to see that C is a substitutable affirmative action rule that satisfies the *monotonicity* axioms.

Next, we show that C satisfies *CTC*. Let $(S,\tau,\succ)$ be a problem and $s,s'\in C^m(S,\tau,\succ)$ be such that $s\succ s'$. Let τ' be obtained from τ by changing the type of s, τ'' be obtained from τ by changing the types of both s and s'. Suppose that $C^m(S,\tau',\succ)\setminus C^m(S,\tau,\succ)\neq\emptyset$. First note that in the lexicographic procedure at $(S,\tau,\succ)$, s must be chosen at a step, say Step k, at which $l(k)=1$ (i.e., the affirmative priority ordering $\succ^a$ is used), since otherwise s would be chosen also at Step k at $(S,\tau',\succ)$ and thus we obtain the contradiction that $C^m(S,\tau',\succ)=C^m(S,\tau,\succ)$. Moreover, for any minority student $m\in C^m(S,\tau,\succ)$ such that $s\succ m$, in the lexicographic procedure at $(S,\tau,\succ)$, m must also be chosen at a step, say Step k', at which $l(k')=1$ since otherwise, for each $k''>k$ such that $l(k'')=1$, the same minority students will be chosen at steps k'' of the lexicographic procedures at $(S,\tau,\succ)$ and at $(S,\tau',\succ)$, contradicting that $C^m(S,\tau',\succ)\setminus C^m(S,\tau,\succ)\neq\emptyset$.

Finally, since $s\succ s'$, in the lexicographic procedure at $(S,\tau,\succ)$, s' is chosen at a step, say Step k', at which $l(k')=1$ and for any $m\in C^m(S,\tau,\succ)$ such that $s'\succ m$, m is chosen at a step, say Step $k''>k'$, at which $l(k'')=1$. But then either $C^m(S,\tau'',\succ)\setminus C^m(S,\tau',\succ)\neq\emptyset$ or $C^m(S,\tau'',\succ)=S^m$. □

Appendix B Independence of Axioms

We show the independence of axioms for Theorem 2.

Example 2 (Violating only *capacity-filling*)**.** Let $\mathcal{S}=\{s_1,s_2,s_3\}$ and $q=2$. Consider the choice rule C such that for each problem $(S,\tau,\succ)$, $C(S,\tau,\succ)$ is a singleton consisting of the highest priority student. It is easy to see that C satisfies all axioms in the statement of Theorem 2 but *capacity-filling*.

Example 3 (Violating only *neutrality*)**.** Let $\mathcal{S}=\{s_1,s_2,s_3\}$ and $q=2$. Consider the choice rule C such that for each problem $(S,\tau,\succ)$, $C(S,\tau,\succ)$ is obtainable as follows: if $s_1\in S$ and $\tau(s_1)=1$, then first choose s_1 and then choose the highest priority students until all seats are filled or no student is left; otherwise, choose the highest priority students until all seats are filled or no student is left. It is easy to see that C satisfies all axioms in the statement of Theorem 2 but *neutrality*.

Example 4 (Violating only *priority compatibility*)**.** Let $\mathcal{S}=\{s_1,s_2,s_3\}$ and $q=2$. Consider the choice rule C such that for each problem $(S,\tau,\succ)$, $C(S,\tau,\succ)$ is obtainable as follows: if the lowest priority student $s \in S$ is a majority student, i.e., $\tau(s) = 0$, then first choose s and then choose the highest priority students until all seats are filled or no student is left; otherwise, choose the highest priority students until all seats are filled or no student is left. It is easy to see that C satisfies all axioms in the statement of Theorem 2 but *priority compatibility*.

Example 5 (Violating only *substitutability*)**.** Let $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$ and $q = 2$. Consider the choice rule C such that for each problem $(S, \tau, \succ)$, $C(S, \tau, \succ)$ is obtainable as follows: if $S = \mathcal{S}$, then first choose the highest priority minority student and then choose the highest priority students until all seats are filled or no student is left; otherwise, choose the highest priority students until all seats are filled or no student is left. It is easy to see that C satisfies all axioms in the statement of Theorem 2 but *substitutability*.

Example 6 (Violating only *MTC*)**.** Let $\mathcal{S} = \{s_1, s_2, s_3, s_4, s_5\}$ and $q = 4$. Consider the choice rule C that admits a reserve representation via the following R. For each $\rho \in Q$,

i. if ρ does not include any minority student, then $R(\rho) = 1$,
ii. if ρ includes one minority student, say m, then $R(\rho) = 1$ if $rank(m, \rho) \leq 3$ and $R(\rho) = 0$ otherwise,
iii. if ρ includes two minority students, then $R(\rho) = 1$,
iv. otherwise, i.e., if ρ includes three or more minority students, $R(\rho) = 0$.

It is easy to see that C is a substitutable affirmative action rule. To see that C satisfies *CTC*, note that if changing the type of a chosen minority student is *effective*, then all minority students must have been chosen. To see that C satisfies *MPI*, the critical case is ρ includes two minority students and the bottom-ranked student m is a minority. Then, by iii., $R(\rho) = 1$ and m is chosen. Now, suppose that ρ' is obtained from ρ by improving the priority rank of m. If the bottom-ranked student is again a minority, then, by iii., $R(\rho') = 1$ and he is chosen; otherwise ρ' includes three minority students and, by iv., chosen minorities remain to be chosen.

To see that C violates *MTC*, let $(S, \tau, \succ)$ be the problem such that $S = \mathcal{S}$, $\tau(s_1) = \tau(s_3) = 0$ and $\tau(s_2) = \tau(s_4) = \tau(s_5) = 1$, and $s_1 \succ \cdots \succ s_5$. Note that $s_5 \in C(S, \tau, \succ)$. Let τ' be obtained from τ by changing the type of s_2. Note that $s_5 \notin C(S, \tau', \succ)$.

Example 7 (Violating only *MPI*)**.** Let $S = \{s_1, s_2, s_3, s_4, s_5\}$ and $q = 4$. Consider the choice rule C that admits a reserve representation via the following R. For each $\rho \in Q$,

i. if ρ does not include any minority student or includes one minority student, then $R(\rho) = 1$,

ii. if ρ includes two minority students, say m_1 and m_2 such that $m_1 \; \rho \; m_2$. Then $R(\rho) = 1$ if $rank(m_1, \rho) = 1$, or $rank(m_1, \rho) = 3$ and $rank(m_2, \rho) = 4$; and $R(\rho) = 0$ otherwise,

iii. otherwise, i.e., if ρ includes three or more minority students, then $R(\rho) = 0$.

It is easy to see that C is a substitutable affirmative action rule. To see that C satisfies *CTC*, note that if improving the priority of a chosen minority student or changing his type is *effective*, then all minority students must have been chosen. To see that C satisfies *MTC*, note that the reserve number never decreases in response to a type change of a chosen minority student.

To see that it violates *MPI*, let $(S, \tau, \succ)$ be the problem such that $S = \mathcal{S}$, $\tau(s_1) = \tau(s_2) = 0$ and $\tau(s_3) = \tau(s_4) = 1$, and $s_1 \succ \cdots \succ s_5$. Let $\succ'$ be obtained from $\succ$ by improving the priority of s_3 to the second rank. Note that $s_5 \in C^m(S, \tau, \succ) \setminus C^m(S, \tau, \succ')$.